UNDERSTANDING OVER-PARAMETERIZATION IN GENERATIVE ADVERSARIAL NETWORKS

Anonymous authors
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ABSTRACT

A broad class of *unsupervised* deep learning methods such as Generative Adversarial Networks (GANs) involve training of over-parameterized models where the parameters of the model exceed the size of the training data set. Indeed, most successful GANs used in practice are trained using over-parameterized generator and discriminator networks, both in terms of depth and width. A large body of work in *supervised* learning have shown the importance of such model over-parameterization in the convergence of the gradient descent (GD) to globally optimal solutions. In contrast, the unsupervised setting and GANs in particular involve non-convex concave mini-max optimization problems that are often trained using Gradient Descent/Ascent (GDA). The role and benefits of model over-parameterization in the convergence of GDA to a global saddle point in non-convex concave problems is far less understood. In this work, we present a comprehensive analysis of the importance of model over-parameterization in GANs both theoretically and empirically. We theoretically show that in an over-parameterized GAN model with a 1-layer neural network generator and a linear discriminator, GDA converges to a global saddle point of the underlying non-convex concave min-max problem. To the best of our knowledge, this is the first result for global convergence of GDA in such settings. Our theory is based on a more general result that holds for a broader class of nonlinear generators and discriminators that obey certain assumptions (including deeper generators and random feature discriminators). Our theory utilizes and builds upon a novel connection with the convergence analysis of linear time-varying dynamical systems which may have broader implications for understanding the convergence behavior of GDA for non-convex concave problems involving over-parameterized models. We also empirically study the role of model over-parameterization in GANs using several large-scale experiments on CIFAR-10 and Celeb-A datasets. Our experiments show that over-parameterization improves the quality of generated samples across various model architectures and datasets. Remarkably, we observe that over-parameterization leads to faster and more stable convergence behavior of GDA across the board.

1 INTRODUCTION

In recent years, we have witnessed tremendous progress in deep generative modeling with some state-of-the-art models capable of generating photo-realistic images of objects and scenes (Brock et al., 2019; Karras et al., 2019; Clark et al., 2019). Three prominent classes of deep generative models include GANs (Goodfellow et al., 2014), VAEs (Kingma & Welling, 2014) and normalizing flows (Dinh et al., 2017). Of these, GANs remain a popular choice for data synthesis especially in the image domain. GANs are based on a two player *min-max* game between a generator network that generates samples from a distribution, and a critic (discriminator) network that discriminates real distribution from the generated one. The networks are optimized using Gradient Descent/Ascent (GDA) to reach a saddle-point of the min-max optimization problem.

One of the key factors that has contributed to the successful training of GANs is model *over-parameterization*. By increasing the complexity of discriminator and generator networks, both in depth and width, recent papers show that GANs can achieve photo-realistic image and video synthesis (Brock et al., 2019; Clark et al., 2019; Karras et al., 2019). While these works empirically

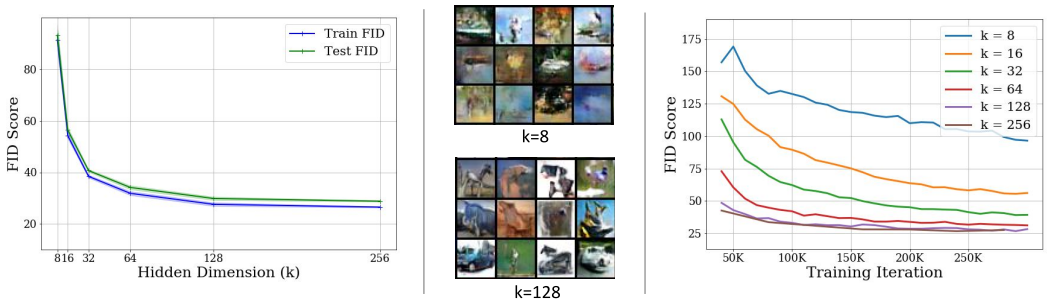


Figure 1: **Over-parameterization in GANs.** We train DCGAN models by varying the size of the hidden dimension k (larger the k, more over-parameterized the models are, see Fig. 6 for details). Over-parameterized GANs enjoy improved training and test FID scores *(the left panel)*, generate high-quality samples *(the middle panel)* and have fast and stable convergence *(the right panel)*.

demonstrate some benefits of over-parameterization, there is lack of a rigorous study explaining this phenomena. In this work, we attempt to provide a comprehensive understanding of the role of over-parameterization in GANs, both theoretically and empirically.

Recently, there has been a large body of work in *supervised* learning (e.g. regression or classification problems) studying the importance of model over-parameterization in gradient descent (GD)'s convergence to globally optimal solutions (Soltanolkotabi et al., 2018; Allen-Zhu et al., 2019; Du et al., 2019; Oymak & Soltanolkotabi, 2019; Zou & Gu, 2019; Oymak et al., 2019). A key observation in these works is that, under some conditions, over-parameterized models experience *lazy training* (Chizat et al., 2019) where optimal model parameters computed by GD remain close to a randomly initialized model. Thus, using a linear approximation of the model in the parameter space, one can show the global convergence of GD in such minimization problems.

In contrast, training GANs often involves solving a non-convex concave *min-max* optimization problem that fundamentally differs from a single minimization problem of classification/regression. The key question is whether over-parameterized GANs also experience lazy training in the sense that over-parameterized generator and discriminator networks remain sufficiently close to their initializations. This may then lead to a general theory of global convergence of GDA for such over-parameterized non-convex concave min-max problems.

In this paper we first theoretically study the role of over-paramterization for a GAN model with a 1-hidden layer generator and a linear discriminator. We study two optimization procedures to solve this problem: (i) using a conventional training procedure in GANs based on GDA in which generator and discriminator networks perform simultaneous steps of gradient descent to optimize their respective models, (ii) using GD to optimize generator's parameters for the optimal discriminator. The latter case corresponds to taking a sufficiently large number of gradient ascent steps to update discriminator's parameters for each GD step of the generator. In both cases, our results show that in an over-parameterized regime, the GAN optimization converges to a global solution. To the best of our knowledge, this is the first result showing the global convergence of GDA in such settings. While in our results we focus on one-hidden layer generators and linear discriminators, our theory is based on analyzing a general class of min-max optimization problems which can be used to study a much broader class of generators and discriminators potentially including deep generators and deep random feature-based discriminators. A key component of our analysis is a novel connection to exponential stability of non-symmetric time varying dynamical systems in control theory which may have broader implications for theoretical analysis of GAN's training.

Having analyzed over-parameterized GANs for relatively simple models, we next provide a comprehensive empirical study of this problem for practical GANs such as DCGAN (Radford et al., 2016) and ResNet GAN (Gulrajani et al., 2017) trained on CIFAR-10 and Celeb-A datasets. For example, the benefit of over-paramterization in training DCGANs on CIFAR-10 is illustrated in Figure 1. We have three key observations: (i) as the model becomes more over-parameterized (e.g. using wider networks), the *training* FID scores that measure the training error, decrease. This phenomenon has been observed in other studied as well (Brock et al., 2019). (ii) Over-parameterization does not

hurt the *test* FID scores (i.e. the generalization gap remains small). This improved test-time performance can also be seen qualitatively in the center panel of Figure 1, where over-parameterized models produce samples of improved quality. (iii) Remarkably, over-parameterized GANs, with a lot of parameters to optimize over, have significantly improved convergence behavior of GDA, both in terms of rate and stability, compared to small GAN models (see the right panel of Figure 1).

In summary, in this paper

- We provide the first theoretical guarantee of simultaneous GDA's global convergence for an over-parametereized GAN with one-hidden neural network generator and a linear discriminator (Theorem 2.1).
- By establishing connections with linear time-varying dynamical systems, we provide a theoretical framework to analyze simultaneous GDA's global convergence for a general over-parametereized GAN (including deeper generators and random feature discriminators), under some general conditions (Theorems 2.3 and A.4).
- We provide a comprehensive empirical study of the role of model over-parameterization in GANs using several large-scale experiments on CIFAR-10 and Celeb-A datasets. We observe over-parameterization improves GANs' training error, generalization error, sample qualities as well as the convergence rate and stability of GDA.

2 Theoretical Results

2.1 Problem Formulation

Given n data points of the form $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in \mathbb{R}^m$, the goal of GAN's training is to find a generator that can mimic sampling from the same distribution as the training data. More specifically, the goal is to find a generator mapping $\mathcal{G}_{\boldsymbol{\theta}}(z) : \mathbb{R}^d \to \mathbb{R}^m$, parameterized by $\boldsymbol{\theta} \in \mathbb{R}^p$, so that $\mathcal{G}_{\boldsymbol{\theta}}(\boldsymbol{z}_1), \mathcal{G}_{\boldsymbol{\theta}}(\boldsymbol{z}_2), \ldots, \mathcal{G}_{\boldsymbol{\theta}}(\boldsymbol{z}_n)$ with $\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n$ generated i.i.d. according to $\mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_d)$ has a similar empirical distribution to $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$[1]. To measure the discrepancy between the data points and the GAN outputs, one typically uses a discriminator mapping $\mathcal{D}_{\widetilde{\theta}} : \mathbb{R}^m \to \mathbb{R}$ parameterized with $\widetilde{\theta} \in \mathbb{R}^{\widetilde{p}}$. The overall training approach takes the form of the following min-max optimization problem which minimizes the worst-case discrepancy detected by the discriminator

$$\min_{\boldsymbol{\theta}} \max_{\widetilde{\boldsymbol{\theta}}} \quad \frac{1}{n}\sum_{i=1}^{n} \mathcal{D}_{\widetilde{\boldsymbol{\theta}}}(\boldsymbol{x}_i) - \frac{1}{n}\sum_{i=1}^{n} \mathcal{D}_{\widetilde{\boldsymbol{\theta}}}(\mathcal{G}_{\boldsymbol{\theta}}(\boldsymbol{z}_i)) + \mathcal{R}(\widetilde{\boldsymbol{\theta}}). \tag{1}$$

Here, $\mathcal{R}(\widetilde{\boldsymbol{\theta}})$ is a regularizer that typically ensures the discriminator is Lipschitz. This formulation mimics the popular Wasserstein GAN (Arjovsky et al., 2017) (or, IPM GAN) formulations. This optimization problem is typically solved by running Gradient Descent Ascent (GDA) on the minmization/maximization variables.

The generator and discriminator mappings $\mathcal{G}$ and $\mathcal{D}$ used in practice are often deep neural networks. Thus, the min-max optimization problem above is highly nonlinear and non-convex concave. Saddle point optimization is a classical and fundamental problem in game theory (Von Neumann & Morgenstern, 1953) and control (Gutman, 1979). However, most of the classical results apply to the convex-concave case (Arrow et al., 1958) while the saddle point optimization of GANs is often *non convex-concave*. If GDA converges to the global (local) saddle points, we say it is globally (locally) stable. For a general min-max optimization, however, GDA can be trapped in a loop or even diverge. Except in some special cases (e.g. (Feizi et al., 2018) for a quadratic GAN formulation or (Lei et al., 2019) for the under-parametrized setup when the generator is a one-layer network), GDA is not globally stable for GANs in general (Nagarajan & Kolter, 2017; Mescheder et al., 2018; Adolphs et al., 2019; Mescheder et al., 2017; Daskalakis et al., 2020).

None of these works, however, study the role of model over-parameterization in the global/local convergence (stability) of GDA. In particular, it has been empirically observed (as we also demonstrate

[1] In general, the number of observed and generated samples can be different. However, in practical GAN implementations, batch sizes of observed and generated samples are usually the same. Thus, for simplicity, we make this assumption in our setup.

in this paper) that when the generator/discriminator contain a large number of parameters (i.e. are sufficiently over-parameterized) GDA does indeed find (near) globally optimal solutions. In this section we wish to demystify this phenomenon from a theoretical perspective.

2.2 Results for one-hidden layer generators and random discriminators

In this section, we discuss our main results on the convergence gradient based algorithms when training GANs in the over-parameterized regime. We focus on the case where the generator takes the form of a single hidden-layer ReLU network with d inputs, k hidden units, and m outputs. Specifically, $\mathcal{G}(\boldsymbol{z}) = \boldsymbol{V} \cdot \text{ReLU}(\boldsymbol{W}\boldsymbol{z})$ with $\boldsymbol{W} \in \mathbb{R}^{k\times d}$ and $\boldsymbol{V} \in \mathbb{R}^{m\times k}$ denoting the input-to-hidden and hidden-to-output weights. We also consider a linear discriminator of the form $\mathcal{D}(\boldsymbol{x}) = \boldsymbol{d}^T\boldsymbol{x}$ with an ℓ_2 regularizer on the weights i.e. $\mathcal{R}(\boldsymbol{d}) = -\|\boldsymbol{d}\|_{\ell_2}^2/2$. The overall min-max optimization problem (equation 1) takes the form

$$\min_{\boldsymbol{W}\in\mathbb{R}^{k\times d}} \max_{\boldsymbol{d}\in\mathbb{R}^m} \quad \mathcal{L}(\boldsymbol{d},\boldsymbol{W}) := \langle \boldsymbol{d}, \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{x}_i - \boldsymbol{V}\text{ReLU}(\boldsymbol{W}\boldsymbol{z}_i))\rangle - \frac{\|\boldsymbol{d}\|_{\ell_2}^2}{2}. \tag{2}$$

The common approach to solve the above optimization problem is to run a Gradient Descent Ascent (GDA) algorithm. At iteration t, GDA takes the form

$$\begin{cases} \boldsymbol{d}_{t+1} = & \boldsymbol{d}_t + \mu\nabla_{\boldsymbol{d}}\mathcal{L}(\boldsymbol{W}_t,\boldsymbol{d}_t) \\ \boldsymbol{W}_{t+1} = & \boldsymbol{W}_t - \eta\nabla_{\boldsymbol{W}}\mathcal{L}(\boldsymbol{W}_t,\boldsymbol{d}_t) \end{cases} \tag{3}$$

Next, we establish the global convergence of GDA for such an over-parameterized model:

Theorem 2.1 *Let $\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n \in \mathbb{R}^m$ be n training data with their mean defined as $\bar{\boldsymbol{x}} := \frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i$. Consider the GAN model with a linear discriminator of the form $\mathcal{D}(\boldsymbol{x}) = \boldsymbol{d}^T\boldsymbol{x}$ parameterized by $\boldsymbol{d}\in\mathbb{R}^m$ and a one hidden layer neural network generator of the form $\mathcal{G}(\boldsymbol{z}) = \boldsymbol{V}\phi(\boldsymbol{W}z)$ parameterized by $\boldsymbol{W} \in \mathbb{R}^{k\times d}$ with $\boldsymbol{V}\in\mathbb{R}^{m\times k}$ a fixed matrix generated at random with i.i.d. $\mathcal{N}(0,\sigma_v^2)$ entries. Also assume the input data to the generator $\{\boldsymbol{z}_i\}_{i=1}^n$ are generated i.i.d. according to $\sim\mathcal{N}(\boldsymbol{0},\sigma_z^2\boldsymbol{I}_d)$. Furthermore, assume the generator weights at initialization $\boldsymbol{W}_0\in\mathbb{R}^{k\times d}$ are generated i.i.d. according to $\mathcal{N}(0,\sigma_w^2)$. Furthermore, assume the standard deviations above obey $\sigma_v\sigma_w\sigma_z \geq \|\bar{\boldsymbol{x}}\|_{\ell_2}/(md^5\log d^{\frac{3}{2}})$. Then, as long as*

$$k \geq C\cdot md^9\log(d)^3$$

with C a fixed constant, running GDA updates per equation 3 starting from the random $\boldsymbol{W}_0$ above and $\boldsymbol{d}_0 = \boldsymbol{0}$[2] with step-sizes obeying $0<\mu<1$ and $\eta = \bar{\eta}\frac{2\mu}{225k\cdot d\cdot\sigma_v^2\cdot\sigma_z^2}$, with $\bar{\eta}\leq 1$, satisfies

$$\left\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{V}\mathit{ReLU}(\boldsymbol{W}_\tau\boldsymbol{z}_i) - \bar{\boldsymbol{x}}\right\|_{\ell_2} \leq 5\left(1 - 5\times10^{-6}\cdot\frac{\bar{\eta}\mu}{d}\right)^\tau \left\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{V}\mathit{ReLU}(\boldsymbol{W}_0\boldsymbol{z}_i) - \bar{\boldsymbol{x}}\right\|_{\ell_2}. \tag{4}$$

This holds with probability at least $1-(2n+4)e^{-\frac{m}{1500}} - 5k\cdot e^{-c_1\cdot n} - (2k+2)e^{-\frac{d}{162}} - ne^{-c_2\cdot md^7\log(d)^2}$ where c_1, c_2 are fixed numerical constants.

To better understand the implications of the above theorem, note that the objective of equation 2 can be simplified by solving the inner maximization in a closed form so that the min-max problem in equation 2 is equivalent to the following single minimization problem:

$$\min_{\boldsymbol{W}} \mathcal{L}(\boldsymbol{W}) := \frac{1}{2}\left\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{V}\text{ReLU}(\boldsymbol{W}_0\boldsymbol{z}_i) - \bar{\boldsymbol{x}}\right\|_{\ell_2}^2, \tag{5}$$

which has a global optimum of zero. As a result equation 4 in Theorem 2.1 guarantees that running simultaneous GDA updates achieves the global optimum. This holds as long as the generator

[2]The zero initialization of $\boldsymbol{d}$ is merely done for simplicity. A similar result can be derived for an arbitrary initialization of the discriminator's parameters with minor modifications. See Theorem 2.3 for such a result.

network is sufficiently over-parameterized in the sense that the number of hidden nodes is polynomially large in its output dimension m and input dimension d. Interestingly, the rate of convergence guaranteed by this result is geometric, guaranteeing fast GDA convergence to the global optima. To the extent of our knowledge, this is the first result that establishes the global convergence of simultaneous GDA for an over-parameterized GAN model.

We acknowledge that the considered GAN formulation of equation 2 is very simpler than GANs used in practice. Specially, since the discriminator is linear, this GAN can be viewed as a moment-matching GAN (Li et al., 2017) pushing first moments of input and generative distributions towards each other. Alternatively, this GAN formulation can be viewed as one instance of the Sliced Wasserstein GAN (Deshpande et al., 2018). Although the maximization on discriminator's parameters is concave, the minimization over the generator's parameters is still non-convex due to the use of a neural-net generator. Thus, the overall optimization problem is a non-trivial non-convex concave min-max problem. From that perspective, our result in Theorem 2.1 *partially* explains the role of model over-parameterization in GDA's convergence for GANs.

