Noisy ℓ^0-Sparse Subspace Clustering on Dimensionality Reduced Data

Abstract

High-dimensional data often lie in or close to low-dimensional subspaces. Sparse subspace clustering methods with sparsity induced by ℓ^0-norm, such as ℓ^0-Sparse Subspace Clustering (ℓ^0-SSC) [Yang et al., 2016], are demonstrated to be more effective than its ℓ^1 counterpart such as Sparse Subspace Clustering (SSC) [Elhamifar and Vidal, 2013]. However, the theoretical analysis of ℓ^0-SSC is restricted to clean data that lie exactly in subspaces. Real data often suffer from noise and they may lie close to subspaces. In this paper, we show that an optimal solution to the optimization problem of noisy ℓ^0-SSC achieves subspace detection property (SDP), a key element with which data from different subspaces are separated, under deterministic and semi-random model. Our results provide theoretical guarantee on the correctness of noisy ℓ^0-SSC in terms of SDP on noisy data for the first time, which reveals the advantage of noisy ℓ^0-SSC in terms of much less restrictive condition on subspace affinity. In order to improve the efficiency of noisy ℓ^0-SSC, we propose Noisy-DR-ℓ^0-SSC which provably recovers the subspaces on dimensionality reduced data. Noisy-DR-ℓ^0-SSC first projects the data onto a lower dimensional space by random projection, then performs noisy ℓ^0-SSC on the dimensionality reduced data for improved efficiency. Experimental results demonstrate the effectiveness of Noisy-DR-ℓ^0-SSC.

1 INTRODUCTION

Clustering is an important unsupervised learning procedure for analyzing a broad class of scientific data. High-dimensional data, such as facial images and gene expression data, often lie in low-dimensional subspaces in many cases, and clustering in accordance to the underlying subspace structure is particularly important. Among various subspace clustering algorithms, the ones that employ sparsity prior, such as Sparse Subspace Clustering (SSC) [Elhamifar and Vidal, 2013] and ℓ^0-Sparse Subspace Clustering (ℓ^0-SSC) [Yang et al., 2016], have been proven to be effective in separating the data in accordance with the subspaces that the data lie in under certain assumptions.

Sparse subspace clustering methods construct the sparse similarity matrix by sparse representation of the data. Subspace detection property (SDP) defined in Section 2.2 ensures that the similarity between data from different subspaces vanishes in the sparse similarity matrix, and applying spectral clustering [Ng et al., 2001] on such sparse similarity matrix leads to compelling clustering performance. Elhamifar and Vidal [Elhamifar and Vidal, 2013] prove that when the subspaces are independent or disjoint, SDP can be satisfied by solving the canonical sparse linear representation problem using data as the dictionary, under certain conditions on the rank, or singular value of the data matrix and the principle angle between the subspaces. Under the independence assumption on the subspaces, low rank representation [Liu et al., 2010, 2013] is also proposed to recover the subspace structures. Relaxing the assumptions on the subspaces to allowing overlapping subspaces, the Greedy Subspace Clustering [Park et al., 2014] and the Low-Rank Sparse Subspace Clustering [Wang et al., 2013] achieve subspace detection property with high probability. The geometric analysis in [Soltanolkotabi and Candés, 2012] shows the theoretical results on subspace recovery by SSC. In the following text, we use the term SSC or ℓ^1-SSC exchangeably to indicate the Sparse Subspace Clustering method in [Elhamifar and Vidal, 2013].

Real data often suffer from noise. The correctness of noisy SSC is analyzed in [Wang and Xu, 2013] which handles noisy data that lie close to disjoint or overlapping subspaces, and the original optimization problem of noisy SSC is proposed in [Elhamifar and Vidal, 2013]. While [Yang et al., 2016, Yang, 2018] prove the correctness of ℓ^0-SSC or its

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dimensionality reduced variant on clean data based on a constrained ℓ^0-minimization problem, it empirically solves an unconstrained ℓ^0-regularized problem to handle noise in data, and they lack theoretical analysis on the correctness of ℓ^0-SSC on noisy data. In this paper, the correctness of noisy ℓ^0-SSC on noisy data in terms of the subspace detection property is established. Our analysis is under both deterministic model and semi-random model, which are the models employed in the geometric analysis of SSC [Soltanolkotabi and Candés, 2012]. Our randomized analysis demonstrates the advantage of noisy ℓ^0-SSC over its ℓ^1 counterpart as more general assumption on data distribution can be adopted. Moreover, we present Noisy Dimensionality Reduced ℓ^0-Sparse Subspace Clustering (Noisy-DR-ℓ^0-SSC), an efficient version of noisy ℓ^0-SSC which also enjoys robustness to noise. Noisy-DR-ℓ^0-SSC first projects the data onto a lower dimensional space by random projection, then performs noisy ℓ^0-SSC on the dimensionality reduced data. Noisy-DR-ℓ^0-SSC provably recovers the underlying subspace structure in the original data from the dimensionality reduced data under deterministic model. Experimental results demonstrate the effectiveness of both noisy ℓ^0-SSC and Noisy-DR-ℓ^0-SSC.

1.1 CONTRIBUTIONS

First, the correctness of noisy ℓ^0-SSC on noisy data in terms of the subspace detection property is established for the first time, which is presented in Section 3 of this paper. Our analysis is under both deterministic model and semi-random model, which are also the models employed by the geometric analysis of SSC [Soltanolkotabi and Candés, 2012]. Our randomized analysis demonstrates the significant advantage of noisy ℓ^0-SSC over its ℓ^1 counterpart in terms of much less restrictive condition on the subspace affinity. Please refer to Remark 3.7 for details.

Second, we propose Noisy Dimensionality Reduced ℓ^0-Sparse Subspace Clustering (Noisy-DR-ℓ^0-SSC) to accelerate noisy ℓ^0-SSC with provable robustness to noise. Noisy-DR-ℓ^0-SSC first projects the data onto a lower dimensional space by random projection, then performs noisy ℓ^0-SSC on the dimensionality reduced data. Two types of random projections are used in Noisy-DR-ℓ^0-SSC, which are the random projection induced by randomized low-rank approximation and the sparse random projection by Oblivious Sparse Norm-Approximating Projection (OSNAP) [Nelson and Nguyen, 2013].

It should be emphasized that [Yang, 2018] also studies dimensionality reduced ℓ^0-SSC. However, the analysis of this work is performed on the following constrained ℓ^0-minimization problem and only for clean data without noise,

$$\min_{\mathbf{Z}} \|\mathbf{Z}\|_0 \quad s.t.\ \tilde{\mathbf{X}} = \tilde{\mathbf{X}}\mathbf{Z},\ \operatorname{diag}(\mathbf{Z}) = \mathbf{0}, \tag{1}$$

where $\tilde{\mathbf{X}}$ is the dimensionality reduced version of clean data by random projection. However, the actual optimization problem [Yang, 2018] solves the unconstrained ℓ^0-regularized problem,

$$\min_{\mathbf{Z}} \left\|\tilde{\mathbf{X}} - \tilde{\mathbf{X}}\mathbf{Z}\right\|_2^2 + \lambda\|\mathbf{Z}\|_0 \quad s.t.\ ,\ \operatorname{diag}(\mathbf{Z}) = \mathbf{0}. \tag{2}$$

In contrast, we analyze noisy ℓ^0-SSC on the unconstrained ℓ^0-regularized problem with noisy data which reveals the advantage of noisy ℓ^0 SSC over ℓ^1-SSC. Our analysis also suggests that a larger λ tends to guarantee the subspace detection property (Remark 3.5), which is verified by experiments. Throughout this paper, we refer to ℓ^0-SSC for noisy data with the unconstrained ℓ^0-regularized problem as noisy ℓ^0-SSC.

1.2 NOTATIONS

We use bold letters for matrices and vectors, and regular lower letter for scalars throughout this paper. The bold letter with superscript indicates the corresponding column of a matrix, e.g. $\mathbf{A}^i$ is the i-th column of matrix $\mathbf{A}$, and the bold letter with subscript indicates the corresponding element of a matrix or vector. $\|\cdot\|_F$ and $\|\cdot\|_p$ denote the Frobenius norm and the vector ℓ^p-norm or the matrix p-norm, and $\|\cdot\|_0$ is the ℓ^0-norm, that is, the number of nonzero elements of a vector. $\operatorname{diag}(\cdot)$ indicates the diagonal elements of a matrix. $\mathbf{H_T} \subseteq \mathbb{R}^d$ indicates the subspace spanned by the columns of $\mathbf{T}$, and $\mathbf{A_I}$ denotes a submatrix of $\mathbf{A}$ whose columns correspond to the nonzero elements of $\mathbf{I}$ (or with indices in $\mathbf{I}$ without confusion). $\sigma_t(\cdot)$ denotes the t-th largest singular value of a matrix, and $\sigma_{\min}(\cdot)$ indicates the smallest singular value of a matrix. $\operatorname{supp}(\cdot)$ is the support of a vector, $\mathbb{P}_{\mathcal{S}'}$ is the operator of orthogonal projection onto the subspace $\mathcal{S}'$. $[n]$ represents all the natural numbers between 1 and n inclusively. $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

2 PROBLEM SETUP

2.1 NOTATIONS

We hereby introduce the notations for subspace clustering on noisy data considered in this paper. The uncorrupted data matrix is denoted by $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n] \in \mathbb{R}^{d\times n}$, where d is the dimensionality and n is the size of the data. The uncorrupted data $\mathbf{Y}$ lie in a union of K distinct subspaces $\{\mathcal{S}_k\}_{k=1}^K$ of dimensions $\{d_k\}_{k=1}^K$ with $d_{\max} := \max_{k\in[K]} d_k$ and $d_{\min} := \min_{k\in[K]} d_k$. The observed noisy data is $\mathbf{X} = \mathbf{Y} + \mathbf{N}$, where $\mathbf{N} = [\mathbf{n}_1, \ldots, \mathbf{n}_n] \in \mathbb{R}^{d\times n}$ is the additive noise. $\mathbf{x}_i = \mathbf{y}_i + \mathbf{n}_i$ is the noisy data point that is corrupted by the noise $\mathbf{n}_i$. We let $\mathbf{Y}^{(k)} \in \mathbb{R}^{d\times n_k}$ denote the data belonging to subspace $\mathcal{S}_k$ with $\sum_{k=1}^K n_k = n$, and denote the corresponding columns in $\mathbf{X}$ by $\mathbf{X}^{(k)}$. Let $\mathbf{U}^{(k)} \in \mathbb{R}^{d\times d_k}$ be the orthogonal basis of $\mathcal{S}_k$ for all $k \in [K]$.

The data $\mathbf{X}$ are normalized such that each column has unit ℓ^2-norm in our deterministic analysis. We consider deterministic noise model where the noise $\mathbf{N}$ is fixed and $\max_{i\in[n]} \|\mathbf{n}_i\|_2 \leq \delta$. Note that our analysis can be extended to a random noise model which is common and also considered by noisy SSC [Wang and Xu, 2013], and the random noise model assumes that columns of $\mathbf{N}$ are i.i.d. samples with maximum ℓ_2-norm bounded by δ with high probability. Note that such random noise model does not require spherical symmetric noise as that in [Wang and Xu, 2013].

2.2 REVISIT ℓ^0-SSC

ℓ^0-SSC [Yang et al., 2016] proposes to solve the following ℓ^0 sparse representation problem

$$\min_{\mathbf{Z}} \|\mathbf{Z}\|_0 \quad s.t.\ \mathbf{X} = \mathbf{X}\mathbf{Z},\ \operatorname{diag}(\mathbf{Z}) = \mathbf{0}, \tag{3}$$

and it proves that the subspace detection property defined in Definition 2.1 is satisfied with the globally optimal solution to (3). In [Yang et al., 2016], the ℓ^0 regularized sparse approximation problem below is solved so as to handle noisy data for ℓ^0-SSC, which is the optimization problem of noisy ℓ^0-SSC:

$$\min_{\mathbf{Z}\in\mathbb{R}^{n\times n},\operatorname{diag}(\mathbf{Z})=\mathbf{0}} L(\mathbf{Z}) = \|\mathbf{X} - \mathbf{X}\mathbf{Z}\|_F^2 + \lambda\|\mathbf{Z}\|_0. \tag{4}$$

The optimization problem of noisy ℓ^0-SSC (4) is separable. For each $1 \leq i \leq n$, the optimization problem with respect to the sparse code $\boldsymbol{\beta}$ of i-th data point is

$$\min_{\boldsymbol{\beta}\in\mathbb{R}^n,\boldsymbol{\beta}_i=0} L(\boldsymbol{\beta}) = \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda\|\boldsymbol{\beta}\|_0. \tag{5}$$

In the following text, we always use $\boldsymbol{\beta}^*$ to denote an optimal solution to (5), and define $r^* := \|\boldsymbol{\beta}^*\|_0$.

The definition of subspace detection property for noisy ℓ^0-SSC and noiseless ℓ^0-SSC, i.e. ℓ^0-SSC on noiseless data, is defined in Definition 2.1 below.

Definition 2.1. (Subspace detection property for noisy and noiseless ℓ^0-SSC) Let $\mathbf{Z}^*$ be an optimal solution to (4). The subspaces $\{\mathcal{S}_k\}_{k=1}^K$ and the data $\mathbf{X}$ satisfy the Subspace Detection Property (SDP) for noisy ℓ^0-SSC if $\mathbf{Z}^i$ is a nonzero vector, and nonzero elements of $\mathbf{Z}^i$ correspond to the columns of $\mathbf{X}$ from the same subspace as $\mathbf{y}_i$ for all $1 \leq i \leq n$. We say that SDP holds for $\mathbf{x}_i$ if nonzero elements of $\mathbf{Z}^{*i}$, which is $\boldsymbol{\beta}^*$ for problem (5), correspond to the data that lie in the same subspace as $\mathbf{y}_i$, for either noisy ℓ^0-SSC or noiseless ℓ^0-SSC.

It should be emphasized that the theoretical analysis of ℓ^0-SSC [Yang et al., 2016] is limited to clean data, as a result, it is not clear if SDP can hold for an optimal solution to the noisy ℓ^0-SSC problem. In the following section, we provide theoretical analysis of

3 THEORETICAL ANALYSIS FOR NOISY ℓ^0-SSC

The theoretical results on the subspace detection property for noisy ℓ^0-SSC are presented in this section under the deterministic model and the semi-random model. These models are introduced in the following subsection.

3.1 MODELS

Similar to [Soltanolkotabi and Candés, 2012], we introduce the deterministic and the semi-random model for the analysis of noisy ℓ^0-SSC.

- **Deterministic Model:** the subspaces and the data in each subspace are fixed.
- **Semi-Random Model:** the subspaces are fixed but the data are independent and identically distributed in each of the subspaces.

The data in the above definitions refer to clean data without noise. Both the deterministic model and the semi-random model are extensively employed to analyze the subspace detection property in the subspace learning literature [Soltanolkotabi and Candés, 2012, Wang et al., 2013, Wang and Xu, 2013, Yining Wang and Singh, 2016].

3.2 NOISY ℓ^0-SSC: DETERMINISTIC ANALYSIS

We introduce the definition of general position and external subspace before our analysis on noisy ℓ^0-SSC.

Definition 3.1. (General position) For any $1 \leq k \leq K$, the data $\mathbf{Y}^{(k)}$ are in general position if any subset of $L \leq d_k$ data points (columns) of $\mathbf{Y}^{(k)}$ are linearly independent. $\mathbf{Y}$ are in general position if $\mathbf{Y}^{(k)}$ are in general position for $1 \leq k \leq K$.

The assumption of general condition is rather mild. In fact, if the data points in $\mathbf{X}^{(k)}$ are independently distributed according to any continuous distribution, then they almost surely in general position.

