IMPROVED CONVERGENCE OF DIFFERENTIAL PRIVATE SGD WITH GRADIENT CLIPPING

Anonymous authors
Paper under double-blind review

ABSTRACT

Differential private stochastic gradient descent (DP-SGD) with gradient clipping (DP-SGD-GC) is an effective optimization algorithm that can train machine learning models with a privacy guarantee. Despite the popularity of DP-SGD-GC, its convergence in unbounded domain without the Lipschitz continuous assumption is less-understood; existing analysis of DP-SGD-GC either impose additional assumptions or end up with an utility bound that involves a non-vanishing bias term. In this work, for smooth and unconstrained problems, we improve the current analysis and show that DP-SGD-GC can achieve a vanishing utility bound without any bias term. Furthermore, when the noise generated from subsampled gradients is light-tailed, we prove that DP-SGD-GC can achieve nearly the same utility bound as DP-SGD applies to the Lipschitz continuous objectives. As a by-product, we propose a new clipping technique, called value clipping, to mitigate the computational overhead caused by the classic gradient clipping. Experiments on standard benchmark datasets are conducted to support our analysis.

1 INTRODUCTION

Training machine learning models that can achieve decent prediction accuracy while preserving data privacy is fundamental in many modern machine learning applications. The concept of differential privacy (DP) from Dwork et al. (Dwork et al., 2006) offers an elegant mathematical framework to characterize the privacy-preserving ability of randomized algorithms. A huge line of works has been devoted to designing differential-private training algorithms for the empirical risk minimization (ERM) problem. Among many proposed methods, differential private stochastic gradient descent (DP-SGD) is an effective optimization algorithm that can solve ERM with a privacy guarantee and achieve a reasonable utility bound. DP-SGD has received substantial interest in recent years due to its simplicity and effectiveness (Song et al., 2013; Bassily et al., 2014; Abadi et al., 2016; Wang et al., 2017; Bassily et al., 2019; Feldman et al., 2020; Asi et al., 2021).

In the classic analysis of DP-SGD, the variance of the Gaussian noise used in each iteration of DP-SGD relies crucially on the ℓ_2-sensitivity of the loss function. Therefore most early works on DP-SGD assume each individual loss function to be Lipschitz continuous in its domain (Song et al., 2013; Bassily et al., 2014). However, many real-world problems are only smooth but not globally Lipschitz continuous; for example, the unconstrained linear regression problem. There are two techniques to circumvent the Lipschitz continuous assumption: (i) imposing an additional bounded domain constraint to the original problem; (ii) clipping each gradients in their 2-norm and using them to update the model (Abadi et al., 2016). In practice, the gradient clipping technique is usually more preferred than imposing a bounded domain constraint because the latter requires prior knowledge of the distance between initialization and solution, which is typically unavailable for unconstrained problems. Indeed, the state-of-the-art implementations of DP-SGD including Opacus (Yousefpour et al., 2021) and TensorFlow Privacy[1] all advocate the gradient clipping technique.

Despite the popularity of DP-SGD with gradient clipping (DP-SGD-GC), the convergence of DP-SGD-GC for unconstrained problems that are not globally Lipschitz continuous has not been well-studied. In fact, recent works (Chen et al., 2020; Song et al., 2021) have reported that DP-SGD-GC can suffer from a constant utility in the worst case. With these negative results on the convergence of DP-SGD-GC, one may consider DP-SGD-GC as an algorithm with a fundamental non-convergence issue. In this work, we show that this is not the case. With a careful choice of the clipping threshold,

[1] https://github.com/tensorflow/privacy. Last accessed Aug 15, 2022.

Algorithms	Domain	Objective function	Light-tail-noise	Utility
DP-SGD (Bassily et al., 2014)	Bounded	Lipschitz cont.	No	$\widetilde{\mathcal{O}}\big(\sqrt{d}/(n\epsilon)\big)$
DP-SGD†	Bounded	Smooth	No	$\widetilde{\mathcal{O}}\big(\sqrt{d}/(n\epsilon)\big)$
DP-SGD (Wang et al., 2022)	Bounded	Hölder smooth	No	$\widetilde{\mathcal{O}}\big(\sqrt{d}/(n\epsilon)\big)$
DP-SGD-GC (Ours)	Unbounded	Smooth	No	$\widetilde{\mathcal{O}}\big(\sqrt{d}/(\sqrt{n}\epsilon)\big)$
DP-SGD-GC (Ours)	Unbounded	Smooth	Yes	$\widetilde{\mathcal{O}}\big(\sqrt{d}/(n\epsilon)\big)$

Table 1: The utility bound and assumptions needed by different algorithms for convex problems, where d is the problem size, n is the number of data points and ϵ measures the privacy-preserving ability; see Section 3 for more details. †is based on a trivial extension of Bassily et al's work (Bassily et al., 2014).

we prove that DP-SGD-GC can achieve the same utility bound as its non-clipped counterpart DP-SGD. Formally, we summarize our contributions as follows.

- For unconstrained problems that are convex and smooth but not necessarily globally Lipschitz continuous, we show that DP-SGD-GC can achieve a $\widetilde{\mathcal{O}}(\sqrt{d}/(\sqrt{n}\epsilon))$ utility bound. When the noise generated from the subsampled gradients is light-tailed (Assumption 4.1)[2], the utility bound can be improved to $\widetilde{\mathcal{O}}(\sqrt{d}/(n\epsilon))$, which is the same as the utility bound of DP-SGD applies to the Lipschitz continuous problems. See Table 1 for a comparison to existing results. Our convergence analysis of DP-SGD-GC for convex, smooth and unconstrained problems, to our knowledge, provide the first utility bound without a non-vanishing bias term.
- We show that our analysis also applies to unconstrained smooth problems that can potentially be nonconvex. Consequently, DP-SGD-GC can achieve a $\widetilde{\mathcal{O}}(\sqrt{d}/(\sqrt{n}\epsilon))$ gradient norm bound for smooth problems. Again, the bound can be improved to $\widetilde{\mathcal{O}}(\sqrt{d}/(n\epsilon))$ with an additional light-tail-noise assumption.
- This work is theoretical in essence but also includes a practical contribution (Section 5). We develop a novel *value clipping* technique for problems that satisfy the growth condition (Definition 3.1). The proposed value clipping technique can be implemented within one forward-backward propagation on existing deep learning platforms and can alleviate the computation overhead caused by gradient clipping. The efficiency of value clipping is demonstrated on real-world datasets.

2 Related work

DP-SGD with gradient clipping was initially proposed by Abadi et al. (Abadi et al., 2016). Gradient clipping and its variants have been widely adopted by many privacy-aware training algorithms (Yousefpour et al., 2021; Andrew et al., 2021). Despite the popularity of gradient clipping, the convergence rate of DP-SGD-GC without the Lipschitz continuous and bounded domain assumptions remains a challenging task; see (Wang et al., 2022, Remark 5) for a short discussion on the hardness of removing the bounded domain assumption. This challenging research question was not carefully studied until the recent works from Chen et al. (Chen et al., 2020) and Song et al. (Song et al., 2021), who provided counter-examples showing that DP-SGD-GC can suffer from a constant utility in the worst case. Chen et al. (Chen et al., 2020) studied the convergence of DP-SGD-GC to a stationary point in the nonconvex setting and showed that an additional assumption on gradient distribution is sufficient to derive a meaningful utility bound. Song et al. (Song et al., 2021) showed that DP-SGD-GC converges to a perturbed objective function for the generalized linear model and can suffer from a constant utility for the original objective in the worst case. Note there are some other recent works that study the convergence of DP-SGD-GC for smooth objective Du et al. (2021); Wu et al. (2021), the rates in these works usually involve a bias term due to clipping.

Another line of work is using gradient clipping to avoid gradient explosion when training neural networks (Zhang et al., 2020b;a). These works clip the averaged gradient instead of clipping each individual gradient in the sampled batch, which is different from the gradient clipping used by

[2]The light-tail assumption is standard for deriving high probability error bound of SGD in the literature.

DP-SGD when the batch size is greater than one. The convergence analysis of clipping averaged gradient can not be applied directly to DP-SGD and does not overlap with this work.

On the practical side, the original implementation of gradient clipping technique was inefficient as one needs to calculate the norm of each individual sample in every iteration. Many works (Goodfellow, 2015; Abadi et al., 2016; Rochette et al., 2019; Bu et al., 2021; Subramani et al., 2021) have been carried on to improve the efficiency of DP-SGD-GC from either engineering or algorithmic perspective. Our proposed value clipping technique can be viewed as an alternative to the classic gradient clipping technique.

Notation Throughout this paper, for any positive integer n, we denote $[n] \coloneqq \{1, 2, \ldots, n\}$. We denote $\|\cdot\|$ to be the vector 2-norm or matrix operator norm if not otherwise specified. We use the notation $\widetilde{\mathcal{O}}(\cdot)$ to hide poly-logarithmic terms.

3 PRELIMINARIES

We consider the empirical-risk minimization (ERM) problem

$$\underset{w\in\mathbb{R}^d}{\text{minimize}} \quad f(w) \coloneqq \frac{1}{n}\sum_{i=1}^{n} f_i(w), \tag{P}$$

where n is the number of training samples, f_i's are differentiable functions and w is the model we wish to train. Throughout the paper, we assume that f is bounded below and its minimum is attainable. We let $\mathcal{W}^*$ to be the set of solutions for problem (P), and denote the optimal function value as f^*. We assume that a lower bound of f^* is known as a prior. Note that this assumption holds in many realistic settings, for example the ERM problems are usually lower bounded by zero.

Next we introduce the *growth condition* (GC), which serves as the cornerstone for our analysis.

Definition 3.1. A function $h : \mathbb{R}^d \to \mathbb{R}$ is (β_1, β_2)-GC for some $\beta_1 > 0, \beta_2 \geq 0$ if

$$\|\nabla h(w)\|^2 \;\leq\; \beta_1\Big(h(w) - \inf_{u\in\mathbb{R}^d} h(u)\Big) + \beta_2 \qquad \forall w \in \mathbb{R}^d.$$

The growth condition bounds the norm of the gradient by a linear function of the objective value. GC is gaining increasing interest in recent years as multiple works have demonstrated that GC and its variants can improve the classic analysis of SGD-type algorithms (Schmidt & Le Roux, 2013; Needell et al., 2016; Vaswani et al., 2019; Gower et al., 2019; Stich, 2019; Khaled & Richtárik, 2020; Fang et al., 2021; Gower et al., 2021). It is easy to show that smooth functions that are bounded below necessarily satisfy GC; the following Lemma makes this precise.

Lemma 3.2. *If a function $h : \mathbb{R}^d \to \mathbb{R}$ is L-smooth for some $L > 0$ and bounded below e.g., $\inf_{w\in\mathbb{R}^d} h(w) > -\infty$. Then h is $(2L, 0)$-GC.*

We stress that GC is not a strong assumption. In fact, a wide range of nonconvex and nonsmooth problems arising from the empirical-risk minimization problem also satisfy GC, for example the Lipschitz continuous model with smooth and convex loss function; see Fang et al. (Fang et al., 2021, § 4.1) and Section E for more details.

We recall the standard definition of differential privacy (DP).

Definition 3.3 (Dwork et al., 2006)**.** A randomized algorithm $\mathcal{A}$ is (ϵ, δ)-differentially private if for all neighboring datasets D, D' and for all events S in the output space of $\mathcal{A}$, we have

$$\Pr[\mathcal{A}(D) \in S] \leq e^{\epsilon} \Pr[\mathcal{A}(D') \in S] + \delta.$$

If $\delta = 0$, then $\mathcal{A}$ is said to be ϵ-differentially private.

The detailed algorithm of DP-SGD-GC is shown in Algorithm 1. It has been shown that DP-SGD-GC is (ϵ, δ)-DP as long as the noise level σ is larger than certain threshold (Theorem 3.4).

Theorem 3.4 (Abadi et al., 2016, Theorem 1)**.** *Let $q = B/n$, where B is the batch size and n is the number of data points. There exist constants c_1 and c_2, such that for any $\epsilon < c_1 q^2 T$, Algorithm 1 is (ϵ, δ)-DP for any $\delta > 0$ if*

$$\sigma \geq c_2 \frac{q\sqrt{T\log(1/\delta)}}{\epsilon}.$$

Algorithm 1 Differential-private SGD with gradient clipping (DP-SGD-GC)

Input: number of iteration $T \in \mathbb{N}$, clipping threshold $C > 0$, noise level $\sigma > 0$, batch size $B \in [1, n]$, learning rate $\eta > 0$, initial iterate $w^{(0)}$.
for $t \leftarrow 0, \ldots, T-1$ **do**
  Uniform randomly sample a batch $\mathcal{B}_t$ with size B from $[n]$;
  $g_i^{(t)} = \nabla f_i(w^{(t)}) \quad \forall i \in \mathcal{B}_t$;
  $\tilde{g}_i^{(t)} = g_i^{(t)} / \max\{1, \|g_i^{(t)}\|/C\}, \quad \forall i \in \mathcal{B}_t$;
  $w^{(t+1)} = w^{(t)} - \eta \frac{1}{B} \left(\sum_{i \in \mathcal{B}_t} \tilde{g}_i^{(t)} + \xi^{(t)} \right), \quad$ where $\xi^{(t)} \sim \mathcal{N}(0, C^2\sigma^2 \mathbf{I}_{d\times d})$;
end for
Return: $w^{(i)}$ where i is uniform randomly sampled from $\{0, 1 \ldots, T\}$.

