Taming Communication and Sample Complexities in Decentralized Policy Evaluation for Cooperative Multi-Agent Reinforcement Learning

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Abstract

Cooperative multi-agent reinforcement learning (MARL) has received increasing attention in recent years and has found many scientific and engineering applications. However, a key challenge arising from many cooperative MARL algorithm designs (e.g., the actor-critic framework) is the policy evaluation problem, which can only be conducted in a *decentralized* fashion. In this paper, we focus on decentralized MARL policy evaluation with nonlinear function approximation, which is often seen in deep MARL. We first show that the empirical decentralized MARL policy evaluation problem can be reformulated as a decentralized nonconvex-strongly-concave minimax saddle point problem. We then develop a decentralized gradient-based descent ascent algorithm called GT-GDA that enjoys a convergence rate of $\mathcal{O}(1/T)$. To further reduce the sample complexity, we propose two decentralized stochastic optimization algorithms called GT-SRVR and GT-SRVRI, which enhance GT-GDA by variance reduction techniques. We show that all algorithms all enjoy an $\mathcal{O}(1/T)$ convergence rate. to a stationary point of the reformulated minimax problem. Moreover, the fast convergence rates of GT-SRVR and GT-SRVRI imply $\mathcal{O}(\epsilon^{-2})$ communication complexity and $\mathcal{O}(m\sqrt{n}\epsilon^{-2})$ sample complexity, where m is the number of agents and n is the length of trajectories. To our knowledge, this paper is the first work that achieves $\mathcal{O}(\epsilon^{-2})$ in both sample and communication complexities in decentralized policy evaluation for cooperative MARL. Our extensive experiments also corroborate the theoretical results of our proposed decentralized policy evaluation algorithms.

1 Introduction

In recent years, multi-agent reinforcement learning (MARL) has found important applications in many scientific and engineering fields, such as robotic network [38, 19, 34], sensor network [6, 33, 36], and power network [5, 10, 13, 15], just to name a few. In MARL, *multiple* agents observe the current joint state over a network, perform their own actions based on the current state, and transition to the next joint state. Each agent can only observe its local reward, which is a function of the joint state and actions. One important paradigm in MARL is the *cooperative* MARL, where the agents share a common goal to find an optimal global policy to achieve the maximum global accumulative reward [47, 46, 12, 30]. As in classical reinforcement learning (RL) problems, a key component in cooperative MARL algorithms is the policy evaluation (PE) problem, whose goal is to evaluate the expected long-term accumulative reward for a given global policy. The PE problem emerges as an essential step for the agents to find an optimal global policy in MARL tasks [3, 20]. For example, in the actor-critic algorithmic framework for MARL, the actors conduct the policy improvement step, while the critics perform the policy evaluation step and estimates the value function. The whole

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actor-critic algorithm tries to find an optimal policy by iterating between the policy evaluation and improvement steps. Thus, developing efficient policy evaluation algorithms is critical to the success of RL algorithms based on the actor-critic framework.

However, developing efficient PE algorithms for MARL is highly non-trivial. On one hand, the global accumulative reward is not directly observable in an MARL system. As a result, the PE problem of MARL can only be solved in a *decentralized* fahsion. On the other hand, modern MARL tasks have been increasingly complex and often not directly computable. As a result, MARL often use highly nonlinear parametric models (e.g., deep neural network (DNN)) for policy approximation. However, it is often hard to guarantee the convergence performance of RL algorithms with such nonlinear function approximations [40]. In light of the growing importance of MARL, in this paper, we focus on addressing the above challenges. The key contributions of this paper are summarized as follows:

- To our knowledge, this work is the *first* to investigate the decentralized PE (DPE) problem for MARL with *nonlinear* function approximation. Via the Fenchel's duality and local reward decomposition, we first reformulate the DPE problem of MARL as a decentralized non-convex-strongly-concave minimax saddle point problem. To solve the minimax problem in a decentralized fashion, we propose a gradient-tracking based gradient descent-ascent (GT-GDA) algorithm. We show that GT-GDA enjoys a convergence rate of $\mathcal{O}(1/T)$, which leads to an $\mathcal{O}(mn\epsilon^{-2})$ sample complexity and $\mathcal{O}(\epsilon^{-2})$ communication complexity, where m is the number of agents, n is the data size, and ϵ is the convergence accuracy.
- To further reduce the sample complexity, we develop two variance reduced algorithms, namely gradient-tracking stochastic recursive variance reduction (GT-SRVR) algorithm and its variant with incremental batch size (GT-SRVRI). We show that both algorithms achieve the same communication complexity $\mathcal{O}(\epsilon^{-2})$ as GT-GDA, but requiring a lower sample complexity $\mathcal{O}(m\sqrt{n}\epsilon^{-2})$.
- It is worth noting that, in our theoretical analysis, we relax the commonly-used compactness conditions of the feasible set with some mild assumptions on objectives. Thus, the solutions found by our algorithms are exactly the stationary points for the original policy evaluation problem. This result may be of independent interest for general RL problems.

The rest of the paper is organized as follows. In Section 2, we first provide the preliminaries of the DPE problem of MARL and discuss related works. In Section 3, we first introduce the GT-GDA algorithm, and then propose two stochastic variance reduced algorithms, namely GT-SRVR and GT-SRVRI. We present their theoretical properties in Section 4. Section 5 provides numerical results to verify our theoretical findings, and Section 6 concludes this paper.

2 Problem formulation and related work

In this section, we first introduce the DPE problem formulation in in Section 2.1. Then in Section 2.2, we review the recent developments of PE algorithms and compare them with our work. In Section 2.3, we highlight the challenges in designing efficient DPE algorithms with nonlinear function approximation and the significance of our work.

2.1 Problem formulation of decentralized policy evaluation for MARL

Consider a multi-agent network system $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ and $\mathcal{L}$ denote the sets of agents and edges, respectively, with $|\mathcal{N}| = m$. In the system, the agents cooperatively to perform a learning task. The agents can communicate with each other through edges in $\mathcal{L}$. An MARL problem is formulated based on the multi-agent Markov decision process (MDP), which is characterized by a quintuple $(\mathcal{S}, \mathcal{A}, \mathcal{P}^{\mathbf{a}}_{\mathbf{s}\mathbf{s}'}, \{\mathcal{R}_i(\mathbf{s}, \mathbf{a})\}_{i=1}^m, \gamma)$, where $\mathcal{S}$ and $\mathcal{A}$ are the state and action spaces, respectively; $\mathbf{s} \in \mathcal{S}$ and $\mathbf{a} \in \mathcal{A}$ are joint state and action; $\mathcal{P}^{\mathbf{a}}_{\mathbf{s}\mathbf{s}'}$ is the transition probability from state $\mathbf{s}$ to state $\mathbf{s}'$ after taking action $\mathbf{a}$; $\mathcal{R}_i(\mathbf{s}, \mathbf{a})$ is the local reward received by agent i after taking action $\mathbf{a}$ in state $\mathbf{s}$; and $\gamma \in (0, 1)$ is a discount factor. Both the joint state $\mathbf{s}$ and action $\mathbf{a}$ are available to all agents, while the local reward $\mathcal{R}_i$ is private to agent i. In the multi-agent system, the global reward function is defined as the average of the local rewards $\frac{1}{m}\sum_{i=1}^m \mathcal{R}_i(\mathbf{s}, \mathbf{a})$. Moreover, a joint policy π specifies sequential decision rules for all agents. Policy $\pi(\mathbf{a}|\mathbf{s})$ is the conditional probability of taking joint action $\mathbf{a}$ given state $\mathbf{s}$. The goal of PE is to estimate the value function of a given policy π, which is defined as the long-term discounted accumulative reward as: $\mathcal{V}^\pi(\mathbf{s}_0) = \mathbb{E}\left[\frac{1}{m}\sum_{t=0}^\infty \gamma^t \sum_{i=1}^m \mathcal{R}_i(\mathbf{s}_t, \mathbf{s}_{t+1})|\mathbf{s}_0, \pi\right]$, where the expectation is taken over all possible state-action trajectories and initial state $\mathbf{s}_0$.

To determine $\mathcal{V}^\pi(\cdot)$, one of the most effective methods is the temporal-difference (TD) learning algorithm, which focuses on solving the Bellman equation for $\mathcal{V}^\pi(\cdot)$: $\mathcal{V}(\mathbf{s}) = \mathcal{T}^\pi \mathcal{V}(\mathbf{s}) \triangleq \frac{1}{m}\sum_{i=1}^m \mathcal{R}_i^\pi(\mathbf{s}) + \gamma \sum_{s'\in\mathcal{S}} \mathcal{P}_{\mathbf{ss}'}^\pi \mathcal{V}(\mathbf{s}')$, where $\mathcal{T}^\pi$ denotes the Bellman operator, $\mathcal{R}_i^\pi(\mathbf{s}) = \mathbb{E}_{\mathbf{a}\sim\pi(\cdot|\mathbf{s})}\mathcal{R}(\mathbf{s},\mathbf{a})$ and $\mathcal{P}_{\mathbf{ss}'}^\pi = \mathbb{E}_{\mathbf{a}\sim\pi(\cdot|\mathbf{s})}\mathcal{P}_{\mathbf{ss}'}^a$. However, $\mathcal{P}_{\mathbf{ss}'}^\pi$ is unknown in MARL and the size of the state space $\mathcal{S}$ could be infinite. To address this challenge, a widely adopted approach is to approximate $\mathcal{V}^\pi(\cdot)$ by a function $\mathcal{V}_{\boldsymbol{\theta}}(\cdot)$ parameterized by $\boldsymbol{\theta} \in \mathbb{R}^p$. According to the formulation in [31, 41, 9, 8], the Bellman equation can be solved by minimizing the following mean-squared projected bellman error (MSPBE): $\text{MSPBE}(\boldsymbol{\theta}) \triangleq \frac{1}{2}\big\|\mathbb{E}_{\mathbf{s}\sim d^\pi}\big[(\mathcal{T}^\pi\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}) - \mathcal{V}_{\boldsymbol{\theta}})\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s})^\top\big]\big\|^2_{\mathbf{K}_{\boldsymbol{\theta}}^{-1}}$, where $\mathbf{K}_{\boldsymbol{\theta}} = \mathbb{E}_{\mathbf{s}\sim d^\pi}[\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s})\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s})^\top] \in \mathbb{R}^{p\times p}$ and d^π is the stationary distribution of the MDP under policy π. From the Fenchel's duality $\|\mathbf{x}\|^2_{\mathbf{A}^{-1}} = \max_{\mathbf{y}\in\mathbb{R}^p} 2\langle \mathbf{x},\mathbf{y}\rangle - \mathbf{y}^\top\mathbf{A}\mathbf{y}$, we can reformulate the MSPBE minimization problem as the following primal-dual minimax problem:

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p}\max_{\boldsymbol{\omega}\in\mathbb{R}^p} \mathcal{L}(\boldsymbol{\theta},\boldsymbol{\omega}) \triangleq \mathbb{E}\big[\langle \boldsymbol{\delta}\cdot\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}),\boldsymbol{\omega}\rangle - \frac{1}{2}\boldsymbol{\omega}^\top[\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s})\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s})^\top]\boldsymbol{\omega}\big], \tag{1}$$

where the expectation is taken over $\mathbf{s}\sim d^\pi(\cdot)$, $\mathbf{a}\sim\pi(\cdot|\mathbf{s})$, $\mathbf{s}'\sim\mathcal{P}_{\mathbf{s}\cdot}^{\mathbf{a}}$, and $\boldsymbol{\delta} = \frac{1}{m}\sum_{i=1}^m \mathcal{R}_i(\mathbf{s},\mathbf{a}) + \gamma\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}') - \mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s})$. In practice, we only have access to a finite dataset with n-step trajectories $\mathcal{D} = \big\{(\mathbf{s}_t,\mathbf{a}_t,\{\mathcal{R}_i(\mathbf{s}_t,\mathbf{a}_t)\}_{i=1}^n,\mathbf{s}_{t+1})\big\}_{t=0}^n$. By replacing the unknown expectation with the finite sample average, we have the following empirical minimax problem:

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p}\max_{\boldsymbol{\omega}\in\mathbb{R}^p} F(\boldsymbol{\theta},\boldsymbol{\omega}) = \frac{1}{m}\sum_{t=1}^m \langle \boldsymbol{\delta}_t\cdot\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t),\boldsymbol{\omega}\rangle - \frac{1}{2}\boldsymbol{\omega}^\top\widehat{\mathbf{K}}_{\boldsymbol{\theta}}\boldsymbol{\omega}, \tag{2}$$

where $\boldsymbol{\delta}_t \triangleq \frac{1}{m}\sum_{i=1}^m \mathcal{R}_i(\mathbf{s}_t,\mathbf{a}_t) + \gamma\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_{t+1}) - \mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t)$ and $\widehat{\mathbf{K}}_{\boldsymbol{\theta}} \triangleq \frac{1}{n}\sum_{t=1}^n \nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t)\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t)^\top$. In this paper, we assume that both $\mathbf{K}_{\boldsymbol{\theta}}$ and its empirical estimate $\widehat{\mathbf{K}}_{\boldsymbol{\theta}}$ are positive definite for all $\boldsymbol{\theta}$. Define $J(\boldsymbol{\theta}) \triangleq F(\boldsymbol{\theta},\boldsymbol{\omega}^*) = \max_{\boldsymbol{\omega}\in\mathbb{R}^p} F(\boldsymbol{\theta},\boldsymbol{\omega})$, where $\boldsymbol{\omega}^* = \arg\max_{\boldsymbol{\omega}\in\mathbb{R}^p} F(\boldsymbol{\theta},\boldsymbol{\omega})$. $J(\boldsymbol{\theta})$ can be viewed as the finite empirical version of MSPBE. Here, we aim to minimize $J(\boldsymbol{\theta})$ by finding a stationary point of $F(\boldsymbol{\theta},\boldsymbol{\omega})$. Recall that in MARL, the local reward is only observable for each individual agent. Thus, it is hard to obtain the global reward $\frac{1}{m}\sum_{i=1}^m \mathcal{R}_i(\mathbf{s}_t,\mathbf{a}_t)$ and $\boldsymbol{\delta}_t$ in a multi-agent network. To address this challenge, we define $\boldsymbol{\delta}_{i,t} = \mathcal{R}_i(\mathbf{s}_t,\mathbf{a}_t) + \gamma\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_{t+1}) - \mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t)$ and decompose the minimax problem in (2) as follows: $\min_{\boldsymbol{\theta}\in\mathbb{R}^p}\max_{\boldsymbol{\omega}\in\mathbb{R}^p} F(\boldsymbol{\theta},\boldsymbol{\omega}) = \frac{1}{m}\sum_{i=1}^m F_i(\boldsymbol{\theta},\boldsymbol{\omega}) = \frac{1}{mn}\sum_{i=1}^m\sum_{t=1}^n f_{it}(\boldsymbol{\theta},\boldsymbol{\omega})$, where $f_{it}(\boldsymbol{\theta},\boldsymbol{\omega}) \triangleq \langle \boldsymbol{\delta}_{i,t}\cdot\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t),\boldsymbol{\omega}\rangle - \frac{1}{2}\boldsymbol{\omega}^\top[\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t)\,\nabla_{\boldsymbol{\theta}}\mathcal{V}_{\boldsymbol{\theta}}(\mathbf{s}_t)^\top]\boldsymbol{\omega}$. We call this step as local reward decomposition. Further, to solve Problem (2) in a *decentralized* fashion, we can rewrite it in the following equivalent form:

$$\begin{aligned}&\min_{\{\boldsymbol{\theta}_i\}_{i=1}^m}\max_{\{\boldsymbol{\omega}_i\}_{i=1}^m} \frac{1}{m}\sum_{i=1}^m F_i(\boldsymbol{\theta}_i,\boldsymbol{\omega}_i) = \frac{1}{mn}\sum_{i=1}^m\sum_{t=1}^n f_{it}(\boldsymbol{\theta}_i,\boldsymbol{\omega}_i),\\ &\text{subject to}\quad \boldsymbol{\theta}_i = \boldsymbol{\theta}_j, \boldsymbol{\omega}_i = \boldsymbol{\omega}_j,\ \ \forall (i,j)\in\mathcal{L},\end{aligned} \tag{3}$$

where $\boldsymbol{\theta}_i$ and $\boldsymbol{\omega}_i$ are the local copies of the original primal-dual parameters at agent i. In (3), the equality constraint ensures that the local copies at all nodes are equal to each other, so the formulation is also referred to as "consensus form." Clearly, Problems (2) and (3) are equivalent. For a fixed $\boldsymbol{\theta}$, each local function $F_i(\boldsymbol{\theta}_i,\cdot)$ is a strongly concave function of $\boldsymbol{\omega}$. For a fixed $\boldsymbol{\omega}$, $F_i(\cdot,\boldsymbol{\omega})$ is a non-convex function of $\boldsymbol{\theta}$. Thus, Problem (3) is a decentralized non-convex-strongly-concave minimax consensus optimization problem. In this paper, we adopt two complexity metrics that are widely used in the decentralized optimization literature (e.g., [39]) to measure the efficiency of an algorithm:

Definition 1 (Sample Complexity)**.** *The sample complexity is defined as the total number of the incremental first-order oracle (IFO) calls required across all the nodes until algorithm converges, where one IFO call evaluates a pair of* $(f_{it}(\boldsymbol{\theta},\boldsymbol{\omega}),\nabla f_{it}(\boldsymbol{\theta},\boldsymbol{\omega}))$ *at node* i*.*

Definition 2 (Communication Complexity)**.** *The communication complexity is defined as the total rounds of communications required until algorithm converges, where each node can send and receive a* p*-dimensional vector with its neighboring nodes in one communication round.*

2.2 Related work on policy evaluation

1) The tabular approach: The study of MARL under the MDP formulation traces its roots to the seminal work by [24]. Motivated by this formalization, several methods have been developed to solve

Table 1: Comparisons among existing policy evaluation algorithms, where m is the number of agents; n is the size of dataset; ϵ^2 is the convergence error. Our proposed algorithms are marked in bold.

Algorithm	Reference	Decentralized Multi-Agent	Nonlinear Approxi.	Convex Sets[1]	Sample Complex.	Commun. Complex.
STSG	[35]	✗	✓	✓	$\mathcal{O}(\epsilon^{-4})$	-
ASTSG		✗	✓	✓	$\mathcal{O}(\epsilon^{-3})$	-
DHPD	[11]	✓	✗	✓	$\mathcal{O}(mn\epsilon^{-2})$	$\mathcal{O}(\epsilon^{-2})$
PD-DistIAG	[42]	✓	✗	✗	$\mathcal{O}(m\log\epsilon^{-2})$	$\mathcal{O}(\log\epsilon^{-2})$
APP-SAG	[37]	✓	✗	✗	$\mathcal{O}(m\log\epsilon^{-2})$	$\mathcal{O}(\log\epsilon^{-2})$
DTDT	[14]	✓	✗	✗	$\mathcal{O}(m\log\epsilon^{-2})$	$\mathcal{O}(\log\epsilon^{-2})$
GT-GDA	Theorem 1	✓	✓	✗	$\mathcal{O}(mn\epsilon^{-2})$	$\mathcal{O}(\epsilon^{-2})$
GT-SRVR	Theorem 3	✓	✓	✗	$\mathcal{O}(m\sqrt{n}\epsilon^{-2})$	$\mathcal{O}(\epsilon^{-2})$
GT-SRVRI	Theorem 4					

[1] The feasible parameter spaces are required to be closed convex sets.

and analyze MARL problems, including [21, 43, 16, 1], etc. However, most of these early works approximate the value function in a tabular form, which only works for cases where the state and action spaces are relatively small. For complex MARL tasks where the state space is large or even infinite, the tabular approach becomes intractable.

