RESNET AFTER ALL: NEURAL ODES AND THEIR NUMERICAL SOLUTION

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ABSTRACT

A key appeal of the recently proposed Neural Ordinary Differential Equation (ODE) framework is that it seems to provide a continuous-time extension of discrete residual neural networks. As we show herein, though, trained Neural ODE models actually depend on the specific numerical method used during training. If the trained model is supposed to be a flow generated from an ODE, it should be possible to choose another numerical solver with equal or smaller numerical error without loss of performance. We observe that if training relies on a solver with overly coarse discretization, then testing with another solver of equal or smaller numerical error results in a sharp drop in accuracy. In such cases, the combination of vector field and numerical method cannot be interpreted as a flow generated from an ODE, which arguably poses a fatal breakdown of the Neural ODE concept. We observe, however, that there exists a critical step size beyond which the training yields a valid ODE vector field. We propose a method that monitors the behavior of the ODE solver during training to adapt its step size, aiming to ensure a valid ODE without unnecessarily increasing computational cost. We verify this adaption algorithm on a common bench mark dataset as well as a synthetic dataset.

1 INTRODUCTION

The choice of neural network architecture is an important consideration in the deep learning community. Among a plethora of options, Residual Neural Networks (ResNets) (He et al., 2016) have emerged as an important subclass of models, as they mitigate the gradient issues (Balduzzi et al., 2017) arising with training deep neural networks by adding skip connections between the successive layers. Besides the architectural advancements inspired from the original scheme (Zagoruyko & Komodakis, 2016; Xie et al., 2017), recently Neural Ordinary Differential Equation (Neural ODE) models (Chen et al., 2018; E, 2017; Lu et al., 2018; Haber & Ruthotto, 2017) have been proposed as an analog of continuous-depth ResNets. While Neural ODEs do not necessarily improve upon the sheer predictive performance of ResNets, they offer the vast knowledge of ODE theory to be applied to deep learning research. For instance, the authors in Yan et al. (2020) discovered that for specific perturbations, Neural ODEs are more robust than convolutional neural networks. Moreover, inspired by the theoretical properties of the solution curves, they propose a regularizer which improved the robustness of Neural ODE models even further. However, if Neural ODEs are chosen for their theoretical advantages, it is essential that the effective model—the combination of ODE problem and its solution via a particular numerical method—is a close approximation of the true analytical, but practically inaccessible ODE solution.

In this work, we study the empirical risk minimization (ERM) problem

$$L_{\mathcal{D}} = \frac{1}{|\mathcal{D}|} \sum_{(x,y)\in\mathcal{D}} l(f(x;w), y) \tag{1}$$

where $\mathcal{D} = \{(x_n, y_n) \mid x_n \in \mathbb{R}^{D_x}, y_n \in \mathbb{R}^{D_y}, n = 1, \ldots, N\}$ is a set of training data, $l : \mathbb{R}^{D_y} \times \mathbb{R}^{D_y} \to \mathbb{R}$ is a (non-negative) loss function and f is a Neural ODE model with weights w, i.e.,

$$f = f_d \circ \varphi_T^{f_v} \circ f_u \tag{2}$$

where $f_x, x \in \{d, v, u\}$ are (suitable) neural networks and u and d denote the upstream and downstream layers respectively. φ is defined to be the (analytical) flow of the dynamical system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_v(z; w_v), \; z(t) = \varphi_t^{f_v}(z(0)). \tag{3}$$

As the vector field f_v of the dynamical system is itself defined by a neural network, evaluating $\varphi_T^{f_v}$ is intractable and we have to resort to a numerical scheme Ψ_t to compute φ_t. Ψ belongs either to a class of fixed step size $h = TK^{-1}$ methods or is an adaptive step size solver as proposed in Chen et al. (2018). The global numerical error e_{train} of the model is the difference between the true, (unknown), analytical solution of the model and the numerical solution $e_{train} = ||\varphi_T(z(0)) - \Psi_T(z(0))||$ at time T. The global numerical error for a given problem can be controlled by adjusting either the step size or the local error tolerance.

Since the numerical solvers play an essential role in the approximation of the solutions of an ODE, it is intuitive to ask: how does the choice of the numerical method affect the training of a Neural ODE model? Specifically, does the discretization of the numerical solver impact the resulting flow of the ODE? To test the effect of the numerical solver on a Neural ODE model, we first train a Neural ODE on a synthetic classification task using a fixed step solver with a small step size and a fixed step solver with a large step size (see Figure 1 (a) and (b) respectively). If the model is trained with a large step size, then the numerically computed trajectories for the individual IVPs cross in phase space (see Figure 1 (b)). Specifically, we observe that trajectories of IVPs belonging to different classes cross. This crossing behavior contradicts the expected behavior of autonomous ODE solutions, as according to the Picard-Lindelöf theorem we expect unique solutions to the IVPs. The reason we observe crossing trajectories is that the discretization error of the solver is so large that the resulting numerical solutions no longer maintain the properties of ODE solutions.

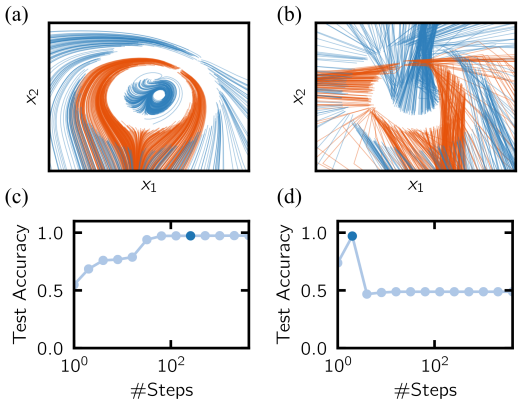

Figure 1: The Neural ODE was trained on a classification task with a small (a) and large step size (b). In (a) the trajectories look smooth and do not cross, whereas in (b) the solutions found by the solver cross. The colors of the trajectories indicate the label for each IVP. Panels (c) and (d) show the test accuracy of the Neural ODE solver using different step sizes, the dark blue circle indicates the number of steps used for training.

We observe that both, the model trained with the small step size and the model trained with the large step size, achieve very high accuracy. This leads us to the conclusion that the step size parameter is not simply like any other hyperparameter, as its chosen value often does not affect the performance of the model. Instead, the step size affects the properties of the model, such as whether the model has a valid ODE interpretation. Crossing trajectories are not bad per se if the performance is all we are interested. If, however, we are interested in applying algorithms whose success is motivated from ODE theory to, for example, increase model robustness (Yan et al., 2020), then the trajectories must not cross.

We argue that if *any* discretization with similar or lesser discretization error yields the same prediction, the trained model corresponds to an ODE that is qualitatively well approximated by the applied discretization. Therefore, in our experiments we evaluate each Neural ODE model with smaller and larger step sizes than the training step size. We notice that the model trained with the small step size achieves the same level of performance when using a solver with smaller discretization error for testing (Figure 1 (c)). For the model trained with the large step size, we observe a significant drop in performance if the model is evaluated using a solver with a smaller discretization error (see Figure 1 (d)). The reason for the drop in model performance is that the decision boundary of the classifier has adapted to the global numerical error e_{train} in the computed solution. For this specific example, correct classification relies on crossing trajectories as a feature. Therefore, the solutions of solvers with a smaller discretization error are no longer assigned the right class by the classifier and the Neural ODE model is a ResNet model without ODE interpretation.

If we are interested in extending ODE theory to Neural ODE models, we have to ensure that the trained Neural ODE model indeed maintains the properties of ODE solutions. In this work we show that the training process of a Neural ODE yields a discrete ResNet without valid ODE interpretation if the discretization is chosen too coarse. With our rigorous Neural ODE experiments on a synthetic dataset as well as CIFAR10 using both fixed step and adaptive step size methods, we show that if the precision of the solver used for training is high enough, the model does not depend on the solver used for testing as long as the test solver has a small enough discretization error. Therefore, such a model allows for a valid ODE interpretation. Based on this observation we propose an algorithm to find the coarsest discretization for which the model is independent of the solver.

