Inverse Reinforcement Learning via Inverse Optimization

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Abstract

Inverse reinforcement learning (IRL) and inverse optimization (IO) for Markov decision processes (MDPs) have developed independently in the literature, despite addressing the same problem. We establish the relationship between the IO framework for MDPs and the convex-analytic view of the apprenticeship learning (AL) formalism proposed by Kamoutsi et al. (2021). Furthermore, we demonstrate that this view of the AL formalism emerges as a relaxation of the IRL problem when observed through the lens of IO. The proposed formulation frames the IRL problem as a regularized min-max problem, extending prior approaches. Notably, the AL formalism is a special case when the regularization term is absent. We solve the regularized-convex-concave-min-max problem using stochastic mirror descent (SMD) and establish convergence bounds for the proposed method. Numerical experiments highlight the critical role of regularization in recovering the true cost vector for IRL problems.

1 Introduction

In scenarios where an agent must learn to navigate in a random or uncertain environment, it is a common practice to model the situation as a Markov decision process (MDP) and apply reinforcement learning (RL). The goal in RL is to find a policy that minimizes the total expected discounted cost for the agent. Usually, it is assumed that the cost function is known; however, specifying this function is difficult for most real-life scenarios (Ng & Russell, 2000). Moreover, an incorrect specification of the cost function can lead to unintended and potentially detrimental effects on the agent's behavior (Amodei et al., 2016; Hadfield-Menell et al., 2020). Consider the problem of driving: should the agent be rewarded for arriving quickly, safely, or cheaply, and how should the importance of each factor be balanced?

Inverse reinforcement learning (IRL) tackles this problem by reducing the work of manually designing the cost function and making use of observations of an expert agent's actions. Specifically, IRL aims to infer the cost function that the expert is optimizing based on recorded behavior and a model of the environment. Returning to the driving example, this approach involves observing an expert driver's behavior and deducing the underlying objective that guides their decisions. However, the goal extends beyond identifying the cost function; in many cases, there is a desire to emulate the expert's actions, much like a student assimilating knowledge from a mentor. For instance, when children learn to run; they are not explicitly given a cost function to optimize, but an expert shows them demonstrations of how they should run. Building on this idea, learning from demonstrations (LfD) or imitation learning (IL) seek to derive a policy that matches or surpasses the expert's performance.

IRL was first informally proposed by Russell (1998), and Ng & Russell (2000)introduced three algorithms for different scenarios: (1) when the policy, transition dynamics, and a finite state space are known; (2) when the state space is infinite; and (3) when the policy is unknown, but sample trajectories are available. Several methods have since been proposed, including a maximum margin approach (Ratliff et al., 2006), Bayesian frameworks (Ramachandran & Amir, 2007), and maximum entropy techniques (Ziebart et al., 2008). However, all of these methods rely on RL as a subroutine within an inner loop, leading to significant computational expenses.

In the context of LfD or IL, the literature often adopts the apprenticeship learning (AL) formalism proposed by Abbeel & Ng (2004), which assumes access to a set of expert demonstrations and that the unknown true

cost function belongs to a specific class of functions. Consequently, this assumption requires identifying these basis functions in advance, which can be nontrivial. There are two main classes of cost functions considered: (1) linear combinations of known basis functions called features (Abbeel & Ng, 2004; Syed & Schapire, 2007; Ziebart et al., 2008) and (2) convex combinations of a set of vectors (Syed et al., 2008; Kamoutsi et al., 2021). Building on the AL formalism, Syed & Schapire (2007) presented a game-theoretic view of AL and solved it using a multiplicative weights algorithm. Later, Syed et al. (2008) proposed a linear programming approach to solve the AL problem without employing IRL or RL as a subroutine. This marked an initial step toward leveraging the tools of mathematical optimization to address the LfD problem. Following this direction, Kamoutsi et al. (2021) introduced a convex-analytic approach to the LfD problem within the AL formalism. They formulated a bilinear min-max problem using Lagrangian duality and solved it using stochastic mirror descent (SMD). Moreover, Ho & Ermon (2016) solved the LfD problem for a general class of cost functions $\mathcal{C} = \mathbb{R}^{\mathcal{S}\times\mathcal{A}}$ solving an entropy-regularized-min-max problem and connected their approach with generative adversarial networks. Nevertheless, this min-max problem is nonconvex-nonconcave, limiting its theoretical understanding.

Contribution. Our study bridges the gap between IRL, LfD, and the AL formalism with inverse optimization (IO). In particular, we formulate an inverse problem for estimating the cost function of an MDP given an optimal policy, building on the ideas of Erkin et al. (2010) and Chan et al. (2023), which leverage complementary slackness and the linear programming formulation of an MDP. Through this approach, we revisit the proof that the inverse-feasible set of this inverse problem is equivalent to the dual problem derived by Kamoutsi et al. (2021) and extend it to a general class of cost functions. Moreover, rather than simply selecting an element from the inverse-feasible set, we incorporate an a priori estimate of the cost function to guide the search over the class of cost functions considered. Using Lagrangian duality, we derive a regularized-convex-concave-min-max problem, which reduces to previous formulations (Kamoutsi et al., 2021) when the regularization term is null. Additionally, we show that the stochastic mirror descent algorithm proposed in Jin & Sidford (2020) to solve ℓ_∞-ℓ_1 games naturally adapts to our problem, and we provide theoretical convergence bounds.

1.1 Notation

We denote the cardinality of a set $\mathcal{S}$ as $|\mathcal{S}|$. The probability simplex over elements n elements is given by $\Delta^{|\mathcal{S}|} = \left\{\boldsymbol{x} \in \mathbb{R}^{|\mathcal{S}|} \mid x_i \geq 0, \sum_{i=1}^{|\mathcal{S}|} x_i = 1\right\}$ and boxes are denoted by $\mathbb{B}_b^n = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\|_\infty \leq b\}$. Componentwise multiplication between two vectors $\boldsymbol{x}, \boldsymbol{y}$ is denoted by $\boldsymbol{x} \circ \boldsymbol{y}$.

2 Preliminaries and problem formulation

In this section, we establish the foundational concepts necessary for our study. We begin by defining the structure of infinite-horizon MDPs. We then introduce the IRL problem and discuss the LfD problem through the AL formalism. Finally, we provide an overview of IO and formally state our problem.

2.1 Infinite Horizon MDPs

A finite MDP is defined as a tuple $(\mathcal{S}, \mathcal{A}, P, \boldsymbol{\nu}_0, \boldsymbol{c}, \gamma)$ where $\mathcal{S}$ is a finite state space, $\mathcal{A}$ a finite action space, and P is a transition law $P = (P(\cdot \mid s, a))_{s,a}$ where $P(\cdot \mid s, a) \in \Delta^{|\mathcal{S}|}$. The initial state distribution is denoted by $\boldsymbol{\nu}_0 \in \Delta^{|\mathcal{S}|}$ and satisfies $\boldsymbol{\nu}_0(s) > 0$ for every $s \in \mathcal{S}$. The cost vector is defined as $\boldsymbol{c} \in \mathcal{C} = \mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}$ and the discount factor is given by $\gamma \in (0, 1)$.

A *stationary Markov policy* is a collection of distributions, indexed by $s \in \mathcal{S}$ and denoted by $(\pi(\cdot \mid s))_{s\in\mathcal{S}}$, where $\pi(\cdot \mid s) \in \Delta^{|\mathcal{A}|}$. We denote the space of stationary Markov policies by Π_0. In this framework, the MDP begins with an initial state $s_0 \sim \boldsymbol{\nu}_0$. At each time-step t, where the current state is s_t: the agent selects an action according to $a_t \sim \pi(\cdot \mid s_t)$, the next state is determined by the transition law $s_{t+1} \sim P(\cdot \mid s_t, a_t)$, and a cost $c(s_t, a_t)$ is incurred. Note that in an infinite horizon model, the process continues indefinitely.

The *normalized value function* $\boldsymbol{V}_{\boldsymbol{c}}^{\pi} \in \mathbb{R}^{|\mathcal{S}|}$ of a policy π given a cost $\boldsymbol{c}$ is given by

$$\boldsymbol{V}_{\boldsymbol{c}}^{\pi}(s) = (1-\gamma)\mathbb{E}_s^{\pi}\left[\sum_{t=0}^{\infty}\gamma^t c(s_t, a_t)\right]$$

where $\mathbb{E}_s^{\pi}[\cdot]$ denotes the expectation with respect to the trajectories generated by π when starting from the state s. The fundamental goal of RL is to find a policy π such that the process $((s_t, a_t))_t$ is optimal in the following way:

$$\rho_{\boldsymbol{c}}^{*} = \min_{\pi\in\Pi_0}(1-\gamma)\mathbb{E}_{\boldsymbol{\nu}_0}^{\pi}\left[\sum_{t=0}^{\infty}\gamma^t c(s_t, a_t)\right]. \tag{RL$_{\boldsymbol{c}}$}$$

Notice that we explicitly highlight the dependence of equation RL$_{\boldsymbol{c}}$ on the cost vector $\boldsymbol{c}$. An equivalent way to write RL$_{\boldsymbol{c}}$ making use of the value functions is $\rho_{\boldsymbol{c}}^{*} = \min_{\pi\in\Pi_0}\langle \boldsymbol{\nu}_0, \boldsymbol{V}_{\boldsymbol{c}}^{\pi}\rangle$.

The *normalized occupancy measure* $\boldsymbol{\mu}_{\pi} \in \Delta^{|\mathcal{S}||\mathcal{A}|}$ of a policy π is defined as

$$\boldsymbol{\mu}_{\pi}(s,a) = (1-\gamma)\sum_{t=0}^{\infty}\gamma^t \mathbb{P}_{\boldsymbol{\nu}_0}^{\pi}[s_t = s,\ a_t = a],$$

where $\mathbb{P}_{\boldsymbol{\nu}_0}^{\pi}[\cdot]$ represents the probability of an event when starting from $s \sim \boldsymbol{\nu}_0$ and following π. The occupancy measure of a state-action pair can be interpreted as the discounted visitation frequency of the pair when following a particular policy. Hence, we can also write $\rho_{\boldsymbol{c}}^{*} = \min_{\pi\in\Pi_0}\langle \boldsymbol{\mu}_{\pi}, \boldsymbol{c}\rangle$.

We also define matrix $\boldsymbol{P} \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}||\mathcal{A}|}$ where $\boldsymbol{P}_{s',(s,a)} = P(s' \mid s, a)$. We also define the set $\mathcal{F} = \{\boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \mid \boldsymbol{T}_{\gamma}\boldsymbol{\mu} = \boldsymbol{\nu}_0,\ \boldsymbol{\mu} \geq \boldsymbol{0}\}$ where $\boldsymbol{T} \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}||\mathcal{A}|}$, $\boldsymbol{T}(s,\ (t,a)) = \delta_{s,t} - \gamma P(s \mid t, a)$, and $\boldsymbol{T}_{\gamma} = \frac{1}{(1-\gamma)}\boldsymbol{T}$. Moreover, $\boldsymbol{T}_{\gamma}\boldsymbol{\mu} = \frac{1}{(1-\gamma)}(\boldsymbol{B} - \gamma\boldsymbol{P})\boldsymbol{\mu}$ where $\boldsymbol{B}$ is a binary matrix that satisfies $\boldsymbol{B}_{s',(s,a)} = 1$ if $s' = s$ and $\boldsymbol{B}_{s',(s,a)} = 0$ otherwise.