2) Policy evaluation with linear function approximation: To address limitation in tabular approaches for MARL, the work in [22] proposed to estimate the value function with a *linear approximation* (i.e., $\mathcal{V}(\mathbf{s}) \approx \boldsymbol{\phi}(\mathbf{s})^\top \boldsymbol{\theta}, \boldsymbol{\theta} \in \mathbb{R}^p$, where $\boldsymbol{\phi} : \mathcal{S} \to \mathbb{R}^p$ is a feature mapping) and developed a distributed gradient temporal-difference (DGTD) algorithm. However, this work only considered asymptotic convergence analysis and required diminishing step-sizes to ensure convergence. In [11], the authors proposed a distributed homotopy primal-dual algorithm (DHPD) for the PE problem in MARL. They also cast MSPBE minimization as a stochastic primal-dual optimization problem, where the objective is convex in primal and strongly-concave in dual. By using an adaptive restarting scheme, DHPD achieves an $\mathcal{O}(1/T)$ convergence rate in finding stationary points. The work in [12] developed a distributed consensus-based TD(0) algorithm, which integrates the network consensus step and local TD(0) updates. They provided a finite-time analysis and showed a convergence rate of $\mathcal{O}(1/T)$. To further improve convergence, the work in [42] proposed a primal-dual distributed incremental aggregated gradient (PD-DistIAG) method to integrate gradient-tracking and incremental aggregated gradient methods to achieve linear convergence. However, a major limitation of the linear approximation approach is that it is not applicable for nonlinear MARL models (e.g., DNN).

3) Policy evaluation with nonlinear function approximation (single-agent): In the literature, PE with nonlinear approximation is by far only limited to *single-agent* RL. Here, we provide a quick overview in this area. To our knowledge, the work in [4] was the first to study the PE problem with nonlinear approximation and developed a nonlinear TD algorithm. However, the proposed algorithms adopted two-timescale step-sizes[1], resulting in a slow convergence performance. Recently, the work in [41] showed that PE with nonlinear approximation in RL is equivalent to a non-convex-strongly-concave minimax optimization problem. To find a stationary point for such minimax problem, they proposed a non-convex primal-dual gradient with variance reduced (nPD-VR) algorithm. However, their algorithm requires an $\mathcal{O}(1/m)$ step-size, where m is the size of the dataset. This is problematic in cases with a large transition dataset. More recently, the authors of [35] proposed two single-time scale first-order stochastic algorithms for the nonconvex-strongly-concave minimax optimization. These two algorithms utilized stochastic gradient with momentum and variance-reduced momentum [7], and achieved $\mathcal{O}(1/\sqrt{T})$ and $\mathcal{O}(1/T^{2/3})$ convergence rates, respectively. However, diminishing step-sizes are required in these two algorithms, which do not work well in practice.

2.3 Significance of our work and challenges of DPE with nonlinear function approximation

To our knowledge, our work is the *first* to solve the DPE problem with *nonlinear* function approximation for MARL. However, such nonlinear approximation imposes several significant challenges on algorithm development and analysis. As discussed in Section 2.1, we propose to reformulate the DPE problem as a *decentralized* nonconvex-strongly-concave minimax optimization problem. In the

[1] A primal-dual algorithm is a two-timescale if $\gamma_t/\eta_t \to 0$ or $\gamma_t/\eta_t \to \infty$ as $t \to \infty$, where γ_t and η_t denote the primal and dual step-sizes at time t, respectively.

Algorithm 1 GT-GDA Algorithm at Agent i.

1: Set prime-dual parameter pair $(\boldsymbol{\theta}_{i,0}, \boldsymbol{\omega}_{i,0}) = (\boldsymbol{\theta}^0, \boldsymbol{\omega}^0)$.
2: Calculate local gradients as $\mathbf{p}_{i,0} = \nabla_{\boldsymbol{\theta}} F_i(\boldsymbol{\theta}_{i,0}, \boldsymbol{\omega}_{i,0})$, and $\mathbf{d}_{i,0} = \nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}_{i,0}, \boldsymbol{\omega}_{i,0})$;
3: **for** $t = 1, \cdots, T$ **do**
4: Update local parameters $(\boldsymbol{\theta}_{i,t+1}, \boldsymbol{\omega}_{i,t+1})$ as in (6);
5: Calculate local gradients $(\mathbf{v}_{i,t+1}, \mathbf{u}_{i,t+1})$ as in (4);
6: Track global gradients $(\mathbf{p}_{i,t+1}, \mathbf{d}_{i,t+1})$ as in (5);
7: **end for**

literature, although a few works [26, 25, 28] have studied similar decentralized minimax problem, their variable updates rely on either stochastic or full gradients, which are not sample/communication-efficient for MARL. To address these limitations, we first propose two decentralized variance-reduced algorithms for solving the nonconvex-strongly-concave DPE minimax problem, for which establishing the convergence rates is highly challenging. Second, we adopt the gradient tracking (GT) technique to reduce network consensus error. Under nonlinear function approximation, our algorithms need to track the gradients for both primal and dual variables. However, such "double gradient tracking" introduces new constraints on algorithm design. Third, many of the existing nonconvex-strongly-concave minimax optimization methods (e.g. [35, 41]) require prior knowledge of the compact domain of model variables, which cannot be assumed in DPE for MARL. Such unboundedness in DPE creates new challenges and necessitates new proof techniques in our algorithm analysis. To conclude this section, we summarize all related work in Table 1.

3 Gradient-tracking gradient descent ascent algorithm.

In this section, we first present a gradient-tracking gradient descent ascent (GT-GDA) method for solving the DPE problem in (3) for MARL, and then provide the its theoretical results.

1) The Algorithm: For the consensus problem in (3), a popular approach is to let agents aggregate their neighbor information through a consensus weight matrix $\mathbf{M} \in \mathbb{R}^{m \times m}$ [32, 42]. Let $[\mathbf{M}]_{ij}$ denote the element in the i-th row and the j-th column in $\mathbf{M}$. $\mathbf{M}$ satisfies the following properties: (a) *Doubly stochastic:* $\sum_{i=1}^m [\mathbf{M}]_{ij} = \sum_{j=1}^m [\mathbf{M}]_{ij} = 1$; (b) *Symmetric:* $[\mathbf{M}]_{ij} = [\mathbf{M}]_{ji}, \forall i, j \in \mathcal{N}$; and (c) *Network-Defined Sparsity:* $[\mathbf{M}]_{ij} > 0$ if $(i, j) \in \mathcal{L}$; otherwise $[\mathbf{M}]_{ij} = 0, \forall i, j \in \mathcal{N}$. The above properties imply that the eigenvalues of $\mathbf{M}$ are real and can be sorted as $-1 < \lambda_m(\mathbf{M}) \leq \cdots \leq \lambda_2(\mathbf{M}) < \lambda_1(\mathbf{M}) = 1$. We define the second-largest eigenvalue in magnitude of $\mathbf{M}$ as $\lambda \triangleq \max\{|\lambda_2(\mathbf{M})|, |\lambda_m(\mathbf{M})|\}$. Our GT-GDA algorithm for each agent i is illustrated in Algorithm 1. Specifically, at the tth iteration, agent i first calculates the local full gradients as follows:

$$\mathbf{v}_{i,t} = \nabla_{\boldsymbol{\theta}} F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}), \quad \mathbf{u}_{i,t} = \nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}). \tag{4}$$

Note that $\mathbf{v}_{i,t}$ and $\mathbf{u}_{i,t}$ only contain the gradient information of the local objective function $F_i(\boldsymbol{\theta}, \boldsymbol{\omega})$. Thus, merely updating with $\mathbf{v}_{i,t}$ and $\mathbf{u}_{i,t}$ cannot guarantee the convergence of the global objective function $F(\boldsymbol{\theta}, \boldsymbol{\omega})$. To address this challenge, we introduce two auxiliary variables, $\mathbf{p}_{i,t}$ and $\mathbf{d}_{i,t}$. The agent updates the two variables by performing the following local weighted aggregation:

$$\mathbf{p}_{i,t} = \sum_{j \in \mathcal{N}_i} [\mathbf{M}]_{ij} \mathbf{p}_{j,t-1} + \mathbf{v}_{i,t} - \mathbf{v}_{i,t-1}, \quad \mathbf{d}_{i,t} = \sum_{j \in \mathcal{N}_i} [\mathbf{M}]_{ij} \mathbf{d}_{j,t-1} + \mathbf{u}_{i,t} - \mathbf{u}_{i,t-1}, \tag{5}$$

where $\mathcal{N}_i \triangleq \{j \in \mathcal{N}, : (i, j) \in \mathcal{L}\}$ denotes the set of agent i's neighbors. Technically, $\mathbf{p}_{i,t}$ and $\mathbf{d}_{i,t}$ track the directions of global gradients. With some derivations, it can be shown that $\sum_{i \in \mathcal{N}} \mathbf{p}_{i,t} = \sum_{i \in \mathcal{N}} \nabla_{\boldsymbol{\theta}} F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t})$ and $\sum_{i \in \mathcal{N}} \mathbf{d}_{i,t} = \sum_{i \in \mathcal{N}} \nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t})$. Lastly, each agent updates local parameters following the conventional decentralized gradient descent and ascent steps:

$$\boldsymbol{\theta}_{i,t+1} = \sum_{j \in \mathcal{N}_i} [\mathbf{M}]_{ij} \boldsymbol{\theta}_{j,t} - \gamma \mathbf{p}_{i,t}, \quad \boldsymbol{\omega}_{i,t+1} = \sum_{j \in \mathcal{N}_i} [\mathbf{M}]_{ij} \boldsymbol{\omega}_{j,t} + \eta \mathbf{d}_{i,t}, \tag{6}$$

where the constants γ and η are the step-sizes for individual primal and dual variables, respectively.

2) Theoretical Results of GT-GDA: In this section, we will establish the convergence behaviors of the proposed GT-GDA algorithm. Toward this end, we first state several assumptions as follows:

Assumption 1. *The function* $F(\boldsymbol{\theta}, \boldsymbol{\omega}) = \frac{1}{m} \sum_{i=1}^m F_i(\boldsymbol{\theta}, \boldsymbol{\omega})$ *and* $J(\boldsymbol{\theta}) = \max_{\boldsymbol{\omega} \in \mathbb{R}^p} F(\boldsymbol{\theta}, \boldsymbol{\omega})$ *satisfy:*

(a) *(Boundness from Below): There exists a finite lower bound* $J^* = \inf_{\boldsymbol{\theta}} J(\boldsymbol{\theta}) > -\infty$;

(b) *(Lipschitz Smoothness): Local objective function* $F_i(\boldsymbol{\theta}, \cdot)$ *is* L_F*-Lipschitz smooth, i.e., there exists a positive constant* L_F *such that the gradient* $\nabla F_i(\boldsymbol{\theta}, \boldsymbol{\omega}) = [\nabla_{\boldsymbol{\theta}} F_i(\boldsymbol{\theta}, \boldsymbol{\omega})^\top, \nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}, \boldsymbol{\omega})^\top]^\top$ *satisfies* $\|\nabla F_i(\boldsymbol{\theta}, \boldsymbol{\omega}) - \nabla F_i(\boldsymbol{\theta}', \boldsymbol{\omega}')\|^2 \leq L_F^2 \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|^2 + L_F^2 \|\boldsymbol{\omega} - \boldsymbol{\omega}'\|^2, \forall\, \boldsymbol{\theta}, \boldsymbol{\theta}', \boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbb{R}^p, i \in [m]$;

(c) *(Strong Concavity in Dual): Local objective function* $F_i(\boldsymbol{\theta}, \cdot)$ *is* μ*-strongly concave for fixed* $\boldsymbol{\theta} \in \mathbb{R}^p$*, i.e., there exists a positive constant* μ *such that* $\|\nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}, \boldsymbol{\omega}) - \nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}, \boldsymbol{\omega}')\| \geq \mu \|\boldsymbol{\omega} - \boldsymbol{\omega}'\|, \forall\, \boldsymbol{\theta}, \boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbb{R}^p, i \in [m]$

(d) *(Bounded Dual Maximizer): For any primal variable* $\boldsymbol{\theta} \in \mathbb{R}^p$*, its associated dual maximizer* $\boldsymbol{\omega}^*(\boldsymbol{\theta}) \triangleq \arg\max_{\boldsymbol{\omega} \in \mathbb{R}^p} F(\boldsymbol{\theta}, \boldsymbol{\omega})$ *is bounded, i.e.,* $\|\boldsymbol{\omega}^*(\boldsymbol{\theta})\| < \infty$;

(e) *(Bounded Gradient at Maximum): The partial derivative at maximum point* $\nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))$ *is bounded, i.e.,* $\|\nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))\| < \infty, \forall\, \boldsymbol{\theta} \in \mathbb{R}^p$.

In these assumptions, (a) and (b) are standard in literature. It can be verified that (c) holds when $\widehat{\mathbf{K}}_{\boldsymbol{\theta}}$ is positive definite; (d)-(e) guarantee that $\nabla J(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))$ (see Lemma 10 in the supplementary material). Note that most of the existing works [23, 35] adopt the compactness assumption to ensure such gradient equivalence. The compactness assumption restricts the feasible parameter space as a closed convex set. Although we also make these boundedness assumptions, the convergence performance of our algorithm is *independent* of the upper bound of $\|\boldsymbol{\omega}^*(\boldsymbol{\theta})\|$ and $\|\nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))\|$. To quantify the convergence rate, we propose to use the following *new* metric, which is the key to the success of establishing all convergence results in this paper:

$$\mathfrak{M}_t \triangleq \|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 + 2\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{1}{m}\sum_{i=1}^{m} \big(\|\boldsymbol{\theta}_{i,t} - \bar{\boldsymbol{\theta}}_t\|^2 + \|\boldsymbol{\omega}_{i,t} - \bar{\boldsymbol{\omega}}_t\|^2\big), \tag{7}$$

where $\boldsymbol{\omega}_t^*$ denotes $\boldsymbol{\omega}^*(\bar{\boldsymbol{\theta}}_t) = \arg\max_{\boldsymbol{\omega} \in \mathbb{R}^p} F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega})$. The first term in (7) measures the convergence of primal variable $\boldsymbol{\theta}$: $\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 = 0$ indicates that $\bar{\boldsymbol{\theta}}_t$ is a first-order stationary point for $J(\cdot)$. The second term in (7) measures $\bar{\boldsymbol{\omega}}_t$'s convergence to the unique maximizer $\boldsymbol{\omega}_t^*$ for $F(\bar{\boldsymbol{\theta}}_t, \cdot)$. The last term in (7) is the average consensus error of local copies. Thus, as $\mathfrak{M}_t \to 0$, we have that the algorithm reaches a consensus first-order stationary point of the original MSPBE problem. Based on the metric in (7), we have the following:

Theorem 1 (Convergence of GT-GDA)**.** *Under Assumption 1, if the step-sizes satisfy that* $\kappa \triangleq \gamma/\eta \leq \mu^2/13L_F^2$ *and* $\eta \leq \min\{k_1, k_2, k_3, k_4\}$*, then GT-GDA has the following convergence result:*

$$\frac{1}{T+1}\sum_{t=0}^{T} \mathbb{E}[\mathfrak{M}_t] \leq \frac{2\mathbb{E}[\mathfrak{P}_0 - J^*]}{\min\{1, L_F^2\}(T+1)\gamma},$$

where $\mathfrak{P}_t$ *is a potential function defined as:*

$$\mathfrak{P}_t \triangleq J(\bar{\boldsymbol{\theta}}_t) + \frac{8\gamma L_F^2}{\mu\eta}\|\bar{\boldsymbol{\omega}}_t - \boldsymbol{\omega}_t^*\|^2 + \frac{1}{m}\sum_{i=1}^{m}\|\boldsymbol{\theta}_{i,t} - \bar{\boldsymbol{\theta}}_t\|^2 + \|\boldsymbol{\omega}_{i,t} - \bar{\boldsymbol{\omega}}_t\|^2 + \gamma\|\mathbf{p}_{i,t} - \bar{\mathbf{p}}_t\|^2 + \eta\|\mathbf{d}_{i,t} - \bar{\mathbf{d}}_t\|^2,$$

and the constants in the step-size η *are as follows:*

$$k_1 = \frac{13L_F^2}{2\mu^2}\Big(L_F + \frac{L_F^2}{\mu} + (1-\lambda)\Big), \quad k_2 = \frac{13L_F^2}{\mu^2\big(1/2 + 1/(1-\lambda)^2\big)}, \quad k_3 = \frac{(1-\lambda)}{6\mu(1+1/\kappa)},$$

$$k_4 = \frac{26(1-\lambda)L_F^2}{(\mu^2 + 144L_F^4 + 4L_F^2\mu^2 + \frac{48\mu^2 L_F^2(1+1/\kappa)}{1-\lambda})}.$$

Remark 1. In Theorem 1, the step-sizes and convergence rate depend on by the network topology. For a sparse network, λ is close to (but not exactly) one (recall that $\lambda = \max\{|\lambda_2|, |\lambda_m|\} < 1$). Therefore, k_2 and k_4 are close to zero in this case. Also, the ratio of the step-sizes $\kappa \triangleq \gamma/\eta$ is required to be a non-zero constant. Either too small or too large value of κ might affect the primal or dual convergence of the algorithm. This restriction is due to the consensus error in the decentralized training. In practice, one can first determine κ and then select η and γ.

From Theorem 1, we immediately have the following complexity results for GT-GDA:

Corollary 2. *Under the same conditions in Theorem 1, to achieve an* ϵ^2*-stationary solution, i.e.,* $\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}[\mathfrak{M}_t] \leq \epsilon^2$*, the total communication rounds are on the order of* $\mathcal{O}(\epsilon^{-2})$ *and the total samples evaluated across the network system is on the order of* $\mathcal{O}(mn\epsilon^{-2})$.

4 Gradient-tracking stochastic variance reduction algorithms

In the GT-GDA algorithm, agents need to evaluate local full gradients in each iteration, which may result in a high sample complexity when the trajectory length n is large in MARL. This limitation motivates us to leverage the stochastic recursive variance-reduced approach (e.g., [44]) to achieve low sample complexity in MARL decentralized policy evaluation.

