AQUILA: COMMUNICATION EFFICIENT FEDERATED LEARNING WITH ADAPTIVE QUANTIZATION OF LAZILY-AGGREGATED GRADIENTS

Anonymous authors
Paper under double-blind review

ABSTRACT

The development and deployment of federated learning (FL) have been bottlenecked by heavy communication overheads of high-dimensional models between the distributed device nodes and the central server. To achieve better error-communication trade-offs, recent efforts have been made to either adaptively reduce the communication frequency by skipping unimportant updates, e.g., lazy aggregation, or adjust the quantization bits for each communication. In this paper, we propose a unifying communication efficient framework for FL based on **a**daptive **qu**ant**i**zation of **l**azily-**a**ggregated gradients (**AQUILA**), which adaptively balances two mutually-dependent factors, the communication frequency, and the quantization level, in a synergistic way. Specifically, we start from a careful investigation of the classical lazy aggregation scheme and formulate AQUILA as an optimization problem where the optimal quantization level is selected by minimizing the model deviation caused by update skipping. Furthermore, we create a new lazy aggregation strategy to fit the novel quantization criterion better and thus keep the communication frequency at an appropriate level. The effectiveness and convergence of the proposed AQUILA framework are theoretically verified. The experimental results demonstrate that AQUILA can reduce around **60%** of overall transmitted bits compared to existing methods while achieving the same model performance in a number of non-homogeneous FL scenarios, including Non-IID data and heterogeneous model architecture. The proposed AQUILA is highly adaptive and compatible with existing FL settings.

1 INTRODUCTION

With the deployment of ubiquitous sensing and computing devices, the Internet of things (IoT) as well as many other distributed systems have gradually grown from concept to reality, bringing dramatic convenience to people's daily life (Du et al., 2020; Liu et al., 2020; Hard et al., 2018). To fully utilize such distributed computing resources, distributed learning provides a promising framework that can achieve comparable performance with the traditional centralized learning scheme. However, the privacy and security of sensitive data during the updating and transmission processes in distributed learning have been a growing concern. In this context, *federated learning (FL)* (McMahan et al., 2017) has been developed, allowing distributed devices to collaboratively learn a global model without privacy leakage by keeping private data isolated and masking transmitted information with secure approaches. On account of its privacy-preserving property and great potentiality in some distributed but privacy-sensitive fields such as finance and health, FL has attracted tremendous attention from both academia and industry in recent years.

Unfortunately, in many FL applications, such as image classification and objective recognition, the trained model tends to be high-dimensional, resulting in significant communication costs. Hence, communication efficiency has become one of the key bottlenecks of FL. To this end, Sun et al. (2020) proposes the lazily-aggregated quantization (LAQ) method to skip some unnecessary parameter uploads by estimating the value of gradient innovation — the difference between the current unquantized gradient and the previously quantized gradient. Moreover, Mao et al. (2021) devises an adaptive quantized gradient (AQG) strategy based on LAQ to dynamically select the quantization level within some artificially given numbers during the training process. Nevertheless, the AQG is

still not sufficiently adaptive because the pre-determined quantization levels are difficult to choose in complicated FL environments. In another separate line of work, Jhunjhunwala et al. (2021) introduces an adaptive quantization rule for FL (AdaQuantFL), which searches in a given range for an optimal quantization level and achieves a better error-communication trade-off.

Most previous research has investigated how to optimize communication frequency or adjust the quantization level in a highly adaptive fashion, but not both. Intuitively, we ask the question, *can we adaptively adjust the quantization level in LAQ to reduce communication rounds further and transmitted bits simultaneously?* A straightforward approach is to train LAQ in conjunction with adaptive quantization methods such as AdaQuantFL. However, the quantization level and the communication frequency are mutually dependent on establishing the convergence of the model and must be adjusted together. For example, lazy aggregation leads to the skip of some gradients that no longer require quantization, while the choice of quantization levels directly affects the quality of a gradient update and thus whether it is selected for transmission and aggregation. Therefore, the key novelty of this literature is to jointly leverage these two complementary yet mutually-dependent degrees of freedom for further optimizing communication efficiency in FL.

In the paper, we intend to select the **optimal quantization level** for every participated device by optimizing the model deviation caused by skipping quantized gradient updates (i.e., lazy aggregation), which gives us a novel quantization criterion cooperated with a new proposed **lazy aggregation strategy** to further reduce overall communication costs while still offering a convergence guarantee. The contributions of this paper are trifold.

1. We propose an innovative FL procedure with **a**daptive **qu**ant**i**zation of **l**azily-**a**ggregated gradients termed AQUILA, which simultaneously adjusts the communication frequency and quantization level in a synergistic fashion.
2. Instead of naively combining LAQ and AdaQuantFL, AQUILA owns a completely different device selection method and quantitative level calculation method. Specifically, we present a new lazy aggregation criterion that is more precise and saves more device storage. Subsequently, we derive an adaptive quantization strategy from a new perspective that minimizes the model deviation introduced by lazy aggregation. Furthermore, we provide a convergence analysis of AQUILA under both the generally non-convex case and Polyak-Łojasiewicz condition.
3. Except for normal FL settings such as independent and identically distributed (IID) data environment, we experimentally evaluate the performance of AQUILA in a number of **non-homogeneous** FL settings, such as non-independent and non-identically distributed (Non-IID) local dataset and various heterogeneous model aggregations. The evaluation results reveal that AQUILA considerably mitigates the communication overhead compared to a variety of state-of-art algorithms.

Notations: Bold fonts denote vectors (e.g., $\boldsymbol{\theta}$). Normal fonts denote scalars (e.g., α). Subscript m is used to describe functions about a local client m (e.g., $f_m(\boldsymbol{\theta})$). A function without subscript is used to describe an average among all clients (e.g., $f(\boldsymbol{\theta})$).

2 Background and Related Work

Consider an FL system with one central parameter server and a device set $\mathcal{M}$ with $M = |\mathcal{M}|$ distributed devices to collaboratively train a global model parameterized by $\boldsymbol{\theta} \in \mathbb{R}^d$. Each device $m \in \mathcal{M}$ has a private local dataset $\mathcal{D}_m = \{(\boldsymbol{x}_1^{(m)}, \boldsymbol{y}_1^{(m)}), \cdots, (\boldsymbol{x}_{n_m}^{(m)}, \boldsymbol{y}_{n_m}^{(m)})\}$ of n_m samples. The federated training process is typically performed by solving the following optimization problem

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^d} f(\boldsymbol{\theta}) = \frac{1}{M} \sum_{m=1}^{M} f_m(\boldsymbol{\theta}) \quad \text{with} \quad f_m(\boldsymbol{\theta}) = \left[l\left(h_{\boldsymbol{\theta}}(\boldsymbol{x}), \boldsymbol{y}\right) \right]_{(\boldsymbol{x},\boldsymbol{y}) \sim \mathcal{D}_m}, \tag{1}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ denotes the empirical risk, $f_m : \mathbb{R}^d \to \mathbb{R}$ denotes the local objective based on the private data $\mathcal{D}_m$ of the device m, l denotes the local loss function, and $h_{\boldsymbol{\theta}}$ denotes the local model. The FL training process is conducted by iteratively performing local updates and global aggregation as proposed in (McMahan et al., 2017). For instance, at communication round k, first each device m

receives the global model $\boldsymbol{\theta}^k \in \mathbb{R}^d$ from the parameter server and trains it with its local data D_m. Subsequently, it sends the local gradient $\nabla f_m(\boldsymbol{\theta}^k)$ to the central server, and the server will update the global model with learning rate α by

$$\boldsymbol{\theta}^{k+1} := \boldsymbol{\theta}^k - \frac{\alpha}{M} \sum_{m \in \mathcal{M}} \nabla f_m(\boldsymbol{\theta}^k). \tag{2}$$

For communication frequency reduction, the **lazy aggregation** strategy allows the device $m \in \mathcal{M}$ upload its newly-quantized gradient innovation $\Delta \boldsymbol{q}_m^k = \mathcal{Q}(\nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1})$ at epoch k only when the change in local gradient is sufficiently larger than a threshold, i.e.,

$$\boldsymbol{q}_m^k := \begin{cases} \boldsymbol{q}_m^{k-1}, & \text{if } \left\|\mathcal{Q}(\nabla f_m\left(\theta^k\right) - \boldsymbol{q}_m^{k-1})\right\|_2^2 \leqslant \text{Threshold} \\ \boldsymbol{q}_m^{k-1} + \Delta \boldsymbol{q}_m^k, & \text{otherwise} \end{cases}, \tag{3}$$

where $\boldsymbol{q}_m^k$ denotes the quantization of the local gradient of client m at epoch k, and $\mathcal{Q}(\boldsymbol{a})$ denotes the quantization operation on $\boldsymbol{a} \in \mathbb{R}^d$.

Definition 2.1 (Quantization error). *The global quantization error $\boldsymbol{\varepsilon}^k$ is defined by the subtraction between the current unquantized gradient $\nabla f(\boldsymbol{\theta}^k)$ and its quantized value $\boldsymbol{q}^{k-1} + \Delta \boldsymbol{q}^k$, i.e.,*

$$\boldsymbol{\varepsilon}^k = \nabla f(\boldsymbol{\theta}^k) - \boldsymbol{q}^{k-1} - \Delta \boldsymbol{q}^k, \tag{4}$$

where $\nabla f(\boldsymbol{\theta}^k) = \sum_{m \in \mathcal{M}} \nabla f_m(\boldsymbol{\theta}^k)$, $\boldsymbol{q}^{k-1} = \sum_{m \in \mathcal{M}} \boldsymbol{q}_m^{k-1}$, $\Delta \boldsymbol{q}^k = \sum_{m \in \mathcal{M}} \Delta \boldsymbol{q}_m^k$.