4 Main theoretical results

We present our main theoretical contributions in this section. Denote w_{priv} as the output of Algorithm 1. We are interested in the upper bound of the *excess empirical risk* $f(w_{\text{priv}}) - f^*$ and the *gradient norm square* $\|\nabla f(w_{\text{priv}})\|^2$ without assuming f_i's to be Lipschitz continuous in $\mathbb{R}^d$. Part of our analysis relies on assuming the noise generated from subsampled gradients is "light-tailed" (sub-Gaussian). Formally, we introduce the following assumption[3].

Assumption 4.1. *There exist $\rho > 0$ such that*

$$\mathbb{E}_i \left[\exp\left(\|\nabla f_i(w) - \nabla f(w)\|^2 / \rho^2 \right) \right] \leq e, \qquad \forall w \in \mathbb{R}^d.$$

We note that the above light-tail assumption is a widely used assumption for the analysis of high probability utility bound of SGD (Nemirovski et al., 2009; Juditsky & Nesterov, 2014; Ghadimi & Lan, 2013). Assumption 4.1 does not imply f_i's to be globally Lipschitz continuous on $\mathbb{R}^d$, and therefore will not trivialize our analysis.

The idea of our proof is concise. We first summarize the proof sketch as follows, and then explain the technical details step by step.

Proof sketch:

- Assuming that the initial objective gap $f(w^{(0)}) - f^*$ is bounded. We are able to prove that, with high probability, the iterates generated from SGD have bounded objective values that only logarithmically depend on T.
- The growth condition allows us to convert the objective value upper bound to the gradient norm upper bound. Therefore, with high probability, gradient clipping will never happen during the optimization process of DP-SGD if the clipping threshold C is chosen appropriately.
- Finally, we can apply the classic convergence analysis of SGD (without gradient clipping) and obtain a non-trivial excess empirical risk or gradient norm upper bound.

To begin with, we develop a uniform upper bound on the objective values, e.g., $f(w^{(t)})$ where $w^{(t)}$'s are the iterates generated from the vanilla SGD algorithm (Algorithm 2) with sub-Gaussian noise.

Proposition 4.2 (Uniform upper bound on objective values of SGD with sub-Gaussian noise)**.** *Assume f is L-smooth for some $L > 0$ and there exist $\tilde{\sigma} > 0$ such that $\mathbb{E}[\exp(\|\zeta^{(t)}\|^2/\tilde{\sigma}^2)] \leq e$ for any $t \in \mathbb{N}$. Denote $\{w^{(t)}\}_{t=0}^T$ as the iterates generated from Algorithm 2 with $\eta \leq \min\left\{\frac{1}{2L}, \frac{1}{\tilde{\sigma}\sqrt{T}}\right\}$. Then for any $\delta \in (0, 1)$,*

$$\max_{t \in \{0,1,\ldots,T\}} \left(f(w^{(t)}) - f^* \right) \;\leq\; 2\left(f(w^{(0)}) - f^* \right) + \mathcal{O}\left(\log(T/\delta)\right)$$

with probability at least $1 - \delta$.

[3]Note that there are several equivalent (up to constant) definitions of sub-Gaussian variable; see Vershynin, 2018, Proposition 2.5.2. These definitions are often used interchangeably in the literature.

Algorithm 2 Stochastic gradient descent

1: **Input:** total iterations $T \in \mathbb{N}$, learning rate $\eta > 0$, initial iterate $w^{(0)}$.
2: **for** $t \leftarrow 0, \ldots, T-1$ **do**
3: $\quad w^{(t+1)} = w^{(t)} - \eta \left(\nabla f(w^{(t)}) + \zeta^{(t)}\right)$;
4: **end for**
5: **Return:** $w^{(i)}$ where i uniform randomly sampled from $\{0, 1 \ldots, T\}$.

The conclusion stated in Proposition 4.2 may seem obvious at first glance as the training algorithm is expected to produce almost decreasing objective values during the optimization process. However, we note that deriving a nearly constant upper bound of $f(w^{(t)}) - f^*$ that holds uniformly over $t \in \{0, 1 \ldots, T\}$ without assuming Lipschitz continuity or bounded domain is nontrivial. Our proof relies on a recently proposed technical tool called the generalized Freedman inequality (Harvey et al., 2019); see Section C.1 for the detailed proof of Proposition 4.2. We also remark that Proposition 4.2 holds for the standard SGD algorithm without considering differential privacy, and thus may be of independent interest.

Based on Proposition 4.2, we can further obtain an upper bound on each individual loss $f_i(w^{(t)}) - f_i^*$ under Assumption 4.1, and therefore also upper bound $\|\nabla f_i(w^{(t)})\|$ via the growth condition.

Proposition 4.3. *Assume f_i's are L-smooth for some $L > 0$ and there exist $\tilde{\sigma} > 0$ such that $\mathbb{E}[\exp(\|\zeta^{(t)}\|^2/\tilde{\sigma}^2)] \leq e$ for any $t \in \mathbb{N}$. Denote $\{w^{(t)}\}_{t=0}^T$ as the iterates generated from Algorithm 2 with $\eta \leq \min\left\{\frac{1}{2L}, \frac{1}{\tilde{\sigma}\sqrt{T}}\right\}$. Then for any $\delta \in (0,1)$,*

$$\max_{i\in[n], t\in[T]} \|\nabla f_i(w^{(t)})\| \;\leq\; \sqrt{2n\beta_1(f(w^{(0)}) - f^*)} + \mathcal{O}\Big(\sqrt{n\log(1/\delta)} + \sqrt{n\log T}\Big) \tag{1}$$

holds with probability at least $1-\delta$. Furthermore, when Assumption 4.1 is satisfied,

$$\max_{i\in[n], t\in[T]} \|\nabla f_i(w^{(t)})\| \;\leq\; \sqrt{2\beta_1(f(w^{(0)}) - f^*)} + \mathcal{O}\Big(\sqrt{\log(1/\delta)} + \sqrt{\log T} + \sqrt{\log n}\Big) \tag{2}$$

holds with probability at least $1-\delta$ for any $\delta \in (0,1)$.

When Assumption 4.1 holds, Proposition 4.3 suggests that the upper bound of the maximum gradient norm logarithmically depends on δ, T and n (eq. (1)). When Assumption 4.1 does not hold, there is an addtional $\sqrt{n}$ term appears in eq. (1), which eventually causes a $\sqrt{n}$ loss in the utility bound.

Now we are ready to present our excess empirical risk bound and gradient norm bound for DP-SGD in terms of conditional expectation. Due to space limitation, we only present the proposition with Assumption 4.1 in the main text and place the proposition without Assumption 4.1 in Appendix B.

Proposition 4.4 (Convergence on conditional expectation)**.** *Assume that Assumption 4.1 holds. Denote w_{priv} as the output of Algorithm 1 and define $\mathcal{E} := \{\|f_i(w^{(t)})\| \leq C \,\forall i \in [n], t \in \{0, 1 \ldots, T\}\}$ as the event of no clipping happens during the training of Algorithm 1. Let $D_f := f(w^{(0)}) - f^*$. Given any $\epsilon > 0$ and $\delta, \delta' \in (0,1)$.*

- *Assume f_i's are convex and L-smooth for some $L > 0$. set $T > \epsilon/c_1$,*

$$\sigma = \frac{c_2 B\sqrt{T\log(1/\delta)}}{n\epsilon},\; C = \sqrt{c_3 + c_4\log(nT/\delta')},\; \eta = \min\left\{\frac{1}{2L}, \frac{c_5 B}{\sqrt{(B\rho^2 + C^2 d\sigma^2)T}}\right\}, \tag{3}$$

where c_1, c_2, c_5 are some absolute constants and c_3, c_4 are constants that depend on L and D_f. We have that Algorithm 1 is (ϵ, δ)-DP, $\Pr[\mathcal{E}] \geq 1 - \delta'$, and

$$\mathbb{E}\left[f(w_{\text{priv}}) - f^* \mid \mathcal{E}\right] \;\leq\; \mathcal{O}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\left(\sqrt{\log(1/\delta')} + \sqrt{\log(Tn)}\right)\sqrt{d\log(1/\delta)}}{n\epsilon}\right).$$

- *Assume f_i's are L-smooth for some $L > 0$. Setting $\epsilon, \sigma, C, \eta$ as in eq. (3). It holds that Algorithm 1 is (ϵ, δ)-DP, $\Pr[\mathcal{E}] \geq 1 - \delta'$, and*

$$\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2 \mid \mathcal{E}\right] \;\leq\; \mathcal{O}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\left(\sqrt{\log(1/\delta')} + \sqrt{\log(Tn)}\right)\sqrt{d\log(1/\delta)}}{n\epsilon}\right).$$

While the bounds stated in Proposition 4.4 are close to our objective, we note that these bounds are expressed in terms of conditional expectation where the conditioning event happens with high probability, which is different from the classic notion of convergence in expectation. Fortunately, we show that it is easy to convert the bounds in Proposition 4.4 to expected utility bound when the random variable $f(w_{\text{priv}}) - f^*$ (or $\|\nabla f(w_{\text{priv}})\|^2$) is sub-exponential, which is true under Assumption 4.1. Lemma A.8 serves as the main technical tool for this conversion.

Theorem 4.5 (Convergence on expectation)**.** *Suppose that Assumption 4.1 holds. Denote w_{priv} as the output of Algorithm 1. Let $D_f \coloneqq f(w^{(0)}) - f^*$. Given any $\epsilon > 0$ and $\delta \in (0, 1)$. Setting T, σ, η in the same way as eq. (3) and let $C = \sqrt{c_3 + c_4 \log(nT)}$, where c_1, c_2, c_5 are some absolute constants and c_3, c_4 are constants that depend on L, D_f.*

- *Assume f_i's are convex and L-smooth for some $L > 0$. Then Algorithm 1 is (ϵ, δ)-DP and*

$$\mathbb{E}\left[f(w_{\text{priv}}) - f^*\right] \;\leq\; \widetilde{\mathcal{O}}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{n\epsilon}\right).$$

  Consequently, we have $\mathbb{E}\left[f(w_{\text{priv}}) - f^*\right] = \widetilde{\mathcal{O}}\left(d^{1/2}(n\epsilon)^{-1}\right)$ by setting $T = \Theta\left(n^2\epsilon^2 d^{-1}\right)$.

- *Assume f_i's are L-smooth for some $L > 0$. Then Algorithm 1 is (ϵ, δ)-DP and*

$$\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2\right] \;\leq\; \widetilde{\mathcal{O}}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{n\epsilon}\right).$$

  Consequently, we have $\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2\right] = \widetilde{\mathcal{O}}\left(d^{1/2}(n\epsilon)^{-1}\right)$ by setting $T = \Theta\left(n^2\epsilon^2 d^{-1}\right)$.

Theorem 4.6 (Convergence without light-tail noise, informal)**.** *Under the same conditions as Theorem 4.5 but without Assumption 4.1. We have $\mathbb{E}\left[f(w_{\text{priv}}) - f^*\right] = \widetilde{\mathcal{O}}\left(d^{1/2}(\sqrt{n}\epsilon)^{-1}\right)$ and $\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2\right] = \widetilde{\mathcal{O}}\left(d^{1/2}(\sqrt{n}\epsilon)^{-1}\right)$ by setting $T = \Theta\left(n\epsilon^2 d^{-1}\right)$.*

Remark 4.1. *Our results suggest that, when the problem is smooth, the Lipschitz continuous and bounded domain assumptions can be removed almost for free when analyzing DP-SGD-GC with light-tailed gradient noise; the only cost is some logarithmic terms. When the gradient noise is not necessarily light-tailed, there is a loss of $\sqrt{n}$ in the utility bound.*

Theorem 4.5 and Theorem 4.6 are the main theoretical contributions of this paper. The bounds stated in Theorem 4.5 nearly match the rate $\mathcal{O}(\log(1/\delta)d^{1/2}(n\epsilon)^{-1})$, which is the best-known bound of DP-SGD with the Lipschitz continuous or bounded domain assumption (Bassily et al., 2014). When the Assumption 4.1 does not hold, the bound stated in Theorem 4.6 is worse than the rate $\mathcal{O}(\log(1/\delta)d^{1/2}(n\epsilon)^{-1})$ with a factor of $\sqrt{n}$. To our knowledge, for unconstrained smooth problems, this is the first utility bound of DP-SGD-GC without a non-vanishing bias term. Note that existing lower bound analyses for DP-SGD either assume the domain is bounded (Bassily et al., 2014) or the loss is Lipschitz continuous (Song et al., 2021). Therefore those lower bounds are not comparable with the upper bounds stated in Theorem 4.5. We leave the lower bound of DP-SGD-GC for unconstrained smooth problems as a future direction to explore.