1) The Algorithms: We first propose an algorithm called gradient-tracking stochastic recursive variance reduction (GT-SRVR) algorithm. Different from GT-GDA, in iteration t and at agent i, GT-SRVR estimates the local gradient with the following estimators:

$$\mathbf{v}_{i,t} = \begin{cases} \nabla_{\boldsymbol{\theta}} F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}), & \text{if } \mathrm{mod}(t,q)=0, \\ \mathbf{v}_{i,t-1} + \frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\big(\nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1})\big), & \text{otherwise,} \end{cases} \tag{8a}$$

$$\mathbf{u}_{i,t} = \begin{cases} \nabla_{\boldsymbol{\omega}} F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}), & \text{if } \mathrm{mod}(t,q)=0, \\ \mathbf{u}_{i,t-1} + \frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\big(\nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1})\big), & \text{otherwise,} \end{cases} \tag{8b}$$

where $\mathcal{S}_{i,t}$ is a local subsample at the tth iteration for agent i. In (8), the algorithm evaluates full gradients $\nabla F_i(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t})$ only every q steps. For other iterations with $\mathrm{mod}(t,q) \neq 0$, the algorithm estimates the local gradients with a mini-batch of gradients $\frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}} \nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t})$ and a recursive correction term $\mathbf{u}_{i,t-1} - \frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}} \nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1})$. It will be shown later that, thanks to the periodic full gradient (when $\mathrm{mod}(t,q)=0$) and recursive correction term, GT-SRVR is able to achieve the *same* convergence rate and communication complexity as GT-GDA. Moreover, because of the $\mathcal{S}_{i,t}$ subsampling, GT-SRVR has a *lower* sample complexity than GT-GDA. The full description of GT-SRVR is shown in Algorithm 2.

Note that in GT-SRVR, full gradients are still required for every q steps, which may still incur a high computational cost. Also, in the initialization phase (before the main loop), agents need to evaluate full gradients, which could be time-consuming. To address these limitations, we propose an enhanced version of GT-SRVR called GT-SRVR with `I`ncremental batch size (GT-SRVR`I`). Specifically, we modify the gradient estimators in (8a) and (8b) for the tth iteration with $\mathrm{mod}(t,q)=0$ as follows :

$$\mathbf{v}_{i,t} = \frac{1}{|\mathcal{R}_{i,t}|}\sum_{j\in\mathcal{R}_{i,t}} \nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}), \quad \mathbf{u}_{i,t} = \frac{1}{|\mathcal{R}_{i,t}|}\sum_{j\in\mathcal{R}_{i,t}} \nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}), \tag{9}$$

where $\mathcal{R}_{i,t}$ is a subsample set (sampling without replacement), whose size is chosen as $|\mathcal{R}_{i,t}| = \min\{(t/q+1)^{\alpha}q, c_{\epsilon}\epsilon^{-2}, n\}$. Here, $\alpha > 0$ is a constant, ϵ is a desired convergence error, and $c_{\epsilon} > 0$ is a constant that depends on ϵ. Our design of $|\mathcal{R}_{i,t}|$ is motivated by the fact that the periodic full gradient evaluation only plays an important role in the later stage of the convergence process for achieving high accuracy. Later, we will see that under some mild assumptions and parameter settings, GT-SRVR`I` has similar convergence performance as GT-SRVR. The full description of GT-SRVR/GT-SRVR`I` is shown in Algorithm 2.

Remark 2. In GT-SRVR`I`, we increase the batch-size as the number of iterations increases. We note that the work in [17] also proposed a batch-size adaptation scheme based on the historical gradient information. Although similar idea can be also adopted in our algorithms, it requires the exact value of the stochastic gradient variance σ^2 for batch-size selection as well as extra memory cost to store the history-gradient information, which is less practical compared to our approach.

2) Theoretical Results of GT-SRVR/GT-SRVR`I`: Now, we establish the convergence performance of GT-SRVR/GT-SRVR`I`. First, we replace Assumption 1(b) with the following individual Lipschitz smoothness assumption:

Assumption 2 (Lipschitz smoothness)**.** *The function $f_{ij}(\boldsymbol{\theta}, \cdot)$ is L_f-Lipschitz smooth, i.e., there exists a constant $L_f > 0$, such that the gradient $\nabla f_{ij}(\boldsymbol{\theta}, \boldsymbol{\omega}) = [\nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}, \boldsymbol{\omega})]^{\top}, \nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}, \boldsymbol{\omega})^{\top})^{\top}$ satisfies $\|\nabla f_{ij}(\boldsymbol{\theta}, \boldsymbol{\omega}) - \nabla f_{ij}(\boldsymbol{\theta}', \boldsymbol{\omega}')\|^2 \leq L_f^2\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|^2 + L_F^2\|\boldsymbol{\omega} - \boldsymbol{\omega}'\|^2$, $\forall\, \boldsymbol{\theta}, \boldsymbol{\theta}', \boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbb{R}^p, i \in [m], j \in [n]$.*

We note that Assumption 2 is a common assumption analyzing stochastic variance reduced methods [44, 39, 35]. Further, we have the following assumption only for GT-SRVR`I` algorithm:

Assumption 3 (Bounded Variance)**.** *There exists a constant $\sigma^2 > 0$, such that $\mathbb{E}\|\nabla f_{ij}(\boldsymbol{\theta}, \boldsymbol{\omega}) - \nabla F_i(\boldsymbol{\theta}, \boldsymbol{\omega})\|^2 \leq \sigma^2$, $\forall\, \boldsymbol{\theta}, \boldsymbol{\omega}, \in \mathbb{R}^p, i \in [m], j \in [n]$.*

Algorithm 2 GT-SRVR/GT-SRVRI Algorithm at Agent i.

If GT-SRVR: $|\mathcal{R}_{i,t}|=n$, $|\mathcal{S}_{i,t}|=q$; If GT-SRVRI: $|\mathcal{R}_{i,t}|=\min\{(t/q+1)^{\alpha}q, c_{\epsilon}\epsilon^{-2}, n\}$, $|\mathcal{S}_{i,t}|=q$.

Set prime-dual parameter pair $(\boldsymbol{\theta}_{i,0}, \boldsymbol{\omega}_{i,0}) = (\boldsymbol{\theta}^0, \boldsymbol{\omega}^0)$.

Draw $\mathcal{R}_{i,0}$ samples without replacement and calculate local stochastic gradient estimators as $\mathbf{p}_{i,0}=\mathbf{v}_{i,0}=\frac{1}{|\mathcal{R}_{i,0}|}\sum_{j\in\mathcal{R}_{i,0}}\nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}_{i,0}, \boldsymbol{\omega}_{i,0})$, and $\mathbf{d}_{i,0}=\mathbf{u}_{i,0}=\frac{1}{|\mathcal{R}_{i,0}|}\sum_{j\in\mathcal{R}_{i,0}}\nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,0}, \boldsymbol{\omega}_{i,0})$;

for $t = 1, \cdots, T$ **do**

  Update local parameters $(\boldsymbol{\theta}_{i,t+1}, \boldsymbol{\omega}_{i,t+1})$ as in (6);

  Calculate local gradient estimators $(\mathbf{v}_{i,t+1}, \mathbf{u}_{i,t+1})$ as in (8);

  Track global gradients $(\mathbf{p}_{i,t+1}, \mathbf{d}_{i,t+1})$ as in (5);

end for

With the metric in (7), the convergence rate performance of GT-SRVR/GT-SRVRI can be characterized as follows:

Theorem 3 (Convergence of GT-SRVR)**.** *Under Assumption 1 (a)&(c)-(e) and Assumption 2, if the step-sizes satisfy $\kappa \triangleq \gamma/\eta \leq \mu^2/13L_f^2$ and $\eta \leq \min\{k_5, k_6, k_7, k_8\}$, then we have the following convergence result for GT-SRVR:*

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}[\mathfrak{M}_t] \leq \frac{2\mathbb{E}[\mathfrak{p}_0 - J^*]}{\min\{1, L_f^2\}(T+1)\gamma},$$

where $\mathfrak{p}_t$ is the potential function defined as:

$$\mathfrak{p}_t \triangleq J(\bar{\boldsymbol{\theta}}_t) + \frac{8\gamma L_f^2}{\mu\eta}\|\bar{\boldsymbol{\omega}}_t - \boldsymbol{\omega}_t^*\|^2 + \frac{1}{m}\sum_{i=1}^{m}\big(\|\boldsymbol{\theta}_{i,t} - \bar{\boldsymbol{\theta}}_t\|^2 + \|\boldsymbol{\omega}_{i,t} - \bar{\boldsymbol{\omega}}_t\|^2 + \gamma\|\mathbf{p}_{i,t} - \bar{\mathbf{p}}_t\|^2 + \eta\|\mathbf{d}_{i,t} - \bar{\mathbf{d}}_t\|^2\big),$$

and the constants in the step-size η are:

$$C_0 = \frac{1}{1-\lambda}(1+\frac{1}{\kappa}) + \frac{1}{2} + \frac{18L_f^2}{\mu^2}, \quad k_5 = \frac{13L_f^2}{\mu^2\big(1/2 + 1/(1-\lambda)^2\big)}, \quad k_6 = \frac{1}{8\mu C_0},$$
$$k_7 = \frac{26(1-\lambda)L_f^2}{(\mu^2 + 144L_f^4 + 4L_f^2\mu^2 + 64C_0L_f^2\mu^2)}, \quad k_8 = \frac{13L_f^2}{2\mu^2}\Big(L_f + \frac{L_f^2}{\mu} + (1-\lambda)\Big).$$

Theorem 4 (Convergence of GT-SRVRI)**.** *Under Assumption 1 (a)&(c)-(e), Assumption 2, and the same parameter settings and potential function as stated in Theorem 3, we have the following convergence result for GT-SRVRI:*

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}[\mathfrak{M}_t] \leq \frac{2\mathbb{E}[\mathfrak{p}_0 - \mathfrak{p}_{T+1}]}{\min\{1, L_f^2\}(T+1)\gamma} + \frac{1}{\min\{1, L_f^2\}} \times$$
$$\big(\frac{12}{\lambda}(1+\frac{1}{\kappa}) + 4 + \frac{144L_f^2}{\mu^2}\big)\big(\frac{\sigma^2\epsilon^2}{c_\epsilon} + \frac{\sigma^2 C(n,q,\alpha)}{T+1}\big), \tag{10}$$

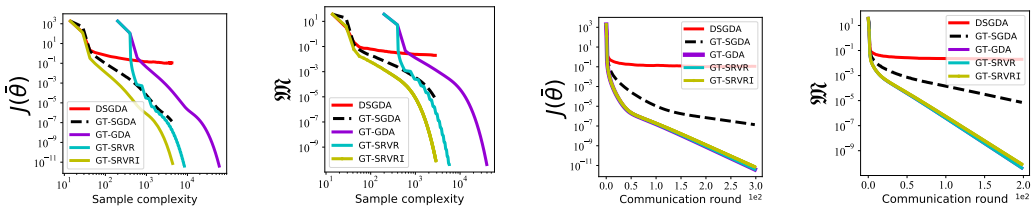
where the constant $C(n, q, \alpha)$ is defined as:

$$C(n,q,\alpha) \triangleq \begin{cases} \frac{1}{1-\alpha}(\frac{n}{q})^{(\frac{1}{\alpha}-1)} - \frac{\alpha}{1-\alpha}, & \text{if } \alpha > 0 \text{ and } \alpha \neq 1 \\ \log(\frac{n}{q}) + 1, & \text{if } \alpha = 1. \end{cases}$$

Remark 3. In Theorems 3 and 4, it can be seen that the step-sizes and convergence rate depend on the network topology and the ratio of the step-sizes κ. Additionally, the convergence performance of GT-SRVRI is affected by the constant $C(n, q, \alpha)$, which depends on the inexact gradient estimation in tth iteration with $\text{mod}(t, q) = 0$.

Following from Theorems 3 and 4, we immediately have the sample and communication complexity results for GT-SRVR/GT-SRVRI:

Corollary 5. *Under the conditions in Theorems 3 and 4, and with $q = \sqrt{n}$, to achieve an ϵ^2-stationary solution (i.e., $\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}[\mathfrak{M}_t] \leq \epsilon^2$) with $\sqrt{n}\epsilon^2 \leq 1$, we have:*



(a) MSPBE value vs. sample complexity. (b) Convergence metric $\mathfrak{M}$ vs. sample complexity. (c) [MSPBE value vs. comm. complexity. (d) Convergence metric $\mathfrak{M}$ vs. comm. complexity.

Figure 1: MARL decentralized policy evaluation performance comparison.

- *for GT-SRVR, the total communication rounds are $\mathcal{O}(\epsilon^{-2})$ and the total samples evaluated across the network are $\mathcal{O}(m\sqrt{n}\epsilon^{-2})$);*
- *GT-SRVR*I *with $\alpha \geq 1$, the total communication rounds are bounded by $\mathcal{O}(\log(\sqrt{n})\epsilon^{-2})$ and the total samples evaluated across the network are bounded by $\mathcal{O}(m\log(\sqrt{n})\sqrt{n}\epsilon^{-2})$).*

Remark 4. From Corollary 5, we can see that GT-SRVR has the same communication complexity as GT-GDA, but the sample complexity is lower than GT-GDA. For GT-SRVRI with $\alpha \geq 1$, the upper bounds of both the complexities have an additional factor $\log(\sqrt{n})$ factor compared with GT-SRVR. Although the theoretical complexity bounds for GT-SRVRI is weaker than GT-SRVR (due to abandoning full gradients completely), we show through experiments in Section 5 that GT-SRVRI empirically outperforms than GT-SRVR in practice.

5 Experimental Results

In this section, we demonstrate the performance of our proposed GT-GDA and GT-SRVR/GT-SRVRI algorithms for cooperative MARL decentralized policy evaluation. We adopt the environment of Cooperative Navigation task in [27], which consists of m agents inhabiting a two-dimensional world with continuous space and discrete time. In the system, agents cooperate with each other to reach their own landmarks. Due to space limitation, the detailed experimental settings and different network topologies are relegated to the supplementary material.

Since there is no existing work in the literature on solving decentralized policy evaluation with *nonlinear* function approximation for MARL, we compare our algorithms with two stochastic algorithms as simple baselines in our experiments: 1) Decentralized Stochastic Gradient Descent Ascent (DSGDA) and 2) Gradient-Tracking-Based Stochastic Gradient Descent Ascent (GT-SGDA) (see the supplementary material for their detailed definitions). We initialize the parameters from the normal distribution for all the algorithms. The learning rates is fixed as $\gamma = 10^{-1}, \eta = 10^{-1}$.

From Figure 1(a) and 1(b), it can be seen that GT-SRVRI converges much faster than other algorithms (GT-GDA, GT-SRVR, GT-SGD and DSGD) in terms of the total number of gradient evaluations. We can also observe that both GT-SRVR and GT-SRVRI have better sample efficiency in attaining high accuracy (error smaller than 10^{-9}) than the other three algorithms thanks to the variance-reduced techniques. As is shown in Figure 1(c) and 1(d), GT-SRVR and GT-SRVRI have the same communication cost as GT-GDA, which is much lower than those of DSGDA and GT-SGDA. Our experimental results confirm our theoretical analysis that GT-SRVR/GT-SRVRI enjoy low sample and communication complexities.

6 Conclusion

In this paper, we studied the MARL decentralized policy evaluation problem with nonlinear function approximation. We first reformulated the problem as a decentralized non-convex-strongly-concave minimax problem and developed a gradient tracking based algorithm called GT-GDA. We showed that GT-GDA algorithm has the communication complexity of $\mathcal{O}(\epsilon^{-2})$ and sample complexity of $\mathcal{O}(mn\epsilon^{-2})$. To further reduce the sample complexity while maintaining the communication complexity, we proposed two stochastic variance-reduced methods called GT-SRVR and GT-SRVRI, both of which can achieve the same communication complexity as GT-GDA but improve the sample complexity to $\mathcal{O}(m\sqrt{n}\epsilon^{-2})$. We have also conducted extensive numerical studies to verify the performance of our proposed algorithms.

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Checklist

The checklist follows the references. Please read the checklist guidelines carefully for information on how to answer these questions. For each question, change the default **[TODO]** to [Yes] , [No] , or [N/A] . You are strongly encouraged to include a **justification to your answer**, either by referencing the appropriate section of your paper or providing a brief inline description. Please do not modify the questions and only use the provided macros for your answers. Note that the Checklist section does not count towards the page limit. In your paper, please delete this instructions block and only keep the Checklist section heading above along with the questions/answers below.

1. For all authors...

(a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes] See Section 1.
(b) Did you describe the limitations of your work? [No] Not applicable.
(c) Did you discuss any potential negative societal impacts of your work? [No] Not applicable.
(d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes] See Section 4.
   (b) Did you include complete proofs of all theoretical results? [Yes] See Appendix.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See Section 5.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 5.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [No]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See Section 5.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] See Section 5.
   (b) Did you mention the license of the assets? [Yes]
   (c) Did you include any new assets either in the supplemental material or as a URL? [No] Not applicable.
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [No] Not applicable.
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [No] Not applicable.

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [No] Not applicable.
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [No] Not applicable.
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [No] Not applicable.

Supplementary Material

A Further Experiments and Additional Results

In the followings, we give the details for our experiments.

1) RL Environment: We adopt the environment of Cooperative Navigation task in [27], which consists of m agents inhabiting a two-dimensional world with continuous space and discrete time. In the system, agents cooperate with each other to reach their own landmarks.

2) Multi-Agent Networks: We use a six-node multi-agent system, with the communication graph $\mathcal{G}$ being generated by the Erdös-Rènyi graph, where edge connectivity probability is $p_c = 0.35$. The network consensus matrix is chosen as $\mathbf{W} = \mathbf{I} - \frac{2}{3\lambda_{\max}(\mathbf{L})}\mathbf{L}$, where $\mathbf{L}$ is the Laplacian matrix of $\mathcal{G}$, and $\lambda_{\max}(\mathbf{L})$ denotes the largest eigenvalue of $\mathbf{L}$. The generated topology is shown in Figure 2.

3) Data Generation and Model: We first obtain a good policy and then generate a trajectory of the state-action pairs. In all experiments, the trajectory lengh is $n = 200$. We set the discount factor $\gamma = 0.95$. Then, $\mathcal{V}_{\boldsymbol{\theta}}(\cdot)$ is parametrized by a 2-hidden-layer neural network with 20 hidden units, where the Sigmoid activation is used at each unit.

4 Compared decentralized stochastic algorithms:

1) *Decentralized Stochastic Gradient Descent Ascent (DSGDA):* This algorithm is motivated by DSGD [32, 18]. Each agent updates its local parameters as $\boldsymbol{\theta}_{i,t+1} = \sum_{j\in\mathcal{N}_i}[\mathbf{M}]_{ij}\boldsymbol{\theta}_{j,t} - \gamma\frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})$ and $\boldsymbol{\omega}_{i,t+1} = \sum_{j\in\mathcal{N}_i}[\mathbf{M}]_{ij}\boldsymbol{\omega}_{j,t} - \eta\frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})$.
2) *Gradient-Tracking-Based Stochastic Gradient Descent Ascent (GT-SGDA)*: This algorithm is motivated by the GT-SGD algorithm [45, 29]. GT-SGDA has the same structure as that of GT-GDA, but it updates $\mathbf{v}_{i,t}$ and $\mathbf{u}_{i,t}$ using stochastic gradients as follows: $\mathbf{v}_{i,t} = \frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\theta}} f_{ij}(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})$ and $\mathbf{u}_{i,t} = \frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\omega}} f_{ij}(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})$.