Given the closed form equation 5, one may wonder what would happen if we run gradient descent on this minimization objective directly. That is running gradient descent updates of the form $\boldsymbol{W}_{\tau+1} = \boldsymbol{W}_\tau - \eta \nabla \mathcal{L}(\boldsymbol{W}_\tau)$ with $\mathcal{L}(\boldsymbol{W})$ given by equation 5. This is equivalent to GDA but instead of running one gradient ascent iteration for the maximization iteration we run infinitely many. Interestingly, in some successful GAN implementations (Gulrajani et al., 2017), often more updates on the discriminator's parameters are run per generator's updates. This is the subject of the next result.

Theorem 2.2 *Consider the setup of Theorem 2.1. Then as long as*

$$k \geq C \cdot md^4 \log(d)^3$$

with C a fixed numerical constant, running GD updates of the form $\boldsymbol{W}_{\tau+1} = \boldsymbol{W}_\tau - \eta \nabla \mathcal{L}(\boldsymbol{W}_\tau)$ on the loss given in equation 5 with step-size $\eta = \frac{16\bar{\eta}}{675k \cdot d \cdot \sigma_v^2 \cdot \sigma_z^2}$, with $\bar{\eta} \leq 1$, satisfies

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{V} ReLU\left(\boldsymbol{W}_\tau \boldsymbol{z}_i\right) - \bar{\boldsymbol{x}} \right\|_{\ell_2} \leq 5 \left(1 - 5 \times 10^{-6} \cdot \frac{\bar{\eta}\mu}{d}\right)^{\tau} \left\| \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{V} ReLU\left(\boldsymbol{W}_0 \boldsymbol{z}_i\right) - \bar{\boldsymbol{x}} \right\|_{\ell_2}. \tag{6}$$

This holds with probability at least $1 - (2n+4)\, e^{-\frac{m}{1500}} - 5k \cdot e^{-c_1 \cdot n} - (2k+2)\, e^{-\frac{d}{162}} - ne^{-c_2 \cdot md^3 \log(d)^2}$ with c_1, c_2 fixed numerical constants.

This theorem states that if we solve the max part of equation 2 in closed form and run GD on the loss function per equation 5 with enough over-parametrization, the loss will decrease at a geometric rate to zero. This result holds again when the model is sufficiently over-parameterized. However, it improves upon the amount of over-parametrization required in Theorem 2.1 by a factor of d^5.

2.3 Can the analysis be extended to general GANs?

In the previous section, we focused on the implications of our results for one-hidden layer generator and linear discriminator. However, as it will become clear in the proofs, our theoretical results are based on analyzing the convergence behavior of GDA on a general min-max problem of the form

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} \max_{\boldsymbol{d} \in \mathbb{R}^m} h(\boldsymbol{\theta}, \boldsymbol{d}) := \langle \boldsymbol{d}, f(\boldsymbol{\theta}) - \boldsymbol{y} \rangle - \frac{\|\boldsymbol{d}\|_{\ell_2}^2}{2}, \tag{7}$$

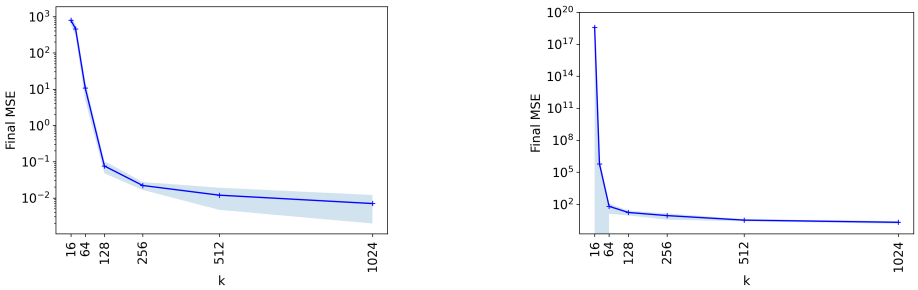
where $f : \mathbb{R}^p \to \mathbb{R}^m$ denoting a general nonlinear mapping.

Theorem 2.3 (Informal version of Theorem A.4) *Consider a general nonlinear mapping $f : \mathbb{R}^p \to \mathbb{R}^m$ with the singular values of its Jacobian mapping around initialization obeying certain assumptions (most notably $\sigma_{\min}(\mathcal{J}(\boldsymbol{\theta}_0)) \geq \alpha$). Then, running GDA iterations of the form*

$$\begin{cases} \boldsymbol{d}_{t+1} = (1-\mu)\, \boldsymbol{d}_t + \mu \nabla_{\boldsymbol{d}} h(\boldsymbol{d}_t, \boldsymbol{\theta}_t) \\ \boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \nabla_{\boldsymbol{\theta}} h(\boldsymbol{d}_t, \boldsymbol{\theta}_t) \end{cases} \tag{8}$$

with sufficiently small step sizes η and μ obeys

$$\|f(\boldsymbol{\theta}_t) - \boldsymbol{y}\|_{\ell_2} \leq \left(1 - \frac{\eta\alpha^2}{2}\right)^t \sqrt{\|f(\boldsymbol{\theta}_0) - \boldsymbol{y}\|_{\ell_2}^2 + \|\boldsymbol{d}_0\|_{\ell_2}^2}.$$



(a) Discriminator trained to optimality

(b) Gradient Descent Ascent

Figure 2: **Convergence plot** a GAN model with linear discriminator and 1-hidden layer generator as the hidden dimension (k) increases. Over-paramterizated models show improved convergence

Note that similar to the previous sections one can solve the maximization problem in equation 7 in closed form so that equation 7 is equivalent to the following minimization problem

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \mathcal{L}(\boldsymbol{\theta}) := \frac{1}{2}\left\|f(\boldsymbol{\theta}) - \boldsymbol{y}\right\|_{\ell_2}^2, \tag{9}$$

with global optima equal to zero. Theorem 2.3 ensures that GDA converges with a fast geometric rate to this global optima. This holds as soon as the model $f(\boldsymbol{\theta})$ is sufficiently over-parameterized which is quantitatively captured via the minimum singular value assumption on the Jacobian at initialization ($\sigma_{\min}(\mathcal{J}(\boldsymbol{\theta}_0)) \geq \alpha$ which can only hold when $m \leq p$). This general result can thus be used to provide theoretical guarantees for a much more general class of generators and discriminators. To be more specific, consider a deep GAN model where the generator $\mathcal{G}_{\boldsymbol{\theta}}$ is a deep neural network with parameters $\boldsymbol{\theta}$ and the discriminator is a deep random feature model of the form $\mathcal{D}_{\boldsymbol{d}}(\boldsymbol{x}) = \boldsymbol{d}^T\psi(\boldsymbol{x})$ parameterized with d and $\psi : \mathbb{R}^d \to \mathbb{R}^m$ a deep neural network with random weights. Then the min-max training optimization problem equation 1 with a regularizer $\mathcal{R}(\boldsymbol{d}) = -\left\|\boldsymbol{d}\right\|_{\ell_2}^2/2$ is a special instance of equation 7 with

$$f(\boldsymbol{\theta}) := \frac{1}{n}\sum_{i=1}^{n}\psi(\mathcal{G}_{\boldsymbol{\theta}}(\boldsymbol{z}_i)) \quad \text{and} \quad \boldsymbol{y} := \frac{1}{n}\sum_{i=1}^{n}\psi(\boldsymbol{x}_i)$$

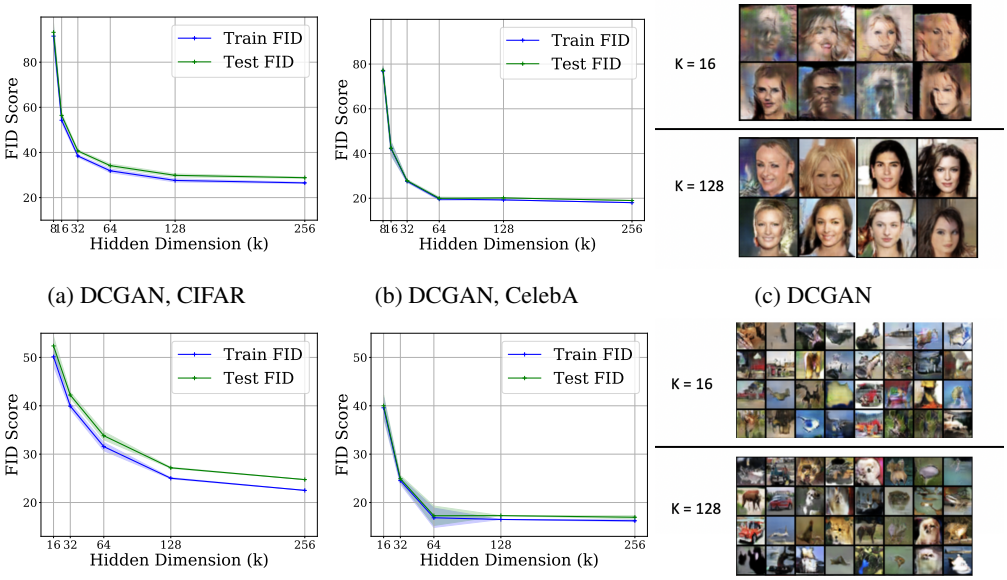
Therefore, the above result can in principle be used to rigorously analyze global convergence of GDA for an over-parameterized GAN problem with a deep generator and a deep random feature discriminator model. However, characterizing the precise amount of over-parameterization required for such a result to hold requires a precise analysis of the minimum singular value of the Jacobian of $f(\boldsymbol{\theta})$ at initialization as well as other singular value related conditions stated in Theorem A.4. We defer such a precise analysis to future works.

Numerical Validations: Next, we numerically study the convergence of GAN model considered in Theorems 2.1 and 2.2 where the discriminator is a linear network while the generator is a one hidden layer neural net. In our experiments, we generate $\boldsymbol{x}_i$'s from an m-dimension Gaussian distribution with mean μ and an identity covariance matrix. The mean vector μ is randomly generated. We train two variants of GAN models using (1) GDA (as considered in Thm 2.1) and (2) GD on generator while solving the discriminator to optimality (as considered in Thm 2.2).

In Fig. 2, we plot the converged loss values of GAN models trained using both techniques (1) and (2) as the hidden dimension k of the generator is varied. The MSE loss between the true data mean and the data mean of generated samples is used as our evaluation metric. We observe that over-parameterized GAN models show improved convergence behavior than the narrower models. Additionally, the MSE loss converges to 0 for larger values of k which shows that with sufficient over-parameterization, GDA converges to a global saddle point.

3 Experiments

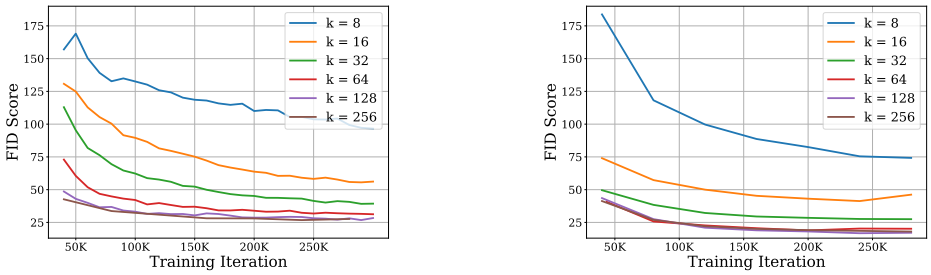
In this section, we demonstrate benefits of over-parameterization in large GAN models. In particular, we train GANs on two benchmark datasets: CIFAR-10 (32×32 resolution) and Celeb-A (64×64



(a) DCGAN, CIFAR (b) DCGAN, CelebA (c) DCGAN

(d) Resnet, CIFAR (e) Resnet, CelebA (f) Resnet

Figure 3: **Over-paramterization Results:** We plot the FID scores (lower, better) of DCGAN and Resnet DCGAN as the hidden dimension k is varied. Results on CIFAR-10 and Celeb-A are shown on the plots on the left and right panels, respectively. Over-parameterization gives better FID scores.



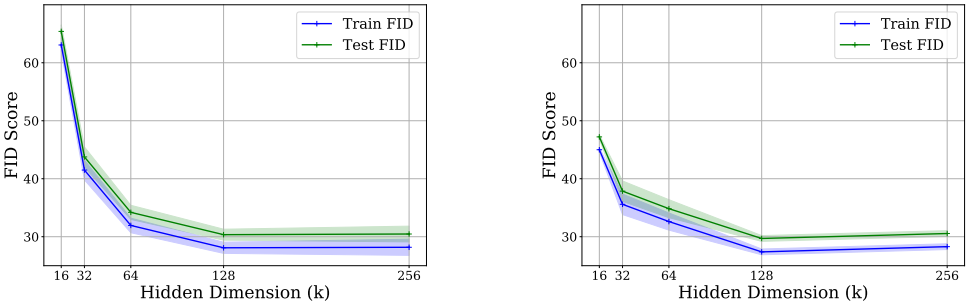
(a) CIFAR (b) CelebA

Figure 4: **DCGAN Training Results:** We plot the FID scores across training iterations of DCGAN on CIFAR-10 and Celeb-A for different values of hidden dimension k. Remarkably, we observe that over-parameterization improves the rate of convergence of GDA and its stability in training.

resolution). We use two commonly used GAN architectures: DCGAN and Resnet-based GAN. For both of these architectures, we train several models, each with a different number of filters in each layer, denoted by k. For simplicity, we refer to k as the hidden dimension. Appendix Fig. 6 illustrates the architectures used in our experiments. Networks with large k are more over-parameterized.

We use the same value of k for both generator and discriminator networks. This is in line with the design choice made in most recent GAN models (Radford et al., 2016; Brock et al., 2019), where the size of generator and discriminator models are roughly maintained the same. We train each model till convergence and evaluate the performance of converged models using FID scores. FID scores measure the Frechet distance between feature distributions of real and generated data distributions (Heusel et al., 2017). A small FID score indicates high-quality synthesized samples. Each experiment is conducted for 5 runs, and mean and the variance of FID scores are reported.

Over-parameterization yields better generative models: In Fig. 3, we show the plot of FID scores as the hidden dimension (k) is varied for DCGAN and Resnet GAN models. We observe a clear trend



(a) Discriminator overparameterization

(b) Generator overparameterization

Figure 5: **Single Component Over-paramterization Results:** We plot FID scores of Resnet GAN as the hidden dimension of one of the components are varied, while the hidden dimension of other component is held fixed. Even in this case, over-parameterization improves model convergence.

where the FID scores are high (i.e. poor) for small values of k, while they improve as models become more over-parameterized. Also, the FID scores saturate beyond $k = 64$ for DCGAN models, and $k = 128$ for Resnet GAN models. Interestingly, these are the standard values used in the existing model architecures (Radford et al., 2016; Gulrajani et al., 2017). This experiment clearly illustrates the importance of over-parameterization in training large-scale generative models.

Over-parameterization leads to improved convergence of GDA: In Fig. 4, we show the plot of FID scores over training iterations for different values for k. We observe that models with larger values of k converge faster and demonstrate a more stable behavior. This agrees with our theoretical results that over-parameterized models have a fast rate of convergence.

Generalization gap in GANs: The evaluation metric we use – FID scores computes the Frechet distance between feature maps of real data and the synthesized GAN samples. To study the generalization gap, we compute the FID scores by using (1) the training-set of real data, which we call *FID train*, and (2) a held-out validation set of real data, which we call *FID test*. In Fig. 3, a plot of FID train *(in blue)* and FID test *(in green)* are shown as the hidden dimension k is varied. We observe that FID test values are consistently higher than the the FID train values. However, the generalization gap does not increase with increasing over-parameterization.

Effect of single component over-parameterization: In the previous results, we discussed the setting where the size of generator and discriminator are held roughly the same (both discriminator and generator uses the same value of k). In this part, we analyze single-component over-parameterization where we study the effect of over-parameterization when one of the components (generator / discriminator) has varying k, while the other component uses the standard value of k (64 for DCGAN and 128 for Resnet GAN). The FID variation of single-component over-parameterization are shown in Fig. 5. We observe similar trends as the previous case where increasing over-parameterization leads to improved FID scores. Interestingly, increasing the value of k beyond the default value used in the other component leads to a slight drop in performance. Hence, choosing comparable sizes of discriminator and generator models is recommended.

4 Conclusion

In this paper, we perform a systematic study of the importance of over-parameterization in training GANs. We first analyze a GAN model with one-hidden layer generator and a linear discriminator optimized using Gradient Descent Ascent (GDA). Under this setup, we prove that with sufficient over-parameterization, GDA converges to a global saddle point. Additionally, our result demonstrate that over-parameterized models have a fast rate of convergence. We then validate our theoretical findings through extensive experiments on DCGAN and Resnet models trained on CIFAR-10 and Celeb-A datasets. We observe over-parameterized models to perform well both in terms of the rate of convergence and the quality of generated samples.

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Appendix

A PROOFS

In this section, we prove Theorems 2.1 and 2.2. First, we provide some notations we use throughout the remainder of the paper in Section A.1. Before proving these specialized results for one hidden layer generators and linear discriminators (Theorems 2.1 and 2.2), we state and prove a more general result (formal version of Theorem 2.3) on the convergence of GDA on a general class of min-max problems in Section A.2. Then we state a few preliminary calculations in Section A.3. Next, we state some key lemmas in Section A.4 and defer their proofs to Appendix B. Finally, we prove Theorems 2.1 and 2.2 in Sections A.5 and A.6, respectively.

A.1 NOTATION

We will use C, c, c_1, etc. to denote positive absolute constants, whose value may change throughout the paper and from line to line. We use $\phi(z) = \text{ReLU}(z) = \max(0, z)$ and its (generalized) derivative $\phi'(z) = \mathbf{1}_{\{z \geq 0\}}$ with $\mathbf{1}$ being the indicator function. $\sigma_{min}(\boldsymbol{X})$ and $\sigma_{max}(\boldsymbol{X}) = \|\boldsymbol{X}\|$ denote the minimum and maximum singular values of matrix $\boldsymbol{X}$. For two matrices $\boldsymbol{A} \in \mathbb{R}^{m \times n_1}$ and $\boldsymbol{B} \in \mathbb{R}^{m \times n_2}$ we define their Khatri-Rao product as the matrix $\boldsymbol{C} = \boldsymbol{A} * \boldsymbol{B}$ with rows $\boldsymbol{C}_i = \boldsymbol{A}_i \otimes \boldsymbol{B}_i$, where $\otimes$ denotes the Kronecker product. The spectral radius of a matrix $\boldsymbol{A} \in \mathbb{C}^{n \times n}$ is defined as $\rho(\boldsymbol{A}) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$, where λ_i's are the eigenvalues of $\boldsymbol{A}$. Throughout the proof we shall assume $\phi := ReLU$ to avoid unnecessarily long expressions.