5 Value Clipping

This section focuses on the practical side of DP-SGD-GC. A well-known implementation issue of Algorithm 1 is that the gradient clipping step (line 4 of Algorithm 1) requires to access the norm of each individual gradient from the sampled batch, and naive implementation of DP-SGD-GC on current deep learning platforms cannot fully exploit the parallelism of GPU; see some attempts that try to mitigate this issue (Goodfellow, 2015; Rochette et al., 2019; Bu et al., 2021). The state-of-the-art

Algorithm 3 Differential-private SGD with value clipping (DP-SGD-VC)

1: **Input:** number of iteration $T \in \mathbb{N}$, clipping threshold $C > 0$, noise level $\sigma > 0$, batch size $B \in [1, n]$, learning rate $\eta > 0$, initial iterate $w^{(0)}$, growth condition parameters $\beta_1 > 0, \beta_2 \geq 0$, $f_{\text{lb}}^* \in \mathbb{R}$ that lower bound f_i^* $\forall i \in [n]$.
2: **for** $t \leftarrow 0, \ldots, T-1$ **do**
3: Uniform randomly sample a batch $\mathcal{B}_t$ with size B from $[n]$;
4: $\tilde{g}_i^{(t)} = \nabla f_i(w^{(t)}) / \max\left\{1, \sqrt{\beta_1(f(w^{(t)}) - f_{\text{lb}}^*) + \beta_2}/C\right\} \qquad \forall i \in \mathcal{B}_t;$
5: $w^{(t+1)} = w^{(t)} - \eta \frac{1}{B}\left(\sum_{i \in \mathcal{B}_t} \tilde{g}_i^{(t)} + \xi^{(t)}\right), \quad$ where $\xi^{(t)} \sim \mathcal{N}(0, C^2\sigma^2 \mathbf{I}_{d\times d})$;
6: **end for**
7: **Return:** $w^{(i)}$ where i uniform randomly sampled from $\{0, 1 \ldots, T\}$.

implementation of DP-SGD-GC is from Subramani et al. (Subramani et al., 2021), who developed a highly engineered approach to exploit language primitives, compilation, and vectorization on certain deep learning platforms. In this section, we propose a *value clipping* technique for functions that satisfy the growth condition (Definition 3.1). The proposed value clipping can be viewed as an alternative to the classic gradient clipping technique that is easy to implement on all existing deep learning platforms such as PyTorch (Paszke et al., 2019), Tensorflow (Abadi et al., 2015) and MXNet (Chen et al., 2015).

The intuition behind value clipping is simple — when f_i's satisfy the growth condition, the norm of their gradients can be bounded by a function of their objective values, e.g., $\|\nabla f_i(w)\| \leq \sqrt{\beta_1(f_i(w) - f_i^*) + \beta_2}$, therefore scaling the gradient by $\sqrt{\beta_1(f_i(w) - f_i^*) + \beta_2}$ ensures the scaled gradient has a bounded norm. Formally,

$$\forall w \in \mathbb{R}^d, \quad \tilde{g} := f_i(w) / \max\left\{1, \frac{\sqrt{\beta_1(f_i(w) - f_i^*) + \beta_2}}{C}\right\} \quad \Longrightarrow \quad \|\tilde{g}\| \leq C.$$

The detailed algorithm of DP-SGD with value clipping, termed DP-SGD-VC, is shown in Algorithm 3. Note that DP-SGD-VC requires knowing the GC parameters β_1, β_2 and a lower bound of f_i^*'s as its input. The GC parameters for simple models, including linear and logistic regression, are easy to calculate. For feed-forward neural networks, the calculation of GC parameters is achievable but more involved; see Appendix E for details. It is easy to verify that DP-SGD-VC is (ϵ, δ)-DP because the norm of the clipped gradient is guaranteed to be bounded by C; the following corollary is a direct consequence of Theorem 3.4 and describes the DP property of DP-SGD-VC.

Corollary 5.1. *Assume f_i's are (β_1, β_2)-GC for some $\beta_1 > 0, \beta_2 \geq 0$. Let $q = B/n$, where B is the batch size and n is the number of data points. There exist constants c_1 and c_2, such that for any $\epsilon < c_1 q^2 T$, Algorithm 3 is (ϵ, δ)-DP for any $\delta > 0$ if $\sigma \geq c_2 q \sqrt{T \log(1/\delta)} \epsilon^{-1}$.*

Remark 5.1. *DP-SGD-VC is easy to implement on existing auto-differentiation based deep learning platforms. The value clipping step (line 4 of Algorithm 3) can be realized within one forward-backward propagation if the GC parameters are given in advance. Therefore DP-SGD-VC can be as fast as the vanilla SGD algorithm.*

6 Numerical Study

We conduct experiments on two standard image classification benchmark datasets: MNIST (LeCun & Cortes, 2011) and CIFAR10 (Krizhevsky, 2009). In Appendix, we also present some experimental results on synthetic data with light-tailed noise. For MNIST, we train a linear classifier and a two-layer MLP with 128 hidden nodes respectively. For CIFAR10, to achieve decent accuracy, we use a pre-trained VGG16 network (Simonyan & Zisserman, 2015) to extract informative high-level features. Based on the 512-dimensional extracted features, we train a linear classifier and a two-layer MLP with 128 hidden nodes respectively.

Implementation details For all experiments, we set the batch size $B = 128$, the noise level $\sigma = 1.0$ and the confidence level $\delta = 10^{-5}$. For MNIST, we try learning rate in $\{2 \times 10^{-3}, 5 \times 10^{-3}, 10^{-2}\}$ for each experiment and report the best result. For CIFAR10 we fix the learning rate to be 0.1. All experiments are conducted on a server with 4 CPUs and one NVIDIA Tesla P100 GPU.

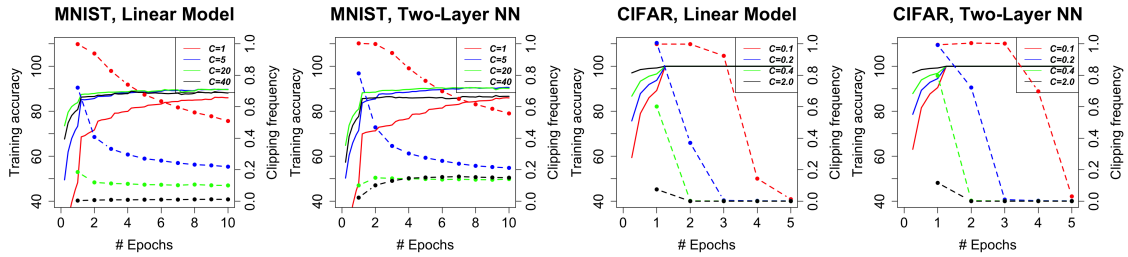

Figure 1: The evolution of training accuracy and clipping frequency during DP-SGD-GC, where the solid line ▬ represent training accuracy and the dashed line - - - denote the clipping frequency per epoch. The left two figures present the results on the MNIST dataset, the algorithm is about $(1.0379, 10^{-5})$-DP. The right two figures show the results on the CIFAR10 dataset (feature extracted by a pretrained VGG16 network), the algorithm is about $(0.9580, 10^{-5})$-DP.

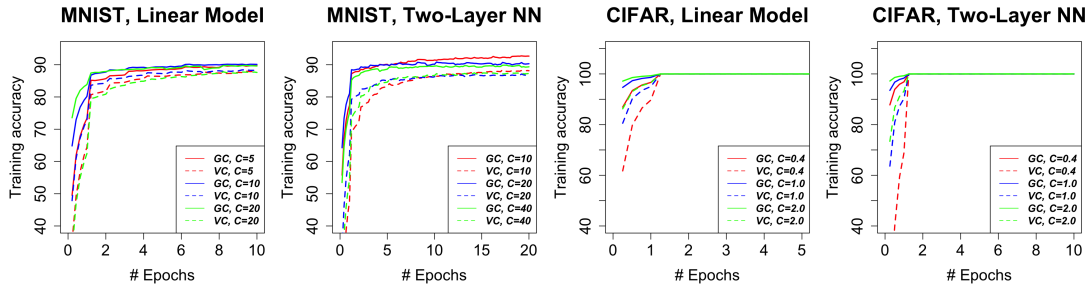

Figure 2: Value clipping (VC) versus gradient clipping (GC) in terms of training accuracy.

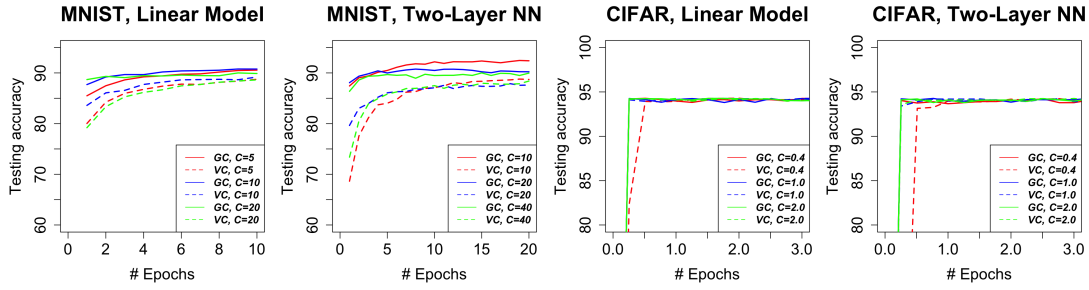

Figure 3: Value clipping (VC) versus gradient clipping (GC) in terms of testing accuracy.

6.1 THE EVOLUTION OF CLIPPING FREQUENCY DURING TRAINING

We run DP-SGD-GC with different clipping thresholds. In particular, we try $C \in \{1, 5, 20, 40\}$ and $C \in \{0.1, 0.2, 0.4, 1.0\}$ for MNIST and CIFAR10 respectively. The evolutions of training accuracy and clipping frequency per epoch with different clipping thresholds are shown in Figure 1. We can observe that, in most cases, the clipping frequency decreases as the training accuracy goes up. This observation aligns with the growth condition as lower training loss implies a smaller average gradient norm and further results in lower clipping frequency. We also see that, in most cases, the clipping frequency can become close to 0 when the clipping threshold is chosen appropriately; this observation is consistent with our theoretical analysis. Another interesting observation is the result of training the two-layer neural network with MNIST and $C \in \{20, 40\}$. The clipping frequency is small initially and becomes stable at 15% instead of decreasing to zero. We conjecture that this phenomenon is because the growth condition parameters of the neural network grow as the training goes and thus prevent some gradient norms from being smaller than the clipping threshold.

6.2 THE EVALUATION OF DP-SGD-VC

We set $f_{\text{lb}}^* = 0$ for all experiments with DP-SGD-VC. The calculation of the GC parameters for a feed-forward neural network with cross-entropy loss is given in Section E. Compared with vanilla SGD, DP-SGD-VC has an additional cost to calculate the spectral norm of each layer in each iteration. As shown in the following content, the overhead of calculating the spectral norm is not significant.

Data	Model	SGD	Micro-batching	GC-Opacus	DP-SGD-VC
MNIST	Linear	6.83s	34.82s	12.80s	**8.62**s
MNIST	2-layer NN	6.91s	43.75s	16.61s	**8.96**s
CIFAR	Linear	0.58s	23.99s	4.13s	**1.25**s
CIFAR	2-layer NN	0.65s	30.86s	7.47s	**1.70**s

Table 2: Per epoch runtime of different methods. SGD without gradient clipping is the baseline method; Micro-batching is the naive implementation of DP-SGD-GC; GC and VC are the classic gradient clipping and the proposed value clipping, respectively. We bold the shortest per epoch runtime among methods besides SGD.

Training and testing accuracy The training and testing accuracy of DP-SGD-GC and DP-SGD-VC with different clipping thresholds are shown in Figure 2 and Figure 3. We can observe that DP-SGD-VC converges slightly slower than DP-SGD-GC in terms of the epoch. This observation should not be surprising as DP-SGD-VC uses an upper bound of the gradient norm for clipping and will result in a smaller effective learning rate than DP-SGD-GC. For CIFAR10, the training and testing are easy as the model is trained on pre-trained features; both DP-SGD-VC and DP-SGD-GC can achieve similar accuracy at the end of training. For MNIST, there is an unfortunate loss of training and testing accuracy. For MNIST with the linear model, there is a $\sim 2\%$ loss in training and testing accuracy. For MNIST with two-layer NN, there is a $2 \sim 3\%$ loss in training and testing accuracy for $C \in \{20, 40\}$ and a $\sim 4\%$ loss for $C = 10$. The gap between DP-SGD-GC and GP-SGD-VC is more obvious when the clipping threshold is small. We conjecture that the loss of training and testing accuracy is due to our estimation of f_i^*. The estimation $f_{\text{lb}}^* = 0$ is accurate for CIFAR10 as the model can almost perfectly fit all pre-trained data. However, the estimation is inaccurate for MNIST and thus results in a loose upper bound of the gradients' norm.

Comparing the computational time per epoch We report the per epoch runtime of different algorithms in Table 2. All experiments are conducted on a server with one NVIDIA Tesla P100 GPU. The vanilla SGD without privacy consideration is the baseline method and is the fastest among all algorithms. Micro-batching is the naive implementation of DP-SGD-GC and is significantly slower than SGD. GC-Opacus is the implementation of DP-SGD-GC from the (highly optimized) Opacus package (Yousefpour et al., 2021); we can see that there is still a gap between the performance of GC-Opacus and the standard non-private SGD algorithm. DP-SGD-VC is our implementation of DP-SGD with the proposed value clipping technique. We can observe that DP-SGD-VC is slightly slower than the standard SGD algorithm and faster than other private training methods.

Limitation Despite the efficiency of DP-SGD-VC, it also has certain limitations: (i) as shown in the experiments, DP-SGD-VC may cause some loss in training/testing accuracy if our estimation for the GC parameters is loose; (ii) DP-SGD-VC requires calculating the GC parameters, which we show is available for feed-forward neural works with cross-entropy loss. However, VC technique can not be applied to more complicated neural network architectures with arbitrary loss in a black-box manner. Overall, DP-SGD-VC is more efficient than DP-SGD-GC and can perform similarly to DP-SGD-GC in certain scenarios but also possibly worse in other situations.