Proposition 1 (Puterman (1994))**.** *It holds that, $\mathcal{F} = \{\boldsymbol{\mu}_{\pi} \mid \pi \in \Pi_0\}$. For every $\pi \in \Pi_0$, we have that $\boldsymbol{\mu}_{\pi} \in \mathcal{F}$. Moreover, for every feasible solution $\boldsymbol{\mu} \in \mathcal{F}$, we can obtain a stationary Markov policy $\pi_{\boldsymbol{\mu}} \in \Pi_0$ by $\pi_{\boldsymbol{\mu}}(a \mid x) = \frac{\mu(x,a)}{\sum_{a'\in\mathcal{A}}\mu(x,a')}$. Then, the induced occupancy measure is exactly μ.*

Hence, the linear programming approach consists of solving the MDP-P$_{\boldsymbol{c}}$ problem

$$\begin{aligned} \min_{\boldsymbol{\mu}\in\Delta^{|\mathcal{S}||\mathcal{A}|}}\ & \langle \boldsymbol{\mu}, \boldsymbol{c}\rangle \\ \text{s.t}\quad & \boldsymbol{T}_{\gamma}\boldsymbol{\mu} = \boldsymbol{\nu}_0, \\ & \boldsymbol{\mu} \geq \boldsymbol{0}, \end{aligned} \tag{MDP-P$_{\boldsymbol{c}}$}$$

and using Proposition 1 to define the policy corresponding to the occupancy measure found.

2.2 Inverse reinforcement learning

The IRL problem aims to uncover the true reward function that an expert agent is optimizing given some information about the expert's behavior: sample trajectories, its real policy, or an estimate of its policy (Ng & Russell, 2000). Formally, given an MDP without a cost vector and with access to information about an expert's policy π_E, which could be the actual policy, an estimate, or a set of demonstrations, the IRL problem is defined by the tuple $(\mathcal{S}, \mathcal{A}, P, \boldsymbol{\nu}_0, \pi_E, \gamma)$. The goal of the IRL problem is to determine a cost vector $\boldsymbol{c}$ for which the policy π_E is optimal for RL$_{\boldsymbol{c}}$ within the MDP $(\mathcal{S}, \mathcal{A}, P, \boldsymbol{\nu}_0, \boldsymbol{c}, \gamma)$.

2.3 Learning from demonstrations and the apprenticeship learning formalism

The goal of learning from demonstrations is to learn a policy that matches or outperforms the expert's policy π_E for an unknown true cost vector $\boldsymbol{c}_{\text{true}}$. The apprenticeship learning formalism (Abbeel & Ng,

2004) has been routinely used in literature for addressing the LfD problem. The AL formalism assumes that the unknown true cost function $\boldsymbol{c}_{\text{true}}$ belongs to a class of functions $\mathcal{C}$ and searches for a policy that solves the following min-max problem

$$\beta^* := \min_{\pi \in \Pi_0} \max_{\boldsymbol{c} \in \mathcal{C}} \langle \boldsymbol{\mu}_\pi, \boldsymbol{c} \rangle - \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} \rangle = \min_{\pi \in \Pi_0} \max_{\boldsymbol{c} \in \mathcal{C}} \langle \boldsymbol{\mu}_\pi - \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} \rangle, \tag{LfD$_{\pi_E}$}$$

An optimal solution to LfD$_{\pi_E}$ is called an apprentice policy π_A and satisfies

$$\langle \boldsymbol{\mu}_{\pi_A}, \boldsymbol{c}_{\text{true}} \rangle \leq \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c}_{\text{true}} \rangle + \beta^*.$$

In this formalism, the learner is provided with a set of m i.i.d. expert sample trajectories $\{(s_0, a_0)^k, ..., (s_H, a_H)^k\}_{k=1}^m$, where (s_i, a_i) is a state-action pair, and no additional queries to the expert are allowed. Typically, it is assumed that the underlying MDP is known, with the exception of the cost function (Abbeel & Ng, 2004; Ramachandran & Amir, 2007; Syed & Schapire, 2007; Syed et al., 2008). However, a more general assumption is that the MDP's transition dynamics are also unknown, and instead, the learner has access to a generative-model oracle (Kamoutsi et al., 2021). This oracle, given a state-action pair (s, a), generates the next state $s' \sim P(\cdot \mid s, a)$ and allows sampling from the initial state distribution $s_0 \sim \boldsymbol{\nu}_0$.

In optimization-focused approaches to LfD, the $\boldsymbol{c}_{\text{true}}$ is assumed to belong to a convex hull

$$\mathcal{C} = \mathcal{C}_{\text{conv}} := \left\{ \boldsymbol{c}_{\boldsymbol{w}} := \sum_{i=1}^{n_c} w_i c_i \;\middle|\; w_i \geq 0, \sum_{i=1}^{n_c} w_i = 1 \right\}$$

(Syed et al., 2008; Kamoutsi et al., 2021) of a set of vectors $\{\boldsymbol{c}_i\}_{i=1}^{n_c} \subseteq \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ where $\|\boldsymbol{c}_i\|_\infty \leq 1$ for each $i = 1, ..., n_c$. It is assumed that this set of vectors is known; however, in practice, an initial estimation step is required to determine this set, a task that is generally nontrivial.

2.4 A primer on inverse optimization

Inverse optimization is a mathematical framework that fits optimization models to decision data. Given an observed optimal solution, it seeks to learn the objectives and constraints of the underlying model. For example, IRL can be thought of as an inverse optimization problem, as it searches for the cost function that an optimal agent is optimizing.

Consider the general forward optimization problem FOP for a given $\theta \in \Gamma$:

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \{ f(\boldsymbol{x}, \theta) \mid \boldsymbol{x} \in X(\theta) \}. \tag{FOP}$$

Given an optimal solution $\hat{\boldsymbol{x}}$, the inverse optimization problem consists of finding a $\theta^* \in \Gamma$ that makes $\hat{\boldsymbol{x}}$ optimal for FOP with $\theta = \theta^*$ and is optimal in some way. For this purpose, define the optimal solution set $X^{\text{opt}}(\theta) := \arg\min_{\boldsymbol{x}} \{ f(\boldsymbol{x}, \theta) \mid \boldsymbol{x} \in X(\theta) \}$ and the inverse-feasible set $\Theta^{\text{inv}}(\hat{\boldsymbol{x}}) := \{ \theta \in \Gamma \mid \hat{\boldsymbol{x}} \in X^{\text{opt}}(\theta) \}$. Naturally, we want to find a $\theta \in \Theta^{\text{inv}}(\hat{\boldsymbol{x}})$, but rather than selecting an arbitrary θ from this set, we aim for one that minimizes a certain criterion.

Hence, the inverse optimization problem INV-OPT is defined as:

$$\min_{\theta \in \Gamma} \{ F(\theta) \mid \theta \in \Theta^{\text{inv}}(\hat{\boldsymbol{x}}) \}, \tag{INV-OPT}$$

where F should convey information about the quality of θ given some prior knowledge and the search space Γ should be appropriately chosen for each instance of the problem.

2.5 Our problem

Suppose that the environment is modeled as an MDP $(\mathcal{S}, \mathcal{A}, P, \boldsymbol{\nu}_0, \boldsymbol{c}_{\text{true}}, \gamma)$ where only the state space $\mathcal{S}$, action space $\mathcal{A}$, and discount factor γ are known. We assume that $\boldsymbol{c}_{\text{true}}$ belongs to a general class of cost

functions $\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}$ and that the learner has access to a generative-model oracle of the expert's occupancy measure $\boldsymbol{\mu}_{\pi_E}$, as well as a generative-model oracle for the MDP's transition dynamics P. Our goal is to solve the previously defined IRL problem (see Subsection 2.2) under both the assumptions of expert optimality and suboptimality.

3 The inverse optimization viewpoint

Let us suppose that the true cost vector $\boldsymbol{c}_{\text{true}}$ the expert is optimizing lies in an arbitrary class of functions $\mathcal{C}$ and that the expert's policy π_E is optimal for $\text{RL}_{\boldsymbol{c}}$, which means that its corresponding occupancy measure $\boldsymbol{\mu}_{\pi_E}$ is optimal for MDP-$\text{P}_{\boldsymbol{c}}$.

Proposition 2 (Complementary slackness)**.** *An element $\boldsymbol{\mu}_\pi$ is an optimal solution to MDP-$P_{\boldsymbol{c}}$ if and only if there exists a vector $\boldsymbol{u} \in \mathbb{R}^{|\mathcal{S}|}$ such that $\boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq \boldsymbol{0}$ and $\langle \boldsymbol{\mu}_\pi, \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \rangle = 0$.*

Therefore, the inverse-feasible set for $\boldsymbol{\mu}_{\pi_E}$ consists of the cost functions in $\mathcal{C}$ for which such a $\boldsymbol{u}$ exists

$$\Theta^{\text{inv}}(\boldsymbol{\mu}_{\pi_E}) := \{\boldsymbol{c} \in \mathcal{C} \mid \exists \boldsymbol{u} \in \mathbb{R}^{|S|} \ : \ \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq \boldsymbol{0},\ \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \rangle = 0\}.$$

Substituting for the inverse-feasible set in INV-OPT and choosing an appropriate function F for comparing cost vectors, we arrive to the inverse reinforcement learning problem through inverse optimization (Chan et al., 2023)

$$\begin{aligned} \min_{\boldsymbol{c} \in \mathcal{C}, \boldsymbol{u} \in \mathbb{R}^{|S|}} \quad & F(\boldsymbol{c}) \\ \text{s.t} \quad & \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq \boldsymbol{0}, \\ & \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \rangle = 0. \end{aligned} \tag{IRL-IO}$$

3.1 Connections to LfD and the AL formalism

Kamoutsi et al. (2021) considered the LfD problem under the assumption that the true cost function $\boldsymbol{c}_{\text{true}}$ belongs to the convex hull $\mathcal{C}_{\text{conv}}$ of a given set of vectors. By applying an epigraphic transformation to LfD_{π_E}, where the validity of this transformation depends on the previous assumption on $\boldsymbol{c}_{\text{true}}$, and deriving its dual, they arrived to the optimization problem D_{π_E}:

$$\max_{\boldsymbol{c}, \boldsymbol{u}} \{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{T}_\gamma^\top \boldsymbol{u} - \boldsymbol{c} \rangle \mid \boldsymbol{c} \in \mathcal{C}_{\text{conv}}, \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq \boldsymbol{0}\}. \tag{D_{π_E}}$$

They focus on this problem and optimize its unconstrained version derived through Lagrangian duality, where the dual variable corresponding to the constraint $\boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq \boldsymbol{0}$ represents the apprentice policy.

