StrategyLLM: Large Language Models as Strategy Generators, Executors, Optimizers, and Evaluators for Problem Solving

Anonymous ACL submission

Abstract

Most existing chain-of-thought (CoT) prompting methods suffer from the issues of generalizability and consistency, as they often rely on instance-specific solutions that may not be applicable to other cases and lack task-level consistency in their reasoning steps. To address these limitations, we propose a comprehensive framework, StrategyLLM, harnessing the capabilities of LLMs to tackle various tasks. The framework improves generalizability by formulating general problem-solving strategies and enhances consistency by producing consistent solutions using these strategies. StrategyLLM employs four LLM-based agents: strategy generator, executor, optimizer, and evaluator, working together to generate, evaluate, and select promising strategies for a given task automatically. The experimental results demonstrate that StrategyLLM outperforms the competitive baseline CoT-SC that requires human-annotated solutions on 13 datasets across 4 challenging tasks without human involvement, including math reasoning (39.2% $\rightarrow$ 43.3%), commonsense reasoning (70.3% $\rightarrow$ 72.5%), algorithmic reasoning (51.7% $\rightarrow$ 62.0%), and symbolic reasoning (30.0% $\rightarrow$ 79.2%)[1].

1 Introduction

Recent advances in large language models (LLMs) have facilitated the development of prompting techniques (Liu et al., 2023a; Wei et al., 2022; Kojima et al., 2022; Chu et al., 2023). In particular, chain-of-thought (CoT) prompting methods (Wei et al., 2022; Chen et al., 2023c; Fu et al., 2023; Wang et al., 2023c), which condition LLMs on a few task examples with step-by-step solutions, guide LLMs to break down complex reasoning processes into simpler steps. These approaches have markedly improved performance compared to standard few-shot prompting across a variety of tasks.

Despite their potential, current CoT approaches employing few-shot prompts with instance-specific solutions may face challenges in terms of *generalizability* and *consistency*. Concerning *generalizability*, the solution can be highly specific to the question in each instance, limiting its applicability to other cases. For example, as illustrated in Figure 1, a solution for a particular system of linear equations with two variables may not provide valuable insights for addressing another system with three variables. Furthermore, the solutions in different instances within the few-shot prompt may exhibit a lack of task-level *consistency*, which complicates the process for LLMs to develop effective strategies for new instances. As demonstrated in Figure 1, the two specific solutions are based on different approaches: Solution 1 employs expression substitution, while Solution 2 utilizes equation subtraction, which may not provide consistent guidance for LLMs to solve new instances. Deriving an effective strategy that can solve all systems of linear equations from them can be challenging. To address these limitations, it is crucial to incorporate *effective problem-solving strategies* and develop *consistent strategy-based solutions* within few-shot prompts. Figure 1 presents an effective strategy, i.e., Gaussian Elimination Method, offering generalizable steps applicable to any system of linear equations. By providing this strategy and consistently applying it across various instances in the few-shot prompt, LLMs can be better equipped to generate effective solutions for new task instances.

In this paper, we propose StrategyLLM, a versatile framework leveraging the capabilities of LLMs to tackle various tasks. StrategyLLM enables LLMs to reason inductively, i.e., deriving general strategies from specific instances, and deductively, i.e., applying general strategies to particular examples, to automatically generate strategy-based few-shot prompts. An example of strategy-based prompts can be seen in Figure 3. The inductive rea-

[1]Code will be publicly available.

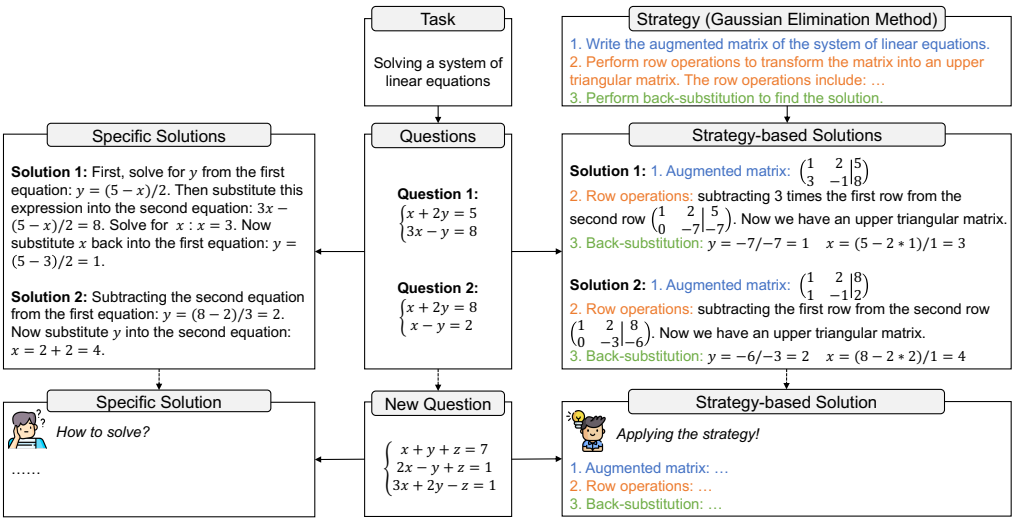


Figure 1: Comparison of specific solutions and strategy-based solutions.

soning process enhances *generalizability* by formulating general problem-solving strategies, while the deductive reasoning process improves *consistency* by producing consistent solutions using a given strategy. Developing effective problem-solving strategies is crucial to the success of our framework. To achieve this, we design StrategyLLM as a multi-agent collaboration framework comprising four LLM-based agents—strategy generator, executor, optimizer, and evaluator. These agents work together to generate, evaluate, and select promising strategies for addressing a given problem. Through the collaboration of these intelligent agents, our framework is capable of autonomously producing effective strategies for various tasks and eliminating the need for human involvement.

We conduct comprehensive evaluations of StrategyLLM on 13 datasets across 4 challenging tasks: math reasoning, commonsense reasoning, algorithmic reasoning, and symbolic reasoning. The experimental results reveal the following key findings: (1) StrategyLLM outperforms competitive baselines on all tasks without using any human-annotated reasoning processes; (2) StrategyLLM is capable of producing multiple complementary strategies for complex and diverse tasks; (3) StrategyLLM is robust to different LLMs and groups of task examples. These findings demonstrate the potential of StrategyLLM as an effective, efficient, and reliable problem-solving framework.

2 StrategyLLM

Our objective is to develop a comprehensive problem-solving framework that can efficiently and effectively address various tasks. To achieve this, we propose a multi-agent collaboration framework that harnesses the power of LLMs to automatically generate and select promising strategies.

2.1 Multi-Agent Collaboration Framework

Overview As presented in Figure 2, our framework consists of four key agents: strategy generator, strategy executor, strategy optimizer, and strategy evaluator. Initially, the strategy generator produces a pool of strategies based on their understanding of the target task. Next, the strategy executor applies each strategy to a set of task examples to obtain its execution result and computes its execution accuracy. Qualified strategies, i.e., strategies whose execution accuracy is larger than or equal to a pre-defined threshold, are cached with their execution results and execution accuracy. If the number of qualified strategies is smaller than a pre-defined number k, the optimizer will optimize unqualified strategies based on their execution results. These updated strategies will be sent to the strategy executor for the next iteration. This process may iterate several times until we have enough qualified strategies or the maximum iteration number is reached. Afterwards, we rank all cached strategies based on their execution accuracy and retain top k candidate strategies. Finally, the strategy evaluator evaluates all candidate strategies using a validation set. The prompts of the strategy generator, executor, and optimizer are in Appendix B. Typically, only a few task examples, i.e., less than 10, are used for strategy generation, execution, and optimization, making our framework highly efficient.

Notations We use p, q, st, so, and a to denote the prompt, question, strategy, solution, and answer,

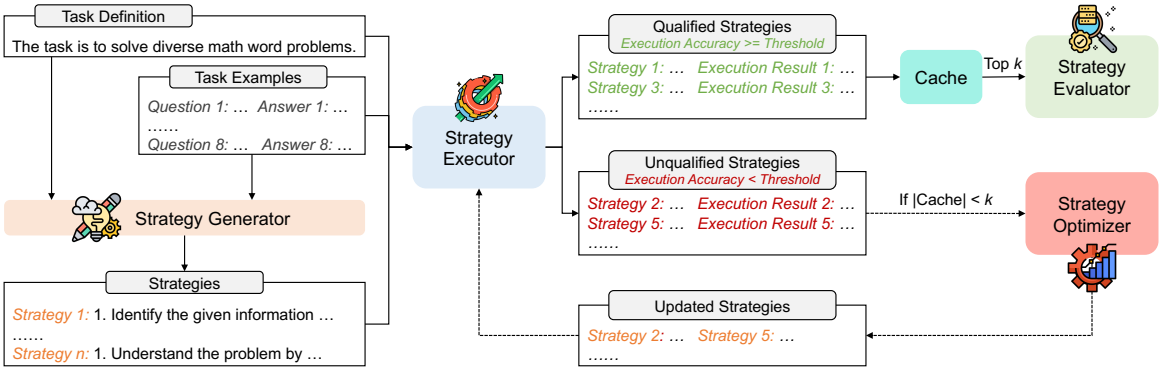


Figure 2: Overview of StrategyLLM. The letter k denotes the pre-defined number of candidate strategies.

respectively. During inference, given a question q, the language model $M : (p, q) \rightarrow (so, a)$ generates a solution so and an answer a for it conditioned on the prompt p. We denote the target task as t, its definition as d, and the set of task examples as $\mathcal{E}$. Each example in $\mathcal{E}$ is a (q, a) pair.

Strategy Generator G The strategy generator, represented as $G : (\mathcal{E}, d, n) \rightarrow \{st_j\}_{j=1}^{n}$, aims to generate n diverse strategies for the target task t based on a set of task examples $\mathcal{E}$ and the task definition d using temperature sampling.

Strategy Executor X The strategy executor, denoted as $X : (\mathcal{E}, d, st) \rightarrow (\mathcal{R}_{st}, eacc_{st})$, writes solutions to a set of task examples $\mathcal{E}$ following the strategy st to obtain the execution result $\mathcal{R}_{st} = \{(q, so, a)\}_{i=1}^{|\mathcal{E}|}$ of st. The execution accuracy $eacc_{st}$ is calculated as the proportion of examples whose solutions yield correct answers, reflecting the degree of alignment between the strategy and task. Therefore, we select strategies with high execution accuracy as qualified strategies.

Strategy Optimizer O The strategy optimizer, represented as $O : (\mathcal{E}, d, st, \mathcal{R}_{st}) \rightarrow st^o$, optimize the strategy st according to its execution result $\mathcal{R}_{st}$ to obtain the updated strategy st^o. Firstly, the strategy optimizer O analyzes why some solutions in $\mathcal{R}_{st}$ are not correct and provides suggestions for improving st. Secondly, it modifies st to obtain st^o based on the analysis and suggestions.

Strategy Evaluator E We select top k candidate strategies according to the execution accuracy. However, to ensure efficiency, we use a limited number of task examples for execution, making the execution accuracy not a very informative metric for choosing strategies. Therefore, we introduce a strategy evaluator to further evaluate the candidate strategies on a validation set $\mathcal{V}$. This process only requires to perform inference once for each candidate strategy and is efficient. The strategy evaluator, denoted as $E : (st, \mathcal{R}_{st}, \mathcal{V}) \rightarrow vacc_{st}$, computes the validation accuracy $vacc_{st}$ of the strategy st on $\mathcal{V}$. To achieve this, it constructs the strategy-based few-shot prompt $p_{st} = (st, \mathcal{R}_{st})$ and conducts inference on $\mathcal{V}$ using p_{st}. An example of strategy-based prompts is presented in Figure 3. The validation accuracy $vacc_{st}$ is calculated as the percentage of validation examples whose answers are correct, reflecting the effectiveness of st in real-world scenarios. Strategies with high validation accuracy can be used for inference.

2.2 Inference

Through collaborative efforts among multiple agents, we have obtained multiple candidate strategies, each with their corresponding validation accuracy. Depending on the task at hand, we can select one or more strategies with high validation accuracy for inference. For simple or specific tasks, a single optimal strategy may exist, making it sufficient to use only one strategy. However, for complex or diverse tasks, it is unlikely to find a strategy with absolute superiority. In such cases, combining multiple strategies for inference is more appropriate, as they may possess complementary strengths. To harness the strengths of multiple strategies, we employ two methods. The first method involves taking a majority vote on all answers obtained by multiple strategies, akin to the self-consistency (SC) method (Wang et al., 2023c). The second method requires LLMs to determine the final answer by comprehensively considering the solutions derived from multiple strategies in a zero-shot (ZS) manner, making it more suitable for complex and diverse tasks. We denote the first and second methods as StrategyLLM-SC and StrategyLLM-ZS, respectively. The prompt for the second approach is pro-

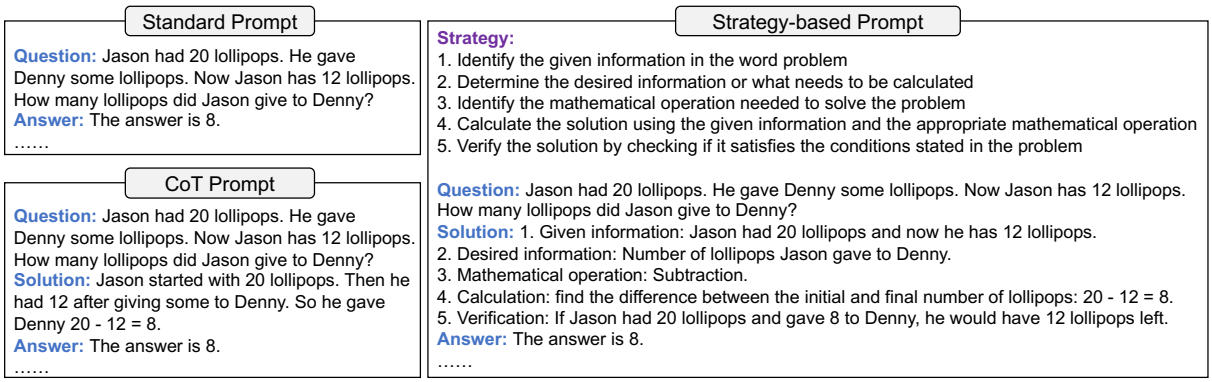


Figure 3: Comparison of the strategy-based, standard, and chain-of-thought (CoT) (Wei et al., 2022) prompt.

vided in Appendix C.