Particularly, the threshold in LAQ is calculated by the summation of the accumulated model difference and quantization error:

$$Threshold_{LAQ} = \frac{1}{\alpha^2 M^2} \sum_{d'=1}^{D} \xi_{d'} \left\|\boldsymbol{\theta}^{k+1-d'} - \boldsymbol{\theta}^{k-d'}\right\|_2^2 + 3(\left(\left\|\boldsymbol{\varepsilon}_m^k\right\|_2^2 + \left\|\hat{\boldsymbol{\varepsilon}}_m^{k-1}\right\|_2^2\right)). \tag{5}$$

Please refer to Sun et al. (2020) for more details on (5). In LAQ, if the quantized gradient innovation $\Delta \boldsymbol{q}_m^k$ is smaller than a threshold involving quantization errors and the global model's innovation, the device m will skip the upload of $\Delta \boldsymbol{q}_m^k$ at communication round k whilst the central server will reuse $\boldsymbol{q}_m^{k-1}$ for lazy aggregation. Therefore, the global model update rule of LAQ can be changed from (2) to:

$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k - \frac{\alpha}{M} \sum_{m \in \mathcal{M}} \boldsymbol{q}_m^k = \boldsymbol{\theta}^k - \frac{\alpha}{M} \sum_{m \in \mathcal{M}^k} \left(\boldsymbol{q}_m^{k-1} + \Delta \boldsymbol{q}_m^k\right) - \frac{\alpha}{M} \sum_{m \in \mathcal{M}_c^k} \boldsymbol{q}_m^{k-1}, \tag{6}$$

where $\mathcal{M}^k$ denotes the subset of devices that upload their quantized gradient innovation, while $\mathcal{M}_c^k = \mathcal{M} \setminus \mathcal{M}^k$ denotes the subset of devices that skip the gradient update and reuse the old quantized gradient at epoch k.

For AdaQuantFL, it is proposed to achieve a better error-communication trade-off by adaptively adjusting the quantization levels during the FL training process. Specifically, AdaQuantFL computes the optimal quantization level $(b^k)^*$ at epoch k based on the following criterion involving the pre-determined initial quantization level b_0:

$$(b^k)^* = \left\lfloor \sqrt{f\left(\boldsymbol{\theta}^0\right) / f\left(\boldsymbol{\theta}^k\right)} \cdot b_0 \right\rfloor, \tag{7}$$

where $f(\boldsymbol{\theta}^0)$ and $f(\boldsymbol{\theta}^k)$ are the global objective loss defined in (1) and calculated with the global model broadcast at the beginning of epoch 0 and k, respectively.

However, AdaQuantFL transmits quantized gradients at **every communication round**. In order to skip unnecessary communication rounds and adaptively adjust the quantization level for each communication jointly, a naive approach is to quantize lazily aggregated gradients with AdaQuantFL. Nevertheless, it fails to achieve efficient communication due to a number of reasons. Firstly, given the descending trend of training loss, AdaQuantFL's criterion (7) may lead to a high quantization bit number even exceeding 32 bits in the training process (assuming a floating point is represented by 32 bits in our case), which is too large for cases where the global convergence is already approaching

and makes the quantification meaningless. Second, a higher quantization level results in a smaller quantization error, leading to a lower communication threshold in the lazy aggregation criterion (3) and thus a higher frequency of transmission.

Consequently, it is desirable to develop a more efficient adaptive quantization method in the lazily-aggregated setting to systematically improve communication efficiency in FL.

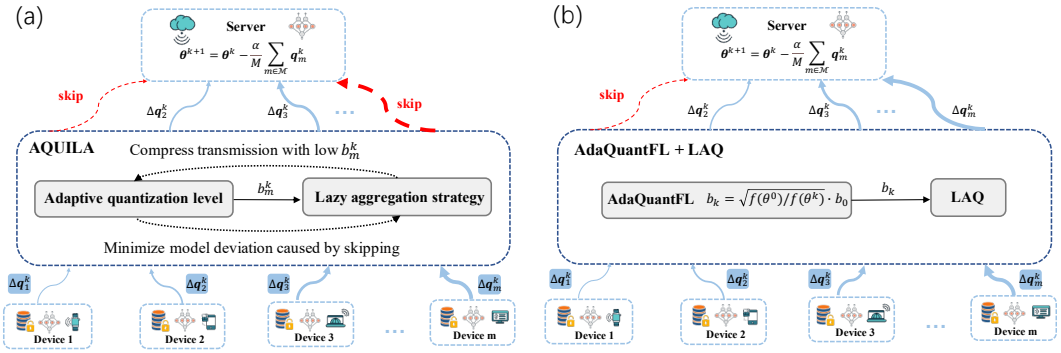


Figure 1: The schematic illustration of the communication-efficient FL with AQUILA in comparison with the naive combination of AdaQuantFL and LAQ. The blue lines indicating the transmission of quantized gradient innovation in AQUILA are drawn in different thicknesses to represent various size of quantized gradient innovation, considering the heterogeneous FL environment as in our evaluation part. For instance, the gradient innovation $\Delta \boldsymbol{q}_3^k$ of a PC is larger than $\Delta \boldsymbol{q}_2^k$ of a mobile phone.

3 Adaptive Quantization of Lazily-Aggregated Gradients

Given the above limitations of the naive joint use of the existing adaptive quantization criterion and lazy aggregation strategy, this paper aims to design a unifying procedure for communication efficiency optimization where the quantization level and communication frequency are considered in a synergistic and interactive way.

Definition 3.1. (Deterministic mid-tread quantizer.) *In order to better compare with LAQ, we utilize the quantizer in LAQ, i.e., every element of the gradient innovation of device m at epoch k is mapped to an integer $\left[\boldsymbol{\psi}_m^k\right]_i$ as*

$$\left[\boldsymbol{\psi}_m^k\right]_i = \left\lfloor \frac{\left[\nabla f_m(\boldsymbol{\theta}^k)\right]_i - \left[\boldsymbol{q}_m^{k-1}\right]_i + R_m^k}{2\tau_m^k R_m^k} + \frac{1}{2} \right\rfloor, \forall i \in \{1, 2, ..., d\}, \tag{8}$$

where $\nabla f(\boldsymbol{\theta}_m^k)$ denotes the current unquantized gradient, $R_m^k = \left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_\infty$ denotes the quantization range, b_m^k denotes the quantization level, and $\tau_m^k := 1/\left(2^{b_m^k} - 1\right)$ denotes the quantization granularity.

From Definition 3.1, we can derive that the relationship between **quantized** gradient innovation $\Delta \boldsymbol{q}_m^k$ and its quantization representation $\boldsymbol{\psi}_m^k$ which applies b_m^k bits for each dimension:

$$\Delta \boldsymbol{q}_m^k = 2\tau_m^k R_m^k \boldsymbol{\psi}_m^k - R_m^k \mathbf{1}, \tag{9}$$

where $\mathbf{1} \in \mathbb{R}^d$ denotes a vector filled with scalar value 1. Note that the quantization level b_m^k of LAQ is always fixed. In this paper, based on a careful investigation of lazy aggregation, we design a new quantization criterion to optimize the model deviation between *unbiased aggregated model* and *lazily aggregated model* in terms of quantization bits as elaborated in (31). It gives AQUILA a adaptive strategy (10) that selects the optimal quantization level for the device m at epoch k based on the quantization range $R_m^k = \left\| \nabla f_m(\boldsymbol{\theta}^k) - \boldsymbol{q}_m^{k-1} \right\|_\infty$, the current gradient $\nabla f_m\left(\boldsymbol{\theta}^k\right)$, and the last

uploaded quantized gradient $\boldsymbol{q}_m^{k-1}$:

$$(b_m^k)^* = \left\lfloor \log_2 \left(\frac{R_m^k \sqrt{d}}{\left\| \nabla f_m \left(\boldsymbol{\theta}^k \right) - \boldsymbol{q}_m^{k-1} \right\|_2} + 1 \right) \right\rfloor . \tag{10}$$

The superiority of (10) comes from the following two aspects. Firstly, since $R_m^k \geqslant \left[\nabla f_m \left(\boldsymbol{\theta}^k \right) \right]_i - \left[\boldsymbol{q}_m^{k-1} \right]_i \geqslant -R_m^k$, the optimal quantization level $(b_m^k)^*$ is greater than or equal to 1. In particular, if all elements of $\boldsymbol{q}_m^{k-1}$ and $\nabla f_m \left(\boldsymbol{\theta}^k \right)$ are equal (which is basically impossible), we will set $(b_m^k)^* = 1$ for rationality. Secondly, the gradient innovation and quantization range R_m^k tend to fluctuate along with the training process instead of keeping descending such as the objective loss value, and thus prevent the quantization level from increasing tremendously compared with AdaQuantFL.

To better fit the larger quantization errors induced by fewer quantization bits in (10), AQUILA possesses a new communication criterion to avoid the potential expansion of the devices group being skipped:

$$\left\| \Delta \boldsymbol{q}_m^k \right\|_2^2 + \gamma \left\| \boldsymbol{\varepsilon}_m^k \right\|_2^2 \leqslant \frac{\beta}{\alpha^2} \left\| \boldsymbol{\theta}^{k+1} - \boldsymbol{\theta}^k \right\|_2^2, \forall m \in \mathcal{M}_c^k, \tag{11}$$

where $\beta \geqslant 0$ is a tuning factor. Furthermore, we provide an intuitive method to choose the value of β in Appendix G and set $\beta = 0.25$ in our experiment part.

The strength of (11) is that AQUILA directly utilizes the global model for two adjacent rounds as the skip condition instead of storing a large number of previous model parameters as LAQ, which considerably reduces the storage pressure of local devices. This is especially important for small-capacity devices (e.g., sensors) in real IoT scenarios.

The cooperation of the novel adaptive quantization criterion (10) and lazy aggregation strategy (11) is illustrated in Fig. 1a. Compared to the naive combination of AdaQuantFL and LAQ where the mutual influence between adaptive quantization and lazy aggregation has not been considered as shown in Fig. 1b, AQUILA adaptively optimizes the allocation of quantization bits throughout training to promote the convergence of lazy aggregation, and at the same time utilizes the lazy aggregation strategy to improve the efficiency of adaptive quantization by compress the transmission with a lower quantization level.

The proposed AQUILA is comprehensively summarized in Appendix A.1. At epoch $k = 0$, each device calculates b_m^0 by setting $\boldsymbol{q}_0^{k-1} = \mathbf{0}$ and uploads $\Delta \boldsymbol{q}_0^k$ to the server since the (11) is not satisfied. At epoch $k \in \{1, 2, ..., K\}$, the server first broadcasts the global model $\boldsymbol{\theta}^k$ to all devices. Each device m computes $\nabla f(\boldsymbol{\theta}_m^k)$ with local training data, and then utilizes it to calculate an optimal quantization level by (10). Subsequently, each device computes its gradient innovation after quantization and determines whether or not to upload based on the communication criterion (11). Finally, the server updates the new global model $\boldsymbol{\theta}^{k+1}$ with up-to-date quantized gradients $\boldsymbol{q}_m^{k-1} + \Delta \boldsymbol{q}_m^k$ for those devices who transmit the uploads at epoch k, while reusing the old quantized gradients $\boldsymbol{q}_m^{k-1}$ for those who skip the uploads.