A.1 Learning rate setting

We use a 6-nodes multi-agent system with a generated topology as shown in Figure 2. In this experiment, we choose the trajectory length $n = 200$, discount factor $\gamma = 0.95$ and mini-batch size $q = \lceil\sqrt{n}\rceil$. $\mathcal{V}_{\boldsymbol{\theta}}(\cdot)$ is parametrized by a 2-hidden-layer neural network with 20 hidden units, where the Sigmoid activation is used at each unit. Fig.3-4 illustrate the objective function $J(\boldsymbol{\theta})$ and convergence metric $\mathfrak{M}$ performance of GT-GDA and GT-SRVR with different learning rates γ and η. Since excessive learning rates will result in larger fluctuation in loss value function. Thus, we fixed a relatively small learning rate $\gamma = 10^{-3}$ while comparing η, and set $\eta = 10^{-3}$ while comparing γ. In this experiment, we observe that methods with a smaller learning rate have a smaller slope in the figure, which leads to a slower convergence.

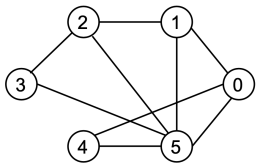
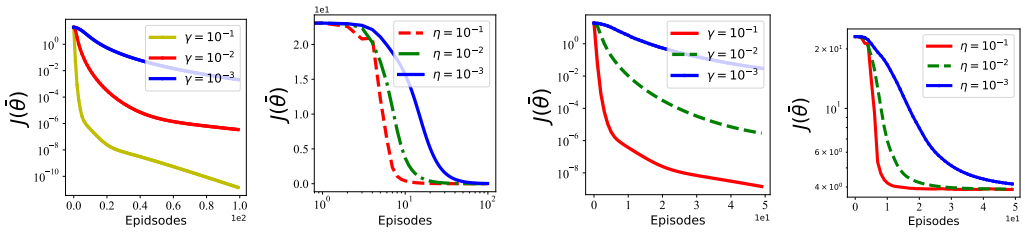
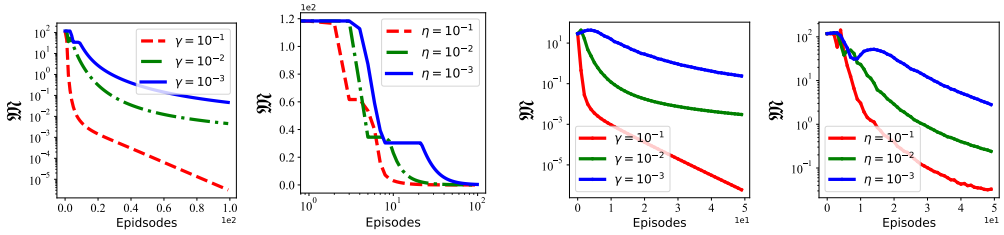


Figure 2: Network topology



(a) Step-size comparison on GT-GDA. (b) Step-size comparison on GT-SRVR.

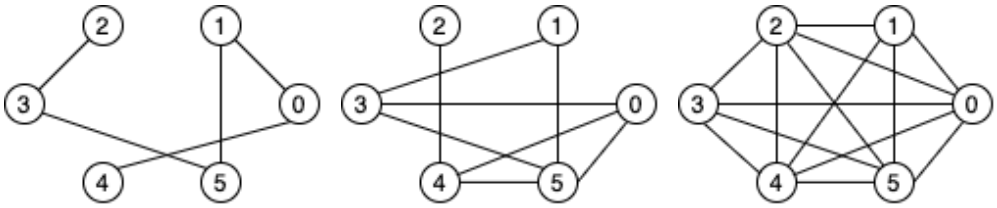
Figure 3: Performance of objective function with different step-size.



(a) Step-size comparison on GT-GDA. (b) Step-size comparison on GT-SRVR.

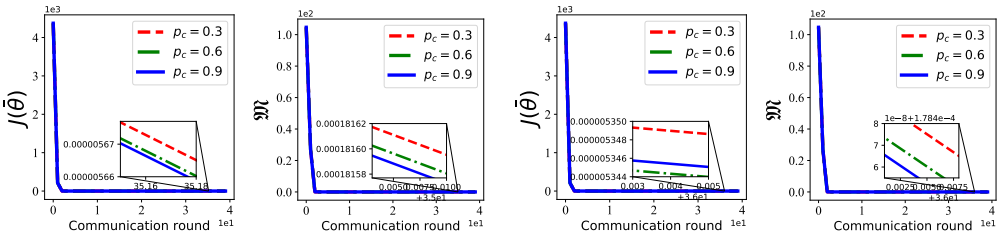
Figure 4: Performance of convergence metric $\mathfrak{M}$ with different step-size.

A.2 Topology setting



(a) Topology sparsity $p_c = 0.3$. (b) Topology sparsity $p_c = 0.6$. (c) Topology sparsity $p_c = 0.9$.

Figure 5: Topology.



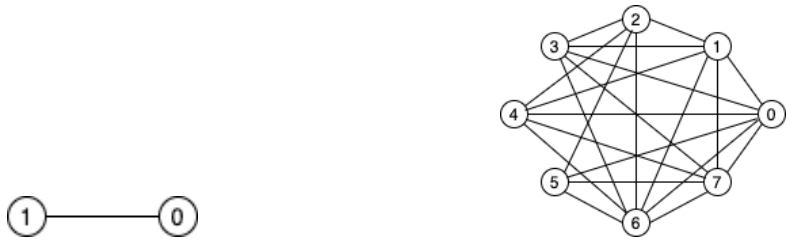
(a) Topology sparsity comparison on GT-GDA. (b) Topology sparsity comparison on GT-SRVR.

Figure 6: Performance with different topology.

We use a 6-nodes multi-agent system and experiments on three different topologies. The generated topology with different sparsity are shown in Fig. 5. The trajectory length is $n = 200$ and we set the discount factor $\gamma = 0.95$, constant learning rate $\gamma = 0.1$, $\eta = 0.1$ and mini-batch size $q = \lceil \sqrt{n} \rceil$. We observe that the objective function $J(\theta)$ and convergence metric $\mathfrak{M}$ is insensitive to network topology. The subplot in Fig.6 show that the $J(\theta)$ and $\mathfrak{M}$ slightly increases as p_c decreases.

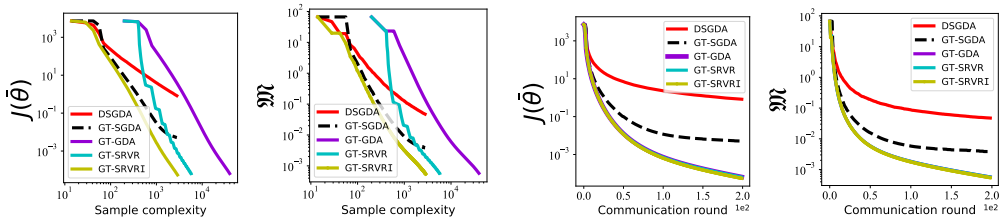
A.3 Node setting

We test the following experiments on different multi-agent systems. The generated topology with a 2-nodes system and a 8-nodes system are shown in Fig. 7. The trajectory length is $n = 200$ and we set the discount factor $\gamma = 0.95$, constant learning rate $\gamma = 0.1$, $\eta = 0.1$ and mini-batch size $q = \lceil \sqrt{n} \rceil$. We compare our proposed algorithm GT-GDA and GT-SRVR/GT-SRVRI with two baseline algorithms GT-SGDA and DSGDA mentioned in Section 5 in terms of MSPBE $J(\boldsymbol{\theta}) = \max_{\boldsymbol{\omega} \in \mathbb{R}^p} F(\boldsymbol{\theta}, \boldsymbol{\omega})$ and the convergence metric in (7). We observe similar results as shown in Section 5. Thus, we can conclude that our proposed algorithms GT-SRVR/GT-SRVRI enjoy low sample and communication complexities in general.



(a) Topology sparsity $p_c = 0.8$ with 2 nodes. (b) Topology sparsity $p_c = 0.8$ with 8 nodes.

Figure 7: Topology.



(a) Sample complexity v.s. MSPBE & convergence metric. (b) Communication complexity v.s. MSPBE & convergence metric.

Figure 8: Performance with 2 Nodes.

B Proof Outline for Theorem 1

In this section, we organize the proof of Theorem 1 into several key lemmas. Before diving in our theoretical analysis, we first define the following notations:

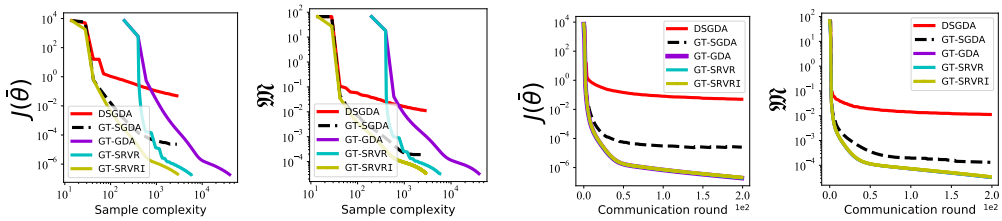
- $\bar{\mathbf{x}}_t = \frac{1}{m}\sum_{i=1}^m \mathbf{x}_{i,t}$ and $\mathbf{x}_t = [\mathbf{x}_{1,t}^\top, \cdots, \mathbf{x}_{m,t}^\top]^\top$ for any vector $\mathbf{x}$;
- $\nabla_{\boldsymbol{\theta}} F_t = [\nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}_{1,t}, \boldsymbol{\omega}_{1,t})^\top, \cdots, \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}_{m,t}, \boldsymbol{\omega}_{m,t})^\top]^\top$;
- $\nabla_{\boldsymbol{\omega}} F_t = [\nabla_{\boldsymbol{\omega}} F(\boldsymbol{\theta}_{1,t}, \boldsymbol{\omega}_{1,t})^\top, \cdots, \nabla_{\boldsymbol{\omega}} F(\boldsymbol{\theta}_{m,t}, \boldsymbol{\omega}_{m,t})^\top]^\top$;
- $\mathcal{E}(\mathbf{x}_t) = \frac{1}{m}\sum_{i=1}^m \|\mathbf{x}_{i,t} - \bar{\mathbf{x}}_t\|^2$ for any vector $\mathbf{x}$.

Our first step is to show the following descent property of GT-GDA algorithm on the function $J(\cdot)$:

Lemma 1 (Descent Inequality on $J(\boldsymbol{\theta})$)**.** *Under Assumption 1, the following descent inequality holds under GT-GDA:*

$$J(\bar{\boldsymbol{\theta}}_{t+1}) - J(\bar{\boldsymbol{\theta}}_t) \leq -\frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2})\|\bar{\mathbf{p}}_t\|^2 + \gamma L_F^2\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \gamma\|\nabla_{\boldsymbol{\theta}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{p}}_t\|^2, \quad (11)$$

where $J(\boldsymbol{\theta}_t) = \max_{\boldsymbol{\omega}} F(\boldsymbol{\theta}_t, \boldsymbol{\omega})$ *and* $\boldsymbol{\omega}_t^* = \arg\max_{\boldsymbol{\omega}} F(\boldsymbol{\theta}_t, \boldsymbol{\omega})$.



(a) Sample complexity v.s. MSPBE & convergence metric. (b) Communication complexity v.s. MSPBE & convergence metric.

Figure 9: Performance with 8 Nodes.

Proof. According to the algorithm update, we have:

$$
\begin{aligned}
&J(\bar{\boldsymbol{\theta}}_{t+1}) - J(\bar{\boldsymbol{\theta}}_t) \overset{(a)}{\leq} \langle \nabla J(\bar{\boldsymbol{\theta}}_t), \bar{\boldsymbol{\theta}}_{t+1} - \bar{\boldsymbol{\theta}}_t \rangle + \frac{L_J}{2}\|\bar{\boldsymbol{\theta}}_{t+1} - \bar{\boldsymbol{\theta}}_t\|^2 \\
\overset{(b)}{=} &- \gamma \langle \nabla J(\bar{\boldsymbol{\theta}}_t), \bar{\mathbf{p}}_t \rangle + \frac{L_J \gamma^2}{2}\|\bar{\mathbf{p}}_t\|^2 \\
= &- \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2})\|\bar{\mathbf{p}}_t\|^2 + \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t) - \bar{\mathbf{p}}_t\|^2 \\
= &- \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2})\|\bar{\mathbf{p}}_t\|^2 + \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t) - \nabla_{\boldsymbol{\theta}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) + \nabla_{\boldsymbol{\theta}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{p}}_t\|^2 \\
\overset{(c)}{\leq} &- \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2})\|\bar{\mathbf{p}}_t\|^2 + \gamma\|\nabla J(\bar{\boldsymbol{\theta}}_t) - \nabla_{\boldsymbol{\theta}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t)\|^2 + \gamma\|\nabla_{\boldsymbol{\theta}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{p}}_t\|^2 \\
\overset{(d)}{\leq} &- \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2})\|\bar{\mathbf{p}}_t\|^2 + \gamma L_F^2\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \gamma\|\nabla_{\boldsymbol{\theta}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{p}}_t\|^2, \qquad (12)
\end{aligned}
$$

where (a) is because of Lipschitz continuous gradients of J, (b) follows from the update rule (5), (c) follows from $\|\mathbf{x} + \mathbf{y}\|^2 \leq 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 \; \forall \mathbf{x}, \mathbf{y}$, and (d) follows from Lemma 10 and Assumption 1 (b). □

Note that in the RHS of (11), there is an error term $\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2$. The following lemma states the contraction property of this term.

Lemma 2 (Error Bound on $\boldsymbol{\omega}^*(\boldsymbol{\theta})$)**.** *Under Assumption 1 and with $\eta \leq 1/2L_F$, the following inequality holds for GT-GDA:*

$$
\begin{aligned}
\|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_{t+1}^*\|^2 \leq (1 - \frac{\mu\eta}{4})\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + &\frac{9\eta}{2\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| \\
&- (1 + \frac{\mu\eta}{4})\frac{\eta^2}{4}\|\bar{\mathbf{d}}_t\|^2 + \frac{5L_{\boldsymbol{\omega}}^2\gamma^2}{\mu\eta}\|\bar{\mathbf{p}}_t\|^2, \qquad (13)
\end{aligned}
$$

where $\boldsymbol{\omega}_t^* = \boldsymbol{\omega}^*(\bar{\boldsymbol{\theta}}_t) = \arg\max_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega})$.

Proof. Define $\boldsymbol{\omega}_t^* = \boldsymbol{\omega}^*(\bar{\boldsymbol{\theta}}_t) = \arg\max_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega})$. We have:

$$
\|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_t^*\|^2 = \|\bar{\boldsymbol{\omega}}_t + \eta\bar{\mathbf{d}}_t - \boldsymbol{\omega}_t^*\|^2 = \|\bar{\boldsymbol{\omega}}_t - \boldsymbol{\omega}_t^*\|^2 + \eta^2\|\bar{\mathbf{d}}_t\|^2 + 2\eta\langle \bar{\boldsymbol{\omega}}_t - \boldsymbol{\omega}_t^*, \bar{\mathbf{d}}_t \rangle. \qquad (14)
$$

To remove the third term in (14), we have:

$$
\begin{aligned}
&F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega}) - F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) + \frac{\mu}{2}\|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_t\|^2 \leq \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t), \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_t \rangle \\
=&\langle \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t), \bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t \rangle. \qquad (15)
\end{aligned}
$$

With $\eta \leq 1/2L_F$ and by Assumption 1 (b), it follows that

$$
\begin{aligned}
&-\frac{1}{4\eta}\|\bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t\|^2 \leq -\frac{L_F}{2}\|\bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t\|^2 \\
&\leq F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega}_{t+1}) - F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t), \bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t \rangle. \qquad (16)
\end{aligned}
$$

Combining (15) and (16), with the update $\bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t = \eta\bar{\mathbf{d}}_t$, we have:

$$
\begin{aligned}
&F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega}) - F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_{t+1}) + \frac{\mu}{2}\|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_t\|^2 \\
\leq&\langle \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \frac{1}{4\eta}\|\bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t\|^2 \\
=&\langle \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_t \rangle + \langle \bar{\mathbf{d}}_t, \bar{\boldsymbol{\omega}}_t - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \frac{\eta}{4}\|\bar{\mathbf{d}}_t\|^2 \\
=&\langle \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_t \rangle - \eta\|\bar{\mathbf{d}}_t\|^2 + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle + \frac{\eta}{4}\|\bar{\mathbf{d}}_t\|^2 \\
=&\langle \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_t \rangle + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t, \boldsymbol{\omega} - \bar{\boldsymbol{\omega}}_{t+1} \rangle - \frac{3\eta}{4}\|\bar{\mathbf{d}}_t\|^2. \qquad (17)
\end{aligned}
$$

Let $\boldsymbol{\omega} = \boldsymbol{\omega}_t^*$, we have

$$
\begin{aligned}
& F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega}_t^*) - F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_{t+1}) + \frac{\mu}{2}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 \\
\leq & \langle \bar{\mathbf{d}}_t, \boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t \rangle + \langle \nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t, \boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_{t+1} \rangle - \frac{3\eta}{4}\|\bar{\mathbf{d}}_t\|^2 \\
\overset{(a)}{\leq} & \langle \bar{\mathbf{d}}_t, \boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t \rangle + \frac{2}{\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| + \frac{\mu}{8}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_{t+1}\| - \frac{3\eta}{4}\|\bar{\mathbf{d}}_t\|^2 \\
\overset{(b)}{\leq} & \langle \bar{\mathbf{d}}_t, \boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t \rangle + \frac{2}{\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| + \frac{\mu}{4}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{\mu}{4}\|\bar{\boldsymbol{\omega}}_t - \bar{\boldsymbol{\omega}}_{t+1}\|^2 - \frac{3\eta}{4}\|\bar{\mathbf{d}}_t\|^2 \\
\overset{(c)}{=} & \langle \bar{\mathbf{d}}_t, \boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t \rangle + \frac{2}{\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| + \frac{\mu}{4}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 - (\frac{3\eta}{4} - \frac{\mu\eta^2}{4})\|\bar{\mathbf{d}}_t\|^2, \qquad (18)
\end{aligned}
$$

where (a) follows from $-\langle \mathbf{x}, \mathbf{y} \rangle \leq \frac{1}{2c}\|\mathbf{x}\|^2 + \frac{c}{2}\|\mathbf{y}\|^2$ and $c = \frac{\mu}{4}$, (b) is due to $\|\mathbf{x} + \mathbf{y}\|^2 \leq 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$, and (c) is from $\bar{\boldsymbol{\omega}}_{t+1} - \bar{\boldsymbol{\omega}}_t = \eta \bar{\mathbf{d}}_t$.