2 Interaction of Neural ODE and ODE solver can lead to discrete dynamics

We want to study how the Neural ODE is affected by the specific solver configuration used for training. To this end, in our experiments we first train each model with a specific step size h_{train} (or a specific tolerance $\text{tol}_{\text{train}}$ in the case of adaptive step size methods). For the remainder of this section we will only consider fixed step solvers, but all points made equally hold for adaptive step methods, as shown by our experiments. Post-training, we evaluate the trained models using different step sizes h_{test} and note how using smaller steps sizes $h_{\text{test}} < h_{\text{train}}$ affects the model performance. We expect that if the model represents a valid ODE, then in the limiting behavior using smaller and smaller step sizes $h_{\text{test}} \to 0$ for testing should still yield good performance. For a model trained with a *small* step size, we find that the numerical solutions do not change drastically if the testing step size h_{test} is decreased (see Figure 1 (c)). But if the step size h_{train} is beyond a critical value, the model accumulates a large global numerical error e_{train}. The decision layer may use these drastically altered solutions as a signal/feature in the downstream computations. In this case, the model is tied to a *specific*, *discrete* flow and the model remains no longer valid in the limit of using smaller and smaller step sizes $h_{\text{test}} \to 0$. To use the reasoning from ODE theory, we expect the model to remain valid in the limit of converging to the true, analytical flow, which is not the case for models trained with a step size h_{train} beyond a critical value.

2.1 The trajectory crossing problem

In this sub-section, we examine the effects that cause the ODE interpretation to break down for models trained with a step size h_{train} beyond a certain critical step size. First, we look at the numerically computed trajectories in phase space of a Neural ODE model trained with a very large step size of $h_{\text{train}} = 1/2$ (see Figure 1 (b)). A key observation is that the trajectories cross in phase space. This crossing happens because the step size h_{train} is much bigger than the length scale at which the vector

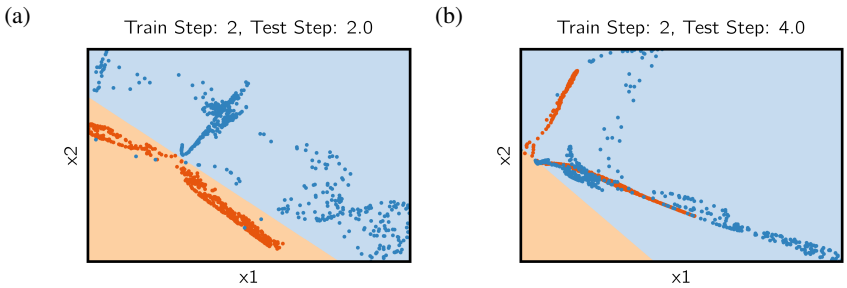

Figure 2: Output of the Neural ODE block for different step sizes. The model was trained with a step size of $1/2$, we used the same model as for Figure 1. The output of the Neural ODE block if tested with a step size of $1/2$ is shown in (a). Here, the points indicate the output of the Neural ODE block, the color of the points shows their true label, and the light color in the background indicates the label assigned by the classifier to this region. In (b) the model was tested with a step size of $1/4$. In this case, many of the points belonging to the orange class are now in the region classified as blue by the classifier.

field changes, thus missing "the curvature" of the true solution. Specifically, we observe that the model exploits this trajectory crossing behavior as a feature to separate observations from different classes. This is a clear indication that these trajectories do not approximate the true analytical solution of an ODE, as according to the Picard-Lindelöf theorem (Hairer et al., 1993, § 1.8), solutions of first order autonomous ODEs do not cross in phase space. Since the numerical solutions using smaller step sizes $h_{\text{test}} < h_{\text{train}}$ no longer maintain the crossing trajectory feature, the classifier cannot separate the data with the learned vector field (see Figure 2).

2.2 Lady Windermere's Fan

Here we show that there are also other, more subtle effects than trajectory crossing which lead to a drop in performance in the limit of using smaller and smaller test step sizes $h_{\text{test}} \to 0$. To understand these effects we introduce an example based on the XOR problem $D = \{((0,0) \mapsto 0), ((1,1) \mapsto 0), ((0,1) \mapsto 1), ((1,0) \mapsto 1)\}$. This dataset cannot be classified correctly in 2D with a linear decision boundary (Goodfellow et al., 2016, § 1.2). Therefore, we consider the ODE

$$z'(t) = \begin{pmatrix} \alpha & 1 \\ -\gamma ||z||^{\delta} & \beta \end{pmatrix} z. \tag{4}$$

The qualitative behavior of the analytical flow are increasing ellipsoids with ever increasing rotational speed. Fig. 3 depicts the numerical solution of this flow with one set of fixed parameters and different step sizes $h = 10^{-2.5}, 10^{-3.5}$. For $h = 10^{-2.5}$ the numerical flow produces a transformation in which the data points can be separated linearly. But for the smaller step size of $h = 10^{-3.5}$, the numerical solution is no longer linearly separable. The problem here is that the numerical solution using the larger step size is not accurate enough to resolve the rotational velocity. However, in Figure 3 (a), the accumulation of error in the numerical solution (coined as Lady Windermere's Fan in Hairer et al. (1993, § 1.7)) results in a valid feature for a linear decision (classification) layer. The reason for this is that the global numerical errors e_{train} are biased. We define as the *fingerprint* of the method the structure in the global numerical error. The decision layer then adapts to this method specific fingerprint. Whether this fingerprint has an effect on the performance of the model when using smaller step size h_{test} is dependent on two main factors. First, does the data remain separable when using smaller step sizes $h_{\text{test}} < h_{\text{train}}$? If not, we will observe a significant drop in performance. Second, how sensitive is the decision layer to changes in the solutions and how much do the numerical solutions change when decreasing the test step size $h_{\text{test}} \to 0$? Essentially, the input sensitivity of the downstream layer should be less than the output sensitivity of $h_{\text{test}} < h_{\text{train}}$. For the decision layer, there should exist a sensitivity threshold d such that $f_d(z(T) + \delta) = f_d(z(T)) \; \forall ||\delta|| < d$. Thus, if two solvers compute the same solution up to δ, the classifier identifies these solutions as the same class and the result of the model is not affected by the interchanging these solvers.

The current implementation of Neural ODEs does not ensure that the model is driven towards continuous semantics as there are no checks in the gradient update ensuring that the model remains a valid ODE nor are there penalties in the loss function if the Neural ODE model becomes tied to a specific numerical configuration. If the discretization is not chosen fine enough then the combination of vector field and decision layer only works for the finite difference case but not for the continuous case.

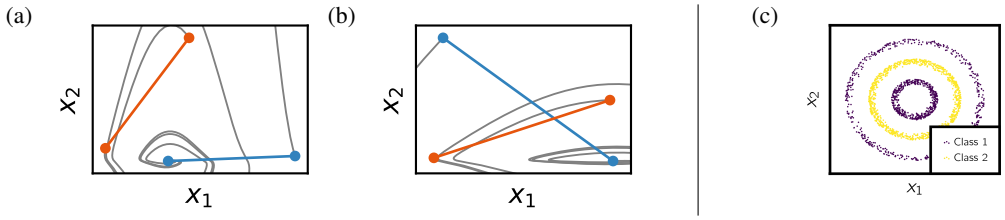

Figure 3: Solutions to Eq. 4 using Euler's method with step size $h = 10^{-2.5}$ (a) and $h = 10^{-3.5}$ (b). The trajectories taken by the numerical solver are shown in light gray. The points indicate where each IVP ends up in phase space and the color indicates the class the solution belongs to. The connecting line between the points is there to indicate whether the two classes are linearly separable or not. (c) The concentric sphere dataset (Sphere2).

2.3 CONCENTRIC SPHERE DATASET (SPHERE2)

For our experiments, we introduce a classification task based on the concentric sphere dataset proposed by Dupont et al. (2019). We use three concentric spheres, where the outer and the inner sphere correspond to the same class (see Figure 3 (c) for a 2 dimensional example). Whether this dataset can be fully described by an autonomous ODE, is dependent on the degrees of freedom introduced by combining the Neural ODE with additional downstream (and upstream) layers.