4 Theoretical Derivation and Analysis of AQUILA

As mentioned before, in this work we bound the model deviation caused by skipping updates with respect to quantization bits. Specifically, if the communication criterion (11) holds for the device m at epoch k, it does not contribute to epoch k's gradient. Otherwise, the loss caused by device m will be minimized with the optimal quantization level selection criterion (10). In this section, the theoretical convergence derivation of AQUILA is based on the following standard assumptions:

Assumption 4.1 (L-smoothness). *Each local objective function f_m is L_m-smooth, i.e., there exist a constant $L_m > 0$, such that $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,*

$$\|\nabla f_m(\mathbf{x}) - \nabla f_m(\mathbf{y})\|_2 \leqslant L_m \|\mathbf{x} - \mathbf{y}\|_2. \tag{12}$$

which implies that the global objective function f is L_f-smooth with $L_f^2 \leq L^2 = \frac{1}{m} \sum_{i=1}^m L_m^2$.

Assumption 4.2 (Uniform lower bound). *For all $\boldsymbol{x} \in \mathbb{R}^d$, there exist $f^* \in \mathbb{R}$ such that $f(\boldsymbol{x}) \geq f^*$.*

Assumption 4.3. *All devices' quantization errors $\boldsymbol{\varepsilon}^k$ will be constrained by the total error of the omitted devices., i.e., $\forall\, k = 0, 1, \cdots, K$, if $\mathcal{M}_c^k \neq \varnothing$, $\exists\, \gamma \geqslant 1$, such that*

$$\left\|\boldsymbol{\varepsilon}^k\right\|_2^2 = \left\|\frac{1}{M}\sum_{m\in\mathcal{M}}\boldsymbol{\varepsilon}_m^k\right\|_2^2 \leqslant \frac{\gamma}{M^2}\left\|\sum_{m\in\mathcal{M}_c^k}\boldsymbol{\varepsilon}_m^k\right\|_2^2, \tag{13}$$

where K denotes the termination time, and $\boldsymbol{\varepsilon}_m^k = \nabla f_m\left(\boldsymbol{\theta}^k\right) - \left(\boldsymbol{q}_m^{k-1} + \Delta\boldsymbol{q}_m^k\right)$,

This assumption is easy to verify when $\mathcal{M}_c^k \neq \varnothing$, a bounded variable (here is $\boldsymbol{\varepsilon}^k$) will always be bounded by a part of itself ($\frac{1}{M}\sum_{m\in\mathcal{M}_c^k}\boldsymbol{\varepsilon}_m^k$) multiplied by a real number (γ).

Lemma 4.1. *Following the assumption that the function f is L-smooth, we have*

$$f\left(\boldsymbol{\theta}^{k+1}\right) - f\left(\boldsymbol{\theta}^k\right) \leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2 + \alpha\left(\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k}\Delta\boldsymbol{q}_m^k\right\|_2^2 + \left\|\boldsymbol{\varepsilon}^k\right\|_2^2\right) + \left(\frac{L}{2} - \frac{1}{2\alpha}\right)\left\|\boldsymbol{\theta}^{k+1} - \boldsymbol{\theta}^k\right\|_2^2. \tag{14}$$

4.1 Convergence analysis for Generally Non-convex Case.

Theorem 4.1. *Suppose Assumption 4.1, 4.2, and 4.3 be satisfied. If $\mathcal{M}_c^k \neq \varnothing$. Then, the global objective function f satisfies*

$$f\left(\boldsymbol{\theta}^{k+1}\right) - f\left(\boldsymbol{\theta}^k\right) \leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2 + \left(\frac{L}{2} - \frac{1}{2\alpha} + \frac{\beta}{\alpha^2}\right)\left\|\boldsymbol{\theta}^{k+1} - \boldsymbol{\theta}^k\right\|_2^2. \tag{15}$$

Corollary 4.1. *Let all the assumptions of Theorem 4.1 hold and*

$$\frac{L}{2} - \frac{1}{2\alpha} + \frac{\beta}{\alpha} \leqslant 0, \tag{16}$$

then the AQUILA requires

$$K = \mathcal{O}\left(\frac{2\omega_0}{\alpha\epsilon^2}\right) \tag{17}$$

communication rounds with $\omega_0 = f\left(\boldsymbol{\theta}^0\right) - f\left(\boldsymbol{\theta}^*\right)$ to achieve $\min_{0\leqslant k\leqslant K-1}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2 \leqslant \epsilon^2$.

The proof of Theorem 4.1 and Corollary 4.1 can be found in Appendix E. Let us discuss the obtained result. First of all, the Assumption 4.3 (13) may not hold when $M_c^k = \varnothing$ since when $\gamma = 0$ does not satisfy the definition of $\gamma \geqslant 1$. Nevertheless, we could still prove the correctness of Corollary 4.1 in this special condition without any extra assumptions, in Appendix D. Secondly, the result of needed communication rounds $\mathcal{O}\left(\frac{2\omega_0}{\alpha\epsilon^2}\right)$ is **independent** of γ in Assumption 4.3, which implies that we can choose the value of γ arbitrarily in this definition domain and it will have no effect on the convergence. Furthermore, with the appropriate selection of α, AQUILA can achieve the same convergence rate order as gradient descent (GD) while empirically requiring substantially fewer communication bits.

4.2 Convergence analysis Under Polyak-Łojasiewicz condition.

Assumption 4.4 ($\mu-$PŁ condition). *Function f satisfies the PL condition with a constant $\mu > 0$, that is,*

$$\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2 \geqslant 2\mu(f(\boldsymbol{\theta}^k) - f(\boldsymbol{\theta}^*)). \tag{18}$$

Theorem 4.2. *Suppose Assumption 4.1, 4.2, 4.3 and 4.4 be satisfied, if the factors satisfy*

$$\frac{L}{2} - \frac{1}{2\alpha} + \frac{\beta}{\alpha} \leqslant 0 \;\;\&\;\; 0 < 1 - \alpha\mu < 1, \tag{19}$$

then the global objective function satisfies

$$f\left(\boldsymbol{\theta}^K\right) - f\left(\boldsymbol{\theta}^*\right) \leqslant (1 - \alpha\mu)^K(f(\boldsymbol{\theta}^0) - f(\boldsymbol{\theta}^*)), \tag{20}$$

and the AQUILA requires

$$K = \mathcal{O}\left(-\frac{1}{\log(1-\alpha\mu)}\log\frac{\omega_0}{\epsilon}\right) \tag{21}$$

communication round with $\omega_0 = f\left(\boldsymbol{\theta}^0\right) - f\left(\boldsymbol{\theta}^*\right)$ *to achieve* $f\left(\boldsymbol{\theta}^K\right) - f\left(\boldsymbol{\theta}^*\right) \leqslant \epsilon$.

The proof of Theorem 4.2 can be found in Appendix F. Theorem 4.2 informs us that AQUILA can achieve a linear convergence rate as in LAG (Chen et al., 2018) and LAQ when certain assumptions are satisfied.

5 Experiments and Discussion

5.1 Experiment Setup

In this paper, we evaluate AQUILA with `CIFAR-10`, `CIFAR-100` (Krizhevsky et al., 2009), and `WikiText-2` dataset (Merity et al., 2016), considering IID, Non-IID data scenario, and heterogeneous model architecture simultaneously.

The FL environment is simulated in Python 3.9 with `PyTorch 11.1` (Paszke et al., 2019) implementation. For the diversity of the neural network structures, we train `ResNet-18` (He et al., 2016) at `CIFAR-10` dataset, `MobileNet-v2` (Sandler et al., 2018) at `CIFAR-100` dataset, and `Transformer` (Vaswani et al., 2017) at `WikiText-2` dataset.

As for the FL system setting, in the majority of our experiments, the whole system exists $M = 10$ total devices. However, considering the large-scale feature of FL, we also validate AQUILA on a larger system of $M = 24$ total devices (owing to the restriction of GPU resources). In addition, we choose the trigger factor $\gamma = 5$ in our evaluation. The hyperparameters and additional details of our experiments are revealed in Appendix A.2.

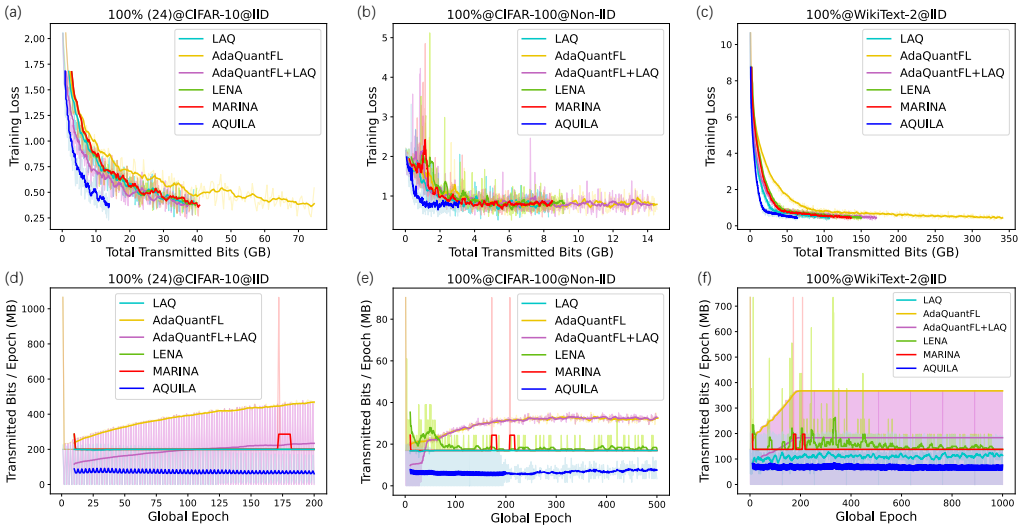


Figure 2: Comparison of AQUILA with other communication-efficient algorithms on IID and Non-IID settings with **homogeneous** model structure. (a)-(c): training loss v.s. total transmitted bits, (d)-(f): transmitted bits per epoch v.s. global epoch.