Since $F(\bar{\boldsymbol{\theta}}_t, \boldsymbol{\omega}_t^*) \geq F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_{t+1})$, we have

$$
\frac{\mu\eta}{2}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 \leq 2\eta\langle \bar{\mathbf{d}}_t, \boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t \rangle + \frac{4\eta}{\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| - (\frac{3\eta^2}{2} - \frac{\mu\eta^3}{2})\|\bar{\mathbf{d}}_t\|^2. \qquad (19)
$$

Combining (14) and (19) and setting $\eta \leq 1/2\mu \leq 1/2L_F$, we have

$$
\|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_t^*\|^2 \leq (1 - \frac{\mu\eta}{2})\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{4\eta}{\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| - \frac{\eta^2}{4}\|\bar{\mathbf{d}}_t\|^2. \qquad (20)
$$

Then, it holds that

$$
\begin{aligned}
& \|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_{t+1}^*\|^2 = \|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_t^* + \boldsymbol{\omega}_t^* - \boldsymbol{\omega}_{t+1}^*\|^2 \\
\overset{(a)}{\leq} & (1 + \frac{\mu\eta}{4})\|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_t^*\|^2 + (1 + \frac{4}{\mu\eta})\|\boldsymbol{\omega}_t^* - \boldsymbol{\omega}_{t+1}^*\|^2 \\
\overset{(b)}{\leq} & (1 + \frac{\mu\eta}{4})\|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_t^*\|^2 + (1 + \frac{4}{\mu\eta})L_{\boldsymbol{\omega}}^2\|\bar{\boldsymbol{\theta}}_t - \bar{\boldsymbol{\theta}}_{t+1}\|^2 \\
\overset{(c)}{\leq} & (1 + \frac{\mu\eta}{4})(1 - \frac{\mu\eta}{2})\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + (1 + \frac{\mu\eta}{4})\frac{4\eta}{\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| \\
& - (1 + \frac{\mu\eta}{4})\frac{\eta^2}{4}\|\bar{\mathbf{d}}_t\|^2 + (1 + \frac{4}{\mu\eta})L_{\boldsymbol{\omega}}^2\|\bar{\boldsymbol{\theta}}_t - \bar{\boldsymbol{\theta}}_{t+1}\|^2 \\
\overset{(d)}{\leq} & (1 - \frac{\mu\eta}{4})\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{9\eta}{2\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| - (1 + \frac{\mu\eta}{4})\frac{\eta^2}{4}\|\bar{\mathbf{d}}_t\|^2 + \frac{5L_{\boldsymbol{\omega}}^2\gamma^2}{\mu\eta}\|\bar{\mathbf{p}}_t\|^2, \qquad (21)
\end{aligned}
$$

where (a) follows from $\|\mathbf{x} + \mathbf{y}\|^2 \leq (1 + 1/c)\|\mathbf{x}\|^2 + (1 + c)\|\mathbf{y}\|^2$ and $c = \mu\eta/4$, (b) follows from Lemma 9, (c) follows fromy plugging (20), and (d) due to the facts that:

$$
\begin{aligned}
& (1 + \frac{\mu\eta}{4})(1 - \frac{\mu\eta}{2}) = 1 + \frac{\mu\eta}{4} - \frac{\mu\eta}{2} - \frac{\mu^2\eta^2}{8} \leq 1 - \frac{\mu\eta}{4}, \\
& (1 + \frac{\mu\eta}{4})\frac{4\eta}{\mu} \leq (1 + \frac{\mu}{4} \cdot \frac{1}{2\mu})\frac{4\eta}{\mu} = \frac{9\eta}{2\mu}, \\
& 1 + \frac{4}{\mu\eta} \leq \frac{1}{\mu\eta} + \frac{4}{\mu\eta} = \frac{5}{\mu\eta} \text{ , and } \bar{\boldsymbol{\theta}}_t - \bar{\boldsymbol{\theta}}_{t+1} = \gamma\bar{\mathbf{p}}_t. \qquad (22)
\end{aligned}
$$

Plugging (22) into (21) yields:

$$
\begin{aligned}
& \|\bar{\boldsymbol{\omega}}_{t+1} - \boldsymbol{\omega}_{t+1}^*\|^2 - \|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 \\
\leq & -\frac{\mu\eta}{4}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{9\eta}{2\mu}\|\nabla_{\boldsymbol{\omega}} F(\bar{\boldsymbol{\theta}}_t, \bar{\boldsymbol{\omega}}_t) - \bar{\mathbf{d}}_t\| - (1 + \frac{\mu\eta}{4})\frac{\eta^2}{4}\|\bar{\mathbf{d}}_t\|^2 + \frac{5L_{\boldsymbol{\omega}}^2\gamma^2}{\mu\eta}\|\bar{\mathbf{p}}_t\|^2. \qquad (23)
\end{aligned}
$$

□

Next, by combining the results from Lemma 1-2, we have the following descent result:

Lemma 3. *Under Assumption 1 and with $\eta \leq 1/2L_F$, the following inequality holds for GT-GDA:*

$$
\begin{aligned}
&J(\bar{\boldsymbol{\theta}}_{T+1})-J(\bar{\boldsymbol{\theta}}_0)+\frac{8\gamma L_F^2}{\mu\eta}\left[\|\bar{\boldsymbol{\omega}}_{T+1}-\boldsymbol{\omega}_{T+1}^*\|^2-\|\boldsymbol{\omega}_0^*-\bar{\boldsymbol{\omega}}_0\|^2\right]\\
\leq&-\frac{\gamma}{2}\sum_{t=0}^{T}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2-\gamma L_F^2\sum_{t=0}^{T}\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2-(\frac{\gamma}{2}-\frac{L_J\gamma^2}{2}-\frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2})\sum_{t=0}^{T}\|\bar{\mathbf{p}}_t\|^2-\frac{2\gamma\eta L_F^2}{\mu}\sum_{t=0}^{T}\|\bar{\mathbf{d}}_t\|^2\\
&+(\frac{72\gamma L_F^4}{\mu^2 m}+\frac{2\gamma L_F^2}{m})\sum_{t=0}^{T}[\mathcal{E}(\boldsymbol{\theta}_t)+\mathcal{E}(\boldsymbol{\omega}_t)]+\frac{2\gamma}{m}\sum_{t=0}^{T}\|\nabla_{\boldsymbol{\omega}}F_t-\mathbf{v}_t\|^2+\frac{72L_F^2\gamma}{m\mu^2}\sum_{t=0}^{T}\|\nabla_{\boldsymbol{\theta}}F_t-\mathbf{u}_t\|^2.
\end{aligned}
$$

Proof. From Lemma 1-2, we have

$$
\begin{aligned}
&J(\bar{\boldsymbol{\theta}}_{t+1})-J(\bar{\boldsymbol{\theta}}_t)+\frac{8\gamma L_F^2}{\mu\eta}\left[\|\bar{\boldsymbol{\omega}}_{t+1}-\boldsymbol{\omega}_{t+1}^*\|^2-\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2\right]\\
\leq&-\frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2-\gamma L_F^2\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2-(\frac{\gamma}{2}-\frac{L_J\gamma^2}{2}-\frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2})\|\bar{\mathbf{p}}_t\|^2-\frac{2\gamma\eta L_F^2}{\mu}\|\bar{\mathbf{d}}_t\|^2\\
&+\gamma\|\nabla_{\boldsymbol{\theta}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\bar{\mathbf{p}}_t\|^2+\frac{36L_F^2\gamma}{\mu^2}\|\nabla_{\boldsymbol{\omega}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\bar{\mathbf{d}}_t\|^2. \qquad (24)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\|\nabla_{\boldsymbol{\theta}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\bar{\mathbf{p}}_t\|^2\\
=&\|\nabla_{\boldsymbol{\theta}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})+\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})-\bar{\mathbf{p}}_t\|^2\\
\leq&2\|\nabla_{\boldsymbol{\theta}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})\|^2+2\|\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})-\bar{\mathbf{p}}_t\|^2\\
\leq&\frac{2}{m}\sum_{i=1}^{m}\|\nabla_{\boldsymbol{\theta}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})\|^2+2\|\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})-\bar{\mathbf{p}}_t\|^2\\
\leq&\frac{2L_f^2}{m}\sum_{i=1}^{m}[\|\bar{\boldsymbol{\theta}}_t-\boldsymbol{\theta}_{i,t}\|^2+\|\bar{\boldsymbol{\omega}}_t-\boldsymbol{\omega}_{i,t}\|^2]+2\|\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})-\bar{\mathbf{p}}_t\|^2. \qquad (25)
\end{aligned}
$$

Similarly, we have:

$$
\begin{aligned}
\|\nabla_{\boldsymbol{\omega}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\bar{\mathbf{d}}_t\|^2\leq&\frac{2L_F^2}{m}\sum_{i=1}^{m}[\|\bar{\boldsymbol{\theta}}_t-\boldsymbol{\theta}_{i,t}\|^2+\|\bar{\boldsymbol{\omega}}_t-\boldsymbol{\omega}_{i,t}\|^2]\\
&+2\|\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\omega}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t})-\bar{\mathbf{d}}_t\|^2. \qquad (26)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&J(\bar{\boldsymbol{\theta}}_{t+1})-J(\bar{\boldsymbol{\theta}}_t)+\frac{8\gamma L_F^2}{\mu\eta}\left[\|\bar{\boldsymbol{\omega}}_{t+1}-\boldsymbol{\omega}_{t+1}^*\|^2-\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2\right]\\
\leq&-\frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2-\gamma L_F^2\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2-(\frac{\gamma}{2}-\frac{L_J\gamma^2}{2}-\frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2})\|\bar{\mathbf{p}}_t\|^2-\frac{2\gamma\eta L_F^2}{\mu}\|\bar{\mathbf{d}}_t\|^2\\
&+\gamma\|\nabla_{\boldsymbol{\theta}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\bar{\mathbf{p}}_t\|^2+\frac{36L_F^2\gamma}{\mu^2}\|\nabla_{\boldsymbol{\omega}}F(\bar{\boldsymbol{\theta}}_t,\bar{\boldsymbol{\omega}}_t)-\bar{\mathbf{d}}_t\|^2\\
\leq&-\frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2-\gamma L_F^2\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2-(\frac{\gamma}{2}-\frac{L_J\gamma^2}{2}-\frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2})\|\bar{\mathbf{p}}_t\|^2-\frac{2\gamma\eta L_F^2}{\mu}\|\bar{\mathbf{d}}_t\|^2\\
&+(\frac{72\gamma L_F^4}{\mu^2 m}+\frac{2\gamma L_F^2}{m})\sum_{i=1}^{m}[\|\bar{\boldsymbol{\theta}}_t-\boldsymbol{\theta}_{i,t}\|^2+\|\bar{\boldsymbol{\omega}}_t-\boldsymbol{\omega}_{i,t}\|^2]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\gamma\|\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\omega}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t}) - \bar{\mathbf{v}}_t\|^2 + \frac{72L_F^2\gamma}{\mu^2}\|\frac{1}{m}\sum_{i=1}^{m}\nabla_{\boldsymbol{\theta}}F_i(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t}) - \bar{\mathbf{u}}_t\|^2 \\
\overset{(a)}{\leq} &- \frac{\gamma}{2}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - \gamma L_F^2\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 - (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2} - \frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2})\|\bar{\mathbf{p}}_t\|^2 - \frac{2\gamma\eta L_F^2}{\mu}\|\bar{\mathbf{d}}_t\|^2 \\
&+ (\frac{72\gamma L_F^4}{\mu^2 m} + \frac{2\gamma L_F^2}{m})[\mathcal{E}(\boldsymbol{\theta}_t) + \mathcal{E}(\boldsymbol{\omega}_t)] + \frac{2\gamma}{m}\|\nabla_{\boldsymbol{\omega}}F_t - \mathbf{v}_t\|^2 + \frac{72L_F^2\gamma}{m\mu^2}\|\nabla_{\boldsymbol{\theta}}F_t - \mathbf{u}_t\|^2, \quad (27)
\end{aligned}
$$

where (a) due to $\|\frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_{i,t} - \bar{x}_t\|^2 \leq \frac{1}{m}\sum_{i=1}^{m}\|\mathbf{x}_{i,t} - \bar{x}_t\|^2$.

Telescoping the above inequality, we have the stated result. □

Next, we prove the contraction of iterations in the following lemma, which is useful in analyzing the decentralized gradient tracking algorithms.

Lemma 4 (Iterates Contraction)**.** *The following contraction properties of the iterates hold:*

$$
\|\boldsymbol{\theta}_t - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_t\|^2 \leq (1+c_1)\lambda^2\|\boldsymbol{\theta}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_{t-1}\|^2 + (1+\frac{1}{c_1})\gamma^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2, \quad (28)
$$

$$
\|\boldsymbol{\omega}_t - \mathbf{1}\otimes\bar{\boldsymbol{\omega}}_t\|^2 \leq (1+c_2)\lambda^2\|\boldsymbol{\omega}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\omega}}_{t-1}\|^2 + (1+\frac{1}{c_2})\eta^2\|\mathbf{d}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{d}}_{t-1}\|^2, \quad (29)
$$

$$
\|\mathbf{p}_t - \mathbf{1}\otimes\bar{\mathbf{p}}_t\|^2 \leq (1+c_1)\lambda^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2 + (1+\frac{1}{c_1})\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2, \quad (30)
$$

$$
\|\mathbf{d}_t - \mathbf{1}\otimes\bar{\mathbf{d}}_t\|^2 \leq (1+c_2)\lambda^2\|\mathbf{d}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{d}}_{t-1}\|^2 + (1+\frac{1}{c_2})\|\mathbf{u}_t - \mathbf{u}_{t-1}\|^2, \quad (31)
$$

where c_1 and c_2 are arbitrary positive constants. Additionally, we have

$$
\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 \leq 8\|(\boldsymbol{\theta}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_{t-1})\|^2 + 4\gamma^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2 + 4\gamma^2 m\|\bar{\mathbf{p}}_{t-1}\|^2, \quad (32)
$$

$$
\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2 \leq 8\|(\boldsymbol{\omega}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\omega}}_{t-1})\|^2 + 4\gamma^2\|\mathbf{d}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{d}}_{t-1}\|^2 + 4\gamma^2 m\|\bar{\mathbf{d}}_{t-1}\|^2. \quad (33)
$$

Proof. First for the iterates $\boldsymbol{\theta}_t$, we have the following contraction:

$$
\|\widetilde{\mathbf{M}}\boldsymbol{\theta}_t - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_t\|^2 = \|\widetilde{\mathbf{M}}(\boldsymbol{\theta}_t - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_t)\|^2 \leq \lambda^2\|\boldsymbol{\theta}_t - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_t\|^2. \quad (34)
$$

This is because $\boldsymbol{\theta}_t - \mathbf{1}\otimes\boldsymbol{\theta}_t$ is orthogonal to $\mathbf{1}$, which is the eigenvector corresponding to the largest eigenvalue of $\widetilde{\mathbf{M}}$, and $\lambda = \max\{|\lambda_2|, |\lambda_m|\}$. Recall that $\bar{\boldsymbol{\theta}}_t = \bar{\boldsymbol{\theta}}_{t-1} - \gamma\bar{\mathbf{p}}_{t-1}$, hence,

$$
\begin{aligned}
\|\boldsymbol{\theta}_t - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_t\|^2 &= \|\widetilde{\mathbf{M}}\boldsymbol{\theta}_{t-1} - \gamma\mathbf{p}_{t-1} - \mathbf{1}\otimes(\bar{\boldsymbol{\theta}}_{t-1} - \gamma\bar{\mathbf{p}}_{t-1})\|^2 \\
&\leq (1+c_1)\|\widetilde{\mathbf{M}}\boldsymbol{\theta}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_{t-1}\|^2 + (1+\frac{1}{c_1})\gamma^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2 \\
&\leq (1+c_1)\lambda^2\|\boldsymbol{\theta}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\theta}}_{t-1}\|^2 + (1+\frac{1}{c_1})\gamma^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2. \quad (35)
\end{aligned}
$$

Similarly, we have the result for $\boldsymbol{\omega}_t$,

$$
\|\boldsymbol{\omega}_t - \mathbf{1}\otimes\bar{\boldsymbol{\omega}}_t\|^2 \leq (1+c_2)\lambda^2\|\boldsymbol{\omega}_{t-1} - \mathbf{1}\otimes\bar{\boldsymbol{\omega}}_{t-1}\|^2 + (1+\frac{1}{c_2})\eta^2\|\mathbf{d}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{d}}_{t-1}\|^2. \quad (36)
$$

For $\mathbf{p}_t$, we have

$$
\begin{aligned}
&\|\mathbf{p}_t - \mathbf{1}\otimes\bar{\mathbf{p}}_t\|^2 = \|\widetilde{\mathbf{M}}\mathbf{p}_{t-1} + \mathbf{v}_t - \mathbf{v}_{t-1} - \mathbf{1}\otimes\big(\bar{\mathbf{p}}_{t-1} + \bar{\mathbf{v}}_t - \bar{\mathbf{v}}_{t-1}\big)\| \\
&\leq (1+c_1)\lambda^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2 + (1+\frac{1}{c_1})\|\mathbf{v}_t - \mathbf{v}_{t-1} - \mathbf{1}\otimes\big(\bar{\mathbf{v}}_t - \bar{\mathbf{v}}_{t-1}\big)\|^2 \\
&\leq (1+c_1)\lambda^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2 + (1+\frac{1}{c_1})\|\big(\mathbf{I} - \frac{1}{n}(\mathbf{1}\mathbf{1}^\top)\otimes\mathbf{I}\big)\big(\mathbf{v}_t - \mathbf{v}_{t-1}\big)\|^2 \\
&\overset{(a)}{\leq} (1+c_1)\lambda^2\|\mathbf{p}_{t-1} - \mathbf{1}\otimes\bar{\mathbf{p}}_{t-1}\|^2 + (1+\frac{1}{c_1})\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2, \quad (37)
\end{aligned}
$$

where (a) is due to $\|\mathbf{I} - \frac{1}{m}(\mathbf{1}\mathbf{1}^\top) \otimes \mathbf{I}\| \leq 1$. Similarly, we have

$$\|\mathbf{d}_t - \mathbf{1} \otimes \bar{\mathbf{d}}_t\|^2 \leq (1 + c_2)\lambda^2 \|\mathbf{d}_{t-1} - \mathbf{1} \otimes \bar{\mathbf{d}}_{t-1}\|^2 + (1 + \frac{1}{c_2})\|\mathbf{u}_t - \mathbf{u}_{t-1}\|^2. \tag{38}$$

According to the update, we have

$$\begin{aligned}
&\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 = \|\widetilde{\mathbf{M}}\boldsymbol{\theta}_{t-1} - \gamma \mathbf{p}_{t-1} - \boldsymbol{\theta}_{t-1}\|^2 \\
=&\|(\widetilde{\mathbf{M}} - \mathbf{I})\boldsymbol{\theta}_{t-1} - \gamma \mathbf{p}_{t-1}\|^2 \leq 2\|(\widetilde{\mathbf{M}} - \mathbf{I})\boldsymbol{\theta}_{t-1}\|^2 + 2\gamma^2\|\mathbf{p}_{t-1}\|^2 \\
=&2\|(\widetilde{\mathbf{M}} - \mathbf{I})(\boldsymbol{\theta}_{t-1} - \mathbf{1} \otimes \bar{\boldsymbol{\theta}}_{t-1})\|^2 + 2\gamma^2\|\mathbf{p}_{t-1}\|^2 \\
\leq&8\|(\boldsymbol{\theta}_{t-1} - \mathbf{1} \otimes \bar{\boldsymbol{\theta}}_{t-1})\|^2 + 4\gamma^2\|\mathbf{p}_{t-1} - \mathbf{1} \otimes \bar{\mathbf{p}}_{t-1}\|^2 + 4\gamma^2 m\|\bar{\mathbf{p}}_{t-1}\|^2 \\
=&8\mathcal{E}(\boldsymbol{\theta}_{t-1}) + 4\gamma^2\mathcal{E}(\mathbf{p}_{t-1}) + 4\gamma^2 m\|\bar{\mathbf{p}}_{t-1}\|^2,
\end{aligned} \tag{39}$$

and also

$$\|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2 \leq 8\mathcal{E}(\boldsymbol{\omega}_{t-1}) + 4\eta^2\mathcal{E}(\mathbf{d}_{t-1}) + 4\eta^2 m\|\bar{\mathbf{d}}_{t-1}\|^2. \tag{40}$$