Let the distance between a point $\mathbf{x} \in \mathbb{R}^d$ and a subspace $\mathcal{S} \subseteq \mathbb{R}^d$ be defined as $d(\mathbf{x}, \mathcal{S}) = \inf_{\mathbf{y}\in\mathcal{S}} \|\mathbf{x} - \mathbf{y}\|_2$, the definition of external subspaces is presented as follows.

Definition 3.2. (External subspace of limited dimension) For a point $\mathbf{y} \in \mathbf{Y}^{(k)}$, a subspace $\mathbf{H}_{\{\mathbf{y}_{i_j}\}_{j=1}^L}$ spanned by a set of linear independent points $\{\mathbf{y}_{i_j}\}_{j=1}^L \subseteq \mathbf{Y}$ is defined to be an external subspace of $\mathbf{y}$ if $\{\mathbf{y}_{i_j}\}_{j=1}^L \not\subseteq \mathbf{Y}^{(k)}$ and $\mathbf{y} \notin \{\mathbf{y}_{i_j}\}_{j=1}^L$. The set of all external subspaces of $\mathbf{y}$ of dimension no greater than r with $r \geq 1$ for $\mathbf{y}$ is denoted by $\mathcal{H}_{\mathbf{y},r}$, that is, $\mathcal{H}_{\mathbf{y},r} = \{\mathbf{H}\colon \mathbf{H} = \mathbf{H}_{\{\mathbf{y}_{i_j}\}_{j=1}^L}, \dim[\mathbf{H}] = L, L \leq r, \{\mathbf{y}_{i_j}\}_{j=1}^L \not\subseteq \mathbf{Y}^{(k)}, \mathbf{y} \notin \{\mathbf{y}_{i_j}\}_{j=1}^L\}$. The point $\mathbf{y}$

is said to be away from its external subspaces of dimension r if $\min_{\mathbf{H}\in\mathcal{H}_{\mathbf{y},r}} d(\mathbf{y},\mathbf{H}) > 0$. All the data points in $\mathbf{Y}^{(k)}$ are said to be away from the external subspaces if each of them is away from the its associated external spaces.

To present our theoretical results of the correctness of noisy ℓ^0-SSC, we need the definitions related to the spectrum of $\mathbf{X}$ and $\mathbf{Y}$, which are defined as follows. In the following analysis, we employ $\boldsymbol{\beta}$ to denote the sparse code of datum $\mathbf{x}_i$ so that a simpler notation other than $\mathbf{Z}^i$ is dedicated to our analysis.

Definition 3.3. The minimum restricted eigenvalue of the uncorrupted data is defined as

$$\sigma_{\mathbf{Y},r} := \min_{\boldsymbol{\beta}:\|\boldsymbol{\beta}\|_0=r,\operatorname{rank}(\mathbf{Y}_{\boldsymbol{\beta}})=\|\boldsymbol{\beta}\|_0} \sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}}) \tag{6}$$

for $r \geq 1$. In addition, the normalized minimum restricted eigenvalue of the uncorrupted data is defined by

$$\bar{\sigma}_{\mathbf{Y},r} := \frac{\sigma_{\mathbf{Y},r}}{\sqrt{r}} \tag{7}$$

Moreover, the following quantities are defined for our analysis. We define

$$\tau_0 := \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}} + \tau_1, \tag{8}$$

where

$$\tau_1 := \frac{\delta}{\bar{\sigma}^*_{\mathbf{Y}} - \delta}, \quad \sigma^*_{\mathbf{X}} := \sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}^*}), \tag{9}$$

with $\delta < \bar{\sigma}^*_{\mathbf{Y}}$, and $\bar{\sigma}^*_{\mathbf{Y}}$ is defined as

$$\bar{\sigma}^*_{\mathbf{Y}} := \min_{1\leq r<r^*} \bar{\sigma}_{\mathbf{Y},r}. \tag{10}$$

Now we present the main result on the correctness of noisy ℓ^0-SSC.

Theorem 3.1. (Subspace detection property holds for noisy ℓ^0-SSC) Let nonzero vector $\boldsymbol{\beta}^*$ be an optimal solution to the noisy ℓ^0-SSC problem (5) for point $\mathbf{x}_i$ with $\|\boldsymbol{\beta}^*\|_0 = r^* \geq 1$, and $c^* := \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2$. Suppose $\mathbf{Y}$ is in general position, $\mathbf{y}_i \in \mathcal{S}_k$ for some $1 \leq k \leq K$, $\delta < \bar{\sigma}^*_{\mathbf{Y}}$, $\lambda > \tau_0$, $\mathbf{B}(\mathbf{y}_i, \delta + c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}) \cap \mathbf{H} = \emptyset$ for any $\mathbf{H} \in \mathcal{H}_{\mathbf{y}_i,r^*}$. Then the subspace detection property holds for $\mathbf{x}_i$ with $\boldsymbol{\beta}^*$. Here τ_0, τ_1, $\bar{\sigma}^*_{\mathbf{Y}}$ and $\sigma^*_{\mathbf{X}}$ are defined in (8), (9) and (10).

Remark 3.2. When $\delta = 0$ and there is no noise in the data $\mathbf{X}$, the conditions for the correctness of noisy ℓ^0-SSC in Theorem 3.1 almost reduce to that for noiseless ℓ^0-SSC. To see this, the conditions are reduced to $\mathbf{B}(\mathbf{y}_i, c^*) \cap \mathbf{H} = \emptyset$, which are exactly the conditions required by noiseless ℓ^0-SSC in Lemma 7.1 in the supplementary, namely data are away from the external subspaces by choosing $\lambda \to 0$ and it follows that $c^* = 0$.

While Theorem 3.1 establishes geometric conditions under which the subspace detection property holds for noisy ℓ^0-SSC, it can be seen that these conditions are often coupled with an optimal solution $\boldsymbol{\beta}^*$ to the noisy ℓ^0-SSC problem (5). In the following theorem, the correctness of noisy ℓ^0-SSC is guaranteed in terms of λ, the weight for the ℓ^0 regularization term in (5), and the geometric conditions independent of an optimal solution to (5).

Let $M_i > 0$ be the minimum distance between $\mathbf{y}_i \in \mathcal{S}_k$ and its external subspaces when $\mathbf{y}_i$ is away from its external subspaces of dimension r, that is,

$$M_i := \min\{d(\mathbf{y}_i, \mathbf{H}) \colon \mathbf{H} \in \mathcal{H}_{\mathbf{y}_i,d_k}\}, \tag{11}$$

The following two quantities related to the spectrum of clean and noisy data, μ_r and $\sigma_{\mathbf{X},r}$, are defined as follows with $r > 1$ for the analysis in Theorem 3.3.

$$\mu_r := \frac{\delta}{\min_{1\leq r'<r} \bar{\sigma}_{\mathbf{Y},r} - \delta}, \tag{12}$$

$$\sigma_{\mathbf{X},r} := \min\{\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}}) \colon 1 \leq \|\boldsymbol{\beta}\|_0 \leq r\} \tag{13}$$

Theorem 3.3. (Subspace detection property holds for noisy ℓ^0-SSC under deterministic model with conditions in terms of λ) Let nonzero vector $\boldsymbol{\beta}^*$ be an optimal solution to the noisy ℓ^0-SSC problem (5) for point $\mathbf{x}_i$ with $\|\boldsymbol{\beta}^*\|_0 = r^* > 1$, $n_k \geq d_k + 1$ for every $k \in [K]$, and there exists $1 < r_0 \leq \lfloor \frac{1}{\lambda} \rfloor$ such that $r^* \leq r_0$. Suppose $\mathbf{Y}$ is in general position, $\mathbf{y}_i \in \mathcal{S}_k$ for some $1 \leq k \leq K$, $\delta < \bar{\sigma}^*_{\mathbf{Y}}$, and $M_{i,\delta} := M_i - \delta$. Suppose

$$M_{i,\delta} > \frac{2\delta}{\sigma_{\mathbf{X},r_0}}, \tag{14}$$

and

$$\mu_{r_0} < 1 - \frac{2\delta}{\sigma_{\mathbf{X},r_0}}. \tag{15}$$

Then if

$$\lambda_0 < \lambda < 1, \tag{16}$$

where $\lambda_0 := \max\{\lambda_1, \lambda_2\}$ and

$$\lambda_1 := \inf\{0 < \lambda < 1 \colon \sqrt{1-\lambda} + \frac{2\delta}{\sigma_{\mathbf{X},r_0}\sqrt{\lambda}} < M_{i,\delta}\}, \tag{17}$$

$$\lambda_2 := \inf\{0 < \lambda < 1 \colon \lambda - \frac{2\delta}{\sigma_{\mathbf{X},r_0}} \frac{1}{\sqrt{\lambda}} > \mu_{r_0}\}, \tag{18}$$

the subspace detection property holds for $\mathbf{x}_i$ with $\boldsymbol{\beta}^*$. Here M_i, μ_{r_0} and $\sigma_{\mathbf{X},r_0}$ are defined in (11), (12) and (13) respectively.

Remark 3.4. The two conditions (14) and (15) are induced by two conditions, $\mathbf{B}(\mathbf{y}_i, \delta + c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}) \cap \mathbf{H} = \emptyset$ for any $\mathbf{H} \in \mathcal{H}_{\mathbf{y}_i,d_k}$ and $\lambda > \tau_0$ respectively, which are required by Theorem 3.1. Note that when (14) and (15) hold, λ_1 and λ_2 can always be chosen in accordance with (17) and (18).

Remark 3.5. It can be observed from condition (16) that noisy ℓ^0-SSC encourages sparse solution by a relatively large λ so as to guarantee the subspace detection property. This theoretical finding is consistent with the empirical study shown in the experimental results.

3.3 NOISY ℓ^0-SSC: RANDOMIZED ANALYSIS

In this subsection, the correctness of noisy ℓ^0-SSC is analyzed under the semi-random model that the data in subspace $\mathcal{S}^{(k)}$ are i.i.d. according to the uniform distribution on the unit sphere, $\mathbb{S}^{d_k-1}$, of $\mathbb{R}^{d_k}$ centered at the origin for all $k \in [K]$. This setting is employed extensively in the subspace learning literature [Soltanolkotabi and Candés, 2012, Wang et al., 2013, Wang and Xu, 2013, Yining Wang and Singh, 2016]. We then have the major theorem below stating the theoretical guarantee of the subspace detection property of noisy ℓ^0-SSC under the semi-random model. Before stating this theorem, we introduce the following definition of subspace affinity, which is widely used in the analysis of semi-random model in the sparse subspace clustering literature.

Definition 3.4. (Subspace affinity) The affinity between two subspaces, $\mathcal{S}_k$ and $\mathcal{S}_l$ with $k, l \in [K]$, is defined by

$$\operatorname{aff}(\mathcal{S}_k, \mathcal{S}_l) = \sqrt{\sum_{t=1}^{\min\{k,l\}} \cos^2 \theta_{kl}^{(t)}}, \tag{19}$$

where $\cos \theta_{kl}^{(t)}$ is the t-th canonical angle between $\mathcal{S}_k$ and $\mathcal{S}_l$ defined in [Soltanolkotabi and Candés, 2012]. Let $\mathbf{U}^{(k)}$ and $\mathbf{U}^{(l)}$ be the orthonormal basis for $\mathcal{S}_k$ and $\mathcal{S}_l$ respectively, then

$$\cos \theta_{kl}^{(t)} = \sup_{\mathbf{u} \in \mathcal{S}_k, \mathbf{v} \in \mathcal{S}_l} \frac{\mathbf{u}^\top \mathbf{v}}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2} = \frac{\mathbf{u}^{t\top} \mathbf{v}^t}{\|\mathbf{u}^t\|_2 \|\mathbf{v}^t\|_2},$$

with the orthogonality constraints $\mathbf{u}^\top \mathbf{u}^j = 0$ and $\mathbf{v}^\top \mathbf{v}^j = 0$ with $j = 1, 2, \ldots, t-1$. It can be verified that $\operatorname{aff}(\mathcal{S}_k, \mathcal{S}_l) = \|\mathbf{U}^{(k)\top} \mathbf{U}^{(l)}\|_F$.

Theorem 3.6. (Subspace detection property holds for noisy ℓ^0-SSC under semi-random model with conditions in terms of λ) Under the semi-random model, let nonzero vector $\boldsymbol{\beta}^*$ be an optimal solution to the noisy ℓ^0-SSC problem (5) for point $\mathbf{x}_i$ with $\|\boldsymbol{\beta}^*\|_0 = r^* > 1$, $n_k \geq d_k + 1$ for every $1 \leq k \leq K$, and there exists $1 < r_0 \leq \lfloor \frac{1}{\lambda} \rfloor$ such that $r^* \leq r_0$. Let $\sigma'(d_{\min}, \frac{1}{\lambda}) := \frac{1}{\sqrt{5}} \left(1 - 14\sqrt[3]{4}(\frac{M^2}{\lambda d_{\min}})^{2/3}\right)$, $c := \sqrt{\frac{\sigma'^2_{\min} - (r_0-1)\operatorname{aff}(\mathcal{S}_{t_1}, \mathcal{S}_{t_2})}{r_0}}$. For $t > 0$ such that $\frac{1}{d_{\max}} - 2t\sqrt{d_{\max} - 1} - t^2 > 0$, suppose

$$\max_{t_1, t_1 \in [K], t_1 \neq t_2} \operatorname{aff}(\mathcal{S}_{t_1}, \mathcal{S}_{t_2}) < \frac{\sigma'(d_{\min}, \frac{1}{\lambda})}{r_0 - 1}, \tag{20}$$

$$\delta < c, \tag{21}$$

$$\delta + \frac{2\delta}{\sqrt{r_0}(c - \delta)} \leq \frac{1}{d_{\max}} - 2t\sqrt{d_{\max} - 1} - t^2, \tag{22}$$

$$\frac{\delta}{c - \delta} + \frac{2\delta}{\sqrt{r_0}(c - \delta)} < 1, \tag{23}$$

and

$$\lambda_0' < \lambda < 1, \tag{24}$$

where $M := \sup_{0 \leq t < 1} \frac{2t^3 \arccos t}{\pi} < 1$, $\lambda_0' := \max\{\lambda_1', \lambda_2'\}$,

$$\lambda_1' := \inf\{0 < \lambda < 1 \colon \sqrt{1 - \lambda} + \frac{2\delta}{\sqrt{r_0}(c - \delta)\sqrt{\lambda}} < \frac{1}{d_{\max}} - 2t\sqrt{d_{\max} - 1} - t^2 - \delta\}, \tag{25}$$

$$\lambda_2' := \inf\{0 < \lambda < 1 \colon \lambda - \frac{2\delta}{\sqrt{r_0}(c - \delta)} \frac{1}{\sqrt{\lambda}} > \frac{\delta}{c - \delta}\}, \tag{26}$$

and $d_{\min} > \left(14\sqrt[3]{4}\right)^{\frac{3}{2}} \frac{M^2}{\lambda}$. Then with probability at least $1 - \sum_{k=1}^{K} \left(\exp(-d_k) + 2n_k \exp\left(-d_k^{0.9}\right)\right) - 8 \sum_{k=1}^{K} n_k \exp(-\frac{d_k t^2}{2})$, the subspace detection property holds for $\mathbf{x}_i$ with $\boldsymbol{\beta}^*$ for all $i \in [n]$.