5.2 Homogeneous Environment

We first evaluate our proposed AQUILA with homogeneous settings where all local models share the same model architecture as the global model. To better demonstrate the effectiveness of AQUILA, its performance is compared with several state-of-the-art methods, including AdaQuantFL, LAQ with fixed levels, LENA (Ghadikolaei et al., 2021), MARINA (Gorbunov et al., 2021), and the naive combination of AdaQuantFL with LAQ. Note that based on this homogeneous setting, we conduct

Table 1: Numerical numbers of total communication bits in the **homogeneous** environment.

Total Comm Bits (GB)		Ada	LAQ	Ada+LAQ	LENA	MARINA	**AQUILA**
CIFAR-10	IID-24	74.65	42.96	37.29	39.86	40.71	**13.89**
	IID	34.19	15.42	17.11	16.39	16.28	**6.32**
	Non-IID	27.07	7.8	23.31	33.29	10.18	**6.77**
CIFAR-100	IID-24	68.21	20.84	34.35	47.00	41.93	**13.08**
	IID	28.68	16.55	14.41	16.74	16.77	**11.24**
	Non-IID	14.54	8.27	14.25	9.19	8.49	**3.11**
WikiText-2	IID-24	839.12	312.93	730.90	338.38	340.77	**144.37**
	IID	340.97	106.92	170.40	150.07	136.31	**63.39**

both IID and Non-IID evaluations on `CIFAR-10` and `CIFAR-100` dataset, and an IID evaluation on `WikiText-2`. To simulate the Non-IID FL setting as (Diao et al., 2020), each device is allocated two classes of data in `CIFAR-10` and 10 classes of data in `CIFAR-100` at most, and the amount of data for each label is balanced.

The experimental results are presented in Fig. 2, where *100%* implies all local models share a similar structure with the global model (i.e., homogeneity), *100% (24)* denotes the experiment is conducted in a 24 devices system, and *AdaQuantFL+LAQ* represents the naive combination of AdaQuantFL and LAQ. For better illustration, the results have been smoothed by their standard deviation. The solid lines represent values after smoothing, and transparent shades of the same colors around them represent the true values. Additionally, Table 1 shows the total number of bits transmitted by all devices throughout the FL training process. The comprehensive experimental results are established in Appendix A.3.

5.3 Non-homogeneous Scenario

In this section, we also evaluate AQUILA with heterogeneous model structures as HeteroFL (Diao et al., 2020), where the structures of local models trained on the device side are heterogeneous. Suppose the global model at epoch k is $\boldsymbol{\theta}^k$ and its size is $d = w_g * h_g$, then the local model of each device m can be selected by $\boldsymbol{\theta}_m^k = \boldsymbol{\theta}^k[: w_m, : h_m]$, where $w_m = r_m w_g$ and $h_m = r_m h_g$, respectively. In this paper, we choose model complexity levels $r_m = 0.5$.

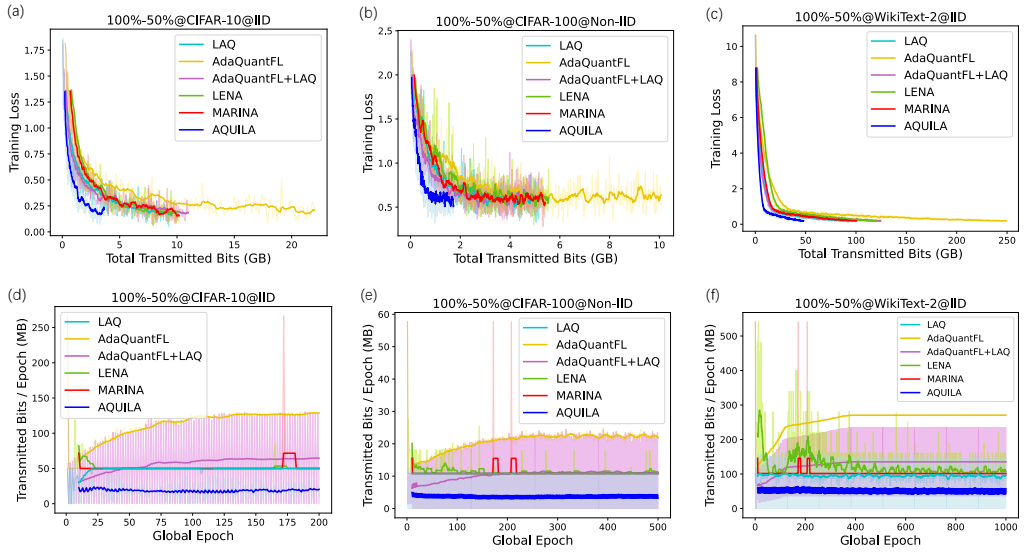


Figure 3: Comparison of AQUILA with other communication-efficient algorithms on IID and Non-IID settings with **heterogeneous** model structure. (a)-(c): training loss v.s. total transmitted bits, (d)-(f): transmitted bits per epoch v.s. global epoch.

Most of the symbols in Fig. 3 are identical to the Fig. 2. *100%-50%* is a newly introduced symbol that implies half of the devices share the same structure with the global model while another half only have 50% * 50% parameters as the global model.

Table 2: Numerical numbers of total communication bits in the **heterogeneous** environment.

Total Comm Bits (GB)		Ada	LAQ	Ada+LAQ	LENA	MARINA	**AQUILA**
CIFAR-10	IID	21.99	11.89	10.98	10.17	10.18	**3.33**
	Non-IID	19.13	4.88	9.52	18.44	16.28	**3.92**
CIFAR-100	IID	19.42	10.56	9.7	10.65	10.7	**3.65**
	Non-IID	10.07	5.28	5.02	5.56	5.42	**1.73**
WikiText-2	IID	248.87	92.74	124.47	119.83	100.38	**47.91**

5.4 Performance Analysis

In the evaluation part, we analyze the performance of AQUILA with various experiment settings including Non-IID data distribution and heterogeneous model architecture. The following analysis can be drawn from the previously established tables and figures.

First of all, AQUILA achieves a significant transmission reduction compared to the naive combination of LAQ and AdaQuantFL in all datasets, which demonstrates the superiority of AQUILA's efficiency. Specifically, Table 1 indicates that AQUILA saves 80.24% of transmitted bits in the system of 24 IID devices at the WikiText-2 dataset and reduces 70.96% of transmitted bits in the system of 10 Non-IID devices at the CIFAR-10 dataset, compared to the naive combination. And other results in Table 2 also show an obvious reduction in terms of the total transmitted bits required for convergence.

Secondly, in Fig. 2 and Fig. 3, the changing trend of AQUILA's communication bits per each round clearly verifies the necessity and effectiveness of our well-designed adaptive quantization level and skip criterion. In these two figures, the number of bits transmitted in each round of AQUILA **fluctuates** a bit, indicating the effectiveness of AQUILA's selection rule. Meanwhile, the value of transmitted bits remains at a quite low level, suggesting that the adaptive quantization principle makes training more efficient. Moreover, the figures also inform that the quantization level selected by AQUILA will not continuously increase during training instead of being as AdaQuantFL. In addition, based on these two figures, we can also conclude that AQUILA converges faster under the same communication costs.

Finally, AQUILA is capable of adapting to a wide range of challenging FL circumstances. In the Non-IID scenario and heterogeneous model structure, AQUILA still outperforms other algorithms by significantly reducing overall transmitted bits while maintaining the same convergence property and objective function value. In particular, AQUILA reduces 60.4% overall communication costs compared to LENA and 57.2% compared to MARINA on average. These experimental results in non-homogeneous FL settings prove that AQUILA can be stably employed in more general and complicated FL scenarios.

6 Conclusions and Future Work

This paper proposes a communication efficient FL procedure to simultaneously adjust two mutually-dependent degrees of freedom: communication frequency and quantization level. With the close cooperation of the novel adaptive quantization and adjusted lazy aggregation strategy derived in this paper, the proposed AQUILA has been proven to be capable of reducing the transmitted costs while maintaining the convergence guarantee and model performance compared to existing methods. The evaluation with Non-IID data distribution and various heterogeneous model architectures demonstrates that AQUILA is compatible in a non-homogeneous FL environment. Future works include further improvements and theoretical guarantees for FL systems with heterogeneity.

Reproducibility

We present the overall theorem statements and proofs for our main results in the Appendix, as well as necessary experimental plotting figures. Furthermore, we submit the code of AQUILA in the

supplementary material part, including all the hyperparameters and a *requirements* to help the public reproduce our experimental results. Our algorithm is straightforward, well-described, and easy to implement.

ETHICS STATEMENT

All evaluations of AQUILA are performed on publicly available datasets for reproducibility purposes. This paper empirically studies the performance of various state-of-art algorithms, therefore probably introduces no new ethical or cultural problems. This paper does not utilize any new dataset.

REFERENCES

Tianyi Chen, Georgios B Giannakis, Tao Sun, and Wotao Yin. Lag: Lazily aggregated gradient for communication-efficient distributed learning. In *Proceedings of Advances in Neural Information Processing Systems*, pp. 1–25, 2018.

Enmao Diao, Jie Ding, and Vahid Tarokh. HeteroFL: Computation and communication efficient federated learning for heterogeneous clients. In *Proceedings of the 8th International Conference on Learning Representations*, 2020.

Zhaoyang Du, Celimuge Wu, Tsutomu Yoshinaga, Kok-Lim Alvin Yau, Yusheng Ji, and Jie Li. Federated learning for vehicular Internet of things: Recent advances and open issues. *IEEE Computer Graphics and Applications*, pp. 45–61, 2020.

Hossein Shokri Ghadikolaei, Sebastian Stich, and Martin Jaggi. Lena: Communication-efficient distributed learning with self-triggered gradient uploads. In *International Conference on Artificial Intelligence and Statistics*, pp. 3943–3951. PMLR, 2021.

Eduard Gorbunov, Konstantin P Burlachenko, Zhize Li, and Peter Richtárik. Marina: Faster non-convex distributed learning with compression. In *International Conference on Machine Learning*, pp. 3788–3798. PMLR, 2021.