□

Lemma 5 (Differential Bound on Estimator for GT-GDA)**.** *Under Assumption 1, the following inequalities holds*

$$\sum_{t=1}^{T} \mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 \leq \sum_{t=1}^{T} 3L_F^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 3L_F^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2, \tag{41}$$

$$\sum_{t=1}^{T} \mathbb{E}\|\mathbf{u}_t - \mathbf{u}_{t-1}\|^2 \leq \sum_{t=1}^{T} 3L_F^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 3L_F^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2. \tag{42}$$

Proof. For $\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2$, we have

$$\begin{aligned}
&\mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 = \mathbb{E}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}}\mathbf{F}_t + \nabla_{\boldsymbol{\theta}}\mathbf{F}_t - \nabla_{\boldsymbol{\theta}}\mathbf{F}_{t-1} + \nabla_{\boldsymbol{\theta}}\mathbf{F}_{t-1} - \mathbf{v}_{t-1}\|^2 \\
\leq&3\mathbb{E}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}}F_t\|^2 + 3\mathbb{E}\|\nabla_{\boldsymbol{\theta}}F_t - \nabla_{\boldsymbol{\theta}}F_{t-1}\|^2 + 3\mathbb{E}\|\nabla_{\boldsymbol{\theta}}F_{t-1} - \mathbf{v}_{t-1}\|^2 \\
\leq&3L_F\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 3L_F^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2.
\end{aligned} \tag{43}$$

Thus, we have: $\sum_{t=1}^{T} \mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 \leq \sum_{t=1}^{T} 3L_F^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 3L_F^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2$, and similarly, $\sum_{t=1}^{T} \mathbb{E}\|\mathbf{u}_t - \mathbf{u}_{t-1}\|^2 \leq \sum_{t=1}^{T} 3L_F^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 3L_F^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2$. □

Now, we show the final step for proving Theorem 1. With Lemmas 3-5 and the defined potential function, we have:

$$\begin{aligned}
\mathbb{E}\mathfrak{P}_{T+1} - \mathfrak{P}_0 \leq& -\frac{\gamma}{2}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - \gamma L_f^2\sum_{t=0}^{T}\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 \\
&- (\frac{\gamma}{2} - \frac{L_J\gamma^2}{2} - \frac{40\gamma^3 L_{\boldsymbol{\omega}}^2 L_F^2}{\mu^2\eta^2})\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{p}}_t\|^2 - \frac{2\gamma\eta L_F^2}{\mu}\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{d}}_t\|^2 \\
&- (1 - (1 + c_1)\lambda^2 - \frac{72\gamma L_F^4}{\mu^2} - 2\gamma L_F^2)\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m} \\
&- (1 - (1 + c_2)\lambda^2 - \frac{72\gamma L_F^4}{\mu^2} - 2\gamma L_F^2)\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m} \\
&- \big(1 - (1 + c_1)\lambda^2 - (1 + \frac{1}{c_1})\gamma\big)\gamma\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{p}_t)}{m} \\
&- \big(1 - (1 + c_2)\lambda^2 - (1 + \frac{1}{c_2})\eta\big)\eta\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{d}_t)}{m}
\end{aligned}$$

$$+\underbrace{\frac{2\gamma}{m}\sum_{t=0}^{T}\mathbb{E}\|\nabla_{\boldsymbol{\omega}}F_t-\mathbf{v}_t\|^2+\frac{72L_F^2\gamma}{m\mu^2}\sum_{t=0}^{T}\mathbb{E}\|\nabla_{\boldsymbol{\theta}}F_t-\mathbf{u}_t\|^2}_{R_1}$$

$$+\underbrace{(1+\frac{1}{c_1})\frac{\gamma}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{v}_t-\mathbf{v}_{t-1}\|^2+(1+\frac{1}{c_2})\frac{\eta}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{u}_t-\mathbf{u}_{t-1}\|^2}_{R_2}. \quad (44)$$

First, we have $R_1=0$ because of the full gradient evaluation. R_2 can be bounded by

$$\begin{aligned}
R_2 &= (1+\frac{1}{c_1})\frac{\gamma}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{v}_t-\mathbf{v}_{t-1}\|^2+(1+\frac{1}{c_2})\frac{\eta}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{u}_t-\mathbf{u}_{t-1}\|^2\\
&\overset{(a)}{\le}\Big((1+\frac{1}{c_1})\frac{\gamma}{m}+(1+\frac{1}{c_2})\frac{\eta}{m}\Big)\sum_{t=1}^{T}3L_F^2\Big(\mathbb{E}\|\boldsymbol{\theta}_{t-1}-\boldsymbol{\theta}_t\|^2+\mathbb{E}\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_t\|^2\Big)\\
&\overset{(b)}{\le}3L_F^2\Big((1+\frac{1}{c_1})\frac{\gamma}{m}+(1+\frac{1}{c_2})\frac{\eta}{m}\Big)\sum_{t=1}^{T}\Big(8\mathcal{E}(\boldsymbol{\theta}_{t-1})+4\gamma^2\mathcal{E}(\mathbf{p}_{t-1})+4\gamma^2m\|\bar{\mathbf{p}}_{t-1}\|^2\Big)\\
&\quad+3L_F^2\Big((1+\frac{1}{c_1})\frac{\gamma}{m}+(1+\frac{1}{c_2})\frac{\eta}{m}\Big)\sum_{t=1}^{T}\Big(8\mathcal{E}(\boldsymbol{\omega}_{t-1})+4\eta^2\mathcal{E}(\mathbf{d}_{t-1})+4\eta^2m\|\bar{\mathbf{d}}_{t-1}\|^2\Big), \quad (45)
\end{aligned}$$

where (a) is from Lemma 5 and (b) is from Lemma 4.

Thus, we have

$$\begin{aligned}
\mathbb{E}\mathfrak{P}_{T+1}-\mathfrak{P}_0 \le& -\frac{\gamma}{2}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2-\gamma L_f^2\sum_{t=0}^{T}\mathbb{E}\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2\\
&-\Big(\frac{\gamma}{2}-\frac{L_J\gamma^2}{2}-\frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2}-12\gamma^2L_F^2\big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\big)\Big)\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{p}}_t\|^2\\
&-\Big(\frac{2\gamma\eta L_F^2}{\mu}-12\eta^2L_F^2\big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\big)\Big)\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{d}}_t\|^2\\
&-\Big(1-(1+c_1)\lambda^2-\frac{72\gamma L_F^4}{\mu^2}-2\gamma L_F^2-24L_F^2\big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\big)\Big)\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m}\\
&-\Big(1-(1+c_2)\lambda^2-\frac{72\gamma L_F^4}{\mu^2}-2\gamma L_F^2-24L_F^2\big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\big)\Big)\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m}\\
&-\Big(1-(1+c_1)\lambda^2-(1+\frac{1}{c_1})\gamma-12\gamma L_F^2\big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\big)\Big)\gamma\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{p}_t)}{m}\\
&-\Big(1-(1+c_2)\lambda^2-(1+\frac{1}{c_2})\eta-12\eta L_F^2\big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\big)\Big)\eta\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{d}_t)}{m}. \quad (46)
\end{aligned}$$

Choosing $c_1=c_2=1/\lambda-1$, we have

$$\begin{aligned}
\mathbb{E}\mathfrak{P}_{T+1}-\mathfrak{P}_0 \le& -\frac{\gamma}{2}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2-\gamma L_f^2\sum_{t=0}^{T}\mathbb{E}\|\boldsymbol{\omega}_t^*-\bar{\boldsymbol{\omega}}_t\|^2-C_{\bar{\mathbf{p}}}\gamma\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{p}}_t\|^2\\
&-C_{\bar{\mathbf{d}}}\gamma\eta\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{d}}_t\|^2-C_{\boldsymbol{\theta}}\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m}-C_{\boldsymbol{\omega}}\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m}-C_{\mathbf{p}}\gamma\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{p}_t)}{m}-C_{\mathbf{d}}\eta\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{d}_t)}{m}, \quad (47)
\end{aligned}$$

where the constants are

$$C_{\bar{\mathbf{p}}}=\frac{1}{2}-\frac{L_J\gamma}{2}-\frac{40\gamma^2L_{\boldsymbol{\omega}}^2L_F^2}{\mu^2\eta^2}-\frac{12\gamma L_F^2(\gamma+\eta)}{1-\lambda}, \quad (48)$$

$$C_{\bar{\mathbf{d}}} = \frac{2L_F^2}{\mu} - \frac{12\eta L_F^2(1+\eta/\gamma)}{1-\lambda}, \tag{49}$$

$$C_{\boldsymbol{\theta}} = C_{\boldsymbol{\omega}} = 1-\lambda - \frac{72\gamma L_F^4}{\mu^2} - 2\gamma L_F^2 - \frac{24L_F^2(\gamma+\eta)}{1-\lambda}, \tag{50}$$

$$C_{\mathbf{p}} = 1-\lambda - \frac{\gamma}{1-\lambda} - \frac{12\gamma L_F^2(\gamma+\eta)}{1-\lambda}, \tag{51}$$

$$C_{\mathbf{d}} = 1-\lambda - \frac{\eta}{1-\lambda} - \frac{12\eta L_F^2(\gamma+\eta)}{1-\lambda}. \tag{52}$$

To ensure $C_{\bar{\mathbf{p}}} \geq 0$, we have

$$\begin{aligned} C_{\bar{\mathbf{p}}} &\overset{(a)}{\geq} \frac{1}{4} - \frac{L_J\gamma}{2} - \frac{12\gamma^2 L_F^2(1+\eta/\gamma)}{1-\lambda} \\ &\overset{(b)}{\geq} \frac{1}{4} - \frac{(L_f + L_f^2/\mu)\gamma}{2} - \frac{(1-\lambda)\gamma}{2} \overset{(c)}{\geq} 0, \end{aligned} \tag{53}$$

where (a) follows from $\kappa := \gamma/\eta \leq \mu^2/13L_F^2$ and Lemma 9, (b) is due to $\gamma \leq (1-\lambda)^2/24L_F^2(1+1/\kappa)$ and Lemma 11, and (c) follows from $\gamma \leq 1/2\big((L_F + L_F^2/\mu) + (1-\lambda)\big)$. By setting $\eta \leq (1-\lambda)/6\mu(1+1/\kappa)$, we have $C_{\bar{\mathbf{d}}} \geq 0$. By setting $\gamma \leq (1-\lambda)/(\frac{1}{2} + \frac{72L_F^4}{\mu^2} + 2L_F^2 + \frac{24L_F^2(1+1/\kappa)}{1-\lambda})$, we have $C_{\boldsymbol{\theta}} = C_{\boldsymbol{\omega}} \geq \gamma/2$. To ensure $C_{\mathbf{p}} \geq 0$,

$$\begin{aligned} C_{\mathbf{p}} &= 1-\lambda - \frac{\gamma}{1-\lambda} - \frac{12\gamma^2 L_F^2(1+\eta/\gamma)}{1-\lambda} \\ &\overset{(a)}{\geq} 1-\lambda - \frac{\gamma}{1-\lambda} - \frac{(1-\lambda)\gamma}{2} \overset{(b)}{\geq} 0, \end{aligned} \tag{54}$$

where (a) is by $\gamma \leq (1-\lambda)^2/24L_F^2(1+1/\kappa)$ and (b) is by $\gamma \leq 1/\big(1/2 + 1/(1-\lambda)^2\big)$. Similarly, with $\eta \leq 1/\big(1/2 + 1/(1-\lambda)^2\big)$, we have $C_{\mathbf{d}} \geq 0$.

To summarize, we need the following conditions to ensure $C_{\bar{\mathbf{p}}} \geq 0$, $C_{\bar{\mathbf{d}}} \geq 0$, $C_{\mathbf{p}} \geq 0$, $C_{\mathbf{d}} \geq 0$, $C_{\boldsymbol{\theta}} \geq \gamma/2$, $C_{\boldsymbol{\omega}} \geq \gamma/2$,

$$\kappa = \gamma/\eta \leq \mu^2/13L_F^2 \tag{55}$$

$$\gamma \leq \min\Big\{\frac{1}{2}\Big(L_F + \frac{L_F^2}{\mu} + (1-\lambda)\Big), \frac{1-\lambda}{(\frac{1}{2} + \frac{72L_F^4}{\mu^2} + 2L_F^2 + \frac{24L_F^2(1+1/\kappa)}{1-\lambda})}, \big(1/2 + 1/(1-\lambda)^2\big)^{-1}\Big\}, \tag{56}$$

$$\eta \leq \min\Big\{\frac{(1-\lambda)}{6\mu(1+1/\kappa)}, \big(1/2 + 1/(1-\lambda)^2\big)^{-1}\Big\}, \tag{57}$$

which is satisfied with

$$\kappa = \gamma/\eta \leq \mu^2/13L_F^2 \tag{58}$$

$$\begin{aligned} \eta \leq \min\Big\{&\frac{13L_F^2}{2\mu^2}\Big(L_F + \frac{L_F^2}{\mu} + (1-\lambda)\Big), \frac{26(1-\lambda)L_F^2}{(\mu^2 + 144L_F^4 + 4L_F^2\mu^2 + \frac{48\mu^2 L_F^2(1+1/\kappa)}{1-\lambda})}, \\ &\frac{(1-\lambda)}{6\mu(1+1/\kappa)}, \frac{13L_F^2}{\mu^2\big(1/2 + 1/(1-\lambda)^2\big)}\Big\}. \end{aligned} \tag{59}$$

Note that $\frac{(1-\lambda)}{6\mu(1+1/\kappa)} \leq \frac{1}{2L_F}$, which satisfies the step-size condition in Lemma 2 and Lemma 3.

Thus, with such parameter settings, we have

$$\frac{\gamma}{2}\sum_{t=0}^{T}\Big(\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 + 2L_f^2\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m} + \frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m}\Big) \leq \mathbb{E}\mathfrak{P}_0 - \mathfrak{P}_{T+1}, \tag{60}$$

which yields the final result by multiplying $2/(T+1)\gamma$ on both sides. Note that $\mathfrak{P}_{T+1} \geq J(\bar{\boldsymbol{\theta}}_{T+1}) \geq J^*$, then we complete the proof.

C Proof for Theorem 3 and Theorem 4

In this section, we provide the proofs of Theorem 3-4. Under Assumption 2, we can easily obtain that funciton F_i satisifies L_f-Lipschitz smoothness. Thus, we have the following modified descending result:

Lemma 6. *Under Assumption 1 and set $\eta \leq 1/2L_f$, the following inequality holds for GT-SRVR and GT-SRVR*I*:*

$$
\begin{aligned}
&J(\bar{\boldsymbol{\theta}}_{T+1}) - J(\bar{\boldsymbol{\theta}}_0) + \frac{8\gamma L_f^2}{\mu\eta}\left[\|\bar{\boldsymbol{\omega}}_{T+1} - \boldsymbol{\omega}_{T+1}^*\|^2 - \|\boldsymbol{\omega}_0^* - \bar{\boldsymbol{\omega}}_0\|^2\right] \\
\leq& -\frac{\gamma}{2}\sum_{t=0}^{T}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - \gamma L_f^2 \sum_{t=0}^{T}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 - \left(\frac{\gamma}{2} - \frac{L_J\gamma^2}{2} - \frac{40\gamma^3 L_{\boldsymbol{\omega}}^2 L_f^2}{\mu^2\eta^2}\right)\sum_{t=0}^{T}\|\bar{\mathbf{p}}_t\|^2 - \frac{2\gamma\eta L_f^2}{\mu}\sum_{t=0}^{T}\|\bar{\mathbf{d}}_t\|^2 \\
&+ \left(\frac{72\gamma L_f^4}{\mu^2 m} + \frac{2\gamma L_f^2}{m}\right)\sum_{t=0}^{T}[\mathcal{E}(\boldsymbol{\theta}_t) + \mathcal{E}(\boldsymbol{\omega}_t)] + \frac{2\gamma}{m}\sum_{t=0}^{T}\|\nabla_{\boldsymbol{\omega}} F_t - \mathbf{v}_t\|^2 + \frac{72 L_f^2\gamma}{m\mu^2}\sum_{t=0}^{T}\|\nabla_{\boldsymbol{\theta}} F_t - \mathbf{u}_t\|^2.
\end{aligned}
$$

Next, we bound the error of the gradient estimators as the follows:

Lemma 7 (Error of Gradient Estimator)**.** *Under Assumption 2, we have the following error bounds for the estimators:*

$$
\begin{aligned}
\sum_{t=0}^{T}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}} F_t\|^2 \leq \sum_{t=1}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}_{(n_t-1)q}, \boldsymbol{\omega}_{(n_t-1)q})\|^2 \\
+ L_f^2\big(\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 + \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2\big),
\end{aligned} \tag{61}
$$

$$
\begin{aligned}
\sum_{t=0}^{T}\|\mathbf{u}_t - \nabla_{\boldsymbol{\omega}} F_t\|^2 \leq \sum_{t=1}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F(\boldsymbol{\theta}_{(n_t-1)q}, \boldsymbol{\omega}_{(n_t-1)q})\|^2 \\
+ L_f^2\big(\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 + \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2\big),
\end{aligned} \tag{62}
$$

where n_t is the largest positive integer that satisfies $(n_t - 1)q \leq t$.