Remark 3.7 (Advantage of Noisy ℓ^0-SSC in terms of Subspace Affinity)**.** It is well known that the difficulty of achieving the subspace detection property increases with larger affinity between subspaces, that is, the subspace are closer to each other. Our analysis reveals the significant advantage of noisy ℓ^0-SSC over ℓ^1-SSC in terms of the maximum subspace affinity. To the best of our knowledge, the best theoretical result of ℓ^1-SSC, including its geometrical analysis [Soltanolkotabi and Candés, 2012] and the subsequent works on noisy or dimensionality-reduced data [Wang and Xu, 2013, Wang et al., 2015], requires that the maximum subspace affinity satisfies

$$\max_{t_1, t_1 \in [K], t_1 \neq t_2} \operatorname{aff}(\mathcal{S}_{t_1}, \mathcal{S}_{t_2}) < \sqrt{d_0} \cdot \frac{\bar{c}\sqrt{\log \rho}}{8\sqrt{2} \log n}, \tag{27}$$

under the setting in [Soltanolkotabi and Candés, 2012] that $d_k = d_0$ and $n_k = \rho d_0 + 1$ for all $k \in [K]$, so that $n = K(\rho d_0 + 1)$. When $n > \exp(d_0^\tau)$ for $\tau \in (0.5, 0.9)$, then (27) requires $\max_{t_1, t_1 \in [K], t_1 \neq t_2} \operatorname{aff}(\mathcal{S}_{t_1}, \mathcal{S}_{t_2}) \to 0$ when $d_0 \to \infty$, while the condition of noisy ℓ^0-SSC, (20), only requires that $\max_{t_1, t_1 \in [K], t_1 \neq t_2} \operatorname{aff}(\mathcal{S}_{t_1}, \mathcal{S}_{t_2}) < \frac{1}{\sqrt{5}} \cdot \frac{1}{r_0 - 1}$ where r_0 can be set to $\lfloor \frac{1}{\lambda} \rfloor$ when $d_0 \to \infty$. Such less restive condition on the maximum subspace affinity reveals the theoretical advantage of noisy ℓ^0-SSC over ℓ^1-SSC.

4 NOISY ℓ^0-SSC ON DIMENSIONALITY REDUCED DATA: NOISY-DR-ℓ^0-SSC

Albeit the theoretical guarantee and compelling empirical performance of noisy ℓ^0-SSC to be shown in the experimental results, the computational cost of noisy ℓ^0-SSC is high

with the high dimensionality of the data. In this section, we propose Noisy Dimensionality Reduced ℓ^0-SSC (Noisy-DR-ℓ^0-SSC) which performs noisy ℓ^0-SSC on dimensionality reduced data. The theoretical guarantee on the correctness of Noisy-DR-ℓ^0-SSC under the deterministic model as well as its empirical performance are presented.

4.1 METHOD

Noisy-DR-ℓ^0-SSC performs subspace clustering by the following two steps: 1) obtain the dimension reduced data $\tilde{\mathbf{X}} = \mathbf{P}\mathbf{X}$ with a linear transformation $\mathbf{P} \in \mathbb{R}^{p\times d}$ $(p < d)$. 2) perform noisy ℓ^0-SSC on the compressed data $\tilde{\mathbf{X}}$:

$$\min_{\tilde{\boldsymbol{\beta}}\in\mathbb{R}^n, \tilde{\boldsymbol{\beta}}_i=0} L(\tilde{\boldsymbol{\beta}}) = \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{X}}\boldsymbol{\beta}\|_2^2 + \tilde{\lambda}\|\tilde{\boldsymbol{\beta}}\|_0. \tag{28}$$

If $p < d$, Noisy-DR-ℓ^0-SSC operates on the compressed data $\tilde{\mathbf{X}}$ rather than on the original data, so that the efficiency is improved. We will introduce two different types of random projection for Noisy-DR-ℓ^0-SSC in the following two subsections.

4.2 RANDOM PROJECT INDUCED BY RANDOMIZED LOW-RANK APPROXIMATION

High-dimensional data often exhibits low-dimensional structures, which often leads to low-rankness of the data matrix. Intuitively, if the data is low rank, then it could be safe to perform noisy ℓ^0-SSC on its dimensionality reduced version by the linear projection $\mathbf{P}$, and it is expected that $\mathbf{P}$ can preserve the information of the subspaces contained in the original data as much as possible, while effectively removing uninformative dimensions.

To this end, we propose to choose $\mathbf{P}$ as a random projection induced by randomized low-rank approximation of the data. The key idea is to obtain an approximate low-rank decomposition of the data. Using the random projection induced by such low-rank approximation as the linear transformation $\mathbf{P}$, the clustering correctness hold for Noisy-DR-ℓ^0-SSC with a high probability.

Randomized algorithms are efficient and they have been extensively studied in the computer science and numerical linear algebra literature. They have been employed to accelerate various numerical matrix computation and matrix optimization problems, including random projection for matrix decomposition [Frieze et al., 2004, Drineas et al., 2004, Sarlos, 2006, Drineas et al., 2006, 2008, Mahoney and Drineas, 2009, Drineas et al., 2011, Lu et al., 2013].

Formally, a random matrix $\mathbf{T} \in \mathbb{R}^{n\times p}$ is generated such that each element $\mathbf{T}_{ij}$ is sampled independently according to the Gaussian distribution $\mathcal{N}(0,1)$. QR decomposition is then performed on $\mathbf{XT}$ to obtain the basis of its column space, namely $\mathbf{XT} = \mathbf{QR}$ where $\mathbf{Q} \in \mathbb{R}^{d\times p}$ is an orthogonal matrix of rank p and $\mathbf{R} \in \mathbb{R}^{p\times p}$ is an upper triangle matrix. The columns of $\mathbf{Q}$ form the orthogonal basis for the sample matrix $\mathbf{XT}$. An approximation of $\mathbf{X}$ is then obtained by projecting $\mathbf{X}$ onto the column space of $\mathbf{XT}$: $\mathbf{QQ}^\top\mathbf{X} = \mathbf{QW} = \widehat{\mathbf{X}}$ where $\mathbf{W} = \mathbf{Q}^\top\mathbf{X} \in \mathbb{R}^{p\times n}$. In this manner, a randomized low-rank decomposition of $\mathbf{X}$ is achieved by

$$\widehat{\mathbf{X}} = \mathbf{QW}. \tag{29}$$

It is proved that the low rank approximation $\bar{\mathbf{X}}$ is close to $\mathbf{X}$ in terms of the spectral norm [Halko et al., 2011]. We present probabilistic result in Theorem 4.1 on the correctness of Noisy-DR-ℓ^0-SSC using the random projection induced by randomized low-rank decomposition of the data $\mathbf{X}$, namely $\mathbf{P} = \mathbf{Q}^\top$. In the sequel, $\tilde{\mathbf{x}} = \mathbf{P}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. To guarantee the subspace detection property on the dimensionality-reduced data $\tilde{\mathbf{X}}$, it is crucial to make sure that the conditions, such as (14) and (15) in Theorem 3.3, still hold after the linear transformation.

Each subspace $\mathcal{S}_k$ is transformed into $\tilde{\mathcal{S}}_k = \mathbf{P}(\mathcal{S}_k)$ with dimension $\tilde{d}_k$. We denote by $\tilde{\boldsymbol{\beta}}^*$ an optimal solution to (28), and define $C_{p,p_0} := \big(1 + 17\sqrt{1+\frac{p_0}{p-p_0}}\big)\sigma_{p_0+1} + \frac{8\sqrt{p}}{p-p_0+1}\big(\sum_{j>p_0}\sigma_j^2\big)^{\frac{1}{2}}$ with $p_0 \geq 2$. We also define the following quantities for the convenience of our analysis, which correspond to M_i, $\bar{\sigma}_{\mathbf{Y},r}$, $\sigma_{\mathbf{X},r}$ and μ_r used in the analysis on the original data:

$$\tilde{M}_i := \min\{d(\tilde{\mathbf{y}}_i, \mathbf{H}) \colon \mathbf{H} \in \mathcal{H}_{\tilde{\mathbf{y}}_i,\tilde{d}_k}\}, \tag{30}$$

where $\mathcal{H}_{\tilde{\mathbf{y}}_i,\tilde{d}_k}$ is all the external subspaces of $\tilde{\mathbf{y}}_i$ with dimension no greater than $\tilde{d}_k$ in the transformed space by $\mathbf{P}$,

$$\bar{\sigma}_{\tilde{\mathbf{Y}},r} := \min_{\boldsymbol{\beta}:\|\boldsymbol{\beta}\|_0=r, \operatorname{rank}(\tilde{\mathbf{Y}}_{\boldsymbol{\beta}})=\|\boldsymbol{\beta}\|_0} \sigma_{\min}(\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}), \tag{31}$$

$$\sigma_{\tilde{\mathbf{X}},r} := \min\{\sigma_{\min}(\tilde{\mathbf{X}}_{\boldsymbol{\beta}}) \colon 1 \leq \|\boldsymbol{\beta}\|_0 \leq r\}, \tag{32}$$

$$\tilde{\mu}_r := \frac{\delta}{\min_{1\leq r'<r}\bar{\sigma}_{\tilde{\mathbf{Y}},r} - \delta}. \tag{33}$$

Theorem 4.1. (Subspace detection property holds for Noisy-DR-ℓ^0-SSC under deterministic model) Let nonzero vector $\boldsymbol{\beta}^*$ be an optimal solution to the noisy ℓ^0-SSC problem (5) for point $\mathbf{x}_i$ with $\|\boldsymbol{\beta}^*\|_0 = r^* > 1$, $n_k \geq d_k + 1$ for every $1 \leq k \leq K$, and there exists $1 < r_0 \leq d_k$ such that $r^* \leq r_0 = \lfloor \frac{1}{\lambda} \rfloor$. Suppose $\mathbf{Y}$ is in general position, $\delta < \min_{1\leq r<r_0}\bar{\sigma}_{\mathbf{Y},r}$, and $\tilde{M}_{i,\delta} := \tilde{M}_i - \delta$. Furthermore, suppose the following conditions hold:

(1)

$$C_{p,p_0} + 2\delta\sqrt{\tilde{d}_{\max}} < \min_{k=1,\ldots,K}\sigma_{\mathbf{Y}}^{(k)}, \tag{34}$$

where $\tilde{d}_{\max} := \max_k \tilde{d}_k$, $\sigma_{\mathbf{Y}}^{(k)} := \min\{\sigma_{\min}(\mathbf{A})\colon \mathbf{A} \subseteq \mathbf{Y}^{(k)}, \mathbf{A} \in \mathbb{R}^{d\times n'}, n' \leq \tilde{d}_k\}$,

(2) $\delta(1+2\sqrt{r_0}) < \min_{1\leq r<r_0} \bar{\sigma}_{\mathbf{Y},r} - C_{p,p_0}$,

(3) $\min_{1\leq r\leq \tilde{d}_k} \sigma_{\mathbf{Y},r} > C_{p,p_0} - 2\delta\sqrt{\tilde{d}_k}$ and

$$M_i - C_{p,p_0}(1 + \frac{1}{\min_{1\leq r\leq \tilde{d}_k} \sigma_{\mathbf{Y},r} - C_{p,p_0} - 2\delta\sqrt{\tilde{d}_k}}) > \delta + \frac{2\delta}{\sigma_{\mathbf{X},r_0} - C_{p,p_0}}, \tag{35}$$

for all $\mathbf{y}_i \in \mathcal{S}_k$ and $1 \leq k \leq K$,

(4) $\min_{1\leq r<r_0} \bar{\sigma}_{\mathbf{Y},r_0} > C_{p,p_0} - 2\delta\sqrt{r_0} - \delta$ and

$$\frac{\delta}{\min_{1\leq r<r_0} \bar{\sigma}_{\mathbf{Y},r_0} - C_{p,p_0} - 2\delta\sqrt{r_0} - \delta} < 1 - \frac{2\delta}{\sigma_{\mathbf{X},r_0} - C_{p,p_0}}. \tag{36}$$

If

$$\tilde{\lambda}_0 < \tilde{\lambda} < 1, \tag{37}$$

where $\tilde{\lambda}_0 = \max\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$ and

$$\tilde{\lambda}_1 = \inf\{0 < \tilde{\lambda} < 1\colon \sqrt{1-\tilde{\lambda}} + \frac{2\delta}{\sigma_{\tilde{\mathbf{X}},r_0}\sqrt{\tilde{\lambda}}} < \tilde{M}_{i,\delta}\}, \tag{38}$$

$$\tilde{\lambda}_2 = \inf\{0 < \tilde{\lambda} < 1\colon \tilde{\lambda} - \frac{2\delta}{\sigma_{\mathbf{X},r_0}}\frac{1}{\sqrt{\tilde{\lambda}}} > \tilde{\mu}_{r_0}\}, \tag{39}$$

then with probability at least $1 - 6e^{-p}$, the subspace detection property holds for $\tilde{\mathbf{x}}_i$ with $\tilde{\boldsymbol{\beta}}^*$. Here $\tilde{M}_i$, $\tilde{\mu}_r$ and $\tilde{\sigma}_{\tilde{\mathbf{X}},r_0}$ are defined in (30), (33) and (32) respectively.

4.3 RANDOM PROJECTION BY OBLIVIOUS SPARSE NORM-APPROXIMATING PROJECTION (OSNAP)

In this subsection, we study the case when the linear transformation $\mathbf{P}$ for the dimensionality reduced ℓ^0-SSC problem (28) is a sparse matrix, that is, the Oblivious Sparse Norm-Approximating Projection (OSNAP) [Nelson and Nguyen, 2013]. We choose $\mathbf{P}$ as one the most efficient version of OSNAP, where each column of $\mathbf{P}$ only has 1 nonzero element. Each column of $\mathbf{P}$ is chosen to have exactly one nonzero element in a random location, and each nonzero element is ± 1 uniformly at random. Such $\mathbf{P}$ enjoys ℓ^2-norm preserving property stated as follows. Let ε be a positive number such that $0 < \varepsilon \leq 1$. By Theorem 3 in [Nelson and Nguyen, 2013], for any subspace $\mathbf{S}$ of dimension d in $\mathbb{R}^n$, $\Pr\big[(1-\varepsilon)\|\mathbf{v}\|_2 \leq \|\mathbf{P}\mathbf{v}\|_2 \leq (1+\varepsilon)\|\mathbf{v}\|_2\big] \geq 1-\delta'$ for all $\mathbf{v} \in \mathbf{S}$ and $\delta' \in (0,1)$ if $p \geq \frac{d^2+d}{\delta'(2\varepsilon-\varepsilon^2)^2}$. We note that while the general Johnson-Lindenstrauss transform possesses the similar ℓ^2-norm preserving property, the sparse matrix $\mathbf{P}$ considerably speeds up the computation of $\tilde{\mathbf{X}} = \mathbf{P}\mathbf{X}$.

Using the ℓ^2-norm preserving property of OSNAP, we have the following theorem about the correctness of Noisy-DR-ℓ^0-SSC when $\mathbf{P}$ is the said OSNAP where each column of $\mathbf{P}$ only has 1 nonzero element.