Theorem 1. *Suppose that $\boldsymbol{\mu}_{\pi_E}$ is an optimal solution for MDP-$P_{\boldsymbol{c}}$ where $\boldsymbol{c} \in \mathcal{C}$. Then the following equality holds:*

$$\Theta^{inv}(\boldsymbol{\mu}_{\pi_E}) = \Pi_1 \left(\arg\max_{(\boldsymbol{c}, \boldsymbol{u})} \{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{T}_\gamma^\top \boldsymbol{u} - \boldsymbol{c} \rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq 0\} \right)$$

where Π_1 denotes the projection in the first component.

This implies that the dual problem D_{π_E} serves as an alternative representation of the inverse-feasible set $\Theta^{\text{inv}}(\boldsymbol{\mu}_{\pi_E})$. In contrast, in problem IRL-IO we choose an element within the inverse-feasible set that minimizes F and is optimal. In this sense, under the assumption of expert's optimality and that $\boldsymbol{c}_{\text{true}} \in \mathcal{C}_{\text{conv}}$, the AL formalism finds an arbitrary element of the inverse-feasible set, whereas IRL-IO has a criterion for searching within this space.

It is important to note that D_{π_E} remains valid even if the expert is not optimal (Kamoutsi et al., 2021). To account for the possibility of suboptimal expert behavior, we can relax the complementary slackness condition in IRL-IO. Weighing the beliefs of the expert's optimality and the quality of the cost function

estimate with parameter $\alpha \in [0,1]$ we arrive to problem IO-AL$_\alpha$:

$$\begin{aligned} \min_{\boldsymbol{c}\in\mathcal{C},\boldsymbol{u}\in\mathbb{R}^{|\mathcal{S}|}} \quad & \alpha\|\boldsymbol{c}-\hat{\boldsymbol{c}}\| + (1-\alpha)\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c}-\boldsymbol{T}_\gamma^\top \boldsymbol{u}\rangle \\ \text{s.t} \quad & \boldsymbol{c}-\boldsymbol{T}_\gamma^\top \boldsymbol{u} \geq \boldsymbol{0}. \end{aligned} \tag{IO-AL$_\alpha$}$$

Here, we use the regularization term $\|\boldsymbol{c}-\hat{\boldsymbol{c}}\|$ where $\hat{\boldsymbol{c}}$ is an estimate of the cost vector, in place of $F(\boldsymbol{c})$. This is the trade-off we propose with respect to the AL formalism: instead of identifying the set of vectors $\{\boldsymbol{c}_i\}_{i=1}^{n_c}$ that define $\mathcal{C}_{\text{conv}}$, we search over a general class of cost vectors $\mathcal{C}$ and define an estimate $\hat{\boldsymbol{c}}$ to guide the search.

3.2 Min-max problem

We aim to reformulate equation IO-AL$_\alpha$ as a convex-concave-min-max problem and solve this unconstrained optimization problem using stochastic mirror descent. To this end, we compute its Lagrangian:

$$\mathcal{L}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\mu}) = \alpha\|\boldsymbol{c}-\hat{\boldsymbol{c}}\| + \langle (1-\alpha)\boldsymbol{\mu}_{\pi_E} - \boldsymbol{\mu}, \boldsymbol{c}-\boldsymbol{T}_\gamma^\top \boldsymbol{u}\rangle$$

where $\boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ and $\boldsymbol{\mu} \geq \boldsymbol{0}$. Observe that $\mathcal{L}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\mu})$ is convex on $(\boldsymbol{c},\boldsymbol{u})$ and concave on $\boldsymbol{\mu}$. Thus, IO-AL$_\alpha$ is equivalent to the min-max problem

$$\min_{\boldsymbol{c}\in\mathcal{C},\boldsymbol{u}\in\mathbb{R}^{|\mathcal{S}|}} \max_{\boldsymbol{\mu}\geq\boldsymbol{0}} \mathcal{L}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\mu}).$$

In our setting, we assume that $\boldsymbol{c}_{\text{true}} \in \mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}$, which is not restrictive because we can scale any cost vector to lie within $[-1,1]^{|\mathcal{S}||\mathcal{A}|}$. Therefore, we know that $\|\boldsymbol{V}^\pi_{\boldsymbol{c}_{\text{true}}}\|_\infty \leq 1$ for any policy $\pi \in \Pi_0$ (see Appendix). Hence, we can search for $(\boldsymbol{c},\boldsymbol{u})$ within the box $\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|} \times \mathbb{B}_1^{|\mathcal{S}|}$. Moreover, as all feasible solutions for MDP-P$_{\boldsymbol{c}}$ belong to the simplex $\Delta^{|\mathcal{S}||\mathcal{A}|}$, we can restrict the search for $\boldsymbol{\mu}$ to the same simplex. The only remaining question is how to choose the norm for the regularizing term $\|\boldsymbol{c}-\hat{\boldsymbol{c}}\|$. For differentiability purposes, we choose the L^2-norm and, henceforth, denote the objective function with this choice as $\mathcal{L}(\boldsymbol{c},\boldsymbol{u},\boldsymbol{\mu})$:

$$\min_{(\boldsymbol{c},\boldsymbol{u})\in\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}\times\mathbb{B}_1^{|\mathcal{S}|}} \max_{\boldsymbol{\mu}\in\Delta^{|\mathcal{S}||\mathcal{A}|}} \alpha\|\boldsymbol{c}-\hat{\boldsymbol{c}}\|_2^2 + \langle (1-\alpha)\boldsymbol{\mu}_{\pi_E} - \boldsymbol{\mu}, \boldsymbol{c}-\boldsymbol{T}_\gamma^\top \boldsymbol{u}\rangle. \tag{RLfD$_\alpha$}$$

Observe that this formulation closely resembles previous min-max formulations of the LfD problem (Kamoutsi et al., 2021). It can be interpreted as a regularized version of this problem, where the search for $\boldsymbol{c}$ is conducted within a general class of cost functions rather than being restricted to a previously specified convex hull.

4 Algorithm

Revisiting the assumptions for our problem, we assume that we have access to a generative-model oracle of the expert's occupancy measure $\boldsymbol{\mu}_{\pi_E}$, as well as a generative-model oracle for the MDP's transition law. In this section, we will focus on solving RLfD$_\alpha$ via stochastic mirror descent. Before attempting to solve this problem, we must first define what constitutes a good solution to RLfD$_\alpha$. We define an ϵ-approximate solution as a pair $(\boldsymbol{c},\boldsymbol{u}),\boldsymbol{\mu}$ such that their duality gap is bounded by $\epsilon > 0$.

Definition 1 (ϵ-approximate solution)**.** *Given $\epsilon > 0$, an ϵ-approximate solution of RLfD$_\alpha$ is a pair of feasible solutions $((\boldsymbol{c}^\epsilon,\boldsymbol{u}^\epsilon),\boldsymbol{\mu}^\epsilon) \in \left(\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}\times\mathbb{B}_1^{|\mathcal{S}|}\right)\times\Delta^{|\mathcal{S}||\mathcal{A}|}$ that satisfy*

$$Gap((\boldsymbol{c}^\epsilon,\boldsymbol{u}^\epsilon),\boldsymbol{\mu}^\epsilon) = \max_{\boldsymbol{\mu}'\in\Delta^{|\mathcal{S}||\mathcal{A}|}} \mathcal{L}((\boldsymbol{c}^\epsilon,\boldsymbol{u}^\epsilon),\boldsymbol{\mu}') - \min_{(\boldsymbol{c}',\boldsymbol{u}')\in\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}\times\mathbb{B}_1^{|\mathcal{S}|}} \mathcal{L}((\boldsymbol{c}',\boldsymbol{u}'),\boldsymbol{\mu}^\epsilon) \leq \epsilon.$$

To minimize the duality gap, we require descent and ascent directions. The gradients of $\mathcal{L}((\boldsymbol{c},\boldsymbol{u}),\boldsymbol{\mu})$ at a given iterate $((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) \in (\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|}\times\mathbb{B}_1^{|\mathcal{S}|})\times\Delta^{|\mathcal{S}||\mathcal{A}|}$ are given by

$$\begin{aligned} g^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) &= \begin{pmatrix} 2\alpha(\boldsymbol{c}_t-\hat{\boldsymbol{c}}) + (1-\alpha)\boldsymbol{\mu}_{\pi_E} - \boldsymbol{\mu}_t \\ \boldsymbol{T}_\gamma\boldsymbol{\mu}_t - (1-\alpha)\boldsymbol{T}_\gamma\boldsymbol{\mu}_{\pi_E} \end{pmatrix}, \\ g^{\boldsymbol{\mu}}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) &= -(-\boldsymbol{c}_t + \boldsymbol{T}_\gamma^\top \boldsymbol{u}_t) = \boldsymbol{c}_t - \boldsymbol{T}_\gamma^\top \boldsymbol{u}_t, \end{aligned}$$

where $g^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) = \nabla_{(\boldsymbol{c},\boldsymbol{u})}\mathcal{L}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t)$ and $g^{\boldsymbol{\mu}}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) = -\nabla_{\boldsymbol{\mu}}\mathcal{L}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t)$. Since explicit access to $\boldsymbol{T}_\gamma$ and $\boldsymbol{\mu}_{\pi_E}$ is unavailable, it is necessary to develop gradient estimators that are compatible with oracle-based queries.

Definition 2 (bounded estimator)**.** *Given the following properties on mean, scale, and variance of an estimator:*

i. unbiasedness: $\mathbb{E}[\tilde{g}] = g$.

ii. bounded maximum entry: $\|\tilde{g}\|_\infty \leq z$ *with probability 1.*

iii. bounded second-moment: $\mathbb{E}[\|\tilde{g}\|^2] \leq v$

we call $\tilde{g}$ *a* $(v, \|\cdot\|)$*-bounded estimator if it satisfies (i) and (iii) and a* $(z, v, \|\cdot\|_{\Delta^m})$*-bounded estimator if it satisfies (i), (ii), (iii) with local norm* $\|\cdot\|_{\boldsymbol{y}}$ *for all* $\boldsymbol{y} \in \Delta^m$.

With this in mind, define the gradient estimator for the $(\boldsymbol{c},\boldsymbol{u})$ side through the following procedure

$$\begin{aligned}
&\text{sample} \quad (s,a) \sim \frac{1}{|\mathcal{S}||\mathcal{A}|},\ (s_t,a_t) \sim \boldsymbol{\mu}_t,\ s'_t \sim P(\cdot \mid s_t,a_t),\ (s_E,a_E) \sim \boldsymbol{\mu}_{\pi_E},\ s'_E \sim P(\cdot \mid s_E,a_E),\\
&\text{set} \quad \tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) = \begin{pmatrix} |\mathcal{S}||\mathcal{A}| \cdot 2\alpha \left(c_t(s,a)\boldsymbol{e}^{(s,a)} - \hat{c}(s,a)\boldsymbol{e}^{(s,a)}\right) + (1-\alpha)\boldsymbol{e}^{(s_E,a_E)} - \boldsymbol{e}^{(s_t,a_t)} \\ \frac{1}{(1-\gamma)}\left(\boldsymbol{e}^{s_t} - \gamma \boldsymbol{e}^{s'_t} - (1-\alpha)(\boldsymbol{e}^{s_E} - \gamma \boldsymbol{e}^{s'_E})\right) \end{pmatrix}. \qquad (1)
\end{aligned}$$

In Lemma 1, we show that this estimator is unbiased and provides a bound for its second moment.