Proof. From the algorithm update, we have:

$$
\begin{aligned}
&\|\underbrace{\mathbf{v}_{i,t} - \nabla_{\boldsymbol{\theta}} F_{i,t}}_{A_{i,t}}\|^2 = \|\mathbf{v}_{i,t-1} + \frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1}) - \nabla_{\boldsymbol{\theta}} F_{i,t}\|^2 \\
=&\|\underbrace{\mathbf{v}_{i,t-1} - \nabla_{\boldsymbol{\theta}} F_{i,t-1}}_{A_{i,t-1}} + \underbrace{\frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\theta}} f_{i,t}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{i,t}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1}) + \nabla_{\boldsymbol{\theta}} F_{i,t-1} - \nabla_{\boldsymbol{\theta}} F_{i,t}}_{B_{i,t}}\|^2 \\
=&\|A_{i,t-1}\|^2 + \|B_{i,t}\|^2 + 2\langle A_{i,t-1}, B_{i,t}\rangle.
\end{aligned} \tag{63}
$$

Note that $\mathbb{E}_t[B_{i,t}] = 0$, where the expectation is taken over the randomness in tth iteration. Thus,

$$
\mathbb{E}_t\|A_{i,t}\|^2 = \|A_{i,t-1}\|^2 + \mathbb{E}_t\|B_{i,t}\|^2. \tag{64}
$$

Also, with $|\mathcal{S}_{i,t}| = q$, we have

$$
\begin{aligned}
\mathbb{E}_t\|B_{i,t}\|^2 =& \mathbb{E}_t\|\frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}}\nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1}) - \nabla_{\boldsymbol{\theta}} F_{i,t} + \nabla_{\boldsymbol{\theta}} F_{i,t-1}\|^2 \\
\leq& \frac{1}{|\mathcal{S}_{i,t}|^2}\sum_{j\in\mathcal{S}_{i,t}}\mathbb{E}_t\|\nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t}, \boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t-1}, \boldsymbol{\omega}_{i,t-1}) - \nabla_{\boldsymbol{\theta}} F_{i,t} + \nabla_{\boldsymbol{\theta}} F_{i,t-1}\|^2 \\
\leq& \frac{L_f^2}{q}\big(\|\boldsymbol{\theta}_{i,t} - \boldsymbol{\theta}_{i,t-1}\|^2 + \|\boldsymbol{\omega}_{i,t} - \boldsymbol{\omega}_{i,t-1}\|^2\big).
\end{aligned} \tag{65}
$$

Taking full expectation and telescoping (65) over t from $(n_t-1)q+1$ to t, where $t \le n_t q - 1$, we have

$$\begin{aligned}\mathbb{E}\|A_t\|^2 &\le \mathbb{E}\|A_{t-1}\|^2 + \frac{L_f^2}{q}\mathbb{E}\big(\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 + \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2\big) \\ &\le \mathbb{E}\|A_{(n_t-1)q}\|^2 + \sum_{r=(n_t-1)q+1}^{t} \frac{L_f^2}{q}\mathbb{E}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big). \qquad (66)\end{aligned}$$

Thus, we have:

$$\begin{aligned}&\sum_{k=0}^{t}\mathbb{E}\|A_k\|^2 = \sum_{k=0}^{q-1}\mathbb{E}\|A_k\|^2 + \cdots + \sum_{k=(n_t-1)q}^{t}\mathbb{E}\|A_k\|^2 \\ \le& q\|A_0\|^2 + \sum_{k=1}^{q-1}\sum_{r=1}^{k}\frac{L_f^2}{q}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\ &+ \cdots \\ &+ \big(t-(n_t-1)q\big)\|A_{(n_t-1)q}\|^2 + \sum_{k=(n_t-1)q+1}^{t}\sum_{r=(n_t-1)q+1}^{k}\frac{L_f^2}{q}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\ \le& q\|A_0\|^2 + \sum_{r=1}^{q-1}\sum_{k=r}^{q-1}\frac{L_f^2}{q}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\ &+ \cdots \\ &+ \big(t-(n_t-1)q\big)\|A_{(n_t-1)q}\|^2 + \sum_{r=(n_t-1)q+1}^{t}\sum_{k=r}^{t}\frac{L_f^2}{q}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\ \le& q\|A_0\|^2 + \sum_{r=1}^{q-1}L_f^2\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\ &+ \cdots \\ &+ \big(t-(n_t-1)q\big)\|A_{(n_t-1)q}\|^2 + \sum_{r=(n_t-1)q+1}^{t}L_f^2\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\ =& \sum_{r=0}^{t}\|A_{(n_r-1)q}\|^2 + \sum_{r=1}^{t}L_f^2\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big). \qquad (67)\end{aligned}$$

Thus, we have:

$$\sum_{t=0}^{T}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}}F_t\|^2 \le \sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}}F_{(n_t-1)q})\|^2 + \sum_{t=1}^{T}L_f^2\big(\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 + \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2\big) \qquad (68)$$

Similarly, we have:

$$\sum_{t=0}^{T}\|\mathbf{u}_t - \nabla_{\boldsymbol{\omega}}F_t\|^2 \le \sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}}F_{(n_t-1)q})\|^2 + \sum_{t=1}^{T}L_f^2\big(\|\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}\|^2 + \|\boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}\|^2\big). \qquad (69)$$

□

Lemma 8 (Differential Bound on Estimator for GT-SRVR and GT-SRVRI)**.** *Under Assumption 2, the following inequalities holds*

$$\sum_{t=1}^{T}\mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 \le \sum_{t=1}^{T}4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2 + \sum_{t=0}^{T}6\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}}F_{(n_t-1)q}\|^2,$$

$$\sum_{t=1}^{T}\mathbb{E}\|\mathbf{u}_t - \mathbf{u}_{t-1}\|^2 \le \sum_{t=1}^{T}4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2 + \sum_{t=0}^{T}6\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}}F_{(n_t-1)q}\|^2.$$

Proof. For $\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2$, we have i) when $t \in \big((n_t-1)q, n_t q - 1\big] \cap \mathbb{Z}$,

$$\begin{aligned}
&\mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 = \sum_{i=1}^m \mathbb{E}\|\frac{1}{|\mathcal{S}_{i,t}|}\sum_{j\in\mathcal{S}_{i,t}} \nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{i,k}(\boldsymbol{\theta}_{i,t-1},\boldsymbol{\omega}_{i,t-1})\|^2 \\
&\le \frac{1}{|\mathcal{S}_{i,t}|^2}\sum_{i=1}^m \sum_{j\in\mathcal{S}_{i,t}} \mathbb{E}\|\nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t},\boldsymbol{\omega}_{i,t}) - \nabla_{\boldsymbol{\theta}} f_{i,j}(\boldsymbol{\theta}_{i,t-1},\boldsymbol{\omega}_{i,t-1})\|^2 \\
&\overset{(a)}{\le} L_f^2 \sum_{i=1}^m \mathbb{E}\|\boldsymbol{\theta}_{i,t-1} - \boldsymbol{\theta}_{i,t}\|^2 + L_f^2 \sum_{i=1}^m \mathbb{E}\|\boldsymbol{\omega}_{i,t-1} - \boldsymbol{\omega}_{i,t}\|^2 \\
&= L_f^2 \mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + L_f^2 \mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2,
\end{aligned} \tag{70}$$

where (a) is by $q \ge 1$ and Assumption 2.

ii) when $t = n_t q$ and $t > 0$,

$$\begin{aligned}
&\mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 = \mathbb{E}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}} F_t + \nabla_{\boldsymbol{\theta}} F_t - \nabla_{\boldsymbol{\theta}} F_{t-1} + \nabla_{\boldsymbol{\theta}} F_{t-1} - \mathbf{v}_{t-1}\|^2 \\
&\le 3\mathbb{E}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}} F_t\|^2 + 3\mathbb{E}\|\nabla_{\boldsymbol{\theta}} F_t - \nabla_{\boldsymbol{\theta}} F_{t-1}\|^2 + 3\mathbb{E}\|\nabla_{\boldsymbol{\theta}} F_{t-1} - \mathbf{v}_{t-1}\|^2 \\
&\overset{(a)}{\le} 3\mathbb{E}\|\mathbf{v}_t - \nabla_{\boldsymbol{\theta}} F_t\|^2 + 3\mathbb{E}\|\nabla_{\boldsymbol{\theta}} F_{t-1} - \mathbf{v}_{t-1}\|^2 + 3L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 3L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2 \\
&\overset{(b)}{\le} 3\mathbb{E}\|\mathbf{v}_{n_t q} - \nabla_{\boldsymbol{\theta}} F_{n_t q}\|^2 + 3\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 \\
&\quad + \sum_{r=(n_t-1)q+1}^{n_t q-1} \frac{3L_f^2}{q}\mathbb{E}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\
&\quad + 3L_f^2\mathbb{E}\|\boldsymbol{\theta}_{n_t q-1} - \boldsymbol{\theta}_{n_t q}\|^2 + 3L_f^2\mathbb{E}\|\boldsymbol{\omega}_{n_t q-1} - \boldsymbol{\omega}_{n_t q}\|^2,
\end{aligned} \tag{71}$$

where (a) is by Assumption 2, and (b) is by setting $t = n_t q$ and Lemma 7.

Telescoping (71) from $r = (n_t-1)q+1$ to $n_t q$,

$$\begin{aligned}
\sum_{r=(n_t-1)q+1}^{n_t q} \mathbb{E}\|\mathbf{v}_r - \mathbf{v}_{r-1}\|^2 \le& \sum_{r=(n_t-1)q+1}^{n_t q-1} L_f^2\mathbb{E}\|\boldsymbol{\theta}_{r-1} - \boldsymbol{\theta}_r\|^2 + L_f^2\mathbb{E}\|\boldsymbol{\omega}_{r-1} - \boldsymbol{\omega}_r\|^2 \\
&+ 3\mathbb{E}\|\mathbf{v}_{n_t q} - \nabla_{\boldsymbol{\theta}} F_{n_t q}\|^2 + 3\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 \\
&+ \sum_{r=(n_t-1)q+1}^{n_t q-1} \frac{3L_f^2}{q}\mathbb{E}\big(\|\boldsymbol{\theta}_r - \boldsymbol{\theta}_{r-1}\|^2 + \|\boldsymbol{\omega}_r - \boldsymbol{\omega}_{r-1}\|^2\big) \\
&+ 3L_f^2\mathbb{E}\|\boldsymbol{\theta}_{n_t q-1} - \boldsymbol{\theta}_{n_t q}\|^2 + 3L_f^2\mathbb{E}\|\boldsymbol{\omega}_{n_t q-1} - \boldsymbol{\omega}_{n_t q}\|^2 \\
\le& \sum_{r=(n_t-1)q+1}^{n_t q} 4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{r-1} - \boldsymbol{\theta}_r\|^2 + 4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{r-1} - \boldsymbol{\omega}_r\|^2 \\
&+ 3\mathbb{E}\|\mathbf{v}_{n_t q} - \nabla_{\boldsymbol{\theta}} F_{n_t q}\|^2 + 3\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2,
\end{aligned} \tag{72}$$

which leads to that

$$\sum_{t=1}^T \mathbb{E}\|\mathbf{v}_t - \mathbf{v}_{t-1}\|^2 \le \sum_{t=1}^T 4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2 + \sum_{t=0}^T 6\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2.$$

Similarly, we have

$$\sum_{t=1}^T \mathbb{E}\|\mathbf{u}_t - \mathbf{u}_{t-1}\|^2 \le \sum_{t=1}^T 4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 + 4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2 + \sum_{t=0}^T 6\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2.$$

□

With the defined potential function $\mathfrak{p}$, we have

$$\mathbb{E}\mathfrak{p}_{T+1} - \mathfrak{p}_0 \le -\frac{\gamma}{2}\sum_{t=0}^T \mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - \gamma L_f^2 \sum_{t=0}^T \mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2$$

$$
\begin{aligned}
&-(\frac{\gamma}{2}-\frac{L_J\gamma^2}{2}-\frac{40\gamma^3L_{\boldsymbol{\omega}}^2L_f^2}{\mu^2\eta^2})\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{p}}_t\|^2-\frac{2\gamma\eta L_f^2}{\mu}\sum_{t=0}^{T}\mathbb{E}\|\bar{\mathbf{d}}_t\|^2\\
&-(1-(1+c_1)\lambda^2-\frac{72\gamma L_f^4}{\mu^2}-2\gamma L_f^2)\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m}\\
&-(1-(1+c_2)\lambda^2-\frac{72\gamma L_f^4}{\mu^2}-2\gamma L_f^2)\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m}\\
&-\big(1-(1+c_1)\lambda^2-(1+\frac{1}{c_1})\gamma\big)\gamma\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{p}_t)}{m}\\
&-\big(1-(1+c_2)\lambda^2-(1+\frac{1}{c_2})\eta\big)\eta\sum_{t=0}^{T}\frac{\mathcal{E}(\mathbf{d}_t)}{m}\\
&+\underbrace{\frac{2\gamma}{m}\sum_{t=0}^{T}\mathbb{E}\|\nabla_{\boldsymbol{\omega}}F_t-\mathbf{v}_t\|^2+\frac{72L_f^2\gamma}{m\mu^2}\sum_{t=0}^{T}\mathbb{E}\|\nabla_{\boldsymbol{\theta}}F_t-\mathbf{u}_t\|^2}_{R_1}\\
&+\underbrace{(1+\frac{1}{c_1})\frac{\gamma}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{v}_t-\mathbf{v}_{t-1}\|^2+(1+\frac{1}{c_2})\frac{\eta}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{u}_t-\mathbf{u}_{t-1}\|^2}_{R_2}. \qquad (73)
\end{aligned}
$$

First, for the term R_1, we have

$$
\begin{aligned}
&\frac{2\gamma}{m}\sum_{t=0}^{T}\mathbb{E}\|\nabla_{\boldsymbol{\omega}}F_t-\mathbf{v}_t\|^2+\frac{72L_f^2\gamma}{m\mu^2}\sum_{t=0}^{T}\mathbb{E}\|\nabla_{\boldsymbol{\theta}}F_t-\mathbf{u}_t\|^2\\
\le&\frac{2\gamma}{m}\mathbb{E}\Big(\sum_{t=0}^{T}\|\mathbf{v}_{(n_t-1)q}-\nabla_{\boldsymbol{\theta}}F_{(n_t-1)q}\|^2+\sum_{t=1}^{T}L_f^2\big(\|\boldsymbol{\theta}_t-\boldsymbol{\theta}_{t-1}\|^2+\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_{t-1}\|^2\big)\Big)\\
&+\frac{72L_f^2\gamma}{m\mu^2}\mathbb{E}\Big(\sum_{t=0}^{T}\|\mathbf{u}_{(n_t-1)q}-\nabla_{\boldsymbol{\omega}}F_{(n_t-1)q}\|^2+\sum_{t=1}^{T}L_f^2\big(\|\boldsymbol{\theta}_t-\boldsymbol{\theta}_{t-1}\|^2+\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_{t-1}\|^2\big)\Big)\\
=&L_f^2(\frac{2\gamma}{m}+\frac{72L_f^2\gamma}{m\mu^2})\sum_{t=1}^{T}\mathbb{E}\big(\|\boldsymbol{\theta}_t-\boldsymbol{\theta}_{t-1}\|^2+\|\boldsymbol{\omega}_t-\boldsymbol{\omega}_{t-1}\|^2\big)\\
&+\frac{2\gamma}{m}\sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q}-\nabla_{\boldsymbol{\theta}}F_{(n_t-1)q}\|^2+\frac{72L_f^2\gamma}{m\mu^2}\sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q}-\nabla_{\boldsymbol{\omega}}F_{(n_t-1)q}\|^2. \qquad (74)
\end{aligned}
$$

Then, for term R_2, we can bound it as follows:

$$
\begin{aligned}
&(1+\frac{1}{c_1})\frac{\gamma}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{v}_t-\mathbf{v}_{t-1}\|^2+(1+\frac{1}{c_2})\frac{\eta}{m}\sum_{t=1}^{T}\mathbb{E}\|\mathbf{u}_t-\mathbf{u}_{t-1}\|^2\\
\le&(1+\frac{1}{c_1})\frac{\gamma}{m}\Big(\sum_{t=1}^{T}4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1}-\boldsymbol{\theta}_t\|^2+4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_t\|^2+\sum_{t=0}^{T}6\mathbb{E}\|\mathbf{v}_{(n_t-1)q}-\nabla_{\boldsymbol{\theta}}F_{(n_t-1)q}\|^2\Big)\\
&+(1+\frac{1}{c_2})\frac{\eta}{m}\Big(\sum_{t=1}^{T}4L_f^2\mathbb{E}\|\boldsymbol{\theta}_{t-1}-\boldsymbol{\theta}_t\|^2+4L_f^2\mathbb{E}\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_t\|^2+\sum_{t=0}^{T}6\mathbb{E}\|\mathbf{u}_{(n_t-1)q}-\nabla_{\boldsymbol{\omega}}F_{(n_t-1)q}\|^2\Big)\\
\le&(1+\frac{1}{c_1})\frac{6\gamma}{m}\sum_{t=0}^{T}6\mathbb{E}\|\mathbf{v}_{(n_t-1)q}-\nabla_{\boldsymbol{\theta}}F_{(n_t-1)q}\|^2+(1+\frac{1}{c_2})\frac{6\eta}{m}\sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q}-\nabla_{\boldsymbol{\omega}}F_{(n_t-1)q}\|^2\\
&+\frac{4L_f^2}{m}\Big((1+\frac{1}{c_1})\gamma+(1+\frac{1}{c_2})\eta\Big)\sum_{t=1}^{T}\mathbb{E}\big(\|\boldsymbol{\theta}_{t-1}-\boldsymbol{\theta}_t\|^2+\|\boldsymbol{\omega}_{t-1}-\boldsymbol{\omega}_t\|^2\big). \qquad (75)
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
R_1 + R_2 \leq & \frac{4L_f^2}{m}\Big((1+\frac{1}{c_1})\gamma + (1+\frac{1}{c_2})\eta + \frac{\gamma}{2} + \frac{18L_f^2\gamma}{\mu^2}\Big)\sum_{t=1}^{T}\mathbb{E}\|\boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t\|^2 \\
& + \frac{4L_f^2}{m}\Big((1+\frac{1}{c_1})\gamma + (1+\frac{1}{c_2})\eta + \frac{\gamma}{2} + \frac{18L_f^2\gamma}{\mu^2}\Big)\sum_{t=1}^{T}\mathbb{E}\|\boldsymbol{\omega}_{t-1} - \boldsymbol{\omega}_t\|^2 \\
& + \Big((1+\frac{1}{c_1})\frac{6\gamma}{m} + \frac{72\gamma}{m}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 \\
& + \Big((1+\frac{1}{c_2})\frac{6\eta}{m} + \frac{72L_f^2\gamma}{m\mu^2}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2 \\
\leq & \frac{4L_f^2}{m}\Big((1+\frac{1}{c_1})\gamma + (1+\frac{1}{c_2})\eta + \frac{\gamma}{2} + \frac{18L_f^2\gamma}{\mu^2}\Big)\sum_{t=1}^{T}\mathbb{E}\Big(8\mathcal{E}(\boldsymbol{\theta}_{t-1}) + 4\gamma^2\mathcal{E}(\mathbf{p}_{t-1}) + 4\gamma^2 m\|\bar{\mathbf{p}}_{t-1}\|^2\Big) \\
& + \frac{4L_f^2}{m}\Big((1+\frac{1}{c_1})\gamma + (1+\frac{1}{c_2})\eta + \frac{\gamma}{2} + \frac{18L_f^2\gamma}{\mu^2}\Big)\sum_{t=1}^{T}\mathbb{E}\Big(8\mathcal{E}(\boldsymbol{\omega}_{t-1}) + 4\eta^2\mathcal{E}(\mathbf{d}_{t-1}) + 4\eta^2 m\|\bar{\mathbf{d}}_{t-1}\|^2\Big) \\
& + \Big((1+\frac{1}{c_1})\frac{6\gamma}{m} + \frac{2\gamma}{m}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 \\
& + \Big((1+\frac{1}{c_2})\frac{6\eta}{m} + \frac{72L_f^2\gamma}{m\mu^2}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2. \qquad (76)
\end{aligned}
$$