7 Conclusion

This paper studied the convergence behavior of a widely used privacy-preserving learning algorithm called DP-SGD-GC. Our analysis extended the convergence of DP-SGD-GC to smooth and unconstrained problems without assuming the objective to be globally Lipschitz continuous. We believe that our theoretical results improved the current understanding of DP-SGD-GC and provided new insights for practitioners and researchers to use DP-SGD-GC and design new algorithms. Our analysis can potentially be used for other privacy-preserving learning algorithms such as adaptive DP-SGD (Asi et al., 2021) and DP-SGD with subspace identification (Zhou et al., 2021; Song et al., 2021).

Our work also suggests some future directions. The light-tail noise condition may not hold in some machine learning applications (Gürbüzbalaban et al., 2021). In these scenarios, our utility bound of DP-SGD-GC (Theorem 4.5) is $\sqrt{n}$ worse than the best-known bound of DP-SGD-GC applied to Lipschitz functions. Whether it is possible to further improve the utility bound is an interesting question from the perspective of theory. Another direction is to explore the lower bound of DP-SGD-GC with a carefully tuned clipping threshold for smooth and unconstrained problems.

ETHIC STATEMENT

This paper studied a class of private-preserving machine learning algorithms called differential private SGD. We provide improved convergence analysis and some new theoretical insights into DP-SGD. We are not aware of any potential negative social impact of this work. The datasets used for the experiments do not contain personally identifiable information or offensive content.

REPRODUCIBILITY STATEMENT

The code to reproduce our experimental results in Section 6 is included as supplementary material in the submission; we will make our code publicly available upon acceptance.

REFERENCES

Martín Abadi, Ashish Agarwal, Paul Barham, Eugene Brevdo, Zhifeng Chen, Craig Citro, Greg S. Corrado, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Ian Goodfellow, Andrew Harp, Geoffrey Irving, Michael Isard, Yangqing Jia, Rafal Jozefowicz, Lukasz Kaiser, Manjunath Kudlur, Josh Levenberg, Dandelion Mané, Rajat Monga, Sherry Moore, Derek Murray, Chris Olah, Mike Schuster, Jonathon Shlens, Benoit Steiner, Ilya Sutskever, Kunal Talwar, Paul Tucker, Vincent Vanhoucke, Vijay Vasudevan, Fernanda Viégas, Oriol Vinyals, Pete Warden, Martin Wattenberg, Martin Wicke, Yuan Yu, and Xiaoqiang Zheng. TensorFlow: Large-scale machine learning on heterogeneous systems, 2015. URL `https://www.tensorflow.org/`. Software available from tensorflow.org.

Martín Abadi, Andy Chu, Ian J. Goodfellow, H. Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In *Proceedings of the ACM SIGSAC Conference on Computer and Communications Security*, pp. 308–318, 2016.

Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via over-parameterization. In *Proceedings of ICML*, pp. 242–252, 2019.

Galen Andrew, Om Thakkar, Brendan McMahan, and Swaroop Ramaswamy. Differentially private learning with adaptive clipping. In *Proceedings of the Advances in Neural Information Processing Systems*, 2021.

Hilal Asi, John C. Duchi, Alireza Fallah, Omid Javidbakht, and Kunal Talwar. Private adaptive gradient methods for convex optimization. In *Proceedings of the International Conference on Machine Learning, ICML*, 2021.

Raef Bassily, Adam D. Smith, and Abhradeep Thakurta. Private empirical risk minimization: Efficient algorithms and tight error bounds. In *IEEE Annual Symposium on Foundations of Computer Science, FOCS*, pp. 464–473, 2014.

Raef Bassily, Vitaly Feldman, Kunal Talwar, and Abhradeep Guha Thakurta. Private stochastic convex optimization with optimal rates. In *Advances in Neural Information Processing Systems (NeurIPS)*, 2019.

Zhiqi Bu, Sivakanth Gopi, Janardhan Kulkarni, Yin Tat Lee, Judy Hanwen Shen, and Uthaipon Tantipongpipat. Fast and memory efficient differentially private-SGD via JL projections. In *Proceedings of the Advances in Neural Information Processing Systems*, 2021.

Tianqi Chen, Mu Li, Yutian Li, Min Lin, Naiyan Wang, Minjie Wang, Tianjun Xiao, Bing Xu, Chiyuan Zhang, and Zheng Zhang. Mxnet: A flexible and efficient machine learning library for heterogeneous distributed systems. *arXiv e-prints*, pp. arXiv:1512.01274, 2015.

Xiangyi Chen, Zhiwei Steven Wu, and Mingyi Hong. Understanding gradient clipping in private SGD: A geometric perspective. In *Proceedings of the Advances in Neural Information Processing Systems*, 2020.

Frank H Clarke. Generalized gradients of lipschitz functionals. *Advances in Mathematics*, 40(1): 52–67, 1981.

Jian Du, Song Li, Xiangyi Chen, Siheng Chen, and Mingyi Hong. Dynamic differential-privacy preserving sgd. *arXiv e-prints*, 2021.

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam D. Smith. Calibrating noise to sensitivity in private data analysis. In *Proceedings of the third Theory of Cryptography Conference, TCC*, volume 3876, pp. 265–284, 2006.

Huang Fang, Zhenan Fan, and Michael Friedlander. Fast convergence of stochastic subgradient method under interpolation. In *Proceedings of ICLR*, 2021.

Vitaly Feldman, Tomer Koren, and Kunal Talwar. Private stochastic convex optimization: optimal rates in linear time. In *Proccedings of ACM Symposium on Theory of Computing, STOC*, 2020.

Saeed Ghadimi and Guanghui Lan. Stochastic first- and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.

Ian Goodfellow. Efficient per-example gradient computations. *arXiv e-prints*, pp. arXiv:1510.01799, 2015.

Robert M. Gower, Nicolas Loizou, Xun Qian, Alibek Sailanbayev, Egor Shulgin, and Peter Richtárik. SGD with arbitrary sampling: General analysis and improved rates. In *Proceedings of ICML*, pp. 5200–5209, 2019.

Robert M. Gower, Othmane Sebbouh, and Nicolas Loizou. SGD for structured nonconvex functions: Learning rates, minibatching and interpolation. In *International Conference on Artificial Intelligence and Statistics, AISTATS*, 2021.

Mert Gürbüzbalaban, Umut Simsekli, and Lingjiong Zhu. The heavy-tail phenomenon in SGD. In *Proceedings of the International Conference on Machine Learning, (ICML)*, volume 139, pp. 3964–3975, 2021.

Nicholas J. A. Harvey, Christopher Liaw, Yaniv Plan, and Sikander Randhawa. Tight analyses for non-smooth stochastic gradient descent. In *Conference on Learning Theory, COLT*, 2019.

Anatoli Juditsky and Yuri Nesterov. Deterministic and stochastic primal-dual subgradient algorithms for uniformly convex minimization. *Stoch. Syst.*, 4(1):44–88, 2014.

Ahmed Khaled and Peter Richtárik. Better theory for sgd in the nonconvex world. *arXiv e-prints*, art. arXiv:2002.03329, Feb 2020.

Alex Krizhevsky. Learning multiple layers of features from tiny images, 2009.

Yann LeCun and Corinna Cortes. The MNIST database. http://yann.lecun.com/exdb/mnist/, 2011.

Deanna Needell, Nathan Srebro, and Rachel Ward. Stochastic gradient descent, weighted sampling, and the randomized kaczmarz algorithm. *Mathematical Programming*, 155:549—-573, 2016.

A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro. Robust stochastic approximation approach to stochastic programming. 19(4):1574–1609, 2009.

Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, Alban Desmaison, Andreas Kopf, Edward Yang, Zachary DeVito, Martin Raison, Alykhan Tejani, Sasank Chilamkurthy, Benoit Steiner, Lu Fang, Junjie Bai, and Soumith Chintala. Pytorch: An imperative style, high-performance deep learning library. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, and R. Garnett (eds.), *Proceedings of the Advances in Neural Information Processing Systems (NeurIPS)*, pp. 8024–8035. 2019.

Gaspar Rochette, Andre Manoel, and Eric W. Tramel. Efficient per-example gradient computations in convolutional neural networks. *arXiv e-prints*, pp. arXiv:1912.06015, Dec 2019.

Mark Schmidt and Nicolas Le Roux. Fast convergence of stochastic gradient descent under a strong growth condition. *arXiv e-prints*, art. arXiv:1308.6370, Aug 2013.

Karen Simonyan and Andrew Zisserman. Very deep convolutional networks for large-scale image recognition. In *International Conference on Learning Representations, ICLR*, 2015.

Shuang Song, Kamalika Chaudhuri, and Anand D. Sarwate. Stochastic gradient descent with differentially private updates. In *IEEE Global Conference on Signal and Information Processing, GlobalSIP*, pp. 245–248, 2013.

Shuang Song, Thomas Steinke, Om Thakkar, and Abhradeep Thakurta. Evading curse of dimensionality in unconstrained private glms via private gradient descent. In *International Conference on Artificial Intelligence and Statistics, AISTATS*, 2021.

Sebastian U. Stich. Unified optimal analysis of the (stochastic) gradient method. *arXiv e-prints*, art. arXiv:1907.04232, July 2019.

Pranav Subramani, Nicholas Vadivelu, and Gautam Kamath. Enabling fast differentially private sgd via just-in-time compilation and vectorization. In *Proceedings of the Advances in Neural Information Processing Systems*, 2021.

Sharan Vaswani, Francis Bach, and Mark Schmidt. Fast and faster convergence of SGD for over-parameterized models and an accelerated perceptron. In *Proceedings of AISTATS*, pp. 1195–1204, 2019.

Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge University Press, 2018.

Di Wang, Minwei Ye, and Jinhui Xu. Differentially private empirical risk minimization revisited: Faster and more general. In *Proceedings of the Advances in Neural Information Processing Systems*, pp. 2722–2731, 2017.

Puyu Wang, Yunwen Lei, Yiming Ying, and Hai Zhang. Differentially private sgd with non-smooth losses. *Applied and Computational Harmonic Analysis*, 56:306–336, 2022.

Xiaoxia Wu, Lingxiao Wang, Irina Cristali, Quanquan Gu, and Rebecca Willett. Adaptive differentially private empirical risk minimization. *arXiv e-prints*, 2021.

Ashkan Yousefpour, Igor Shilov, Alexandre Sablayrolles, Davide Testuggine, Karthik Prasad, Mani Malek, John Nguyen, Sayan Ghosh, Akash Bharadwaj, Jessica Zhao, Graham Cormode, and Ilya Mironov. Opacus: User-friendly differential privacy library in PyTorch. *arXiv preprint arXiv:2109.12298*, 2021.

Bohang Zhang, Jikai Jin, Cong Fang, and Liwei Wang. Improved analysis of clipping algorithms for non-convex optimization. In *Proceedings of the Advances in Neural Information Processing Systems (NeurIPS)*, 2020a.

Jingzhao Zhang, Tianxing He, Suvrit Sra, and Ali Jadbabaie. Why gradient clipping accelerates training: A theoretical justification for adaptivity. In *International Conference on Learning Representations (ICLR)*, 2020b.

Yingxue Zhou, Steven Wu, and Arindam Banerjee. Bypassing the ambient dimension: Private SGD with gradient subspace identification. In *International Conference on Learning Representations, ICLR*, 2021.

8 APPENDIX

APPENDIX A LEMMAS

A.1 STANDARD FACTS

Lemma A.1. *For any $a, b \in \mathbb{R}$, it is true that $(a+b)^2 \le 2a^2 + 2b^2$.*

Lemma A.2. *It holds that $\sum_{i=1}^{\infty}(i/2^i) = 2$.*

A.2 SUB-GAUSSIAN AND SUB-EXPONENTIAL PROPERTIES

Lemma A.3 (Sub-Gaussian properties, Vershynin, 2018, Proposition 2.5.2)**.** *Let X be a random variable. Then the following properties are equivalent; the parameters $K_i, i = 1, \ldots, 4$ differ from each other by at most an absolute constant factor.*

- *The tails of X satisfy*
$$\Pr[|X| \ge t] \le 2\exp(-t^2/K_1^2) \qquad \forall t \ge 0.$$
- *The moment generating function (MGF) of X^2 satisfies*
$$\mathbb{E}[\exp(\lambda X^2)] \le \exp(K_2^2\lambda) \qquad \forall \lambda \in [0, 1/K_2^2].$$
- *The MGF of X^2 is bounded at some point, e.g.,*
$$\mathbb{E}[\exp(X^2/K_3^2)] \le e.$$
- $\mathbb{E}[X] = 0$ *and the MGF of X satisfies*
$$\mathbb{E}[\exp(\lambda X)] \le \exp(K_4^2\lambda^2) \qquad \forall \lambda \in \mathbb{R}.$$

Lemma A.4 (Harvey et al., 2019, Lemma A.4)**.** *Let $X_1, X_2, \ldots, X_n$ be random variables. Assume that there exist $K_1, K_2, \ldots, K_n > 0$ such that $\mathbb{E}[\exp(\lambda X_i)] \le \exp(\lambda K_i)$ for all $0 \le \lambda \le 1/K_i$. Then $\mathbb{E}[\exp(\lambda \sum_{i=1}^n X_i)] \le \exp(\lambda \sum_{i=1}^n K_i)$ for all $0 \le \lambda \le 1/\sum_{i=1}^n K_i$.*

Lemma A.5 (Harvey et al., 2019, Claim A.7)**.** *Suppose X is a random variable such that there exists constants c and K such that $\mathbb{E}[\exp(\lambda X)] \le c\exp(\lambda K)$ for all $0 \le \lambda \le 1/K$. Then for any $\delta \in (0,1)$, $\Pr[X \ge K\log(1/\delta)] \le ce\delta$.*

Note that the original lemmas from Harvey et al. (Harvey et al., 2019) assume $\mathbb{E}[\exp(\lambda X)] \le \exp(\lambda K)$ for all $\lambda \le 1/K$. It is easy to check that their results also hold for X such that $\mathbb{E}[\exp(\lambda X)] \le \exp(\lambda K)$ for all $0 \le \lambda \le 1/K$.

