SYNTHETIC CONTROLS AS BALANCING SCORES

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ABSTRACT

We outline the factors under which conditioning on Synthetic Control (SC) weights emulates a randomized control trial where the treatment status is independent of potential outcomes. Specifically, we demonstrate that if there exist SC weights such that the treatment effects are exactly identified, and these weights are uniformly and cumulatively bounded, then *SC weights are balancing scores*.

INTRODUCTION

Synthetic control (SC) and balancing score methods are critical tools for causal inference (Abadie et al., 2010; 2015; Rosenbaum & Rubin, 1983; Hansen, 2008). The theoretical properties of these approaches are well-studied in the literature (Abadie et al., 2015; Ferman & Pinto, 2016; 2017; Shi et al., 2021; Rosenbaum & Rubin, 1983; Hansen, 2008). However, the connections between them are not well understood. This short note bridges this gap by proving that SC weights are balancing scores under a sufficient set of assumptions. It is of general interest because conditioning on balancing scores guarantees ignorability (i.e., that potential outcomes and treatment assignment are mutually independent) – ensuring consistent estimation of treatment effects (Johnson, 2013).

SYNTHETIC CONTROL: SETUP, ASSUMPTIONS AND IDENTIFICATION

Consider a finite population of n units, denoted by $\mathcal{S}_n$. For each unit $i \in \mathcal{S}_n$, we observe the following three quantities: (i) Z_i, the binary treatment indicator, (ii) $\{X_{i,t}\}_{t=0}^{t_0}$, the time-series of pre-treatment outcomes, and (iii) $\{Y_{i,t}\}_{t=t_0+1}^{T}$, the time-series of post-treatment outcomes. Our setup assumes that exactly one of the units is treated. Without loss of generality, let $Z_1 = 1$ and $\forall j > 1, Z_j = 0$. We assume that these timeseries are generated using the following factor model:

$$\text{for } t \leq t_0: \qquad X_{i,t} = \delta_t + \lambda_t \mu_i + \epsilon_{i,t} \tag{1}$$

$$\text{for } t > t_0: \qquad Y_{i,t} = \delta_t + \lambda_t \mu_i + Z_i \alpha_{i,t} + \epsilon_{i,t} \tag{2}$$

where, (i) δ_t is an unobserved common time-trend across units, (ii) μ_i is an unobserved unit specific factor, (iii) λ_t is the time specific factor loading, (iv) $\epsilon_{i,t} \sim \mathcal{N}(0, \sigma_{i,t}^2)$ is the noise at time t and (v) $\alpha_{i,t}$ is the treatment effect for unit i at time t. We are interested in identifying $\{\alpha_{1,t}\}_{t=t_0+1}^{T}$. This setup is similar to the one discussed in Ferman & Pinto (2017).

We will use bold letters to denote the collection of observations across units: $\boldsymbol{Z} = \{Z_i\}_{i=1}^n$, $\boldsymbol{X} = \{\{X_{i,t}\}_{t=0}^{t_0}\}_{i=1}^n$ and $\boldsymbol{Y} = \{\{Y_{i,t}\}_{t=t_0+1}^{T}\}_{i=1}^n$. Consider, the scenario where the probability of observing treatment assignments is a function of the unit specific factors $\boldsymbol{\mu} = \{\mu_i\}_{i=1}^n$ i.e.

$$\boldsymbol{Z} \sim f(\boldsymbol{\mu}) \tag{3}$$

for some (unknown) distribution f such that $\mathbf{1}^T \boldsymbol{Z} = 1$. Figure 1(a) graphically demonstrates the causal dependencies described in equations 1, 2, and 3.

Consider the following assumption:
A.1. ***(feasibility)*** $\exists \boldsymbol{\beta}$ *s.t.* $\mathbb{E}\left(X_{1,t} - \sum_{i=2}^n \beta_i X_{i,t}\right) = 0$.

Identification. Assumption A.1. implies that $\mu_1 = \sum_{i=2}^n \beta_i \mu_i$. If we choose β_i's such that the condition in A.1. are satisfied, then

$$\mathbb{E}\left(Y_{1,t} - \sum_{i=2}^n \beta_i Y_{i,t}\right) = \lambda(\mu_1 - \sum_{i=2}^n \beta_i \mu_i) + \alpha_{1,t} = \alpha_{1,t}.$$