Theorem 4.2. (Subspace detection property holds for Noisy-DR-ℓ^0-SSC under deterministic model with $\mathbf{P}$ being the OSNAP) Let nonzero vector $\boldsymbol{\beta}^*$ be the optimal solution to the noisy ℓ^0-SSC problem (5) for point $\mathbf{x}_i$ with $\|\boldsymbol{\beta}^*\|_0 = r^*$, $n_k \geq d_k + 1$ for every $1 \leq k \leq K$, and there exists $1 < r_0 \leq \lfloor \frac{1}{\lambda} \rfloor$ such that $1 < r^* \leq r_0$. Suppose $\mathbf{Y}$ is in general position, $\mathbf{y}_i \in \mathcal{S}_k$ for some $1 \leq k \leq K$, $\delta < \min_{1\leq r<r_0} \bar{\sigma}_{\mathbf{Y},r}$. Let $M_{i,\delta} := M_i - \delta$, ε be a positive number such that $0 < \varepsilon \leq 1$. Suppose

$$M_{i,\delta} > \frac{2(1+\varepsilon)^3\delta}{\sigma_{\mathbf{X},r_0}}, \tag{40}$$

and

$$\mu_{r,\varepsilon} := \frac{\delta}{\frac{\min_{1\leq r'<r} \bar{\sigma}_{\mathbf{Y},r}}{(1+\varepsilon)^2} - \delta} < 1 - \frac{2(1+\varepsilon)^2\delta}{\sigma_{\mathbf{X},r_0}}. \tag{41}$$

Then if

$$\tilde{\lambda}_0 < \tilde{\lambda} < 1, \tag{42}$$

where $\tilde{\lambda}_0 := \max\{\lambda_1, \lambda_2\}$ and

$$\lambda_1 := \inf\{0 < \tilde{\lambda} < 1\colon \sqrt{1+\varepsilon-\tilde{\lambda}} + \frac{2\delta}{\sigma_{\tilde{\mathbf{X}},r_0}\sqrt{\tilde{\lambda}}} < M_{i,\delta}\}, \tag{43}$$

$$\lambda_2 := \inf\{0 < \lambda < 1\colon \lambda - \frac{2\delta}{\sigma_{\tilde{\mathbf{X}},r_0}}\frac{1}{\sqrt{\lambda}} > \tilde{\mu}_{r_0}\}, \tag{44}$$

then with probability at least $1 - K\delta'$ for all $\delta' \in (0, \frac{1}{K})$, the subspace detection property holds for $\tilde{\mathbf{x}}_i$ with $\tilde{\boldsymbol{\beta}}^*$. Here $\tilde{\mu}_{r_0}$ and $\sigma_{\tilde{\mathbf{X}},r_0}$ are defined in (33) and (32) respectively. $\tilde{\boldsymbol{\beta}}^*$ is the optimal solution to (28) with $\mathbf{P}$ being the OSNAP described in the beginning of this subsection with $p \geq \frac{d_{\max}^2+d_{\max}}{\delta'(2\varepsilon-\varepsilon^2)^2}$.

4.4 THE ALGORITHM OF NOISY-DR-ℓ^0-SSC

We denote by Noisy-DR-ℓ^0-SSC-LR the Noisy-DR-ℓ^0-SSC with random projection induced by randomized low-rank approximation in Section 4.2, and denote by Noisy-DR-ℓ^0-SSC-OSNAP the Noisy-DR-ℓ^0-SSC with OSNAP serving as the random projection in Section 4.3. Noisy-DR-ℓ^0-SSC-LR is described by Algorithm 1. The algorithm of Noisy-DR-ℓ^0-SSC-OSNAP is similar to Algorithm 1 except that OSNAP serves as the random projection $\mathbf{P}$, so its details are omitted here. Algorithm 2 in Section 7.4 of the supplementary describes how to solve the noisy ℓ^0-SSC problem (5).

Table 1: Clustering results on various data sets, with the best three results in bold

Data Set	Measure	KM	SC	Noisy SSC	Noisy DR-SSC	SMCE	SSC-OMP	Noisy ℓ^0-SSC	Noisy-DR-ℓ^0-SSC-LR	Noisy-DR-ℓ^0-SSC-OSNAP
COIL-20	AC	0.6554	0.4278	0.7854	0.7764	0.7549	0.3389	**0.8472** $\pm$ 0.0031	**0.8479** $\pm$ 0.0023	**0.8472** $\pm$ 0.0019
	NMI	0.7630	0.6217	0.9148	0.9219	0.8754	0.4853	**0.9428** $\pm$ 0.0082	**0.9433** $\pm$ 0.0063	**0.9429** $\pm$ 0.0037
COIL-100	AC	0.4996	0.2835	0.5275	0.5013	0.5639	0.1667	**0.7683** $\pm$ 0.0020	**0.7039** $\pm$ 0.0087	**0.7046** $\pm$ 0.0083
	NMI	0.7539	0.5923	0.8041	0.8019	0.8064	0.3757	**0.9182** $\pm$ 0.0096	**0.8706** $\pm$ 0.0109	**0.8708** $\pm$ 0.0117
Yale-B	AC	0.0954	0.1077	0.7850	0.7255	0.3293	0.7789	**0.8480** $\pm$ 0.0091	**0.8231** $\pm$ 0.0173	**0.8318** $\pm$ 0.0112
	NMI	0.1258	0.1485	0.7760	0.7311	0.3812	0.7024	**0.8612** $\pm$ 0.0072	**0.8533** $\pm$ 0.0294	**0.8593** $\pm$ 0.0133
MPIE S1	AC	0.1164	0.1285	0.5892	0.3588	0.1721	0.1695	**0.6741** $\pm$ 0.0413	**0.6741** $\pm$ 0.0938	**0.6744** $\pm$ 0.0662
	NMI	0.5049	0.5292	0.7653	0.6806	0.5514	0.3395	**0.8622** $\pm$ 0.0533	**0.8622** $\pm$ 0.0834	**0.8548** $\pm$ 0.0931
MPIE S2	AC	0.1315	0.1410	0.6994	0.4611	0.1898	0.2093	**0.7527** $\pm$ 0.0115	**0.7533** $\pm$ 0.0596	**0.7517** $\pm$ 0.0813
	NMI	0.4834	0.5128	0.8149	0.7086	0.5293	0.4292	**0.8939** $\pm$ 0.0389	**0.8926** $\pm$ 0.0742	**0.8910** $\pm$ 0.0454
MPIE S3	AC	0.1291	0.1459	0.6316	0.4841	0.1856	0.1787	**0.7050** $\pm$ 0.0277	**0.7123** $\pm$ 0.0812	**0.7184** $\pm$ 0.1045
	NMI	0.4811	0.5185	0.7858	0.7340	0.5155	0.3415	**0.8750** $\pm$ 0.0157	**0.8455** $\pm$ 0.0693	**0.8457** $\pm$ 0.0913
MPIE S4	AC	0.1308	0.1463	0.6803	0.5511	0.1823	0.1680	**0.7246** $\pm$ 0.0147	**0.7137** $\pm$ 0.0605	**0.7250** $\pm$ 0.0443
	NMI	0.4866	0.5280	0.8063	0.7955	0.5294	0.3345	**0.8837** $\pm$ 0.0212	**0.8847** $\pm$ 0.0781	**0.8834** $\pm$ 0.0517
MNIST	AC	0.5236	0.3504	0.5714	0.5123	**0.6542**	0.5561	0.6259 $\pm$ 0.0249	**0.6296** $\pm$ 0.1522	**0.6310** $\pm$ 0.1031
	NMI	0.4770	0.3607	0.6091	0.5026	**0.6796**	0.5986	**0.6501** $\pm$ 0.0196	0.6440 $\pm$ 0.0259	**0.6497** $\pm$ 0.0313

Algorithm 1 Noisy Dimensionality Reduced ℓ^0-Sparse Subspace Clustering by Randomized Low-Rank Approximation (Noisy-DR-ℓ^0-SSC-LR)

1: Generate a Gaussian random matrix $\mathbf{T} \in \mathbb{R}^{n \times p}$ where each element $\mathbf{T}_{ij}$ is sampled independently according to the standard Gaussian distribution $\mathcal{N}(0, 1)$
2: Perform QR decomposition on $\mathbf{XT}$, $\mathbf{XT} = \mathbf{QR}$ where $\mathbf{Q} \in \mathbb{R}^{d \times p}$
3: Set the linear transformation $\mathbf{P} = \mathbf{Q}^\top$, and obtain the dimensionality reduced data $\tilde{\mathbf{X}} = \mathbf{PX}$
4: Perform noisy ℓ^0-SSC on $\tilde{\mathbf{X}}$ using Algorithm 2

5 EXPERIMENTAL RESULTS

We demonstrate the performance of Noisy-DR-ℓ^0-SSC-LR and Noisy-DR-ℓ^0-SSC-OSNAP, with comparison to other competing clustering methods including K-means (KM), Spectral Clustering (SC), noisy SSC, Sparse Manifold Clustering and Embedding (SMCE) [Elhamifar and Vidal, 2011] and SSC-OMP [Dyer et al., 2013] in this section. We will use Noisy-DR-ℓ^0-SSC to refer to its two variants. With the coefficient matrix $\mathbf{Z}$ obtained by the optimization of noisy ℓ^0-SSC or Noisy-DR-ℓ^0-SSC, a sparse similarity matrix is built by $\mathbf{W} = \frac{|\mathbf{Z}|+|\mathbf{Z}^\top|}{2}$, and spectral clustering is performed on $\mathbf{W}$ to obtain the clustering results. Two measures are used to evaluate the performance of different clustering methods, i.e. the Accuracy (AC) and the Normalized Mutual Information (NMI) [Zheng et al., 2004].

We use randomized rank-p decomposition of the data matrix in Noisy-DR-ℓ^0-SSC-LR with $p = \frac{\min\{d,n\}}{10}$. It can be observed that noisy ℓ^0-SSC and Noisy-DR-ℓ^0-SSC always achieve better performance than other methods in Table 1, including the noisy SSC on dimensionality reduced data (Noisy DR-SSC) [Wang et al., 2015]. Note that noisy ℓ^0-SSC has the same performance as ℓ^0-SSC [Yang et al., 2016]. Throughout all the experiments we find that the best clustering accuracy is achieved whenever λ is chosen by $0.5 < \lambda < 0.95$, justifying our theoretical finding claimed in Remark 3.5 and (16) in Theorem 3.3. For all the methods that involve random projection, we conduct the experiments for 30 times and report the average performance. Note that the cluster accuracy of SSC-OMP on the extended Yale-B data set is reported according to [You et al., 2016]. We randomly sample 1000 images from each class of the MNIST data set so as to collect a total number of 10000 images on which clustering is performed, and the average performance of 10 random sampling is reported for this data set. The time complexity of noisy ℓ^0-SSC and the two variants of Noisy-DR-ℓ^0-SSC are analyzed in Section 7.3 of the supplementary. The actual running time of both algorithms confirms such time complexity, and we observe that Noisy-DR-ℓ^0-SSC-LR is always 8.7 times faster than noisy ℓ^0-SSC with the same number of iterations, and the acceleration is boosted to 9.6 times by Noisy-DR-ℓ^0-SSC-OSNAP due to the sparse random projection.

5.1 JUSTIFICATION OF THEORETICAL ANALYSIS FOR NOISY ℓ^0-SSC

We further demonstrates the practical implication of our theoretical Analysis for noisy ℓ^0-SSC. As mentioned in Remark 3.5, a relatively large λ tends to preserve the subspace detection property. This theoretical finding is consistent with the empirical study shown in this subsection. We add Gaussian noise of zero mean and different choices of variance σ^2 to the extended Yale-B data set. In Section 7.5 of the supplementary, Figure 2a to Figure 2f illustrate SDP violation with respect to λ for different noise levels with σ^2 ranging over $10, 20, 30, 40, 50, 60$. The SDP violation is defined in [Wang and Xu, 2013] which is the percentage of pairs of data points which are mistakenly put in the same subspace by the similarity matrix $\mathbf{W}$, namely the percentage of pairs $(\mathbf{x}_i, \mathbf{x}_j)$ with nonzero $\mathbf{W}_{ij}$ while they are in fact not in the same subspace. It can be observed that increasing λ effectively reduces SDP violation for noisy ℓ^0-SSC, Noisy-DR-ℓ^0-SSC-LR and Noisy-DR-ℓ^0-OSNAP, confirming our theoretical prediction.

6 CONCLUSION

In this paper, we prove that noisy ℓ^0-SSC recovers subspaces from noisy data through ℓ^0-induced sparsity. We then propose Noisy-DR-ℓ^0-SSC to improve the efficiency of noisy ℓ^0-SSC, which performs noisy ℓ^0-SSC on dimensionality reduced data and still provably recovers the underlying subspaces with effectiveness verified by experiments.

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7 SUPPLEMENTARY

7.1 PROOFS

We provide proofs to the lemmas and theorems in the paper in this subsection.

7.1.1 Lemma 7.1 and Its Proof

Lemma 7.1. (Subspace detection property holds for noiseless ℓ^0-SSC under the deterministic model) It can be verified that the following statement is true. Under the deterministic model, suppose data is noiseless, $n_k \geq d_k + 1$, $\mathbf{Y}^{(k)}$ is in general position. If all the data points in $\mathbf{Y}^{(k)}$ are away from the external subspaces for any $1 \leq k \leq K$, then the subspace detection property for ℓ^0-SSC holds with an optimal solution $\mathbf{Z}^*$ to (3).

Proof. Let $\mathbf{x}_i \in \mathcal{S}_k$. Note that $\mathbf{Z}^{*i}$ is an optimal solution to the following ℓ^0 sparse representation problem

$$\min_{\mathbf{Z}^i} \|\mathbf{Z}^i\|_0 \quad s.t.\ \mathbf{x}_i = [\mathbf{X}^{(k)} \setminus \mathbf{x}_i \quad \mathbf{X}^{(-k)}]\mathbf{Z}^i,\ \mathbf{Z}_{ii} = 0, \tag{45}$$

where $\mathbf{X}^{(-k)}$ denotes the data that lie in all subspaces except $\mathcal{S}_k$. Let $\mathbf{Z}^{*i} = \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix}$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are sparse codes corresponding to $\mathbf{X}^{(k)} \setminus \mathbf{x}_i$ and $\mathbf{X}^{(-k)}$ respectively.

Suppose $\boldsymbol{\beta} \neq \mathbf{0}$, then $\mathbf{x}_i$ belongs to a subspace $\mathcal{S}^{'} = \mathbf{H}_{\mathbf{X}_{\mathbf{Z}^{*i}}}$ spanned by the projected data points corresponding to nonzero elements of $\mathbf{Z}^{*i}$, and $\mathcal{S}^{'} \neq \mathcal{S}_k$, $\dim[\mathcal{S}^{'}] \leq d_k$. To see this, if $\mathcal{S}^{'} = \mathcal{S}_k$, then the data corresponding to nonzero elements of $\boldsymbol{\beta}$ belong to $\mathcal{S}_k$, which is contrary to the definition of $\mathbf{X}^{(-k)}$. Also, if $\dim[\mathcal{S}^{'}] > d_k$, then any d_k points in $\mathbf{X}^{(k)}$ can be used to linearly represent $\mathbf{x}_i$ by the condition of general position, contradicting with the optimality of $\mathbf{Z}^{*i}$. Since the data points (or columns) in $\mathbf{X}_{\mathbf{Z}^{*i}}$ are linearly independent, it follows that $\mathbf{x}_i$ lies in an external subspace $\mathbf{H}_{\mathbf{X}_{\mathbf{Z}^{*i}}}$ spanned by linearly independent points in $\mathbf{X}_{\mathbf{Z}^{*i}}$, and $\dim[\mathbf{H}_{\mathbf{X}_{\mathbf{Z}^{*i}}}] = \dim[\mathcal{S}^{'}] \leq d_k$. This contradicts with the assumption that $\mathbf{x}_i$ is away from the external subspaces. Therefore, $\boldsymbol{\beta} = \mathbf{0}$. Perform the above analysis for all $1 \leq i \leq n$, we can prove that the subspace detection property holds for all $1 \leq i \leq n$.

□

7.1.2 Proof of Theorem 3.1

Before proving this theorem, we introduce the following perturbation bound for the distance between a data point and the subspaces spanned by noisy and noiseless data, which is useful to establish the conditions when the subspace detection property holds for noisy ℓ^0-SSC.

Lemma 7.2. Let $\boldsymbol{\beta} \in \mathbb{R}^n$ and $\mathbf{Y}_{\boldsymbol{\beta}}$ has full column rank. Suppose $\delta < \bar{\sigma}_{\mathbf{Y},r}$ where $r = \|\boldsymbol{\beta}\|_0$, then $\mathbf{X}_{\boldsymbol{\beta}}$ is a full column rank matrix, and

$$|d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}}}) - d(\mathbf{x}_i, \mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}}})| \leq \frac{\delta}{\bar{\sigma}_{\mathbf{Y},r} - \delta} \tag{46}$$

for any $1 \leq i \leq n$.