Lemma A.6. *Let $X_1, X_2, \ldots, X_T$ be random variables such that $\mathbb{E}[\exp(\lambda X_i)] \le \exp(\lambda K)\ \forall \lambda \in [0, K]$ for all $i \in \{1, 2, \ldots, T\}$. Then there exists absolute positive constant c such that $\Pr\left[\sum_{i=1}^T X_i \ge TK\log(1/\delta)\right] \le ce\delta$ for any $\delta \in (0,1)$.*

Proof. By Lemma A.4,
$$\mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^T X_i\right)\right] \le \exp(TK\lambda) \qquad \forall \lambda \in [0, (TK)^{-1}].$$
Then Lemma A.5 gives
$$\Pr\left[\sum_{i=1}^T X_i \ge TK\log(1/\delta)\right] \le ce\delta \qquad \forall \delta \in (0,1).$$
The above finishes the proof. □

Lemma A.7 (Tail bound for the maximum of sub-Gaussian variables)**.** *Let $X_1, X_2, \ldots, X_n$ be random variables. Assume that there exist $K > 0$ such that $\mathbb{E}[\exp(\lambda X_i)] \le \exp(\lambda^2 K^2)$ for all $\lambda \in \mathbb{R}$ and $i \in [n]$. Then there exist some absolute positive constant c such that*
$$\Pr\left[\max_{i\in[n]} |X_i| \ge cK\sqrt{\log(2n)} + ct\right] \le \exp(-t^2/K^2) \qquad \forall t \ge 0.$$

Proof. By Lemma A.3, there exist some absolute constant $c > 0$ such that

$$\Pr[|X_i| \geq t] \leq 2\exp\left(-t^2/(c^2K^2)\right) \qquad \forall t \geq 0, i \in [n].$$

Then we simply apply union bound to obtain the conclusion. For any $t \geq 0$,

$$\begin{aligned}
& \Pr\left[\max_{i\in[n]} |X_i| \geq cK\sqrt{\log(2n)} + ct\right] \\
\leq\ & \sum_{i=1}^{n} \Pr\left[|X_i| \geq cK\sqrt{\log(2n)} + ct\right] \\
\leq\ & 2n\exp\left(-(cK\sqrt{\log(2n)} + ct)^2/(c^2K^2)\right) \\
\leq\ & \exp(\log(2n))\exp\left(-\frac{K^2\log(2n)}{K^2} - \frac{2tK\sqrt{\log(2n)}}{K^2} - \frac{t^2}{K^2}\right) \\
=\ & \exp\left(-\frac{2tK\sqrt{\log(2n)}}{K^2} - \frac{t^2}{K^2}\right) \\
\leq\ & \exp(-t^2/K^2),
\end{aligned}$$

which yields the desired result. □

Lemma A.8. *Let X be a random variable and $\mathcal{E}$ be a random event such that $\mathbb{E}[|X| \mid \mathcal{E}] \leq \alpha$ and $\Pr[\mathcal{E}] \geq 1 - \delta$ for some $\alpha > 0, \delta \in (0,1)$. If X satisfies $\Pr[|X| - \beta \geq t] \leq 2\exp(-t/K)\ \forall t \geq 0$ for some $\beta, K > 0$. Then*

$$\mathbb{E}\left[|X|\right] \leq \alpha + \delta\beta + \delta\log(8/\delta)K.$$

Proof. Let $Q : [0,1] \to \mathbb{R} \cup \{\infty\}$ be the quantile function for the random variable $|X|$, e.g.,

$$Q(p) = \inf\left\{x \in \mathbb{R} \,\middle|\, p \leq \Pr[|X| \leq x]\right\} \qquad \forall p \in [0,1].$$

By the assumption that $|X| - \beta$ is sub-exponential, we have that

$$Q(1-\delta') \ \leq\ \beta + K\log(2/\delta') \qquad \forall \delta' \in (0,1). \tag{4}$$

Now we are ready to prove the conclusion. First notice that

$$\begin{aligned}
\mathbb{E}[|X|] \ &=\ \Pr[\mathcal{E}]\mathbb{E}[|X| \mid \mathcal{E}] + \Pr[\mathcal{E}^c]\mathbb{E}[|X| \mid \mathcal{E}^c] \\
&\leq\ \alpha + \mathbb{E}[|X| \cdot \mathbf{1}_{\mathcal{E}^c}],
\end{aligned} \tag{5}$$

where $\mathbf{1}_{\mathcal{E}^c}$ is the indicator function with the event $\mathcal{E}^c$. The remaining is to bound $\mathbb{E}[|X| \cdot \mathbf{1}_{\mathcal{E}^c}]$. Define a new set of events $A_i := \left\{|X| \in \left[Q\left(1 - \frac{\delta}{2^{i-1}}\right), Q\left(1 - \frac{\delta}{2^i}\right)\right]\right\}, i = 1, 2, \ldots$, and denote the probability measure space as $(\Omega, \mathcal{F}, \mu)$. Then

$$\begin{aligned}
\mathbb{E}[|X| \cdot \mathbf{1}_{\mathcal{E}^c}] \ &=\ \int_\Omega |X| \cdot \mathbf{1}_{\mathcal{E}^c} d\mu(\omega) \\
&\leq\ \int_\Omega |X| \cdot \mathbf{1}_{|X| \geq Q(1-\delta)} d\mu(\omega) \\
&\leq\ \sum_{i=1}^{\infty} \int_\Omega |X| \cdot \mathbf{1}_{A_i} d\mu(\omega) \\
&\leq\ \sum_{i=1}^{\infty} \frac{\delta}{2^i} Q\left(1 - \frac{\delta}{2^i}\right) \\
&\overset{\text{(i)}}{\leq}\ \delta\beta + \sum_{i=1}^{\infty} \frac{\delta}{2^i} K\log\left(\frac{2^{i+1}}{\delta}\right) \\
&\leq\ \delta\beta + \delta K\left(\sum_{i=1}^{\infty} \frac{1}{2^i}((i+1)\log(2) + \log(1/\delta))\right) \\
&\overset{\text{(ii)}}{\leq}\ \delta\beta + \delta K(3\log(2) + \log(1/\delta)),
\end{aligned}$$

where (i) is by eq. (4) and (ii) is by the standard scalar inequality (Lemma A.2). The above together with eq. (5) yields the desired result. □

A.3 THE GENERALIZED FREEDMAN INEQUALITY

We restate the generalized Freedman inequality and its corollary from Harvey et al.(Harvey et al., 2019).

Lemma A.9 (Generalized Freedman Inequality, Harvey et al., 2019, Theorem 3.2)**.** *Let $\{d_i, \mathcal{F}_i\}_{i=1}^T$ be a martingale difference sequence. Suppose $v_{i-1} \geq 0, \forall i \in [T]$ are $\mathcal{F}_{i-1}$-measurable random variables such that $\mathbb{E}[\exp(\lambda d_i) \mid \mathcal{F}_{i-1}] \leq \exp(\frac{\lambda^2}{2} v_{i-1})$ for all $i \in [T], \lambda > 0$. Let $S_t = \sum_{i=1}^t d_i$ and $V_t = \sum_{i=1}^t v_{i-1}$. Let $\alpha_i \geq 0$ and $\alpha = \max_{i \in [T]} \alpha_i$. Then*

$$\Pr\left[\bigcup_{t=1}^T \left\{S_t \geq x \quad \textit{and} \quad V_t \leq \sum_{i=1}^t \alpha_i d_i + \beta\right\}\right] \;\leq\; \exp\left(-\frac{x}{4\alpha + 8\beta/x}\right) \quad \forall x, \beta > 0.$$

The following lemma is an immediate consequence from the generalized Freedman inequality.

Lemma A.10. *Let $\{d_i, \mathcal{F}_i\}_{i=1}^T$ be a martingale difference sequence. Suppose $v_{i-1} \geq 0, \forall i \in [T]$ are $\mathcal{F}_{i-1}$-measurable random variables such that $\mathbb{E}[\exp(\lambda d_i) \mid \mathcal{F}_{i-1}] \leq \exp(\frac{\lambda^2}{2} v_{i-1})$ for all $i \in [T], \lambda > 0$. Let $S_t = \sum_{i=1}^t d_i$ and $V_t = \sum_{i=1}^t v_{i-1}$. Let $\delta \in (0,1)$ and suppose there are positive values $R(\delta) > 0$ and non-negative values $\{\alpha_i^{(t)}, i = 1, 2, \ldots, T, t = 1, 2, \ldots, T\}$ such that*

$$\Pr\left[\bigcap_{t=1}^T \left\{V_t \leq \sum_{i=1}^t \alpha_i^{(t)} d_i + R(\delta)\right\}\right] \;\geq\; 1 - \delta.$$

Let $\alpha = \max_{i \in [T], t \in [T]} \alpha_i^{(t)}$. Then

$$\Pr\left[\bigcup_{t=1}^T \{S_t \geq x\}\right] \;\leq\; \delta + T \exp\left(-\frac{x}{4\alpha + 8R(\delta)/x}\right) \quad \forall x, \beta > 0.$$

Proof. Given $\delta \in (0,1), x \in \mathbb{R}$. Define the events $A_t := \{S_t \geq x\}$ and $B_t := \left\{V_t \leq \sum_{i=1}^t \alpha_i^{(t)} d_i + R(\delta)\right\}$. Then

$$\begin{aligned}
\Pr\left[\bigcup_{t=1}^T \{S_t \geq x\}\right] &= \Pr\left[\bigcup_{t=1}^T A_t\right] \\
&= \Pr\left[\left(\bigcup_{t=1}^T A_t\right) \bigcap \left(\bigcap_{t=1}^T B_t\right)\right] + \Pr\left[\left(\bigcup_{t=1}^T A_t\right) \bigcap \left(\bigcap_{t=1}^T B_t\right)^c\right] \\
&\overset{\text{(i)}}{\leq} \Pr\left[\left(\bigcup_{t=1}^T A_t\right) \bigcap \left(\bigcap_{t=1}^T B_t\right)\right] + \delta \\
&\leq \Pr\left[\bigcup_{t=1}^T \left(A_t \bigcap \left(\bigcap_{i=1}^T B_i\right)\right)\right] + \delta \\
&\leq \Pr\left[\bigcup_{t=1}^T \left(A_t \bigcap B_t\right)\right] + \delta \\
&\overset{\text{(ii)}}{\leq} \sum_{t=1}^T \Pr\left[A_t \bigcap B_t\right] + \delta \\
&\overset{\text{(iii)}}{\leq} \delta + T \exp\left(-\frac{x}{4\alpha + 8R(\delta)/x}\right) \qquad \forall x, \beta > 0,
\end{aligned}$$

where (i) is by the assumption $\Pr\left[\cap_{t=1}^T B_t\right] \geq 1 - \delta$, (ii) is by union bound (note that we can not directly use Lemma A.9 since V_t relies on $\alpha_1^{(t)}, \ldots, \alpha_t^{(t)}$ instead of $\alpha_1, \ldots, \alpha_t$), (iii) is by applying Lemma A.9 for T times. □

Algorithm 4 Stochastic gradient descent

1: **Input:** total iterations $T \in \mathbb{N}$, learning rate $\eta > 0$, initial iterate $w^{(0)}$.
2: **for** $t \leftarrow 0, \ldots, T-1$ **do**
3: $\quad w^{(t+1)} = w^{(t)} - \eta \left(\nabla f(w^{(t)}) + \xi^{(t)}\right)$;
4: **end for**
5: **Return:** $w^{(i)}$ where i uniform randomly sampled from $\{0, 1 \ldots, T\}$.

A.4 Other lemmas

Recall Lemma 3.2.

Proof of Lemma 3.2. By the smoothness of h, we have

$$h(v) \;\le\; h(u) + \langle \nabla h(u), v-u\rangle + \frac{L}{2}\|v-u\|^2 \qquad u, v \in \mathbb{R}^d.$$

Making the identification that $v = u - \nabla h(u)/L$. We obtain

$$\begin{aligned} & h(u - \nabla h(u)/L) \;\le\; h(u) - \frac{1}{2L}\|\nabla h(u)\|^2 \\ \implies & \inf_{v\in\mathbb{R}^d} h(v) \;\le\; h(u - \nabla h(u)/L) \;\le\; h(u) - \frac{1}{2L}\|\nabla h(u)\|^2 \\ \implies & \|\nabla h(u)\|^2 \le 2L\left(h(u) - \inf_{v\in\mathbb{R}^d} h(v)\right) \qquad \forall u \in \mathbb{R}^d. \end{aligned}$$

□

Next, we review the standard convergence rate of SGD algorithm in the following lemma.

