How Over-Parameterization Slows Down Gradient Descent in Matrix Sensing: The Curses of Symmetry and Initialization

Nuoya Xiong NUOYAXIONG@GMAIL.COM, *IIIS, Tsinghua University*
Lijun Ding LDING47@WISC.EDU, *University of Wisconsin-Madison*
Simon S. Du SSDU@CS.WASHINGTON.EDU, *University of Washington*

Abstract

This paper rigorously shows how over-parameterization dramatically changes the convergence behaviors of gradient descent (GD) for the matrix sensing problem, where the goal is to recover an unknown low-rank ground-truth matrix from near-isotropic linear measurements. First, we consider the symmetric setting with the symmetric parameterization where $M^* \in \mathbb{R}^{n\times n}$ is a positive semi-definite unknown matrix of rank $r \ll n$, and one uses a symmetric parameterization $XX^\top$ to learn M^*. Here $X \in \mathbb{R}^{n\times k}$ with $k > r$ is the factor matrix. We give a novel $\Omega\left(1/T^2\right)$ *lower bound* of randomly initialized GD for the over-parameterized case ($k > r$) where T is the number of iterations. This is in stark contrast to the exact-parameterization scenario ($k = r$) where the convergence rate is $\exp\left(-\Omega\left(T\right)\right)$. Next, we study asymmetric setting where $M^* \in \mathbb{R}^{n_1\times n_2}$ is the unknown matrix of rank $r \ll \min\{n_1, n_2\}$, and one uses an asymmetric parameterization $FG^\top$ to learn M^* where $F \in \mathbb{R}^{n_1\times k}$ and $G \in \mathbb{R}^{n_2\times k}$. We give the first global exact convergence result of randomly initialized GD for the exact-parameterization case ($k = r$) with an $\exp\left(-\Omega\left(T\right)\right)$ rate. Furthermore, we give the first global exact convergence result for the over-parameterization case ($k > r$) with an $\exp\left(-\Omega\left(\alpha^2 T\right)\right)$ rate where α is the initialization scale. This linear convergence result in the over-parameterization case is especially significant because one can apply the asymmetric parameterization to the symmetric setting to speed up from $\Omega\left(1/T^2\right)$ to linear convergence. Therefore, we identify a surprising phenomenon: *asymmetric parameterization can exponentially speed up convergence.* Equally surprising is our analysis that highlights the importance of *imbalance* between F and G. This is in sharp contrast to prior works which emphasize balance. We further give an example showing the dependency on α in the convergence rate is unavoidable in the worst case. On the other hand, we propose a novel method that only modifies one step of GD and obtains a convergence rate independent of α, recovering the rate in the exact-parameterization case. We provide empirical studies to verify our theoretical findings.

1. Introduction

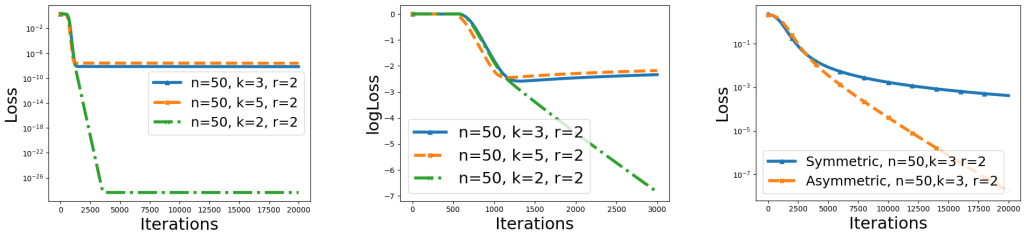
A line of recent work showed over-parameterization plays a key role in optimization, especially for neural networks [1, 6, 10, 12, 14, 22, 24, 28, 35, 51]. However, our understanding of the impact of over-parameterization on optimization is far from complete. In this paper, we focus on the canonical matrix sensing problem and show that over-parameterization qualitatively changes the convergence behaviors of gradient descent (GD).

Matrix sensing aims to recover a low-rank unknown matrix M^* from m linear measurements,

$$y_i = \mathcal{A}_i(M^*) = \langle A_i, M^*\rangle = \mathrm{tr}(A_i^\top M^*), \text{ for } i = 1, \ldots, m, \tag{1.1}$$

where $\mathcal{A}_i$ is a linear measurement operator and A_i is the measurement matrix of the same size as M^*. This is a classical problem with numerous real-world applications, including signal processing

© N. Xiong, L. Ding & S.S. Du.



(a) Loss curve

(b) Curve of logarithm loss: $\log_T \|X_t X_t^\top - M^*\|_F^2$

(c) Loss curve of symmetric and asymmetric parameterization

Figure 1: Experiments on symmetric setting. The first two figures show that the convergence rate of symmetric matrix factorization in the over-parameterized setting is about $\Theta(1/T^2)$, while the rate of the exact-parameterized setting is linear. Figure 1(c)subfigure shows that using asymmetric parameterization is exponentially faster than symmetric parameterization. See Appendix L for experimental details.