Lemma 7.3 shows that an optimal solution to the noisy ℓ^0-SSC problem (5) is also that to a ℓ^0-minimization problem with tolerance to noise.

Lemma 7.3. Let nonzero vector $\boldsymbol{\beta}^*$ be an optimal solution to the noisy ℓ^0-SSC problem (5) for point $\mathbf{x}_i$ with $\|\boldsymbol{\beta}^*\|_0 = r^* > 1$. If $\lambda > \tau_0$ where τ_0 is defined as

$$\tau_0 := \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}} + \tau_1,$$

where

$$\tau_1 := \frac{\delta}{\bar{\sigma}^*_{\mathbf{Y}} - \delta}, \quad \sigma^*_{\mathbf{X}} := \sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}^*}),$$

with $\delta < \bar{\sigma}^*_{\mathbf{Y}}$, and $\bar{\sigma}^*_{\mathbf{Y}}$ is defined as

$$\bar{\sigma}^*_{\mathbf{Y}} := \min_{r \in [r^*]} \bar{\sigma}_{\mathbf{Y},r},$$

then $\boldsymbol{\beta}^*$ is an optimal solution to the following sparse approximation problem with the uncorrupted data as the dictionary:

$$\min_{\boldsymbol{\beta}} \|\boldsymbol{\beta}\|_0 \quad s.t. \ \|\mathbf{x}_i - \mathbf{Y}\boldsymbol{\beta}\|_2 \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}, \ \boldsymbol{\beta}_i = 0, \tag{47}$$

where $c^* := \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2$.

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We first show that $d(\mathbf{x}_i, \mathcal{S}_k) \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}$. To see this, $\sigma^*_{\mathbf{X}} = \sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}^*}) \leq 1$ as the columns of $\mathbf{X}$ have unit ℓ^2-norm. It follows that

$$c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}} \geq 2\delta\sqrt{r^*} \geq 2\delta > \|\mathbf{x}_i - \mathbf{y}_i\| \geq d(\mathbf{x}_i, \mathcal{S}_k). \tag{48}$$

By Lemma 7.3, it can be verified that $\boldsymbol{\beta}^*$ is an optimal solution to the following problem

$$\min_{\boldsymbol{\beta}} \|\boldsymbol{\beta}\|_0 \quad s.t. \ \|\mathbf{x}_i - \mathbf{Y}\boldsymbol{\beta}\|_2 \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}, \ \boldsymbol{\beta}_i = 0. \tag{49}$$

Let $\mathbf{x}'$ be the projection of $\mathbf{x}_i$ onto $\mathbf{H}_{\overline{\mathbf{Y}^{(i)}}}$, and let the columns of $\overline{\mathbf{Y}^{(i)}}$ have column indices $\mathbf{I}$ in $\mathbf{Y}^{(k)}$, that is, $\mathbf{Y}^{(k)}_{\mathbf{I}} = \overline{\mathbf{Y}^{(i)}}$. Then there exists $\boldsymbol{\beta}' \in \mathbb{R}^n$ and $\boldsymbol{\beta}'_j = 0$ for all $j \notin \mathbf{I}$ such that $\mathbf{x}' = \mathbf{Y}\boldsymbol{\beta}'$ and $\|\boldsymbol{\beta}'\|_0 \leq r^*$. It is clear that $\boldsymbol{\beta}'$ is a feasible solution to (49) because $d(\mathbf{x}_i, \mathbf{H}_{\overline{\mathbf{Y}^{(i)}}}) = \|\mathbf{x}_i - \mathbf{Y}\boldsymbol{\beta}'\|_2 \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}$ and it satisfies SDP for $\mathbf{x}_i$.

Suppose that there is an optimal solution $\boldsymbol{\beta}''$ to (49) which does not satisfy SDP for $\mathbf{x}_i$, then $\|\boldsymbol{\beta}''\|_0 \leq r^*$. Then the subspace spanned by $\mathbf{Y}_{\boldsymbol{\beta}''}$, $\mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}''}}$, is an external subspace of $\mathbf{y}_i$ and $\mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}''}} \in \mathcal{H}_{\mathbf{y}_i, r^*}$, and it follows that $d(\mathbf{x}_i, \mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}''}}) > c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}$. However, since $\boldsymbol{\beta}''$ is a feasible solution, $d(\mathbf{x}_i, \mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}''}}) \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma^*_{\mathbf{X}}}$. This contradiction shows that every optimal solution to the noisy ℓ^0-SSC problem (5) satisfies SDP for $\mathbf{x}_i$.

□

7.1.3 Proof of Lemma 7.2

The following lemma is used for proving Lemma 7.2.

Lemma 7.4. (Perturbation of distance to subspaces) Let $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{m \times n}$ are two matrices and $\text{rank}(\mathbf{A}) = r$, $\text{rank}(\mathbf{B}) = s$. Also, $\mathbf{E} = \mathbf{A} - \mathbf{B}$ and $\|\mathbf{E}\|_2 \leq C$, where $\|\cdot\|_2$ indicates the spectral norm. Then for any point $\mathbf{x} \in \mathbb{R}^m$, the difference of the distance of $\mathbf{x}$ to the column space of $\mathbf{A}$ and $\mathbf{B}$, i.e. $|d(\mathbf{x}, \mathbf{H}_{\mathbf{A}}) - d(\mathbf{x}, \mathbf{H}_{\mathbf{B}})|$, is bounded by

$$|d(\mathbf{x}, \mathbf{H}_{\mathbf{A}}) - d(\mathbf{x}, \mathbf{H}_{\mathbf{B}})| \leq \frac{C\|\mathbf{x}\|_2}{\min\{\sigma_r(\mathbf{A}), \sigma_s(\mathbf{B})\}}. \tag{50}$$

Proof. Note that the projection of $\mathbf{x}$ onto the subspace $\mathbf{H}_{\mathbf{A}}$ is $\mathbf{A}\mathbf{A}^+\mathbf{x}$ where $\mathbf{A}^+$ is the Moore-Penrose pseudo-inverse of the matrix $\mathbf{A}$, so $d(\mathbf{x}, \mathbf{H}_{\mathbf{A}})$ equals to the distance between $\mathbf{x}$ and its projection, namely $d(\mathbf{x}, \mathbf{H}_{\mathbf{A}}) = \|\mathbf{x} - \mathbf{A}\mathbf{A}^+\mathbf{x}\|_2$. Similarly, $d(\mathbf{x}, \mathbf{H}_{\mathbf{B}}) = \|\mathbf{x} - \mathbf{B}\mathbf{B}^+\mathbf{x}\|_2$.

It follows that

$$\begin{aligned} |d(\mathbf{x}, \mathbf{H}_{\mathbf{A}}) - d(\mathbf{x}, \mathbf{H}_{\mathbf{B}})| &= |\|\mathbf{x} - \mathbf{A}\mathbf{A}^+\mathbf{x}\|_2 - \|\mathbf{x} - \mathbf{B}\mathbf{B}^+\mathbf{x}\|_2| \\ &\leq \|\mathbf{A}\mathbf{A}^+\mathbf{x} - \mathbf{B}\mathbf{B}^+\mathbf{x}\|_2 \leq \|\mathbf{A}\mathbf{A}^+ - \mathbf{B}\mathbf{B}^+\|_2\|\mathbf{x}\|_2. \end{aligned} \tag{51}$$

According to the perturbation bound on the orthogonal projection in [Chen et al., 2016, Stewart, 1977],

$$\|\mathbf{A}\mathbf{A}^+ - \mathbf{B}\mathbf{B}^+\|_2 \leq \max\{\|\mathbf{E}\mathbf{A}^+\|_2, \|\mathbf{E}\mathbf{B}^+\|_2\}. \tag{52}$$

Since $\|\mathbf{E}\mathbf{A}^+\|_2 \leq \|\mathbf{E}\|_2\|\mathbf{A}^+\|_2 \leq \frac{C}{\sigma_r(\mathbf{A})}$, $\|\mathbf{E}\mathbf{B}^+\|_2 \leq \|\mathbf{E}\|_2\|\mathbf{B}^+\|_2 \leq \frac{C}{\sigma_s(\mathbf{B})}$, combining (51) and (52), we have

$$|d(\mathbf{x}, \mathbf{H}_{\mathbf{A}}) - d(\mathbf{x}, \mathbf{H}_{\mathbf{B}})| \quad \leq \max\{\frac{C}{\sigma_r(\mathbf{A})}, \frac{C}{\sigma_s(\mathbf{B})}\}\|\mathbf{x}\|_2$$

$$= \frac{C\|\mathbf{x}\|_2}{\min\{\sigma_r(\mathbf{A}), \sigma_s(\mathbf{B})\}}. \tag{53}$$

So that (46) is proved. □

Proof of Lemma 7.2. We have $\mathbf{y}_i = \mathbf{x}_i - \mathbf{n}_i$, and $\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}}^\top \mathbf{Y}_{\boldsymbol{\beta}}) = \left(\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})\right)^2 \geq \sigma_{\mathbf{Y},r}^2$.

By Weyl [Weyl, 1912], $|\sigma_i(\mathbf{Y}_{\boldsymbol{\beta}}) - \sigma_i(\mathbf{X}_{\boldsymbol{\beta}})| \leq \|\mathbf{N}_{\boldsymbol{\beta}}\|_2 \leq \|\mathbf{N}_{\boldsymbol{\beta}}\|_F \leq \sqrt{r}\delta$. Since $\sqrt{r}\delta < \sigma_{\mathbf{Y},r} \leq \sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}}) \leq \sigma_i(\mathbf{Y}_{\boldsymbol{\beta}})$, $\sigma_i(\mathbf{X}_{\boldsymbol{\beta}}) \geq \sigma_i(\mathbf{Y}_{\boldsymbol{\beta}}) - \sqrt{r}\delta \geq \sigma_{\mathbf{Y},r} - \sqrt{r}\delta > 0$ for $1 \leq i \leq \min\{d, r\}$. It follows that $\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}}) \geq \sigma_{\mathbf{Y},r} - \sqrt{r}\delta > 0$ and $\mathbf{X}_{\boldsymbol{\beta}}$ has full column rank.

Also, $\|\mathbf{X}_{\boldsymbol{\beta}} - \mathbf{Y}_{\boldsymbol{\beta}}\|_2 \leq \|\mathbf{X}_{\boldsymbol{\beta}} - \mathbf{Y}_{\boldsymbol{\beta}}\|_F \leq \sqrt{r}\delta$. According to Lemma 7.4,

$$\begin{aligned} &|d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}}}) - d(\mathbf{x}_i, \mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}}})| \\ &\leq \frac{\sqrt{r}\delta}{\min\{\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}}), \sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})\}} \\ &\leq \frac{\sqrt{r}\delta}{\sigma_{\mathbf{Y},r} - \sqrt{r}\delta} = \frac{\delta}{\bar{\sigma}_{\mathbf{Y},r} - \delta}. \end{aligned} \tag{54}$$

□

7.1.4 Proof of Lemma 7.3

Proof of Lemma 7.3. We have

$$\begin{aligned} &\|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2^2 + \lambda\|\boldsymbol{\beta}^*\|_0 \leq \|\mathbf{x}_i - \mathbf{X}\mathbf{0}\|_2^2 + \lambda\|\mathbf{0}\|_0 = 1 \\ &\Rightarrow c^* = \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2 \leq \sqrt{1 - \lambda r^*} < 1. \end{aligned}$$

We first prove that $\boldsymbol{\beta}^*$ is an optimal solution to the sparse approximation problem

$$\min_{\boldsymbol{\beta}} \|\boldsymbol{\beta}\|_0 \quad s.t.\ \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}\|_2 \leq c^*,\ \boldsymbol{\beta}_i = 0. \tag{55}$$

To see this, if $r^* = 1$, then β^* must be an optimal solution to (55). If $r^* > 1$, suppose there is a vector $\boldsymbol{\beta}'$ such that $\|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}'\|_2 \leq c^*$ and $\|\boldsymbol{\beta}'\|_0 < \|\boldsymbol{\beta}^*\|_0$, then $L(\boldsymbol{\beta}') < c^* + \lambda\|\boldsymbol{\beta}^*\|_0 = L(\boldsymbol{\beta}^*)$, contradicting the fact that $\boldsymbol{\beta}^*$ is an optimal solution to (5).

Note that $\mathbf{X}_{\boldsymbol{\beta}^*}$ is a full column rank matrix, otherwise a sparser solution to (5) can be obtained as vector whose support corresponds to the maximal linear independent set of columns of $\mathbf{X}_{\boldsymbol{\beta}^*}$.

Also, the distance between $\mathbf{x}_i$ and the subspace spanned by columns of $\mathbf{X}_{\boldsymbol{\beta}^*}$ equals to c^*, i.e. $d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) = c^*$. To see this, it is clear that $d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) \leq c^*$. If there is a vector $\mathbf{y} = \mathbf{X}\tilde{\boldsymbol{\beta}}$ in $\mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}$ with $\text{supp}(\tilde{\boldsymbol{\beta}}) \subseteq \text{supp}(\boldsymbol{\beta}^*)$, and $\|\mathbf{x}_i - \mathbf{y}\|_2 < c^*$, then $L(\tilde{\boldsymbol{\beta}}) < L(\boldsymbol{\beta}^*)$ which contradicts the optimality of $\boldsymbol{\beta}^*$. Therefore, $d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) \geq c^*$, and it follows that $d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) = c^*$.

Since $\|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2 \leq 1$, $\|\mathbf{X}\boldsymbol{\beta}^*\|_2 \leq 2$. Also,

$$\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}^*}^\top \mathbf{X}_{\boldsymbol{\beta}^*})\|\boldsymbol{\beta}^*\|_2^2 \leq \|\mathbf{X}\boldsymbol{\beta}^*\|_2^2 \leq 4,$$

it follows that $\|\boldsymbol{\beta}^*\|_2^2 \leq \frac{4}{\sigma_{\mathbf{X}}^{*2}}$. By Cauchy-Schwarz inequality, $\|\boldsymbol{\beta}^*\|_1 \leq \frac{2\sqrt{r^*}}{\sigma_{\mathbf{X}}^*}$ and $\|\mathbf{N}\boldsymbol{\beta}^*\|_2 \leq \|\boldsymbol{\beta}^*\|_1\delta \leq \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*}$. Therefore,

$$\begin{aligned} &\|\mathbf{x}_i - \mathbf{Y}\boldsymbol{\beta}^*\|_2 = \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^* + \mathbf{N}\boldsymbol{\beta}^*\|_2 \\ &\leq \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2 + \|\mathbf{N}\boldsymbol{\beta}^*\|_2 \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*}, \end{aligned}$$

so that $\boldsymbol{\beta}^*$ is a feasible for problem (47).

To prove that $\boldsymbol{\beta}^*$ is an optimal solution to (47), we first note that $\boldsymbol{\beta}^*$ must be an optimal solution to (47) if $r^* = 1$. This is because $c^* \leq \sqrt{1 - \lambda r^*} \leq 1 - \lambda$ and $\lambda > \tau_0 > \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*}$ so that $c^* + \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*} < 1$, and it follows that $\mathbf{0}$ is not feasible to (47).