Lemma A.11 (Ghadimi & Lan, 2013, Theorem 2.1)**.** *Denote $\hat{w}$ as the output of Algorithm 4. Assume that f_i's are L-smooth for some $L > 0$, $\eta \le 1/L$ and there exist $\sigma \ge 0$ such that $\mathbb{E}[\|\xi^{(t)}\|^2] \le \sigma^2$ for all $t \in \mathbb{N}$.*

- *If f_i's are further convex, then*

$$\mathbb{E}[f(\hat{w}) - f^*] \;\le\; \frac{\inf_{w\in\mathcal{W}^*}\|w^{(0)} - w\|^2}{(T+1)\eta} + \sigma^2\eta,$$

  where $\mathcal{W}^*$ is the set of solutions.

- *If f_i's are not necessarily convex, then*

$$\mathbb{E}\left[\|\nabla f(\hat{w})\|^2\right] \;\le\; \frac{2\left(f(w^{(0)}) - f^*\right)}{(T+1)\eta} + L\sigma^2\eta.$$

The following lemma connects the GC parameters of f_i's to the GC parameters of f.

Lemma A.12. *Assume that f_i's are (β_1, β_2)-GC for all $i \in [n]$. Then $f := \frac{1}{n}\sum_{i=1}^n f_i$ is $(\beta_1, \beta_2 + \Gamma\beta_1)$-GC, where $\Gamma = \frac{1}{n}\sum_{i=1}^n (f^* - f_i^*)$.*

Proof.

$$\begin{aligned} \|\nabla f(w)\|^2 &\overset{\text{(i)}}{\le} \frac{1}{n}\sum_{i=1}^n \|\nabla f_i(w)\|^2 \\ &\overset{\text{(ii)}}{\le} \frac{1}{n}\sum_{i=1}^n \beta_1(f_i(w) - f_i^*) + \beta_2 \\ &\le \beta_1\left(f(w) - f^*\right) + \Gamma\beta_1 + \beta_2, \end{aligned}$$

where (i) is by the convexity of $\|\cdot\|^2$ and (ii) is by the assumption that f_i's are (β_1, β_2)-GC. □

APPENDIX B MISSING PROPOSITIONS AND THEOREMS

The convergence of DP-SGD-GC with Assumption 4.1 is given in Section 4. Here we present the propositions and theorems without Assumption 4.1.

Proposition B.1 (Convergence on conditional expectation)**.** *Denote w_{priv} as the output of Algorithm 1 and define $\mathcal{E} := \{\|f_i(w^{(t)})\| \le C \ \forall i \in [n], t \in \{0, 1 \ldots, T\}\}$ as the event of no clipping happens during the training of Algorithm 1. Let $D_f := f(w^{(0)}) - f^*$. Given any $\epsilon > 0$ and $\delta, \delta' \in (0, 1)$.*

- *Assume f_i's are convex and L-smooth for some $L > 0$. set $T > \epsilon/c_1$,*

$$\sigma = \frac{c_2 B\sqrt{T\log(1/\delta)}}{n\epsilon},\ C = \sqrt{c_3 n + c_4 n \log(T/\delta')},\ \eta = \min\left\{\frac{1}{2L}, \frac{c_5 B}{\sqrt{C^2 d\sigma^2 T}}\right\}, \tag{6}$$

  where c_1, c_2, c_5 are some absolute constants and c_3, c_4 are constants that depend on L and D_f. We have that Algorithm 1 is (ϵ, δ)-DP, $\Pr[\mathcal{E}] \ge 1 - \delta'$, and

$$\mathbb{E}\left[f(w_{\text{priv}}) - f^* \mid \mathcal{E}\right] \ \le\ \mathcal{O}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\left(\sqrt{\log(1/\delta')} + \sqrt{\log(T)}\right)\sqrt{d\log(1/\delta)}}{\sqrt{n}\epsilon}\right).$$

- *Assume f_i's are L-smooth for some $L > 0$. Setting $\epsilon, \sigma, C, \eta$ as in eq. (6). It holds that Algorithm 1 is (ϵ, δ)-DP, $\Pr[\mathcal{E}] \ge 1 - \delta'$, and*

$$\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2 \mid \mathcal{E}\right] \ \le\ \mathcal{O}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\left(\sqrt{\log(1/\delta')} + \sqrt{\log(T)}\right)\sqrt{d\log(1/\delta)}}{\sqrt{n}\epsilon}\right).$$

Theorem B.2 (Convergence on expectation)**.** *Denote w_{priv} as the output of Algorithm 1. Let $D_f := f(w^{(0)}) - f^*$. Given any $\epsilon > 0$ and $\delta \in (0, 1)$. Setting T, σ, η in the same way as eq. (6) and let $C = \sqrt{c_3 n + c_4 n \log(T)}$, where c_1, c_2, c_5 are some absolute constants and c_3, c_4 are constants that depend on L, D_f.*

- *Assume f_i's are convex and L-smooth for some $L > 0$. Then Algorithm 1 is (ϵ, δ)-DP and*

$$\mathbb{E}\left[f(w_{\text{priv}}) - f^*\right] \ \le\ \widetilde{\mathcal{O}}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{\sqrt{n}\epsilon}\right).$$

  Consequently, we have $\mathbb{E}\left[f(w_{\text{priv}}) - f^*\right] = \widetilde{\mathcal{O}}\left(d^{1/2}(n\epsilon)^{-1}\right)$ by setting $T = \Theta\left(n\epsilon^2 d^{-1}\right)$.

- *Assume f_i's are L-smooth for some $L > 0$. Then Algorithm 1 is (ϵ, δ)-DP and*

$$\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2\right] \ \le\ \widetilde{\mathcal{O}}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{\sqrt{n}\epsilon}\right).$$

  Consequently, we have $\mathbb{E}\left[\|\nabla f(w_{\text{priv}})\|^2\right] = \widetilde{\mathcal{O}}\left(d^{1/2}(n\epsilon)^{-1}\right)$ by setting $T = \Theta\left(n\epsilon^2 d^{-1}\right)$.

APPENDIX C PROOFS FOR SECTION 4

C.1 PROOF OF PROPOSITION 4.2

Proof. We begin with the smoothness of f,

$$\begin{aligned}
f(w^{(t+1)}) \ &\le\ f(w^{(t)}) + \langle \nabla f(w^{(t)}), w^{(t+1)} - w^{(t)}\rangle + \frac{L}{2}\|w^{(t+1)} - w^{(t)}\|^2 \\
&=\ f(w^{(t)}) - \eta\|\nabla f(w^{(t)})\|^2 - \eta\langle \nabla f(w^{(t)}), \zeta^{(t)}\rangle + \frac{L\eta^2}{2}\|\nabla f(w^{(t)}) + \zeta^{(t)}\|^2 \\
&=\ f(w^{(t)}) - \underbrace{\left(\eta - \frac{L\eta^2}{2}\right)}_{\ge 0}\|\nabla f(w^{(t)})\|^2 + \underbrace{\frac{L\eta^2}{2}\|\zeta^{(t)}\|^2}_{z_t} + \underbrace{(L\eta^2 - \eta)\langle \nabla f(w^{(t)}), \zeta^{(t)}\rangle}_{u_t}.
\end{aligned} \tag{7}$$

Let

$$Z_t \coloneqq \sum_{i=0}^{t} z_t, \qquad U_t \coloneqq \sum_{i=0}^{t} u_t.$$

Recursively apply eq. (7) gives

$$f(w^{(t)}) \;\le\; f(w^{(0)}) + Z_{T-1} + U_{t-1}, \qquad \forall t = 0, 1, \ldots, T. \tag{8}$$

All we need is to show that Z_{T-1} and U_{t-1} are bounded above with high probability. First, we bound Z_{T-1}. Notice that $\zeta^{(t)}$'s are sub-Gaussian variables, therefore Z_{T-1} is sub-exponential. We apply Lemma A.6, there exist some absolute constant $c_1 > 0$ such that for any $\delta' \in (0, 0.5)$

$$\begin{aligned} Z_{T-1} \;&\le\; \frac{L\eta^2}{2}\left(c_1 T\tilde{\sigma}^2 \log(1/\delta')\right) \\ &\le\; \frac{c_1 L \log(1/\delta')}{2} \qquad \text{(By the definition of } \eta) \end{aligned} \tag{9}$$

with probability at least $1 - \delta'$.

Next we bound the term U_{t-1} by the generalized Freedman inequality (Lemma A.9). Denote $\mathcal{F}_{t-1}$ to be the σ-Algebra generated from $\{w^{(1)}, \ldots, w^{(t)}\}$. Then by the definition of $\zeta^{(t)}$, we have

$$\mathbb{E}\left[\exp\left(u_t^2/((L\eta^2 - \eta)^2\tilde{\sigma}^2\|\nabla f(w^{(t)})\|^2)\right) \;\middle|\; \mathcal{F}_{t-1}\right] \;\le\; e.$$

By the properties of sub-Gaussian variable (Lemma A.3), there exist some absolute constant $c_2 > 0$ such that

$$\mathbb{E}\left[\exp\left(\lambda u_t\right) \mid \mathcal{F}_{t-1}\right] \;\le\; \exp\left(\frac{\lambda^2}{2} c_2 (L\eta^2 - \eta)^2 \tilde{\sigma}^2 \|\nabla f(w^{(t)})\|^2\right) \quad \forall \lambda \in \mathbb{R}.$$

Let $v_{t-1} \coloneqq c_2(L\eta^2 - \eta)^2\tilde{\sigma}^2\|\nabla f(w^{(t)})\|^2$. Then

$$\begin{aligned} v_{t-1} \;&\le\; 2c_2 L\eta^2\tilde{\sigma}^2(f(w^{(t)}) - f^*) && \text{(By the growth condition)} \\ &\le\; \frac{2c_2 L}{T}(f(w^{(t)}) - f^*) && \text{(By } \eta \le 1/(\tilde{\sigma}\sqrt{T})) \quad . \end{aligned}$$

Furthermore, for any $t = 0, 1, \ldots, T-1$,

$$\begin{aligned} V_t \;&\coloneqq\; \sum_{i=0}^{t} v_{i-1} \\ &\le\; c_2 \sum_{i=0}^{t}\left(\frac{2L}{T}(f(w^{(i)}) - f^*)\right) \\ &\le\; c_2 \sum_{i=0}^{t} \frac{2L}{T}\left(f(w^{(0)}) - f^* + Z_{T-1} + U_{i-1}\right) && \text{(By eq. (8))} \\ &\le\; 2c_2 L(f(w^{(0)}) - f^* + Z_{T-1}) + \frac{2c_2 L}{T}\sum_{i=0}^{t}\sum_{j=0}^{i-1} u_j \\ &\le\; 2c_2 L(f(w^{(0)}) - f^* + Z_{T-1}) + \frac{2c_2 L}{T}\sum_{i=0}^{t}(t-i)u_i \\ &\le\; 2c_2 L(f(w^{(0)}) - f^* + Z_{T-1}) + 2c_2 L\sum_{i=0}^{t}\frac{t-i}{T}u_i. \end{aligned}$$

Combining with eq. (9), with probability $1 - \delta'$

$$V_t \;\le\; 2c_2 L\left(f(w^{(0)}) - f^* + c_1 L\log(1/\delta')/2\right) + 2c_2 L\sum_{i=0}^{t}\frac{t-i}{T}u_i \qquad \forall t = 0, 1, \ldots, T-1.$$

Now we are ready to apply Lemma A.10. Making the identification

$$d_i = u_i,\ \alpha_i^{(t)} = 2c_2 L \frac{t-i}{T},\ \alpha = 2c_2 L,\ R(\delta') = 2c_2 L\left(f(w^{(0)}) - f^* + c_1 L \log(1/\delta')/2\right)$$

and apply Lemma A.10 gives

$$\Pr\left[\bigcup_{t=0}^{T-1}\{U_t \geq x\}\right] \leq \delta' + T\exp\left(-\frac{x}{4\alpha + 8R(\delta')/x}\right) \qquad \forall x > 0.$$

It is easy to verify that with the choice $x = \max\left\{4\sqrt{R(\delta')\log(T/\delta')}, 8\alpha\log(T/\delta')\right\}$, we have

$$\Pr\left[\bigcup_{t=0}^{T-1}\{U_t \geq x\}\right] \leq 2\delta'. \tag{10}$$

Therefore, with probability at least $1 - 2\delta'$,

$$\begin{aligned}
&\max_{t=0,1,\ldots,T} f(w^{(t)}) - f^* \\
\overset{\text{(i)}}{\leq}\ & f(w^{(0)}) - f^* + c_1 L\log(1/\delta')/2 \\
&+ \max\left\{4\sqrt{\left(2c_2 L(f(w^{(0)}) - f^* + c_1 L\log(1/\delta')/2)\right)\log(T/\delta')}, 16c_2 L\log(T/\delta')\right\} \\
\overset{\text{(ii)}}{\leq}\ & \left(\sqrt{f(w^{(0)}) - f^* + c_1 L\log(1/\delta')/2} + 4\sqrt{c_2 L\log(T/\delta')}\right)^2 \\
\overset{\text{(iii)}}{\leq}\ & 2\left(f(w^{(0)}) - f^* + c_1 L\log(1/\delta')/2\right) + 32c_2 L\log(T/\delta'),
\end{aligned}$$

where (i) is by eq. (8) and eq. (10), (ii) is by the fact that $\max\{a, b\} \leq a + b$ and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b \geq 0$, (iii) is by Lemma A.1. Substitute δ' with $\delta/2$, we obtain the desired result. □

C.2 Proof of Proposition 4.3

Proof. By the assumption that f_i's are L-smooth, we know that f is also L-smooth. Therefore f is $(2L, 0)$-GC (By Lemma 3.2), e.g.,

$$\|\nabla f(w^{(t)})\| \leq \sqrt{2L\left(f(w^{(t)}) - f^*\right)} \qquad \forall t = 0, 1, \ldots, T.$$

Apply Proposition 4.2. We obtain that

$$\max_{t=0,1,\ldots,T}\|\nabla f(w^{(t)})\| \leq \sqrt{4L\left(f(w^{(0)}) - f^* + c_1 L\log(2/\delta)/2\right) + 64c_2 L^2\log(2T/\delta)} \tag{11}$$

with probability at least $1 - \delta$ for some absolute positive constants c_1, c_2.