A.2 ANALYSIS OF GDA: A CONTROL THEORY PERSPECTIVE

In this section we will focus on solving a general min-max optimization problem of the form

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^n} \max_{\boldsymbol{d} \in \mathbb{R}^m} h(\boldsymbol{\theta}, \boldsymbol{d}) := \langle \boldsymbol{d}, f(\boldsymbol{\theta}) - \boldsymbol{y} \rangle - \frac{\|\boldsymbol{d}\|_{\ell_2}^2}{2}, \tag{10}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a general nonlinear mapping. In particular, we focus on analyzing the convergence behavior of Gradient Descent/Ascent (GDA) on the above loss, starting from initial estimates $\boldsymbol{\theta}_0$ and $\boldsymbol{d}_0$. In this case the GDA updates take the following form

$$\begin{cases} \boldsymbol{d}_{t+1} = (1 - \mu)\, \boldsymbol{d}_t + \mu\, (f(\boldsymbol{\theta}_t) - \boldsymbol{y}) \\ \boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t - \eta \mathcal{J}^T(\boldsymbol{\theta}_t)\, \boldsymbol{d}_t \end{cases}. \tag{11}$$

We note that solving the inner maximization problem in equation 10 would yield

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^n} \frac{1}{2} \|f(\boldsymbol{\theta}) - \boldsymbol{y}\|_{\ell_2}^2. \tag{12}$$

In this section, our goal is to show that the when running the GDA updates equation 11, the norm of the residual vector defined as $\boldsymbol{r}_t := f(\boldsymbol{\theta}_t) - \boldsymbol{y}$ goes to zero and hence we reach a global optimum of equation 12 (and in turn equation 10).

Our proof will build on ideas from the control theory and dynamical systems literature. For that, we are first going to rewrite the equations 11 in a more convenient way. We define the average Jacobian along the path connecting two points $\boldsymbol{x}$,$\boldsymbol{y} \in \mathbb{R}^n$ as

$$\mathcal{J}(\boldsymbol{y}, \boldsymbol{x}) = \int_0^1 \mathcal{J}(\boldsymbol{x} + \alpha(\boldsymbol{y} - \boldsymbol{x}))\, d\alpha,$$

where $\mathcal{J}(\theta) \in \mathbb{R}^{m \times n}$ is the Jacobian associated with the nonlinear mapping f. Next, from the fundamental theorem of calculus it follows that

$$\begin{aligned} \boldsymbol{r}_{t+1} = f(\boldsymbol{\theta}_{t+1}) - \boldsymbol{y} &= f\left(\boldsymbol{\theta}_t - \eta \mathcal{J}_t^T \boldsymbol{d}_t\right) - \boldsymbol{y} \\ &= f(\boldsymbol{\theta}_t) - \eta \mathcal{J}_{t+1,t} \mathcal{J}_t^T \boldsymbol{d}_t - \boldsymbol{y} \\ &= \boldsymbol{r}_t - \eta \mathcal{J}_{t+1,t} \mathcal{J}_t^T \boldsymbol{d}_t, \end{aligned} \tag{13}$$

where we used the shorthands $\mathcal{J}_t := \mathcal{J}(\boldsymbol{\theta}_t)$ and $\mathcal{J}_{t+1,t} := \mathcal{J}(\boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_t)$ for exposition purposes. Next, we combine the updates $\boldsymbol{r}_t$ and $\boldsymbol{d}_t$ into a state vector of the form $\boldsymbol{z}_t := \begin{bmatrix} \boldsymbol{r}_t \\ \boldsymbol{d}_t \end{bmatrix} \in \mathbb{R}^{2m}$. Using this notation the relationship between the state vectors from one iteration to the next takes the form

$$\boldsymbol{z}_{t+1} = \underbrace{\begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_{t+1,t} \mathcal{J}_t^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix}}_{=:\boldsymbol{A}_t} \boldsymbol{z}_t, \quad t \geq 0, \tag{14}$$

which resembles a time-varying linear dynamical system with transition matrix $\boldsymbol{A}_t$. Now note that to show convergences of $\boldsymbol{r}_t$ to zero it suffices to show convergence of $\boldsymbol{z}_t$ to zero. To do this we utilize the following notion of uniform exponential stability, which will be crucial in analyzing the solutions of equation 14. (See Rugh (1996) for a comprehensive overview on stability notions in discrete state equations.)

Definition 1 *A linear state equation of the form $\boldsymbol{z}_{t+1} = \boldsymbol{A}_t \boldsymbol{z}_t$ is called uniformly exponentially stable if for every $t \geq 0$ we have $\|\boldsymbol{z}_t\|_{\ell_2} \leq \gamma \lambda^t \|\boldsymbol{z}_0\|_{\ell_2}$, where $\gamma \geq 1$ is a finite constant and $0 \leq \lambda < 1$.*

Using the above definition to show the convergence of the state vector $\boldsymbol{z}_t$ to zero at a geometric rate it suffices to show the state equation 14 is exponentially stable.[3] For that, we are first going to analyze a state equation which results from linearizing the nonlinear function $f(\boldsymbol{\theta})$ around the initialization $\boldsymbol{\theta}_0$. In the next step, we are going to show that the behavior of these two problems are similar, provided we stay close to initialization (which we are also going to prove). Specifically, we consider the linearized problem

$$\min_{\widetilde{\boldsymbol{\theta}} \in \mathbb{R}^n} \max_{\widetilde{\boldsymbol{d}} \in \mathbb{R}^m} h_{\text{lin}}\left(\widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{d}}\right) := \left\langle \widetilde{\boldsymbol{d}}, f(\boldsymbol{\theta}_0) + \mathcal{J}_0\left(\widetilde{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) - \boldsymbol{y} \right\rangle - \frac{\left\|\widetilde{\boldsymbol{d}}\right\|_{\ell_2}^2}{2}. \tag{15}$$

We first analyze GDA on this linearized problem starting from the same initialization as the original problem, i.e. $\widetilde{\boldsymbol{\theta}}_0 = \boldsymbol{\theta}_0$ and $\widetilde{\boldsymbol{d}}_0 = \boldsymbol{d}_0$. The gradient descent update for $\widetilde{\boldsymbol{\theta}}_t$ takes the form

$$\widetilde{\boldsymbol{\theta}}_{t+1} = \widetilde{\boldsymbol{\theta}}_t - \eta \mathcal{J}_0^T \widetilde{\boldsymbol{d}}_t \tag{16}$$

and the gradient ascent update for $\widetilde{\boldsymbol{d}}_t$ takes the form

$$\begin{aligned} \widetilde{\boldsymbol{d}}_{t+1} &= \widetilde{\boldsymbol{d}}_t + \mu\left(f(\boldsymbol{\theta}_0) + \mathcal{J}_0\left(\widetilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_0\right) - \boldsymbol{y} - \widetilde{\boldsymbol{d}}_t\right) \\ &= (1-\mu)\widetilde{\boldsymbol{d}}_t + \mu \widetilde{\boldsymbol{r}}_t, \end{aligned} \tag{17}$$

where we used the linear residual defined as $\widetilde{\boldsymbol{r}}_t = f(\boldsymbol{\theta}_0) + \mathcal{J}_0\left(\widetilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_0\right)$. Moreover, the residual from one iterate to the next can be written as follows

$$\begin{aligned} \widetilde{\boldsymbol{r}}_{t+1} &= f(\boldsymbol{\theta}_0) + \mathcal{J}_0\left(\widetilde{\boldsymbol{\theta}}_{t+1} - \boldsymbol{\theta}_0\right) \\ &= f(\boldsymbol{\theta}_0) + \mathcal{J}_0\left(\widetilde{\boldsymbol{\theta}}_t - \eta \mathcal{J}_0^T \widetilde{\boldsymbol{d}}_t - \boldsymbol{\theta}_0\right) \\ &= \widetilde{\boldsymbol{r}}_t - \eta \mathcal{J}_0 \mathcal{J}_0^T \widetilde{\boldsymbol{d}}_t. \end{aligned} \tag{18}$$

Again, we define a new vector $\widetilde{\boldsymbol{z}}_t = \begin{bmatrix} \widetilde{\boldsymbol{r}}_t \\ \widetilde{\boldsymbol{d}}_t \end{bmatrix} \in \mathbb{R}^{2m}$ and by putting together equations 17 and equation 18 we arrive at

$$\widetilde{\boldsymbol{z}}_{t+1} = \begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_0 \mathcal{J}_0^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix} \widetilde{\boldsymbol{z}}_t = \boldsymbol{A} \widetilde{\boldsymbol{z}}_t, \quad t \geq 0, \tag{19}$$

[3] We note that technically the dynamical system equation 14 is not linear. However, we still use exponential stability with some abuse of notation to refer to the property that $\|\boldsymbol{z}_t\|_{\ell_2} \leq \gamma \lambda^t \|\boldsymbol{z}_0\|_{\ell_2}$ holds. As we will see in the forth-coming paragraphs, our formal analysis is via a novel perturbation analysis of a linear dynamical system and therefore keeping this terminology is justified.

which is of the form of a linear time-invariant state equation. As a first step in our proof, we are going to show that the linearized state equations are uniformly exponentially stable. First, recall the following well-known lemma, which characterizes uniform exponential stability in terms of the eigenvalues of $\boldsymbol{A}$.

Lemma A.1 *(Rugh, 1996, Theorem 22.11) A linear state equation of the form $\widetilde{\boldsymbol{z}}_{t+1} = \boldsymbol{A}\widetilde{\boldsymbol{z}}_t$ with $\boldsymbol{A}$ a fixed matrix is uniformly exponentially stable if and only if all eigenvalues of $\boldsymbol{A}$ have magnitudes strictly less than one, i.e. $\rho(\boldsymbol{A}) < 1$. In this case, it holds for all $t \geq 0$ and all $\boldsymbol{z}$ that*

$$\|\boldsymbol{A}^t \boldsymbol{z}\| \leq \gamma \rho(\boldsymbol{A})^t \|\boldsymbol{z}\|,$$

where $\gamma \geq 1$ is an absolute constant, which only depends on $\boldsymbol{A}$.

In the next lemma, we prove that under suitable assumptions on $\mathcal{J}_0$ and the step sizes μ and η the state equations 19 indeed fulfill this condition.

Lemma A.2 *Assume that $\alpha \leq \sigma_{min}(\mathcal{J}_0) \leq \sigma_{max}(\mathcal{J}_0) \leq \beta$ and consider the matrix $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_0 \mathcal{J}_0^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix}$. Assume that $\frac{\mu}{\eta} \geq 4\beta^2$ and $\mu \leq 2$. Then it holds that $\rho(\boldsymbol{A}) \leq 1 - \eta\alpha^2$.*

Proof Suppose that λ is an eigenvalue of $\boldsymbol{A}$. Hence, there is an eigenvector $[\boldsymbol{x}, \boldsymbol{y}]^T \neq \boldsymbol{0}$ such that

$$\begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_0 \mathcal{J}_0^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}$$

holds. By a direct calculation we observe that this yields the equation

$$\eta \mathcal{J}_0 \mathcal{J}_0^T \boldsymbol{x} = \left(\frac{-(1-\lambda)^2}{\mu} + (1-\lambda) \right) \boldsymbol{x}.$$

In particular, $\boldsymbol{x}$ must be an eigenvector of $\mathcal{J}_0 \mathcal{J}_0^T$. Denoting the corresponding eigenvalue with s, we obtain the identity

$$\frac{(1-\lambda)^2}{\mu} - (1-\lambda) + \eta s = 0.$$

Hence, we must have

$$\lambda \in \left\{ 1 - \frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} - \mu\eta s};\ 1 - \frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} - \mu\eta s} \right\}.$$

Note that the square root is indeed well-defined, since

$$\frac{\mu^2}{4} - \mu\eta s \geq \mu\eta\beta^2 - \mu\eta s \geq 0,$$

where in the first inequality we used the assumption $\frac{\mu}{\eta} \geq 4\beta^2$ and in the second line we used that $s \leq \beta^2$, which is a consequence of our assumption on the singular values of $\mathcal{J}_0$. Hence, it follows by the reverse triangle inequality that

$$|\lambda| - \left(1 - \frac{\mu}{2}\right) \leq \left|\lambda - \left(1 - \frac{\mu}{2}\right)\right| = \sqrt{\left(\frac{\mu}{2}\right)^2 - \mu\eta s} \leq \frac{\mu}{2} - \sqrt{\mu\eta s} < \frac{\mu}{2} - \eta s \leq \frac{\mu}{2} - \eta\alpha^2,$$

where the third inequality holds because of $\mu \geq 4\eta\beta^2 \geq 4\eta s$. In the last inequality we used the fact that $\alpha^2 \leq s$, which is a consequence of our assumption on the singular values of $\mathcal{J}_0$. By rearranging terms, we obtain that $|\lambda| < 1 - \eta\alpha^2$. Since λ was an arbitrary eigenvalue of $\boldsymbol{A}$, the result follows. ■

Since the last lemma shows that under suitable conditions it holds that $\rho(\boldsymbol{A}) < 1$, Lemma A.3 yields uniform exponential stability of our state equations. However, this will not be sufficient for our purposes. The reason is that Lemma A.3 does not specify the constant γ and that in order to deal with the time-varying dynamical system we will need a precise estimate. The next lemma shows that for the state equations 19 we have, under suitable assumptions, $\gamma \leq 5$.

Lemma A.3 *Consider the linear, time invariant system of equations*

$$\widetilde{\boldsymbol{z}}_{t+1} = \begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_0 \mathcal{J}_0^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix} \widetilde{\boldsymbol{z}}_t = \boldsymbol{A}\widetilde{\boldsymbol{z}}_t, \quad t \geq 0.$$

Furthermore, assume that $\alpha \leq \sigma_{min}(\mathcal{J}_0) \leq \sigma_{max}(\mathcal{J}_0) \leq \beta$ and suppose that the condition $\frac{\mu}{\eta} \geq 8\beta^2$ as well as $\mu \leq 2$ is satisfied. Then there is a constant $\gamma \leq 5$ such that for all $t \geq 0$ it holds that

$$\|\widetilde{\boldsymbol{z}}_t\|_{\ell_2} \leq \gamma \left(1 - \eta\alpha^2\right)^t \|\widetilde{\boldsymbol{z}}_0\|_{\ell_2}.$$

Proof Denote the SVD decomposition of $\mathcal{J}_0$ by $\boldsymbol{W}\boldsymbol{\Sigma}\boldsymbol{V}^T$ and note that

$$\begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_0 \mathcal{J}_0^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\eta \boldsymbol{\Sigma}\boldsymbol{\Sigma}^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{W}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}^T. \end{bmatrix}$$

This means, we can write

$$\begin{bmatrix} \boldsymbol{I} & -\eta \mathcal{J}_0 \mathcal{J}_0^T \\ \mu \boldsymbol{I} & (1-\mu)\boldsymbol{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W} \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \boldsymbol{C}_1 & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & \boldsymbol{C}_m \end{bmatrix} \boldsymbol{P}^T \begin{bmatrix} \boldsymbol{W}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}^T \end{bmatrix},$$

where $\boldsymbol{P}$ is a permutation matrix and the matrices $\boldsymbol{C}_i$ are of the form $\boldsymbol{C}_i = \begin{bmatrix} 1 & -\eta\sigma_i^2 \\ \mu & (1-\mu) \end{bmatrix}$, for $1 \leq i \leq m$, where the σ_i's denote the singular values of $\boldsymbol{J}$. Using this decomposition we can deduce

$$\|\widetilde{\boldsymbol{z}}_t\|_{\ell_2} = \left\|\boldsymbol{A}^t \widetilde{\boldsymbol{z}}_0\right\|_{\ell_2} \leq \left\|\boldsymbol{A}^t\right\| \|\widetilde{\boldsymbol{z}}_0\|_{\ell_2} = \left(\max_{1\leq i\leq m} \left\|\boldsymbol{C}_i^t\right\|\right) \|\widetilde{\boldsymbol{z}}_0\|_{\ell_2}.$$

Now suppose that $\boldsymbol{V}_i \boldsymbol{D}_i \boldsymbol{V}_i^{-1}$ is the eigenvalue decomposition of $\boldsymbol{C}_i$, where the columns of $\boldsymbol{V}_i$ contain the eigenvectors and $\boldsymbol{D}_i$ is a diagonal matrix consisting of the eigenvalues. (Note that it follows from our assumptions on μ and η that the matrix $\boldsymbol{C}_i$ is diagonalizable.) We have

$$\left\|\boldsymbol{C}_i^t\right\| = \left\|\boldsymbol{V}_i \boldsymbol{D}_i^t \boldsymbol{V}_i^{-1}\right\| \leq \|\boldsymbol{V}_i\| \left\|\boldsymbol{D}_i^t\right\| \left\|\boldsymbol{V}_i^{-1}\right\| = \kappa_i \cdot \rho\left(\boldsymbol{C}_i\right)^t,$$

where we defined $\kappa_i := \|\boldsymbol{V}_i\| \left\|\boldsymbol{V}_i^{-1}\right\|$. From Lemma A.2 we know that the assumption $\frac{\mu}{\eta} \geq 4\beta^2$ results in $\rho(\boldsymbol{A}) \leq 1 - \eta\alpha^2$. Therefore, defining $\gamma := \max_{1\leq i\leq m} \kappa_i$ and noting $\rho(\boldsymbol{A}) = \max_{1\leq i\leq m} \rho(\boldsymbol{C}_i)$, we obtain that

$$\|\widetilde{\boldsymbol{z}}_t\|_{\ell_2} \leq \left(\max_{1\leq i\leq m} \left\|\boldsymbol{C}_i^t\right\|\right) \|\widetilde{\boldsymbol{z}}_0\|_{\ell_2} \leq \gamma \left(1 - \eta\alpha^2\right)^t \|\widetilde{\boldsymbol{z}}_0\|_{\ell_2}.$$

In order to finish the proof we need to show that $\gamma \leq 5$. For that, note that calculating the eigenvectors of $\boldsymbol{C}_i$ directly reveals that we can represent this matrix as

$$\boldsymbol{V}_i = \begin{bmatrix} \frac{1+\sqrt{1-4\frac{\eta\sigma_i^2}{\mu}}}{2} & \frac{1-\sqrt{1-4\frac{\eta\sigma_i^2}{\mu}}}{2} \\ 1 & 1 \end{bmatrix}.$$

Since $\|\boldsymbol{V}_i\| = \sqrt{\lambda_{\max}\left(\boldsymbol{V}_i \boldsymbol{V}_i^T\right)}$ and $\left\|\boldsymbol{V}_i^{-1}\right\| = \sqrt{\lambda_{\min}\left(\boldsymbol{V}_i \boldsymbol{V}_i^T\right)}$, we calculate $\boldsymbol{V}_i \boldsymbol{V}_i^T$, which yields

$$\boldsymbol{V}_i \boldsymbol{V}_i^T = \begin{bmatrix} 1 - 2\frac{\eta\sigma_i^2}{\mu} & 1 \\ 1 & 2 \end{bmatrix}.$$

This representation allows us to directly calculate the two eigenvalues of $\boldsymbol{V}_i\boldsymbol{V}_i^T$, which shows that

$$\begin{aligned}
\kappa_i &= \sqrt{\frac{\lambda_{\max}\left(\boldsymbol{V}_i\boldsymbol{V}_i^T\right)}{\lambda_{\min}\left(\boldsymbol{V}_i\boldsymbol{V}_i^T\right)}} \\
&= \sqrt{\frac{3-2\frac{\eta\sigma_i^2}{\mu}+\sqrt{\left(1+2\frac{\eta\sigma_i^2}{\mu}\right)^2+4}}{3-2\frac{\eta\sigma_i^2}{\mu}-\sqrt{\left(1+2\frac{\eta\sigma_i^2}{\mu}\right)^2+4}}} \\
&= \frac{3-2\frac{\eta\sigma_i^2}{\mu}+\sqrt{\left(1+2\frac{\eta\sigma_i^2}{\mu}\right)^2+4}}{2\sqrt{1-4\frac{\eta\sigma_i^2}{\mu}}} \\
&\leq \frac{6}{2\sqrt{1-4\frac{\eta\sigma_i^2}{\mu}}} \\
&< 5,
\end{aligned}$$

where the last two inequalities holds because of $\frac{\eta\sigma_i^2}{\mu} \leq \frac{\eta\beta^2}{\mu} \leq \frac{1}{8}$. Since $\gamma = \max_{1\leq i\leq m} \kappa_i$, this finishes the proof. ■

Now that we have shown that the linearized iterates converge to the global optima we turn our attention to showing that the nonlinear iterates 14 are close to its linear counterpart 19. For that, we make the following assumptions.