Lemma 1. *Gradient estimator* $\tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t)$ *is a* $(v^{(\boldsymbol{c},\boldsymbol{u})}, \|\cdot\|_2)$*-bounded estimator, with*

$$v^{(\boldsymbol{c},\boldsymbol{u})} = 64\alpha^2 \cdot |\mathcal{S}||\mathcal{A}| + 4(1-\alpha)^2 + 4 + 2 \cdot \frac{1 + \gamma^2 + (1-\alpha)^2 + (1-\alpha)^2\gamma^2}{(1-\gamma)^2}.$$

For the $\boldsymbol{\mu}$ side, define the gradient estimator by

$$\begin{aligned}
&\text{sample} \quad (s,a) \sim \frac{1}{|\mathcal{S}||\mathcal{A}|}, \quad s' \sim P(\cdot \mid s,a),\\
&\text{set} \quad \tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t) = |\mathcal{S}||\mathcal{A}| \left(c_t(s,a)\boldsymbol{e}^{(s,a)} - \frac{1}{(1-\gamma)}(u_t(s)\boldsymbol{e}^{(s,a)} - \gamma u_t(s')\boldsymbol{e}^{(s,a)})\right) \qquad (2)
\end{aligned}$$

As before, we will demonstrate unbiasedness and bound its second moment; however, this time we will also calculate a bound on its maximum entry.

Lemma 2. *Gradient estimator* $\tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t)$ *is a* $(z^{\boldsymbol{\mu}}, v^{\boldsymbol{\mu}}, \|\cdot\|_2)$*-bounded estimator, with*

$$z^{\boldsymbol{\mu}} = \frac{2|\mathcal{S}||\mathcal{A}|}{(1-\gamma)} \text{ and } v^{\boldsymbol{\mu}} = |\mathcal{S}||\mathcal{A}| \left(2 + \frac{4(1+\gamma^2)}{(1-\gamma)^2}\right).$$

Using these gradient estimators and the bounds established above, we adapt the SMD algorithm originally designed for solving MDPs in Jin & Sidford (2020). Algorithm 1 presents the SMD method for IRL. This algorithm iteratively computes bounded gradient estimators (Lines 3 and 5) by sampling from the occupancy measures and querying the oracle (Lines 2 and 4). The updates are then obtained using mirror descent steps followed by a projection (Lines 6 and 7). After T it-

erations, the algorithm returns the average of the iterates as an ϵ-approximate solution to RLfD$_\alpha$.

Algorithm 1: Stochastic Mirror Descent for Inverse Reinforcement Learning

Parameters: Step-size $\eta^{(\boldsymbol{c},\boldsymbol{u})}$, $\eta^{\boldsymbol{\mu}}$, number of iterations T, accuracy level ϵ.
Input: State space $\mathcal{S}$, action space $\mathcal{A}$, transition oracle P, initial state distribution $\boldsymbol{\nu}_0$, discount factor γ, initial $((\boldsymbol{c}_0, \boldsymbol{u}_0), \boldsymbol{\mu}_0) \in \mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|} \times \Delta^{|\mathcal{S}||\mathcal{A}|}$.
Output: An expected ϵ-approximate solution $((\boldsymbol{c}^\epsilon, \boldsymbol{u}^\epsilon), \boldsymbol{\mu}^\epsilon)$ for RLfD$_\alpha$.

for $t \leftarrow 0$ **to** $T-1$ **do**

`/* (c, u) gradient estimation */`

Sample $(s_t, a_t) \sim \boldsymbol{\mu}_t$, $s_t' \sim P(\cdot \mid s_t, a_t)$, $(s_E, a_E) \sim \boldsymbol{\mu}_{\pi_E}$, $s_E' \sim P(\cdot \mid s_E, a_E)$

Compute:

$$\tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t) = \begin{pmatrix} 2\alpha(\boldsymbol{c}_t - \hat{\boldsymbol{c}}) + (1-\alpha)\boldsymbol{\mu}_{\pi_E} - \boldsymbol{\mu}_t \\ \frac{1}{(1-\gamma)}\left(\boldsymbol{e}^{s_t} - \gamma \boldsymbol{e}^{s_t'} - (1-\alpha)(\boldsymbol{e}^{s_E} - \gamma \boldsymbol{e}^{s_E'})\right) \end{pmatrix}$$

`/* μ gradient estimation */`

Sample $(s, a) \sim \frac{1}{|\mathcal{S}||\mathcal{A}|}$, $s' \sim P(\cdot \mid s, a)$

Compute:

$$\tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t) = |\mathcal{S}||\mathcal{A}| \left(c_t(s,a)\boldsymbol{e}^{(s,a)} - \frac{1}{(1-\gamma)}(u_t(s)\boldsymbol{e}^{(s,a)} - \gamma u_t(s')\boldsymbol{e}^{(s,a)}) \right)$$

`/* Mirror descent steps */`

$(\boldsymbol{c}_t, \boldsymbol{u}_t) \leftarrow \Pi_{\mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|} \times \mathbb{B}_1^{|\mathcal{S}|}} \left((\boldsymbol{c}_{t-1}, \boldsymbol{u}_{t-1}) - \eta^{(\boldsymbol{c},\boldsymbol{u})} \tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_{t-1}, \boldsymbol{u}_{t-1}), \boldsymbol{\mu}_{t-1}) \right)$

$\boldsymbol{\mu}_t \leftarrow \Pi_{\Delta^{|\mathcal{S}||\mathcal{A}|}} (\boldsymbol{\mu}_{t-1} \circ \exp(-\eta^{\boldsymbol{\mu}} \tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_{t-1}, \boldsymbol{u}_{t-1}), \boldsymbol{\mu}_{t-1})))$

return $((\boldsymbol{c}^\epsilon, \boldsymbol{u}^\epsilon), \boldsymbol{\mu}^\epsilon) \leftarrow \frac{1}{T} \sum_{t=1}^{T} ((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t)$

Theorem 2. *Given $\epsilon \in (0,1)$, Algorithm 1 with step-size*

$$\eta^{(\boldsymbol{c},\boldsymbol{u})} = \frac{\epsilon}{4v^{(\boldsymbol{c},\boldsymbol{u})}}, \quad \eta^{\boldsymbol{\mu}} = \frac{\epsilon}{4v^{\boldsymbol{\mu}}},$$

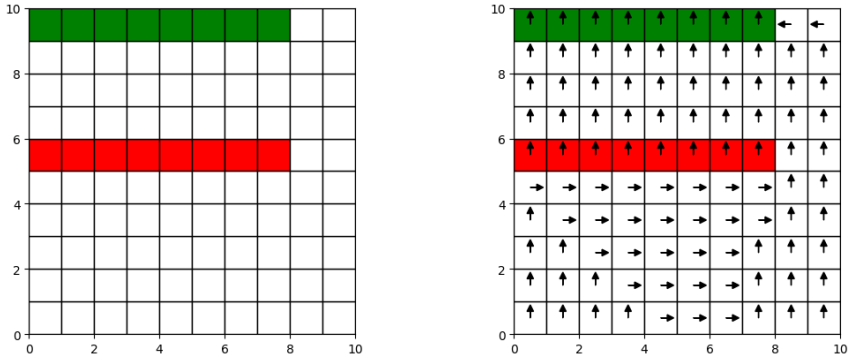
and gradient estimators defined in equations 1 and 2 finds an expected ϵ-approximate solution within any iteration number

$$T \geq \max\left\{ \mathcal{O}\left(\frac{\alpha^2 |\mathcal{S}|^3 |\mathcal{A}|^2}{\epsilon^2} \right), \mathcal{O}\left(\frac{|\mathcal{S}||\mathcal{A}| \log(|\mathcal{S}||\mathcal{A}|)}{\epsilon^2} \right) \right\}.$$

The number of iterations scales quadratically with the number of actions and cubically with the number of states. Theoretically, the number of iterations required depends on the parameter $\alpha \in [0,1]$. When $\alpha = 0$, the number of iterations decreases significantly as the $|\mathcal{S}|^3$ and $|\mathcal{A}|^2$ terms vanish from the initial expression. This suggests that introducing the regularization $\alpha\|\boldsymbol{c} - \hat{\boldsymbol{c}}\|$ increases the complexity of the problem. Nevertheless, we will see in the next section that the regularization term helps to guide the search to uncover the true cost function.

5 Numerical experiments

We use a standard $H \times W$ Gridworld environment (Figure 1(a)), where each cell is a unique state. Obstacles (shown in red) incur a cost of 1, and terminal cells (shown in green) incur a cost (reward) of -1. The action set is $\{\text{up}, \text{down}, \text{left}, \text{right}\}$, but a "wind" introduces a 20% chance of drifting left or otherwise altering the intended move. If the resulting move is out of bounds, the agent remains in the same cell. The discount factor is 0.7, and initial states are chosen uniformly. We solve the corresponding MDP-P$_{\boldsymbol{c}}$ with this cost to obtain the expert's policy (Figure 1(b)).



(a) Gridworld environment. (b) Expert's policy.

Figure 1: Illustration of the Gridworld environment and the expert's policy.

5.1 Implementation details

We executed Algorithm 1 N times with randomly generated initial values for T iterations and averaged the resulting outputs. With reproducibility in mind, we provide a Python package (see suplemmentary material) with a framework that can handle general discounted Markov decision processes.

5.2 Regularization effect

We study the effect of the regularization term $\alpha\|\mathbf{c} - \hat{\mathbf{c}}\|$ in solving problem RLfD_α via SMD. To this end, we define a cost vector $\hat{\boldsymbol{c}}$ that is zero everywhere except for a randomly selected subset of obstacle and goal states. This choice was made based on practical considerations, as in most real-world scenarios, we rarely have access to a highly accurate estimate of the cost vector. Specifically, for obstacles, $\hat{\boldsymbol{c}}$ is set to 1 for a randomly selected 50% of the obstacle states, while for goal states, $\hat{\mathbf{c}}$ is set to -1 for a randomly chosen 50% of them. Moreover, regarding the algorithm's parameters, we chose $N = 20$, $T = 10^5$, and both step-sizes as 10^{-2}.

Figure 5 displays the apprentice policy obtained using SMD for various values of α. Notably, when α is set to 0.005 or 0.001, the apprentice policy closely resembles the one derived by exclusively weighting the expert's policy (i.e., $\alpha = 0$). This observation is significant, as subsequent analysis demonstrates that incorporating regularization yields a cost vector for the apprentice policies that aligns more closely with the true cost vector.