If $r^* > 1$ and suppose $\boldsymbol{\beta}^*$ is not an optimal solution to (47), then an optimal solution to (47) is a vector $\boldsymbol{\beta}'$ such that $\|\mathbf{x}_i - \mathbf{Y}\boldsymbol{\beta}'\|_2 \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*}$ and $\|\boldsymbol{\beta}'\|_0 = r < r^*$. $\mathbf{Y}_{\boldsymbol{\beta}'}$ is a full column rank matrix, otherwise a sparser solution can be obtained as vector whose support corresponds to the maximal linear independent set of columns of $\mathbf{Y}_{\boldsymbol{\beta}'}$. We have

$$d(\mathbf{x}_i, \mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}'}}) \leq \|\mathbf{x}_i - \mathbf{Y}\boldsymbol{\beta}'\|_2 \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*}.$$

According to Lemma 7.2, we have

$$\begin{aligned} &|d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}'}}) - d(\mathbf{x}_i, \mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}'}})| \leq \frac{\sqrt{r}\delta}{\sigma_{\mathbf{Y},r} - \sqrt{r}\delta} \\ &= \frac{\delta}{\bar{\sigma}_{\mathbf{Y},r} - \delta} \leq \frac{\delta}{\bar{\sigma}_{\mathbf{Y}}^* - \delta} \\ &\Rightarrow d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}'}}) \leq c^* + \frac{2\delta\sqrt{r^*}}{\sigma_{\mathbf{X}}^*} + \frac{\delta}{\bar{\sigma}_{\mathbf{Y}}^* - \delta} = c^* + \tau_0. \end{aligned} \tag{56}$$

However, according to the optimality of $\boldsymbol{\beta}^*$ in the noisy ℓ^0-SSC problem (5), we have

$$\begin{aligned} &d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}'}}) - c^* = d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}'}}) - d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) \\ &\overset{①}{\geq} (r^* - r)\lambda > \tau_0. \end{aligned} \tag{57}$$

To see ① holds, let $\boldsymbol{\beta}'' \in \mathbb{R}^d$, $\text{supp}(\boldsymbol{\beta}'') \subseteq \text{supp}(\boldsymbol{\beta}')$ such that $\|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}''\|_2 = d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}'}})$. Then by the optimality of $\boldsymbol{\beta}^*$,

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}''\|_2 &\geq d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) + \lambda r^* - \lambda\,|\text{supp}(\boldsymbol{\beta}'')| \\ &\geq d(\mathbf{x}_i, \mathbf{H}_{\mathbf{X}_{\boldsymbol{\beta}^*}}) + (r^* - r)\lambda. \end{aligned}$$

The contradiction between (56) and (57) shows that $\boldsymbol{\beta}^*$ is an optimal solution to (47). □

7.1.5 Proof of Theorem 3.3

Proof of Theorem 3.3. This theorem can be proved by checking that the conditions in Theorem 3.1 are satisfied. □

7.1.6 Proof of Theorem 3.6

In order to prove this theorem, the following lemma is presented and it provides the geometric concentration inequality for the distance between a point $\mathbf{y} \in \mathbf{Y}^{(k)}$ and any of its external subspaces. It renders a lower bound for M_i, namely the minimum distance between $\mathbf{y}_i \in \mathcal{S}_k$ and its external subspaces.

Lemma 7.5. Under semi-random model, given $1 \le k \le K$ and $\mathbf{y} \in \mathbf{Y}^{(k)}$, suppose $\mathbf{H} \in \mathcal{H}_{\mathbf{y}_i, d_k}$ is any external subspace of $\mathbf{y}$. Then for any $t > 0$,

$$\Pr[d(\mathbf{y}, \mathbf{H}) \ge 1 - 2t\sqrt{d_k - 1} - t^2] \ge 1 - 8\exp(-\frac{d_k t^2}{2}). \tag{58}$$

Proof of Lemma 7.5. Let $\mathbf{H}$ be a fixed subspace of dimension $d_e \le d_k$, and $\mathbf{y} \notin \mathbf{H}$. Since $\mathbf{y} \in \mathcal{S}_k$ and $\mathbf{y} \notin \mathbf{H}$. Let $\mathbf{U}_{\mathcal{S}_k} = \begin{bmatrix} \mathbf{I}_{d_k} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{d \times d_k}$ be the orthonormal basis of $\mathcal{S}_k$ under which the isotropic random vector $\mathbf{y}$ in $\mathcal{S}_k$ satisfies $\mathbb{E}[\mathbf{y}\mathbf{y}^\top] = \begin{bmatrix} \mathbf{I}_{d_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. It follows that more columns vectors can be added to $\mathbf{U}_{\mathcal{S}_k}$ to form a orthonormal basis $\mathbf{U} \in \mathbb{R}^{d \times d'}$ for the minimum subspace that contains $\mathcal{S}_k$ and $\mathbf{H}$. It can be verified that $d_k + 1 \le d' \le \min\{d_k + d_e, d\}$ because $\mathbf{H} \ne \mathcal{S}_k$. Note that $\mathbf{U}$ can be represented as a block matrix as $\mathbf{U} = \begin{bmatrix} \mathbf{I}_{d_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}' \end{bmatrix}$ where $\mathbf{U}' \in \mathbb{R}^{(d-d_k)\times(d'-d_k)}$ has orthonormal columns. It can be verified that the basis of $\mathbf{H}$ can be represented as $\mathbf{U}_\mathbf{H} = \begin{bmatrix} \mathbf{I}_{d_e - d' + d_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}' \end{bmatrix}$. Note that if $d_e - d' + d_k = 0$, $\mathbf{U}_\mathbf{H} = \begin{bmatrix} \mathbf{0} \\ \mathbf{U}' \end{bmatrix}$. Then $\mathbb{P}_\mathbf{H}(\mathbf{y}) = \mathbf{U}_\mathbf{H}\mathbf{U}_\mathbf{H}^\top \mathbf{y}$, and we have

$$\begin{aligned}
&\mathbb{E}[\|\mathbb{P}_\mathbf{H}(\mathbf{y})\|_2^2] = \mathbb{E}[\mathbf{y}^\top \mathbf{U}_\mathbf{H}\mathbf{U}_\mathbf{H}^\top \mathbf{U}_\mathbf{H}\mathbf{U}_\mathbf{H}^\top \mathbf{y}] \\
&= \mathbb{E}[\mathrm{Tr}(\mathbf{y}^\top \mathbf{U}_\mathbf{H}\mathbf{U}_\mathbf{H}^\top \mathbf{y})] \\
&= \mathbb{E}[\mathrm{Tr}(\mathbf{U}_\mathbf{H}^\top \mathbf{y}\mathbf{y}^\top \mathbf{U}_\mathbf{H})] \\
&= \mathrm{Tr}(\mathbf{U}_\mathbf{H}^\top \mathbb{E}[\mathbf{y}\mathbf{y}^\top]\mathbf{U}_\mathbf{H}) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \mathbf{I}_{d_e - d' + d_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}' \end{bmatrix}^\top \begin{bmatrix} \mathbf{I}_{d_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{d_e - d' + d_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}' \end{bmatrix} \right) \\
&= d_e - d' + d_k \le d_e - 1 \le d_k - 1
\end{aligned} \tag{59}$$

According to the concentration inequality in section 5.2 of [Aubrun and Szarek, 2017], for any $t > 0$,

$$\Pr[|\|\mathbb{P}_\mathbf{H}(\mathbf{y})\|_2 - \sqrt{d_e - d' + d_k}| \ge t] \le 8\exp(-\frac{d_k t^2}{2}) \tag{60}$$

Now let $\mathbf{H}$ be spanned by data from $\mathbf{Y}$, i.e. $\mathbf{H} = \mathbf{H}_{\{\mathbf{y}_{i_j}\}_{j=1}^{d_e}}$, where $\{\mathbf{y}_{i_j}\}_{j=1}^{d_e}$ are any d_e linearly independent points that does not contain $\mathbf{y}$. For any fixed points $\{\mathbf{y}_{i_j}\}_{j=1}^{d_e}$, (60) holds. Let A be the event that $|\mathbb{P}_\mathbf{H}(\mathbf{y}) - \sqrt{d_e - d' + d_k}| \ge t$, we aim to integrate the indicator function $\mathbb{I}_A$ with respect to the random vectors, i.e. $\mathbf{y}$ and $\{\mathbf{y}_{i_j}\}_{j=1}^{d_e}$, to obtain the probability that A happens over these random vectors. Let $\mathbf{y} = \mathbf{y}_i$, using Fubini theorem, we have

$$\begin{aligned}
\Pr[A] &= \int_{\times_{j=1}^n \mathcal{S}^{(j)}} \mathbb{I}_A \otimes_{j=1}^n d\mu^{(j)} \\
&= \int_{\times_{j\ne i}\mathcal{S}^{(j)}} \Pr[A|\{\mathbf{y}_j\}_{j\ne i}] \otimes_{j\ne i} d\mu^{(j)} \\
&\le \int_{\times_{j\ne i}\mathcal{S}^{(j)}} 8\exp(-\frac{d_k t^2}{2}) \otimes_{j\ne i} d\mu^{(j)} = 8\exp(-\frac{d_k t^2}{2})
\end{aligned} \tag{61}$$

where $\mathcal{S}^{(j)} \in \{\mathcal{S}_k\}_{k=1}^K$ is the subspace that $\mathbf{y}_j$ lies in, and $\mu^{(j)}$ is the probabilistic measure of the distribution in $\mathcal{S}^{(j)}$. The last inequality is due to (60).

Note that for any $\mathbf{y}$'s external subspace $\mathbf{H} = \mathbf{H}_{\{\mathbf{y}_{i_j}\}_{j=1}^{d_e}}$, $d(\mathbf{y}, \mathbf{H}) = \sqrt{\|\mathbf{y}\|_2^2 - \|\mathbb{P}_\mathbf{H}(\mathbf{y})\|_2^2} = \sqrt{d_k - \|\mathbb{P}_\mathbf{H}(\mathbf{y})\|_2^2}$. According to (61), we have

$$\Pr[d(\mathbf{y}, \mathbf{H}) \ge 1 - 2t\sqrt{d_k - 1} - t^2] \ge 1 - 8\exp(-\frac{d_k t^2}{2}). \tag{62}$$

□

The following lemma shows the lower bound for any submatrix of $\mathbf{Y}^{(k)}$.

Lemma 7.6. Let $\mathbf{Y} \in \mathbb{R}^{d \times r}$ be a submatrix of $\mathbf{Y}^{(k)}$ with $\mathrm{rank}(\mathbf{Y}) = r$ and $r \le d_k$, $k \in [K]$. Define $M := \sup_{0 \le t < 1} \frac{2t^3 \arccos t}{\pi} < 1$. Then with probability at least $1 - \exp(-d_k) - 2n_k \exp\left(-d_k^{0.9}\right)$, $\sigma_{\min}(\mathbf{Y}_\boldsymbol{\beta}) \ge \frac{1}{\sqrt{5}}\left(1 - 14\sqrt[3]{4}\left(\frac{M^2 r}{d_k}\right)^{2/3}\right) := \sigma'(d_k, r)$ for any $\boldsymbol{\beta} \in \mathbb{R}^n$ such that $\mathbf{Y}_\boldsymbol{\beta}$ is a submatrix of $\mathbf{Y}^{(k)}$ and $\|\boldsymbol{\beta}\|_0 = r$.

Proof. Let $\mathbf{Y} = \mathbf{U}^{(k)}\boldsymbol{\alpha}$, $\boldsymbol{\alpha} \in \mathbb{R}^{d_k \times r}$, and each α^i is uniformly distributed on $\mathbb{S}^{d_k - 1}$ for $i \in [r]$. Let $\mathbf{S} \in \mathbb{R}^{r \times r}$ and $\mathbf{S}_{ii} = \left\|\boldsymbol{\alpha}^i\right\|_2$ for $i \in [r]$, and define $\mathbf{C} := \mathbf{S}\boldsymbol{\alpha}^\top$. It can be verified that columns of $\mathbf{C}$ are independent isotropic random vectors, and $\left\|\boldsymbol{\alpha}^i\right\|_2^2$ follows χ^2-distribution, then it can also be verified that M satisfies the following condition: for any $t > 0$, any $i \in [n]$ and any vector $\mathbf{v}$ with unit ℓ^2-norm,

$$\Pr[|\langle \mathbf{C}^i, \mathbf{v}\rangle| > t] \le \frac{M}{t^3}. \tag{63}$$

According to Theorem 2.1 in [Yaskov, 2015] with condition (63), with probability at least $1-\exp(-d_k)$, $\sigma_{\min}(\mathbf{C}) \geq \sqrt{d_k}\left(1-14\sqrt[3]{4}(\frac{M^2 r}{d_k})^{2/3}\right) = \sigma'_{\min}$. By the concentration property of χ^2-distribution, with probability at least $1-2n_k\exp\left(-d_k^{0.9}\right)$, $\mathbf{S}_{ii} \in [\sqrt{d_k-2d_k^{0.95}}, \sqrt{d_k+2d_k^{0.95}+2d_k^{0.9}}]$ for all $i\in[r]$ and any submatrix $\mathbf{Y}$ of $\mathbf{Y}^{(k)}$. As a result, with probability at least $1-\exp(-d_k)-2n_k\exp\left(-d_k^{0.9}\right)$, $\sigma_{\min}(\mathbf{Y}) = \sigma_{\min}(\boldsymbol{\alpha}) \geq \frac{1}{\sqrt{5}}\left(1-14\sqrt[3]{4}(\frac{M^2 r}{d_k})^{2/3}\right)$.

□

Proof of Theorem 3.6. Let $\mathbf{Y}_{\boldsymbol{\beta}}$ for any $\boldsymbol{\beta}\in\mathbb{R}^n$ with $\|\boldsymbol{\beta}\|_0 = r_0$. Noting that $\mathbf{Y}_{\boldsymbol{\beta}}$ have columns from at most r_0 subspaces, let $\boldsymbol{\beta} = \sum_{r=1}^{r_0}\boldsymbol{\beta}^{(r)}$, $\{\boldsymbol{\beta}^{(r)}\}_{r=1}^{r_0}$ have non-coverlapping support, each $\mathbf{Y}_{\boldsymbol{\beta}^{(r)}}$ is a submatrix of $\mathbf{Y}_{\boldsymbol{\beta}}$ and columns of $\mathbf{Y}_{\boldsymbol{\beta}^{(r)}}$ are from the same subspace. For any $\mathbf{u}\in\mathbb{R}^{r_0}$ with $\|\mathbf{u}\|_2=1$, we can write $\mathbf{u}$ as $\mathbf{u}=\sum_{r=1}^{r_0}\mathbf{u}^{(r)}$ where $\{\mathbf{u}^{(r)}\}_{r=1}^{r_0}$ have non-overlapping support and $\mathbf{u}^{(r)}$ corresponds to $\mathbf{Y}_{\boldsymbol{\beta}^{(r)}}$ for $r\in[r_0]$. Because $r_0 \leq \lfloor\frac{1}{\lambda}\rfloor$, with $d_{\min}$ sufficiently large we have $d_{\min}\geq\frac{1}{\lambda}\geq r_0$ and $\sigma'(d_k, r_0)\geq\sigma'(d_{\min},\frac{1}{\lambda})>0$ where $\sigma'(\cdot,\cdot)$ is defined in Lemma 7.6. Define

$$\sigma'_{\min} := \sigma'(d_{\min},\frac{1}{\lambda}) = \frac{1}{\sqrt{5}}\left(1-14\sqrt[3]{4}(\frac{M^2}{\lambda d_{\min}})^{2/3}\right),$$
$$\mathrm{aff}_{\max} := \max_{t_1,t_2\in[K]:\, t_1\neq t_2}\mathrm{aff}\left(\mathcal{S}_{t_1},\mathcal{S}_{t_2}\right).$$

By Lemma 7.6, $\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}^{(r)}})\geq\sigma'_{\min}$ for $r\in[r_0]$. We have

$$\begin{aligned}
&\|\mathbf{Y}_{\boldsymbol{\beta}}\mathbf{u}\|_2^2\\
&=\sum_{r=1}^{r_0}\left\|\mathbf{Y}_{\boldsymbol{\beta}^{(r)}}\mathbf{u}^{(r)}\right\|_2^2+\sum_{s,t\in[r_0]:\, s<t}\mathbf{u}^{(s)\top}\mathbf{Y}_{\boldsymbol{\beta}^{(s)}}^\top\mathbf{Y}_{\boldsymbol{\beta}^{(t)}}\mathbf{u}^{(t)}\\
&\geq\sigma'^2_{\min}\|\mathbf{u}\|_2^2-\left\|\mathbf{u}^{(s)}\right\|_2\left\|\mathbf{u}^{(t)}\right\|_2\\
&\geq\left(\sigma'^2_{\min}-(r_0-1)\mathrm{aff}_{\max}\right)\|\mathbf{u}\|_2^2\\
&=\sigma'^2_{\min}-(r_0-1)\mathrm{aff}_{\max}.
\end{aligned}\tag{64}$$

It follows that $\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})\geq\sigma'^2_{\min}-(r_0-1)\mathrm{aff}(\mathcal{S}_{t_1},\mathcal{S}_{t_2})$. By Weyl [Weyl, 1912], $|\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}})-\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})|\leq\|\mathbf{N}_{\boldsymbol{\beta}}\|_2\leq\delta\sqrt{r_0}$. Therefore,

$$\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}})\geq\sigma'^2_{\min}-(r_0-1)\mathrm{aff}(\mathcal{S}_{t_1},\mathcal{S}_{t_2})-\delta\sqrt{r_0}>0,\tag{65}$$

if $\delta<\frac{\sigma'^2_{\min}-(r_0-1)\mathrm{aff}(\mathcal{S}_{t_1},\mathcal{S}_{t_2})}{\sqrt{r_0}}=c$. It can be verified that (22), (23) and (24) guarantee (14), (15) and (16) in Theorem 3.3 respectively, therefore, the conclusion holds. □

7.1.7 Proof of Theorem 4.1

We need the following lemmas before presenting the proof of Theorem 4.1. Lemma 7.7 shows that the low rank approximation $\bar{\mathbf{X}}$ is close to $\mathbf{X}$ in terms of the spectral norm [Halko et al., 2011]. Lemma 7.8 presents a perturbation bound for the distance between a data point and a subspace before and after the projection $\mathbf{P}$.