Assumption 1: The singular values of the Jacobian at initialization are bounded from below

$$\sigma_{\min}\left(\mathcal{J}\left(\boldsymbol{\theta}_0\right)\right) \geq \alpha$$

for some positive constants α and β.
Assumption 2: The Jacobian mapping associated with f obeys

$$\left\|\mathcal{J}\left(\boldsymbol{\theta}\right)\right\| \leq \beta$$

for all $\boldsymbol{\theta} \in \mathbb{R}^p$.
Assumption 3: In a neighborhood with radius R around the initialization, the spectral norm of the Jacobian varies no more than ϵ in the sense that

$$\left\|\mathcal{J}\left(\boldsymbol{\theta}\right) - \mathcal{J}\left(\boldsymbol{\theta}_0\right)\right\| \leq \epsilon$$

for all $\boldsymbol{\theta} \in \mathcal{B}_R\left(\boldsymbol{\theta}_0\right)$, where $\mathcal{B}_R\left(\boldsymbol{\theta}_0\right) := \{\boldsymbol{\theta} \in \mathbb{R}^p : \left\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\right\|_{\ell_2} \leq R\}$.

With these assumptions in place, we are ready to state the main theorem.

Theorem A.4 *Consider the GDA updates for the min-max optimization problem 10*

$$\begin{bmatrix}\boldsymbol{d}_{t+1}\\ \boldsymbol{\theta}_{t+1}\end{bmatrix} = \begin{bmatrix}\left(1-\mu\right)\boldsymbol{d}_t + \mu\nabla_{\boldsymbol{d}}h(\boldsymbol{d}_t,\boldsymbol{\theta}_t)\\ \boldsymbol{\theta}_t - \eta\nabla_{\boldsymbol{\theta}}h(\boldsymbol{d}_t,\boldsymbol{\theta}_t)\end{bmatrix} \tag{20}$$

and consider the GDA updates of the linearized problem 19

$$\begin{bmatrix}\widetilde{\boldsymbol{d}}_{t+1}\\ \widetilde{\boldsymbol{\theta}}_{t+1}\end{bmatrix} = \begin{bmatrix}\left(1-\mu\right)\widetilde{\boldsymbol{d}}_t + \mu\nabla_{\boldsymbol{d}}h_{lin}(\boldsymbol{d}_t,\boldsymbol{\theta}_t)\\ \widetilde{\boldsymbol{\theta}}_t - \eta\nabla_{\boldsymbol{\theta}}h_{lin}(\widetilde{\boldsymbol{d}}_t,\widetilde{\boldsymbol{\theta}}_t)\end{bmatrix}. \tag{21}$$

Set $\boldsymbol{z}_t := \begin{bmatrix}\boldsymbol{r}_t\\ \boldsymbol{d}_t\end{bmatrix}$ *and* $\widetilde{\boldsymbol{z}}_t := \begin{bmatrix}\widetilde{\boldsymbol{r}}_t\\ \widetilde{\boldsymbol{d}}_t\end{bmatrix}$, *where* $\boldsymbol{r}_t := f\left(\boldsymbol{\theta_t}\right) - \boldsymbol{y}$ *and* $\widetilde{\boldsymbol{r}}_t = f\left(\boldsymbol{\theta}_0\right) + \mathcal{J}_0\left(\widetilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_0\right)$ *denote the residuals.*
Assume that the step sizes of the gradient descent ascent updates satisfy $\frac{\mu}{\eta} \geq 8\beta^2$ *as well as* $\mu \leq 2$. *Moreover, assume that the assumptions 1-3 hold for the Jacobian* $\mathcal{J}\left(\boldsymbol{\theta}\right)$ *of* $f\left(\boldsymbol{\theta}\right)$ *around the initialization* $\boldsymbol{\theta}_0 \in \mathbb{R}^n$ *with parameters* α, β, ϵ, *and*

$$R := 2\gamma\frac{\beta^2}{\alpha^2}\left\|\begin{bmatrix}\mathcal{J}_0^\dagger & 0\\ 0 & \mathcal{J}_0^\dagger\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2} + \frac{18\epsilon\beta^2\gamma^2}{\alpha^4}\left\|\boldsymbol{z}_0\right\|_{\ell_2}, \tag{22}$$

which satisfy $4\gamma\beta\epsilon \leq \alpha^2$. *Here,* $1 \leq \gamma \leq 5$ *is a constant, which only depends on* μ, η, *and* $\mathcal{J}_0$. *By* $\mathcal{J}_0^\dagger$ *we denote the pseudo-inverse of the Jacobian at initialization* $\mathcal{J}_0$. *Then, assuming the same initialization* $\boldsymbol{\theta}_0 = \widetilde{\boldsymbol{\theta}}_0$, $\boldsymbol{d}_0 = \widetilde{\boldsymbol{d}}_0$ *(and, hence,* $\boldsymbol{z}_0 = \widetilde{\boldsymbol{z}}_0$*), the following holds for all iterations* $t \geq 0$.

- $\|\boldsymbol{z_t}\|_{\ell_2}$ *converges to* 0 *with a geometric rate, i.e.*

$$\|\boldsymbol{z}_t\|_{\ell_2} \leq \gamma \left(1 - \frac{\eta\alpha^2}{2}\right)^t \|\boldsymbol{z}_0\|_{\ell_2}. \tag{23}$$

- *The trajectories of* $\boldsymbol{z}_t$ *and* $\widetilde{\boldsymbol{z}}_t$ *stay close to each other and converge to the same limit, i.e.*

$$\begin{aligned} \|\boldsymbol{z}_t - \widetilde{\boldsymbol{z}}_t\|_{\ell_2} &\leq 2\eta\gamma^2\beta\epsilon \cdot t \left(1 - \frac{\eta\alpha^2}{2}\right)^{t-1} \|\boldsymbol{z}_0\|_{\ell_2} \\ &\leq \frac{4\gamma^2\beta\epsilon}{e\left(15\ln\frac{16}{15}\right)\alpha^2} \|\boldsymbol{z}_0\|_{\ell_2}. \end{aligned} \tag{24}$$

- *The parameters of the original and linearized problems stay close to each other, i.e.*

$$\left\|\widetilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_t\right\|_{\ell_2} \leq \frac{9\epsilon\beta^2\gamma^2}{\alpha^4} \|\boldsymbol{z}_0\|_{\ell_2}, \tag{25}$$

- *The parameters of the original problem stay close to the initialization, i.e.*

$$\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_0\|_{\ell_2} \leq \frac{R}{2}. \tag{26}$$

Theorem A.4 will be the main ingredient in the proof of Theorems 2.1 and 2.2. However, as discussed in Section 2.3 we believe that this meta theorem can be used to deal with a much richer class of generators and discriminators.

A.2.1 Proof of Theorem A.4

We will prove the statements in the theorem by induction. The base case for $\tau = 0$ is trivial. Now assume that the equations equation 23 to equation 26 hold for $\tau = 0, \ldots, t-1$. Our goal is to show that they hold for iteration t as well.

Part I: First, we are going to show that $\boldsymbol{\theta}_t \in \mathcal{B}_R(\boldsymbol{\theta}_0)$. Note that by the triangle inequality and the induction assumption we have that

$$\begin{aligned} \|\boldsymbol{\theta}_t - \boldsymbol{\theta}_0\|_{\ell_2} &\leq \|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|_{\ell_2} + \|\boldsymbol{\theta}_{t-1} - \theta_0\|_{\ell_2} \\ &\leq \|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|_{\ell_2} + \frac{R}{2}. \end{aligned}$$

Hence, in order to prove the claim it remains to show that $\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|_{\ell_2} \leq \frac{R}{2}$. For that, we compute

$$\begin{aligned} \frac{1}{\eta}\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|_{\ell_2} &= \left\|\mathcal{J}^T(\boldsymbol{\theta}_{t-1})\boldsymbol{d}_{t-1}\right\|_{\ell_2} \\ &\leq \left\|\mathcal{J}^T(\boldsymbol{\theta}_{t-1})\widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} + \left\|\mathcal{J}^T(\boldsymbol{\theta}_{t-1})\right\| \left\|\boldsymbol{d}_{t-1} - \widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} \\ &\leq \left\|\mathcal{J}_0^T\widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} + \left\|\mathcal{J}(\boldsymbol{\theta}_{t-1}) - \mathcal{J}_0\right\| \left\|\widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} + \left\|\mathcal{J}^T(\boldsymbol{\theta}_{t-1})\right\| \left\|\boldsymbol{d}_{t-1} - \widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} \\ &\overset{(i)}{\leq} \gamma \left\|\begin{bmatrix} \mathcal{J}_0^T & 0 \\ 0 & \mathcal{J}_0^T \end{bmatrix} \boldsymbol{z}_0\right\|_{\ell_2} + \epsilon\cdot\gamma\|\boldsymbol{z}_0\|_{\ell_2} + \frac{4\beta^2\epsilon\gamma^2}{e\left(15\ln\frac{16}{15}\right)\alpha^2}\|\boldsymbol{z}_0\|_{\ell_2} \\ &\overset{(ii)}{\leq} \gamma\beta^2 \left\|\begin{bmatrix} \mathcal{J}_0^\dagger & 0 \\ 0 & \mathcal{J}_0^\dagger \end{bmatrix} \boldsymbol{z}_0\right\|_{\ell_2} + \frac{3\beta^2\epsilon\gamma^2}{\alpha^2}\|\boldsymbol{z}_0\|_{\ell_2}, \end{aligned}$$

where $\gamma \leq 5$ is a constant. Let us verify the last two inequalities. Inequality (ii) holds because $1 \leq \gamma$, $1 \leq \frac{\beta^2}{\alpha^2}$, and

$$\begin{aligned}\left\|\begin{bmatrix}\mathcal{J}_0^T & 0\\ 0 & \mathcal{J}_0^T\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2} &= \left\|\begin{bmatrix}\boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{W}^T & 0\\ 0 & \boldsymbol{V}\boldsymbol{\Sigma}^T\boldsymbol{W}^T\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2}\\ &= \sqrt{\sum_{i=1}^n \sigma_i^2\left(\langle \boldsymbol{w}_i, \boldsymbol{r}_0\rangle^2 + \langle \boldsymbol{w}_i, \boldsymbol{d}_0\rangle^2\right)}\\ &\leq \beta^2\sqrt{\sum_{i=1}^n \frac{1}{\sigma_i^2}\left(\langle \boldsymbol{w}_i, \boldsymbol{r}_0\rangle^2 + \langle \boldsymbol{w}_i, \boldsymbol{d}_0\rangle^2\right)} = \beta^2\left\|\begin{bmatrix}\mathcal{J}_0^\dagger & 0\\ 0 & \mathcal{J}_0^\dagger\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2}. \end{aligned} \tag{27}$$

Also (i) follows from assumptions 1-3, $\left\|\boldsymbol{d}_{t-1} - \widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} \leq \|\boldsymbol{z}_{t-1} - \widetilde{\boldsymbol{z}}_{t-1}\|_{\ell_2}$ together with induction assumption equation 24, $\left\|\widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} \leq \|\widetilde{\boldsymbol{z}}_{t-1}\|_{\ell_2} \leq \|\boldsymbol{z}_0\|_{\ell_2}$, and

$$\begin{aligned}\left\|\mathcal{J}_0^T\widetilde{\boldsymbol{d}}_{t-1}\right\|_{\ell_2} &\leq \left\|\begin{bmatrix}\mathcal{J}_0^T\widetilde{\boldsymbol{r}}_{t-1}\\ \mathcal{J}_0^T\widetilde{\boldsymbol{d}}_{t-1}\end{bmatrix}\right\|_{\ell_2}\\ &= \left\|\begin{bmatrix}\boldsymbol{I} & -\eta\mathcal{J}_0^T\mathcal{J}_0\\ \mu\boldsymbol{I} & (1-\mu)\boldsymbol{I}\end{bmatrix}\begin{bmatrix}\mathcal{J}_0^T\widetilde{\boldsymbol{r}}_{t-2}\\ \mathcal{J}_0^T\widetilde{\boldsymbol{d}}_{t-2}\end{bmatrix}\right\|_{\ell_2}\\ &= \left\|\begin{bmatrix}\boldsymbol{I} & -\eta\mathcal{J}_0^T\mathcal{J}_0\\ \mu\boldsymbol{I} & (1-\mu)\boldsymbol{I}\end{bmatrix}^{t-1}\begin{bmatrix}\mathcal{J}_0^T\widetilde{\boldsymbol{r}}_0\\ \mathcal{J}_0^T\widetilde{\boldsymbol{d}}_0\end{bmatrix}\right\|_{\ell_2} \leq \gamma\left(1-\eta\alpha^2\right)^{t-1}\left\|\begin{bmatrix}\mathcal{J}_0^T & 0\\ 0 & \mathcal{J}_0^T\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2}, \end{aligned} \tag{28}$$

where in the last inequality we applied Lemma A.3. Finally, by using $\eta \leq \frac{1}{8\beta^2}$ we arrive at

$$\begin{aligned}\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|_{\ell_2} &\leq \gamma\eta\beta^2\left\|\begin{bmatrix}\mathcal{J}_0^\dagger & 0\\ 0 & \mathcal{J}_0^\dagger\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2} + \frac{3\eta\beta^2\epsilon\gamma^2}{\alpha^2}\|\boldsymbol{z}_0\|_{\ell_2}\\ &\leq \frac{\gamma}{8}\left\|\begin{bmatrix}\mathcal{J}_0^\dagger & 0\\ 0 & \mathcal{J}_0^\dagger\end{bmatrix}\boldsymbol{z}_0\right\|_{\ell_2} + \frac{3\epsilon\gamma^2}{8\alpha^2}\|\boldsymbol{z}_0\|_{\ell_2}\\ &\leq \frac{R}{2}, \end{aligned}$$

where the last line is directly due to inequality (22), $\gamma \leq 5$, and $\alpha \leq \beta$. Hence, we have established $\boldsymbol{\theta}_t \in \mathcal{B}_R(\boldsymbol{\theta}_0)$.

Part II: In Lemma A.3 we showed that the time invariant system of state equations $\widetilde{\boldsymbol{z}}_{t+1} = \boldsymbol{A}\widetilde{\boldsymbol{z}}_t$ is uniformly exponentially stable, i.e. $\|\widetilde{\boldsymbol{z}}_t\|_{\ell_2}$ goes down to zero exponentially fast. Now by using the assumption that the Jacobian remains close to the Jacobian at the initialization $\mathcal{J}_0$, we aim to show the exponential stability of the time variant system of the state equations 14. For that, we compute

$$\begin{aligned}\boldsymbol{z}_t = \boldsymbol{A}_{t-1}\boldsymbol{z}_{t-1} &= \begin{bmatrix}\boldsymbol{I} & -\eta\mathcal{J}_{t,t-1}\mathcal{J}_{t-1}^T\\ \mu\boldsymbol{I} & (1-\mu)\boldsymbol{I}\end{bmatrix}\boldsymbol{z}_{t-1}\\ &= \begin{bmatrix}\boldsymbol{I} & -\eta\mathcal{J}_0\mathcal{J}_0^T\\ \mu\boldsymbol{I} & (1-\mu)\boldsymbol{I}\end{bmatrix}\boldsymbol{z}_{t-1} + \begin{bmatrix}\eta\left(\mathcal{J}_0\mathcal{J}_0^T - \mathcal{J}_{t,t-1}\mathcal{J}_{t-1}^T\right)\boldsymbol{d}_{t-1}\\ \boldsymbol{0}\end{bmatrix}\\ &=: \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\Delta}_{t-1}. \end{aligned}$$

Now set $\lambda := 1-\eta\alpha^2$. By induction, we obtain the relation $\boldsymbol{z}_t = \boldsymbol{A}^t \boldsymbol{z}_0 + \sum_{i=0}^{t-1} \boldsymbol{A}^{t-1-i}\boldsymbol{\Delta}_i$. Hence,

$$\begin{aligned}
\|\boldsymbol{z}_t\|_{\ell_2} &= \left\| \boldsymbol{A}^t \boldsymbol{z}_0 + \sum_{i=0}^{t-1} \boldsymbol{A}^{t-1-i}\boldsymbol{\Delta}_i \right\|_{\ell_2} \\
&\leq \left\|\boldsymbol{A}^t \boldsymbol{z}_0\right\|_{\ell_2} + \sum_{i=0}^{t-1} \left\|\boldsymbol{A}^{t-1-i}\boldsymbol{\Delta}_i\right\|_{\ell_2} \\
&\leq \gamma\lambda^t \|\boldsymbol{z}_0\|_{\ell_2} + \sum_{i=0}^{t-1} \gamma\lambda^{t-1-i} \left\|\eta\left(\mathcal{J}_0\mathcal{J}_0^T - \mathcal{J}_{i+1,i}\mathcal{J}_i^T\right)\right\| \|\boldsymbol{d}_i\|_{\ell_2} \\
&\leq \gamma\lambda^t \|\boldsymbol{z}_0\|_{\ell_2} + \sum_{i=0}^{t-1} \eta\gamma\lambda^{t-1-i} (2\beta\epsilon) \|\boldsymbol{z}_i\|_{\ell_2}. \qquad (29)
\end{aligned}$$

The second inequality holds because of Lemma A.3. The last inequality holds because by combining our assumptions 1 to 3 with $\boldsymbol{\theta}_t \in \mathcal{B}_R(\boldsymbol{\theta}_0)$ and the induction assumption 26 for $0 \leq i \leq t-1$, we have that

$$\begin{aligned}
\left\|\mathcal{J}_0\mathcal{J}_0^T - \mathcal{J}_{i+1,i}\mathcal{J}_i^T\right\| &= \left\|\mathcal{J}_0\mathcal{J}_0^T - \mathcal{J}_0\mathcal{J}_i^T + \mathcal{J}_0\mathcal{J}_i^T - \mathcal{J}_{i+1,i}\mathcal{J}_i^T\right\| \\
&\leq \|\mathcal{J}_0\| \|\mathcal{J}_0 - \mathcal{J}_i\| + \|\mathcal{J}_0 - \mathcal{J}_{i+1,i}\| \|\mathcal{J}_i\| \\
&\leq \beta \|\mathcal{J}_0 - \mathcal{J}_i\| + \beta \|\mathcal{J}_0 - \mathcal{J}_{i+1,i}\| \\
&\leq 2\beta\epsilon. \qquad (30)
\end{aligned}$$

In order to deal with inequality 29, we will rely on the following lemma.