3 Experiments

3.1 Experimental Setup

Evaluation Tasks and Datasets We evaluate StrategyLLM on a variety of tasks:

- **Math Reasoning**: We use the GSM8K dataset (Cobbe et al., 2021) and seven datasets of different subjects, namely, Algebra (AL), Prealgebra (PA), Intermediate Algebra (IA), Counting and Probability (CP), Number Theory (NT), Geometry (GE), and Precalculus (PC), from the MATH benchmark (Hendrycks et al., 2021). GSM8K contains diverse grade school math problems, while MATH comprises challenging problems from mathematics competitions that require more than standard K-12 mathematics tools.

- **Commonsense Reasoning**: We employ StrategyQA (Geva et al., 2021) and the Date Understanding (DU) task from Big-Bench Hard (Suzgun et al., 2023; collaboration, 2023). StrategyQA necessitates inferring a multi-hop strategy to answer questions, while the DU task involves deducing a date from a given context.

- **Algorithmic Reasoning**: We adopt the Word Sorting (WS) task and the Multi-step Arithmetic (MA) task from Big-Bench Hard (Suzgun et al., 2023; collaboration, 2023). The WS task involves sorting a list of words lexicographically, and the MA task requires solving multi-step equations with basic arithmetic operations.

- **Symbolic Reasoning**: We utilize the Last Letter Concatenation (LLC) task from Wei et al. (2022), which requires concatenating the last letters of words in a sequence. In the few-shot prompt, the model only sees examples with two words. To evaluate the generalization abilities of different methods, we construct three out-of-distribution test sets (LLC-4, LLC-8, and LLC-16) with 4, 8, and 16 words in a sequence, respectively.

Baselines We conduct experiments in the few-shot setting and compare StrategyLLM with the following baselines:

- **Standard Prompting (SP)**: SP is the most direct approach for problem-solving. In SP, the prompt p contains a set of question-answer pairs without intermediate reasoning steps.

- **Chain-of-Thought (CoT) Prompting** (Wei et al., 2022): CoT incorporates step-by-step solutions for questions in the prompt p to elicit the multi-step reasoning capabilities of LLMs. We use few-shot CoT prompts from Wei et al. (2022) for GSM8K, StrategyQA, DU, and LLC, and prompts from Suzgun et al. (2023) for WS and MA. For MATH datasets, we create few-shot CoT prompts by randomly sampling 4 examples from each dataset's training set since these datasets contain human-annotated solutions. The CoT prompts for these datasets are in Appendix F.

- **Self-Consistency with CoT (CoT-SC)** (Wang et al., 2023c): CoT-SC generates a set of solutions using CoT via temperature sampling to obtain multiple answers. Subsequently, it takes a majority vote over these answers to determine the final answer. For experiments, we sample 3 reasoning paths using temperature sampling with a temperature of 0.7.

- **SolutionLLM**: We construct this baseline to leverage LLMs to directly write the solution for each example in the few-shot prompts using greedy decoding, without using any strategies. The prompt of SolutionLLM is in Appendix D.

Methods	GSM8K	AL	PA	IA	CP	NT	GE	PC	Avg
SP	31.0	24.5	50.0	17.5	27.0	20.5	21.0	20.5	26.5
SolutionLLM	64.5	59.5	56.5	13.5	33.0	33.0	28.0	19.5	38.4
CoT	69.0	53.5	57.5	15.0	33.5	28.0	23.0	20.0	37.4
CoT-SC	**76.0**	57.0	62.0	16.5	34.5	28.0	24.5	15.0	39.2
StrategyLLM	72.5	52.0	57.5	18.0	35.0	29.5	24.5	22.5	38.9
StrategyLLM-SC	75.0	58.5	61.5	18.0	38.5	30.5	28.0	**24.0**	41.8
StrategyLLM-ZS	**76.0**	**63.0**	**65.5**	**19.0**	**39.0**	**32.5**	**28.5**	22.5	**43.3**

Table 1: Experimental results on the math reasoning task.

Methods	Commonsense			Algorithmic			Symbolic			
	StrategyQA	DU	Avg	WS	MA	Avg	LLC-4	LLC-8	LLC-16	Avg
SP	56.5	48.5	52.5	73.3	0.7	37.0	0	0	0	0
SolutionLLM	59.5	52.0	55.8	74.7	26.7	50.7	81.5	25.5	0	35.7
CoT	64.0	70.5	67.3	67.2	42.0	54.6	68.5	22.0	0	30.2
CoT-SC	70.0	70.5	70.3	61.3	42.0	51.7	68.0	22.0	0	30.0
StrategyLLM	67.5	68.5	68.0	**80.0**	41.3	60.7	**98.0**	86.5	51.5	78.7
StrategyLLM-SC	**71.0**	**74.0**	**72.5**	79.3	**44.7**	**62.0**	**98.0**	**87.0**	**52.5**	**79.2**
StrategyLLM-ZS	70.0	72.5	71.3	78.7	42.0	60.4	**98.0**	86.0	44.0	76.0

Table 2: Experimental results on the commonsense, algorithmic, and symbolic reasoning tasks.

Implementation Details We employ GPT-3.5 (gpt-3.5-turbo-16k-0613) (OpenAI, 2023b) as the language model for our experiments, serving as the backend for the strategy generator, executor, optimizer, and evaluator. For a fair comparison with baselines such as CoT, we use the same examples in their few-shot prompts for strategy generation, execution, and optimization. Typically, the number of task examples $|\mathcal{E}|$ ranges from 3 to 8. The strategy generator creates $n = 30$ strategies using temperature sampling with a temperature of 1. The strategy executor, optimizer, and evaluator employ greedy decoding with a temperature of 0. The threshold of the execution accuracy for all datasets is set to 0.75, except for the CP and GE datasets. Their thresholds are set to 0.5 due to the challenging examples in their few-shot prompts. We retain $k = 10$ candidate strategies for validation, selecting the top 1 or 3 strategies with the highest validation accuracy for inference. This allows us to demonstrate the performance of the optimal strategy and the benefits of using multiple strategies. We adopt greedy decoding for inference. Details of the strategies for each dataset can be found in Appendix E. The validation set size is 100 for all the datasets. For datasets with over 200 test examples, we randomly sample 200 examples for testing to reduce API costs.

3.2 Main Results

Tables 1 and 2 present the experimental results of StrategyLLM and several baselines across four reasoning tasks. We have the following observations:

- *StrategyLLM is an effective and efficient framework for problem-solving*. StrategyLLM using multiple strategies, i.e., StrategyLLM-SC and StrategyLLM-ZS, outperforms all baselines across the four reasoning tasks. Furthermore, StrategyLLM employing the optimal strategy consistently outperforms CoT. Moreover, StrategyLLM automatically generates and selects effective strategies for tackling various tasks without human expertise, while CoT relies on human-annotated examples for each task.

- *Explicitly incorporating effective strategies significantly enhance the out-of-distribution (OOD) generalization abilities of LLMs*. StrategyLLM substantially outperforms CoT on the three OOD test sets of the LLC task, demonstrating the generalizability of effective strategies.

- *Combining multiple strategies brings obvious benefits on complex or diverse tasks*. The performance of StrategyLLM is significantly improved by combining multiple strategies on the math

Methods	AL-1	AL-2	AL-3	AL-Avg	CP-1	CP-2	CP-3	CP-Avg
SP	24.5	34.0	24.5	27.7	27.0	23.5	31.0	27.2
SolutionLLM	59.5	53.5	55.5	56.2	33.0	31.0	31.5	31.8
CoT	53.5	55.0	53.5	54.0	33.5	31.0	34.0	32.8
CoT-SC	57.0	58.5	**60.0**	58.5	34.5	32.5	31.0	32.7
StrategyLLM	52.0	56.5	52.5	53.7	35.0	39.0	37.5	37.2
StrategyLLM-SC	58.5	60.0	59.0	59.2	38.5	40.5	**39.5**	39.5
StrategyLLM-ZS	**63.0**	**60.5**	58.0	**60.5**	**39.0**	**41.5**	**39.5**	**40.0**

Table 3: Experimental results on two math reasoning datasets, namely AL and CP, with different groups of examples.

and commonsense reasoning tasks. The benefits of leveraging multiple strategies on simpler or more specific tasks, i.e., symbolic and algorithmic reasoning, is less significant. These observations indicate that there are multiple complementary strategies for diverse or complex tasks. Furthermore, StrategyLLM-ZS outperforms StrategyLLM-SC on the math reasoning task, showing that allowing LLMs to determine the answer is more appropriate for intricate tasks.

- *Compared with random sampling, explicitly using multiple strategies to obtain multiple solutions is more reliable*. StrategyLLM-SC achieves consistently better performance than CoT-SC on all tasks. Moreover, CoT-SC using temperature sampling to obtain multiple solutions suffers from the randomness issue, while the results of StrategyLLM-SC are reproducible.

4 Analysis

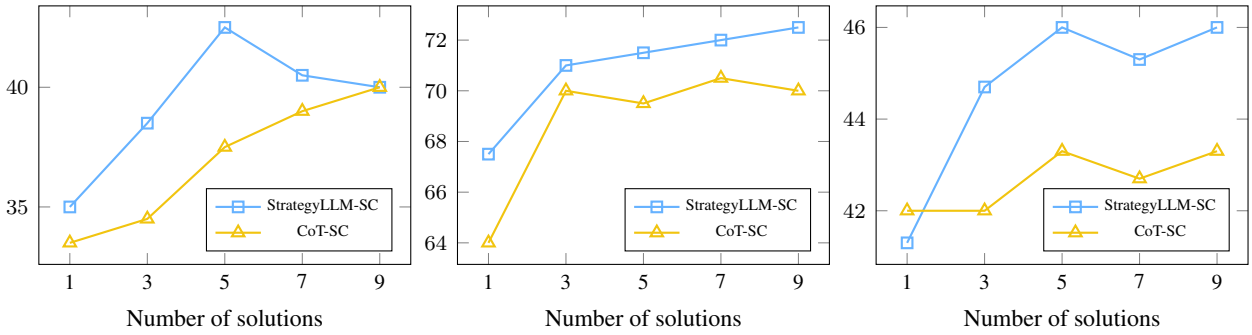
Evaluating the robustness of StrategyLLM We conduct an investigation to assess the robustness of our StrategyLLM framework with respect to varying groups of examples. For this purpose, we select two math reasoning datasets with diverse examples, namely AL and CP, and randomly sample three distinct groups of examples from their respective training sets. The results, as presented in Table 3, indicate that StrategyLLM consistently delivers satisfactory performance across all three example groups on both datasets. Notably, StrategyLLM-ZS outperforms all other methods under consideration. These findings suggest that StrategyLLM is a robust and reliable framework for problem-solving.

Examining the optimization process In our primary experiments, the strategy generator initially produces 30 strategies, and we aim to obtain 10 qualified strategies for validation. For complex datasets, it is difficult to directly acquire 10 qualified strategies without optimization. Table 4 presents the optimization process for three challenging datasets: CP from the mathematical reasoning task, StrategyQA from the commonsense reasoning task, and MA from the algorithmic reasoning task, along with the iteration in which the optimal strategy (i.e., the strategy with the highest validation accuracy) is obtained. It is evident that the strategy optimizer plays a vital role in obtaining more qualified strategies and superior strategies, indicating its significance in our framework.

Iteration	CP	StrategyQA	MA
1	7	9	6
2	12	10	9
3	-	-	10
Optimal	2	1	2

Table 4: Total count of qualified strategies achieved in each iteration of the optimization process and the specific iteration when the optimal strategy is attained.

Assessing the complementarity of strategies To this end, we combine multiple strategies to derive various solutions and ascertain the answer by majority voting. Specifically, we employ the top 1, 3, 5, 7, and 9 strategies for the CP, StrategyQA, and MA datasets. Figure 4 unveils the following observations: (1) StrategyLLM-SC consistently surpasses CoT-SC on all three datasets when employing multiple solutions, suggesting that explicitly introducing diverse strategies to obtain solutions is beneficial; (2) Combining multiple strategies outperforms the utilization of a single strategy across all three datasets, implying that multiple complementary strategies exist in most scenarios; (3) Incorporating additional strategies generally demonstrates advantageous. However, this does not guarantee enhancement in performance, as it relies on the effectiveness of the newly introduced strategies and



(a) CP (b) StrategyQA (c) MA

Figure 4: Performance of StrategyLLM-SC and CoT-SC on the CP, StrategyQA, and MA datasets.

Methods	CP	StrategyQA	MA	Avg
Inconsistent Prompt	29.0	56.5	37.3	40.9
StrategyLLM	**35.0**	**67.5**	**41.3**	**47.9**

Table 5: Results of StrategyLLM using the optimal strategy and the method employing inconsistent prompts.