2.4 EXPERIMENTS

In this subsection, we present results from the experiments performed on Sphere2 and CIFAR10 datasets using fixed and adaptive step solvers. The aim of these experiments is to analyze the dynamics of Neural ODEs and show its dependence on the specific solver used during training by testing the model with a different solver configuration. In the main experiments presented in the paper, we choose to back-propagate through the numerical solver. The results pertaining to the adjoint method (Chen et al., 2018) are provided in the Supplementary Material. For all our experiments, we do not use an upstream block f_u similar to the architectures proposed in Dupont et al. (2019). We chose such an architectural scheme to maximize the modeling contributions of the ODE block.

For training the Neural ODE with fixed step solvers, Euler's method and a 4th order Runge–Kutta (rk4) method were used (detailed descriptions of these methods can be found in Hairer et al. (1993)). The trained Neural ODE was then tested with different step sizes and solvers. For a Neural ODE trained with Euler's method, the model was tested with Euler's method, the midpoint method and the rk4 method. The testing step size was chosen as a factor of 0.5, 0.75, 1, 1.5, and 2 of the original step size used for training. For rk4, we only tested using the rk4 method with different step sizes. Likewise, the adaptive step solver experiments were performed using Fehlberg21 and Dopri54. The models were trained and tested using different tolerances and solvers. The models trained using

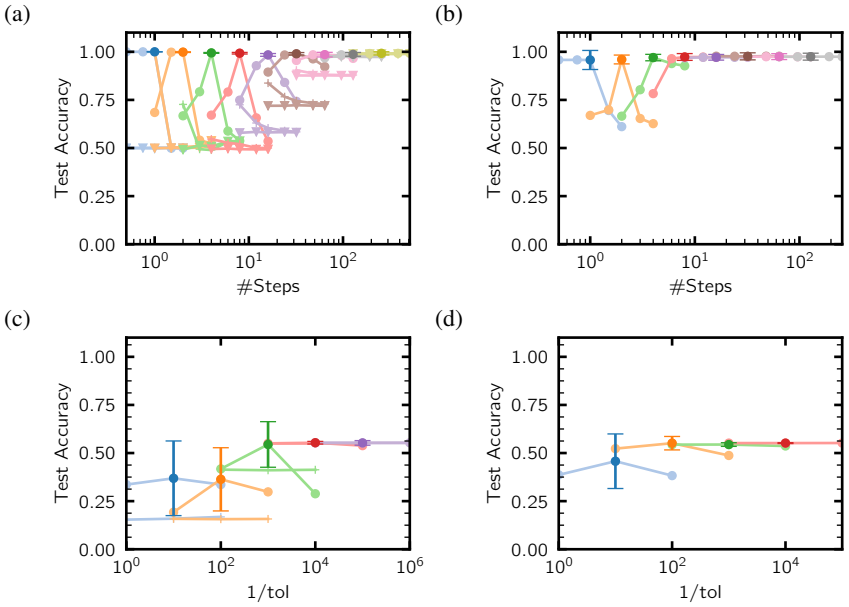

Figure 4: A Neural ODE was trained with different step sizes (plotted in different colors) on Sphere2 (a), (b) and on CIFAR10 (c), (d). The model was tested with different solvers and different step sizes. In (a) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles). In (c) the model was trained using the Fehlberg21 method. Circles correspond to Fehlberg21 method, cross to the Dopri54 method. In (d) Dopri54 was used for training (dark circles) and testing (light circles)

Fehlberg21 were tested using Fehlberg21 and Dopri54, whereas the models trained using Dopri54 were only tested using Dopri54. The testing tolerance was chosen as a factor of 0.1, 1, and 10 of the original tolerance used for training. We report an average over five runs, where we used an aggregation of seeds for which the Neural ODE model trained successfully (the results for all seeds are disclosed in the Supplementary Material). We did not tune all hyper-parameters to reach best performance for each solver configuration. Rather, we focused on hyper-parameters that worked well across the entire range of step sizes and tolerances used for training (see Supplementary Material for more details of the choice of hyper parameters and the architecture of the Neural ODE).

As shown in Figure 4 (a), (b), when training and testing the model with the same step size (dark circles), the test accuracy does not show any clear dependence on the step size for both datasets. Since we did not tune the learning rate for each step size, any visible trends could be due to this choice. Indeed, many different solver configurations work well in practice, but only for small enough step sizes the model represents a valid ODE. On both datasets, we observe similar behavior for dependence of the test accuracy on the test solver: when using large step sizes for training, the Neural ODE shows dependence on the solver used for testing. But there exists some critical step size below which the model shows no clear dependence on the test solver as long as this test solver has equal or smaller numerical error than the solver used for training (as seen in Figure 4). For additional results on CIFAR10 and on higher dimensional versions of the concentric sphere dataset we refer the reader to the Supplementary Material Section A.1.

The aforementioned dynamics of Neural ODE were also verifiable in the adaptive step solver experiments (see Figure 4, (c) and (d)). In this case, the trained model's test accuracy was dependent on the configuration of the test solver below a critical tolerance value. For additional results on the Sphere2 dataset we refer the reader to the Supplementary Material Section A.2.

3 ALGORITHM FOR STEP SIZE ADAPTION

Algorithm 2: Step and tolerance adaption algorithm

```
initialize accuracy_param;
while Training do
    if Iteration % 50 == 0 then
        test_acc =
        calculate_accuracy_higher_order_solver();

        if |train_acc-test_acc| > 0.1 then
            new_accuracy_param = 0.5
              accuracy_param;
        else
            new_accuracy_param = 1.1
              accuracy_param;
        end
    end
end
```

Although the Neural ODE achieves good accuracy for a large variety of solver configurations, if theoretic results of ODEs are to be applicable to Neural ODEs, it is paramount to find a solution corresponding to an ODE flow. To ensure this, we propose an algorithm that checks whether the Neural ODE remains independent of the specific train solver configuration and adapts the step size for fixed step solver and the tolerance for adaptive solvers if necessary. It is important to note that adaptive step size methods with one fixed tolerance parameter do not solve this issue, as embedded methods can severely underestimate the local numerical error if the vector field is not sufficiently smooth (Hairer et al., 1993). In contrast to the common application of such methods, in the case of Neural ODEs we cannot choose the appropriate solver and tolerance for a given problem as the vector field of the Neural ODE block is changing throughout training.

So far, there does not exist any other algorithm that we are aware of which solves the issue. The aim of the proposed algorithms is not to achieve state of the art results, but rather be a first step towards ensuring that trained Neural ODE models can be viewed independently of the solver configuration used for training. Here we will describe the algorithm for the fixed step solvers, which shows promising results. We also developed an equivalent algorithm for adaptive methods, which shows good initial results on synthetic data (description and preliminary results in the Supplementary Material Section B). Pseudo-code for both settings are presented in Alg. 2.