Lemma 7.7. (Corollary 10.9 in [Halko et al., 2011]) Let $p_0\geq 2$ be an integer and $p'=p-p_0\geq 4$, then with probability at least $1-6e^{-p}$, the spectral norm of $\mathbf{X}-\widehat{\mathbf{X}}$ is bounded by

$$\|\mathbf{X}-\widehat{\mathbf{X}}\|_2\leq C_{p,p_0}\tag{66}$$

where

$$C_{p,p_0} := \left(1+17\sqrt{1+\frac{p_0}{p'}}\right)\sigma_{p_0+1}+\frac{8\sqrt{p}}{p'+1}\left(\sum_{j>p_0}\sigma_j^2\right)^{\frac{1}{2}}\tag{67}$$

and $\sigma_1\geq\sigma_2\geq\ldots$ are the singular values of $\mathbf{X}$.

Lemma 7.8. Let $\boldsymbol{\beta}\in\mathbb{R}^n$, $\tilde{\mathbf{y}}_i=\mathbf{P}\mathbf{y}_i$, $\mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}}}$ is an external subspace of $\mathbf{y}_i$, $\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}=\mathbf{P}(\mathbf{Y}_{\boldsymbol{\beta}})$ and $\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}$ has full column rank. Then

$$\begin{aligned}
&|d(\mathbf{y}_i,\mathbf{H}_{\mathbf{Y}_{\boldsymbol{\beta}}})-d(\tilde{\mathbf{y}}_i,\mathbf{H}_{\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}})|\\
&\leq C_{p,p_0}\left(1+\frac{1}{\min_{1\leq r\leq\tilde{d}_k}\sigma_{\mathbf{Y},r}-C_{p,p_0}-2\delta\sqrt{\tilde{d}_k}}\right)
\end{aligned}\tag{68}$$

for any $1\leq i\leq n$ and $\mathbf{y}_i\in\mathcal{S}_k$.

Proof. This lemma can be proved by applying Lemma 7.4. □

Proof of Theorem 4.1. For any matrix $\mathbf{A}\in\mathbb{R}^{p\times q}$, we first show that multiplying $\mathbf{Q}$ to the left of $\mathbf{A}$ would not change its spectrum. To see this, let the singular value decomposition of $\mathbf{A}$ be $\mathbf{A}=\mathbf{U}_{\mathbf{A}}\boldsymbol{\Sigma}\mathbf{V}_{\mathbf{A}}^\top$ where $\mathbf{U}_{\mathbf{A}}$ and $\mathbf{V}_{\mathbf{A}}$ have orthonormal columns with $\mathbf{U}_{\mathbf{A}}^\top\mathbf{U}_{\mathbf{A}}=\mathbf{V}_{\mathbf{A}}^\top\mathbf{V}_{\mathbf{A}}=\mathbf{I}$. Then $\mathbf{Q}\mathbf{A}=\mathbf{U}_{\mathbf{QA}}\boldsymbol{\Sigma}\mathbf{V}_{\mathbf{QA}}$ is the singular value decomposition of $\mathbf{QA}$ with $\mathbf{U}_{\mathbf{QA}}=\mathbf{Q}\mathbf{U}_{\mathbf{A}}$ and $\mathbf{V}_{\mathbf{QA}}=\mathbf{V}_{\mathbf{A}}$. This is because the columns of $\mathbf{U}_{\mathbf{QA}}$ are orthonormal since the columns $\mathbf{Q}$ are orthonormal: $\mathbf{U}_{\mathbf{QA}}^\top\mathbf{U}_{\mathbf{QA}}=\mathbf{U}_{\mathbf{A}}^\top\mathbf{Q}^\top\mathbf{Q}\mathbf{U}_{\mathbf{A}}=\mathbf{I}$, and $\boldsymbol{\Sigma}$ is a diagonal matrix with nonnegative diagonal elements. It follows that $\sigma_{\min}(\mathbf{QA})=\sigma_{\min}(\mathbf{A})$ for any $\mathbf{A}\in\mathbb{R}^{p\times q}$.

For a point $\mathbf{x}_i=\mathbf{y}_i+\mathbf{n}_i$, after projection via $\mathbf{P}$, we have the projected noise $\tilde{\mathbf{n}}_i=\mathbf{P}\mathbf{n}_i$. Because

$$\|\tilde{\mathbf{n}}_i\|_2=\|\mathbf{P}\mathbf{n}_i\|_2=\|\mathbf{Q}^\top\mathbf{n}_i\|_2\leq\|\mathbf{Q}\|_2\|\mathbf{n}_i\|_2\leq\|\mathbf{n}_i\|_2\leq\delta,\tag{69}$$

the magnitude of the noise in the projected data is also bounded by δ. Also,

$$\|\tilde{\mathbf{x}}_i\|_2=\|\mathbf{Q}^\top\mathbf{x}_i\|_2\leq\|\mathbf{x}_i\|_2\leq 1,\tag{70}$$

Let $\boldsymbol{\beta} \in \mathbb{R}^n$, $\tilde{\mathbf{Y}}_{\boldsymbol{\beta}} = \mathbf{P}\mathbf{Y}_{\boldsymbol{\beta}}$ with $\|\boldsymbol{\beta}\|_0 = r$. Then $\sigma_{\min}(\mathbf{Q}\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}) = \sigma_{\min}(\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}))$. Since

$$\begin{aligned}
&|\sigma_{\min}(\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}) - \sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})| = |\sigma_{\min}(\mathbf{Q}\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}) - \sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})| \\
&\leq \|\mathbf{Q}\tilde{\mathbf{Y}}_{\boldsymbol{\beta}} - \mathbf{Y}_{\boldsymbol{\beta}}\|_2 \\
&= \|\mathbf{Q}\mathbf{Q}^\top\mathbf{Y}_{\boldsymbol{\beta}} - \mathbf{Y}_{\boldsymbol{\beta}}\|_2 \\
&= \|\mathbf{Q}\mathbf{Q}^\top\mathbf{X}_{\boldsymbol{\beta}} - \mathbf{X}_{\boldsymbol{\beta}} + \mathbf{N}_{\boldsymbol{\beta}} - \mathbf{Q}\mathbf{Q}^\top\mathbf{N}_{\boldsymbol{\beta}}\|_2 \\
&\leq C_{p,p_0} + \|\mathbf{N}_{\boldsymbol{\beta}}\|_F + \|\mathbf{Q}\mathbf{Q}^\top\mathbf{N}_{\boldsymbol{\beta}}\|_F \\
&\leq C_{p,p_0} + 2\delta\sqrt{r}
\end{aligned} \tag{71}$$

Therefore, it follows from (71) that if

$$C_{p,p_0} + 2\delta\sqrt{\tilde{d}_{\max}} < \min_{k=1,\ldots,K} \sigma_{\mathbf{Y}}^{(k)}, \tag{72}$$

then $\tilde{\mathbf{Y}}$ is also in general position.

In addition, since $\lambda \geq \frac{1}{r_0}$, we have $\lambda\|\tilde{\boldsymbol{\beta}}^*\|_0 \leq L(\mathbf{0}) \leq 1$, and it follows that $\|\tilde{\boldsymbol{\beta}}^*\|_0 \leq r_0 \leq \lfloor \frac{1}{\lambda} \rfloor$.

Based on (71) we have

$$|\bar{\sigma}_{\tilde{\mathbf{Y}},r} - \bar{\sigma}_{\mathbf{Y},r}| \leq C_{p,p_0} + 2\delta\sqrt{r_0}, \tag{73}$$

it follows that $\delta < \min_{1\leq r<r_0} \bar{\sigma}_{\tilde{\mathbf{Y}},r}$ because $\delta < \min_{1\leq r<r_0} \bar{\sigma}_{\mathbf{Y},r} - C_{p,p_0} - 2\delta\sqrt{r_0}$.

Again, for $\boldsymbol{\beta} \in \mathbb{R}^n$ with $\|\boldsymbol{\beta}\|_0 = r \leq r_0$, we have

$$\begin{aligned}
&|\sigma_{\min}(\tilde{\mathbf{X}}_{\boldsymbol{\beta}}) - \sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}})| = |\sigma_{\min}(\mathbf{Q}\tilde{\mathbf{X}}_{\boldsymbol{\beta}}) - \sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}})| \\
&\leq \|\mathbf{Q}\tilde{\mathbf{X}}_{\boldsymbol{\beta}} - \mathbf{X}_{\boldsymbol{\beta}}\|_2 \\
&= \|\mathbf{Q}\mathbf{Q}^\top\mathbf{X}_{\boldsymbol{\beta}} - \mathbf{X}_{\boldsymbol{\beta}}\|_2 = \|\widehat{\mathbf{X}} - \mathbf{X}_{\boldsymbol{\beta}}\|_2 \\
&\leq C_{p,p_0}
\end{aligned} \tag{74}$$

It can be verified that

$$|\sigma_{\tilde{\mathbf{X}},r} - \sigma_{\mathbf{X},r}| \leq C_{p,p_0} \tag{75}$$

Combining (75) and Lemma 7.8, noting that $\sigma_{\mathbf{X},r_0} - C_{p,p_0}$, since

$$\begin{aligned}
&M_i - C_{p,p_0}\Big(1 + \frac{1}{\min_{1\leq r\leq \tilde{d}_k} \sigma_{\mathbf{Y},r} - C_{p,p_0} - 2\delta\sqrt{\tilde{d}_k}}\Big) \\
&> \delta + \frac{2\delta}{\sigma_{\mathbf{X},r_0} - C_{p,p_0}},
\end{aligned} \tag{76}$$

we have

$$\tilde{M}_{i,\delta} := \tilde{M}_i - \delta > \frac{2\delta}{\tilde{\sigma}_{\tilde{\mathbf{X}},r_0}}, \tag{77}$$

where $\mathbf{y}_i \in \mathcal{S}_k$.

Based on (73) and (75), we have

$$\tilde{\mu}_{r_0} < 1 - \frac{2\delta}{\sigma_{\tilde{\mathbf{X}},r_0}}, \tag{78}$$

because

$$\begin{aligned}
&\frac{\delta}{\min_{1\leq r<r_0} \bar{\sigma}_{\mathbf{Y},r_0} - C_{p,p_0} - 2\delta\sqrt{r_0} - \delta} \\
&< 1 - \frac{2\delta}{\sigma_{\mathbf{X},r_0} - C_{p,p_0}}
\end{aligned} \tag{79}$$

□

7.1.8 Proof of Theorem 4.2

Proof of Theorem 4.2. It can be verified that $\tilde{M}_i \geq \frac{M_i}{1+\varepsilon}$. Let $\boldsymbol{\beta} \in \mathbb{R}^n$, $\tilde{\mathbf{Y}}_{\boldsymbol{\beta}} = \mathbf{P}\mathbf{Y}_{\boldsymbol{\beta}}$ with $\|\boldsymbol{\beta}\|_0 = r$ and $\operatorname{rank}(\mathbf{Y}_{\boldsymbol{\beta}}) = r$, then for any $\mathbf{u} \in \mathbb{R}^r$, $\|\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}\mathbf{u}\|_2 = \|\mathbf{P}\mathbf{Y}_{\boldsymbol{\beta}}\mathbf{u}\|_2 \geq (1-\varepsilon)\|\mathbf{Y}_{\boldsymbol{\beta}}\mathbf{u}\|_2 \geq (1-\varepsilon)\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})\|\mathbf{u}\|_2$. It follows that $\sigma_{\min}(\tilde{\mathbf{Y}}_{\boldsymbol{\beta}}) \geq (1-\varepsilon)\sigma_{\min}(\mathbf{Y}_{\boldsymbol{\beta}})$, and $\bar{\sigma}_{\tilde{\mathbf{Y}},r} \geq (1-\varepsilon)\bar{\sigma}_{\mathbf{Y},r}$. Similarly, $\sigma_{\min}(\tilde{\mathbf{X}}_{\boldsymbol{\beta}}) \geq (1-\varepsilon)\sigma_{\min}(\mathbf{X}_{\boldsymbol{\beta}})$ for $\boldsymbol{\beta} \in \mathbb{R}^n, \|\boldsymbol{\beta}\|_0 = r$ and $\operatorname{rank}(\mathbf{X}_{\boldsymbol{\beta}}) = r$. It follows that $\sigma_{\tilde{\mathbf{X}},r} \geq (1-\varepsilon)\sigma_{\mathbf{X},r}$. Since (40)-(44) hold, the conditions (14)-(18) required by Theorem 3.3 on the projected data ($\tilde{\mathbf{Y}}$ and $\tilde{\mathbf{X}}$) also hold. Therefore, the subspace detection property holds with $\tilde{\boldsymbol{\beta}}^*$ for $\tilde{\mathbf{x}}_i$ with probability at least $1 - K\delta$ by the union bound when $p \geq \frac{d^2+d}{\delta'(2\varepsilon-\varepsilon^2)^2}$. □

7.2 BOUND FOR SUBOPTIMAL AND GLOBALLY OPTIMAL SOLUTIONS FOR NOISY ℓ^0-SSC AND NOISY-DR-ℓ^0-SSC

While our theoretical analysis for noisy ℓ^0-SSC and Noisy-DR-ℓ^0-SSC is based on optimal solution to the ℓ^0 regularized problem (5), in this subsection we prove that the bound for the suboptimal solution $\widehat{\boldsymbol{\beta}}$ obtained by Algorithm 2 is in fact close to an optimal solution to (5), justifying the theoretical findings of noisy ℓ^0-SSC and Noisy-DR-ℓ^0-SSC.