Methods	CP	StrategyQA	MA	Avg
SP	29.0	71.5	1.3	33.9
SolutionLLM	52.0	75.5	46.7	58.1
CoT	49.5	80.5	46.0	58.7
CoT-SC	54.5	**83.5**	46.7	61.6
StrategyLLM	52.5	81.5	**48.0**	60.7
StrategyLLM-SC	56.0	**83.5**	**48.0**	62.5
StrategyLLM-ZS	**58.0**	83.0	**48.0**	**63.0**

Table 6: Experimental results of GPT-4 on the CP, StrategyQA, and MA datasets.

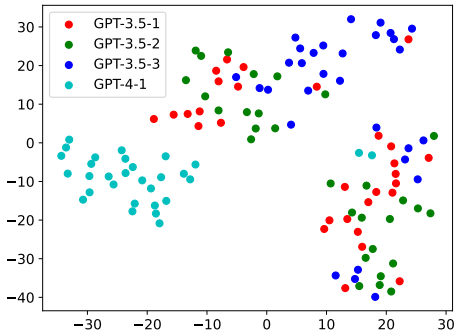
their complementarity with pre-existing strategies.

Examining the impact of prompt consistency The influence of prompt consistency is examined across the CP, StrategyQA, and MA datasets. For each test example within a particular dataset, an inconsistent few-shot prompt is constructed by incorporating multiple strategies dynamically. Specifically, the top 3 strategies are applied randomly and uniformly to the examples in the prompt. As a result, the prompt does not provide consistent guidance for LLMs, and LLMs need to select the most suitable strategy for each test example based on their comprehension of these strategies and the test example itself. Table 5 displays the comparative results between this approach and StrategyLLM employing the optimal strategy. It is evident that the method utilizing inconsistent prompts performs significantly worse than StrategyLLM, suggesting that automatically determining the most appropriate strategy for each test example is quite challenging. Consequently, consistently applying an effective strategy to diverse examples within the prompt is advantageous.

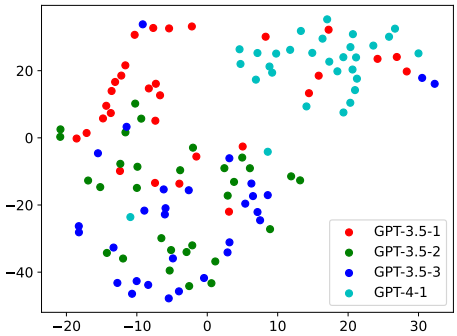
Evaluating the effectiveness of StrategyLLM with advanced LLM To explore this aspect, we leverage GPT-4 (gpt-4-0613) (OpenAI, 2023a) as the underlying LLM of our StrategyLLM framework. We conduct experiments on the CP, StrategyQA, and MA datasets from three distinct reasoning tasks. The results, as illustrated in Table 6, reveal that StrategyLLM consistently outperforms baselines across all three datasets, indicating that integrating effective strategies for constructing few-shot prompts can bring considerable advantages, even when applied to the most advanced LLM.

Identifying the influential factors in strategy generation We examine two crucial factors: the LLM and the task examples using the AL and CP datasets, as they contain diverse examples. Firstly, we apply GPT-3.5 and GPT-4 to the same set of examples, denoted as GPT-3.5-1 and GPT-4-1, respectively. Secondly, we apply GPT-3.5 to three distinct sets of examples, denoted as GPT-3.5-1, GPT-3.5-2, and GPT-3.5-3, respectively. For each combination of an LLM and a set of examples, we employ the strategy generator to produce 30 strategies and map them to high-dimensional embeddings using the OpenAI embedding model API[2]. Following this, we employ t-SNE (van der Maaten and Hinton, 2008) to visualize these embeddings in a two-dimensional map. Figure 5 reveals the following insights: (1) The points corresponding to

[2] https://platform.openai.com/docs/guides/embeddings/what-are-embeddings. Specifically, we utilize the text-embedding-ada-002 model.



(a) AL



(b) CP

Figure 5: Visualization of strategy embeddings corresponding to different LLMs and groups of examples.

the strategies of GPT-3.5-1, GPT-3.5-2, and GPT-3.5-3 are intermingled; (2) The points corresponding to GPT-4-1 are relatively distinct compared to the previous three sets of points. These observations imply that the LLM exerts a more significant influence on strategy generation than the examples.

5 Related Work

Prompting LLMs for Problem Solving The prominent chain-of-thought (CoT) prompting approach (Wei et al., 2022) has inspired a variety of prompting methods aimed at enhancing the problem-solving abilities of LLMs. These methods include using programming languages to describe the reasoning process (Chen et al., 2023c; Gao et al., 2023; Lyu et al., 2023), representing the reasoning process with complex structures such as trees or graphs (Yao et al., 2023; Besta et al., 2023; Sel et al., 2023; Zhang et al., 2023a), applying task decomposition (Zhou et al., 2023a; Khot et al., 2023; Press et al., 2023; Chen et al., 2023a), implementing self-correction with automatic feedback (Li et al., 2023a; Madaan et al., 2023; Miao et al., 2023; Chen et al., 2023b,d), and combining different prompting techniques (Liu et al., 2023b; Zhou et al., 2023b). However, most of these approaches require manual annotation of reasoning processes, limiting their generalizability and flexibility. By comparison, our StrategyLLM framework can automatically construct strategy-based few-shot prompts for any task, ensuring generalizable and consistent solutions following effective strategies. This approach sets it apart from existing automatic prompt construction methods (Zhang et al., 2023b; Shum et al., 2023; Xu et al., 2023), which may generate inconsistent solutions within the prompt. The plan-and-solve prompting method (Wang et al., 2023b) aims to address missing-step errors by requesting LLMs to generate a plan before solving a specific example in a zero-shot manner. The plan is instance-specific and significantly different from the task-solving strategy. The learning-to-program approach (Guo et al., 2023) learns a text program for each task to assist LLMs in problem-solving. The program, which is not necessarily a strategy, can be any instruction helpful for solving the task. Moreover, it demands a large training and validation set during the learning process, resulting in high costs. In contrast, our framework is efficient and cost-effective.

LLM-based Autonomous Agents The adoption of autonomous agents driven by LLMs across various disciplines is revolutionizing our methodologies for tackling problems, making decisions, and fostering innovation (Wang et al., 2023a; Xi et al., 2023). These agents have been utilized to enhance the reasoning capabilities of LLMs (Wang et al., 2023d; Liang et al., 2023; Du et al., 2023), contribute to social simulation (Park et al., 2023; Li et al., 2023b; Lin et al., 2023; Kovač et al., 2023), and advance software development (Qian et al., 2023; Hong et al., 2023; Dong et al., 2023). In this paper, we employ multiple LLM-based agents to collaborate in the generation, execution, optimization, and evaluation of problem-solving strategies.

6 Conclusion

In this paper, we propose StrategyLLM, a multi-agent collaboration framework that harnesses the power of LLMs for effective and efficient problem-solving. Comprehensive evaluations on four challenging tasks, namely math reasoning, commonsense reasoning, algorithmic reasoning and symbolic reasoning, demonstrate the effectiveness and reliability of our framework. Additional analysis shows that our framework is robust to different LLMs and groups of task examples.

Limitations

A limitation of our StrategyLLM framework is that the generated strategy-based few-shot prompts can often be longer than traditional CoT prompts, which may result in higher token usage during inference. Developing techniques to compress the generated strategy-based prompts without losing their effectiveness could help reduce token usage during inference. This may involve exploring more concise representations of strategies or utilizing summarization techniques to shorten the prompts while preserving their essential information. Moreover, the current framework generates strategies based on the provided task examples. However, it may be possible to further refine these strategies using additional information, such as expert knowledge or external resources. Future work could investigate methods for incorporating such information to improve the quality of the generated strategies.

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Methods	CP	StrategyQA	MA	Avg
GPT-3.5 → GPT-3.5	**38.5**	**71.0**	44.7	**51.4**
GPT-4 → GPT-3.5	34.0	**71.0**	**46.7**	50.6
GPT-4 → GPT-4	**56.0**	**83.5**	**48.0**	**62.5**
GPT-3.5 → GPT-4	**56.0**	79.5	43.3	59.6

Table 7: Performance of StrategyLLM-SC of applying the optimal strategies of an LLM to another.

A More Analysis

Upper limit of accuracy with multiple strategies The upper limit of utilizing multiple strategies can be determined by calculating the *coverage*, which is defined as the percentage of examples that can be accurately solved by at least one strategy. The coverage represents the maximum potential accuracy achievable with multiple strategies. Figure 6 illustrates the coverage and accuracy (i.e., through a majority vote) of the StrategyLLM using multiple strategies on the CP, StrategyQA, and MA datasets. The following observations can be made: (1) There is a significant gap between accuracy and coverage, indicating that substantial performance improvements could be realized if the optimal strategy for each test example is selected rather than relying on a simple majority vote. (2) For complex or diverse tasks, such as mathematical or commonsense reasoning, the incorporation of additional strategies considerably enhances the coverage. However, for relatively specific tasks, such as algorithmic reasoning, the coverage plateaus after a limited number of strategies. This observation highlights the complementary nature of multiple strategies for complex or diverse tasks.

Assessing the transferability of strategies between different LLMs To address this, we apply the optimal strategy-based prompts of an LLM to another for inference. Specifically, we evaluate StrategyLLM-SC by applying the optimal strategy-based prompts of GPT-4 to GPT-3.5 and vice versa. From Table 7, we can make the following observations: (1) Employing optimal strategies from a different LLM, even if it is more advanced, yields suboptimal results on average, underscoring the necessity of selecting strategies tailored to the target LLM; (2) The transferability of strategies created by GPT-4 surpasses that of GPT-3.5, as evidenced by the more pronounced decline in performance when applying strategies from GPT-3.5 to GPT-4.

B Prompts of Strategy Generator, Executor, and Optimizer

The prompts for the strategy generator and executor are presented in Figure 7 and Figure 8, respectively. For optimizing a strategy, the strategy optimizer conducts the following tasks: (1) constructing the examination result for each task example by comparing the answer extracted from its execution result and the gold answer, which is demonstrated in Figure 9; (2) analyzing the examination results of all provided task examples and providing suggestions for improving the strategy using the prompt in Figure 10; (3) modifying the strategy to obtain an updated strategy based on the feedback, i.e., the analysis and suggestions, leveraging the prompt in Figure 11.

C Prompt for Answer Determination

The prompt is shown in Figure 12.

D Prompt of SolutionLLM

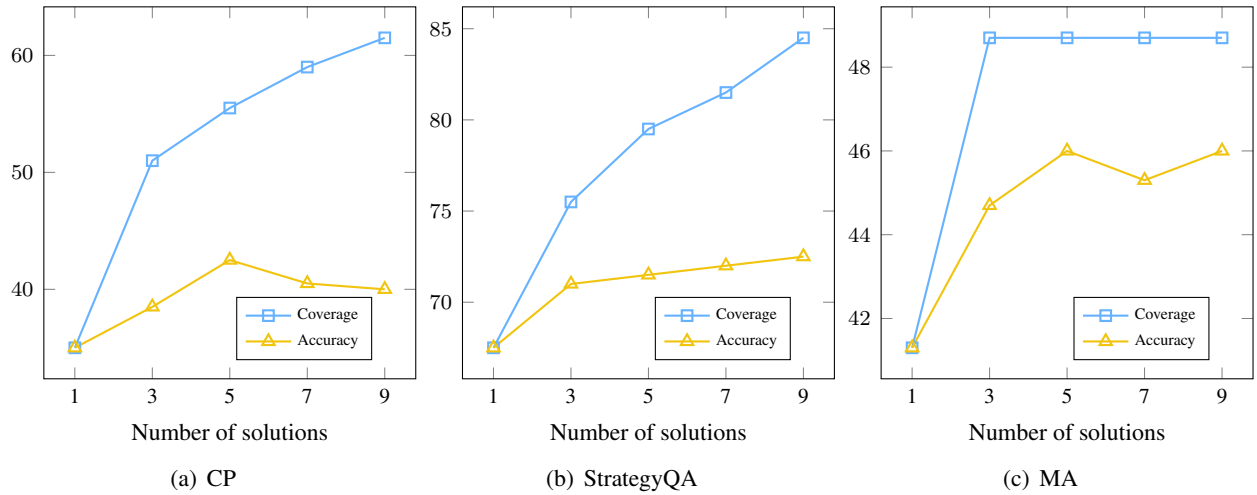
The prompt is presented in Figure 13.

E Strategies and Execution Results

Figures 14-26 provide the optimal strategies for various datasets and their execution results on task examples obtained by GPT-3.5. Figures 27-29 present the optimal strategies for various datasets and their execution results on task examples obtained by GPT-4.

F Chain-of-Thought Prompts for MATH Datasets

Figures 30-38 demonstrate the chain-of-thought prompts for the seven datasets from the MATH benchmark.



(a) CP (b) StrategyQA (c) MA

Figure 6: Coverage and accuracy of StrategyLLM using multiple strategies on the CP, StrategyQA, and MA datasets.

Strategy Generator

Task:
{task definition}

Some examples of the task are as follows:
{task examples}

Let's understand the task and write a strategy that consists of a sequence of subtasks to solve the task. For writing, you must satisfy the following requirements:
- Include all necessary subtasks.
- All subtasks are easy to solve.
- Subtasks are in an appropriate order.
- Do not include specific information in the examples.
- Make sure the strategy is general and concise.