Lemma A.5 *(Rugh, 1996, Lemma 24.5) Consider two real sequences $p(t)$ and $\phi(t)$, where $p(t) \geq 0$ for all $t \geq 0$ and*

$$\phi(t) \leq \begin{cases} \psi, & \text{if } t = 0 \\ \psi + \eta \sum_{i=0}^{t-1} p(i)\,\phi(i), & \text{if } t \geq 1 \end{cases}$$

with η and ψ are constants with $\eta \geq 0$. Then for all $t \geq 1$ we have that

$$\phi(t) \leq \psi \prod_{i=0}^{t-1} (1 + \eta \cdot p(i)).$$

Now we define $\phi_t = \frac{\|\boldsymbol{z}_t\|_{\ell_2}}{\lambda^t}$ and rewrite inequality 29 as

$$\phi_t \leq \gamma\phi_0 + \sum_{i=0}^{t-1} \frac{2\eta\gamma\beta\epsilon}{\lambda}\phi_i.$$

Hence, Lemma A.5 yields that

$$\begin{aligned}
\phi_t &\leq \gamma\phi_0 \prod_{i=0}^{t-1} \left(1 + \frac{2\eta\gamma\beta\epsilon}{\lambda}\right) \\
&= \gamma\phi_0 \left(1 + \frac{2\eta\gamma\beta\epsilon}{\lambda}\right)^t \\
&\overset{(i)}{\leq} \gamma\phi_0 \left(1 + \frac{\eta\alpha^2}{2\lambda}\right)^t \\
&\overset{(ii)}{=} \gamma\phi_0 \left(\frac{1 - \frac{\eta\alpha^2}{2}}{1 - \eta\alpha^2}\right)^t,
\end{aligned}$$

where (i) follows from $4\gamma\beta\epsilon \leq \alpha^2$ and (ii) holds by inserting $\lambda = 1 - \eta\alpha^2$. Inserting the definition of ϕ_0 and ϕ_t we obtain that

$$\|\boldsymbol{z}_t\|_{\ell_2} \leq \gamma \left(1 - \frac{\eta\alpha^2}{2}\right)^t \|\boldsymbol{z}_0\|_{\ell_2}.$$

This completes the proof of Part II.

Part III: In this part, our aim is to show that the error vector $\boldsymbol{e}_t := \boldsymbol{z}_t - \widetilde{\boldsymbol{z}}_t$ obeys inequality 24. First, note that

$$\begin{aligned}
\boldsymbol{e}_t &= \boldsymbol{z}_t - \widetilde{\boldsymbol{z}}_t \\
&\overset{(*)}{=} (\boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\Delta}_{t-1}) - \boldsymbol{A}\widetilde{\boldsymbol{z}}_{t-1} \\
&= \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{\Delta}_{t-1},
\end{aligned}$$

where in $(*)$ we used the same notation as in Part II for $\boldsymbol{\Delta}_{t-1}$. Using a recursive argument as well as $\boldsymbol{e}_0 = 0$ we obtain that

$$\begin{aligned}
\|\boldsymbol{e}_t\|_{\ell_2} &= \left\| \sum_{i=0}^{t-1} \boldsymbol{A}^{t-1-i} \boldsymbol{\Delta}_i \right\|_{\ell_2} \\
&\leq \sum_{i=0}^{t-1} \eta\gamma \left(1 - \eta\alpha^2\right)^{t-1-i} \|\boldsymbol{\Delta}_i\|_{\ell_2} \\
&\overset{(i)}{\leq} \sum_{i=0}^{t-1} \eta\gamma \left(1 - \eta\alpha^2\right)^{t-1-i} \left\|\mathcal{J}_0\mathcal{J}_0^T - \mathcal{J}_{i+1,i}\mathcal{J}_i^T\right\| \|\boldsymbol{d}_i\|_{\ell_2} \\
&\overset{(ii)}{\leq} \sum_{i=0}^{t-1} 2\eta\beta\epsilon\gamma \left(1 - \eta\alpha^2\right)^{t-1-i} \|\boldsymbol{z}_i\|_{\ell_2}.
\end{aligned}$$

The first inequality follows from the triangle inequality and Lemma A.3. Inequality (i) follows from the definition of $\boldsymbol{\Delta}_i$. Inequality (ii) follows from inequality 30. Setting $c := 2\eta\beta\epsilon$ we continue

$$\begin{aligned}
\|\boldsymbol{e}_t\|_{\ell_2} &\leq \sum_{i=0}^{t-1} c\gamma \left(1 - \eta\alpha^2\right)^{t-i-1} \|\boldsymbol{z}_i\|_{\ell_2} \\
&\overset{(iii)}{\leq} \sum_{i=0}^{t-1} c\gamma^2 \left(1 - \eta\alpha^2\right)^{t-i-1} \left(1 - \frac{\eta\alpha^2}{2}\right)^i \|\boldsymbol{z}_0\|_{\ell_2} \\
&\overset{(iv)}{\leq} \sum_{i=0}^{t-1} c\gamma^2 \left(1 - \frac{\eta\alpha^2}{2}\right)^{t-1} \|\boldsymbol{z}_0\|_{\ell_2} \\
&= 2\eta\gamma^2\beta\epsilon \cdot t \left(1 - \frac{\eta\alpha^2}{2}\right)^{t-1} \|\boldsymbol{z}_0\|_{\ell_2}.
\end{aligned}$$

Here (iii) holds because of our induction hypothesis 23 and (iv) follows simply from $1 - \eta\alpha^2 \leq 1 - \frac{\eta\alpha^2}{2}$. This shows of the first part of equation 24 for iteration t. Finally, to derive the second part of equation 24 we observe that for all $t \geq 0$ and $0 < x \leq \frac{1}{16}$ we have $t\left(1 - x\right)^{t-1} \leq \frac{1}{e\left(15 \ln \frac{16}{15}\right)x}$. Since $\frac{\eta\alpha^2}{2} \leq \frac{\mu\alpha^2}{16\beta^2} \leq \frac{1}{16}$ we can use this estimate, which yields

$$\begin{aligned}
\|\boldsymbol{e}_t\|_{\ell_2} &\leq 2\eta\gamma^2\beta\epsilon \cdot t \left(1 - \frac{\eta\alpha^2}{2}\right)^{t-1} \|\boldsymbol{z}_0\|_{\ell_2} \\
&\leq \frac{4\gamma^2\beta\epsilon}{e\left(15 \ln \frac{16}{15}\right)\alpha^2} \|\boldsymbol{z}_0\|_{\ell_2}.
\end{aligned}$$

Hence, we have shown equation 24.

Part IV: In this part, we aim to show that the parameters of the original and linearized problems are close. For that, we compute that

$$\begin{aligned}
\frac{1}{\eta}\left\|\boldsymbol{\theta}_t-\widetilde{\boldsymbol{\theta}}_t\right\|_{\ell_2} &= \left\|\sum_{i=0}^{t-1}\nabla_{\boldsymbol{\theta}}h\left(\boldsymbol{\theta}_i,\boldsymbol{d}_i\right)-\nabla_{\boldsymbol{\theta}}h_{\text{lin}}\left(\boldsymbol{\theta}_i,\boldsymbol{d}_i\right)\right\|_{\ell_2} \\
&= \left\|\sum_{i=0}^{t-1}\mathcal{J}^T\left(\boldsymbol{\theta}_i\right)\boldsymbol{d}_i-\mathcal{J}_0^T\widetilde{\boldsymbol{d}}_i\right\|_{\ell_2} \\
&\leq \sum_{i=0}^{t-1}\left\|\left(\mathcal{J}^T\left(\boldsymbol{\theta}_i\right)-\mathcal{J}_0^T\right)\widetilde{\boldsymbol{d}}_i\right\|_{\ell_2}+\sum_{i=0}^{t-1}\left\|\mathcal{J}^T\left(\boldsymbol{\theta}_i\right)\left(\boldsymbol{d}_i-\widetilde{\boldsymbol{d}}_i\right)\right\|_{\ell_2} \\
&\overset{(i)}{\leq} \sum_{i=0}^{t-1}\epsilon\left\|\widetilde{\boldsymbol{z}}_i\right\|_{\ell_2}+\beta\sum_{i=0}^{t-1}\left\|\boldsymbol{e}_i\right\|_{\ell_2} \\
&\overset{(ii)}{\leq} \gamma\epsilon\sum_{i=0}^{t-1}\left(1-\eta\alpha^2\right)^i\left\|\boldsymbol{z}_0\right\|_{\ell_2}+2\eta\gamma^2\beta^2\epsilon\sum_{i=0}^{t-1}i\left(1-\frac{\eta\alpha^2}{2}\right)^{i-1}\left\|\boldsymbol{z}_0\right\|_{\ell_2}.
\end{aligned}$$

Here (i) follows from assumptions 2 and 3, and (ii) holds because of Lemma A.3 and our induction hypothesis 24. Hence, using the formula $\sum_{i=0}^{t} ix^i = \frac{x\left(1+tx^{t+1}-(t+1)x^t\right)}{(x-1)^2}$ we obtain that

$$\begin{aligned}
\frac{1}{\eta}\left\|\boldsymbol{\theta}_t-\widetilde{\boldsymbol{\theta}}_t\right\|_{\ell_2} &\leq \gamma\epsilon\left\|\boldsymbol{z}_0\right\|_{\ell_2}\left(\frac{1-\left(1-\eta\alpha^2\right)^t}{\eta\alpha^2}+2\eta\beta^2\gamma\frac{1-t\left(1-\frac{\eta\alpha^2}{2}\right)^{t-1}+(t-1)\left(1-\frac{\eta\alpha^2}{2}\right)^t}{\left(\frac{\eta\alpha^2}{2}\right)^2}\right) \\
&\leq \gamma\epsilon\left\|\boldsymbol{z}_0\right\|_{\ell_2}\left(\frac{1}{\eta\alpha^2}+2\eta\beta^2\gamma\frac{1}{\left(\frac{\eta\alpha^2}{2}\right)^2}\right) \\
&\overset{(iii)}{\leq} \gamma\epsilon\left\|\boldsymbol{z}_0\right\|_{\ell_2}\left(\frac{\beta^2\gamma}{\eta\alpha^4}+\frac{8\beta^2\gamma}{\eta\alpha^4}\right) \\
&= \frac{9\epsilon\beta^2\gamma^2}{\eta\alpha^4}\left\|\boldsymbol{z}_0\right\|_{\ell_2},
\end{aligned}$$

where (iii) holds due to $1 \leq \gamma$ and $1 \leq \frac{\beta^2}{\alpha^2}$. Hence, we have established inequality 25 for iteration t.

Part V: In this part, we are going to prove equation 26 for iteration t. First, it follows from the triangle inequality that

$$\begin{aligned}
\left\|\boldsymbol{\theta}_t-\boldsymbol{\theta}_0\right\|_{\ell_2} &\leq \left\|\widetilde{\boldsymbol{\theta}}_t-\boldsymbol{\theta}_0\right\|_{\ell_2}+\left\|\boldsymbol{\theta}_t-\widetilde{\boldsymbol{\theta}}_t\right\|_{\ell_2} \\
&\leq \left\|\widetilde{\boldsymbol{\theta}}_t-\boldsymbol{\theta}_0\right\|_{\ell_2}+\frac{9\epsilon\beta^2\gamma^2}{\alpha^4}\left\|\boldsymbol{z}_0\right\|_{\ell_2},
\end{aligned}$$

where in the second inequality we have used Part IV. Now we bound $\left\|\widetilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_0\right\|_{\ell_2}$ from above as follows

$$\begin{aligned}
\left\|\widetilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_0\right\|_{\ell_2} &= \eta \left\|\sum_{i=0}^{t-1} \mathcal{J}_0^T \widetilde{\boldsymbol{d}}_i\right\|_{\ell_2} \\
&\leq \eta \sum_{i=0}^{t-1} \left\|\mathcal{J}_0^T \widetilde{\boldsymbol{d}}_i\right\|_{\ell_2} \\
&\overset{(i)}{\leq} \eta\gamma \sum_{i=0}^{t-1} \left(1-\eta\alpha^2\right)^i \left\|\begin{bmatrix} \mathcal{J}_0^T & 0 \\ 0 & \mathcal{J}_0^T \end{bmatrix} \boldsymbol{z}_0\right\|_{\ell_2} \\
&= \eta\gamma \frac{1-\left(1-\eta\alpha^2\right)^t}{\eta\alpha^2} \left\|\begin{bmatrix} \mathcal{J}_0^T & 0 \\ 0 & \mathcal{J}_0^T \end{bmatrix} \boldsymbol{z}_0\right\|_{\ell_2} \\
&\overset{(ii)}{\leq} \gamma \frac{\beta^2}{\alpha^2} \left\|\begin{bmatrix} \mathcal{J}_0^\dagger & 0 \\ 0 & \mathcal{J}_0^\dagger \end{bmatrix} \boldsymbol{z}_0\right\|_{\ell_2},
\end{aligned}$$

where (i) holds by equation 28 and (ii) holds by equation 27. Hence, it follows from the definition of R (equation 22) that

$$\begin{aligned}
\left\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_0\right\|_{\ell_2} &\leq \gamma \frac{\beta^2}{\alpha^2} \left\|\begin{bmatrix} \mathcal{J}_0^\dagger & 0 \\ 0 & \mathcal{J}_0^\dagger \end{bmatrix} \boldsymbol{z}_0\right\|_{\ell_2} + \frac{9\epsilon\beta^2\gamma^2}{\alpha^4} \left\|\boldsymbol{z}_0\right\|_{\ell_2} \\
&= \frac{R}{2}.
\end{aligned}$$

This completes the proof.

A.3 PRELIMINARIES FOR PROOFS OF RESULTS WITH ONE-HIDDEN LAYER GENERATORS AND LINEAR DISCRIMINATOR

In this section, we gather some preliminary results that will be useful in proving the main results i.e. Theorems 2.1 and 2.2. We begin by noting that Theorem 2.1 is an instance of Theorem A.4 with $f(\boldsymbol{W}) = \frac{1}{n}\sum_{i=1}^n \boldsymbol{V} \cdot \phi(\boldsymbol{W}\boldsymbol{z}_i)$. We thus begin this section by noting that $f(\boldsymbol{W})$ can be rewritten as follows

$$f(\boldsymbol{W}) = \boldsymbol{V} \cdot \begin{bmatrix} \frac{1}{n}\sum_{i=1}^n \phi\left(\boldsymbol{w}_1^T \boldsymbol{z}_i\right) \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{n}\sum_{i=1}^n \phi\left(\boldsymbol{w}_k^T \boldsymbol{z}_i\right). \end{bmatrix}$$

Furthermore, the Jacobian of this mapping $f(\boldsymbol{W})$ takes the form

$$\mathcal{J}(\boldsymbol{W}) = \frac{1}{n}\sum_{i=1}^n \left(\boldsymbol{V} \cdot \operatorname{diag}\left(\phi'\left(\boldsymbol{W}\boldsymbol{z}_i\right)\right)\right) * \left(\boldsymbol{1}\boldsymbol{z}_i^T\right). \tag{31}$$

To characterize the spectral properties of this Jacobian it will also be convenient to write down the expression for $\mathcal{J}(\boldsymbol{W})\mathcal{J}(\boldsymbol{W})^T$ which has the compact form

$$\begin{aligned}
\mathcal{J}(\boldsymbol{W})\mathcal{J}(\boldsymbol{W})^T &= \frac{1}{n^2}\sum_{i,j} \left(\boldsymbol{V}\operatorname{diag}\left(\phi'\left(\boldsymbol{W}\boldsymbol{z}_i\right)\right)\operatorname{diag}\left(\phi'\left(\boldsymbol{W}\boldsymbol{z}_j\right)\right)\boldsymbol{V}^T\right) \odot \left(\boldsymbol{z}_i^T\boldsymbol{z}_j\boldsymbol{1}\boldsymbol{1}^T\right) \\
&= \left(\boldsymbol{V} \operatorname*{diag}_{\ell=1,\ldots,k} \left(\left\|\frac{1}{n}\sum_{i=1}^n \boldsymbol{z}_i \phi'\left(\boldsymbol{w}_\ell^T\boldsymbol{z}_i\right)\right\|_{\ell_2}^2\right)\boldsymbol{V}^T\right) \\
&= \frac{1}{n^2}\boldsymbol{V}\cdot\boldsymbol{D}^2\cdot\boldsymbol{V}^T,
\end{aligned}$$

where $\boldsymbol{D}$ is a diagonal matrix with entries

$$D_{\ell\ell} = \left\|\sum_{i=1}^n \boldsymbol{z}_i \phi'(\boldsymbol{w}_\ell^T\boldsymbol{z}_i)\right\|_{\ell_2} = \left\|\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w}_\ell)\right\|_{\ell_2}, \tag{32}$$

and $\boldsymbol{Z} \in \mathbb{R}^{n\times d}$ contains the $\boldsymbol{z}_i$'s in its rows. The next lemma establishes concentration of the diagonal entries of matrix $\boldsymbol{D}$ around their mean, which will be used in the future lemmas regarding the spectrum of the Jacobian mapping. The proof is deferred to Appendix B.1.