The algorithm for fixed step solvers chooses the initial step size using an algorithm described by Hairer et al. (1993)[p. 169]. This algorithm ensures that the Neural ODE chooses an appropriate

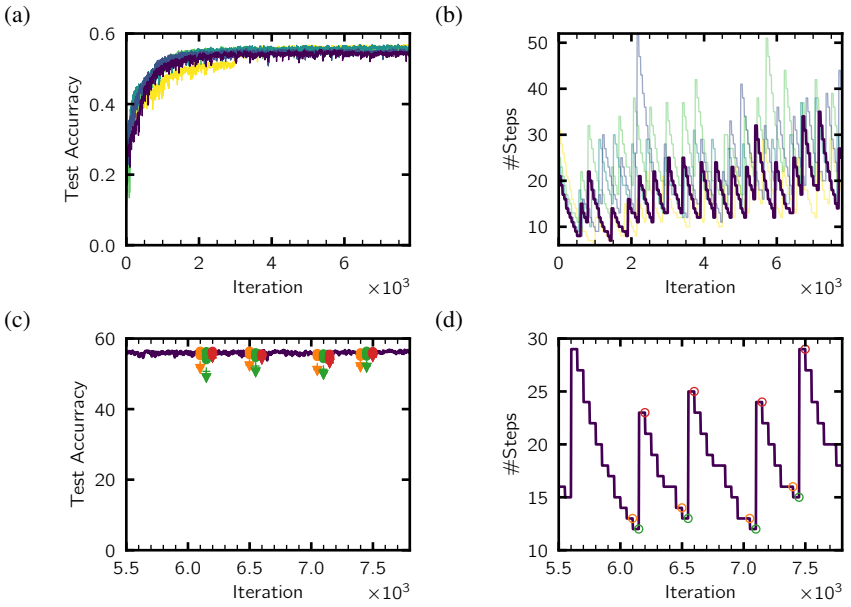

Figure 5: Using the step adaption algorithm for training on CIFAR10 (a), (b). (a) shows the test accuracy over the course of training for five different seeds. (b) shows the number of steps chosen by the algorithm over the course of training. (c) shows the test accuracy. At certain points in time (also marked in (d)), the model is evaluated with solvers of smaller discretization error (orange green and red data points). Triangles correspond to a 4th order Runge Kutta method, crosses to the midpoint method. (d) shows the number of steps chosen by the algorithm.

step size for all neural networks and initializations. We found that the initial step size suggested by the algorithm is not too small, which makes the algorithm useful in practice. The Neural ODE starts training with the proposed step size h. After a predefined number of steps (we chose $k = 50$), the algorithm checks whether the model can still be interpreted as a valid ODE: the accuracy is calculated over one batch with the train solver and with a test solver, where the test solver has a smaller discretization error than the train solver. If test and train solver show a significant difference in performance, we decrease the step size and let the model train a couple of iterations to regain valid ODE dynamics. If performance of test and train solver agree up to a threshold, we cautiously increase the step size. Unlike in ODE solvers, the difference between train and test accuracy does not tell by how much the step size needs to be adapted, so we choose some constant multiplicative factor that works well in practice (see Algorithm 2 for a simplified version and the Supplementary Material for details). The algorithm was robust against small changes to the constants in the algorithm.

3.1 Experiments

We test the step adaption algorithm on two different datasets: the synthetic dataset and on CIFAR10 (results for synthetic data are found in the Supplementary Material). We use Euler's method as the train solver and the midpoint method as the test solver. On all datasets, we observe that the number of steps taken by the solver fluctuate over the course of training (see Figure 11). The reason for such a behavior is that the algorithm increases the step size until the step size is too large and training with this step size leads to an adaption of the vector field to this particular step size. Upon continuing training with a smaller step size, this behavior is corrected (see Figure 11 (c) and (d)) and the algorithm starts increasing the step size again. We also observe this behavior for the tolerance adaption algorithm for adaptive methods (see Figures in the Supplementary Material).

To compare the results of the step adaption algorithm to the results of the grid search, we detail accuracy as well as number of average function evaluations (NFE) per iteration. For the grid search, we determine the critical number of steps using the same method as in the step adaption algorithm. We report the two step sizes above and below the critical step size which were part of the grid

	Grid search		Step adaption algorithm	
Data set	NFE	Accuracy	NFE	Accuracy
Concentric spheres 2d	65-129	$98.7 \pm 1.0\%$	100.5	$98.9 \pm 0.6\%$
Cifar10	17-33	$54.7 \pm 0.3\%$	21.9	$55.0 \pm 0.8\%$

Table 1: Results for the accuracy and the number of function evaluations to achieve time continuous dynamics using a grid search and the proposed step adaption algorithm. For the grid search, we report the accuracy of the run with the smallest step size above the critical threshold.

search. For the step adaption algorithm we calculate the NFE per iteration by including all function evaluations over course of training (see Table 1).

The achieved accuracy and step size found by our algorithm is on par with the smallest step size above the critical threshold thereby eliminating the need for a grid search.

4 Related Work

The connections between ResNets and ODEs have been discussed in E (2017); Lu et al. (2018); Haber & Ruthotto (2017); Sonoda & Murata (2019). The authors in Behrmann et al. (2018) use similar ideas to build an invertible ResNet. Likewise, additional knowledge about the ODE solvers can be exploited to create more stable and robust architectures with a ResNet backend (Haber & Ruthotto, 2017; Haber et al., 2019; Chang et al., 2018; Ruthotto & Haber, 2019; Ciccone et al., 2018; Cranmer et al., 2020; Benning et al., 2019).

Continuous-depth deep learning was first proposed in Chen et al. (2018); E (2017). Although ResNets are universal function approximators (Lin & Jegelka, 2018), Neural ODEs require specific architectural choices to be as expressive as their discrete counterparts (Dupont et al., 2019; Zhang et al., 2019a; Li et al., 2019). In this direction, one common approach is to introduce a time-dependence for the weights of the neural network (Zhang et al., 2019c; Thorpe & van Gennip, 2018; Avelin & Nyström, 2020; Choromanski et al., 2020; Queiruga et al., 2020). Other solutions include, novel Neural ODE models (Lu et al., 2020; Massaroli et al., 2020) with improved training behavior, and variants based on kernels (Owhadi & Yoo, 2019) and Gaussian processes (Hegde et al., 2019). Adaptive ResNet architectures have been proposed in Veit & Belongie (2018); Chang et al. (2017). The dynamical systems view of ResNets has lead to the development of methods using time step control as a part of the ResNet architecture (Yang et al., 2020; Zhang et al., 2019b).

In a similar vein as our work, Queiruga et al. (2020) study how the solver influences the Neural ODE model, showing that a model trained with Euler's method can have significantly lower performance when tested with a higher order solver. To avoid this issue, they propose to use higher order solvers for training Neural ODEs.

5 Conclusion

We have shown that the step size of fixed step solvers and the tolerance for adaptive methods used for training Neural ODEs impacts whether the resulting model maintains properties of ODE solutions. As a simple test that works well in practice, we conclude that the model only corresponds to an continuous ODE flow, if the performance does not depend on the exact solver configuration. We illustrated that the reasons for the model to become dependent on a specific train solver configuration are the use of the bias in the numerical global errors as a feature by the classifier, and the sensitivity of the classifier to changes in the numerical solution. We have verified this behavior on CIFAR10 as well as a synthetic dataset using fixed step and adaptive methods. Based on these observations, we developed step size and tolerance adaption algorithms, which maintain a continuous ODE interpretation throughout training. For minimal loss in accuracy and computational efficiency, our step adaption algorithm eliminates a massive grid search. In future work, we plan to eliminate the oscillatory behavior of the adaption algorithm and improve the tolerance adaption algorithm to guarantee robust training on many datasets.

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A ADDITIONAL EXPERIMENTAL RESULTS

A.1 ADDITIONAL RESULTS FOR FIXED STEP SOLVERS

A.1.1 RESULTS FOR CIFAR10

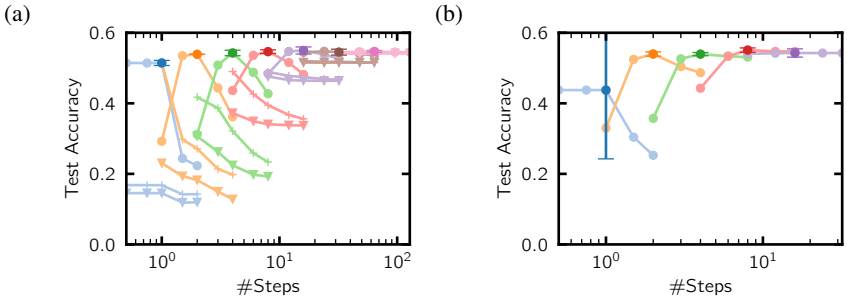

Figure 6: A Neural ODE was trained with different step sizes (plotted in different colors) on CIFAR10 (a), (b) . The model was tested with different solvers and different step sizes. In (a) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles).