The result must be a numbered list in the following format:
1. First subtask
2. Second subtask

Figure 7: Prompt of the strategy generator.

Strategy Executor

Task:
{task definition}

Example of the task:
{example}

Strategy:
{strategy}

The strategy consists of a sequence of subtasks for solving the task. Please execute the strategy on the provided example. For executing, you need to write a step-by-step solution to the example based on the subtasks. The solution must satisfy the following requirements:
- Adjust and execute these subtasks for this example.
- Compute as many intermediate results as possible.
- The answer obtained from the solution must be the same as the original answer.

The result must be in the following format:
Question: Question in the provided example
Solution: Solution obtained based on the subtasks in the strategy
Answer: Answer in the provided example, which must include the string 'The answer is '

Figure 8: Prompt of the strategy executor.

Strategy Optimizer - Construction of Examination Result

Example:
{example}

Execution result obtained by executing the strategy on the example:
{execution result}

The answer extracted from the execution result is {predicted answer}, and the correct answer is {gold answer}. They are {different or the same}.

Figure 9: Construction of the examination result for a task example.

Strategy Optimizer - Analysis & Suggestion

Task:
{task definition}

Strategy:
{strategy}

Examination results obtained by executing the strategy on the provided examples of the task and examining the execution results:
{examination results}

We can see that we do not get the correct answer after executing this strategy on some of the provided examples. Please carefully analyze why the answers extracted from the execution results of these examples are incorrect and provide suggestions for improving the strategy.

Figure 10: Analysis & Suggestion Prompt of the strategy optimizer.

Strategy Optimizer - Modification

Task:
{task definition}

Some examples of the task are as follows:
{task examples}

Original strategy to solve the task:
{original strategy}

Feedback:
{feedback}

You need to modify the original strategy based on the feedback to obtain a better strategy. The newly obtained strategy must be a numbered list in the following format:
1. First subtask
2. Second subtask

Figure 11: Modification Prompt of the strategy optimizer.

Answer Determination

Question:
{question}

Candidate Solutions:
{candidate solutions}

Please determine the final answer for the question based on the candidate solutions and their corresponding answers. The output should be in the following format:
Final Answer: The final answer, which must include the string 'The final answer is '.

Figure 12: Prompt for answer determination.

SolutionLLM

Task:
{task definition}

Example of the task:
{example}

Please write a solution to the provided example. The answer obtained from the solution must be the same as the original answer. The result must be in the following format:
Question: Question in the provided example
Solution: Solution to the question
Answer: Answer in the provided example, which must include the string 'The answer is '

Figure 13: Prompt of SolutionLLM.

GSM8K

Strategy:
1. Identify the given information in the word problem (e.g. number of trees, number of cars, amount of money).
2. Determine the desired information or what needs to be calculated (e.g. number of trees planted, total number of cars, remaining amount of money).
3. Identify the mathematical operation needed to solve the problem (e.g. addition, subtraction, multiplication).
4. Calculate the solution using the given information and the appropriate mathematical operation.
5. Verify the solution by checking if it satisfies the conditions stated in the problem.
6. Express the solution in the form of a sentence or statement that answers the question.
7. Repeat the process for each word problem given.

Executing the strategy on a task example:
Question: Shawn has five toys. For Christmas, he got two toys each from his mom and dad. How many toys does he have now?
Solution:
1. Given information: Shawn has five toys.
2. Desired information: How many toys does Shawn have now?
3. Mathematical operation: Addition.
4. Calculation:
- Shawn received 2 toys from his mom: 5 + 2 = 7 toys
- Shawn received 2 toys from his dad: 7 + 2 = 9 toys
5. Verification: The solution satisfies the conditions stated in the problem.
6. Solution: Shawn has 9 toys now.
Answer: The answer is 9.

Figure 14: Strategy generated by GPT-3.5 for GSM8K and its execution on a task example.

MATH-Algebra

Strategy:
1. Identify the given information in the problem. This includes any equations, values, or variables that are provided.
2. Determine the goal of the problem. What is the unknown variable or calculation that needs to be solved?
3. Simplify any given equations or expressions by performing any necessary operations or simplifications, such as combining like terms or factoring.
4. Solve the equations or expressions to find the value of the unknown variable(s). This may involve isolating the variable, applying algebraic rules, or using formulas.
5. Check the obtained solution(s) by substituting them back into the original equation(s) to see if they satisfy the given conditions.
6. If required, apply any additional algebraic techniques, such as manipulating equations, graphing, or using systems of equations, to solve for the unknown variable(s).
7. Write the final answer(s) in proper mathematical notation or use appropriate units if necessary.

Executing the strategy on a task example:
Question: Compute $\frac{x^8+12x^4+36}{x^4+6}$ when $x = 5$.
Solution:
1. Given information: $x = 5$
2. Goal: Compute $\frac{x^8+12x^4+36}{x^4+6}$
3. Simplify the expression:
$\frac{x^8+12x^4+36}{x^4+6} = \frac{(x^4+6)^2}{x^4+6}$
4. Solve the expression:
$\frac{(x^4+6)^2}{x^4+6} = x^4 + 6$
Substitute $x = 5$:
$5^4 + 6 = 631$
5. Check the obtained solution:
Substitute $x = 5$ into the original expression:
$\frac{5^8+12(5^4)+36}{5^4+6} = \frac{39061}{31} = 631$ The obtained solution satisfies the original expression.
6. Final Answer: The answer is 631.
Answer: The answer is 631.

Figure 15: Strategy generated by GPT-3.5 for MATH-Algebra and its execution on a task example.

MATH-Prealgebra

Strategy:
1. Understand the problem: Read and carefully analyze the given problem to understand the requirements and constraints.
2. Identify the known and unknown information: Determine what information is provided and what needs to be calculated or determined.
3. Translate the problem into mathematical equations or statements: Use appropriate mathematical symbols, equations, or statements to represent the problem.
4. Solve the equations or statements: Use algebraic techniques to solve the mathematical equations or statements.
5. Verify the Solution: Substitute the found values back into the original problem to check if they satisfy the given conditions.
6. Express the solution in the required format: Answer the question or provide the solution in the appropriate form, such as a fraction, decimal, or percentage.

Executing the strategy on a task example:
Question: Pete thinks of a number. He doubles it, adds 10, multiplies by 4, and ends up with 120. What was his original number?
Solution:
1. Understand the problem: Pete starts with a number, performs a series of operations on it, and ends up with a final result. We need to determine the original number.
2. Identify the known and unknown information:
Known information: Pete ends up with 120 after doubling the number, adding 10, and multiplying by 4.
Unknown information: The original number Pete started with.
3. Translate the problem into mathematical equations or statements:
Let x be the original number Pete started with.
Pete doubles the number: 2x
Pete adds 10: 2x + 10
Pete multiplies by 4: 4(2x + 10)
Pete ends up with 120: 4(2x + 10) = 120
4. Solve the equations or statements:
4(2x + 10) = 120
Distribute the 4: 8x + 40 = 120
Subtract 40 from both sides: 8x = 80
Divide both sides by 8: x = 10
The original number Pete started with is 10.
5. Verify the Solution: Substitute the found value back into the original problem.
Pete doubles 10: 2(10) = 20
Pete adds 10: 20 + 10 = 30
Pete multiplies by 4: 4(30) = 120
The final result is indeed 120, which matches the given information.
6. Express the solution in the required format:
Answer: The answer is 10.

Figure 16: Strategy generated by GPT-3.5 for MATH-Prealgebra and its execution on a task example.

MATH-Intermediate Algebra

Strategy:
1. Begin by identifying the type of problem you are given: quadratic polynomial, algebraic equation, geometric problem, etc.
2. If the problem involves a polynomial, start by determining the degree of the polynomial and then proceed to solve for the unknown coefficients using the given conditions or given equations.
3. If the problem involves algebraic equations, simplify the expressions and solve for the unknowns by isolating the variable. Make use of algebraic rules such as distributive property, solving equations with fractions, and solving systems of equations if necessary.
4. If the problem involves geometric concepts, analyze the given information and understand the geometric relationships involved. Apply the appropriate formulas or theorems and solve for the unknown quantities.
5. If the problem involves inequalities, solve for the range of possible values by analyzing the conditions or constraints given. Make use of algebraic manipulation and reasoning skills to determine the range or the maximum/minimum values.
6. If the problem involves functions, analyze the properties of the function and use any given information to determine the unknown variables, such as finding the maximum/minimum values, solving for x-intercepts or y-intercepts, or finding asymptotes.
7. Check your solutions by substituting them back into the original equation or problem statement to ensure they satisfy the given conditions.

Executing the strategy on a task example:
Question: Find the quadratic polynomial $p(x)$ such that $p(-3) = 10$, $p(0) = 1$, and $p(2) = 5$..
Solution:
1. Identify the type of problem: quadratic polynomial.
2. Determine the degree of the polynomial and solve for the unknown coefficients using the given conditions.
Let the quadratic polynomial be $p(x) = ax^2 + bx + c$.
Using the given conditions:
$p(-3) = 10 \Rightarrow a(-3)^2 + b(-3) + c = 10$
$p(0) = 1 \Rightarrow a(0)^2 + b(0) + c = 1$
$p(2) = 5 \Rightarrow a(2)^2 + b(2) + c = 5$
Simplifying the equations:
$9a - 3b + c = 10$
$c = 1$
$4a + 2b + c = 5$
Substituting $c = 1$ into the equations:
$9a - 3b + 1 = 10$
$4a + 2b + 1 = 5$
Simplifying further:
$9a - 3b = 9$
$4a + 2b = 4$
3. Solve the system of equations using algebraic manipulation.
Multiplying the second equation by 3:
$9a - 3b = 9$
$12a + 6b = 12$
Adding the equations together: $21a = 21$
Dividing both sides by 21: $a = 1$
Substituting $a = 1$ into the second equation:
$4(1) + 2b = 4$
$4 + 2b = 4$
$2b = 0$
Dividing both sides by 2: $b = 0$
Substituting $a = 1$ and $b = 0$ into the first equation:
$9(1) - 3(0) = 9$
$9 = 9$
Therefore, $a = 1$, $b = 0$, and $c = 1$.
4. Write the quadratic polynomial $p(x)$.
$p(x) = 1x^2 + 0x + 1$
Simplifying:
$p(x) = x^2 + 1$
Answer: The answer is $x^2 + 1$.

Figure 17: Strategy generated by GPT-3.5 for MATH-Intermediate Algebra and its execution on a task example.

MATH-Counting and Probability

Strategy:
1. Understand the problem statement:
- Read and understand the given problem.
- Identify what needs to be calculated or determined.
- Make note of any special conditions or restrictions mentioned.
2. Identify the counting or probability concept:
- Determine which counting or probability concept is needed to solve the problem.
- Common concepts include permutations, combinations, probability rules, etc.
3. Break down the problem:
- Break down the problem into smaller subproblems or cases if needed.
- Identify any patterns or symmetries that can simplify the calculations.
- Consider any specific conditions or restrictions mentioned in the problem.
4. Calculate the required counting or probability:
- Apply the appropriate counting or probability formula or rule to calculate the desired quantity.
- Use combinations or permutations for counting problems.
- Apply probability rules like the multiplication rule, addition rule, conditional probability, etc., for probability problems.
5. Simplify or reduce the problem:
- Look for ways to simplify or reduce the problem.
- Utilize symmetry or equivalence to eliminate redundant cases.
- Consider any combinatorial identities or properties that can simplify calculations.
6. Verify and check the Solution:
- Double-check the calculations to ensure accuracy.
- Check if the solution satisfies any specific conditions or restrictions mentioned in the problem.
- Compare the solution with any provided answers or results.
7. Communicate the solution clearly:
- Present the final solution in a clear and concise manner.
- Include any necessary explanations or steps taken to arrive at the solution.
- Express the final answer in the required units or as a fraction, decimal, or whole number, as specified by the problem.
8. Practice and review:
- Solve additional problems on counting and probability to reinforce the concepts.
- Review the solved problems to identify any common patterns or strategies that can be applied to similar problems.

Executing the strategy on a task example:
Question: Given that $\binom{17}{7} = 19448$, $\binom{17}{8} = 24310$ and $\binom{17}{9} = 24310$, calculate $\binom{19}{9}$.
Solution:
1. Understand the problem statement:
- We are given three values of binomial coefficients: $\binom{17}{7} = 19448$, $\binom{17}{8} = 24310$, and $\binom{17}{9} = 24310$.
- We need to calculate $\binom{19}{9}$.
2. Identify the counting or probability concept:
- We need to use the concept of combinations to solve this problem.
3. Break down the problem:
- We can use the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ to break down the problem.
- Using this identity, we can express $\binom{19}{9}$ in terms of the given values: $\binom{19}{9} = \binom{18}{8} + \binom{18}{9}$.
4. Calculate the required counting or probability:
- Using the given values, we can calculate $\binom{18}{8}$ and $\binom{18}{9}$.
- $\binom{18}{8} = \binom{17}{7} + \binom{17}{8} = 19448 + 24310 = 43758$
- $\binom{18}{9} = \binom{17}{8} + \binom{17}{9} = 24310 + 24310 = 48620$
5. Simplify or reduce the problem:
- We have calculated the values of $\binom{18}{8}$ and $\binom{18}{9}$.
- We can now substitute these values into the expression for $\binom{19}{9}$: $\binom{19}{9} = \binom{18}{8} + \binom{18}{9} = 43758 + 48620 = 92378$
6. Verify and check the Solution:
- The solution obtained is 92378, which matches the provided answer.
7. Communicate the solution clearly:
- The value of $\binom{19}{9}$ is 92378.
Answer: The answer is 92378.