We further present the bound for the gap between $\widehat{\boldsymbol{\beta}}$ and $\boldsymbol{\beta}^*$, $\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_2$, based on Theorem 5 in [Yang and Yu, 2019]. Let $g(\boldsymbol{\beta}) = \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}\|_2^2$ and $\boldsymbol{\beta}^*$ be the globally optimal solution to (5), $\mathbf{S}^* = \operatorname{supp}(\boldsymbol{\beta}^*)$, $\widehat{\boldsymbol{\beta}}$ be the suboptimal solution to (5) obtained by PGD, $\widehat{\mathbf{S}} = \operatorname{supp}(\widehat{\boldsymbol{\beta}})$. The following theorem presents the bound for $\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_2$.

Theorem 7.9. (Theorem 5 in [Yang and Yu, 2019]) Suppose $\mathbf{X}_{\mathbf{S}\cup\mathbf{S}^*}$ has full column rank with $\kappa_0 := \sigma_{\min}(\mathbf{X}_{\mathbf{S}\cup\mathbf{S}^*}) > 0$ where $\mathbf{S}$ is the support of the initialization for PGD on problem (5). Let $\kappa > 0$ such that $2\kappa_0^2 > \kappa$ and b is chosen according to (80) as below:

$$\begin{aligned}
0 < b < \min\{&\min_{j\in\widehat{\mathbf{S}}} |\widehat{\boldsymbol{\beta}}_j|, \frac{\lambda}{\max_{j\notin\widehat{\mathbf{S}}} |\frac{\partial g}{\partial \boldsymbol{\beta}_j}|_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}}|}, \\
&\min_{j\in\mathbf{S}^*} |\boldsymbol{\beta}_j^*|, \frac{\lambda}{\max_{j\notin\mathbf{S}^*} |\frac{\partial g}{\partial \boldsymbol{\beta}_j}|_{\boldsymbol{\beta}=\boldsymbol{\beta}^*}|}\}.
\end{aligned} \tag{80}$$

Let $\mathbf{F} = (\widehat{\mathbf{S}} \setminus \mathbf{S}^*) \cup (\mathbf{S}^* \setminus \widehat{\mathbf{S}})$ be the symmetric difference between $\widehat{\mathbf{S}}$ and $\mathbf{S}^*$, then

$$\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_2 \leq \frac{1}{2\kappa_0^2 - \kappa}\Big(\sum_{j \in \mathbf{F} \cap \widehat{\mathbf{S}}} (\max\{0, \frac{\lambda}{b} - \kappa|\widehat{\boldsymbol{\beta}}_j - b|\})^2 + \sum_{j \in \mathbf{F} \setminus \widehat{\mathbf{S}}} (\max\{0, \frac{\lambda}{b} - \kappa b\})^2\Big)^{\frac{1}{2}} \tag{81}$$

Remark 7.10. It is observed that the gap $\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_2$ is small when $\frac{\lambda}{b} - \kappa|\widehat{\boldsymbol{\beta}}_j - b|$ for $j \in \mathbf{F} \cap \widehat{\mathbf{S}}$ and $\frac{\lambda}{b} - \kappa b$ are small. Based on this observation, Theorem 7.11 establishes the conditions under which $\widehat{\boldsymbol{\beta}}$ is also an optimal solution to (5), i.e. $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*$.

Define $\mathbf{S}^* = \operatorname{supp}(\boldsymbol{\beta}^*)$, $H^* = \max_{1 \leq j \leq n} \operatorname{dist}(\boldsymbol{\beta}, \mathbf{H}_{\mathbf{X}_{\mathbf{S}^* \setminus \{j\}}})$, $\mu = \max\{H^* + \|\boldsymbol{\beta}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2, 2\|\mathbf{x}_i - \mathbf{X}\widehat{\boldsymbol{\beta}}\|_2, 2\|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}^*\|_2\}$, $\kappa_0 = \sigma_{\min}(\mathbf{X}_{\mathbf{S} \cup \mathbf{S}^*}) > 0$ where $\mathbf{S} = \operatorname{supp}(\boldsymbol{\beta}^{(0)})$. The following theorem demonstrates that $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*$ if λ is two-side bounded and $\widehat{\boldsymbol{\beta}}_{\min} = \min_{t:\widehat{\boldsymbol{\beta}}_t \neq 0} |\widehat{\boldsymbol{\beta}}_t|$ is sufficiently large.

Theorem 7.11. (Conditions that the suboptimal solution by PGD is also globally optimal) If

$$\widehat{\boldsymbol{\beta}}_{\min} \geq \frac{\mu}{\kappa_0^2} \tag{82}$$

and

$$\frac{\mu^2}{2\kappa_0^2} \leq \lambda \leq (\widehat{\boldsymbol{\beta}}_{\min} - \frac{\mu}{2\kappa_0^2})\mu, \tag{83}$$

then $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*$.

Sketch of Proof. It can be verified that $\max\{0, \frac{\lambda}{b} - \kappa|\widehat{\boldsymbol{\beta}}_j - b|\} = 0$ and $\max\{0, \frac{\lambda}{b} - \kappa b\} = 0$ under the conditions (82) and (83), therefore, $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*$ by applying Theorem 7.9. □

7.3 TIME COMPLEXITY OF NOISY ℓ^0-SSC, NOISY-DR-ℓ^0-SSC-LR, NOISY-DR-ℓ^0-SSC-OSNAP

The time complexity of running PGD by Algorithm 2 for noisy ℓ^0-SSC is $\mathcal{O}(Tnd)$, where T is the maximum iteration number. The time complexity of running Algorithm 1 for Noisy-DR-ℓ^0-SSC-LR is comprised of two parts. The first part is the time complexity of steps 1-3 with matrix multiplication and QR decomposition, which is $\mathcal{O}(dp^2 + pdn)$. The second part is the time complexity of step 4, which is $\mathcal{O}(Tnp)$. The overall time complexity of Noisy-DR-ℓ^0-SSC is $\mathcal{O}(dp^2 + pdn + Tnp)$. In practice, p is much smaller than $\min\{d, n, T\}$, so Noisy-DR-ℓ^0-SSC-LR is more efficient than noisy ℓ^0-SSC. Noisy-DR-ℓ^0-SSC-OSNAP is even more efficient than both noisy ℓ^0-SSC and Noisy-DR-ℓ^0-SSC, whose time complexity is $\mathcal{O}(pdn + Tnp)$. This is because the linear transformation $\mathbf{P}$ obtained by OSNAP does require QR decomposition.

7.4 PROXIMAL GRADIENT DESCENT (PGD) FOR NOISY ℓ^0-SSC

Algorithm 1 describes how to perform Noisy-DR-ℓ^0-SSC-LR for data clustering. Note that Noisy-DR-ℓ^0-SSC performs noisy ℓ^0-SSC on the dimensionality reduced data $\tilde{\mathbf{X}}$. Proximal Gradient Descent (PGD) is employed to optimize the objective function of noisy ℓ^0-SSC for every data point $\mathbf{x}_i$, which is desribed in Algorithm 2. In the k-th iteration of PGD for problem (5), the variable $\boldsymbol{\beta}$ is updated according to

$$\boldsymbol{\beta}^{(k+1)} = T_{\sqrt{2\lambda s}}(\boldsymbol{\beta}^{(k)} - s\nabla g(\boldsymbol{\beta}^{(k)})), \tag{84}$$

where s is a positive step size, $g(\boldsymbol{\beta}) = \|\mathbf{x}_i - \mathbf{X}\boldsymbol{\beta}\|_2^2$, T_θ is an element-wise hard thresholding operator:

$$[T_\theta(\mathbf{u})]_j = \begin{cases} 0 & : \ |\mathbf{u}_j| \leq \theta \\ \mathbf{u}_j & : \ \text{otherwise} \end{cases}, \quad 1 \leq j \leq n.$$

It is proved in [Yang et al., 2017] that the sequence $\{\boldsymbol{\beta}^{(k)}\}$ generated by PGD converges to a critical point of (5).

Algorithm 2 Proximal Gradient Descent (PGD) for noisy ℓ^0-SSC problem (5)

Input:
The initialization $\boldsymbol{\beta}^{(0)}$, step size $s > 0$, parameter λ, maximum iteration number T, stopping threshold ε.

1: **for** $1 \leq i \leq n$ **do**
2: $\quad \tilde{\boldsymbol{\beta}}^{(t)} = \boldsymbol{\beta}^{(t-1)} - s\nabla g(\boldsymbol{\beta}^{(t-1)})$
3: $\quad \boldsymbol{\beta}^{(t)} = T_{\sqrt{2\lambda s}}(\tilde{\boldsymbol{\beta}}^{(t)})$
4: $\quad$ **if** $|L(\boldsymbol{\beta}^{(t)}) - L(\boldsymbol{\beta}^{(t-1)})| < \varepsilon$ **then**
5: $\quad\quad$ **break**
6: $\quad$ **end if**
7: **end for**

Output: $\widehat{\boldsymbol{\beta}}$ which is the suboptimal solution to (5)

7.5 ADDITIONAL EXPERIMENTAL RESULTS

We present more results of Noisy-DR-ℓ^0-SSC-LR and Noisy-DR-ℓ^0-SSC-OSNAP in Table 2 with different projection dimension p. Figure 1 show how the accuracy and NMI varies with respect to λ on the Extended Yale-B data set.

Figure 2a to Figure 2f illustrate SDP violation with respect to λ for different noise levels, justifying our theoretical finding that a large λ tends to preserve the subspace detection property for noisy ℓ^0-SSC, Noisy-DR-ℓ^0-SSC-LR and Noisy-DR-ℓ^0-OSNAP.

Table 2: Clustering results on various data sets, with different values of p for the linear transformation $\mathbf{P}$ and the best two results in bold

Data Set	Measure	Noisy ℓ^0-SSC	Noisy-DR-ℓ^0-SSC-LR			Noisy-DR-ℓ^0-SSC-OSNAP		
p			$p = \min\{d, n\}/5$	$p = \min\{d, n\}/10$	$p = \min\{d, n\}/15$	$p = \min\{d, n\}/5$	$p = \min\{d, n\}/10$	$p = \min\{d, n\}/15$
COIL-20	AC	0.8472	0.8479	0.8479	**0.8479**	**0.8486**	0.8472	0.8472
	NMI	0.9428	0.9433	0.9433	**0.9433**	**0.9439**	0.9428	0.9428
COIL-100	AC	**0.7683**	0.6992	**0.7276**	0.7043	**0.5404**	0.7046	0.7233
	NMI	**0.9182**	0.8626	**0.8919**	0.8636	0.7819	0.8708	**0.8726**
Yale-B	AC	**0.8480**	0.8219	0.8231	0.8289	**0.8500**	0.8318	0.8277
	NMI	0.8612	0.8519	0.8527	0.8534	0.8538	**0.8593**	**0.8594**

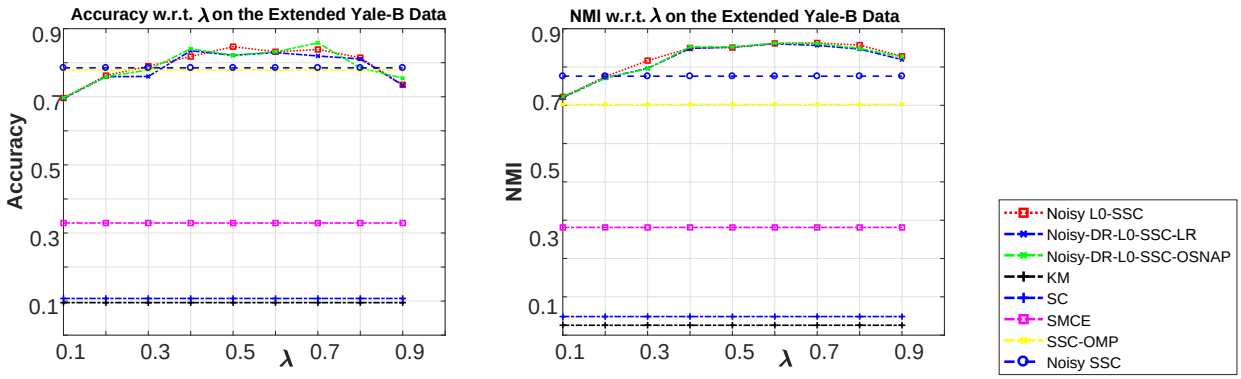
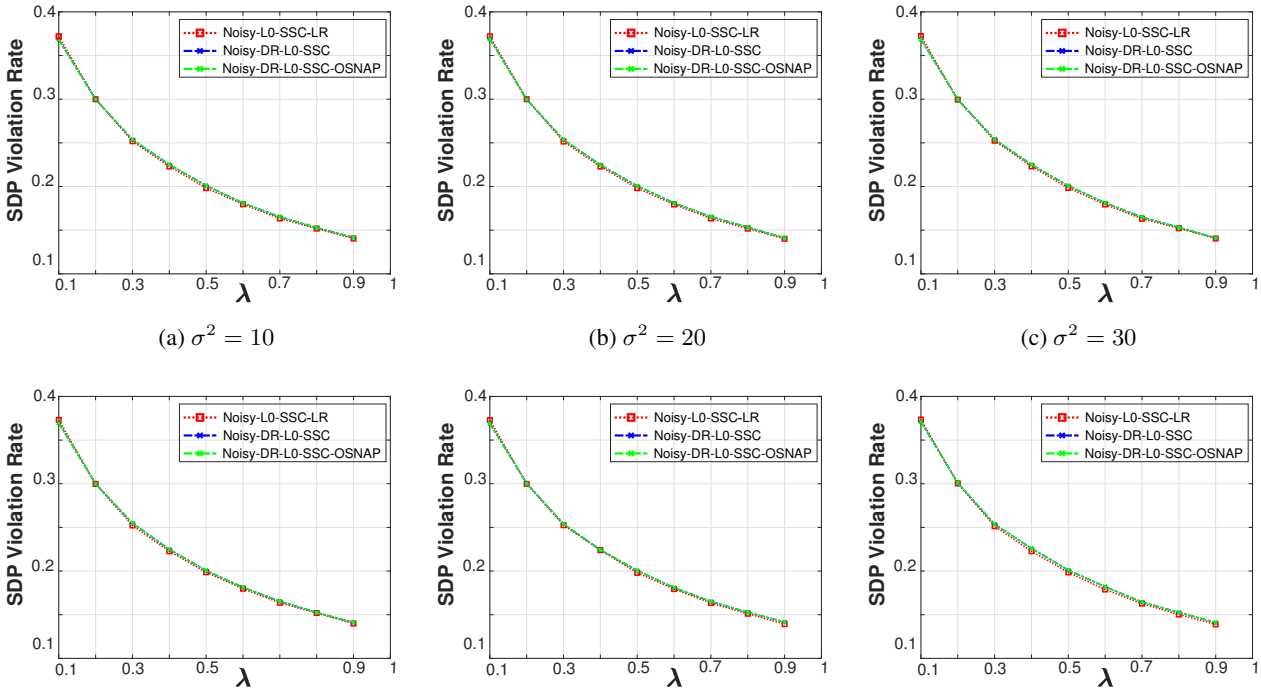


Figure 1: Accuracy (left) and NMI (right) with respect to different values of λ on the Extended Yale-B data set



(a) $\sigma^2 = 10$ (b) $\sigma^2 = 20$ (c) $\sigma^2 = 30$

(d) $\sigma^2 = 40$ (e) $\sigma^2 = 50$ (f) $\sigma^2 = 60$

Figure 2: The SDP violation rate with respect to λ for noisy ℓ^0-SSC, Noisy-DR-ℓ^0-SSC and Noisy-DR-ℓ^0-SSC-OSNAP. The SDP violation rate for Noisy-DR-ℓ^0-SSC and that for Noisy-DR-ℓ^0-SSC-OSNAP are the same, so their curves overlap each other.