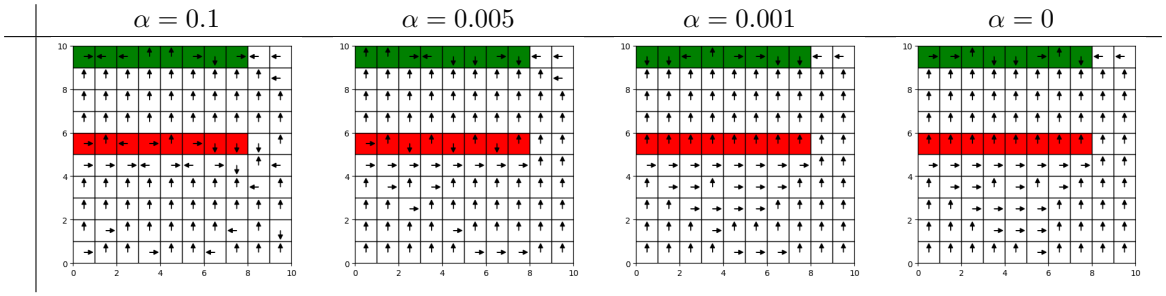


Figure 2: Effect of regularization on the apprentice policy.

Figure 5 depicts the learned cost vectors for each action $\{\text{up}, \text{down}, \text{left}, \text{right}\}$ under varying levels of regularization α. As α increases, the cost vectors display more white regions, indicating near-zero cost values, and more accurately highlight the main obstacles. In turn, this leads to a better approximation of the true cost structure. However, when the regularization is too strong, it may overly penalize costs associated with

obstacle areas that are less frequently demonstrated, thereby ignoring parts of the environment that do not appear in the demonstration data. Consequently, selecting an appropriate α is crucial to achieve a cost vector that balances fidelity to the true environment and robustness in identifying cost structures that are not present in the estimate $\hat{\boldsymbol{c}}$.

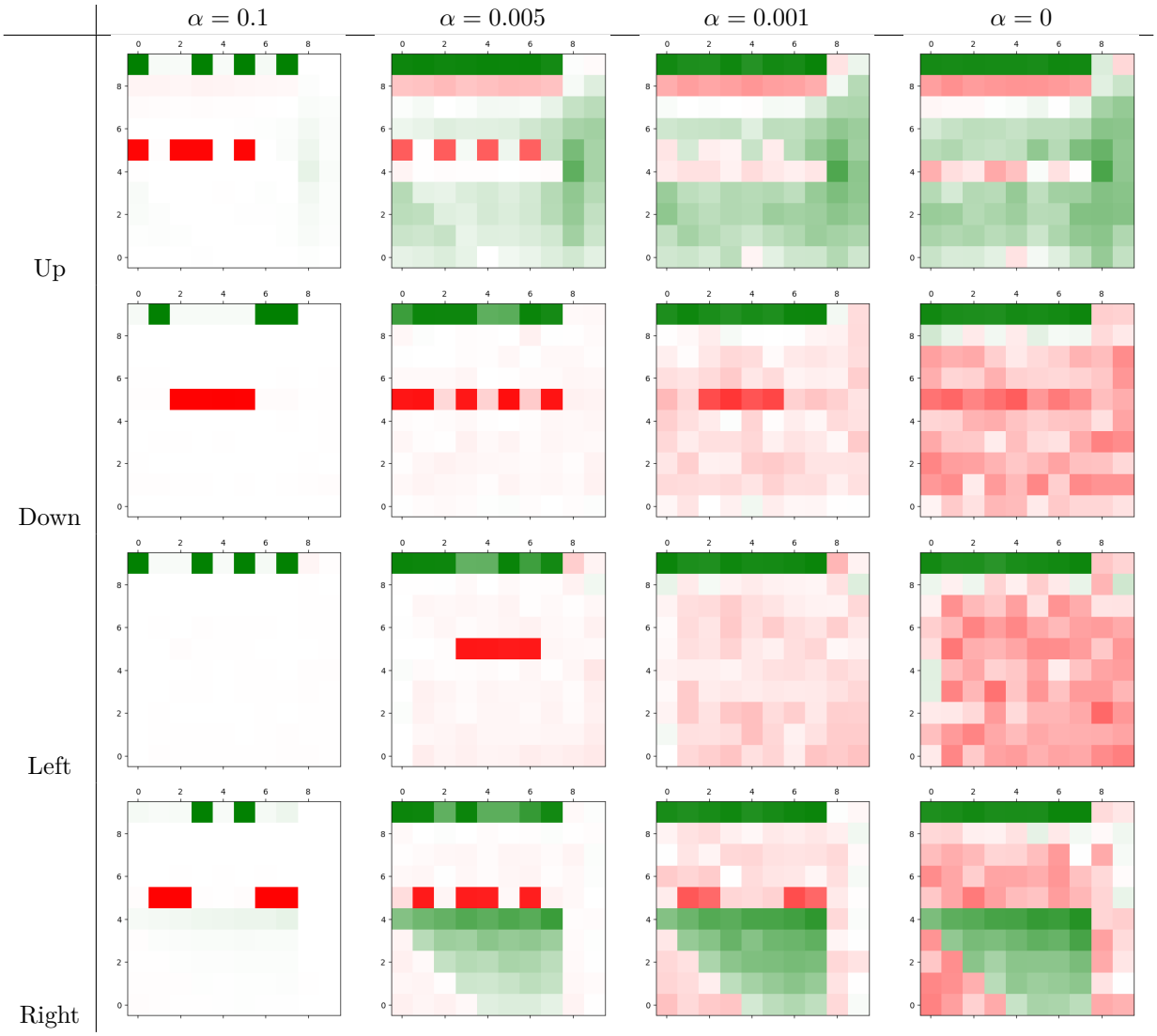


Figure 3: Effect of the regularization on the cost vector.

5.3 Convergence

In the same experimental setting, we examined the convergence of the solutions up to iteration $t < T$ by computing the L^1-norm of the difference between the solutions at iterations t and $t-1$. We plotted this norm for each of the α values considered in the previous experiments. According to this sense of convergence, all α values exhibit comparable convergence rates for $\boldsymbol{u}$ and $\boldsymbol{\mu}$. However, for $\boldsymbol{c}$, the convergence rate at $\alpha = 0.1$ is faster than that observed for smaller values of α. This behavior aligns with the increased penalty imposed for deviating from $\hat{\boldsymbol{c}}$ under stronger regularization, thereby accelerating convergence.

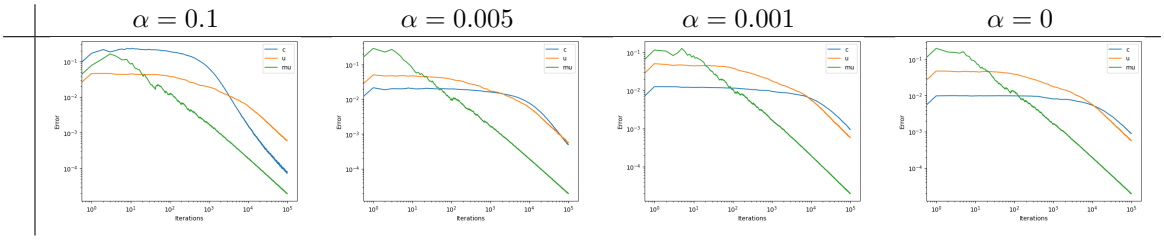


Figure 4: Effect of regularization on the difference of solutions found between iterations.

However, it is important to note that Theorem 2 characterizes convergence in terms of the duality gap. To assess this, we compute the duality gap of the solution every 25 iterations making use of IPOPT (Wächter & Biegler, 2006). The algorithm parameters remain unchanged, except that, due to computational constraints, we reduce the grid size to 6×6 and limit the execution to $T = 10^4$ iterations. Again, the stronger the regularization, the smaller the incentive to deviate from $\hat{\boldsymbol{c}}$, and therefore the faster the gap reduction at early iterations of the algorithm.

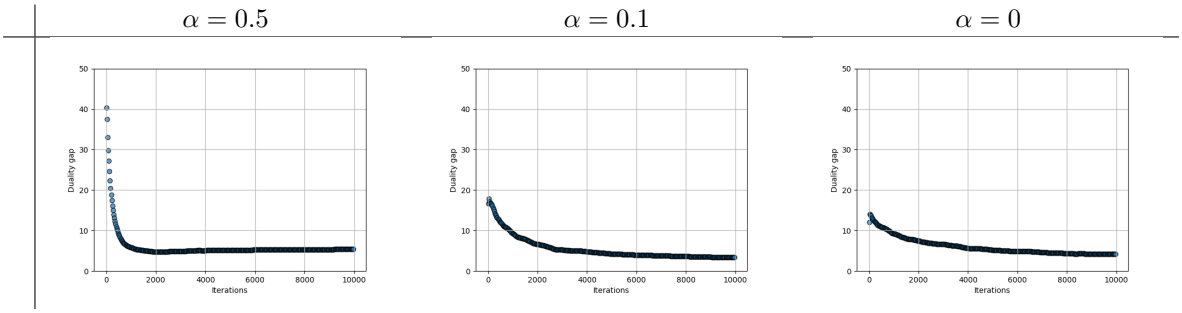


Figure 5: Duality gap convergence.

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A Appendix

A.1 MDPs

Lemma 3. *Given a DMDP with discount factor $\gamma \in (0,1)$, then the value function satisfies*

$$V_{\boldsymbol{c}}^{\pi}(s) \leq \|\boldsymbol{c}\|_{\infty} = 1$$

for any policy $\pi \in \Pi_0$ and any state $s \in \mathcal{S}$.

Proof. Given a policy $\pi \in \Pi_0$ and a state $s \in \mathcal{S}$ the value function

$$\begin{aligned} V_{\boldsymbol{c}}^{\pi}(s) &= (1-\gamma)\mathbb{E}_s^{\pi}\left[\sum_{t=0}^{\infty}\gamma^t c(s_t,a_t)\right] \\ &\leq (1-\gamma)\mathbb{E}_s^{\pi}\left[\sum_{t=0}^{\infty}\gamma^t \|\boldsymbol{c}\|_{\infty}\right] \\ &= (1-\gamma)\frac{1}{(1-\gamma)}\|\boldsymbol{c}\|_{\infty} \\ &= 1. \end{aligned}$$

□

A.2 IO, LfD, and AL

Theorem 1. *Suppose that $\boldsymbol{\mu}_{\pi_E}$ is an optimal solution for equation MDP-$P_{\boldsymbol{c}}$ where $\boldsymbol{c} \in \mathcal{C}$. Then the following equality holds:*

$$\Theta^{inv}(\boldsymbol{\mu}_{\pi_E}) = \Pi_1\left(\arg\max_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} - \boldsymbol{c}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq 0\}\right)$$

where Π_1 denotes the projection in the first component.