Figure 18: Strategy generated by GPT-3.5 for MATH-Counting and Probability and its execution on a task example.

MATH-Number Theory

Strategy:
1. Understand the problem: Read the problem carefully and identify what is being asked and what information is given.
2. Apply basic number theory concepts: Use basic number theory principles such as modular arithmetic, divisibility rules, prime factorization, etc., to simplify the problem or find relevant information.
3. Simplify expressions: Simplify any given expressions or equations to make them easier to work with.
4. Solve congruences: If the problem involves solving congruences (e.g., finding the remainder when a number is divided by another number), use the Chinese Remainder Theorem or specific methods for solving modular equations to find the solution.
5. Study patterns and properties: Look for patterns or properties in the given problem that might help in finding the solution. This could involve using modular arithmetic properties, Euler's totient function, or other relevant principles.
6. Use prime factorization: If the problem involves finding factors or multiples of numbers, use the prime factorization of the numbers involved to simplify the calculations.
7. Apply number theory theorems: If the problem relates to specific number theory theorems, such as Fermat's Little Theorem or Wilson's Theorem, apply those theorems to find the solution.
8. Check for divisibility: Use divisibility rules to check if a number is divisible by another number, or find other relevant information that can help in solving the problem.
9. Solve step-by-step: Break down the problem into smaller steps and solve them one by one to find the final solution.
10. Check the Answer: Always double-check your final answer and ensure that it satisfies the given conditions or requirements of the problem.

Executing the strategy on a task example:
Question: What is the remainder when 2007 is divided by 81?
Solution:
1. Understand the problem: We are asked to find the remainder when 2007 is divided by 81.
2. Apply basic number theory concepts: We can use the concept of modular arithmetic to solve this problem. In modular arithmetic, we find the remainder when a number is divided by another number.
3. Simplify expressions: There are no expressions to simplify in this problem.
4. Solve congruences: To find the remainder when 2007 is divided by 81, we can write it as a congruence: $2007 \equiv x \pmod{81}$, where x is the remainder we want to find.
5. Study patterns and properties: There are no specific patterns or properties to consider in this problem.
6. Use prime factorization: We can use the prime factorization of 81 to simplify the calculations. $81 = 3^4$.
7. Apply number theory theorems: There are no specific number theory theorems to apply in this problem.
8. Check for divisibility: We can check if 2007 is divisible by 81. If it is divisible, then the remainder will be 0. Otherwise, we need to find the remainder.
9. Solve step-by-step:
a. Divide 2007 by 81: $2007 \div 81 = 24$ with a remainder of 63.
b. The remainder is 63.
10. Check the Answer: The remainder we found, 63, is the same as the original answer.
Answer: The answer is 63.

Figure 19: Strategy generated by GPT-3.5 for MATH-Number Theory and its execution on a task example.

MATH-Geometry

Strategy:
1. To find the area of a shaded region in a circle sector overlap problem:
a. Calculate the area of each sector using the formula $A_{\text{sector}} = \frac{1}{2}r^2\theta$, where r is the radius and θ is the central angle.
b. Determine the area of the overlapping region by subtracting the area of the triangle formed by the two radii from the sum of the areas of the two sectors.
c. Double-check the calculations and verify the obtained answer against the correct answer.
2. To calculate the perimeter of a quadrilateral formed by cutting a square with a line:
a. Find the coordinates of the intersection points between the line and the square.
b. Calculate the lengths of the sides of the quadrilateral using the distance formula.
c. Add up the lengths of the sides to find the perimeter.
d. Double-check the calculations and verify the obtained answer against the correct answer.
3. To determine the height of a cylinder inscribed in a hemisphere:
a. Use the Pythagorean theorem to find the radius of the cylinder's base by subtracting the height of the cylinder from the radius of the hemisphere.
b. The height of the cylinder is equal to the radius of the hemisphere minus the radius of the cylinder's base.
c. Double-check the calculations and verify the obtained answer against the correct answer.
4. To reflect a parallelogram across the x-axis:
a. Change the sign of the y-coordinate of each vertex to obtain the reflected image of the parallelogram.
b. Double-check the coordinates of the reflected vertices and verify against the correct answer.
5. To reflect a parallelogram across a given line:
a. Find the equation of the given line.
b. Calculate the distance between each vertex of the parallelogram and the given line.
c. Reflect each vertex across the given line by moving it the same distance on the opposite side of the line.
d. Connect the reflected vertices to form the reflected parallelogram.
e. Double-check the coordinates of the reflected vertices and verify against the correct answer.
6. To find the ordered pair of a reflected point:
a. Determine the equation of the line of reflection.
b. Use the formula for the reflection of a point across a line to find the coordinates of the reflected point.
c. Double-check the coordinates of the reflected point and verify against the correct answer.
7. Double-check all calculations and results to ensure accuracy.
a. Verify the obtained answers against the correct answers.
b. Review all steps and calculations to identify any errors or discrepancies.
c. Make necessary corrections and adjustments to improve the accuracy of the solution.

Executing the strategy on a task example:
Question: Parallelogram $ABCD$ with $A(2,5)$, $B(4,9)$, $C(6,5)$, and $D(4,1)$ is reflected across the x-axis to $A'B'C'D'$ and then $A'B'C'D'$ is reflected across the line $y = x + 1$ to $A''B''C''D''$. This is done such that D' is the image of D, and D'' is the image of D'. What is the ordered pair of D'' in the coordinate plane?
Solution:
1. Reflect parallelogram $ABCD$ across the x-axis to obtain parallelogram $A'B'C'D'$: a. Change the sign of the y-coordinate of each vertex: - $A'(2,-5)$ - $B'(4,-9)$ - $C'(6,-5)$ - $D'(4,-1)$
2. Reflect parallelogram $A'B'C'D'$ across the line $y = x + 1$ to obtain parallelogram $A''B''C''D''$: a. Find the equation of the line $y = x + 1$: - The slope of the line is 1, and the y-intercept is 1. - The equation of the line is $y = x + 1$.
b. Calculate the distance between each vertex of parallelogram $A'B'C'D'$ and the line $y = x + 1$: - Distance from A' to the line $y = x + 1$: - Substitute the coordinates of A' into the equation of the line: - $-5 = 2 + 1$ - $-5 = 3$ - The distance is $|-5-3| = 8$.
- Distance from B' to the line $y = x + 1$: - Substitute the coordinates of B' into the equation of the line: - $-9 = 4 + 1$ - $-9 = 5$ - The distance is $|-9-5| = 14$.
- Distance from C' to the line $y = x + 1$: - Substitute the coordinates of C' into the equation of the line: - $-5 = 6 + 1$ - $-5 = 7$ - The distance is $|-5-7| = 12$.
- Distance from D' to the line $y = x + 1$: - Substitute the coordinates of D' into the equation of the line: - $-1 = 4 + 1$ - $-1 = 5$ - The distance is $|-1-5| = 6$.
c. Reflect each vertex of parallelogram $A'B'C'D'$ across the line $y = x + 1$ by moving it the same distance on the opposite side of the line: - Reflect A' across the line $y = x + 1$: - Move A' 8 units in the opposite direction of the line's normal vector: - $A''(2-8, -5+8) = (-6, 3)$
- Reflect B' across the line $y = x + 1$: - Move B' 14 units in the opposite direction of the line's normal vector: - $B''(4-14, -9+14) = (-10, 5)$
- Reflect C' across the line $y = x + 1$: - Move C' 12 units in the opposite direction of the line's normal vector: - $C''(6-12, -5+12) = (-6, 7)$
- Reflect D' across the line $y = x + 1$: - Move D' 6 units in the opposite direction of the line's normal vector: - $D''(4-6, -1+6) = (-2, 5)$
3. The ordered pair of D'' is $(-2, 5)$.
Answer: The answer is $(-2, 5)$.

Figure 20: Strategy generated by GPT-3.5 for MATH-Geometry and its execution on a task example.

MATH-Precalculus

Strategy:
1. Identify the problem or concept being tested in the given question.
2. Define any necessary terms or formulas related to the problem.
3. Simplify or manipulate any given expressions or equations to solve the problem.
4. Solve the problem step by step with the help of appropriate formulas or techniques.
5. Verify the solution or answer to ensure its correctness.
6. Repeat steps 1-5 for each problem in the precalculus subject.

Executing the strategy on a task example:
Question: The function

$$f(z) = \frac{(-1 + i\sqrt{3})z + (-2\sqrt{3} - 18i)}{2}$$

represents a rotation around some complex number c. Find c.
Solution:
1. Identify the problem or concept being tested in the given question.
- The problem is asking us to find the complex number c that represents the center of rotation for the given function.
2. Define any necessary terms or formulas related to the problem.
- The formula for a rotation around a complex number c is given by $f(z) = e^{i\theta}(z - c) + c$, where θ is the angle of rotation.
3. Simplify or manipulate any given expressions or equations to solve the problem.
- The given function can be rewritten as $f(z) = \frac{-z+i\sqrt{3}z-2\sqrt{3}-18i}{2}$.
4. Solve the problem step by step with the help of appropriate formulas or techniques.
- Comparing the given function with the formula for rotation, we can see that $e^{i\theta} = \frac{-1+i\sqrt{3}}{2}$ and $c = \frac{-2\sqrt{3}-18i}{2}$. - To find θ, we can take the argument of $e^{i\theta}$, which is $\theta = \arg\left(\frac{-1+i\sqrt{3}}{2}\right)$. - Simplifying the expression inside the argument, we get $\frac{-1+i\sqrt{3}}{2} = \frac{-2+2i\sqrt{3}}{4} = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$. - The argument of this complex number is $\theta = \arctan\left(\frac{\sqrt{3}}{-1}\right) = \arctan(-\sqrt{3})$. - Using the unit circle, we can determine that $\theta = -\frac{\pi}{3}$. - Therefore, the angle of rotation is $-\frac{\pi}{3}$.
5. Verify the solution or answer to ensure its correctness.
- Substituting $\theta = -\frac{\pi}{3}$ into the formula for rotation, we get $f(z) = e^{-i\frac{\pi}{3}}(z - c) + c$. - Simplifying this expression, we have $f(z) = \frac{1}{2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)(z-c)+c$. - Using the trigonometric identities $\cos(-\frac{\pi}{3}) = \frac{1}{2}$ and $\sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$, we can further simplify the expression to $f(z) = \frac{1}{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(z-c)+c$. - Distributing and simplifying, we get $f(z) = \frac{1}{4}(z-c)-\frac{\sqrt{3}}{4}i(z-c)+c$. - Combining like terms, we have $f(z) = \left(\frac{1}{4} - \frac{\sqrt{3}}{4}i\right)z + \left(\frac{3}{4} + \frac{\sqrt{3}}{4}i\right)c$. - Comparing this expression with the given function, we can see that $\frac{1}{4} - \frac{\sqrt{3}}{4}i = \frac{-1+i\sqrt{3}}{2}$ and $\frac{3}{4} + \frac{\sqrt{3}}{4}i = \frac{-2\sqrt{3}-18i}{2}$. - Therefore, our solution is correct.
Answer: The answer is $\sqrt{3} - 5i$.

Figure 21: Strategy generated by GPT-3.5 for MATH-Precalculus and its execution on a task example.

StrategyQA

Strategy:
1. Read and understand the question fully by carefully analyzing the given information and requirements.
2. Identify the type of question being asked, whether it is a simple "yes or no" question or if there is a need to provide a specific answer.
3. Determine the key elements and concepts mentioned in the question that need to be addressed.
4. Break down the question into smaller parts, if necessary, to identify any hidden information or assumptions.
5. Look for any clues or hints within the question that can help in finding the answer.
6. Decide on the approach to be used, whether it requires logical reasoning, knowledge of specific facts, comparison, or analysis.
7. Use prior knowledge and general understanding to answer the question, if applicable.
8. If additional information is required, either use external sources or gather relevant data from memory.
9. Organize the information and answer the question clearly and concisely.
10. Review the answer to ensure it accurately addresses the question and provides a correct response.
11. If necessary, double-check the answer to ensure it aligns with the requirements and request.
12. Finally, present the answer in a clear and understandable manner, either as a "yes or no" response or as a specific answer as required by the question.

Executing the strategy on a task example:
Question: Could Brooke Shields succeed at University of Pennsylvania?
Solution:
1. Read and understand the question fully by carefully analyzing the given information and requirements.
- The question is asking if Brooke Shields could succeed at the University of Pennsylvania.
2. Identify the type of question being asked, whether it is a simple "yes or no" question or if there is a need to provide a specific answer.
- The question requires a simple "yes or no" answer.
3. Determine the key elements and concepts mentioned in the question that need to be addressed.
- Key elements: Brooke Shields, University of Pennsylvania, succeed.
4. Break down the question into smaller parts, if necessary, to identify any hidden information or assumptions.
- No need to break down the question further.
5. Look for any clues or hints within the question that can help in finding the answer.
- The question does not provide any specific clues or hints.
6. Decide on the approach to be used, whether it requires logical reasoning, knowledge of specific facts, comparison, or analysis.
- The approach requires logical reasoning and knowledge of the University of Pennsylvania.
7. Use prior knowledge and general understanding to answer the question, if applicable.
- The University of Pennsylvania is a prestigious university known for its rigorous academic programs. Brooke Shields, as a successful actress and model, may have the potential to succeed at the university.
8. If additional information is required, either use external sources or gather relevant data from memory.
- No additional information is required.
9. Organize the information and answer the question clearly and concisely.
- The answer is yes, Brooke Shields could succeed at the University of Pennsylvania.
10. Review the answer to ensure it accurately addresses the question and provides a correct response.
- The answer accurately addresses the question and provides a correct response.
11. If necessary, double-check the answer to ensure it aligns with the requirements and request.
- The answer aligns with the requirements and request.
12. Finally, present the answer in a clear and understandable manner, either as a "yes or no" response or as a specific answer as required by the question.
Answer: The answer is yes.