A.1.2 RESULTS FOR N-DIMENSIONAL CONCENTRIC SPHERE DATASET

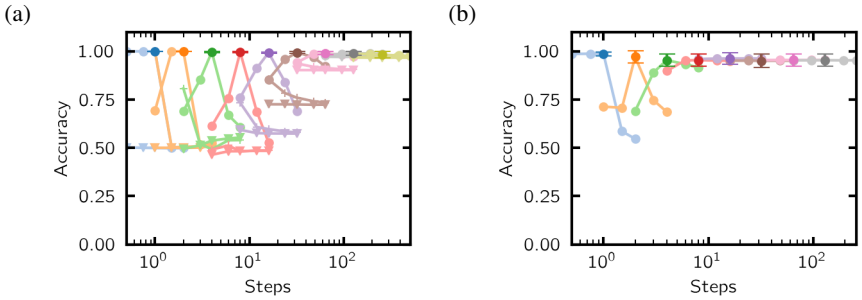

Figure 7: A Neural ODE was trained with different step sizes (plotted in different colors) on the 3 dimensional concentric sphere dataset (a), (b) and on the potential landscape dataset (c), (d). The model was tested with different solvers and different step sizes. In (a), (c) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles).

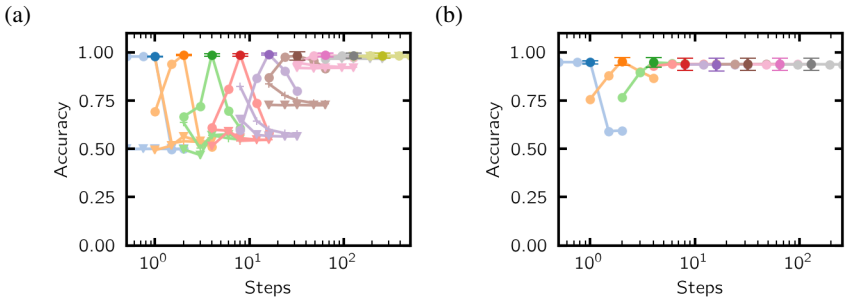

Figure 8: A Neural ODE was trained with different step sizes (plotted in different colors) on the 10 dimensional concentric sphere dataset (a), (b) and on the potential landscape dataset (c), (d). The model was tested with different solvers and different step sizes. In (a), (c) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles).

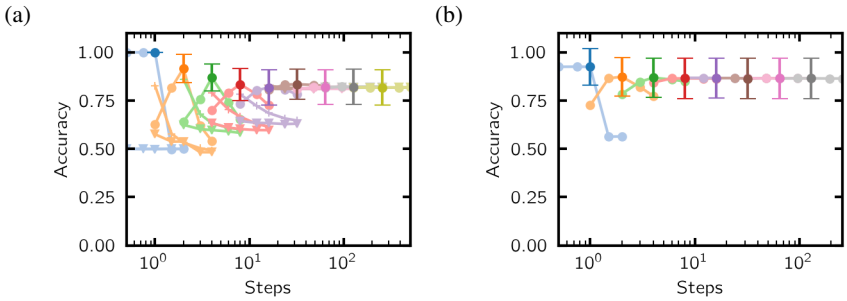

Figure 9: A Neural ODE was trained with different step sizes (plotted in different colors) on the 900 dimensional concentric sphere dataset (a), (b) and on the potential landscape dataset (c), (d). The model was tested with different solvers and different step sizes. In (a), (c) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles).

A.2 ADDITIONAL RESULTS FOR ADAPTIVE SOLVERS

A.2.1 RESULTS FOR SPHERE2

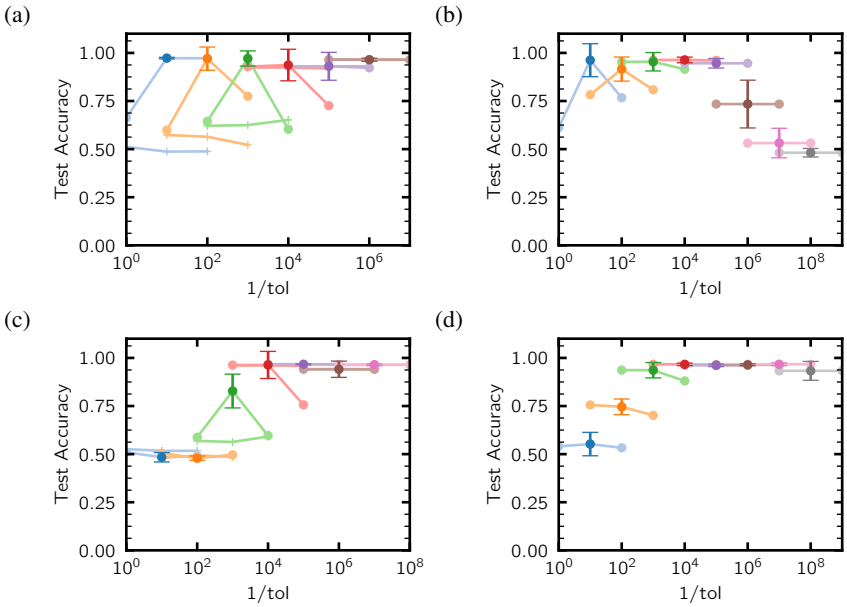

Figure 10: A Neural ODE was trained with different tolerances (plotted in different colors)on Sphere2 (a), (b) and on Sphere2 using the adjoint method (c), (d). The model was tested with different solvers and different tolerances. In (a), (c) the model was trained using the Fehlberg21 method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Fehlberg21 method, cross to the Dopri54 method. In (b), (d) Dopri54 was used for training (dark circles) and testing (light circles).

B ADDITIONAL INFORMATION ABOUT STEP SIZE AND TOLERANCE ADAPTION ALGORITHM

B.1 EXTENDED STEP ADAPTION ALGORITHM

Algorithm 4: Step adaption algorithm

```
initialize starting step_size according to (Hairer et al., 1993, p. 169);
while Training do
    if Iteration % 50 == 0 then
        test_acc = calculate_accuracy_higher_order_solver();
        if |train_acc-test_acc| > 0.1 then
            new_step_size = 0.5 step_size;
        else
            new_step_size = 1.1 step_size;
            if new_step_size > T then
                // Avoid increasing the step size indefinitely
                new_step_size = T;
            end
            new_acc = calculate_accuracy_with_new_step_size();
            if |test_acc-new_acc| > 0.1 then
                new_step_size = step_step;
            end
        end
    end
end
```

B.2 RESULTS WITH STEP ADAPTION ALGORITHM ON SPHERE2

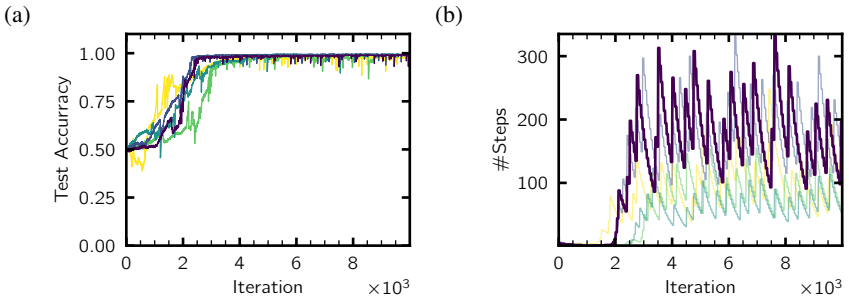

Figure 11: Using the step adaption algorithm for training on Sphere2 (a), (b). (a) shows the test accuracy over the course of training for five different seeds. (b) shows the number of steps chosen by the algorithm over the course of training.