Lemma A.6 *Suppose $\boldsymbol{w} \in \mathbb{R}^d$ is a fixed vector, $\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_n \in \mathbb{R}^d$ are distributed as $\mathcal{N}(0, \sigma_z^2 \boldsymbol{I}_d)$ and constitute the rows of $\boldsymbol{Z} \in \mathbb{R}^{n\times d}$. Then for any $0 \leq \delta \leq \frac{3}{2}$ the random variable $D = \left\|\boldsymbol{Z}^T \phi\left(\boldsymbol{Z}\boldsymbol{w}\right)\right\|_{\ell_2}$ satisfies*

$$(1-\delta)\,\mathbb{E}\left(D^2\right) \leq D^2 \leq (1+\delta)\,\mathbb{E}\left(D^2\right)$$

with probability at least $1-2\left(e^{-\frac{n\delta^2}{18}} + e^{-\frac{d\delta^2}{54}} + e^{-c_1 n\delta}\right)$ where c_1 is a positive constant. Moreover we have

$$\mathbb{E}\left(D^2\right) = \sigma_z^2\left(\frac{nd}{2} + \frac{n(n-1)}{2\pi}\right),$$

Furthermore, using the above we have

$$\mathbb{E}\left[\mathcal{J}(\boldsymbol{W})\mathcal{J}(\boldsymbol{W})^T\right] = \frac{\sigma_z^2\left(d + \frac{n-1}{\pi}\right)}{2n}\boldsymbol{V}\boldsymbol{V}^T.$$

A.4 Lemmas regarding the initial misfit and the spectrum of the Jacobian

In this section, we state some lemmas regarding the spectrum of the Jacobian mapping and the initial misfit, and defer their proofs to Appendix B. First, we state a result on the minimum singular value of the Jacobian mapping:

Lemma A.7 ***(Minimum eigenvalue of the Jacobian at initialization)*** *Consider our GAN model with a linear discriminator and the generator being a one hidden layer neural network of the form $\boldsymbol{z} \rightarrowtail \boldsymbol{V}\phi(\boldsymbol{W}z)$, where we have n independent data points $\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_n \in \mathbb{R}^d$ distributed as $\mathcal{N}(0, \sigma_z^2 \boldsymbol{I}_d)$ and aggregated as the rows of a matrix $\boldsymbol{Z} \in \mathbb{R}^{n\times d}$, and $\boldsymbol{V} \in \mathbb{R}^{m\times k}$ has i.i.d $\mathcal{N}(0, \sigma_v^2)$ entries. We also assume that $\boldsymbol{W}_0 \in \mathbb{R}^{k\times d}$ has i.i.d $\mathcal{N}(0, \sigma_w^2)$ entries and all entries of $\boldsymbol{W}_0$, $\boldsymbol{V}$, and $\boldsymbol{Z}$ are independent. Then the Jacobian matrix at the initialization point obeys*

$$\sigma_{min}\left(\mathcal{J}(\boldsymbol{W}_0)\right) \geq \left(\sqrt{(1-\delta)^2 k - (1+\delta)^2} - \sqrt{m}\,(1+\eta)\,(1+\delta)\right)\sigma_v\sigma_z\sqrt{\frac{d}{2n} + \frac{1}{4\pi}}, \quad 0 \leq \delta \leq \frac{3}{2}$$

with probability at least $1 - 3e^{-\frac{\eta^2 m}{8}} - 2k\cdot\left(e^{-\frac{n\delta^2}{18}} + e^{-\frac{d\delta^2}{54}} + e^{-c_1 n\delta}\right)$, where c_1 is a positive constant.

The next two lemmas are adapted from Van Veen et al. (2018) and allow us to bound the variations in the Jacobian matrix around initialization as well as the spectral norm of the Jacobian matrix.

Lemma A.8 ***(single-sample Jacobian perturbation)*** *Let $\boldsymbol{V} \in \mathbb{R}^{m\times k}$ be a matrix with i.i.d. $\mathcal{N}\left(0, \sigma_v^2\right)$ entries, $\boldsymbol{W} \in \mathbb{R}^{k\times d}$, and define the Jacobian mapping $\mathcal{J}(\boldsymbol{W}; z) = \left(\boldsymbol{V}\,diag\left(\phi'\left(\boldsymbol{W}z\right)\right)\right) * \left(\boldsymbol{1}z^T\right)$. Then, by taking $\boldsymbol{W}_0$ to be a random matrix with i.i.d. $\mathcal{N}\left(0, \sigma_w^2\right)$ entries, we have*

$$\|\mathcal{J}(\boldsymbol{W}; z) - \mathcal{J}(\boldsymbol{W}_0; z)\| \leq \sigma_v \|z\|_{\ell_2}\left(2\sqrt{m} + \sqrt{6\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}} log\left(\frac{k}{3\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}}\right)}\right)$$

for all $\boldsymbol{W} \in \mathbb{R}^{k\times d}$ obeying $\|\boldsymbol{W} - \boldsymbol{W}_0\| \leq R$ with probability at least $1 - e^{-\frac{m}{2}} - e^{-\frac{\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}}{6}}$.

Lemma A.9 ***(Spectral norm of single-sample Jacobian)*** *Consider $\boldsymbol{V} \in \mathbb{R}^{m\times k}$ with i.i.d. $\mathcal{N}\left(0, \sigma_v^2\right)$ entries, $\boldsymbol{W} \in \mathbb{R}^{k\times d}$, and the Jacobian mapping $\mathcal{J}(\boldsymbol{W}; z) = \left(\boldsymbol{V}\,diag\left(\phi'\left(\boldsymbol{W}z\right)\right)\right) * \left(\boldsymbol{1}z^T\right)$. Then we have*

$$\|\mathcal{J}\left(\boldsymbol{W}; z\right)\| \leq \sigma_v\left(\sqrt{k} + 2\sqrt{m}\right)\|z\|_{\ell_2}$$

for all $\boldsymbol{W} \in \mathbb{R}^{k\times d}$ with probability at least $1 - e^{-\frac{m}{2}}$.

Our final key lemma bounds the initial misfit $f(\boldsymbol{W}_0) - \boldsymbol{y} := \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{V}\phi\left(\boldsymbol{W}_0 \boldsymbol{z_i}\right) - \bar{\boldsymbol{x}}$.

Lemma A.10 ***(Initial misfit)*** *Consider our GAN model with a linear discriminator and the generator being a one hidden layer neural network of the form $\boldsymbol{z} \rightarrowtail \boldsymbol{V}\phi(\boldsymbol{W}z)$, where we have n independent data points $\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_n \in \mathbb{R}^d$ distributed as $\mathcal{N}(0, \sigma_z^2 \boldsymbol{I}_d)$ and aggregated as the rows of a matrix $\boldsymbol{Z} \in \mathbb{R}^{n\times d}$, and $\boldsymbol{V} \in \mathbb{R}^{m\times k}$ has i.i.d $\mathcal{N}(0, \sigma_v^2)$ entries. We also assume that the initial $\boldsymbol{W}_0 \in \mathbb{R}^{k\times d}$ has i.i.d $\mathcal{N}(0, \sigma_w^2)$ entries. Then the following event*

$$\left\| \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{V}\phi\left(\boldsymbol{W}_0 \boldsymbol{z_i}\right) - \bar{\boldsymbol{x}} \right\|_{\ell_2} \leq (1+\delta)\frac{1}{\sqrt{2\pi}}\sigma_v\sigma_w\sigma_z\sqrt{kdm} + \|\bar{\boldsymbol{x}}\|_{\ell_2}, \quad 0 \leq \delta \leq 3$$

holds with probability at least $1 - \left(k \cdot e^{-c_2 n(\delta/27)^2} + e^{-\frac{(\delta/9)^2 m}{2}} + e^{-\frac{(\delta/3)^2 kd}{2}}\right)$, with c_2 a fixed constant.

A.5 Proof of Theorem 2.1

In this section, we prove Theorem 2.1 by using our general meta Theorem A.4. To do this we need to check that Assumptions 1-3 are satisfied with high probability. Specifically, in our case the parameter $\boldsymbol{\theta}$ is the matrix $\boldsymbol{W}$ and the non-linear mapping f is given by $f(\boldsymbol{W}) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{V}\phi\left(\boldsymbol{W}\boldsymbol{z_i}\right)$. We note that in our result $\boldsymbol{d}_0 = \boldsymbol{0}$ and thus $\|\boldsymbol{z}_0\|_{\ell_2} = \|\boldsymbol{r}_0\|_{\ell_2}$, which simplifies our analysis.
To prove Assumption 1 note that by setting $\delta = \frac{1}{2}$ and $\eta = \frac{1}{3}$ in Lemma A.7, we have

$$\begin{aligned}\sigma_{\min}\left(\mathcal{J}\left(\boldsymbol{W}_0\right)\right) &\geq \sigma_v\sigma_z\left(\frac{1}{2}\sqrt{k-9} - 2\sqrt{m}\right)\sqrt{\frac{d}{2n} + \frac{1}{4\pi}} \\ &=: \alpha.\end{aligned}$$

This holds with probability at least $1 - 3e^{-\frac{m}{72}} - 4k \cdot e^{-c\cdot n} - 2k \cdot e^{-\frac{d}{216}}$, concluding the proof of Assumption 1. Next, by using Lemma A.9 and triangle inequality we have

$$\begin{aligned}\|\mathcal{J}\left(\boldsymbol{W}\right)\| &= \frac{1}{n}\left\|\sum_{i=1}^{n}\mathcal{J}\left(\boldsymbol{W}; z_i\right)\right\| \\ &\leq \frac{\sigma_v}{n}\left(\sqrt{k} + 2\sqrt{m}\right)\left(\sum_{i=1}^{n}\|z_i\|_{\ell_2}\right) \\ &\overset{(i)}{\leq} \sigma_v\frac{\|\boldsymbol{Z}\|_F}{\sqrt{n}}\left(\sqrt{k} + 2\sqrt{m}\right) \\ &\overset{(ii)}{\leq} \frac{5}{4}\sigma_v\sigma_z\sqrt{d}\left(\sqrt{k} + 2\sqrt{m}\right),\end{aligned} \tag{33}$$

where (i) holds by Cauchy–Schwarz inequality, and (ii) holds because for a Gaussian matrix $\boldsymbol{Z}$ with probability at least

$$\mathbb{P}\left(\|\boldsymbol{Z}\|_F \leq \frac{5}{4}\sigma_z\sqrt{nd}\right) \geq \mathbb{P}\left(\|\boldsymbol{Z}\|_F^2 \leq \frac{3}{2}\sigma_z^2 nd\right) \geq 1 - e^{-\frac{nd}{24}}.$$

Therefore, Assumption 2 holds with

$$\beta = \frac{5}{4}\sigma_v\sigma_z\sqrt{d}\left(\sqrt{k} + 2\sqrt{m}\right),$$

with probability at least $1 - ne^{-\frac{m}{2}} - e^{-\frac{nd}{24}}$.

Finally, to show Assumption 3 holds, we use the single-sample Jacobian perturbation result of Lemma A.8 combined with the triangle inequality to conclude that

$$\begin{aligned}
\|\mathcal{J}(\boldsymbol{W}) - \mathcal{J}(\boldsymbol{W}_0)\| &= \left\| \frac{1}{n} \left(\sum_{i=1}^{n} \mathcal{J}(\boldsymbol{W}; z_i) - \mathcal{J}(\boldsymbol{W}_0; z_i) \right) \right\| \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \|\mathcal{J}(\boldsymbol{W}; z_i) - \mathcal{J}(\boldsymbol{W}_0; z_i)\| \\
&\leq \frac{\sigma_v}{n} \left(\sum_{i=1}^{n} \|z_i\|_{\ell_2} \right) \left(2\sqrt{m} + \sqrt{6 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}} \log \left(\frac{k}{3 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}}} \right)} \right) \\
&\leq \sigma_v \frac{\|\boldsymbol{Z}\|_F}{\sqrt{n}} \left(2\sqrt{m} + \sqrt{6 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}} \log \left(\frac{k}{3 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}}} \right)} \right) \\
&\leq \frac{5}{4} \sigma_v \sigma_z \sqrt{d} \left(2\sqrt{m} + \sqrt{6 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}} \log \left(\frac{k}{3 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}}} \right)} \right). \qquad (34)
\end{aligned}$$

Now we set $\epsilon = \frac{\alpha^2}{4\gamma\beta}$ and show that Assumption 3 holds with this choice of ϵ and with radius $\widetilde{R}$, whose value will define later in the proof. First, note that

$$\begin{aligned}
\epsilon &= \frac{\alpha^2}{4\gamma\beta} \\
&= \frac{\sigma_v^2 \sigma_z^2 \left(\frac{1}{2}\sqrt{k-9} - 2\sqrt{m} \right)^2 \left(\frac{d}{2n} + \frac{1}{4\pi} \right)}{5\gamma\sigma_v\sigma_z\sqrt{d}\left(\sqrt{k} + 2\sqrt{m} \right)} \\
&\overset{(i)}{\geq} \frac{\sigma_v\sigma_z \left(\frac{1}{8}\sqrt{k} \right)^2 \cdot \frac{1}{4\pi}}{25\sqrt{d}\left(3\sqrt{k} \right)} \\
&\geq \frac{\sigma_v\sigma_z\sqrt{k}}{1000\sqrt{d}},
\end{aligned}$$

where (i) holds by assuming $k \geq C \cdot m$ with C being a large positive constant. Combining the last inequality with equation 34, we observe that a sufficient condition for assumption 3 to hold is

$$\frac{5}{4}\sigma_v\sigma_z\sqrt{d} \left(2\sqrt{m} + \sqrt{6 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}} \log \left(\frac{k}{3 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}}} \right)} \right) \leq \frac{\sigma_v\sigma_z\sqrt{k}}{1000\sqrt{d}},$$

which is equivalent to

$$1250\sqrt{md^2} + 1250 \cdot d \cdot \sqrt{6 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}} \log \left(\frac{k}{3 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}}} \right)} \leq \sqrt{k}.$$

Now the first term in the L.H.S. is upper bounded by $\frac{1}{2}\sqrt{k}$ if $k \geq (2500)^2 md^2$, and for the second term we need

$$2500 \cdot d \cdot \sqrt{6 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}} \log \left(\frac{k}{3 \left(\frac{2kR}{\sigma_w} \right)^{\frac{2}{3}}} \right)} \leq \sqrt{k},$$

which by defining $x = 3d^2 \left(\frac{2R}{\sigma_w \sqrt{k}}\right)^{\frac{2}{3}}$ is equivalent to

$$x \log \frac{d^2}{x} \leq \frac{1}{2 \cdot 2500^2}.$$

This last inequality holds for $x \leq \frac{c}{\log d}$ with $c < 1$ a sufficiently small positive constant, which translates into

$$R \leq c \frac{\sigma_w \sqrt{k}}{(d^2 \log d)^{\frac{3}{2}}}. \tag{35}$$

So far we have shown that Assumption 3 holds with $\epsilon = \frac{\alpha^2}{4\gamma\beta}$ and with radius $\widetilde{R}$ defined as $\widetilde{R} := c \frac{\sigma_w \sqrt{k}}{(d^2 \log d)^{\frac{3}{2}}}$, and we conclude that it holds for any radius R less than $\widetilde{R}$ as well. Now we work with the definition of R in equation 22 to show that $R \leq \widetilde{R}$:

$$\begin{aligned}
\frac{R}{2} &= \gamma \frac{\beta^2}{\alpha^2} \left\| \begin{bmatrix} \mathcal{J}_0^\dagger & 0 \\ 0 & \mathcal{J}_0^\dagger \end{bmatrix} \boldsymbol{z}_0 \right\|_{\ell_2} + \frac{9\epsilon\beta^2\gamma^2}{\alpha^4} \|\boldsymbol{z}_0\|_{\ell_2} \\
&\overset{(i)}{\leq} \gamma \frac{\beta^2}{\alpha^3} \|\boldsymbol{r}_0\|_{\ell_2} + \frac{9\frac{\alpha^2}{4\gamma\beta}\beta^2\gamma^2}{\alpha^4} \|\boldsymbol{r}_0\|_{\ell_2} \\
&= \gamma \|\boldsymbol{r}_0\|_{\ell_2} \left(\frac{\beta^2}{\alpha^3} + \frac{9}{4} \frac{\beta}{\alpha^2} \right) \\
&\overset{(ii)}{\leq} 20 \frac{\beta^2}{\alpha^3} \|\boldsymbol{r}_0\|_{\ell_2} \\
&= 20 \frac{\left(\frac{5}{4} \sigma_v \sigma_z \sqrt{d} \left(\sqrt{k} + 2\sqrt{m} \right) \right)^2}{\left(\sigma_v \sigma_z \left(\frac{1}{2}\sqrt{k-9} - 2\sqrt{m} \right) \sqrt{\frac{d}{2n} + \frac{1}{4\pi}} \right)^3} \|\boldsymbol{r}_0\|_{\ell_2} \\
&\overset{(iii)}{\leq} C \cdot \frac{d}{\sigma_v \sigma_z \sqrt{k}} \cdot \left(\frac{2}{3} \sigma_v \sigma_w \sigma_z \sqrt{k \cdot d \cdot m} + \|\bar{\boldsymbol{x}}\|_{\ell_2} \right)
\end{aligned}$$

where (i) holds because $\left\|\mathcal{J}_0^\dagger\right\| \leq \frac{1}{\alpha}$ and $4\gamma\beta\epsilon = \alpha^2$, (ii) holds as $1 \leq \frac{\beta}{\alpha}$ and as we substitute $\gamma = 5$ from Lemma A.3, and (iii) follows from $k \geq C \cdot md^2$ and using $\delta = \frac{1}{3}$ in Lemma A.10. Now a sufficient condition for equation 35 to hold is that

$$\frac{d}{\sigma_v \sigma_z \sqrt{k}} \cdot \left(\frac{2}{3} \sigma_v \sigma_w \sigma_z \sqrt{k \cdot d \cdot m} + \|\bar{\boldsymbol{x}}\|_{\ell_2} \right) \leq c \frac{\sigma_w \sqrt{k}}{(d^2 \log d)^{\frac{3}{2}}},$$

which is equivalent to

$$\frac{2}{3} \sigma_v \sigma_w \sigma_z \cdot d^4 (\log d)^{\frac{3}{2}} \sqrt{k \cdot d \cdot m} + d^4 (\log d)^{\frac{3}{2}} \|\bar{\boldsymbol{x}}\|_{\ell_2} \leq c \cdot k \sigma_v \sigma_w \sigma_z.$$

Finally, this inequality is satisfied if $k \geq C \cdot md^9 \log(d)^3$ and by setting $\sigma_v \sigma_w \sigma_z \geq \frac{\|\bar{\boldsymbol{x}}\|_{\ell_2}}{md^5 \log d^{\frac{3}{2}}}$. This shows that assumption 3 holds with probability at least $1 - ne^{-\frac{m}{2}} - ne^{-c \cdot md^7 \log(d)^2} - k \cdot e^{-c \cdot n} - e^{-\frac{m}{1500}} - e^{-\frac{kd}{162}}$, concluding the proof of Theorem 2.1.