Proof. Note that

$$\begin{aligned} &\arg\max_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} - \boldsymbol{c}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}\} \\ =&\arg\min_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}\}, \end{aligned}$$

thus, we will prove that

$$\Theta^{\text{inv}}(\boldsymbol{\mu}_{\pi_E}) = \Pi_1\left(\arg\min_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}\}\right).$$

If $\boldsymbol{\mu}_{\pi_E} \in \mathcal{F}$, then $\boldsymbol{\mu}_{\pi_E} \geq \boldsymbol{0}$. Using the restriction $\boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}$, we have that

$$\min_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}\} \geq 0.$$

If $\boldsymbol{c}_0 \in \Theta^{\text{inv}}(\boldsymbol{\mu}_{\pi_E})$, then there exists $\boldsymbol{u}_0 \in \mathbb{R}^{|\mathcal{S}|}$ that satisfies $\boldsymbol{c}_0 - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}_0 \geq \boldsymbol{0}$ and $\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c}_0 - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}_0\rangle = 0$. This implies that

$$(\boldsymbol{c}_0, \boldsymbol{u}_0) \in \arg\min_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}\}.$$

On the other hand, if $\boldsymbol{c}^* \in \Pi_1\left(\arg\min_{(\boldsymbol{c},\boldsymbol{u})}\{\langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}\rangle \mid \boldsymbol{c} \in \mathcal{C},\ \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}\}\right)$, then $\boldsymbol{c}^* \in \mathcal{C}$ and there exists $\boldsymbol{u}^*$ such that

$$\boldsymbol{c}^* - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}^* \geq \boldsymbol{0} \text{ and } \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c}^* - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}^*\rangle \leq \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}\rangle,$$

for every pair $(\boldsymbol{c}, \boldsymbol{u})$ such that $\boldsymbol{c} \in \mathcal{C}$ and $\boldsymbol{c} - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u} \geq \boldsymbol{0}$. In particular, this is true for $(\hat{\boldsymbol{c}}, \hat{\boldsymbol{u}})$ where $\boldsymbol{\mu}_{\pi_E}$ is optimal for MDP-$P_{\hat{\boldsymbol{c}}}$ and $\hat{\boldsymbol{u}}$ is its dual optimal. Therefore, by complementary slackness we have that

$$0 \leq \langle \boldsymbol{\mu}_{\pi_E}, \boldsymbol{c}^* - \boldsymbol{T}_{\gamma}^{\top}\boldsymbol{u}^*\rangle \leq \langle \boldsymbol{\mu}_{\pi_E}, \hat{\boldsymbol{c}} - \boldsymbol{T}_{\gamma}^{\top}\hat{\boldsymbol{u}}\rangle = 0,$$

and we conclude that $\boldsymbol{c}^* \in \Theta^{\text{inv}}(\boldsymbol{\mu}_{\pi_E})$. □

A.3 Gradient estimation

Lemma 1. *Gradient estimator* $\tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c}_t,\boldsymbol{u}_t),\boldsymbol{\mu}_t)$ *is a* $(v^{(\boldsymbol{c},\boldsymbol{u})},\|\cdot\|_2)$*-bounded estimator, with*

$$v^{(\boldsymbol{c},\boldsymbol{u})} = 64\alpha^2\cdot|\mathcal{S}||\mathcal{A}| + 4(1-\alpha)^2 + 4 + 2\cdot\frac{1+\gamma^2+(1-\alpha)^2+(1-\alpha)^2\gamma^2}{(1-\gamma)^2}.$$

Proof. The unbiasedness follows from the following observations

$$\begin{aligned}
&\sum_{s,a}\frac{|\mathcal{S}||\mathcal{A}|}{|\mathcal{S}||\mathcal{A}|}\cdot 2\alpha\left(c_t(s,a)\boldsymbol{e}^{(s,a)} - \hat{c}(s,a)\boldsymbol{e}^{(s,a)}\right)\\
&+\sum_{s_E,a_E}(1-\alpha)\mu_{\pi_E}(s_E,a_E)\boldsymbol{e}^{(s_E,a_E)} - \sum_{s_t,a_t}\mu_t(s_t,a_t)\boldsymbol{e}^{(s_t,a_t)}\\
&= 2\alpha(\boldsymbol{c}_t-\hat{\boldsymbol{c}}) + (1-\alpha)\boldsymbol{\mu}_{\pi_E} - \boldsymbol{\mu}_t
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{(1-\gamma)}\sum_{s_t',a_t,s_t}\mu_t(s_t,a_t)P(s_t'\mid s_t,a_t)(\boldsymbol{e}^{(s_t)}-\gamma\boldsymbol{e}^{(s_t')})\\
&\quad= \frac{1}{(1-\gamma)}\left(\sum_{s_t,a_t}\mu_t(s_t,a_t)\boldsymbol{e}^{(s_t)} - \gamma\sum_{s_t',a_t,s_t}\mu_t(s_t,a_t)P(s_t'\mid s_t,a_t)\boldsymbol{e}^{(s_t')}\right)\\
&\quad= \frac{1}{(1-\gamma)}(\boldsymbol{B}\boldsymbol{\mu}-\gamma\boldsymbol{P}\boldsymbol{\mu})\\
&\quad= \boldsymbol{T}_\gamma\boldsymbol{\mu}.
\end{aligned}$$

Thus, we obtain

$$\mathbb{E}\left[\tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c},\boldsymbol{u}),\boldsymbol{\mu})\right] = \begin{pmatrix}2\alpha(\boldsymbol{c}_t-\hat{\boldsymbol{c}})+(1-\alpha)\boldsymbol{\mu}_{\pi_E}-\boldsymbol{\mu}_t\\ \boldsymbol{T}_\gamma\boldsymbol{\mu}-(1-\alpha)\boldsymbol{T}_\gamma\boldsymbol{\mu}_{\pi_E}\end{pmatrix}.$$

For proving the bound on the second-moment, note that

$$\begin{aligned}
&\mathbb{E}\left[\left\||\mathcal{S}||\mathcal{A}|\cdot 2\alpha\left(c_t(s,a)\boldsymbol{e}^{(s,a)}-\hat{c}(s,a)\boldsymbol{e}^{(s,a)}\right)+(1-\alpha)\boldsymbol{e}^{(s_E,a_E)}-\boldsymbol{e}^{(s_t,a_t)}\right\|_2^2\right]\\
&\le 2\mathbb{E}\left[4|\mathcal{S}||\mathcal{A}|\alpha^2\left\|c_t(s,a)\boldsymbol{e}^{(s,a)}-\hat{c}(s,a)\boldsymbol{e}^{(s,a)}\right\|_2^2+\left\|(1-\alpha)\boldsymbol{e}^{(s_E,a_E)}-\boldsymbol{e}^{(s_t,a_t)}\right\|_2^2\right]\\
&\le 2\mathbb{E}\left[4|\mathcal{S}||\mathcal{A}|\alpha^2\cdot 4+(1-\alpha)^2+1\right]\\
&\le 32|\mathcal{S}||\mathcal{A}|\alpha^2+2(1-\alpha)^2+2,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\left\|\frac{1}{(1-\gamma)}\left(\boldsymbol{e}^{(s_t)}-\gamma\boldsymbol{e}^{(s_t')}-(1-\alpha)(\boldsymbol{e}^{(s_E)}-\gamma\boldsymbol{e}^{(s_E')})\right)\right\|_2^2\right] &\le \mathbb{E}\left[\frac{1+\gamma^2+(1-\alpha)^2+(1-\alpha)^2\gamma^2}{(1-\gamma)^2}\right]\\
&= \frac{1+\gamma^2+(1-\alpha)^2+(1-\alpha)^2\gamma^2}{(1-\gamma)^2}.
\end{aligned}$$

Hence, we can provide the bound

$$\begin{aligned}
\mathbb{E}\left[\left\|\tilde{g}^{(\boldsymbol{c},\boldsymbol{u})}((\boldsymbol{c},\boldsymbol{u}),\boldsymbol{\mu})\right\|_2^2\right] &\overset{(i)}{\le} 2\left[(32|\mathcal{S}||\mathcal{A}|\alpha^2+2(1-\alpha)^2+2)+\frac{1+\gamma^2+(1-\alpha)^2+(1-\alpha)^2\gamma^2}{(1-\gamma)^2}\right]\\
&= 64\alpha^2\cdot|\mathcal{S}||\mathcal{A}|+4(1-\alpha)^2+4+2\cdot\frac{1+\gamma^2+(1-\alpha)^2+(1-\alpha)^2\gamma^2}{(1-\gamma)^2}.
\end{aligned}$$

where in (i) we used $\|\boldsymbol{x}+\boldsymbol{y}\|^2\le 2[\|\boldsymbol{x}\|^2+\|\boldsymbol{y}\|^2]$. □

Lemma 2. *Gradient estimator $\tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t)$ is a $(z^{\boldsymbol{\mu}}, v^{\boldsymbol{\mu}}, \|\cdot\|_2)$-bounded estimator, with*

$$z^{\boldsymbol{\mu}} = \frac{2|\mathcal{S}||\mathcal{A}|}{(1-\gamma)} \text{ and } v^{\boldsymbol{\mu}} = |\mathcal{S}||\mathcal{A}|\left(2 + \frac{4(1+\gamma^2)}{(1-\gamma)^2}\right).$$

Proof. The unbiasedness follows from

$$\begin{aligned}
\mathbb{E}[\tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t)] &= \sum_{s,a,s'} \frac{1}{|\mathcal{S}||\mathcal{A}|} P(s' \mid s,a) \cdot |\mathcal{S}||\mathcal{A}| \left(c_t(s,a)\boldsymbol{e}^{(s,a)} - \frac{1}{(1-\gamma)}(u_t(s)\boldsymbol{e}^{(s,a)} - \gamma u_t(s')\boldsymbol{e}^{(s,a)})\right) \\
&= \sum_{s,a} c_t(s,a)\boldsymbol{e}^{(s,a)} - \frac{1}{(1-\gamma)}\left(\sum_{s,a} u_t(s)\boldsymbol{e}^{(s,a)} - \gamma \sum_{s,a}\left(\sum_{s'} P(s' \mid s,a) u_t(s')\right)\boldsymbol{e}^{(s,a)}\right) \\
&= \boldsymbol{c}_t - \frac{1}{(1-\gamma)}\left(\boldsymbol{B}^\top \boldsymbol{u} - \gamma \boldsymbol{P}^\top \boldsymbol{u}\right) \\
&= \boldsymbol{c}_t - \boldsymbol{T}_\gamma^\top \boldsymbol{u}.
\end{aligned}$$

For the bound on the maximum entry observe that

$$\begin{aligned}
\|\tilde{g}^{\boldsymbol{\mu}}((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t)\|_\infty &= |\mathcal{S}||\mathcal{A}| \max_{s,a,s'} \left\{\left| c_t(s,a) - \frac{u_t(s) - \gamma u_t(s')}{(1-\gamma)}\right|\right\} \\
&\leq |\mathcal{S}||\mathcal{A}| \left(\|\boldsymbol{c}_t\|_\infty + \|\boldsymbol{u}_t\|_\infty \frac{(1+\gamma)}{(1-\gamma)}\right) \\
&= \frac{2|\mathcal{S}||\mathcal{A}|}{(1-\gamma)}
\end{aligned}$$