B.3 PRELIMINARY TOLERANCE ADAPTION ALGORITHM

Algorithm 6: Tolerance adaption algorithm

```
initialize tolerance tol= 10^-6;
while Training do
    if Iteration % 50 == 0 then
        test_acc = calculate_accuracy_higher_order_solver();
        if |train_acc-test_acc| > 0.1 then
            new_tol= 0.5 tol;
        else
            new_tol = 1.1 tol;
            if steps <= min_steps then
                // Avoid increasing the tolerance indefinitely
                new_tol = tol;
            else
                new_acc = calculate_accuracy_with_new_tol();
                if |test_acc-new_acc| > 0.1 then
                    new_tol = tol;
                end
            end
        end
    end
end
```

B.4 PRELIMINARY RESULTS WITH TOLERANCE ADAPTION ALGORITHM ON SPHERE2

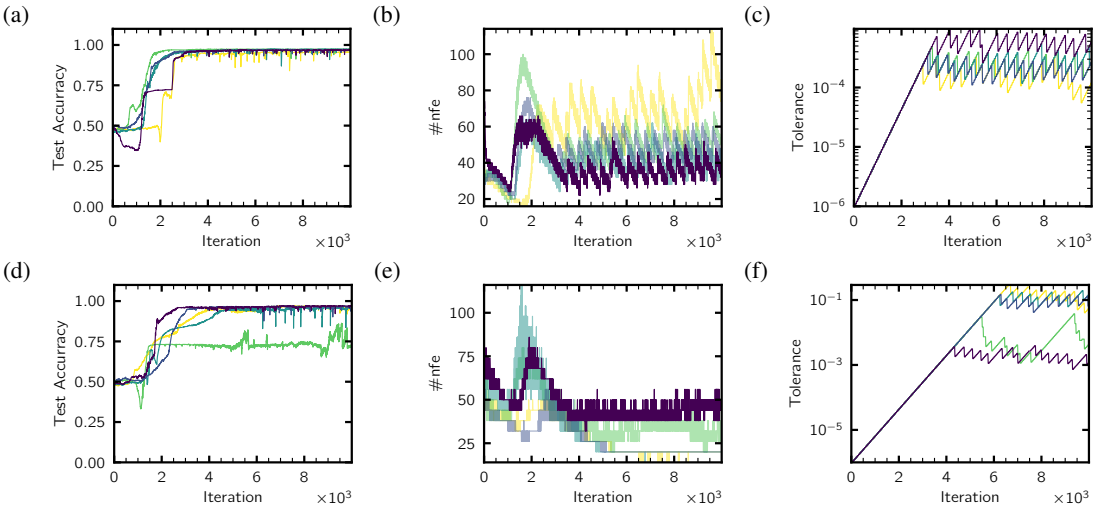

Figure 12: Using the tolerance adaption algorithm for training on Sphere2 for Fehlberg21 (a), (b), (c) and Dopri54 (d), (e), (f). (a), (d) shows the test accuracy over the course of training for five different seeds. (b), (e) shows the number of function evaluations the course of training. (c), (f) show the tolerance chosen by the algorithm over the course of training

C ARCHITECTURE AND HYPER-PARAMETERS

We chose the architecture for our network similar to the architecture proposed by Dupont et al. (2019). We tried to find hyperparameters which worked well for all step sizes. The same hyper-parameters were used for the grid search and for training with the step adaption algorithm:

C.1 ARCHITECTURE AND HYPER-PARAMETERS USED FOR CIFAR10

Neural ODE Block

- Conv2D(3, 128, Kernel 1x1, padding 0) + ReLu
- Conv2D(128, 128, Kernel 3x3, padding 1) + ReLu
- Conv2D(128, 3, Kernel 1x1, padding 0)

Classifier

- Flatten + LinearLayer(3072,10) + SoftMax

Hyper-parameters

- Batch size: 256
- Optimizer: Adam
- Learning rate: 1e-3
- Iterations used for training: 7800

C.2 ARCHITECTURE USED FOR CONCENTRIC SPHERE 2D DATASET

Neural ODE Block

- Conv1D(1, 32, Kernel 1x1, padding 0) + ReLu
- Conv1D(32, 32, Kernel 3x3, padding 1) + ReLu
- Conv1D(32, 1, Kernel 1x1, padding 0)

Classifier

- Flatten + LinearLayer(2,2) + SoftMax

Hyper-parameters

- Batch size: 128
- Optimizer: Adam
- Learning rate: 1e-4
- Iterations used for training: 10000

D EXTENDED RESULTS

D.1 CONCENTRIC SPHERE 2D

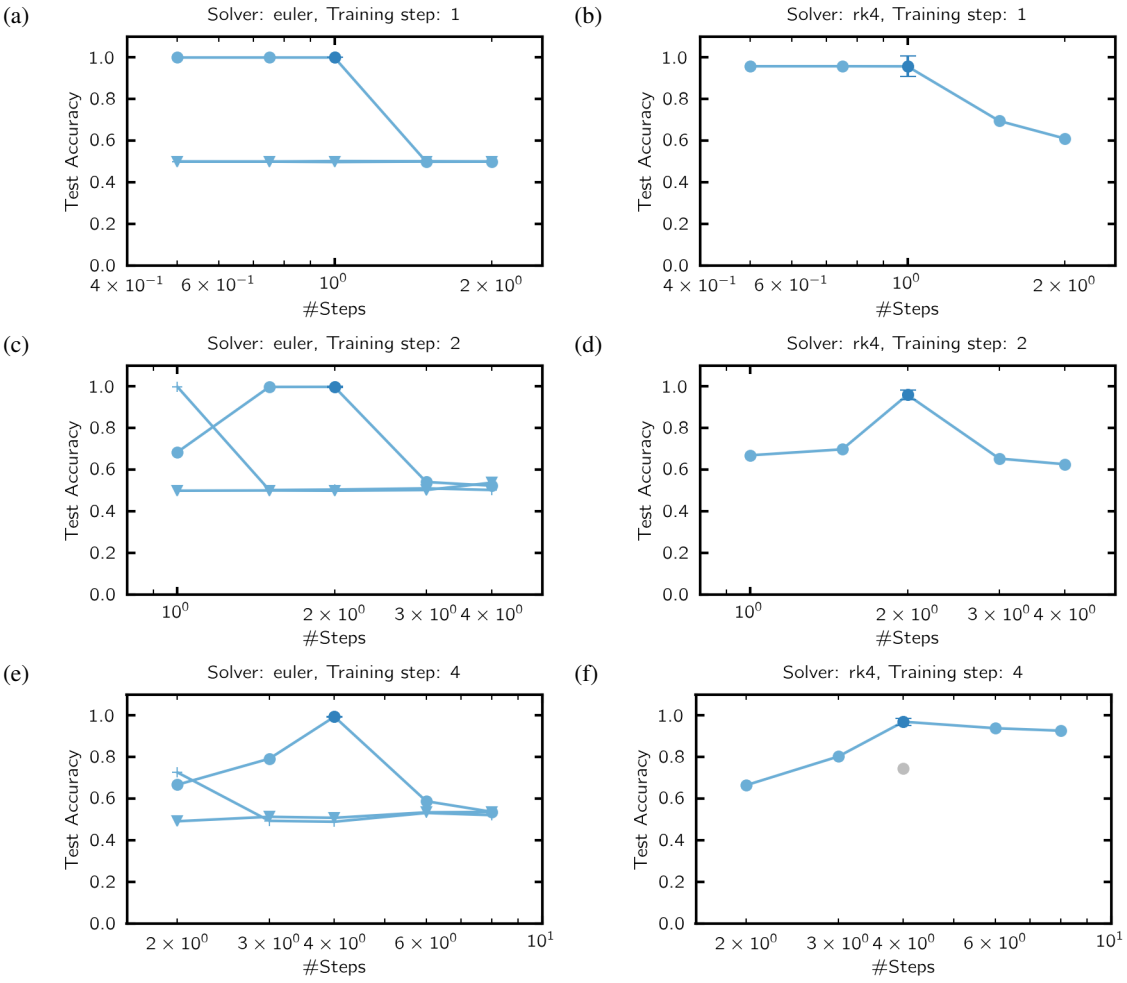

Figure 13: A Neural ODE was trained with different step sizes ((a), (b) 1 step, (c),(d) 2 steps, (e), (f) 4 steps) on the two dimensional concentric sphere dataset. The model was tested with different solvers and different step sizes. In (a), (c), (e) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d), (f) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles). Excluded seeds are shown as grey circles.