Andrew Hard, Kanishka Rao, Rajiv Mathews, Swaroop Ramaswamy, Françoise Beaufays, Sean Augenstein, Hubert Eichner, Chloé Kiddon, and Daniel Ramage. Federated learning for mobile keyboard prediction. *arXiv preprint arXiv:1811.03604*, 2018.

Kaiming He, Xiangyu Zhang, Shaoqing Ren, and Jian Sun. Deep residual learning for image recognition. In *Proceedings of the 29th IEEE Conference on Computer Vision and Pattern Recognition*, pp. 770–778, 2016.

Divyansh Jhunjhunwala, Advait Gadhikar, Gauri Joshi, and Yonina C Eldar. Adaptive quantization of model updates for communication-efficient federated learning. In *Proceedings of the 2021 IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 3110–3114, 2021.

Alex Krizhevsky, Geoffrey Hinton, et al. Learning multiple layers of features from tiny images. 2009. URL `https://www.cs.toronto.edu/~kriz/cifar.html`.

Yang Liu, Anbu Huang, Yun Luo, He Huang, Youzhi Liu, Yuanyuan Chen, Lican Feng, Tianjian Chen, Han Yu, and Qiang Yang. Fedvision: An online visual object detection platform powered by federated learning. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence*, pp. 13172–13179, 2020.

Yuzhu Mao, Zihao Zhao, Guangfeng Yan, Yang Liu, Tian Lan, Linqi Song, and Wenbo Ding. Communication efficient federated learning with adaptive quantization. *arXiv preprint arXiv:2104.06023*, 2021.

Brendan McMahan, Eider Moore, Daniel Ramage, Seth Hampson, and Blaise Aguera y Arcas. Communication-efficient learning of deep networks from decentralized data. In *Artificial Intelligence and Statistics*, pp. 1273–1282, 2017.

Stephen Merity, Caiming Xiong, James Bradbury, and Richard Socher. Pointer sentinel mixture models. *arXiv preprint arXiv:1609.07843*, 2016. URL https://blog.salesforceairesearch.com/the-wikitext-long-term-dependency-language-modeling-dataset/].

Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, et al. Pytorch: An imperative style, high-performance deep learning library. *Advances in neural information processing systems*, 32, 2019.

Mark Sandler, Andrew Howard, Menglong Zhu, Andrey Zhmoginov, and Liang-Chieh Chen. Mobilenetv2: Inverted residuals and linear bottlenecks. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 4510–4520, 2018.

Jun Sun, Tianyi Chen, Georgios B Giannakis, Qinmin Yang, and Zaiyue Yang. Lazily aggregated quantized gradient innovation for communication-efficient federated learning. *IEEE Transactions on Pattern Analysis & Machine Intelligence*, pp. 1–15, 2020.

Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Łukasz Kaiser, and Illia Polosukhin. Attention is all you need. *Advances in neural information processing systems*, 30, 2017.

A APPENDIX

The appendix includes supplementary experimental results, mathematical proof of the aforementioned theorems, and a detailed derivation of the novel adaptive quantization criterion and lazy aggregation strategy. Compared to Fig. 2 and Fig. 3 in the main text, the result figures in the appendix show a more comprehensive evaluation with AQUILA, which contains more detailed information including but not limited to *accuracy vs steps* and *training loss vs steps* curves.

A.1 OVERALL ALGORITHM OF AQUILA

Algorithm 1 Communication Efficient FL with AQUILA

Input: the number of communication rounds K, the learning rate α.
Initialize: the initial global model parameter $\boldsymbol{\theta}^0$.

Server broadcasts $\boldsymbol{\theta}^0$ to all devices. ▷ For the first round $k = 0$.
for each device $m \in \mathcal{M}$ **in parallel do**
    Calculates local gradient $\nabla f(\boldsymbol{\theta}_m^0)$.
    Compute $(b_m^0)*$ by setting $\boldsymbol{q}_m^{k-1} = \mathbf{0}$ in (10) and the quantized gradient innovation $\Delta \boldsymbol{q}_m^0$ by (9), and transmits it back to the server side.
end for
for $k = 1, 2, ..., K$ **do**
    Server broadcasts $\boldsymbol{\theta}^k$ to all devices.
    for each device $m \in \mathcal{M}$ **in parallel do**
        Calculates local gradient $\nabla f(\boldsymbol{\theta}_m^k)$.
        Computes the optimal local quantization level $(b_m^k)^*$ by (10) and the quantized gradient innovation $\Delta \boldsymbol{q}_m^k$ by (9).
        if (11) holds for device m **then**
            device m sends nothing.
        else
            device m transmits $\Delta \boldsymbol{q}_m^k$ to the server.
        end if
    end for
    Server updates $\boldsymbol{\theta}^{k+1}$ by the saving previous global quantized gradient $\boldsymbol{q}_m^{k-1}$ and the received quantized gradient innovation $\Delta \boldsymbol{q}_m^k$: $\boldsymbol{\theta}^{k+1} := \boldsymbol{\theta}^k - \alpha \left(\boldsymbol{q}^{k-1} + 1/M \sum_{m \in \mathcal{M}^k} \Delta \boldsymbol{q}_m^k \right) = \boldsymbol{\theta}^k - \alpha/M \sum_{m \in \mathcal{M}} \boldsymbol{q}_m^k$.
    Server saves the average quantized gradient $\boldsymbol{q}^k$ for the next aggregation.
end for

A.2 EXPERIMENT SETUP

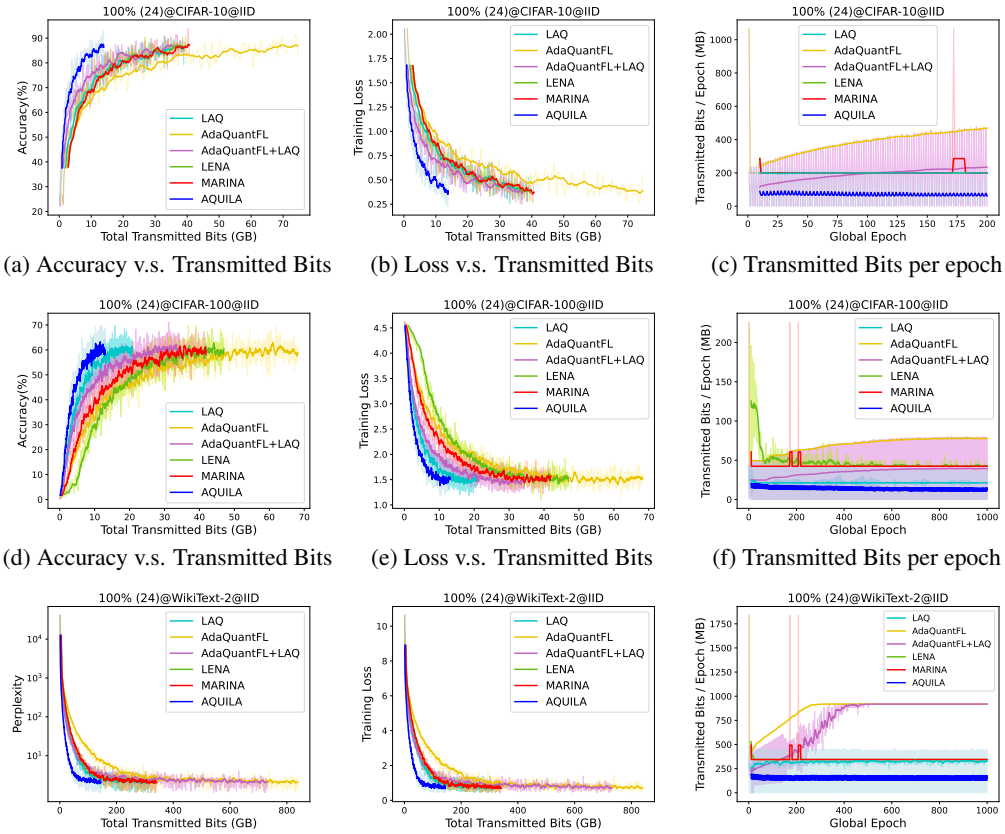
In this section, we provide some extra hyperparameter settings for our evaluation. For the LAQ, we set $D = 10$ and $\xi_1 = \xi_2 = \cdots = \xi_D = 0.8/D$ as the same as the setting in their paper. For LENA, we set $\beta_{LENA} = 40$ in their trigger condition. And for MARINA, we calculate the uploading probability of Bernoulli distribution as $p = \xi_Q/d$ as announced in their paper. In addition, we choose the CrossEntropy function as our objective function in the experiment part. Table 3 shows the hyperparameter details of our evaluation.

A.3 COMPREHENSIVE EXPERIMENT RESULTS

This section will cover all the experimental results in our paper.

Table 3: The hyperparameters for `CIFAR-10`, `CIFAR-100`, and `WikiText-2` datasets in the FL training process.