Finally, the bound on the second-moment can be obtained by:

$$\begin{aligned}
\mathbb{E}\left[\|\tilde{g}^{(\boldsymbol{\mu})}((\boldsymbol{c}_t, \boldsymbol{u}_t), \boldsymbol{\mu}_t)\|^2_{\boldsymbol{\mu}'}\right] &\overset{(i)}{\leq} |\mathcal{S}|^2|\mathcal{A}|^2 \cdot \mathbb{E}\left[2\|c_t(s,a)\boldsymbol{e}^{(s,a)}\|^2_{\boldsymbol{\mu}'} + \frac{4}{(1-\gamma)^2}(\|u_t(s)\boldsymbol{e}^{(s,a)}\|^2_{\boldsymbol{\mu}'} + \|\gamma u_t(s')\boldsymbol{e}^{(s,a)}\|^2_{\boldsymbol{\mu}'})\right] \\
&= |\mathcal{S}|^2|\mathcal{A}|^2 \cdot \mathbb{E}\left[\mu'(s,a)\left(2(c_t(s,a))^2 + \frac{4}{(1-\gamma)^2}(u_t(s))^2 + \frac{4\gamma^2}{(1-\gamma)^2}(u_t(s'))^2\right)\right] \\
&= |\mathcal{S}||\mathcal{A}|\left[\sum_{s,a} \mu'(s,a)\left(2(c_t(s,a))^2 + \frac{4}{(1-\gamma)^2}(u_t(s))^2\right)\right. \\
&\quad \left. + \sum_{s',s,a} \mu'(s,a)P(s' \mid s,a)\frac{4\gamma^2}{(1-\gamma)^2}(u_t(s'))^2\right] \\
&\overset{(ii)}{\leq} |\mathcal{S}||\mathcal{A}|\left[\left(2 + \frac{4}{(1-\gamma)^2}\right)\sum_{s,a}\mu'(s,a) + \frac{4\gamma^2}{(1-\gamma)^2}\sum_{s',s,a}\mu'(s,a)P(s' \mid s,a)\right] \\
&= |\mathcal{S}||\mathcal{A}|\left(2 + \frac{4(1+\gamma^2)}{(1-\gamma)^2}\right),
\end{aligned}$$

where we used $\|\boldsymbol{x}+\boldsymbol{y}\|^2 \leq 2[\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2]$ two times for (i) and that $(\boldsymbol{c}_t, \boldsymbol{u}_t) \in \mathbb{B}_1^{|\mathcal{S}||\mathcal{A}|} \times \mathbb{B}_1^{|\mathcal{S}|}$ for (ii). □

A.4 Algorithm convergence

We will follow Jin & Sidford (2020) and show how their results accommodates to our problem.

Definition 3 (ℓ_∞-ℓ_1 convex-concave min-max problem)**.** *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a differentiable function that is convex in $\boldsymbol{x} \in \mathbb{R}^n$ and concave in $\boldsymbol{y} \in \mathbb{R}^m$. We define the ℓ_∞-ℓ_1 convex-concave min-max problem as*

$$\min_{\boldsymbol{x} \in \mathbb{B}_b^n} \max_{\boldsymbol{y} \in \Delta^m} f(\boldsymbol{x}, \boldsymbol{y}).$$

Furthermore, define the operator

$$G(\boldsymbol{z}) = G(\boldsymbol{x}, \boldsymbol{y}) = [\nabla_{\boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{y}), -\nabla_{\boldsymbol{y}} f(\boldsymbol{x}, \boldsymbol{y})] = [g^{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}), g^{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})].$$

Lemma 3 (cf. Appendix A.1 in Carmon et al. (2019))**.** *For every $\boldsymbol{z}_1, ..., \boldsymbol{z}_K \in \mathcal{Z}$ it holds that*

$$Gap\left(\frac{1}{K}\sum_{k=1}^{K} \boldsymbol{z}_k\right) \leq \sup_{\boldsymbol{u}\in\mathcal{Z}} \frac{1}{K}\sum_{k=1}^{K} \langle G(\boldsymbol{z}_k), \boldsymbol{z}_k - \boldsymbol{u}\rangle.$$

Proof. For all $\boldsymbol{z} \in \mathcal{Z}$, $f(\boldsymbol{z}^{\boldsymbol{x}}, \boldsymbol{u}^{\boldsymbol{y}})$ is concave in $\boldsymbol{u}^{\boldsymbol{y}}$ and $-f(\boldsymbol{u}^{\boldsymbol{x}}, \boldsymbol{z}^{\boldsymbol{y}})$ is concave in $\boldsymbol{u}^{\boldsymbol{x}}$. Therefore, $\text{gap}(\boldsymbol{z}, \boldsymbol{u})$ is concave in $\boldsymbol{u}$ for every $\boldsymbol{z}$ and we have

$$\begin{aligned}
\text{gap}(\boldsymbol{z}, \boldsymbol{u}) &\leq \text{gap}(\boldsymbol{z}, \boldsymbol{z}) + \langle \nabla_{\boldsymbol{u}} \text{gap}(\boldsymbol{z}, \boldsymbol{z}), \boldsymbol{u} - \boldsymbol{z}\rangle \\
&= \langle \nabla_{\boldsymbol{u}} \text{gap}(\boldsymbol{z}, \boldsymbol{z}), \boldsymbol{u} - \boldsymbol{z}\rangle \\
&= \langle -G(\boldsymbol{z}), \boldsymbol{u} - \boldsymbol{z}\rangle \\
&= \langle G(\boldsymbol{z}), \boldsymbol{z} - \boldsymbol{u}\rangle.
\end{aligned}$$

Similarly, $\text{gap}(\boldsymbol{z}; \boldsymbol{u})$ is convex in $\boldsymbol{z}$ for every $\boldsymbol{u}$. Therefore,

$$\text{gap}\left(\frac{1}{K}\sum_{k=1}^{K} \boldsymbol{z}_k; \boldsymbol{u}\right) \leq \frac{1}{K}\sum_{k=1}^{K} \text{gap}(\boldsymbol{z}_k; \boldsymbol{u}) \leq \frac{1}{K}\sum_{k=1}^{K} \langle G(\boldsymbol{z}_k), \boldsymbol{z}_k - \boldsymbol{u}\rangle,$$

where the first inequality follows from convexity in $\boldsymbol{z}$ and the second inequality from above's result. Taking the supremum over the inequality yields the result. □

The two divergences that we will use in the stochastic mirror descent algorithm for the ℓ_∞-ℓ_1 convex-concave min-max problem are the following:

1. given the euclidean distance $h(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|_2^2$, we obtain the divergence $V_{\boldsymbol{x}}(\boldsymbol{x}') = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}'\|_2^2$;
2. given $h(\boldsymbol{y}) = \sum_i y_i \log y_i$, we obtain the Kullback-Leibler divergence $V_{\boldsymbol{y}}(\boldsymbol{y}') = \sum y_i \log\left(\frac{y_i'}{y_i}\right)$.

Theorem 3. *Given a ℓ_∞-ℓ_1 convex-concave-min-max problem 3, desired accuracy ϵ, $(v^{\boldsymbol{x}}, \|\cdot\|_2)$-bounded estimators $\tilde{g}^{\boldsymbol{x}}$ of $g^{\boldsymbol{x}}$, and $(\frac{2v^{\boldsymbol{y}}}{\epsilon}, v^{\boldsymbol{y}}, \|\cdot\|_{\Delta^m}^2)$-bounded estimators $\tilde{g}^{\boldsymbol{y}}$ of $g^{\boldsymbol{y}}$. Algorithm 1 with choice of parameters $\eta_{\boldsymbol{x}} \leq \frac{\epsilon}{4v^{\boldsymbol{x}}}$, $\eta_{\boldsymbol{y}} \leq \frac{\epsilon}{4v^{\boldsymbol{y}}}$ outputs an ϵ-approximate optimal solution within any iteration number $T \geq \max\{\frac{16nb^2}{\epsilon\eta_x}, \frac{8\log(m)}{\epsilon\eta_y}\}$.*

Proof. Note that $V_{\boldsymbol{x}}$ is 1-strongly convex. Since $\eta_{\boldsymbol{y}} \leq \frac{\epsilon}{4v^{\boldsymbol{y}}}$, we have that

$$\|\eta^{\boldsymbol{y}}\tilde{g}_t^{\boldsymbol{y}}\|_\infty \leq \frac{\epsilon}{4v^{\boldsymbol{y}}} \cdot \|\tilde{g}_t^{\boldsymbol{y}}\|_\infty \leq \frac{\epsilon}{4v^{\boldsymbol{y}}} \cdot \frac{2v^{\boldsymbol{y}}}{\epsilon} = \frac{1}{2}.$$

Hence, by Lemma 1 and Lemma 2 in Jin & Sidford (2020) we know that

$$\begin{aligned}
\sum_{t\in[T]} \langle \eta^{\boldsymbol{x}}\tilde{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \boldsymbol{x}\rangle &\leq V_{\boldsymbol{x}_1}(\boldsymbol{x}) + \frac{{\eta^{\boldsymbol{x}}}^2}{2}\sum_{t\in[T]} \|\tilde{g}_t^{\boldsymbol{x}}\|_2^2, \\
\sum_{t\in[T]} \langle \eta^{\boldsymbol{y}}\tilde{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \boldsymbol{y}\rangle &\leq V_{\boldsymbol{y}_1}(\boldsymbol{y}) + \frac{{\eta^{\boldsymbol{y}}}^2}{2}\sum_{t\in[T]} \|\tilde{g}_t^{\boldsymbol{y}}\|_{\boldsymbol{y}_t}^2.
\end{aligned}$$

Now, define $\hat{g}_t^{\boldsymbol{x}} := g_t^{\boldsymbol{x}} - \tilde{g}_t^{\boldsymbol{x}}$, $\hat{g}_t^{\boldsymbol{y}} := g_t^{\boldsymbol{y}} - \tilde{g}_t^{\boldsymbol{y}}$, and the sequences $\hat{\boldsymbol{x}}_1, ..., \hat{\boldsymbol{x}}_T$ and $\hat{\boldsymbol{y}}_1, ..., \hat{\boldsymbol{y}}_T$ by

$$\begin{aligned}
\hat{\boldsymbol{x}}_1 = \boldsymbol{x}_1,\ \hat{\boldsymbol{x}}_{t+1} &= \arg\min_{\boldsymbol{x}\in\mathbb{B}_b^n} \langle \eta^{\boldsymbol{x}}\hat{g}_t^{\boldsymbol{x}}, \boldsymbol{x}\rangle + V_{\hat{\boldsymbol{x}}_t}(\boldsymbol{x}), \\
\hat{\boldsymbol{y}}_1 = \boldsymbol{y}_1,\ \hat{\boldsymbol{y}}_{t+1} &= \arg\min_{\boldsymbol{y}\in\Delta^m} \langle \eta^{\boldsymbol{y}}\hat{g}_t^{\boldsymbol{y}}, \boldsymbol{y}\rangle + V_{\hat{\boldsymbol{y}}_t}(\boldsymbol{y}).
\end{aligned}$$

In a similar way to $\eta^{\boldsymbol{y}} g_t^{\boldsymbol{y}}$, we can bound the ℓ_∞-norm of $\eta^{\boldsymbol{y}} \hat{g}_t^{\boldsymbol{y}}$

$$\|\eta^{\boldsymbol{y}} \hat{g}_t^{\boldsymbol{y}}\|_\infty \leq \|\eta^{\boldsymbol{y}} \tilde{g}_t^{\boldsymbol{y}}\|_\infty + \|\eta^{\boldsymbol{y}} g_t^{\boldsymbol{y}}\|_\infty = \|\eta^{\boldsymbol{y}} \tilde{g}_t^{\boldsymbol{y}}\|_\infty + \|\mathbb{E}[\eta^{\boldsymbol{y}} \tilde{g}_t^{\boldsymbol{y}}]\|_\infty \leq 2\|\eta^{\boldsymbol{y}} \tilde{g}_t^{\boldsymbol{y}}\|_\infty \leq 1.$$