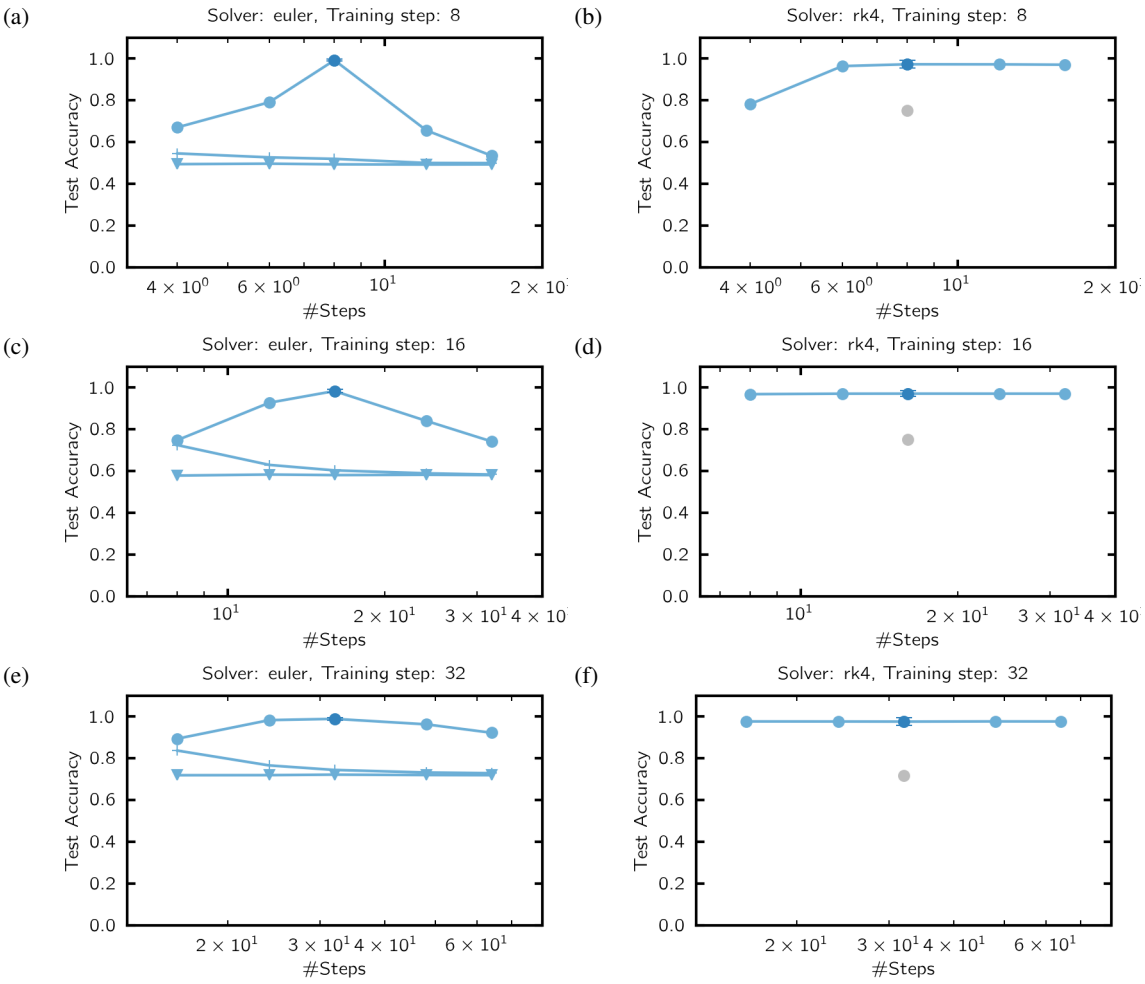

Figure 14: A Neural ODE was trained with different step sizes ((a), (b) 8 steps, (c),(d) 16 steps, (e), (f) 32 steps) on the two dimensional concentric sphere dataset. The model was tested with different solvers and different step sizes. In (a), (c), (e) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d), (f) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles). Excluded seeds are shown as grey circles.

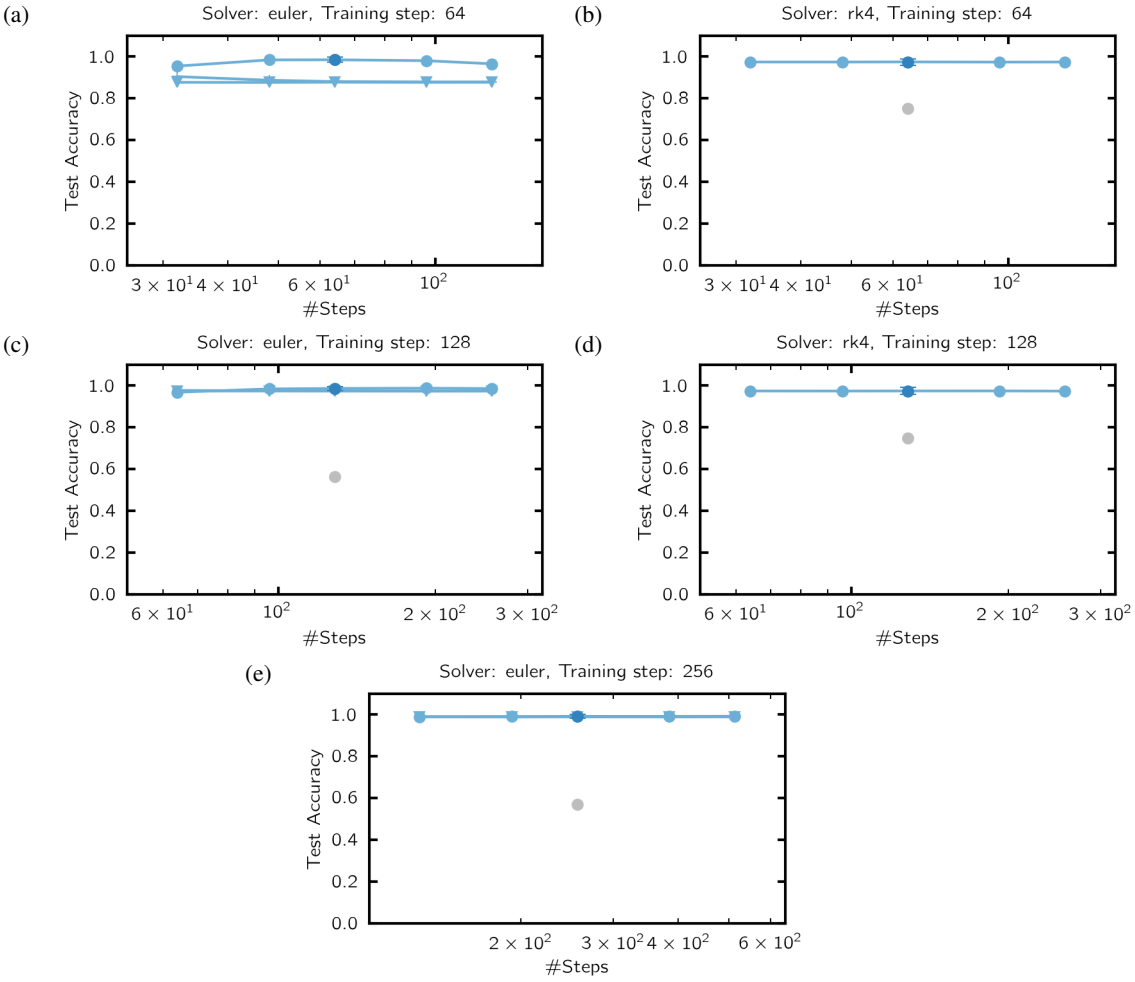

Figure 15: A Neural ODE was trained with different step sizes ((a), (b) 64 steps, (c),(d) 128 steps, (e) 256 steps) on the two dimensional concentric sphere dataset. The model was tested with different solvers and different step sizes. In (a), (c), (e) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles). Excluded seeds are shown as grey circles.