The remaining is to bound the individual gradient norm $\|\nabla f_i(w^{(t)})\|$. When Assumption 4.1 holds, we have that

$$\max_{i\in[n],t=0,1,\ldots,T}\|\nabla f_i(w^{(t)})\| \leq \max_{t\in\{0,1\ldots,T\}}\|\nabla f(w^{(t)})\| + \max_{i\in[n],t=0,1,\ldots,T}\|\nabla f_i(w^{(t)}) - f(w^{(t)})\|.$$

By Assumption 4.1, we know that $\|\nabla f_i(w) - \nabla f(w)\|$ is sub-Gaussian for all $w \in \mathbb{R}^d$. Therefore we can apply Lemma A.7 to bound the maximum of sub-Gaussian variables, which gives

$$\max_{i\in[n],t=0,1,\ldots,T}\|\nabla f_i(w^{(t)}) - f(w^{(t)})\| < c_3\rho\sqrt{\log(n(T+1))} + c_3\sqrt{\rho\log(1/\delta)} \tag{12}$$

for some absolute positive constant $c_3 > 0$ and any $\delta \in (0, 1)$.

Combining eq. (11) and eq. (12). We obtain that

$$\begin{aligned}
\max_{i\in[n],t=0,1,\ldots,T}\|\nabla f_i(w^{(t)})\| \leq\ & \sqrt{4L\left(f(w^{(0)}) - f^* + c_1 L\log(2/\delta)/2\right) + 64c_2 L^2\log(2T/\delta)} \\
&+ c_3\rho\sqrt{\log(n(T+1))} + c_3\sqrt{\rho\log(1/\delta)}
\end{aligned}$$

with probability at least $1-2\delta$, which yields the desired result eq. (2) under Assumption 4.1.

When Assumption 4.1 does not hold.

$$\begin{aligned}
& \max_{i\in[n],t=0,1,\ldots,T} \|\nabla f_i(w^{(t)})\| \\
\leq\ & \max_{i\in[n],t=0,1,\ldots,T} \sqrt{\sum_{i=1}^{n} \|f_i(w^{(t)})\|^2} \\
\leq\ & \max_{i\in[n],t=0,1,\ldots,T} \sqrt{2nL(f(w^{(t)})-f^*)} \\
\leq\ & \sqrt{2nL\left(\left(f(w^{(0)})-f^*+c_1L\log(2/\delta)/2\right)+32c_2L\log(2T/\delta)\right)},
\end{aligned}$$

where the last line is by Proposition 4.2. This completes the proof for eq. (1). □

C.3 Proof of Proposition 4.4

Proof. We first consider the case that f_i's are convex and smooth. By Lemma 3.2, we know that f_i's are $(2L,0)$-GC in this scenario. Setting

$$\begin{aligned}
C &= \sqrt{4L\left(f(w^{(0)})-f^*+c_1L\log(2/\delta')/2+16c_2L\log(2T/\delta')\right)} \\
&\quad + c_3\rho\sqrt{\log(n(T+1))}+c_3\sqrt{\rho\log(1/\delta')}, \\
\sigma &= c_4\frac{q\sqrt{T\log(1/\delta)}}{\epsilon},
\end{aligned} \tag{13}$$

where c_1, c_2, c_3 correspond to the absolute positive constants that appeared in the proof of Proposition 4.2, c_4 is some positive constant that appeared in Theorem 3.4.

Next we need to apply Proposition 4.3. Denote $\mathcal{B}_t$ as the batch sampled at the t-th iteration and let

$$\zeta^{(t)} := \frac{1}{B}\left(\sum_{i\in\mathcal{B}_t}\nabla f_i(w^{(t)})+\xi^{(t)}\right)-\nabla f(w^{(t)}).$$

By the definition of $\xi^{(t)}$ and Assumption 4.1, $\zeta^{(t)}$'s satisfy

$$\mathbb{E}\left[\|\zeta^{(t)}\|^2/\left(\frac{c_5\rho^2}{B}+\frac{c_5C^2d\sigma^2}{B^2}\right)\right]\leq e,$$

where c_5 is some positive absolute constant. Let

$$\tilde{\sigma}^2 = c_5\left(\frac{\rho^2}{B}+\frac{C^2d\sigma^2}{B^2}\right), \quad \eta=\min\left\{\frac{1}{2L},\frac{1}{\tilde{\sigma}\sqrt{T}}\right\}. \tag{14}$$

Then Proposition 4.3 tells us that Algorithm 2 with the above setup of $\tilde{\sigma}$ and η will produce iterates such that

$$\|\nabla f_i(w^{(t)})\| \leq C, \qquad \forall i\in[n], t=0,1,\ldots,T,$$

with probability $1-\delta'$. Therefore Algorithm 1 with the above parameter setup (eq. (13) and η defined as in eq. (14)) will be equivalent to Algorithm 2 (with $\tilde{\sigma},\eta$ defined as in eq. (14)) with probability at least $1-\delta'$. Thus it is straightforward that $\Pr[\mathcal{E}]\geq 1-\delta'$ (recall that $\mathcal{E}$ is the event of no gradient clipping happens) .

Conditioning on the event $\mathcal{E}$, Algorithm 1 reduces to Algorithm 2. Notice that the term $\mathbb{E}[\|\xi^{(t)}\|^2]$ in Lemma A.11 is bounded by $\tilde{\sigma}^2$, then the classic convergence rate of SGD (Lemma A.11) gives that

$$
\begin{aligned}
& \mathbb{E}\left[f(w_{\text{priv}}) - f^* \mid \mathcal{E}\right] \\
\leq\ & \frac{D_w^2}{(T+1)\eta} + \tilde{\sigma}^2\eta \qquad \text{(by Lemma A.11)} \\
\overset{\text{(i)}}{\leq}\ & \frac{2LD_w^2}{T} + \frac{D_w^2\tilde{\sigma}}{\sqrt{T}} + \frac{\tilde{\sigma}}{\sqrt{T}} \\
\overset{\text{(ii)}}{\leq}\ & \mathcal{O}\left(\frac{2LD_w^2}{T} + \frac{D_w^2\rho}{\sqrt{BT}} + \frac{D_w^2 C\sigma\sqrt{d}}{B\sqrt{T}} + \frac{\rho}{\sqrt{TB}} + \frac{C\sigma\sqrt{d}}{B\sqrt{T}}\right) \\
\overset{\text{(iii)}}{\leq}\ & \mathcal{O}\left(\frac{2LD_w^2}{T} + \frac{D_w^2\rho}{\sqrt{BT}} + \frac{D_w^2 Cq\sqrt{d\log(1/\delta)}}{B\epsilon} + \frac{\rho}{\sqrt{TB}} + \frac{Cq\sqrt{d\log(1/\delta)}}{B\epsilon}\right) \\
\overset{\text{(iv)}}{=}\ & \mathcal{O}\left(\frac{2LD_w^2}{T} + \frac{D_w^2\rho}{\sqrt{BT}} + \frac{D_w^2 C\sqrt{d\log(1/\delta)}}{n\epsilon} + \frac{\rho}{\sqrt{TB}} + \frac{C\sqrt{d\log(1/\delta)}}{n\epsilon}\right) \\
\overset{\text{(v)}}{=}\ & \mathcal{O}\left(\frac{D_w^2}{T} + \frac{D_w^2\rho}{\sqrt{BT}} + \frac{D_w^2(\log(1/\delta') + \log(n) + \log(T))\sqrt{d\log(1/\delta)}}{n\epsilon}\right),
\end{aligned}
$$

where (i) is by the definition of η, (ii) is by the definition of $\tilde{\sigma}$, (iii) is by the definition of σ (eq. (13)), (iv) is by the definition of $q = B/n$, (v) is by noticing $C = \mathcal{O}(\log(1/\delta') + \log(n) + \log(T))$ from eq. (13).

When f_i's are L-smooth but not necessarily convex. We set the parameters the same as for the convex case. Apply the convergence rate of SGD for smooth but not necessarily convex functions (Lemma A.11). Following the same proof template as for the convex case, can we obtain the desired result. □

C.4 Proof of Proposition B.1

Proof. The proof is almost idential as the proof of Proposition 4.4, the difference is that we need to set

$$
\begin{aligned}
C &= \sqrt{4nL\left(f(w^{(0)}) - f^* + c_1 L\log(2/\delta')/2 + 16c_2 L\log(2T/\delta')\right)} \\
\tilde{\sigma} &= c'\left(\frac{C^2}{B^2} + \frac{C^2 d\sigma^2}{B^2}\right)
\end{aligned}
$$

where c' is some positive absolute constant. Note that the clipping threshold C now involves an additional $\sqrt{n}$ term. The following the same proof template as Proposition 4.4, we get obtain

$$
\mathbb{E}\left[f(w_{\text{priv}}) - f^* \mid \mathcal{E}\right] \ \leq\ \mathcal{O}\left(\frac{D_w^2}{T} + \frac{D_w^2(\log(1/\delta') + \log(T))\sqrt{d\log(1/\delta)}}{\sqrt{n}\epsilon}\right),
$$

which yields the desired result. Follow the same proof template, the above bound also holds for $\mathbb{E}[\|f(w_{\text{priv}})\|^2]$ when f is nonconvex. □

C.5 Proof of Theorem 4.5

Proof. The outline of the proof is summarized as follows.

- First we show that the excess empirical risk e.g., $f(w_{\text{priv}}) - f^*$ or gradient norm square e.g.,$\|\nabla f(w_{\text{priv}})\|^2$ are sub-exponential random variables. We also derive a pessimistic upper bound of their sub-exponential parameters (Lemma C.1)[4].

[4]We do not try to optimize the bound in Lemma C.1 as we show that a loose bound is enough for our purpose. The bound in Lemma C.1 should be improvable.

- Next, we develop a technical tool to convert conditional expected error bound (the conditioning event happens with high probability) to expected error bound for sub-exponential random variables (Lemma A.8).
- Finally, by tuning δ' (the probability that gradient clipping happens during training; see Proposition 4.4), Lemma A.8 together with Lemma C.1 and Proposition 4.4 yield the desired result.

Now we proceed to the detailed proof. Consider the case that f_i's are convex and L-smooth. Let T, σ, C, η satisfy *eq.* (3). We apply Lemma A.8 by making the identification

$$\begin{aligned} X &\coloneqq f(w_{\text{priv}}) - f^*; \\ \alpha &\coloneqq \mathcal{O}\left(\frac{D_w^2}{T} + \frac{D_w^2}{\sqrt{BT}} + \frac{D_w^2\left(\sqrt{\log(1/\delta')} + \sqrt{\log(Tn)}\right)\sqrt{d\log(1/\delta)}}{n\epsilon}\right) \\ \beta &\coloneqq 2(f(w^{(0)}) - f^*) + \frac{T^2C^2}{L} \\ K &\coloneqq \frac{T^2C^2d\sigma^2}{c_1LB^2}, \end{aligned}$$

where the choice of α is by Proposition 4.4 and the choice of β and K is by Lemma C.1. Then Lemma A.8 gives that

$$\begin{aligned} \mathbb{E}[|X|] &= \alpha + \delta'\beta + \delta'\log(8/\delta')K \\ &\overset{\text{(i)}}{=} \widetilde{\mathcal{O}}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{n\epsilon}\right), \end{aligned}$$

where (i) is true by setting

$$\delta' = \min\left\{\beta^{-1}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{n\epsilon}\right), K^{-1}\left(\frac{1}{T} + \frac{1}{\sqrt{BT}} + \frac{\sqrt{d}}{n\epsilon}\right)\right\}.$$

Note that the above proof template works as long as we can construct β, K that polynomially depends on T, n, d, ϵ — we can tune δ' to convert all polynomial terms in β and K to polylogarithm terms by Lemma A.8. We find Lemma A.8 a quite convenient technical tool for our analysis and we are not aware if this result exist in the literature.