A.6 Proof of Theorem 2.2

Consider a nonlinear least-squares optimization problem of the form

$$\min_{\theta \in \mathbb{R}^p} \mathcal{L}(\theta) := \frac{1}{2} \|f(\theta) - y\|_{\ell_2}^2$$

with $f : \mathbb{R}^p \mapsto \mathbb{R}^m$ and $y \in \mathbb{R}^m$. Suppose the Jacobian mapping associated with f satisfies the following three assumptions.

Assumption 1 We assume $\sigma_{min}\left(\mathcal{J}\left(\theta_0\right)\right) \geq 2\alpha$ for a fixed point $\theta_0 \in \mathbb{R}^p$.

Assumption 2 Let $\|\cdot\|$ be a norm dominated by the Frobenius norm i.e. $\|\theta\| \leq \|\theta\|_F$ holds for all $\theta \in \mathbb{R}^p$. Fix a point θ_0 and a number $R > 0$. For any θ satisfying $\|\theta - \theta_0\| \leq R$, we have $\|\mathcal{J}\left(\theta\right) - \mathcal{J}\left(\theta_0\right)\| \leq \frac{\alpha}{3}$.

Assumption 3 We assume for all $\theta \in \mathbb{R}^p$ obeying $\|\theta - \theta_0\| \leq R$, we have $\|\mathcal{J}\left(\theta\right)\| \leq \beta$.

With these assumptions in place we are now ready to state the following result from Oymak & Soltanolkotabi (2020):

Theorem A.11 *Given $\theta_0 \in \mathbb{R}^p$, suppose assumptions 1, 2, and 3 hold with*

$$R = \frac{3\left\|f\left(\theta_0\right) - y\right\|_{\ell_2}}{\alpha}. \tag{36}$$

Then, using a learning rate $\eta \leq \frac{1}{3\beta^2}$, all gradient descent updates obey

$$\left\|f\left(\theta_\tau\right) - y\right\|_{\ell_2} \leq \left(1 - \eta\alpha^2\right)^\tau \left\|f\left(\theta_0\right) - y\right\|_{\ell_2}. \tag{37}$$

We are going to apply this theorem in our case where the parameter is $\boldsymbol{W}$, the nonlinear mapping is $f\left(\boldsymbol{W}\right) = \frac{1}{n}\sum_{i=1}^n \boldsymbol{V}\phi\left(\boldsymbol{W}\boldsymbol{z_i}\right)$ with $\phi = ReLU$, and the norm $\|\cdot\|$ set to the operator norm.

Again by using Lemma A.7, we see that with probability at least $1 - 3e^{-\frac{m}{72}} - 4k \cdot e^{-c \cdot n} - 2k \cdot e^{-\frac{d}{16}}$ Assumption 1 is satisfied with

$$2\alpha = \sigma_v \sigma_z \left(\frac{1}{2}\sqrt{k-9} - 2\sqrt{m}\right)\sqrt{\frac{d}{2n} + \frac{1}{4\pi}}.$$

Next we show that assumption 2 is valid for α as defined in the previous line and for radius $\widetilde{R}$ defined later. First, by making use of equation 34 we try to show

$$\frac{5}{4}\sigma_v\sigma_z\sqrt{d}\left(2\sqrt{m} + \sqrt{6\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}\log\left(\frac{k}{3\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}}\right)}\right) \leq \frac{\alpha}{3},$$

which after substituting α is equivalent to

$$C\sqrt{d}\left(2\sqrt{m} + \sqrt{6\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}\log\left(\frac{k}{3\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}}\right)}\right) \leq \left(\frac{1}{2}\sqrt{k-9} - 4\sqrt{m}\right)\sqrt{\frac{d}{2n} + \frac{1}{4\pi}}.$$

Now by assuming k to be large enough that $k - 9 \geq \frac{k}{4}$ and $k \geq 256m$, which are satisfied if $k \geq C \cdot m$, it suffices to show that

$$C\sqrt{md} + C\sqrt{d} \cdot \sqrt{6\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}\log\left(\frac{k}{3\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}}\right)} \leq \sqrt{k}.$$

The first term in the L.H.S. of the inequality above is upper bounded by $\frac{1}{2}\sqrt{k}$ if $k \geq C \cdot md$. For upper bounding the second term it is sufficient to show that

$$C\sqrt{d}\sqrt{6\left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}\log\left(\frac{k}{3 \cdot \left(\frac{2kR}{\sigma_w}\right)^{\frac{2}{3}}}\right)} \leq \sqrt{k},$$

which by defining $x = 3d\left(\frac{2R}{\sigma_w\sqrt{k}}\right)^{\frac{2}{3}}$ is equivalent to $x \cdot \log\left(\frac{d}{x}\right) \leq C$. Now this last inequality holds if we have $x \leq \frac{c}{\log(d)}$ for a sufficiently small constant c, which by rearranging terms amounts to showing that $R \leq c \cdot \frac{\sigma_w\sqrt{k}}{(d\cdot\log(d))^{\frac{3}{2}}}$. Hence up to this point, we have shown that assumption 2 holds with radius $\widetilde{R} := c \cdot \frac{\sigma_w\sqrt{k}}{(d\cdot\log(d))^{\frac{3}{2}}}$, and this implies that it holds for all values of R less than $\widetilde{R}$. Therefore, we work with the definition of R in equation 36 to show that $R \leq \widetilde{R}$ as follows:

$$\begin{aligned} R &= \frac{3\left\|f\left(\theta_0\right)-y\right\|_{\ell_2}}{\alpha} \\ &\overset{(i)}{\leq} \frac{3}{\alpha}\left(\frac{2}{3}\sigma_v\sigma_w\sigma_z\sqrt{k\cdot d\cdot m}+\left\|\bar{\boldsymbol{x}}\right\|_{\ell_2}\right) \\ &= \frac{2\left(2\sigma_v\sigma_w\sigma_z\sqrt{k\cdot m\cdot d}+3\left\|\bar{\boldsymbol{x}}\right\|_{\ell_2}\right)}{\sigma_v\sigma_z\left(\frac{1}{2}\sqrt{k-9}-2\sqrt{m}\right)\sqrt{\frac{d}{2n}+\frac{1}{4\pi}}}, \end{aligned}$$

where in (i) we used Lemma A.10 with $\delta = \frac{1}{3}$. Hence for showing $R \leq \widetilde{R}$ it suffices to show that

$$\frac{2\left(2\sigma_v\sigma_w\sigma_z\sqrt{k\cdot m\cdot d}+3\left\|\bar{\boldsymbol{x}}\right\|_{\ell_2}\right)}{\sigma_v\sigma_z\left(\frac{1}{2}\sqrt{k-9}-2\sqrt{m}\right)\sqrt{\frac{d}{2n}+\frac{1}{4\pi}}} \leq c\cdot\frac{\sigma_w\sqrt{k}}{\left(d\cdot\log\left(d\right)\right)^{\frac{3}{2}}},$$

which by assuming $k \geq C \cdot md$ simplifies to

$$\sigma_v\sigma_w\sigma_z\left(d\cdot\log\left(d\right)\right)^{\frac{3}{2}}\sqrt{k\cdot m\cdot d}+\left(d\cdot\log\left(d\right)\right)^{\frac{3}{2}}\left\|\bar{\boldsymbol{x}}\right\|_{\ell_2} \leq C\cdot k\sigma_v\sigma_w\sigma_z.$$

Now this last inequality holds if $k \geq C \cdot md^4\log\left(d\right)^3$ and by setting $\sigma_v\sigma_w\sigma_z \geq \frac{\|\bar{\boldsymbol{x}}\|_{\ell_2}}{md^{\frac{5}{2}}\log d^{\frac{3}{2}}}$. Therefore Assumption 2 holds for radius R defined in equation 36 with probability at least $1 - ne^{-\frac{m}{2}} - ne^{-c\cdot md^3\log(d)^2} - k\cdot e^{-c\cdot n} - e^{-\frac{m}{1500}} - e^{-\frac{kd}{162}}$. Finally by using equation 33 we have

$$\left\|\mathcal{J}\left(\boldsymbol{W}\right)\right\| \leq \frac{5}{4}\sigma_v\sigma_z\sqrt{d}\left(\sqrt{k}+2\sqrt{m}\right),$$

which shows that Assumption 3 holds with $\beta = \frac{5}{4}\sigma_v\sigma_z\sqrt{d}\left(\sqrt{k}+2\sqrt{m}\right)$ and with probability at least $1 - ne^{-\frac{m}{2}} - e^{-\frac{nd}{24}}$, finishing the proof of Theorem 2.2.

B Proofs of the auxiliary lemmas

In this section, we first provide a proof of Lemma A.6 and next go over the proofs of the key lemmas stated in Section A.4.

B.1 Proof of Lemma A.6

We note that

$$\begin{aligned} \mathcal{J}(\boldsymbol{W})\mathcal{J}(\boldsymbol{W})^T &= \frac{1}{n^2}\sum_{i,j}\left(\boldsymbol{V}diag(\phi'(\boldsymbol{W}\boldsymbol{z_i}))\text{diag}(\phi'(\boldsymbol{W}\boldsymbol{z_j})\boldsymbol{V}^T\right)\odot\left(\boldsymbol{z_i^T}\boldsymbol{z_j}\boldsymbol{1}\boldsymbol{1^T}\right) \\ &= \frac{1}{n^2}\boldsymbol{V}\underset{\ell=1,\ldots,k}{\text{diag}}\left(\left\|\sum_{i=1}^n \boldsymbol{z}_i\phi'(\boldsymbol{w}_\ell^T\boldsymbol{z}_i)\right\|_{\ell_2}^2\right)\boldsymbol{V}^T = \frac{1}{n^2}\boldsymbol{V}\cdot\boldsymbol{D}^2\cdot\boldsymbol{V}^T, \end{aligned}$$

where $\boldsymbol{D}$ is a diagonal matrix with entries

$$D_{\ell\ell} = \left\|\sum_{i=1}^n \boldsymbol{z}_i\phi'(\boldsymbol{w}_\ell^T\boldsymbol{z}_i)\right\|_{\ell_2} = \left\|\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w}_\ell)\right\|_{\ell_2}. \tag{38}$$

The matrix $\boldsymbol{Z} \in \mathbb{R}^{n\times d}$ contains the $\boldsymbol{z}_i$'s in its rows. In order to proceed we are going to analyse the entries of the diagonal matrix $\boldsymbol{D}^2$. We observe that

$$\left\|\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w})\right\|_{\ell_2}^2 = \underbrace{\left\|(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w})\right\|_{\ell_2}^2}_{A} + \underbrace{\left\|\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2}\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w})\right\|_{\ell_2}^2}_{B}.$$

First, we compute the expectation of A. We observe that

$$A = \left\|\sum_{i=1}^n(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{z}_i\phi'(\boldsymbol{w}^T\boldsymbol{z}_i)\right\|_{\ell_2}^2.$$

Conditioned on $\boldsymbol{w}$, $\left(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{\|\boldsymbol{w}\|^2}\right)\boldsymbol{z}_i$ is distributed as $\mathcal{N}\left(0,\sigma_z^2\left(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{\|\boldsymbol{w}\|^2}\right)\right)$ and $\boldsymbol{w}^T\boldsymbol{z}_i$ has distribution $\mathcal{N}\left(0,\sigma_z^2\|\boldsymbol{w}\|^2\right)$. Moreover, these two random variables are independent, because $\boldsymbol{w}$ is in the null space of $\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2}$. This observation yields

$$\begin{aligned}
\mathbb{E}(A) &= \mathbb{E}\left\|\sum_{i=1}^n(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{z}_i\phi'(\boldsymbol{w}^T\boldsymbol{z}_i)\right\|_{\ell_2}^2 \\
&= \sum_{i=1}^n\sum_{j=1}^n\mathbb{E}\left(\left\langle(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{z}_i,(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{z}_j\right\rangle\phi'(\boldsymbol{w}^T\boldsymbol{z}_i)\phi'(\boldsymbol{w}^T\boldsymbol{z}_j)\right) \\
&= \sum_{i=1}^n\mathbb{E}\left(\left\|(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{z}_i\right\|_{\ell_2}^2\right)\mathbb{E}\left(\left[\phi'(\boldsymbol{w}^T\boldsymbol{z}_j)\right]^2\right) \\
&= \sum_{i=1}^n\frac{1}{2}(d-1)\sigma_z^2 = \frac{n}{2}(d-1)\sigma_z^2.
\end{aligned}$$

Next we show that A is concentrated around its mean. Because $\left(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2}\right)\boldsymbol{z}_i$ is independent from $\boldsymbol{w}^T\boldsymbol{z}_i$, we use $\boldsymbol{z}_i'$ as an independent copy of $\boldsymbol{z}_i$. Hence, we can write

$$\begin{aligned}
A &= \left\|(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w})\right\|_{\ell_2}^2 \\
&= \left\|(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{Z}^T\phi'(\boldsymbol{Z}'\boldsymbol{w})\right\|_{\ell_2}^2 \\
&= \left\|\sum_{i=1}^n(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2})\boldsymbol{z}_i\phi'\left(\boldsymbol{w}^T\boldsymbol{z}_i'\right)\right\|_{\ell_2}^2 = \left\|\sum_{i=1}^n\boldsymbol{g}_iu_i\right\|_{\ell_2}^2,
\end{aligned}$$

where $\boldsymbol{g}_i = \left(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{\|\boldsymbol{w}\|^2}\right)\boldsymbol{z}_i \sim \mathcal{N}\left(0,\sigma_z^2\left(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{\|\boldsymbol{w}\|^2}\right)\right)$ and $u_i = \phi'\left(\boldsymbol{w}^T\boldsymbol{z}_i'\right) \sim bern(\frac{1}{2})$,[4] are all independent from each other. Note that $\|\sum_{i=1}^n\boldsymbol{g}_iu_i\|_{\ell_2}^2$ has the same distribution as $\|\boldsymbol{g}\|_{\ell_2}^2\cdot\|\boldsymbol{u}\|_{\ell_2}^2$, where $\boldsymbol{g}\sim\mathcal{N}\left(0,\sigma_z^2\left(\boldsymbol{I}-\frac{\boldsymbol{w}\boldsymbol{w}^T}{\|\boldsymbol{w}\|^2}\right)\right)$ and $\boldsymbol{u}$ is a vector with entries u_i. Note that for the norm of $\boldsymbol{u}$, the event

$$\frac{n}{2}(1-\delta)\le\|\boldsymbol{u}\|_{\ell_2}^2\le\frac{n}{2}(1+\delta)$$

holds with probability at least $1-2e^{-\frac{n\delta^2}{2}}$. Recall that for $\boldsymbol{g}\sim\mathcal{N}(0,\sigma^2\boldsymbol{I}_d)$ and $0<\delta\le\frac{1}{2}$ we have that

$$\begin{aligned}
\mathbb{P}\left(\|\boldsymbol{g}\|_{\ell_2}^2\ge(1+\delta)\mathbb{E}\left(\|\boldsymbol{g}\|_{\ell_2}^2\right)\right)&\le e^{-\frac{d\delta^2}{6}}, \\
\mathbb{P}\left(\|\boldsymbol{g}\|_{\ell_2}^2\le(1-\delta)\mathbb{E}\left(\|\boldsymbol{g}\|_{\ell_2}^2\right)\right)&\le e^{-\frac{d\delta^2}{4}}.
\end{aligned} \tag{39}$$

[4]Here, $bern(\frac{1}{2})$ means that the random variable takes values 0 and 1 each with probability $1/2$.

By applying the union bound and noting that $\mathbb{E}\left(\|\boldsymbol{g}\|_{\ell_2}^2\right) = (d-1)\,\sigma_z^2$, for $0 < \delta \leq \frac{3}{2}$, we obtain that the event

$$|\,\|\boldsymbol{g}\|_{\ell_2}^2\,\|\boldsymbol{u}\|_{\ell_2}^2 - \frac{n}{2}(d-1)\sigma_z^2| \leq \delta\frac{n}{2}(d-1)\sigma_z^2$$

holds with probability at least $1 - 2e^{-\frac{n\delta^2}{18}} - 2e^{-\frac{d\delta^2}{54}}$.
In order to analyse B, we note first that

$$\begin{aligned} B = \left\|\frac{\boldsymbol{w}\boldsymbol{w}^T}{||\boldsymbol{w}||^2}\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w})\right\|_{\ell_2}^2 &= \left|\frac{\boldsymbol{w}^T}{||\boldsymbol{w}||}\boldsymbol{Z}^T\phi'(\boldsymbol{Z}\boldsymbol{w})\right|^2 \\ &= \left|\left\langle \boldsymbol{Z}\frac{\boldsymbol{w}}{||\boldsymbol{w}||}, \phi'(\boldsymbol{Z}\boldsymbol{w})\right\rangle\right|^2 \\ &= \left|\,\langle \boldsymbol{g}, \phi'(||\boldsymbol{w}||\boldsymbol{g})\rangle\,\right|^2 \\ &= \left|\sum_{i=1}^n \boldsymbol{g}_i \cdot \mathbb{1}_{(\boldsymbol{g}_i \geq 0)}\right|^2 = \left(\sum_{i=1}^n ReLU(\boldsymbol{g}_i)\right)^2, \end{aligned}$$

where $\boldsymbol{g}_i = \boldsymbol{z}_i^T \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} \sim \mathcal{N}\left(0, \sigma_z^2\right)$. It follows that

$$\begin{aligned} \mathbb{E}\left(B\right) &= \mathbb{E}\left(\sum_{i=1}^n ReLU(\boldsymbol{g}_i)\right)^2 \\ &= \sum_{i=1}^n \mathbb{E}\Big(ReLU^2(\boldsymbol{g}_i)\Big) + \sum_{i\neq j} \mathbb{E}\Big(ReLU(\boldsymbol{g}_i)ReLU(\boldsymbol{g}_j)\Big) \\ &= \sigma_z^2\left(\frac{n}{2} + \frac{n(n-1)}{2\pi}\right). \end{aligned}$$

Next, in order to show that B concentrates around its mean, we note that $ReLU\left(\boldsymbol{g}_i\right)$ is a sub-Gaussian random variable with ψ_2-norm $C\sigma_z$, where C is a fixed constant. Therefore $X = \sum_{i=1}^n ReLU\left(\boldsymbol{g}_i\right)$ is sub-Gaussian with ψ_2-norm $C\sqrt{n}\sigma_z$. By the sub-exponential tail bound for $X^2 - \mathbb{E}(X^2)$ we obtain that

$$\mathbb{P}\left(|B - \mathbb{E}\left(B\right)| \geq t\right) \leq 2e^{-c\frac{t}{n\sigma_z^2}}.$$

Finally, by putting these results together, we obtain that

$$\mathbb{E}\left(\boldsymbol{D}_{\ell\ell}^2\right) = \mathbb{E}\left(A\right) + \mathbb{E}\left(B\right) = \sigma_z^2\left(\frac{nd}{2} + \frac{n(n-1)}{2\pi}\right), \quad 1 \leq \ell \leq k.$$

The union bounds yields that

$$\mathbb{P}\left\{\left|\boldsymbol{D}_{\ell\ell}^2 - \mathbb{E}\left(\boldsymbol{D}_{\ell\ell}^2\right)\right| \geq \delta\mathbb{E}\left(\boldsymbol{D}_{\ell\ell}^2\right)\right\} \leq 2e^{-\frac{n\delta^2}{18}} + 2e^{-\frac{d\delta^2}{54}} + 2e^{-c_1 n\delta}, \quad 0 \leq \delta \leq \frac{3}{2}.$$

This finishes the proof.