Plugging the above results, we have

$$
\begin{aligned}
\mathbb{E}\mathfrak{p}_{T+1} - \mathfrak{p}_0 \leq & -\frac{\gamma}{2}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 - \gamma L_f^2\sum_{t=0}^{T}\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 \\
& - c_{\bar{\mathbf{p}}}\sum_{t=0}^{T}\gamma\mathbb{E}\|\bar{\mathbf{p}}_t\|^2 - c_{\bar{\mathbf{d}}}\sum_{t=0}^{T}\gamma\eta\mathbb{E}\|\bar{\mathbf{d}}_t\|^2 - c_{\boldsymbol{\theta}}\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m} \\
& - c_{\boldsymbol{\omega}}\sum_{t=0}^{T}\frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m} - c_{\mathbf{p}}\sum_{t=0}^{T}\frac{\gamma\mathcal{E}(\mathbf{p}_t)}{m} - c_{\mathbf{d}}\sum_{t=0}^{T}\frac{\eta\mathcal{E}(\mathbf{d}_t)}{m} \\
& + \Big((1+\frac{1}{c_1})\frac{6\gamma}{m} + \frac{2\gamma}{m}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 \\
& + \Big((1+\frac{1}{c_2})\frac{6\eta}{m} + \frac{72L_f^2\gamma}{m\mu^2}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2, \qquad (77)
\end{aligned}
$$

where

$$
c_{\bar{\mathbf{p}}} = \frac{1}{2} - \frac{L_J\gamma}{2} - \frac{40\gamma^2 L_{\boldsymbol{\omega}}^2 L_f^2}{\mu^2\eta^2} - 16L_f^2\gamma^2\Big((1+\frac{1}{c_1}) + (1+\frac{1}{c_2})\frac{\eta}{\gamma} + \frac{1}{2} + \frac{18L_f^2}{\mu^2}\Big), \qquad (78)
$$

$$
c_{\bar{\mathbf{d}}} = \frac{2L_f^2}{\mu} - 16L_f^2\eta\Big((1+\frac{1}{c_1}) + (1+\frac{1}{c_2})\frac{\eta}{\gamma} + \frac{1}{2} + \frac{18L_f^2}{\mu^2}\Big), \qquad (79)
$$

$$
c_{\boldsymbol{\theta}} = 1 - (1+c_1)\lambda^2 - \frac{72\gamma L_f^4}{\mu^2} - 2\gamma L_f^2 - 32L_f^2\gamma\Big((1+\frac{1}{c_1}) + (1+\frac{1}{c_2})\frac{\eta}{\gamma} + \frac{1}{2} + \frac{18L_f^2}{\mu^2}\Big), \qquad (80)
$$

$$
c_{\boldsymbol{\omega}} = 1 - (1+c_2)\lambda^2 - \frac{72\gamma L_f^4}{\mu^2} - 2\gamma L_f^2 - 32L_f^2\gamma\Big((1+\frac{1}{c_1}) + (1+\frac{1}{c_2})\frac{\eta}{\gamma} + \frac{1}{2} + \frac{18L_f^2}{\mu^2}\Big), \qquad (81)
$$

$$c_{\mathbf{p}} = 1-(1+c_1)\lambda^2-(1+\frac{1}{c_1})\gamma-16L_f^2\gamma^2\big((1+\frac{1}{c_1})+(1+\frac{1}{c_2})\frac{\eta}{\gamma}+\frac{1}{2}+\frac{18L_f^2}{\mu^2}\big), \tag{82}$$

$$c_{\mathbf{d}} = 1-(1+c_2)\lambda^2-(1+\frac{1}{c_2})\eta-16L_f^2\eta\gamma\Big((1+\frac{1}{c_1})+(1+\frac{1}{c_2})\frac{\eta}{\gamma}+\frac{1}{2}+\frac{18L_f^2}{\mu^2}\Big). \tag{83}$$

Choose $c_1 = c_2 = 1/\lambda - 1$, and define $C_0 = \frac{1}{1-\lambda}(1+\frac{\eta}{\gamma})+\frac{1}{2}+\frac{18L_f^2}{\mu^2}$. It follows that

$$c_{\bar{\mathbf{p}}} = \frac{1}{2}-\frac{L_J\gamma}{2}-\frac{40\gamma^2L_{\boldsymbol{\omega}}^2L_f^2}{\mu^2\eta^2}-16C_0L_f^2\gamma^2, \tag{84}$$

$$c_{\bar{\mathbf{d}}} = \frac{2L_f^2}{\mu}-16C_0L_f^2\eta, \tag{85}$$

$$c_{\boldsymbol{\theta}} = c_{\boldsymbol{\omega}} = 1-\lambda-\frac{72\gamma L_f^4}{\mu^2}-2\gamma L_f^2-32C_0L_f^2\gamma, \tag{86}$$

$$c_{\mathbf{p}} = 1-\lambda-\frac{\gamma}{1-\lambda}-16C_0L_f^2\gamma^2, \tag{87}$$

$$c_{\mathbf{d}} = 1-\lambda-\frac{\eta}{1-\lambda}-16C_0L_f^2\eta\gamma. \tag{88}$$

To ensure $c_{\bar{\mathbf{p}}} \geq 0$, we have

$$\begin{aligned} c_{\bar{\mathbf{p}}} &\overset{(a)}{\geq} \frac{1}{4}-\frac{L_J\gamma}{2}-16C_0L_f^2\gamma^2 \\ &\overset{(b)}{\geq} \frac{1}{4}-\frac{(L_f+L_f^2/\mu)\gamma}{2}-\frac{(1-\lambda)\gamma}{2} \overset{(c)}{\geq} 0, \end{aligned} \tag{89}$$

where (a) follows from $\kappa := \gamma/\eta \leq \mu^2/13L_f^2$ and Lemma 9, (b) is due to $\gamma \leq (1-\lambda)/32C_0L_f^2$ and Lemma 11, and (c) is from $\gamma \leq 1/2\big((L_f+L_f^2/\mu)+(1-\lambda)\big)$. By setting $\eta \leq 1/8\mu C_0$, we have $c_{\bar{\mathbf{d}}} \geq 0$. By setting $\gamma \leq (1-\lambda)/(\frac{1}{2}+\frac{72L_f^4}{\mu^2}+2L_f^2+32C_0L_f^2)$, we have $c_{\boldsymbol{\theta}} = c_{\boldsymbol{\omega}} \geq \gamma/2$. To ensure $c_{\mathbf{p}} \geq 0$,

$$\begin{aligned} c_{\mathbf{p}} &= 1-\lambda-\frac{\gamma}{1-\lambda}-16C_0L_f^2\gamma^2 \\ &\overset{(a)}{\geq} 1-\lambda-\frac{\gamma}{1-\lambda}-\frac{(1-\lambda)\gamma}{2} \overset{(b)}{\geq} 0, \end{aligned} \tag{90}$$

where (a) follows from $\gamma \leq (1-\lambda)/32C_0L_f^2$ and (b) is due to $\gamma \leq 1/\big(1/2+1/(1-\lambda)^2\big)$. Similarly, with $\eta \leq 1/\big(1/2+1/(1-\lambda)^2\big)$, we have $c_{\mathbf{d}} \geq 0$.

To summarize, we need the following conditions to ensure $c_{\bar{\mathbf{p}}} \geq 0$, $C_{\bar{\mathbf{d}}} \geq 0$, $C_{\mathbf{p}} \geq 0$, $C_{\mathbf{d}} \geq 0$, $C_{\boldsymbol{\theta}} \geq \gamma/2$, $c_{\boldsymbol{\omega}} \geq \gamma/2$,

$$\kappa = \gamma/\eta \leq \mu^2/13L_f^2, \tag{91}$$

$$\gamma \leq \min\Big\{\frac{1}{2}\Big(L_f+\frac{L_f^2}{\mu}+(1-\lambda)\Big), \frac{1-\lambda}{(\frac{1}{2}+\frac{72L_f^4}{\mu^2}+2L_f^2+32C_0L_f^2)}, \big(1/2+1/(1-\lambda)^2\big)^{-1}\Big\}, \tag{92}$$

$$\eta \leq \min\Big\{\frac{1}{8\mu C_0}, \big(1/2+1/(1-\lambda)^2\big)^{-1}\Big\}, \tag{93}$$

which can be satisfied by

$$\kappa = \gamma/\eta \leq \mu^2/13L_f^2, \tag{94}$$

$$\begin{aligned} \eta \leq \min\Big\{&\frac{13L_f^2}{2\mu^2}\Big(L_f+\frac{L_f^2}{\mu}+(1-\lambda)\Big), \frac{26(1-\lambda)L_f^2}{(\mu^2+144L_f^4+4L_f^2\mu^2+64C_0L_f^2\mu^2)}, \\ &\frac{1}{8\mu C_0}, \frac{13L_f^2}{\mu^2\big(1/2+1/(1-\lambda)^2\big)}\Big\}, \end{aligned} \tag{95}$$

where the constant C_0 is defined as $C_0 = \frac{1}{1-\lambda}(1+\frac{1}{\kappa}) + \frac{1}{2} + \frac{18L_f^2}{\mu^2}$. Also, it can be easily verified that $\frac{1}{8\mu C_0} \leq \frac{1}{2L_f}$.

With the above conditions, we have:

$$\begin{aligned}
&\frac{\gamma}{2}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 + 2L_f^2\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m} + \frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m} \\
\leq &\mathbb{E}[\mathfrak{p}_0 - \mathfrak{p}_{T+1}] + \Big(\frac{6\gamma}{m\lambda} + \frac{2\gamma}{m}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 \\
&+ \Big(\frac{6\eta}{m\lambda} + \frac{72L_f^2\gamma}{m\mu^2}\Big)\sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2. \qquad (96)
\end{aligned}$$

For GT-SRVR, the outer loop calculates the full gradients. Thus, we have $\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 = \mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2 = 0$. Then, we have the stated result in Theorem 3:

$$\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 + 2L_f^2\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m} + \frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m} \leq \frac{2\mathbb{E}[\mathfrak{p}_0 - \mathfrak{p}_{T+1}]}{(T+1)\gamma}. \qquad (97)$$

For GT-SRVRI, we have that

$$\begin{aligned}
&\sum_{t=0}^{T}\mathbb{E}\|\mathbf{v}_{(n_t-1)q} - \nabla_{\boldsymbol{\theta}} F_{(n_t-1)q}\|^2 = \sum_{t=0}^{T}\mathbb{E}\|\mathbf{u}_{(n_t-1)q} - \nabla_{\boldsymbol{\omega}} F_{(n_t-1)q}\|^2 \\
&= \sum_{t=0}^{T}\frac{m\sigma^2\mathbb{1}_{(|\mathcal{R}_{i,(n_t-1)q}|)}}{|\mathcal{R}_{i,(n_t-1)q}|} \leq \sum_{t=0}^{T}\frac{m\sigma^2\mathbb{1}_{(|\mathcal{R}_{i,\lfloor t/q\rfloor q}|<n)}}{\min\{(\lfloor t/q\rfloor+1)^\alpha q, \lceil c_\epsilon\epsilon^{-2}\rceil\}} \\
&\leq \sum_{t=0}^{T} m\sigma^2\max\{\frac{\mathbb{1}_{((\lfloor t/q\rfloor+1)^\alpha q<n)}}{(\lfloor t/q\rfloor+1)^\alpha q}, \frac{\epsilon^2}{c_\epsilon}\} \leq m\sigma^2(\sum_{r=1}^{\infty}\frac{\mathbb{1}_{(\tau^\alpha q<n)}}{\tau^\alpha} + \frac{\epsilon^2(T+1)}{c_\epsilon}) \\
&\leq m\sigma^2(1 + \int_1^{\frac{n}{q}^{1/\alpha}}\frac{1}{\tau^\alpha}d\tau + \frac{\epsilon^2(T+1)}{c_\epsilon}) \\
&\leq \begin{cases} m\sigma^2(\frac{1}{1-\alpha}(\frac{n}{q})^{(\frac{1}{\alpha}-1)} - \frac{\alpha}{1-\alpha} + \frac{\epsilon^2(T+1)}{c_\epsilon}), & \text{if } \alpha>0 \text{ and } \alpha\neq 0, \\ m\sigma^2(1+\log(\frac{n}{q}) + \frac{\epsilon^2(T+1)}{c_\epsilon}), & \text{if } \alpha = 1. \end{cases} \qquad (98)
\end{aligned}$$

Thus, for GT-SRVRI, we have the following convergence results:

$$\begin{aligned}
&\frac{1}{T+1}\sum_{t=0}^{T}\mathbb{E}\|\nabla J(\bar{\boldsymbol{\theta}}_t)\|^2 + 2L_f^2\mathbb{E}\|\boldsymbol{\omega}_t^* - \bar{\boldsymbol{\omega}}_t\|^2 + \frac{\mathcal{E}(\boldsymbol{\theta}_t)}{m} + \frac{\mathcal{E}(\boldsymbol{\omega}_t)}{m} \\
\leq &\frac{2\mathbb{E}[\mathfrak{p}_0 - \mathfrak{p}_{T+1}]}{(T+1)\gamma} + \big(\frac{12}{\lambda}(1+\frac{1}{r}) + 4 + \frac{144L_f^2}{\mu^2}\big)\big(\frac{\epsilon^2}{c_\epsilon} + \frac{C(n,q,\alpha)}{T+1}\big)\sigma^2, \qquad (99)
\end{aligned}$$

where the constant $C(n,q,\alpha)$ is defined as

$$C(n,q,\alpha) \triangleq \begin{cases} \frac{1}{1-\alpha}(\frac{n}{q})^{(\frac{1}{\alpha}-1)} - \frac{\alpha}{1-\alpha}, & \text{if } \alpha>0 \text{ and } \alpha\neq 1 \\ \log(\frac{n}{q}) + 1, & \text{if } \alpha = 1. \end{cases} \qquad (100)$$

With $\mathfrak{p}_{T+1} \geq J^*$, we reach the conclusion.

D Supporting Lemmas

Lemma 9. *Under Assumption 1, $\boldsymbol{\omega}^*(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\omega}} F(\boldsymbol{\theta},\boldsymbol{\omega})$ is Lipschitz continuous, i.e., there exists a positive constant $L_{\boldsymbol{\omega}}$, such that*

$$\|\boldsymbol{\omega}^*(\boldsymbol{\theta}) - \boldsymbol{\omega}^*(\boldsymbol{\theta}')\| \leq L_{\boldsymbol{\omega}}\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|, \ \ \forall \boldsymbol{\theta}, \boldsymbol{\theta}' \in \mathbb{R}^d, \qquad (101)$$

where the Lipschitz constant is $L_{\boldsymbol{\omega}} = L_F/\mu$ for Algorithm 1 and $L_{\boldsymbol{\omega}} = L_f/\mu$ for Algorithm 2.

Proof. See Lemma 4.3 in [23]. □

Lemma 10. *Under Assumption 1, the function $J(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))$ satisfies that $\nabla J(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))$.*

Proof. Since $J(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))$,by chain rule, we have

$$dJ(\boldsymbol{\theta}) = \frac{\partial F(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial \boldsymbol{\theta}}\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}^*(\boldsymbol{\theta})} \cdot d\boldsymbol{\theta} + \frac{\partial F(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial \boldsymbol{\omega}}\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}^*(\boldsymbol{\theta})} \cdot \frac{\partial \omega^*(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot d\boldsymbol{\theta}, \tag{102}$$

where $\partial F(\boldsymbol{\theta}, \boldsymbol{\omega})/\partial \boldsymbol{\theta}$ and $\partial F(\boldsymbol{\theta}, \boldsymbol{\omega})/\partial \boldsymbol{\omega}$ are respectively the partial differential of F w.r.t the first variate $\boldsymbol{\theta}$ and the second variate $\boldsymbol{\omega}$. Note that $\boldsymbol{\omega}^*(\boldsymbol{\theta})$ is the unique optimal point such that $F(\boldsymbol{\theta}, \boldsymbol{\omega})$ reaches the maximums. So, it follows that $\frac{\partial F(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=\boldsymbol{\omega}^*(\boldsymbol{\theta})} = 0$ for all $\boldsymbol{\theta}$. Also, from Lemma 9, we have $\partial \omega^*(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$ is bounded. Thus, it follows that

$$dJ(\boldsymbol{\theta}) = \frac{\partial F(\boldsymbol{\theta}, \boldsymbol{\omega})}{\partial \boldsymbol{\theta}}\Big|_{\boldsymbol{\omega}=\boldsymbol{\omega}^*(\boldsymbol{\theta})} \cdot d\boldsymbol{\theta}, \tag{103}$$

which is $\nabla J(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \boldsymbol{\omega}^*(\boldsymbol{\theta}))$.

Additionally, we can follow the detailed derivation of Theorem D2 in [2] to give a more rigorous proof for Eq (103). The difference between our Lemma 10 and Theorem D2 in [2] is that [2] adopted the uniformly bounded gradient assumption (Hypotheses D2.2 in [2]). In the followings, we show that under our assumptions, Eq (103) can be still obtained with similar derivation in [2]. Here we will adopt the notations in [2]. Because of the strong concavity, there is only one set $\{\boldsymbol{\nu}_n\}$ in $\mathcal{W}(\mathbf{u})$ for all $\mathbf{u}$ and $\boldsymbol{\nu}^* = \lim_{n\to} \boldsymbol{\nu}_n$ exists. From Assumption 1 (e), $D_1 J(\mathbf{u}, \boldsymbol{\nu}; h)$ is bounded. Thus, due to Assumption 1 (b), there exists a N_1, with $n \geq N_1$, $D_1 J(\mathbf{u}, \boldsymbol{\nu}_n; h)$ is also bounded. So, Proposition 2 in [2] holds. Meanwhile, because of the continuity of $D_1 J(\mathbf{u}, \boldsymbol{\nu})$ and $D_1 J(\mathbf{u}, \boldsymbol{\nu}^*) < \infty$, there exists N_2 with $n \geq \max\{N_1, N_2\}$, the function $t \mapsto J(\mathbf{u} + th, \boldsymbol{\nu}_n)$ has a bounded directional derivative for all $t \in [0, t_n]$. Thus, Proposition 3 in [2] holds. Because Proposition 2 and Proposition 3 in [2] still holds under our assumptions, we can then reach Eq (103) in Lemma 10. □

Lemma 11. *Under Assumption 1, the funciton $J(\boldsymbol{\theta}) = F(\boldsymbol{\theta}, \omega^*(\boldsymbol{\theta}))$ w.r.t $\boldsymbol{\theta}$ is Lipschitz smooth, i.e., there exists a positive constant L_J, such that*

$$\|\nabla J(\boldsymbol{\theta}) - \nabla J(\boldsymbol{\theta}')\| \leq L_J \|\boldsymbol{\theta} - \boldsymbol{\theta}'\|, \ \ \forall \boldsymbol{\theta}, \boldsymbol{\theta}' \in \mathbb{R}^d, \tag{104}$$

where the Lipschitz constant is $L_J = L_F + L_F^2/\mu$ for Algorithm 1 and $L_J = L_f + L_f^2/\mu$ for Algorithm 2.

Proof. The lemma follows immediately from Lemma 4.3 in [23] and Lemma 10. □