When f_i's are smooth but not necessarily convex. The proof follows exactly the same as the convex case (just need to let $X \coloneqq \|\nabla f(w_{\text{priv}})\|^2, \beta \coloneqq 4L(f(w^{(0)}) - f^*) + 2T^2C^2, K = \frac{2T^2C^2d\sigma^2}{c_1B^2}$ and apply eq. (16) instead of eq. (15)). We omit the details to avoid tedious repetition. □

Lemma C.1 (A pessimistic bound for DP-SGD-GC)**.** *Assume that f_i's are L-smooth. Let w_{priv} be the output from Algorithm 1 with $\eta \le 1/L$. Then*

$$\Pr\left[f(w_{\text{priv}}) - f^* \ge 2(f(w^{(0)}) - f^*) + \frac{T^2C^2}{L} + t\right] \le 2\exp\left(-\frac{tc_1LB^2}{T^2C^2d\sigma^2}\right) \quad \forall t \ge 0. \tag{15}$$

Furthermore,

$$\Pr\left[\|\nabla f(w_{\text{priv}})\|^2 \ge 4L(f(w^{(0)}) - f^*) + 2T^2C^2 + t\right] \le 2\exp\left(-\frac{tc_1B^2}{2T^2C^2d\sigma^2}\right) \quad \forall t \ge 0. \tag{16}$$

Proof. We begin with the smoothness of f, for any $t \in \{0, 1, \ldots, T\}$

$$
\begin{aligned}
f(w^{(t)}) &\leq f(w^{(0)}) + \langle \nabla f(w^{(0)}), w^{(t)} - w^{(0)} \rangle + \frac{L}{2}\|w^{(t)} - w^{(0)}\|^2 \\
&\overset{\text{(i)}}{\leq} f(w^{(0)}) + \frac{1}{2L}\|\nabla f(w^{(0)})\|^2 + L\|w^{(t)} - w^{(0)}\|^2 \\
&\leq f(w^{(0)}) + \frac{1}{2L}\|\nabla f(w^{(0)})\|^2 + L\left\|\eta \sum_{i=0}^{t-1}\left(\frac{1}{B}\sum_{j\in\mathcal{B}_t}\tilde{g}_j^{(i)} + \frac{1}{B}\xi^{(i)}\right)\right\|^2 \\
&\overset{\text{(ii)}}{\leq} f(w^{(0)}) + \frac{1}{2L}\|\nabla f(w^{(0)})\|^2 + L\eta^2 T \sum_{i=0}^{T-1}\left\|\frac{1}{B}\sum_{j\in\mathcal{B}_t}\tilde{g}_j^{(i)} + \frac{1}{B}\xi^{(i)}\right\|^2 \\
&\overset{\text{(iii)}}{\leq} f(w^{(0)}) + \frac{1}{2L}\|\nabla f(w^{(0)})\|^2 + L\eta^2 T^2 C^2 + \frac{L\eta^2 T}{B^2}\sum_{i=0}^{T-1}\|\xi^{(i)}\|^2 \\
&\overset{\text{(iv)}}{\leq} f(w^{(0)}) + (f(w^{(0)}) - f^*) + L\eta^2 T^2 C^2 + \frac{L\eta^2 T}{B^2}\sum_{i=0}^{T-1}\|\xi^{(i)}\|^2 \\
&\overset{\text{(v)}}{\leq} f(w^{(0)}) + (f(w^{(0)}) - f^*) + \frac{T^2C^2}{L} + \underbrace{\frac{T}{LB^2}\sum_{i=0}^{T-1}\|\xi^{(i)}\|^2}_{Z_T},
\end{aligned} \tag{17}
$$

where (i) is by the fact that $\langle a, b\rangle \leq \frac{1}{2\lambda}\|a\|^2 + \frac{\lambda}{2}\|b\|^2 \ \forall \lambda > 0$, (ii) is by the convexity of $\|\cdot\|^2$, (iii) is because of $\|\tilde{g}_j^{(i)}\| \leq C$ due to gradient clipping, (iv) is by the growth condition, (v) is by the assumption on learning rate ($\eta \leq 1/L$).

By the definition of $\xi^{(i)}$, Z_T follows from the sub-exponential distribution. By Lemma A.3, there exist some absolute constant $c_1 > 0$ such that

$$\Pr\left[|Z_T| \geq t\right] \leq 2\exp\left(-\frac{tc_1LB^2}{T^2C^2d\sigma^2}\right).$$

Combining with eq. (17), we have that

$$\Pr\left[f(w_{\text{priv}}) - f^* \geq 2(f(w^{(0)}) - f^*) + \frac{T^2C^2}{L} + t\right] \ \leq \ 2\exp\left(-\frac{tc_1LB^2}{T^2C^2d\sigma^2}\right) \qquad \forall t \geq 0,$$

which finishes the proof for eq. (15).

By the growth condition, we further have that

$$\|\nabla f(w_{\text{priv}})\|^2 \leq 2L(f(w_{\text{priv}}) - f^*)$$

Therefore

$$\Pr\left[\|\nabla f(w_{\text{priv}})\|^2 \geq 4L(f(w^{(0)}) - f^*) + 2T^2C^2 + 2Lt\right] \ \leq \ 2\exp\left(-\frac{tc_1LB^2}{T^2C^2d\sigma^2}\right) \quad \forall t \geq 0,$$

which finishes the proof for eq. (16). □

C.6 Proof of Theorem 4.6

Proof. The proof follows the same proof template of Theorem 4.5, we only need to change the setting of C and $\tilde{\sigma}$. □

Appendix D Missing experiments on synthetic data

We conduct experiments on synthetic data in this section. The synthetic data has $10,000$ samples and each sample is generated from a 256-dimensional standard Gaussian distribution and a ground truth linear model. To simulate stochastic gradient with light-tailed noise, we perform linear regression and add standard Gaussian noise (light-tailed) for the true gradient. The experimental results are shown in Figure 4. The conclusion is the same as in Section 6: the evolution of clipping frequency has a decreasing trend and our value clipping technique can achieve similar accuracy as gradient clipping.

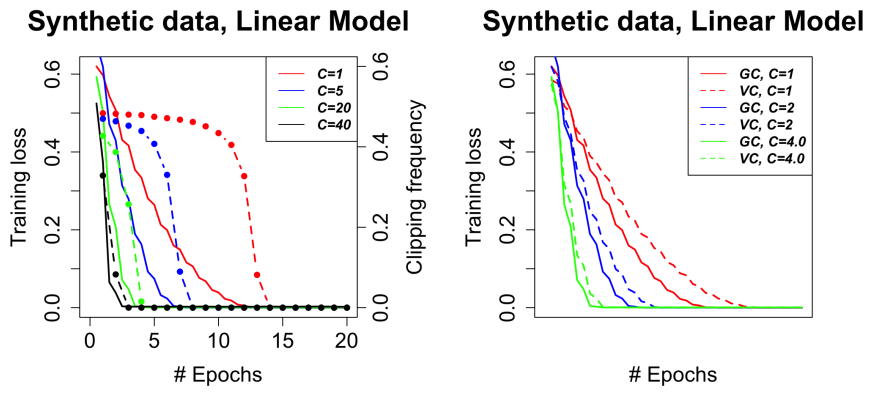

Figure 4: The evolution of clipping frequency and the comparison of value clipping and gradient clipping.

APPENDIX E MORE ON THE GROWTH CONDITION

We review some existing results and describe the growth condition for feed-forward neural networks with cross-entropy loss. First, we review the *generalized growth condition* (Fang et al., 2021, Proposition 4.1). Consider the objective

$$\min_{w\in\mathbb{R}^d} f(w) := \frac{1}{n}\sum_{i=1}^{n} f_i(w) := \frac{1}{n}\sum_{i=1}^{n} \ell(h_i(w)), \tag{18}$$

where $\ell : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a *nonnegative* 1-dimensional loss function that is *convex*, 1-*smooth*, and satisfies $\inf \ell = 0$. The functions h_i's are assumed to be β-Lipschitz continuous for some $\beta > 0$. Then f_i's and f are shown to satisfy the following growth condition.

Lemma E.1 (Fang et al., 2021, Proposition 4.1)**.** *For any $w \in \mathbb{R}^d$,*

$$\|\partial f_i(w)\|^2 \leq 2\beta^2 f_i(w) \qquad \forall i \in [n], \qquad \textit{and} \qquad \|\partial f(w)\|^2 \leq 2\beta^2 f(w),$$

where $\partial f(w)$ is the Clarke's generalized gradient (Clarke, 1981).

We refer readers to Fang et al. (Fang et al., 2021) for more details on the derivation of the above lemma. Note that f_i's and f in eq. (18) can be nonconvex, and the above lemma suggests that the growth condition holds for a certain class of nonconvex ERM.

E.1 GROWTH CONDITION FOR FEED-FORWARD NEURAL NETWORKS WITH CROSS-ENTROPY LOSS

We give a simple extension of Lemma E.1 to feed-forward neural networks with cross-entropy loss.

We denote the number of classes as K. For simplicity, we consider a feed-forward neural network with fixed width m. We denote the parameter of a H-layer feed-forward neural network as $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_H)$, where $\mathbf{W}_1 \in \mathbb{R}^{m\times d}, \mathbf{W}_H \in \mathbb{R}^{K\times m}$ and $\mathbf{W}_i \in \mathbb{R}^{m\times m}, i = 2, 3, \ldots, H-1$. We further denote $\|\mathbf{W}\|_F^2 = \sum_{i=1}^{H} \|\mathbf{W}_i\|_F^2$. With a little abuse of notation, we denote σ as the ReLU activation, e.g., $\sigma(x) = (\max\{x_1, 0\}, \ldots, \max\{x_m, 0\})$. Note that neural networks with the ReLU activation is not differentiable everywhere on its domain. We define the "gradient" of a ReLU neural networks in the same way as in (Allen-Zhu et al., 2019, Fact 2.6).

We consider a single training sample x (which is enough for our purpose) and define the architecture of the H-layer feed-forward neural network as

$$\begin{aligned} &h_0 = x, \\ &h_j = \sigma(z_j),\ z_j = \mathbf{W}_j h_{j-1} \qquad \forall j \in [H-1], \\ &\hat{y} = \mathbf{W}_H h_{H-1}, \end{aligned}$$

where h_j is the hidden variables of the j-th layer, $\hat{y}$ is the prediction produced by the network.

Without loss of generality, we assume that the label for our training sample is the first class. Then by the definition of the cross-entropy loss, we have that

$$f(\mathbf{W}) \;=\; -\log\left(\frac{\exp(\hat{y}_1)}{\sum_{i=1}^{K}\exp(\hat{y}_i)}\right) \;=\; \log\left(1+\sum_{i=2}^{K}\exp(\hat{y}_i-\hat{y}_1)\right).$$

Making the identification

$$\ell(\alpha)=\log(1+\exp(\alpha)) \qquad \text{and} \qquad g(\hat{y})=\log\left(\sum_{i=2}^{K}\exp(\hat{y}_i-\hat{y}_1)\right).$$

Then

$$f(\mathbf{W}) \;=\; \ell(g(\hat{y})).$$

It is obvious that ℓ satisfy the assumptions made by Lemma E.1. In order to apply Lemma E.1, all the remaining is to bound $\left\|\frac{\partial g(\hat{y})}{\partial \mathbf{W}}\right\|$. First, we notice that

$$\begin{aligned}
&\nabla_{\hat{y}} g(\hat{y}) = \left[-1, \frac{\exp(\hat{y}_2-\hat{y}_1)}{\sum_{i=2}^{K}\exp(\hat{y}_i-\hat{y}_1)}, \frac{\exp(\hat{y}_3-\hat{y}_1)}{\sum_{i=2}^{K}\exp(\hat{y}_i-\hat{y}_1)}, \ldots, \frac{\exp(\hat{y}_K-\hat{y}_1)}{\sum_{i=2}^{K}\exp(\hat{y}_i-\hat{y}_1)}\right] \\
\implies\; &\|\nabla_{\hat{y}} g(\hat{y})\|_1 = 2 \\
\implies\; &\|\nabla_{\hat{y}} g(\hat{y})\|_2 \le 2.
\end{aligned} \tag{19}$$

Then we need to bound $\left\|\frac{\partial \hat{y}}{\partial \mathbf{W}}\right\|$. Directly apply (Allen-Zhu et al., 2019, Fact 2.6) gives that

$$\left\|\frac{\partial \hat{y}}{\partial \mathbf{W}_i}\right\|_F \;\le\; \|h_{i-1}\|_2 \prod_{j=i+1}^{H}\|\mathbf{W}_j\|_2 \;\le\; \|x\|_2 \prod_{j=1,j\neq i}^{H}\|\mathbf{W}_j\|_2 \tag{20}$$

for all $i \in [H]$.

Combining Lemma E.1, eq. (19) and eq. (20), we obtain the following growth condition

$$\|\partial f(\mathbf{W})\|_F^2 \;=\; \sum_{i=1}^{H}\|\partial_{\mathbf{W}_i} f(\mathbf{W})\|_F^2 \;\le\; 8\|x\|_2^2 \sum_{i=1}^{H}\prod_{j=1,j\neq i}^{H}\|\mathbf{W}_j\|_2^2 f(\mathbf{W}).$$

Furthermore, by $\|\nabla_{\hat{y}} g(\hat{y})\|_2 \le 2$ and (Allen-Zhu et al., 2019, Fact 2.6). We also have

$$\|\partial f(\mathbf{W})\|_F^2 \;=\; \sum_{i=1}^{H}\|\partial_{\mathbf{W}_i} f(\mathbf{W})\|_F^2 \;\le\; 4\|x\|_2^2 \sum_{i=1}^{H}\prod_{j=1,j\neq i}^{H}\|\mathbf{W}_j\|_2^2.$$

To sum up, we obtain

$$\|\partial f(\mathbf{W})\|_F^2 \;=\; \sum_{i=1}^{H}\|\partial_{\mathbf{W}_i} f(\mathbf{W})\|_F^2 \;\le\; 4\|x\|_2^2 \sum_{i=1}^{H}\prod_{j=1,j\neq i}^{H}\|\mathbf{W}_j\|_2^2 \min\{1, 2f(\mathbf{W})\}.$$

To access the GC-parameters, we need to compute $\|\mathbf{W}_j\|_2, j \in [H]$ in each iteration, the overhead can be amortized by increasing batch size.