Dataset	CIFAR-10		CIFAR-100		WikiText-2
Model	ResNet-18		MobileNet-v2		Transformer
Data Distribution	IID	Non-IID	IID	Non-IID	IID
Global Epoch	200	200	1000	500	1000
Local Batch Size	256	256	256	256	64
Optimizer	SGD	SGD	SGD	SGD	SGD
Momentum	0.9	0.9	0.9	0.9	0.9
Weight Decay	5.00E-04	5.00E-04	5.00E-04	5.00E-04	5.00E-04
Learning Rate α	0.01	0.01	0.01	0.01	0.5

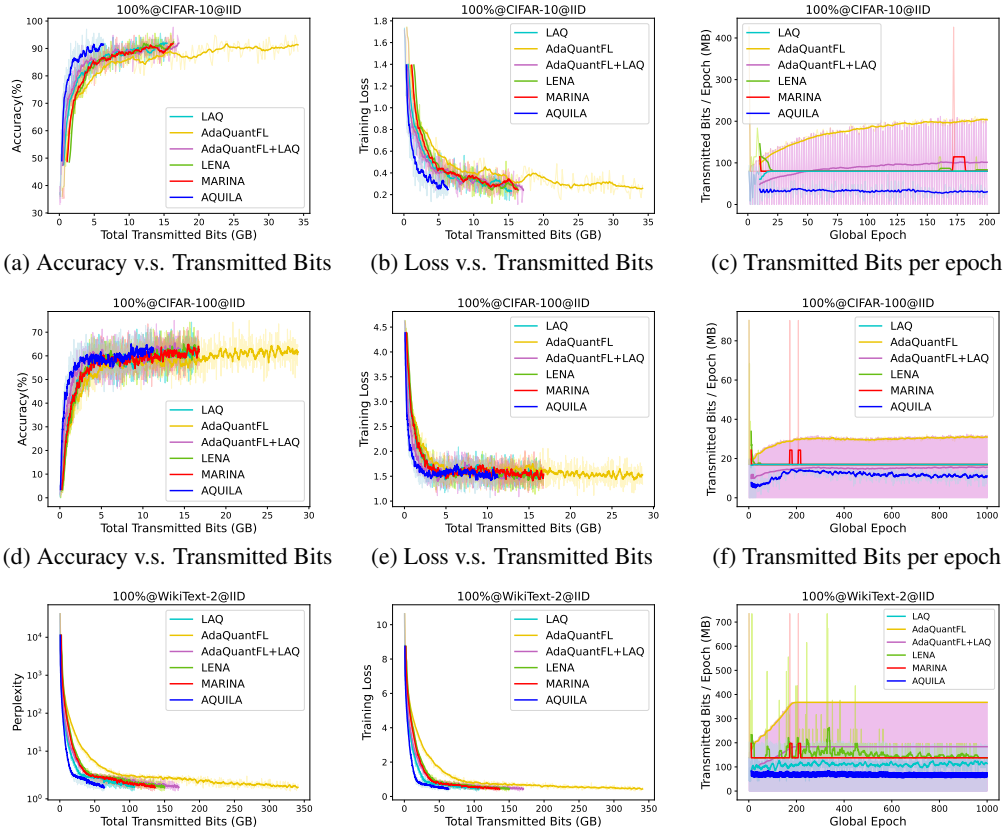


(a) Accuracy v.s. Transmitted Bits (b) Loss v.s. Transmitted Bits (c) Transmitted Bits per epoch

(d) Accuracy v.s. Transmitted Bits (e) Loss v.s. Transmitted Bits (f) Transmitted Bits per epoch

(g) Perplexity v.s. Transmitted Bits (h) Loss v.s. Transmitted Bits (i) Transmitted Bits per epoch

Figure 4: Comparison on 24 devices at three datasets in the IID, homogeneous scenario.

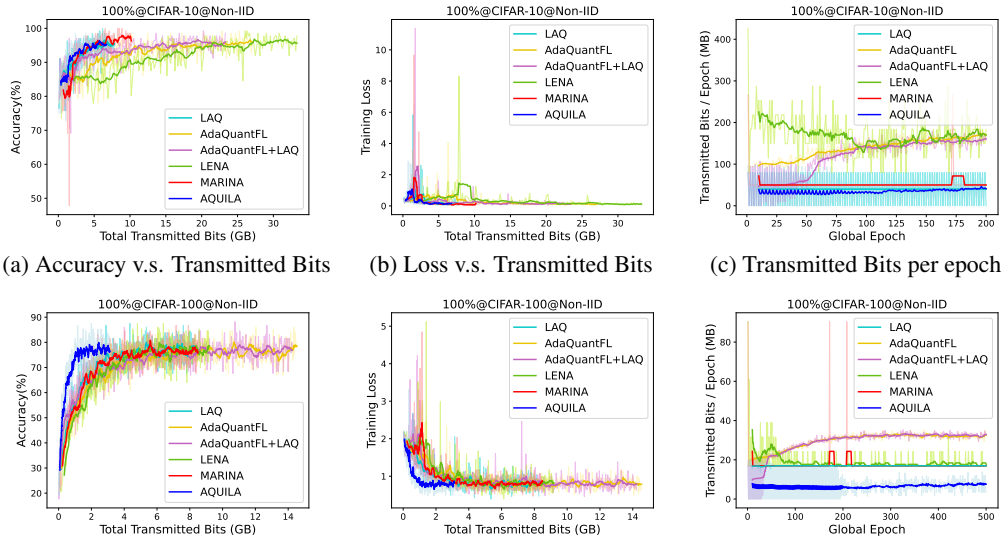


(a) Accuracy v.s. Transmitted Bits (b) Loss v.s. Transmitted Bits (c) Transmitted Bits per epoch

(d) Accuracy v.s. Transmitted Bits (e) Loss v.s. Transmitted Bits (f) Transmitted Bits per epoch

(g) Perplexity v.s. Transmitted Bits (h) Loss v.s. Transmitted Bits (i) Transmitted Bits per epoch

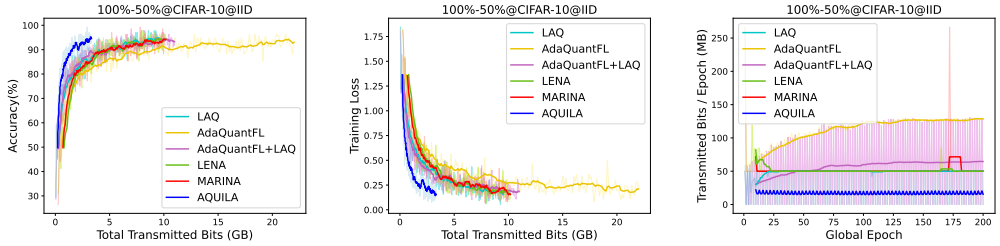
Figure 5: Comparison on 10 devices at three datasets in the IID, homogeneous scenario.



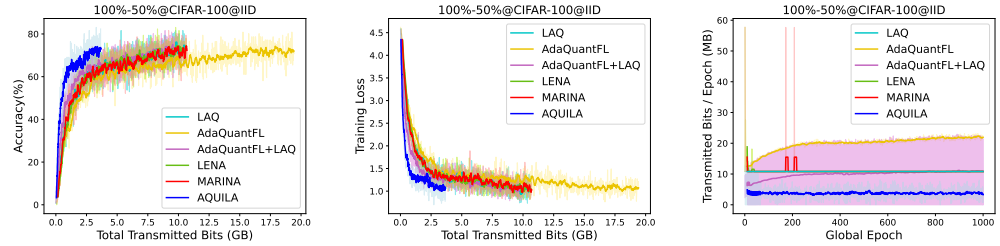
(a) Accuracy v.s. Transmitted Bits (b) Loss v.s. Transmitted Bits (c) Transmitted Bits per epoch

(d) Accuracy v.s. Transmitted Bits (e) Loss v.s. Transmitted Bits (f) Transmitted Bits per epoch

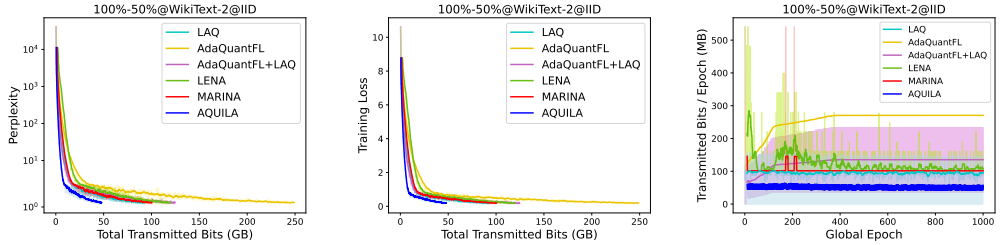
Figure 6: Comparison on 10 devices at three datasets in the Non-IID, homogeneous scenario..



(a) Accuracy v.s. Transmitted Bits (b) Loss v.s. Transmitted Bits (c) Transmitted Bits per epoch

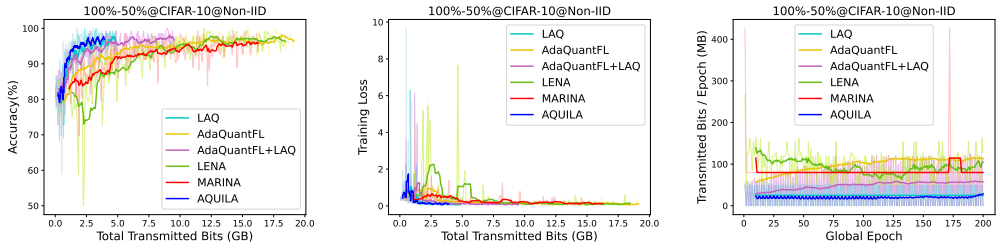


(d) Accuracy v.s. Transmitted Bits (e) Loss v.s. Transmitted Bits (f) Transmitted Bits per epoch

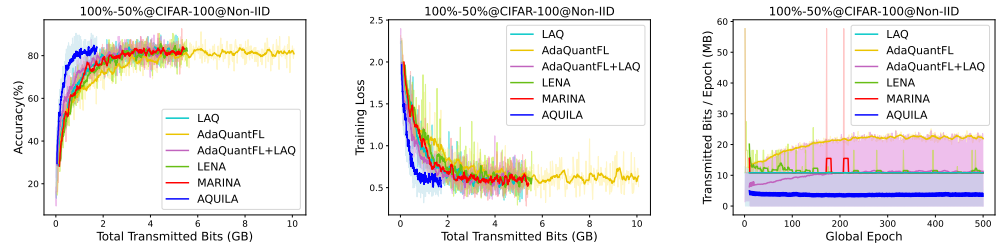


(g) Perplexity v.s. Transmitted Bits (h) Loss v.s. Transmitted Bits (i) Transmitted Bits per epoch

Figure 7: Comparison on 10 devices at three datasets in the IID, heterogeneous scenario.



(a) Accuracy v.s. Transmitted Bits (b) Loss v.s. Transmitted Bits (c) Transmitted Bits per epoch



(d) Accuracy v.s. Transmitted Bits (e) Loss v.s. Transmitted Bits (f) Transmitted Bits per epoch

Figure 8: Comparison on 10 devices at three datasets in the Non-IID, heterogeneous scenario.