Figure 22: Strategy generated by GPT-3.5 for StrategyQA and its execution on a task example.

Date Understanding

Strategy:
1. Determine the current date based on the given information.
2. Calculate the number of hours or days given in the question.
3. Determine if the time period given is in the past or future.
4. Adjust the current date based on the time period and direction (past or future).
5. Convert the adjusted date to the required format (MM/DD/YYYY) as specified in the question.
6. Provide the final date as the answer to the question.

Executing the strategy on a task example:
Question: The concert was scheduled to be on 06/01/1943, but was delayed by one day to today. What is the date 10 days ago in MM/DD/YYYY?
Solution:
1. Determine the current date based on the given information.
- The concert was delayed by one day, so the current date is 06/02/1943.
2. Calculate the number of days given in the question.
- The question asks for the date 10 days ago.
3. Determine if the time period given is in the past or future.
- The time period is in the past.
4. Adjust the current date based on the time period and direction (past or future).
- Subtract 10 days from the current date: 06/02/1943 - 10 days = 05/23/1943.
5. Convert the adjusted date to the required format (MM/DD/YYYY) as specified in the question.
- The adjusted date is already in the required format: 05/23/1943.
6. Provide the final date as the answer to the question.
Answer: The answer is 05/23/1943.

Figure 23: Strategy generated by GPT-3.5 for Date Understanding and its execution on a task example.

Word Sorting

Strategy:
1. Identify the list of words that needs to be sorted.
2. Split the list of words into individual words.
3. Sort the individual words alphabetically.
4. Combine the sorted words back into a single list.
5. Present the sorted list of words as the answer.

Executing the strategy on a task example:
Question: Sort the following words alphabetically: List: oven costume counterpart
Solution:
1. Identify the list of words that needs to be sorted.
- The list of words is "oven costume counterpart".
2. Split the list of words into individual words.
- The individual words are "oven", "costume", and "counterpart".
3. Sort the individual words alphabetically.
- The sorted words are "costume", "counterpart", and "oven".
4. Combine the sorted words back into a single list.
- The combined sorted list is "costume counterpart oven".
5. Present the sorted list of words as the answer.
Answer: The answer is costume counterpart oven.

Figure 24: Strategy generated by GPT-3.5 for Word Sorting and its execution on a task example.

Multi-step Arithmetic

Strategy:
1. Identify the innermost operations enclosed in parentheses.
2. Evaluate the innermost operations first.
3. Replace the evaluated expression with the result.
4. Repeat steps 1-3 until all parentheses are removed.
5. Evaluate the remaining arithmetic operations according to the order of operations (multiplication and division before addition and subtraction).
6. Evaluate all multiplication and division operations from left to right.
7. Replace the evaluated expression with the result.
8. Repeat steps 5-7 until there are no more multiplication and division operations to be performed.
9. Evaluate all addition and subtraction operations from left to right.
10. Replace the evaluated expression with the result.
11. Repeat steps 9-10 until there are no more addition and subtraction operations to be performed.
12. The final result is the solution to the multi-step arithmetic problem.

Executing the strategy on a task example:
Question: ((-9 * 7 * 7 * -9) + (4 * -9 - 8 - -4)) =
Solution:
Step 1: Identify the innermost operations enclosed in parentheses.
The innermost operation is (4 * -9 - 8 - -4).
Step 2: Evaluate the innermost operations first.
4 * -9 = -36
-36 - 8 = -44
-44 - -4 = -40
Step 3: Replace the evaluated expression with the result.
(((-9 * 7 * 7 * -9) + -40)
Step 4: Repeat steps 1-3 until all parentheses are removed.
The innermost operation is (-9 * 7 * 7 * -9).
Step 5: Evaluate the innermost operations first.
-9 * 7 = -63
-63 * 7 = -441
-441 * -9 = 3969
Step 6: Replace the evaluated expression with the result.
(3969 + -40)
Step 7: Repeat steps 5-6 until there are no more multiplication and division operations to be performed.
There are no more multiplication and division operations to be performed.
Step 8: Evaluate all addition and subtraction operations from left to right.
3969 + -40 = 3929
Step 9: Replace the evaluated expression with the result.
Answer: The answer is 3929.

Figure 25: Strategy generated by GPT-3.5 for Multi-step Arithmetic and its execution on a task example.

Last Letter Concatenation

Strategy:
1. Identify the given input, which will be a string containing multiple words separated by spaces.
2. Split the input string into individual words.
3. Initialize an empty string to store the last letters of the words.
4. Iterate through each word in the list of words.
5. Get the last letter of each word.
6. Concatenate the last letter to the string from step 3.
7. Continue steps 4-6 until all words have been processed.
8. Print or return the final concatenated string.

Executing the strategy on a task example:
Question: Take the last letters of the words in "Sergey Brin" and concatenate them.
Solution:
1. Identify the given input: "Sergey Brin"
2. Split the input string into individual words: ["Sergey", "Brin"]
3. Initialize an empty string: last_letters = ""
4. Iterate through each word in the list of words:
- For the word "Sergey":
- Get the last letter: last_letter = "y"
- Concatenate the last letter to the string: last_letters = "y"
- For the word "Brin":
- Get the last letter: last_letter = "n"
- Concatenate the last letter to the string: last_letters = "yn"
5. Print or return the final concatenated string.
Answer: The answer is yn.

Figure 26: Strategy generated by GPT-3.5 for Last Letter Concatenation and its execution on a task example.

MATH-Counting and Probability

Strategy:
1. Identify the type of the counting or probability problem presented.
2. If the problem is regarding permutations or combinations, identify all possible outcomes and arrange or select the required number of objects accordingly.
3. If the problem is about probability, assess the sample space, the number of possible outcomes and the number of favorable outcomes.
4. For probability questions, use the formula of probability, which is the number of favorable outcomes divided by the number of total outcomes.
5. If the problem involves conditions or restrictions, ensure to factor in these when calculating probability or counting the possible arrangements or selections.
6. If the problem involves complex calculations, break them down into smaller manageable calculations.
7. Apply the suitable rule or formula according to the type of counting or probability problem, such as multiplication rule, addition rule, arrangement rules, rules of permutations and combinations, or probability rules.
8. Compute the result using arithmetic calculation or mathematical formulas, and write down the answer.
9. Review the process and the answer to ensure accuracy.

Executing the strategy on a task example:
Question: Given that $\binom{17}{7} = 19448$, $\binom{17}{8} = 24310$ and $\binom{17}{9} = 24310$, calculate $\binom{19}{9}$.
Solution:
1. The problem is about combinations, which is a type of counting problem.
2. The problem is asking for the number of ways to choose 9 items from a set of 19, which is represented by $\binom{19}{9}$.
3. We are given the values of $\binom{17}{7}$, $\binom{17}{8}$, and $\binom{17}{9}$, which can be used to calculate $\binom{19}{9}$.
4. The formula for combinations is $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, where n is the total number of items, r is the number of items to choose, and '!' denotes factorial.
5. However, we can use the property of combinations that $\binom{n}{r} = \binom{n}{n-r}$, and the relationship between $\binom{n+1}{r}$ and $\binom{n}{r-1}$ and $\binom{n}{r}$, which is $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.
6. Using these properties, we can express $\binom{19}{9}$ as $\binom{18}{8} + \binom{18}{9}$.
7. Similarly, $\binom{18}{8}$ can be expressed as $\binom{17}{7} + \binom{17}{8}$, and $\binom{18}{9}$ can be expressed as $\binom{17}{8} + \binom{17}{9}$.
8. Substituting the given values, we get $\binom{19}{9} = (19448 + 24310) + (24310 + 24310)$.
9. Performing the arithmetic calculation, we get $\binom{19}{9} = 43758 + 48620 = 92378$.
Answer: The answer is 92378.

Figure 27: Strategy generated by GPT-4 for MATH-Counting and Probability and its execution on a task example.

StrategyQA

Strategy:
1. Identify the main subject(s) or concept(s) in the question.
2. Identify the relationship(s) or action(s) associated with the main subject(s) or concept(s).
3. Break down the question into sub-questions based on the identified subjects or concepts and their associated relationships or actions, ensuring each sub-question leads to additional information required to answer the main question.
4. For each sub-question, infer or retrieve relevant information. This can come from provided information, general knowledge, or external resources.
5. Compare and synthesize the information gathered from each sub-question.
6. Based on the synthesis, determine if the information satisfies or contradicts the main question.
7. Formulate a 'yes' or 'no' answer to the main question based on the evaluation.
8. Confirm that the answer accurately addresses the main question. If not, revisit the sub-questions and retrieval process.
9. Record the answer.

Executing the strategy on a task example:
Question: Could Brooke Shields succeed at University of Pennsylvania?
Solution:
1. The main subject in the question is Brooke Shields and the main concept is University of Pennsylvania.
2. The relationship associated with the main subject and concept is Brooke Shields' potential success at the University of Pennsylvania.
3. Sub-questions based on the identified subjects or concepts and their associated relationships or actions could be:
- Who is Brooke Shields?
- What is the University of Pennsylvania?
- What does it mean to succeed at the University of Pennsylvania?
4. For each sub-Question:
- Brooke Shields is a well-known American actress and model.
- The University of Pennsylvania is an Ivy League research university in Philadelphia. It is one of the most prestigious universities in the world.
- To succeed at the University of Pennsylvania would likely mean to graduate with a degree, possibly with honors or high academic achievement.
5. Comparing and synthesizing the information gathered from each sub-question, we know that Brooke Shields is a successful actress and model, and the University of Pennsylvania is a prestigious university. Success at the university would likely require dedication, intelligence, and hard work.
6. Based on this synthesis, it is possible that Brooke Shields could succeed at the University of Pennsylvania, given her demonstrated success in her career. However, without specific information about her academic abilities or aspirations, we cannot definitively say.
7. Given the information available, we can infer a 'yes' answer to the main question.
8. The answer 'yes' accurately addresses the main question, given the information available.
9. The answer is recorded.
Answer: The answer is yes.

Figure 28: Strategy generated by GPT-4 for StrategyQA and its execution on a task example.

Multi-step Arithmetic

Strategy:
1. Begin by identifying all the numbers and operations in the given problem.
2. Apply the mathematical rules of order operations. The correct order of operations is Parentheses, Exponents, Multiplication and Division (from left to right), Addition and Subtraction (from left to right). This rule is often abbreviated as PEMDAS.
3. Start solving the problem from the inside of the parentheses first, keeping PEMDAS rule in mind.
4. While doing multiplication, division, addition, and subtraction (in the order), make sure to change the sign of the number if there is a negative sign before it.
5. After solving the operations inside the parentheses, proceed solving the operations outside the parentheses. Remember to still follow the rules and order of operations.
6. Continue to calculate step by step until you reach the final answer.

Executing the strategy on a task example:
Question: ((-9 * 7 * 7 * -9) + (4 * -9 - 8 - -4)) =
Solution:
1. Identify all the numbers and operations in the given problem. The numbers are -9, 7, -9, 4, -9, 8, -4. The operations are multiplication (*), addition (+), and subtraction (-).
2. Apply the mathematical rules of order operations (PEMDAS).
3. Start solving the problem from the inside of the parentheses first.
- The first parentheses is (-9 * 7 * 7 * -9). According to the order of operations, we should do multiplication first. So, -9 * 7 = -63, -63 * 7 = -441, -441 * -9 = 3969.
- The second parentheses is (4 * -9 - 8 - -4). According to the order of operations, we should do multiplication first and then subtraction. So, 4 * -9 = -36, -36 - 8 = -44, -44 - -4 = -40.
4. After solving the operations inside the parentheses, proceed solving the operations outside the parentheses. So, 3969 + -40 = 3929.
5. Continue to calculate step by step until you reach the final answer.
Answer: The answer is 3929.

Figure 29: Strategy generated by GPT-4 for Multi-step Arithmetic and its execution on a task example.

CoT Prompt for MATH-Algebra

Question: Find w, such that $5^6 5^w = 25$.
Solution: The laws of exponents give us $5^6 5^w = 5^{6+w}$. And, because $25 = 5^2$, we have $5^{6+w} = 5^2$. It follows that $6 + w = 2$. Subtracting 6 from both sides gives us $w = -4$.
Answer: The answer is -4

Question: Compute $\frac{x^8+12x^4+36}{x^4+6}$ when $x = 5$.
Solution: Note that $\left(x^4 + 6\right)^2 = x^8 + 12x^4 + 36$. So $\frac{x^8+12x^4+36}{x^4+6} = \frac{\left(x^4+6\right)^2}{x^4+6} = x^4 + 6$. Our answer is therefore $5^4 + 6 = 625 + 6 = 631$.
Answer: The answer is 631

Question: If $x + y = 9$ and $xy = 10$, what is the value of $x^3 + y^3$?
Solution: If we cube both sides of the first equation, we find that $x^3 + 3x^2y + 3xy^2 + y^3 = 729$, so $x^3 + y^3 = 729 - (3x^2y + 3xy^2)$. Since $3x^2y + 3xy^2 = 3(xy)(x + y) = 3(10)(9)$, we see that $x^3 + y^3 = 729 - (3x^2y + 3xy^2) = 729 - 270 = 459$.
Answer: The answer is 459

Question: What is the intersection point of the line $y = 2x + 5$ and the line perpendicular to it that passes through the point $(5, 5)$?
Solution: The slope of $y = 2x + 5$ is 2, which means the slope of any line perpendicular to it is $-\frac{1}{2}$. Using the point-slope equation for a line we can find the equation of the second line to be $y - 5 = -\frac{1}{2}(x - 5)$. To find the intersection of this with the first line, we plug $y = 2x + 5$ into the second equation to get $2x + 5 - 5 = -\frac{1}{2}(x - 5) \Rightarrow \frac{5}{2}x = \frac{5}{2} \Rightarrow x = 1$. Therefore $y = 2 \cdot 1 + 5 = 7$ making the intersection at $(1, 7)$.
Answer: The answer is (1,7)

Figure 30: Chain-of-thought (CoT) prompt for MATH-Algebra.