B.2 Proof of lemma A.7

Our main tool for bounding the minimum singular value of the Jacobian mapping will be the following lemma from Soltanolkotabi (2019):

Lemma B.1 *Let $\boldsymbol{d} \in \mathbb{R}^k$ be a fixed vector with nonzero entries and $\boldsymbol{D} = diag\left(\boldsymbol{d}\right)$. Also, let $\boldsymbol{A} \in \mathbb{R}^{k\times m}$ have i.i.d. $\mathcal{N}\left(0,1\right)$ entries and $\mathcal{T} \subseteq \mathbb{R}^m$. Define*

$$b_k\left(\boldsymbol{d}\right) = \mathbb{E}\left[\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2}\right],$$

where $\boldsymbol{g} \sim \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{I}_k\right)$. Also define

$$\sigma\left(\mathcal{T}\right) := \max_{\boldsymbol{v}\in\mathcal{T}} \|\boldsymbol{v}\|_{\ell_2}.$$

Then for all $\boldsymbol{u} \in \mathcal{T}$ *we have*

$$\left| \|\boldsymbol{D}\boldsymbol{A}\boldsymbol{u}\|_{\ell_2} - b_k(\boldsymbol{d}) \|\boldsymbol{u}\|_{\ell_2} \right| \leq \|\boldsymbol{d}\|_{\ell_\infty} \omega(\mathcal{T}) + \eta$$

with probability at least $1 - 6e^{\frac{-\eta^2}{8\|\boldsymbol{d}\|_{\ell_\infty}^2 \sigma^2(\mathcal{T})}}$.

In order to apply this lemma, we set the elements of $\boldsymbol{d}$ to be $D_{\ell\ell}$ as in equation 38 and choose $\mathcal{T} = S^{m-1}$ and $\boldsymbol{A} = \boldsymbol{V}^T \in \mathbb{R}^{k \times m}$ with $\mathcal{N}(0, \sigma_v^2)$ entries. It follows that

$$b_k(\boldsymbol{d}) = \mathbb{E}\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2} = \sqrt{\mathbb{E}\left(\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2}^2\right) - \text{Var}\left(\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2}\right)},$$

where

$$\mathbb{E}\left(\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2}^2\right) = \|\boldsymbol{d}\|_{\ell_2}^2 = \sum_{\ell=1}^{k} \boldsymbol{D}_{\ell\ell}^2.$$

We are going to use the fact that for a B-Lipschitz function ϕ and normal random variable $g \sim \mathcal{N}(0,1)$, based on the Poincare inequality (Ledoux, 2001) we have $\text{Var}(\phi(g)) \leq B^2$. By noting that for a diagonal matrix $\boldsymbol{D}$

$$\left| \|\boldsymbol{D}x\|_{\ell_2} - \|\boldsymbol{D}y\|_{\ell_2} \right| \leq \|\boldsymbol{D}\boldsymbol{x} - \boldsymbol{D}\boldsymbol{y}\|_{\ell_2} \leq \|\boldsymbol{d}\|_{\ell_\infty} \|\boldsymbol{x} - \boldsymbol{y}\|_{\ell_2},$$

we get

$$\begin{aligned}\mathbb{E}\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2} &= \sqrt{\mathbb{E}\Big(\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2}^2\Big) - \text{Var}(\|\boldsymbol{D}\boldsymbol{g}\|_{\ell_2})} \\ &\geq \sqrt{\|\boldsymbol{d}\|_{\ell_2}^2 - \|\boldsymbol{d}\|_{\ell_\infty}^2},\end{aligned}$$

Combined with $\omega\left(S^{m-1}\right) \leq \sqrt{m}$ and Lemma B.1 this yields that the event

$$\sigma_{min}(\boldsymbol{V}\boldsymbol{D}) \geq \sigma_v \left(\sqrt{\|\boldsymbol{d}\|_{\ell_2}^2 - \|\boldsymbol{d}\|_{\ell_\infty}^2} - \|\boldsymbol{d}\|_{\ell_\infty} \sqrt{m} - \eta \right) \tag{40}$$

holds with probability at least $1 - 3e^{\frac{-\eta^2}{8\|\boldsymbol{d}\|_{\ell_\infty}^2}}$.

Next, using the concentration bound for $\boldsymbol{D}_{\ell\ell}^2$, which we obtained in Section B.1, we bound $\|\boldsymbol{d}\|_{\ell_2}^2$ and $\|\boldsymbol{d}\|_{\ell_\infty}$, where we have set $\boldsymbol{d}_i = \boldsymbol{D}_{ii}$ for $1 \leq i \leq k$. For $0 \leq \delta \leq \frac{3}{2}$ we compute that

$$\begin{aligned}\mathbb{P}\left(\max_{1 \leq i \leq k} \boldsymbol{d}_i \geq (1+\delta)\sqrt{\mathbb{E}\left[\boldsymbol{d}_i^2\right]}\right) &= \mathbb{P}\left(\bigcup_{i=1}^{k} \boldsymbol{d}_i^2 \geq (1+\delta)^2 \mathbb{E}\left[\boldsymbol{d}_i^2\right]\right) \\ &\leq k \cdot \mathbb{P}\left(\boldsymbol{d}_i^2 \geq (1+\delta)^2 \mathbb{E}\left[\boldsymbol{d}_i^2\right]\right) \\ &\leq k \cdot \mathbb{P}\left(\boldsymbol{d}_i^2 \geq (1+\delta) \mathbb{E}\left[\boldsymbol{d}_i^2\right]\right) \leq k \cdot \left(e^{-\frac{n\delta^2}{18}} + e^{-\frac{d\delta^2}{54}} + e^{-c_1 n\delta}\right),\end{aligned} \tag{41}$$

as well as

$$\begin{aligned}\mathbb{P}\left(\|\boldsymbol{d}\|_{\ell_2} \leq (1-\delta)\sqrt{k}\sqrt{\mathbb{E}\left[\boldsymbol{d}_i^2\right]}\right) &\leq \mathbb{P}\left(\bigcup_{i=1}^{k} \boldsymbol{d}_i^2 \leq (1-\delta)^2 \mathbb{E}\left[\boldsymbol{d}_i^2\right]\right) \\ &\leq k \cdot \mathbb{P}\left(\boldsymbol{d}_i^2 \leq (1-\delta)^2 \mathbb{E}\left[\boldsymbol{d}_i^2\right]\right) \\ &\leq k \cdot \mathbb{P}\left(\boldsymbol{d}_i^2 \leq (1-\delta) \mathbb{E}\left[\boldsymbol{d}_i^2\right]\right) \leq k \cdot \left(e^{-\frac{n\delta^2}{18}} + e^{-\frac{d\delta^2}{54}} + e^{-c_1 n\delta}\right).\end{aligned} \tag{42}$$

Finally by replacing η with $\eta \|\boldsymbol{d}\|_{\ell_\infty} \sqrt{m}$ in equation 40, combined with equation 41 and equation 42, for a random $\boldsymbol{W}_0$ with i.i.d. $\mathcal{N}\left(0, \sigma_w^2\right)$ entries we have:

$$\begin{aligned}\sigma_{min}(\mathcal{J}(\boldsymbol{W}_0)) &= \frac{1}{n}\sigma_{min}(\boldsymbol{V}\boldsymbol{D}) \\ &\geq \frac{\sigma_v}{n}\left(\sqrt{(1-\delta)^2 k - (1+\delta)^2} - \sqrt{m}(1+\eta)(1+\delta)\right)\sqrt{\mathbb{E}\left[\boldsymbol{d}_i^2\right]} \\ &\geq \left(\sqrt{(1-\delta)^2 k - (1+\delta)^2} - \sqrt{m}(1+\eta)(1+\delta)\right)\sigma_v \sigma_z \sqrt{\frac{d}{2n} + \frac{1}{4\pi}}, \quad 0 \leq \delta \leq \frac{3}{2},\end{aligned}$$

with probability at least $1-3e^{-\frac{\eta^2 m}{8}}-2k\cdot\left(e^{-\frac{n\delta^2}{18}}+e^{-\frac{d\delta^2}{54}}+e^{-c_1 n\delta}\right)$. This completes the proof of Lemma A.7.

B.3 Proof of Lemma A.10

First, note that if $\boldsymbol{W}$ has i.i.d. $\mathcal{N}\left(0,\sigma_w^2\right)$ entries and $\boldsymbol{V},\boldsymbol{W},\boldsymbol{Z}$ are all independent, then $\|f\left(\boldsymbol{W}\right)\|_{\ell_2}=\frac{1}{n}\left\|\boldsymbol{V}\phi\left(\boldsymbol{W}\boldsymbol{Z}^T\right)\boldsymbol{1}_{n\times 1}\right\|_{\ell_2}$ has the same distribution as $\|\boldsymbol{v}\|_{\ell_2}\|\boldsymbol{a}\|_{\ell_2}$, where $\boldsymbol{v}\sim\mathcal{N}\left(0,\sigma_v^2\boldsymbol{I}_m\right)$ and $\boldsymbol{a}=\frac{1}{n}\phi\left(\boldsymbol{W}\boldsymbol{Z}^T\right)\boldsymbol{1}$ has independent sub-Gaussian entries, so its ℓ_2-norm is concentrated. Note that conditioned on $\boldsymbol{W}$, $a_i=\frac{1}{n}\sum_{j=1}^n ReLU\left(\boldsymbol{z}_j^T\boldsymbol{w}_i\right)$ is sub-Gaussian with $\|a_i\|_{\psi_2}=C\frac{\|\boldsymbol{w}_i\|_{\ell_2}\sigma_z}{\sqrt{n}}$, and it is concentrated around $\mathbb{E}a_i=\frac{1}{\sqrt{2\pi}}\|\boldsymbol{w}_i\|_{\ell_2}\sigma_z$. This gives

$$\mathbb{P}\left\{a_i\le(1+\delta)\mathbb{E}a_i\right\}\ge 1-e^{-c\frac{\delta^2(\mathbb{E}a_i)^2}{\|a_i\|_{\psi_2}^2}}=1-e^{-cn\delta^2},$$

which implies that

$$\mathbb{P}\left\{a_i^2\le(1+3\delta)(\mathbb{E}a_i)^2\right\}\ge\mathbb{P}\left\{a_i^2\le(1+\delta)^2(\mathbb{E}a_i)^2\right\}\ge 1-e^{-cn\delta^2},\quad 0\le\delta\le 1.$$

Due to the union bound we get that

$$\begin{aligned}\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}^2\ge(1+\delta)\sum_{i=1}^k(\mathbb{E}a_i)^2\right\}&\le\mathbb{P}\left\{\bigcup_{i=1}^k a_i^2\ge(1+\delta)\left(\mathbb{E}a_i\right)^2\right\}\\&\le\sum_{i=1}^k\mathbb{P}\left\{a_i^2\ge(1+\delta)\left(\mathbb{E}a_i\right)^2\right\}\le k\cdot e^{-cn(\delta/3)^2},\quad 0\le\delta\le 3.\end{aligned}$$

By substituting $\sum_{i=1}^k(\mathbb{E}a_i)^2=\frac{1}{2\pi}\sigma_z^2\|\boldsymbol{W}\|_F^2$ this shows

$$\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}\le(1+\delta)\frac{1}{\sqrt{2\pi}}\sigma_z\|\boldsymbol{W}\|_F\right\}\ge\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}^2\le(1+\delta)\frac{1}{2\pi}\sigma_z^2\|\boldsymbol{W}\|_F^2\right\}\ge 1-k\cdot e^{-cn(\delta/3)^2},\quad 0\le\delta\le 3.$$

We also have the following result for $\boldsymbol{v}\sim\mathcal{N}(0,\sigma_v^2\boldsymbol{I}_m)$

$$\mathbb{P}\left\{\|\boldsymbol{v}\|_{\ell_2}\le(1+\delta)\sigma_v\sqrt{m}\right\}\ge 1-e^{-\frac{\delta^2 m}{2}}.$$

By combining the above results we obtain

$$\begin{aligned}\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}\|\boldsymbol{v}\|_{\ell_2}\le(1+\delta)\frac{1}{\sqrt{2\pi}}\sigma_v\sigma_z\sqrt{m}\|\boldsymbol{W}\|_F\right\}&\ge\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}\|\boldsymbol{v}\|_{\ell_2}\le(1+\delta/3)^2\frac{1}{\sqrt{2\pi}}\sigma_v\sigma_z\sqrt{m}\|\boldsymbol{W}\|_F\right\}\\&\ge 1-k\cdot e^{-cn(\delta/9)^2}-e^{-\frac{(\delta/3)^2 m}{2}},\quad 0\le\delta\le 3.\end{aligned}$$

Furthermore, we can bound $\|\boldsymbol{W}\|_F$ by the tail inequality

$$\mathbb{P}\left\{\|\boldsymbol{W}\|_F\le(1+\delta)\sigma_w\sqrt{kd}\right\}\ge 1-e^{-\frac{\delta^2 kd}{2}}.$$

Hence, by combining the last two results we have that

$$\begin{aligned}\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}\|\boldsymbol{v}\|_{\ell_2}\le(1+\delta)\frac{1}{\sqrt{2\pi}}\sigma_v\sigma_z\sigma_w\sqrt{k\cdot d\cdot m}\right\}&\ge\mathbb{P}\left\{\|\boldsymbol{a}\|_{\ell_2}\|\boldsymbol{v}\|_{\ell_2}\le(1+\delta/3)^2\frac{1}{\sqrt{2\pi}}\sigma_v\sigma_z\sigma_w\sqrt{k\cdot d\cdot m}\right\}\\&\ge 1-k\cdot e^{-cn(\delta/27)^2}-e^{-\frac{(\delta/9)^2 m}{2}}-e^{-\frac{(\delta/3)^2 kd}{2}},\quad 0\le\delta\le 3.\end{aligned}$$

Therefore, due to the triangle inequality the event

$$\|f(\boldsymbol{W}_0)-\bar{\boldsymbol{x}}\|_{\ell_2}\le(1+\delta)\frac{1}{\sqrt{2\pi}}\sigma_v\sigma_w\sigma_z\sqrt{k\cdot d\cdot m}+\|\bar{\boldsymbol{x}}\|_{\ell_2},\quad 0\le\delta\le 3$$

holds with probability at least $1-k\cdot e^{-c_2 n(\delta/27)^2}-e^{-\frac{(\delta/9)^2 m}{2}}-e^{-\frac{(\delta/3)^2 kd}{2}}$ for some positive constant c_2, completing the proof of Lemma A.10.

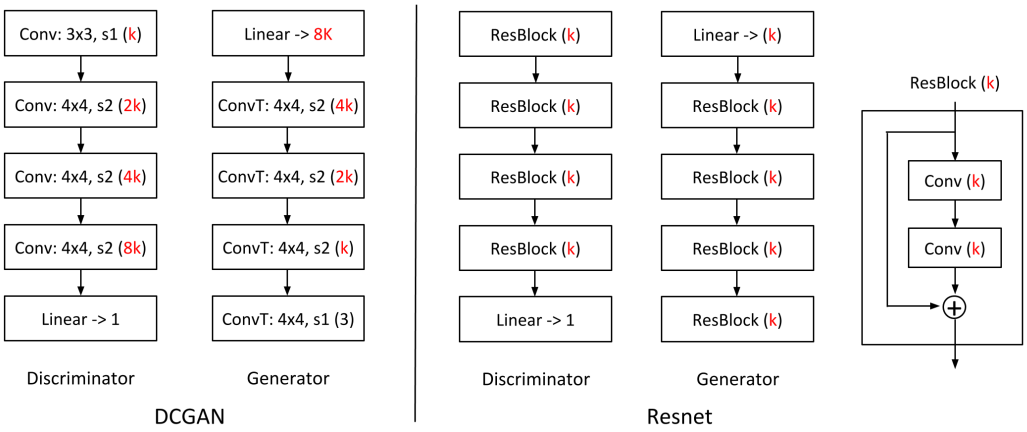

Figure 6: **Architectures used in over-parameterization experiments.** The number of out-channels in convolutional layers is indicated in red. Parameter k controls the width of the architectures – larger the k, more over-parameterized the models are.

C EXPERIMENTAL DETAILS

The model architectures we use this in the experiments are shown in Figure 6. In both DCGAN and Resnet-based GANs, the papemeter k controls the number of convolutional filters in each layer. The larger the value of k is, the more over-parameterized the models are.

Optimization: Both DCGAN and Resnet-based GAN models are optimized using the commonly used hyper-parameters: Adam with learning rate 0.0001 and betas $(0.5, 0.999)$ for DCGAN, gradient penalty of 10 and 5 critic iterations per generator's iteration for both DCGAN and Resnet-based GAN models. Models are trained for $300,000$ iterations with a batch size of 64.