B BASIC FACTS AND SOME LEMMAS

Frequently used norm inequalities

Suppose $n \in \mathbb{N}^+$ and $\|\cdot\|$ denotes the ℓ^2-norm. For p in $\mathbb{R}^+$, $\boldsymbol{x}_i, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, there holds

1. **Norm summation inequality.**

$$\left\|\sum_{i=1}^{n} \boldsymbol{x}_i\right\|_2^2 \leqslant n \sum_{i=1}^{n} \|\boldsymbol{x}_i\|_2^2 . \tag{22}$$

2. **Inner-product inequality.**

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \frac{1}{2}\left(\|\boldsymbol{a}\|_2^2 + \|\boldsymbol{b}\|_2^2 - \|\boldsymbol{a}-\boldsymbol{b}\|_2^2\right). \tag{23}$$

3. **Young's Inequality.**

$$\|\boldsymbol{a}+\boldsymbol{b}\|_2^2 \leqslant (1+p)\|\boldsymbol{a}\|_2^2 + (1+p^{-1})\|\boldsymbol{b}\|_2^2 . \tag{24}$$

4. **Minkowski's Inequality.**

$$\|\boldsymbol{a}+\boldsymbol{b}\|_2 \leqslant \|\boldsymbol{a}\|_2 + \|\boldsymbol{b}\|_2 . \tag{25}$$

Lemma A.1. *The summation of quantized gradient innovation and quantization error is bounded by the global model difference:*

$$\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k} \Delta \boldsymbol{q}_m^k\right\|_2^2 + \left\|\boldsymbol{\varepsilon}^k\right\|_2^2 \leqslant \frac{\beta}{\alpha^2}\left\|\boldsymbol{\theta}^{k+1} - \boldsymbol{\theta}^k\right\|_2^2, \tag{26}$$

Proof.

$$\begin{aligned}
&\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k} \Delta \boldsymbol{q}_m^k\right\|_2^2 + \left\|\boldsymbol{\varepsilon}^k\right\|_2^2 \\
\overset{(a)}{\leqslant}& \left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k} \Delta \boldsymbol{q}_m^k\right\|_2^2 + \gamma\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k} \boldsymbol{\varepsilon}_m^k\right\|_2^2 \\
\overset{(22)}{\leqslant}& |\mathcal{M}_c^k| \sum_{m\in\mathcal{M}_c^k}\left\|\frac{1}{M}\Delta \boldsymbol{q}_m^k\right\|_2^2 + \gamma|\mathcal{M}_c^k| \sum_{m\in\mathcal{M}_c^k}\left\|\frac{1}{M}\boldsymbol{\varepsilon}_m^k\right\|_2^2 \\
\leqslant& \frac{|\mathcal{M}_c^k|}{M^2}\sum_{m\in\mathcal{M}_c^k}\left(\left\|\Delta \boldsymbol{q}_m^k\right\|_2^2 + \gamma\left\|\boldsymbol{\varepsilon}_m^k\right\|_2^2\right) \\
\overset{(b)}{\leqslant}& \frac{\beta|\mathcal{M}_c^k|^2}{\alpha^2 M^2}\left\|\boldsymbol{\theta}^{k+1} - \boldsymbol{\theta}^k\right\|_2^2 \\
\leqslant& \frac{\beta}{\alpha^2}\left\|\boldsymbol{\theta}^{k+1} - \boldsymbol{\theta}^k\right\|_2^2,
\end{aligned} \tag{27}$$

where (a) follows Assumption 4.3, and (b) utilizes our novel trigger condition (11). □

C DERIVATION OF b_m^k

The unbiased aggregated model without gradient skipping at epoch k should be:

$$\bar{\boldsymbol{\theta}}^{k+1} = \boldsymbol{\theta}^k - \frac{\alpha}{M}\sum_{m\in\mathcal{M}}\left(\boldsymbol{q}_m^{k-1} + \Delta \boldsymbol{q}_m^k\right). \tag{28}$$

With lazy aggregation, the actual aggregated model at epoch k is:

$$\boldsymbol{\theta}^{k+1} = \boldsymbol{\theta}^k - \frac{\alpha}{M} \sum_{m \in \mathcal{M}^k} \left(\boldsymbol{q}_m^{k-1} + \Delta \boldsymbol{q}_m^k \right) - \frac{\alpha}{M} \sum_{m \in \mathcal{M}_c^k} \boldsymbol{q}_m^{k-1}. \tag{29}$$

Suppose Δ_m^k denotes the quantization loss of device m at epoch k and $\boldsymbol{\psi}_m^k$ denotes the quantization representation of local gradient innovation as in Definition 3.1, i.e.,

$$\Delta_m^k = \boldsymbol{\psi}_m^k - \frac{\nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} + R_m^k \mathbf{1}}{2\tau_m^k R_m^k} - \frac{1}{2}\mathbf{1} \tag{30}$$

With (28), (29), and (30), the model deviation $\left\|\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}^k\right\|_2^2$ caused by skipping gradients can be written as:

$$\begin{aligned}
\left\|\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}^k\right\|_2^2 &= \left\| \frac{\alpha}{M} \sum_{m \in \mathcal{M}_c^k} \Delta \boldsymbol{q}_m^k \right\|_2^2 \\
&= \left\| \frac{\alpha}{M} \sum_{m \in \mathcal{M}_c^k} \left(2\tau_m^k R_m^k \boldsymbol{\psi}_m^k - R_m^k \mathbf{1} \right) \right\|_2^2 \\
&\overset{(22)}{\leqslant} \frac{\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left\| 2\tau_m^k R_m^k \boldsymbol{\psi}_m^k - R_m^k \mathbf{1} \right\|_2^2 \\
&\overset{(30)}{\leqslant} \frac{\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} + R_m^k \mathbf{1} + \tau_m^k R_m^k \mathbf{1} + \Delta_m^k - R_m^k \mathbf{1} \right\|_2^2 \right) \\
&\overset{(30)}{\leqslant} \frac{2\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} + \tau_m^k R_m^k \mathbf{1} \right\|_2^2 + \left\| \Delta_m^k \right\|_2^2 \right) \\
&\overset{(a)}{\leqslant} \frac{2\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} + \tau_m^k R_m^k \mathbf{1} \right\|_2^2 + d \right) \\
&\overset{(25)}{\leqslant} \frac{2\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_2 + \left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 \right)^2 + d \right) \\
&= \frac{2\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_2 - \left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 + 2 \left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 \right)^2 + d \right) \\
&\leqslant \frac{4\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_2 - \left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 \right)^2 + 4 \left(\left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 \right)^2 + \frac{d}{2} \right) \\
&\overset{(b)}{\leqslant} \frac{4\alpha^2 |\mathcal{M}_c^k|}{M^2} \sum_{m \in \mathcal{M}_c^k} \left(\left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_2 - \left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 \right)^2 + 4 (R_m^k)^2 d + \frac{d}{2} \right),
\end{aligned} \tag{31}$$

where $\mathbf{1} \in \mathbb{R}^d$ denotes the vector filled with scalar value 1, (a) $\Delta_m^k \in (-1, 0]$, (b) $R_m^k \geqslant \tau_m^k R_m^k \geqslant 0$. Since R_m^k is independent of τ_m^k, we can formulate an optimization problem about τ_m^k for device m at communication round k as follows:

$$\min_{0 < \tau_m^k \leqslant 1} \quad \left(\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_2 - \left\| \tau_m^k R_m^k \mathbf{1} \right\|_2 \right)^2 \tag{32}$$

Therefore, the optimal solution of τ_m^k in (32) is

$$(\tau_m^k)^* = \frac{\left\| \nabla f_m\left(\boldsymbol{\theta}^k\right) - \boldsymbol{q}_m^{k-1} \right\|_2}{R_m^k \sqrt{d}}. \tag{33}$$

Then, the optimal adaptive quantization level $(b_m^k)^*$ is equal to

$$\begin{aligned}(b_m^k)^* &= \left\lfloor \log_2(\frac{1}{(\tau_m^k)^*}+1)\right\rfloor \\ &= \left\lfloor \log_2\left(\frac{R_m^k\sqrt{d}}{\left\|\nabla f_m\left(\boldsymbol{\theta}^k\right)-\boldsymbol{q}_m^{k-1}\right\|_2}+1\right)\right\rfloor\end{aligned} \tag{34}$$

Notice that $(b_m^k)^* \geqslant 1$ is always true since $(\tau_m^k)^* \leqslant 1$

D Proof of descent upper bound when $M_c^k = \varnothing$.

Proof. Since the skipping subset of devices are the empty set, from (6) we have

$$\begin{aligned}\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^k &= -\frac{\alpha}{M}\sum_{m\in\mathcal{M}^k}\left(\boldsymbol{q}_m^{k-1}+\Delta\boldsymbol{q}_m^k\right)-\frac{\alpha}{M}\sum_{m\in\mathcal{M}_c^k}\boldsymbol{q}_m^{k-1} \\ &= -\frac{\alpha}{M}\sum_{m\in\mathcal{M}}\left(\boldsymbol{q}_m^{k-1}+\Delta\boldsymbol{q}_m^k\right) \\ &\overset{(4)}{=} -\frac{\alpha}{M}\sum_{m\in\mathcal{M}}\left(\nabla f_m(\boldsymbol{\theta}^k)-\boldsymbol{\varepsilon}_m^k\right) \\ &= -\alpha\left(\nabla f(\boldsymbol{\theta}^k)-\boldsymbol{\varepsilon}^k\right).\end{aligned} \tag{35}$$

From (14) we have:

$$\begin{aligned}&f\left(\theta^{k+1}\right)-f\left(\theta^k\right) \\ &\leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\alpha\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k}\Delta\boldsymbol{q}_m^k\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}\right)\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^k\right\|_2^2+\alpha\left\|\boldsymbol{\varepsilon}^k\right\|_2^2 \\ &\leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}\right)\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^k\right\|_2^2+\alpha\left\|\boldsymbol{\varepsilon}^k\right\|_2^2 \\ &\overset{(24)}{\leqslant} -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}\right)\alpha^2\left((1+p)\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\left(1+p^{-1}\right)\left\|\boldsymbol{\varepsilon}^k\right\|_2^2\right)+\alpha\left\|\boldsymbol{\varepsilon}^k\right\|_2^2 \\ &= -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\frac{1}{2}\left(\alpha^2L-\alpha\right)(1+p)\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\frac{1}{2}\left(\alpha^2L-\alpha\right)\left(1+p^{-1}\right)\left\|\boldsymbol{\varepsilon}^k\right\|_2^2+\alpha\left\|\boldsymbol{\varepsilon}^k\right\|_2^2 \\ &= \frac{\alpha}{2}\left((\alpha L-1)(1+p)-1\right)\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2+\frac{\alpha}{2}\left((\alpha L-1)\left(1+p^{-1}\right)+2\right)\left\|\boldsymbol{\varepsilon}^k\right\|_2^2.\end{aligned} \tag{36}$$

If the factor of $\left\|\boldsymbol{\varepsilon}^k\right\|_2^2$ in (36) is less than or equal to 0, that is,

$$(\alpha L-1)\left(1+p^{-1}\right)+2\leqslant 0, \tag{37}$$

then the factor of $\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2$ will be less than $-\frac{\alpha}{2}$, which indicates that

$$f\left(\boldsymbol{\theta}^{k+1}\right)-f\left(\boldsymbol{\theta}^k\right)\leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^k\right)\right\|_2^2. \tag{38}$$

Note that it is not difficult to demonstrate that (37) and (16) can actually be satisfied at the same time. For instance, we can set $p=0.1, \alpha=0.1, \beta=0.25, L=2.5$ that satisfies both (37) and (16). □

E Proof of Theorem 4.1 and Corollary 4.1.