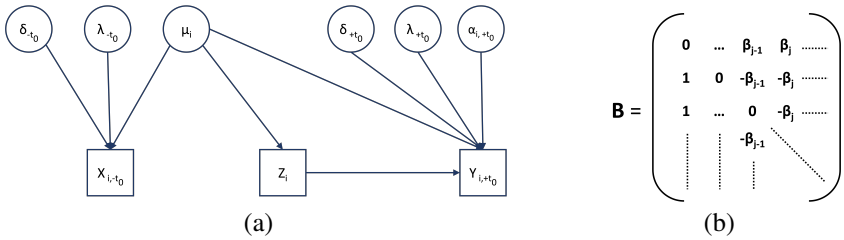

Figure 1: (a) A simplified causal directed acyclic graph for the discussed exposition. Here, with the slight abuse of notation, $-t_0$ refers to the time before intervention and $+t_0$ refers to the time after the intervention. (b) Construction of $\mathbf{B}$ matrix such that $\mathbf{B}_{i,i} = 0$, $\mathbf{B}_{i,1} = 1$ (for $i > 1$), $\mathbf{B}_{1,j} = \beta_j$ (for $j > 1$) and $\mathbf{B}_{i,j} = -\beta_j$ (for $j > 1$ and $i > 1$).

Thus, identifying $\boldsymbol{\beta}$ such that $\mathbb{E}\left(X_{1,t} - \sum_{i=2}^{n} \beta_i X_{i,t}\right) = 0$ leads to the identification of the treatment effect $\alpha_{1,t}$.

We, further, assume that there exists *feasible* weights $\boldsymbol{\beta} = \{\beta_2, ..., \beta_n\}$ such that:
A.2. ***(uniformly bounded)*** $\forall i$, $0 \leq \beta_i \leq 1$,
A.3. ***(cumulatively bounded)*** $\sum_{i=2}^{n} \beta_i = 1$ *i.e.* $\mathbf{1}^T \boldsymbol{\beta} = 1$.

We refer to these as *oracle weights* because it may not be feasible to exactly identify them using the observed finite population, especially, when the noise is heteroskedastic[1] (Ferman & Pinto, 2017). For the rest of the argument, we will assume that we have knowledge of these oracle weights. However, one can always estimate approximate oracle weights by fitting a regularized linear regression on the pre-treatment outcomes (Abadie et al., 2010; 2015).

Result: SC Weights are Balancing Score

For our setup, if there exists a function b such that $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} | b(\mathbf{X})$ then b is a *balancing score*. However, consider the causal graph in Figure 1(a): it is not obvious that $b(\mathbf{X}) = \mathbf{X}$ is a balancing score.

We know that $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} | \boldsymbol{\mu}$; thus, $\boldsymbol{\mu}$ is a balancing score. However, one must note that $\boldsymbol{\mu}$ is unobserved and hence it is not a very useful balancing score. In this discussion, we show that the (oracle) SC weights are also balancing scores i.e. $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} | \boldsymbol{\beta}$.

Theorem 1 *Given assumptions A.1 - A.3,* $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} \mid \boldsymbol{\beta}$.

Proof. Consider an $n \times n$ matrix $\mathbf{B}$ such that: (a) $\mathbf{B}_{i,i} = 0$, (b) $\mathbf{B}_{i,1} = 1$ (for $i > 1$), (c) $\mathbf{B}_{1,j} = \beta_j$ (for $j > 1$), and (d) $\mathbf{B}_{i,j} = -\beta_j$ (for $j > 1$ and $i > 1$) (see Figure 1(b)). Now, *we will show*, $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} | \mathbf{B}$. We know that $\mathbb{E}(\boldsymbol{X}_t) = \delta_t + \lambda_t \boldsymbol{\mu}$, $\mathbb{E}(X_{1,t}) = \sum_{i=2}^{n} \beta_i \mathbb{E}(X_{i,t})$ and $\sum_{i=2}^{n} \beta_i = 1$. These results imply that $\mu_1 = \sum_{i=2}^{n} \beta_i \mu_i$. We now observe that $\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\mu}$, i.e. $\boldsymbol{\mu}$ is one of the eigenvectors of the matrix $\mathbf{B}$ with corresponding eigenvalue equal to 1. Hence, $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} | \mathbf{B}$ and $\mathbf{B} = g(\boldsymbol{\beta})$ where g is the discussed matrix construction. *QED*.

Note that as $\boldsymbol{\beta}$ is a deterministic function of $\mathbf{X}$, $\mathbf{X}$ is also a balancing score i.e. $\{\boldsymbol{Y}_t(\boldsymbol{z})\}_t \perp \boldsymbol{Z} | \mathbf{X}$.

Conclusion and Discussion

In this short paper, we discuss the set of sufficient conditions under which SC weights are balancing scores. This understanding allows for a deeper understanding of the validity of SC inferences and furthering the methodological development. We are extending this work to develop a doubly robust SC method that allows for efficient and robust inference. More work is needed to understand if this result holds in different scenarios where synthetic controls have been used for estimation.

[1]To estimate, these weights, in practice, one fits a model that uses the outcomes of the control units to predict the contemporary outcome of the treated unit in the pretreatment period: `regress y=`$X_{i,t}$, `x=`$\{X_{2,t} \ldots X_{n,t}\}$. Typically, the regularization ensures that the weights are bounded between 0 and 1 and the sum of the weights is one.

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