Therefore, using the lemmas as above we get

$$\sum_{t\in[T]} \langle \eta^{\boldsymbol{x}} \hat{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \boldsymbol{x}\rangle \leq V_{\boldsymbol{x}_1}(\boldsymbol{x}) + \frac{{\eta^{\boldsymbol{x}}}^2}{2} \sum_{t\in[T]} \|\hat{g}_t^{\boldsymbol{x}}\|_2^2,$$
$$\sum_{t\in[T]} \langle \eta^{\boldsymbol{y}} \hat{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \boldsymbol{y}\rangle \leq V_{\boldsymbol{y}_1}(\boldsymbol{y}) + \frac{{\eta^{\boldsymbol{y}}}^2}{2} \sum_{t\in[T]} \|\hat{g}_t^{\boldsymbol{y}}\|_{\boldsymbol{y}_t}^2.$$

Since $g_t^{\boldsymbol{x}} = \hat{g}_t^{\boldsymbol{x}} + \tilde{g}_t^{\boldsymbol{x}}$ and $g_t^{\boldsymbol{y}} = \hat{g}_t^{\boldsymbol{y}} + \tilde{g}_t^{\boldsymbol{y}}$,

$$\begin{aligned}
&\sum_{t\in[T]} [\langle g_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \boldsymbol{x}\rangle + \langle g_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \boldsymbol{y}\rangle] \\
=&\sum_{t\in[T]} \left[\frac{1}{\eta^{\boldsymbol{x}}}\langle \eta^{\boldsymbol{x}} \tilde{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \boldsymbol{x}\rangle + \frac{1}{\eta^{\boldsymbol{y}}}\langle \eta^{\boldsymbol{y}} \tilde{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \boldsymbol{y}\rangle\right] + \sum_{t\in[T]} \left[\frac{1}{\eta^{\boldsymbol{x}}}\langle \eta^{\boldsymbol{x}} \hat{g}_t^{\boldsymbol{x}}, \hat{\boldsymbol{x}}_t - \boldsymbol{x}\rangle + \frac{1}{\eta^{\boldsymbol{y}}}\langle \eta^{\boldsymbol{y}} \hat{g}_t^{\boldsymbol{y}}, \hat{\boldsymbol{y}}_t - \boldsymbol{y}\rangle\right] \\
&+ \sum_{t\in[T]} [\langle \hat{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\rangle + \langle \hat{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t\rangle] \\
=&\frac{1}{\eta^{\boldsymbol{x}}} \sum_{t\in[T]} [\langle \eta^{\boldsymbol{x}} \tilde{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \boldsymbol{x}\rangle] + \frac{1}{\eta^{\boldsymbol{x}}} \sum_{t\in[T]} [\langle \eta^{\boldsymbol{x}} \hat{g}_t^{\boldsymbol{x}}, \hat{\boldsymbol{x}}_t - \boldsymbol{x}\rangle] + \frac{1}{\eta^{\boldsymbol{y}}} \sum_{t\in[T]} [\langle \eta^{\boldsymbol{y}} \tilde{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \boldsymbol{y}\rangle] + \frac{1}{\eta^{\boldsymbol{y}}} \sum_{t\in[T]} [\langle \eta^{\boldsymbol{y}} \hat{g}_t^{\boldsymbol{y}}, \hat{\boldsymbol{y}}_t - \boldsymbol{y}\rangle] \\
&+ \sum_{t\in[T]} [\langle \hat{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\rangle + \langle \hat{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t\rangle] \\
\leq&\frac{2}{\eta^{\boldsymbol{x}}} V_{\boldsymbol{x}_1}(\boldsymbol{x}) + \frac{\eta^{\boldsymbol{x}}}{2} \sum_{t\in[T]} \left[\|\tilde{g}_t^{\boldsymbol{x}}\|_2^2 + \|\hat{g}_t^{\boldsymbol{x}}\|_2^2\right] + \sum_{t\in[T]} \langle \hat{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\rangle \\
&+ \frac{2}{\eta^{\boldsymbol{y}}} V_{\boldsymbol{y}_1}(\boldsymbol{y}) + \frac{\eta^{\boldsymbol{y}}}{2} \sum_{t\in[T]} \left[\|\tilde{g}_t^{\boldsymbol{y}}\|_{\boldsymbol{y}_t}^2 + \|\hat{g}_t^{\boldsymbol{y}}\|_{\boldsymbol{y}_t}^2\right] + \sum_{t\in[T]} \langle \hat{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t\rangle.
\end{aligned}$$

Consider the operator $G(\boldsymbol{z}) := [g^{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}), -g^{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y})]$. As min-max problem 3 is convex in its first argument and concave in its second argument, by Lemma 3, if we show that

$$\sup_{\boldsymbol{u}\in\mathcal{Z}} \frac{1}{T} \sum_{t\in[T]} \langle G(\boldsymbol{z}_t), \boldsymbol{z}_t - \boldsymbol{u}\rangle \leq \epsilon,$$

we obtain that $\mathrm{Gap}(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{z}_t) \leq \epsilon$. Now take the expectation on both sides:

$$\begin{aligned}
\mathbb{E}\sup_{\boldsymbol{u}\in\mathcal{Z}}\frac{1}{T}\sum_{t\in[T]}\langle G(\boldsymbol{z}_t), \boldsymbol{z}_t - \boldsymbol{u}\rangle &= \mathbb{E}\frac{1}{T}\sup_{(\boldsymbol{x},\boldsymbol{y})}\left[\sum_{t\in[T]}\langle g^{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{x}_t - \boldsymbol{x}\rangle + \sum_{t\in[T]}\langle g^{\boldsymbol{y}}(\boldsymbol{x},\boldsymbol{y}), \boldsymbol{y}_t - \boldsymbol{y}\rangle\right] \\
&\overset{(i)}{\leq} \frac{1}{T}\mathbb{E}\sup_{(\boldsymbol{x},\boldsymbol{y})}\left[\frac{2}{\eta^{\boldsymbol{x}}}V_{\boldsymbol{x}_1}(\boldsymbol{x}) + \frac{\eta^{\boldsymbol{x}}}{2}\sum_{t\in[T]}\left[\|\tilde{g}_t^{\boldsymbol{x}}\|_2^2 + \|\hat{g}_t^{\boldsymbol{x}}\|_2^2\right]\right. \\
&\quad \left. + \frac{2}{\eta^{\boldsymbol{y}}}V_{\boldsymbol{y}_1}(\boldsymbol{y}) + \frac{\eta^{\boldsymbol{y}}}{2}\sum_{t\in[T]}\left[\|\tilde{g}_t^{\boldsymbol{y}}\|_2^2 + \|\hat{g}_t^{\boldsymbol{y}}\|_2^2\right]\right] \\
&\overset{(ii)}{\leq} \frac{1}{T}\mathbb{E}\sup_{(\boldsymbol{x},\boldsymbol{y})}\left[\frac{2}{\eta^{\boldsymbol{x}}}V_{\boldsymbol{x}_1}(\boldsymbol{x}) + \eta^{\boldsymbol{x}}\sum_{t\in[T]}\|\tilde{g}_t^{\boldsymbol{x}}\|_2^2 + \frac{2}{\eta^{\boldsymbol{y}}}V_{\boldsymbol{y}_1}(\boldsymbol{y}) + \eta^{\boldsymbol{y}}\sum_{t\in[T]}\|\tilde{g}_t^{\boldsymbol{y}}\|_2^2\right] \\
&\overset{(iii)}{\leq} \sup_{\boldsymbol{x}}\frac{2}{\eta^{\boldsymbol{x}}T}V_{\boldsymbol{x}_1}(\boldsymbol{x}) + \eta^{\boldsymbol{x}}v^{\boldsymbol{x}} + \sup_{\boldsymbol{y}}\frac{2}{\eta^{\boldsymbol{y}}T}V_{\boldsymbol{y}_1}(\boldsymbol{y}) + \eta^{\boldsymbol{y}}v^{\boldsymbol{y}} \\
&\overset{(iv)}{\leq} \frac{4nb^2}{\eta^{\boldsymbol{x}}T} + \eta^{\boldsymbol{x}}v^{\boldsymbol{x}} + \frac{2\log m}{\eta^{\boldsymbol{y}}T} + \eta^{\boldsymbol{y}}v^{\boldsymbol{y}} \\
&\overset{(v)}{\leq} \epsilon,
\end{aligned}$$

where in (i) we used that $\mathbb{E}[\langle\hat{g}_t^{\boldsymbol{x}}, \boldsymbol{x}_t - \hat{\boldsymbol{x}}_t\rangle \mid 1, ..., T] = \mathbb{E}[\langle\hat{g}_t^{\boldsymbol{y}}, \boldsymbol{y}_t - \hat{\boldsymbol{y}}_t\rangle \mid 1, ..., T] = 0$; (ii) $\mathbb{E}[\|\hat{g}_t^{\boldsymbol{x}}\|_2^2] \leq \mathbb{E}[\|\tilde{g}_t^{\boldsymbol{x}}\|_2^2]$ and $\mathbb{E}[\sum_i [\hat{y}_t]_i [\hat{g}_t^y]_i^2] \leq \mathbb{E}[\sum_i [\hat{y}_t]_i [\tilde{g}_t^y]_i^2]$ due to $\mathbb{E}[(X - \mathbb{E}[X])^2] \leq \mathbb{E}[X^2]$; (iii) due to the assumptions on the estimators; (iv) by properties of KL-divergence and that $\frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}_0\|_2^2 \leq 2nb^2$; (v) the choice of $\eta^{\boldsymbol{x}} = \frac{\epsilon}{4v^{\boldsymbol{x}}}$, $\eta^{\boldsymbol{y}} = \frac{\epsilon}{4v^{\boldsymbol{y}}}$, and $T \geq \max\{\frac{16nb^2}{\epsilon\eta^{\boldsymbol{x}}}, \frac{8\log m}{\epsilon\eta^{\boldsymbol{y}}}\}$. □

Theorem 2. *Given $\epsilon \in (0, 1)$, Algorithm 1 with step-size*

$$\eta^{(\boldsymbol{c},\boldsymbol{u})} = \frac{\epsilon}{4v^{(\boldsymbol{c},\boldsymbol{u})}}, \quad \eta^{\boldsymbol{\mu}} = \frac{\epsilon}{4v^{\boldsymbol{\mu}}},$$

and gradient estimators defined in equation 1 and 2 finds an expected ϵ-approximate solution within any iteration number

$$T \geq \max\left\{\mathcal{O}\left(\frac{\alpha^2|\mathcal{S}|^3|\mathcal{A}|^2}{\epsilon^2}\right), \mathcal{O}\left(\frac{|\mathcal{S}||\mathcal{A}|\log(|\mathcal{S}||\mathcal{A}|)}{\epsilon^2}\right)\right\}.$$

Proof. This follows directly from the bounds of the gradient estimators and Theorem 3. □