Proof. Suppose Assumption 4.1, 4.2, and 4.3 are satisfied and $M_c^k \neq \varnothing$. For the simplicity of the convergence proof, we assume $\Phi^k = \frac{1}{M}\sum_{m\in\mathcal{M}_c^k}\Delta\boldsymbol{q}_m^k$.

$$
\begin{aligned}
& f\left(\boldsymbol{\theta}^{k+1}\right)-f\left(\boldsymbol{\theta}^{k}\right) \\
\leqslant & \left\langle \nabla f\left(\boldsymbol{\theta}^{k}\right), \boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\rangle+\frac{L}{2}\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2 \\
= & \left\langle \nabla f\left(\boldsymbol{\theta}^{k}\right), -\alpha\left(\nabla f\left(\boldsymbol{\theta}^{k}\right)-\boldsymbol{\varepsilon}^{k}-\Phi^{k}\right)\right\rangle+\frac{L}{2}\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2 \\
= & -\alpha\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\alpha\left\langle \nabla f\left(\boldsymbol{\theta}^{k}\right), \boldsymbol{\varepsilon}^{k}+\Phi^{k}\right\rangle+\frac{L}{2}\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2 \\
\overset{(23)}{=} & -\alpha\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\frac{\alpha}{2}\left(\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\left\|\boldsymbol{\varepsilon}^{k}+\Phi^{k}\right\|_2^2-\frac{1}{\alpha^2}\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2\right)+\frac{L}{2}\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2 \\
\leqslant & -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\frac{\alpha}{2}\left\|\boldsymbol{\varepsilon}^{k}+\Phi^{k}\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}\right)\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2 \\
\overset{(22)}{\leqslant} & -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\alpha\left\|\boldsymbol{\varepsilon}^{k}\right\|_2^2+\alpha\left\|\Phi^{k}\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}\right)\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2 \\
\overset{(27)}{\leqslant} & -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}+\frac{\beta}{\alpha}\right)\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2
\end{aligned}
\tag{39}
$$

For $\left(\frac{L}{2}-\frac{1}{2\alpha}+\frac{\beta}{\alpha}\right) \leqslant 0$, we have

$$
f\left(\boldsymbol{\theta}^{k+1}\right)-f\left(\boldsymbol{\theta}^{k}\right) \leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 \tag{40}
$$

Notice that inequality (40) holds for both $M_c^k \neq \varnothing$ and $M_c^k = \varnothing$. Sum it up for $k = 0, 1, \cdots, T-1$, we have

$$
f\left(\boldsymbol{\theta}^{K}\right)-f\left(\boldsymbol{\theta}^{0}\right) \leqslant -\frac{\alpha}{2}\sum_{k=0}^{K-1}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 \tag{41}
$$

Therefore, considering the minimum of $\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2$:

$$
\begin{aligned}
\min_{k=0,1,\cdots,K-1}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 & \leqslant \frac{1}{K}\sum_{k=0}^{K-1}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 \\
& \overset{(41)}{\leqslant} \frac{2}{\alpha K}\left(f\left(\boldsymbol{\theta}^{0}\right)-f\left(\boldsymbol{\theta}^{K}\right)\right)
\end{aligned}
\tag{42}
$$

For $\left(\frac{L}{2}-\frac{1}{2\alpha}+\frac{\beta}{\alpha}\right) \leqslant 0$, we have $f\left(\boldsymbol{\theta}^{K}\right)-f\left(\boldsymbol{\theta}^{0}\right)$ is non-increasing and

$$
\min_{0\leqslant k\leqslant K-1}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 \leqslant \frac{2}{\alpha K}\left(f\left(\boldsymbol{\theta}^{0}\right)-f\left(\boldsymbol{\theta}^{*}\right)\right) \leqslant \epsilon^2, \tag{43}
$$

which demonstrates AQUILA requires $K = \mathcal{O}\left(\frac{2\omega_0}{\alpha\epsilon^2}\right)$ communication round with $\omega_0 = f\left(\boldsymbol{\theta}^{0}\right) - f\left(\boldsymbol{\theta}^{*}\right)$ to achieve $\min_{0\leqslant k\leqslant K-1}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 \leqslant \epsilon^2$. □

F PROOF OF THEOREM 4.2.

Proof. From (40), we have

$$
\begin{aligned}
f\left(\boldsymbol{\theta}^{k+1}\right)-f\left(\boldsymbol{\theta}^{k}\right) & \leqslant -\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2 \\
& \overset{(a)}{\leqslant} -\alpha\mu(f(\boldsymbol{\theta}^{k})-f(\boldsymbol{\theta}^{*})),
\end{aligned}
\tag{44}
$$

For $\left(\frac{L}{2}-\frac{1}{2\alpha}+\frac{\beta}{\alpha}\right)\leqslant 0$ and 0 ¡ $1-\alpha\mu<1$, which indicates that

$$f\left(\boldsymbol{\theta}^{k+1}\right)-f\left(\boldsymbol{\theta}^{*}\right)\leqslant(1-\alpha\mu)(f(\boldsymbol{\theta}^{k})-f(\boldsymbol{\theta}^{*})) \tag{45}$$

we have $f\left(\boldsymbol{\theta}^{K}\right)-f\left(\boldsymbol{\theta}^{0}\right)$ is non-increasing and

$$f\left(\boldsymbol{\theta}^{K}\right)-f\left(\boldsymbol{\theta}^{*}\right)\leqslant(1-\alpha\mu)^{K}(f(\boldsymbol{\theta}^{0})-f(\boldsymbol{\theta}^{*}))\leqslant\epsilon, \tag{46}$$

which indicates AQUILA requires $K=\mathcal{O}\left(-\frac{1}{\log(1-\alpha\mu)}\log\frac{\omega_0}{\epsilon}\right)$ communication round with with $\omega_0=f\left(\boldsymbol{\theta}^{0}\right)-f\left(\boldsymbol{\theta}^{*}\right)$ to achieve $f\left(\boldsymbol{\theta}^{K}\right)-f\left(\boldsymbol{\theta}^{*}\right)\leqslant\epsilon$. □

G AN INTUITIVE METHOD TO CHOOSE β.

From the vanilla gradient descent update, we know it yields the following descent:

$$f\left(\boldsymbol{\theta}^{k+1}\right)-f\left(\boldsymbol{\theta}^{k}\right)\leqslant\Delta_{GD}^{k}, \tag{47}$$

where $\Delta_{GD}^{k}=-(1-\frac{\alpha L}{2})\alpha\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2$, α denotes the step size, and $L>0$ denotes the Lipschitz constant.

At the same time, we denote Δ_{AQ}^{k} by $-\frac{\alpha}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\left(\frac{L}{2}-\frac{1}{2\alpha}\right)\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2+\alpha\left\|\boldsymbol{\varepsilon}^{k}\right\|_2^2+\alpha\left\|\frac{1}{M}\sum_{m\in\mathcal{M}_c^k}\Delta\boldsymbol{q}_m^k\right\|_2^2$ according to (14). For simplicity, we temporarily set $\alpha=1/L$.

If we want AQUILA to be more efficient than the vanilla gradient descent, then

$$\frac{\Delta_{AQ}^{k}}{|\mathcal{M}^{k}|}\leqslant\frac{\Delta_{GD}^{k}}{M}, \tag{48}$$

that is,

$$\begin{aligned}\frac{-\alpha/2\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2+\alpha\left\|\Phi^{k}\right\|_2^2+\alpha\left\|\boldsymbol{\varepsilon}^{k}\right\|_2^2}{|\mathcal{M}^{k}|}&\leqslant\frac{-\alpha/2\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2}{M}\\ \left\|\Phi^{k}\right\|_2^2+\left\|\boldsymbol{\varepsilon}^{k}\right\|_2^2&\leqslant\frac{|\mathcal{M}_c^k|}{2M}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2\\ \frac{|\mathcal{M}_c^k|}{M^2}\sum_{m\in\mathcal{M}_c^k}\left(\left\|\Delta\boldsymbol{q}_m^k\right\|_2^2+\gamma\left\|\boldsymbol{\varepsilon}_m^k\right\|_2^2\right)&\overset{(27)}{\leqslant}\frac{|\mathcal{M}_c^k|}{2M}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2\\ \sum_{m\in\mathcal{M}_c^k}\left(\left\|\Delta\boldsymbol{q}_m^k\right\|_2^2+\gamma\left\|\boldsymbol{\varepsilon}_m^k\right\|_2^2\right)&\overset{(27)}{\leqslant}\frac{M}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2.\end{aligned} \tag{49}$$

Thus, for each device m at communication round k, if it satisfies

$$\begin{aligned}\left\|\Delta\boldsymbol{q}_m^k\right\|_2^2+\gamma\left\|\boldsymbol{\varepsilon}_m^k\right\|_2^2&\leqslant\frac{1}{2}\left\|\nabla f\left(\boldsymbol{\theta}^{k}\right)\right\|_2^2\\ &\leqslant\frac{1}{2\alpha^2}\left\|\boldsymbol{\theta}^{k+1}-\boldsymbol{\theta}^{k}\right\|_2^2,\end{aligned} \tag{50}$$

then (49) will also be satisfied, that is, $\beta\leqslant 0.5$ compared to (11). In our paper, we set $\beta=0.25$ while evaluating the performance of AQUILA.