D.2 CIFAR10

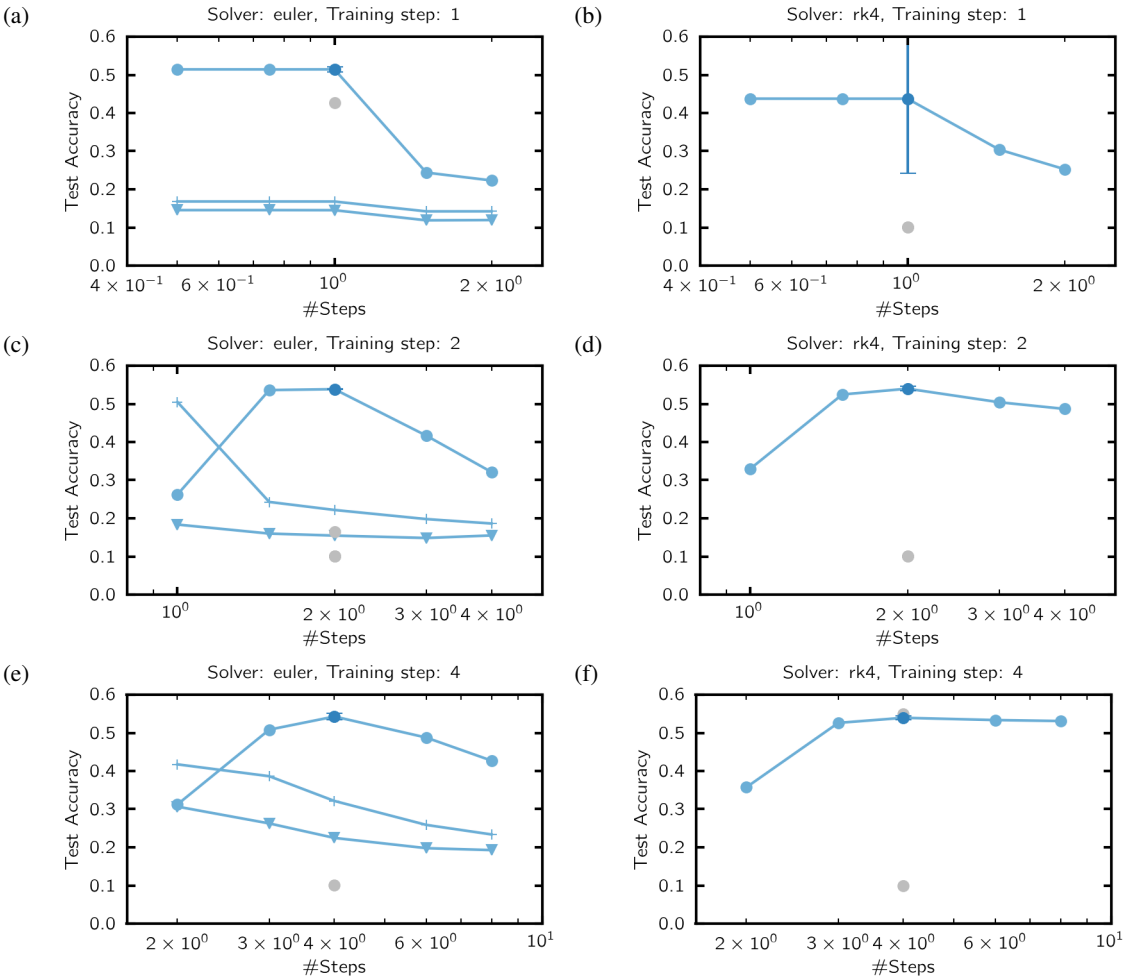

Figure 16: A Neural ODE was trained with different step sizes ((a), (b) 1 step, (c),(d) 2 steps, (e), (f) 4 steps) on the cifar10 dataset. The model was tested with different solvers and different step sizes. In (a), (c), (e) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d), (f) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles). Excluded seeds are shown as grey circles.

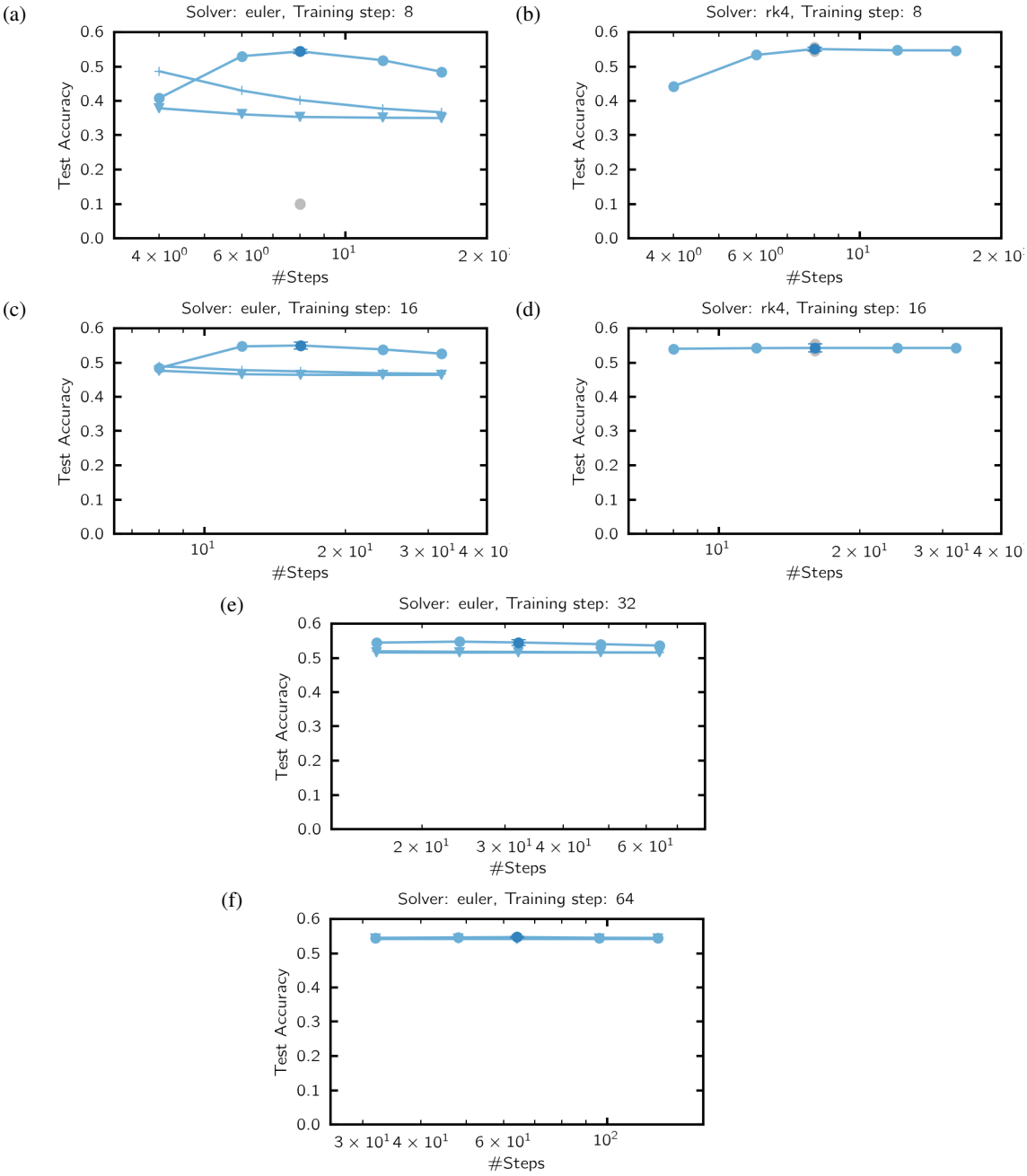

Figure 17: A Neural ODE was trained with different step sizes ((a), (b) 8 step, (c),(d) 16 steps, (e) 32 steps, (f) 64 steps) on the cifar10 dataset. The model was tested with different solvers and different step sizes. In (a), (c), (e), (f) the model was trained using Euler's method. Results obtained by using the same solver for training and testing are marked by dark circles. Light data indicated different step sizes used for testing. Circles correspond to Euler's method, cross to the midpoint method and triangles to a 4th order Rung Kutta method. In (b), (d) a 4th order Runge Kutta methods was used for training (dark circles) and testing (light circles). Excluded seeds are shown as grey circles.