CoT Prompt for MATH-Prealgebra

Question: Pete thinks of a number. He doubles it, adds 10, multiplies by 4, and ends up with 120. What was his original number?
Solution: Let Pete's original number be x. If he double it and adds 10, he will have $2x + 10$. After multiplying by 4, Pete ended up with 120. From this information, we have the equation: $4(2x + 10) = 120$. Expanding the left side and solving, we find:

$$\begin{aligned} 8x + 40 &= 120 \\ \Rightarrow \quad 8x &= 80 \\ \Rightarrow \quad x &= 10. \end{aligned}$$

Answer: The answer is 10

Question: A standard deck of 52 cards has 13 ranks (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King) and 4 suits (♠, ♡, ♢, and ♣), such that there is exactly one card for any given rank and suit. Two of the suits (♠ and ♣) are black and the other two suits (♡ and ♢) are red. The deck is randomly arranged. What is the probability that the top card is a ♡?
Solution: There are 13 ♡'s and 52 cards total, so the probability that the top card is a ♡ is $\dfrac{13}{52} = \dfrac{1}{4}$.
Answer: The answer is $\frac{1}{4}$

Question: A number in the set $\{50, 51, 52, 53, ..., 999\}$ is randomly selected. What is the probability that it is a two-digit number? Express your answer as a common fraction.
Solution: To count the number of numbers in this set, we subtract 49 from all of the numbers, giving the set $\{1, 2, 3, \ldots, 950\}$, making it obvious that there are 950 numbers total. Furthermore, the set $\{50, 51, 52, \ldots, 98, 99\}$ corresponds to the more easily counted $\{1, 2, 3, \ldots, 49, 50\}$ by subtracting 49. So, the probability of selecting a two-digit number is $\frac{50}{950} = \frac{1}{19}$.
Answer: The answer is $\frac{1}{19}$

Question: A pentagon is drawn by placing an isosceles right triangle on top of a square as pictured. What percent of the area of the pentagon is the area of the right triangle?
[asy] size(50); draw((0,0)–(0,-1)–(1,-1)–(1,0)–(0,0)–(.5,.5)–(1,0)); [/asy]
Solution: Let the leg length of the isosceles right triangle be x, so the hypotenuse of the triangle has length $x\sqrt{2}$. The hypotenuse of the triangle is a side of the square, so the area of the square is $(x\sqrt{2})^2 = 2x^2$. The area of the triangle is $(x)(x)/2 = x^2/2$. So, the area of the pentagon is

$$\frac{x^2}{2} + 2x^2 = \frac{5x^2}{2}.$$

Therefore, the fraction of the pentagon's area that is inside the triangle is

$$\frac{x^2/2}{5x^2/2} = \frac{x^2}{2} \cdot \frac{2}{5x^2} = \frac{1}{5} = 20\%.$$

Answer: The answer is 20

Figure 31: Chain-of-thought (CoT) prompt for MATH-Prealgebra.

CoT Prompt for MATH-Intermediate Algebra

Question: Find the quadratic polynomial $p(x)$ such that $p(-3) = 10$, $p(0) = 1$, and $p(2) = 5$.
Solution: Let $p(x) = ax^2 + bx + c$. Then from the given information,

$$\begin{aligned} 9a - 3b + c &= 10, \\ c &= 1, \\ 4a + 2b + c &= 5. \end{aligned}$$

Then $9a - 3b = 9$ and $4a + 2b = 4$, which reduce to $3a - b = 3$ and $2a + b = 2$. Adding, we get $5a = 5$, so $a = 1$. Then $4 + 2b = 4$, so $b = 0$. Therefore, $p(x) = x^2 + 1$.
Answer: The answer is $x^2 + 1$

Question: Calculate: $\frac{5}{3} \times \frac{6}{10} \times \frac{15}{9} \times \frac{12}{20} \times \frac{25}{15} \times \frac{18}{30} \times \frac{35}{21} \times \frac{24}{40}$
Solution: Each of the fractions $\frac{5}{3}$, $\frac{15}{9}$, $\frac{25}{15}$, $\frac{35}{21}$ reduce to $\frac{5}{3}$, and each of the fractions $\frac{6}{10}$, $\frac{12}{20}$, $\frac{18}{30}$, $\frac{24}{40}$ reduce to $\frac{3}{5}$. Therefore, the product of all eight fractions is 1.
Answer: The answer is 1

Question: For $-25 \le x \le 25$, find the maximum value of $\sqrt{25 + x} + \sqrt{25 - x}$.
Solution: By QM-AM,

$$\frac{\sqrt{25 + x} + \sqrt{25 - x}}{2} \le \sqrt{\frac{25 + x + 25 - x}{2}} = 5,$$

so $\sqrt{25 + x} + \sqrt{25 - x} \le 10$. Equality occurs at $x = 0$, so the maximum value is 10.
Answer: The answer is 10

Question: The hyperbolas

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

and

$$\frac{y^2}{18} - \frac{x^2}{N} = 1$$

have the same asymptotes. Find N.
Solution: In general, for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the asymptotes are $\frac{x}{a} = \pm\frac{y}{b}$, or $y = \pm\frac{b}{a}x$. Therefore, the asymptotes of the first hyperbola are $y = \pm\frac{3}{2}x$.
For the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, the asymptotes are $\frac{y}{a} = \pm\frac{x}{b}$, or $y = \pm\frac{a}{b}x$. Therefore, the asymptotes of the second hyperbola are $y = \pm\frac{3\sqrt{2}}{\sqrt{N}}x$.
For the two hyperbolas to have the same asymptotes, we must have $\frac{3}{2} = \frac{3\sqrt{2}}{\sqrt{N}}$. Solving for N gives $N = 8$.
[asy] void axes(real x0, real x1, real y0, real y1) { draw((x0,0)–(x1,0),EndArrow); draw((0,y0)–(0,y1),EndArrow); label("x",(x1,0),E); label("y",(0,y1),N); for (int i=floor(x0)+1; i<x1; ++i) draw((i,.1)–(i,-.1)); for (int i=floor(y0)+1; i<y1; ++i) draw((.1,i)–(-.1,i)); } path[] yh(real a, real b, real h, real k, real x0, real x1, bool upper=true, bool lower=true, pen color=black) { real f(real x) return $k + a/b * sqrt(b^2 + (x - h)^2)$; real g(real x) return $k - a/b * sqrt(b^2 + (x - h)^2)$; if (upper) draw(graph(f, x0, x1),color, Arrows); if (lower) draw(graph(g, x0, x1),color, Arrows); path [] arr = graph(f, x0, x1), graph(g, x0, x1); return arr; } void xh(real a, real b, real h, real k, real y0, real y1, bool right=true, bool left=true, pen color=black) { path [] arr = yh(a, b, k, h, y0, y1, false, false); if (right) draw(reflect((0,0),(1,1))*arr[0],color, Arrows); if (left) draw(reflect((0,0),(1,1))*arr[1],color, Arrows); } void e(real a, real b, real h, real k) { draw(shift((h,k))*scale(a,b)*unitcircle); } size(8cm); axes(-8,8, -10, 10); xh(2, 3, 0, 0, -8, 8); yh(3*sqrt(2),sqrt(8),0,0,-5,5); draw((-6,9)–(6,-9)h6,9)–(-6,-9),dotted); [/asy]
Answer: The answer is 8

Figure 32: Chain-of-thought (CoT) prompt for MATH-Intermediate Algebra.

CoT Prompt for MATH-Counting and Probability

Question: The digits 2, 3, 5 and 7 are arranged randomly to form a four-digit number. What is the probability that the number is odd? Express your answer as a common fraction.
Solution: The number formed is odd if and only if its units digit is not 2. Since the digits 2, 3, 5, and 7 are arranged randomly, the probability that 2 is the units digit is 1/4. Therefore, the probability that the number is odd is $1 - 1/4 = \frac{3}{4}$.
Answer: The answer is $\frac{3}{4}$

Question: Eight congruent equilateral triangles, each of a different color, are used to construct a regular octahedron. How many distinguishable ways are there to construct the octahedron? (Two colored octahedrons are distinguishable if neither can be rotated to look just like the other.)
[asy] import three; import math; unitsize(1.5cm); currentprojection=orthographic(2,0.2,1); triple A=(0,0,1); triple B=(sqrt(2)/2,sqrt(2)/2,0); triple C=(sqrt(2)/2,-sqrt(2)/2,0); triple D=(-sqrt(2)/2,-sqrt(2)/2,0); triple E=(-sqrt(2)/2,sqrt(2)/2,0); triple F=(0,0,-1); draw(A–B–E–cycle); draw(A–C–D–cycle); draw(F–C–B–cycle); draw(F–D–E–cycle,dotted+linewidth(0.7)); [/asy]
(A) 210 **(B)** 560 **(C)** 840 **(D)** 1260 **(E)** 1680
Solution: Since the octahedron is indistinguishable by rotations, without loss of generality fix a face to be red.
[asy] size(8cm); defaultpen(0.5); import three; import math; currentprojection=orthographic(2,0.2,1); triple A=(0,0,1); triple B=(sqrt(2)/2,sqrt(2)/2,0); triple C=(sqrt(2)/2,-sqrt(2)/2,0); triple D=(-sqrt(2)/2,-sqrt(2)/2,0); triple E=(-sqrt(2)/2,sqrt(2)/2,0); triple F=(0,0,-1); draw(A–B–E–cycle); draw(A–C–D–cycle); draw(F–C–B–cycle); draw(F–D–E–cycle,dotted+linewidth(0.7)); draw(surface(A–B–C–cycle),rgb(1,.6,.6),nolight);[/asy] There are 7! ways to arrange the remaining seven colors, but there still are three possible rotations about the fixed face, so the answer is $7!/3 = 1680$.
[asy] size(8cm); defaultpen(0.5); import three; import math; currentprojection=orthographic(2,0,1); triple A=(0,0,1); triple B=(sqrt(2)/2,sqrt(2)/2,0); triple C=(sqrt(2)/2,-sqrt(2)/2,0); triple D=(-sqrt(2)/2,-sqrt(2)/2,0); triple E=(-sqrt(2)/2,sqrt(2)/2,0); triple F=(0,0,-1); triple right=(0,1,0); picture p = new picture, r = new picture, s = new picture; draw(p,A–B–E–cycle); draw(p,A–C–D–cycle); draw(p,F–C–B–cycle); draw(p,F–D–E–cycle,dotted+linewidth(0.7)); draw(p,surface(A–B–C–cycle),rgb(1,.6,.6),nolight); draw(p,surface(A–B–E–cycle),rgb(1,1,.6),nolight); add(scale3(2.2)*p); draw(r,A–B–E–cycle); draw(r,A–C–D–cycle); draw(r,F–C–B–cycle); draw(r,F–D–E–cycle,dotted+linewidth(0.7)); draw(r,surface(A–B–C–cycle),rgb(1,.6,.6),nolight); draw(r,surface(A–C–D–cycle),rgb(1,1,.6),nolight); add(scale3(2.2)*shift(2*right)*r); draw(s,A–B–E–cycle); draw(s,A–C–D–cycle); draw(s,F–C–B–cycle); draw(s,F–D–E–cycle,dotted+linewidth(0.7)); draw(s,surface(A–B–C–cycle),rgb(1,.6,.6),nolight); draw(s,surface(B–C–F–cycle),rgb(1,1,.6),nolight); add(scale3(2.2)*shift(4*right)*s); [/asy]
Answer: The answer is 1680

Question: The Gnollish language consists of 3 words, "splargh," "glumph," and "amr." In a sentence, "splargh" cannot come directly before "glumph"; all other sentences are grammatically correct (including sentences with repeated words). How many valid 3-word sentences are there in Gnollish?
Solution: We proceed by counting the complement, or the number of invalid 3-word sentences. A sentence is invalid precisely when it is of the form "(word) splargh glumph" or "splargh glumph (word)." There are 3 choices for the missing word in each sentence, and since each case is exclusive, we have a total of 6 invalid sentences. Since there are $3 \cdot 3 \cdot 3 = 27$ possible 3-word sentences with no restrictions, there are $27 - 6 = 21$ that satisfy the restrictions of the problem.
Answer: The answer is 21

Question: Given that $\binom{17}{7} = 19448$, $\binom{17}{8} = 24310$ and $\binom{17}{9} = 24310$, calculate $\binom{19}{9}$.
Solution: We can apply Pascal's identity to get that $\binom{19}{9} = \binom{18}{8} + \binom{18}{9}$. From here, we can apply it twice more to get that $\binom{19}{9} = \binom{18}{8} + \binom{18}{9} = \left(\binom{17}{7} + \binom{17}{8}\right) + \left(\binom{17}{8} + \binom{17}{9}\right)$. Substituting the provided values of $\binom{17}{7}$, $\binom{17}{8}$, and $\binom{17}{9}$ gives us $\binom{19}{9} = 19448 + 2(24310) + 24310 = 92378$.
Answer: The answer is 92378

Figure 33: Chain-of-thought (CoT) prompt for MATH-Counting and Probability.

CoT Prompt for MATH-Number Theory

Question: Let x be a positive integer such that $9x \equiv 1 \pmod{25}$. What is the remainder when $11 + x$ is divided by 25?
Solution: The given information can be expressed by writing $x \equiv 9^{-1} \pmod{25}$. Thus we wish to compute $11 + 9^{-1} \pmod{25}$.
Modulo 25, we can write 11 as $11 \cdot (9 \cdot 9^{-1}) \equiv (11 \cdot 9) \cdot 9^{-1} \equiv 99 \cdot 9^{-1}$. Thus

$$\begin{aligned} 11 + 9^{-1} &\equiv 99 \cdot 9^{-1} + 1 \cdot 9^{-1} \\ &\equiv 100 \cdot 9^{-1} \\ &\equiv 0 \cdot 9^{-1} \\ &\equiv 0 \pmod{25}, \end{aligned}$$

so the remainder when $11 + x$ is divided by 25 is 0.
Notice that the trick we used here is analogous to using a common denominator to add fractions.
Answer: The answer is 0

Question: Let m be the product of all positive integers less than $4!$ which are invertible modulo $4!$. Find the remainder when m is divided by $4!$.(Here $n!$ denotes $1 \times \cdots \times n$ for each positive integer n.)
Solution: We compute that $4! = 1 \times 2 \times 3 \times 4 = 2^3 \times 3 = 24$. So we want exactly the numbers in the set $\{1, \ldots, 24\}$ which are divisible by neither 2 nor 3, since an integer a is invertible modulo n for some positive integer n if and only if $\gcd(a, n) = 1$. These turn out to be $\{1, 5, 7, 11, 13, 17, 19, 23\}$. Then

$$\begin{aligned} m &\equiv 1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \\ &\equiv 1 \cdot 5 \cdot 7 \cdot 11 \cdot (-11) \cdot (-7) \cdot (-5) \cdot (-1) \\ &\equiv (5 \cdot 7 \cdot 11)^2 \\ &\equiv (35 \cdot 11)^2 \\ &\equiv (11 \cdot 11)^2 \\ &\equiv (121)^2 \\ &\equiv 1^2 \\ &\equiv 1 \pmod{24} \end{aligned}$$

Answer: The answer is 1

Question: Find $2^{-1} \pmod{185}$, as a residue modulo 185. (Give an answer between 0 and 184, inclusive.)
Solution: Since $2 \cdot 93 \equiv 186 \equiv 1 \pmod{185}$, $2^{-1} \equiv 93 \pmod{185}$.
Answer: The answer is 93

Question: What is the remainder when 2007 is divided by 81?
Solution: Dividing using long division, we find that $2007 = 81 \cdot 24 + 63$, so the remainder is 63.
Answer: The answer is 63

Figure 34: Chain-of-thought (CoT) prompt for MATH-Number Theory.

CoT Prompt for MATH-Geometry (Part 1)

Question: Two sectors of a circle of radius 12 overlap as shown, with P and R as the centers of the respective circles. Determine the area of the shaded region.

[asy] draw((0,0)–(10.3923,-6)–(20.7846,0)–(10.3923,6)–cycle,black+linewidth(1)); filldraw((10.3923,6)..(12,0)..(10.3923,-6)–cycle,gray,black+linewidth(1)); filldraw((10.3923,6)..(8.7846,0)..(10.3923,-6)–cycle,gray,black+linewidth(1)); label("P",(0,0),W); label("Q",(10.3923,6),N); label("R",(20.7846,0),E); label("S",(10.3923,-6),S); label("60°",(0,0),2E); label("60°",(20.7846,0),2W); [/asy]
Solution: By symmetry, the areas of the two parts of the shaded region are equal. Consider the right part of the shaded region and the left triangle.
[asy] draw((0,0)–(10.3923,-6)–(10.3923,6)–cycle,black+linewidth(1)); filldraw((10.3923,6)..(12,0)..(10.3923,-6)–cycle,gray,black+linewidth(1)); draw((0,0)–(10.3923,0),black+linewidth(1)); draw((10.3923,0)–(9.3923,0)–(9.3923,1)–(10.3923,1),black+linewidth(1)); label("P",(0,0),W); label("Q",(10.3923,6),N); label("S",(10.3923,-6),S); label("Z",(10.3923,0),SW); [/asy]
The shaded area is equal to the area of sector PQS minus the area of triangle PQS.
Since $\angle PQS = 60^\circ$ and $PQ = 12$, the area of sector PQS is

$$\frac{1}{6} \cdot 12^2 \cdot \pi = 24\pi.$$

Also, triangle PQS is equilateral with side length 12, so its area is

$$\frac{\sqrt{3}}{4} \cdot 12^2 = 36\sqrt{3}.$$

Thus, the area of the right part of the shaded region is $24\pi - 36\sqrt{3}$, so the area of the entire shaded region is

$$2(24\pi - 36\sqrt{3}) = 48\pi - 72\sqrt{3}.$$

Answer: The answer is $48\pi - 72\sqrt{3}$

Question: The square with vertices $(-a, -a), (a, -a), (-a, a), (a, a)$ is cut by the line $y = x/2$ into congruent quadrilaterals. The perimeter of one of these congruent quadrilaterals divided by a equals what? Express your answer in simplified radical form.
Solution: The line $y = \frac{x}{2}$ will intersect the two vertical sides of the square, as shown below:
[asy] real f(real x) { return x/2; } import graph; size(6cm); real a = 8; pair A=(-a,a), B=(a,a), C=(a,-a), D=(-a,-a); draw(A–B–C–D–cycle); draw(graph(f,-11,11),Arrows); axes(Arrows(4)); dot("$(-a,a)$",A,N); dot("(a,a)",B,N); dot("$(a,-a)$",C,S); dot("$(-a,-a)$",D,S); real eps=0.2; dot((8,4)h-8,-4)); draw(shift((10,0))*"$2a$",(-a+eps,-a/2-.5)–(a-eps,-a/2-.5),Arrows); draw(shift((0,10))*"a",(a+2*eps,-a/2)–(a+2*eps,a/2),Arrows);[/asy] The equation of the right side of the square is $x = a$, so we have $y = \frac{x}{2} = \frac{a}{2}$, which means that the intersection point with the right side of the square is $\left(a, \frac{a}{2}\right)$. Similarly, the equation of the left side of the square is $x = -a$, so we have $y = \frac{x}{2} = -\frac{a}{2}$, which means that the intersection point with the left side of the square is $\left(-a, -\frac{a}{2}\right)$. It follows that the sides of each quadrilateral have lengths $\frac{a}{2}$, $2a$, $\frac{3a}{2}$, and $\sqrt{a^2 + (2a)^2} = a\sqrt{5}$, by the Pythagorean theorem. Hence, the perimeter of the quadrilateral is

$$\frac{a}{2} + 2a + \frac{3a}{2} + a\sqrt{5} = \left(4 + \sqrt{5}\right) a,$$

and when this is divided by a, we get $4 + \sqrt{5}$.
Answer: The answer is $4 + \sqrt{5}$

Figure 35: Chain-of-thought (CoT) prompt for MATH-Geometry (Part 1).

CoT Prompt for MATH-Geometry (Part 2)

Question: A right circular cylinder with radius 2 is inscribed in a hemisphere with radius 5 so that its bases are parallel to the base of the hemisphere. What is the height of this cylinder?
Solution: We draw and label a diagram as follows:
[asy] size(110); pair O = (0,0); pair A = (.3,.94); pair B = (.3,.075); draw(O–A–B–cycle,heavycyan); label("O",O,W); label("A",A,N); label("B",B,S); import solids; import three; defaultpen(linewidth(0.8)); currentprojection = orthographic(5,0,1.3); revolution c = cylinder((0,0,0), .4, .91); draw(c,black); draw(scale(1,.25)*arc((0,0),1,0,180),dashed); draw(scale(1,.25)*arc((0,0),1,180,360)); draw(Arc((0,0),1,0,180)); [/asy]
Let the center of the hemisphere be O, and let A be a point on the circumference of the top circle of the cylinder. Since the cylinder is inscribed in the hemisphere, A lies on the hemisphere as well, so $OA = 5$. We drop a perpendicular from A to the base of the hemisphere and let it intersect the base of the hemisphere at B. Since the cylinder is right and AB is a height of the cylinder, $\angle OBA$ is a right angle, and B lies on the circumference of the bottom circle of the cylinder. Thus, OB is a radius of the cylinder, so $OB = 2$. We have that $\triangle OBA$ is right, so by the Pythagorean theorem, we have

$$AB = \sqrt{OA^2 - OB^2} = \sqrt{5^2 - 2^2} = \sqrt{21}.$$

Thus, the height of the cylinder is $\sqrt{21}$.
Answer: The answer is $\sqrt{21}$

Question: Parallelogram $ABCD$ with $A(2,5)$, $B(4,9)$, $C(6,5)$, and $D(4,1)$ is reflected across the x-axis to $A'B'C'D'$ and then $A'B'C'D'$ is reflected across the line $y = x + 1$ to $A''B''C''D''$. This is done such that D' is the image of D, and D'' is the image of D'. What is the ordered pair of D'' in the coordinate plane?
Solution: Reflecting a point across the x-axis multiplies its y-coordinate by -1. Therefore, $D' = (4, -1)$. To reflect D' across the line $y = x + 1$, we first translate both the line and the point down one unit so that the equation of the translated line is $y = x$ and the coordinates of the translated point are $(4, -2)$. To reflect across $y = x$, we switch the x-coordinate and y-coordinate to obtain $(-2, 4)$. Translating this point one unit up, we find that $D'' = (-2, 5)$.
Answer: The answer is (-2,5)

Figure 36: Chain-of-thought (CoT) prompt for MATH-Geometry (Part 2).

CoT Prompt for MATH-Precalculus (Part 1)

Question: The function

$$f(z) = \frac{(-1 + i\sqrt{3})z + (-2\sqrt{3} - 18i)}{2}$$

represents a rotation around some complex number c. Find c.
Solution: Since a rotation around c fixes c, the complex number c must satisfy $f(c) = c$. In other words,

$$c = \frac{(-1 + i\sqrt{3})c + (-2\sqrt{3} - 18i)}{2}$$

Then $2c = (-1 + i\sqrt{3})c + (-2\sqrt{3} - 18i)$, so

$$(3 - i\sqrt{3})c = -2\sqrt{3} - 18i.$$

Then

$$\begin{aligned} c &= \frac{-2\sqrt{3} - 18i}{3 - i\sqrt{3}} \\ &= \frac{(-2\sqrt{3} - 18i)(3 + i\sqrt{3})}{(3 - i\sqrt{3})(3 + i\sqrt{3})} \\ &= \frac{-6\sqrt{3} - 6i - 54i + 18\sqrt{3}}{12} \\ &= \frac{12\sqrt{3} - 60i}{12} \\ &= \sqrt{3} - 5i. \end{aligned}$$

Answer: The answer is $\sqrt{3} - 5i$

Question: Convert the point $\left(8, \frac{7\pi}{6}\right)$ in polar coordinates to rectangular coordinates.
Solution: In rectangular coordinates, $\left(8, \frac{7\pi}{6}\right)$ becomes

$$\left(8\cos\frac{7\pi}{6}, 8\sin\frac{7\pi}{6}\right) = (-4\sqrt{3}, -4).$$

Answer: The answer is $(-4\sqrt{3}, -4)$

Figure 37: Chain-of-thought (CoT) prompt for MATH-Precalculus (Part 1).

CoT Prompt for MATH-Precalculus (Part 2)

Question: Compute $\begin{pmatrix} 2 & 0 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 8 & -2 \\ 1 & 1 \end{pmatrix}$.
Solution: We have that

$$\begin{pmatrix} 2 & 0 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 8 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (2)(8) + (0)(1) & (2)(-2) + (0)(1) \\ (5)(8) + (-3)(1) & (5)(-2) + (-3)(1) \end{pmatrix} = \begin{pmatrix} 16 & -4 \\ 37 & -13 \end{pmatrix}.$$

Answer: The answer is $\begin{pmatrix} 16 & -4 \\ 37 & -13 \end{pmatrix}$

Question: A line is parameterized by a parameter t, so that the vector on the line at $t = 2$ is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and the vector on the line at $t = 3$ is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Find the vector on the line at $t = -7$.
Solution: Let the line be

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{a} + t\mathbf{d}.$$

Then from the given information,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \mathbf{a} + 2\mathbf{d},$$
$$\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \mathbf{a} + 3\mathbf{d}.$$

We can treat this system as a linear set of equations in $\mathbf{a}$ and $\mathbf{d}$. Accordingly, we can solve to get $\mathbf{a} = \begin{pmatrix} -3 \\ 20 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 20 \end{pmatrix} + t \begin{pmatrix} 2 \\ -8 \end{pmatrix}.$$

Taking $t = -7$, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 20 \end{pmatrix} - 7 \begin{pmatrix} 2 \\ -8 \end{pmatrix} = \begin{pmatrix} -17 \\ 76 \end{pmatrix}.$$

Answer: The answer is $\begin{pmatrix} -17 \\ 76 \end{pmatrix}$

Figure 38: Chain-of-thought (CoT) prompt for MATH-Precalculus (Part 2).