[36] and face recognition [5], image reconstruction [25, 45]. Moreover, this problem can serve as a test-bed of convergence behaviors in deep learning theory since it is non-convex and retains many key phenomena [2, 17, 19, 20, 29]. We primarily focus on the *over-parameterized* case where we use a model with rank larger than that of M^* in the learning process. This case is particularly relevant because rank(M^*) is usually unknown in practice.

Related Work Matrix sensing has been widely studied via the nuclear norm minimization approach [3, 21, 26, 37], spectral method [23, 32] and landscape analysis [13, 48]. The most relevant line of work considered global convergence of gradient descent [29, 30, 32, 50]. We compare our results with them in Table 1. The detailed discussions are also deferred to Appendix A.

2. Model and Prelimanaries

Norm and Big-$\mathcal{O}$ Notations. Given a vector v, we use $\|v\|$ to denote its Euclidean norm. For a matrix M, we use $\|M\|$ to denote its spectral norm and $\|M\|_F$ Frobenius norm. The notations $\mathcal{O}(\cdot), \Theta(\cdot)$, and $\Omega(\cdot)$ in the rest of the paper only omit absolute constants.

Asymmetric Matrix Sensing. Our primary goal is to recover an unknown fixed rank r matrix $M^\star \in \mathbb{R}^{n_1 \times n_2}$ from data (y_i, A_i), $i = 1, \ldots, m$ satisfying $y_i = \langle A_i, M^\star \rangle = \mathrm{tr}(A_i^\top M^\star), i = 1, \ldots, m,$ or compactly $y = \mathcal{A}(M^\star)$, where $y \in \mathbb{R}^m$ and $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ is a linear map with $[\mathcal{A}(M)]_i = \mathrm{tr}(A_i^\top M)$. We further denote the singular values of $M^\star$ as $\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = 0 = \cdots = \sigma_n$, the condition number $\kappa = \frac{\sigma_1}{\sigma_r}$, and the diagonal singular value matrix as Σ with $(\Sigma)_{ii} = \sigma_i$. To recover $M^\star$, we minimize the following loss function:

$$L_{\mathrm{tr}}(F, G) = \frac{1}{2}\|\mathcal{A}(FG^\top) - y\|^2, \tag{2.1}$$

where $F, G \in \mathbb{R}^{n \times k}$, where $k \geq r$ is the user-specified rank. The gradient descent update rule with a step size $\eta > 0$ with respect to loss (2.1) can be written explicitly as

$$F_{t+1} = F_t - \eta \mathcal{A}^* \mathcal{A}(F_t G_t^\top - \Sigma) G_t, \quad G_{t+1} = G_t - \eta \mathcal{A}^* \mathcal{A}(F_t G_t^\top - \Sigma) F_t, \tag{2.2}$$

where $\mathcal{A} : \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is the adjoint map of $\mathcal{A}$ and admits an explicit form: $\mathcal{A}^*(z) = \sum_{i=1}^m z_i A_i$.

From [4], if the matrix A_i has i.i.d. $N(0, \frac{1}{m})$, the operator $\mathcal{A}$ has RIP of order $2k+1$ with constant $\delta \in (0,1)$ when $m = \widetilde{\Omega}\left(\frac{nk}{\delta^2}\right)$.

Symmetric Matrix Sensing. When $F_t = G_t = X_t$ are the same at the initialization phase, the problem above becomes a symmetric matrix sensing problem. In this setting, we should further assume M^* is symmetric and positive semidefinite. See Appendix B for more details.

3. Symmetric Matrix Sensing with Symmetric Parameterization

We first consider the symmetric matrix sensing setting, where $M^* \in \mathbb{R}^{n\times n}$ is a positive semi-definite matrix of rank $r \ll n$. A standard approach is to use a factored form $XX^\top$ to learn M^* where $X \in \mathbb{R}^{n\times k}$. We call this *symmetric parameterization* because $XX^\top$ is always symmetric and positive semi-definite. We will also introduce the *asymmetric* parameterization soon. We call the case when $k = r$ the *exact-parameterization* because the rank of $XX^\top$ matches that of M^*. However, in practice, r is often unknown, so one may choose some large enough $k > r$ to ensure the expressiveness of $XX^\top$, and we call this case *over-parameterization*.

We consider using gradient descent to minimize the standard L_2 loss for training: $L_{\mathrm{tr}}(X) = \frac{1}{2m}\sum_{i=1}^m \left(y_i - \langle A_i, XX^\top\rangle\right)^2$. We use the Frobneius norm of the reconstruction error as the performance metric:

$$L(X) = \frac{1}{2}\|XX^\top - M^*\|_F^2. \tag{3.1}$$

We note that $L(X)$ is also the *matrix factorization* loss and can be viewed as a special case of L_{tr} when $\{A_i\}_{i=1}^m$ are random Gaussian matrices and the number of linear measurements goes to infinity.

Contribution 1: $\Omega(1/T^2)$ Lower Bound for Symmetric Over-Parameterization. Our first contribution is a rigorous exponential separation between the exact-parameterization and over-paramete rization by proving an $\Omega(1/T^2)$ convergence rate *lower bound* for the symmetric setting with the symmetric over-parameterization.

Theorem 1 (Informal Version of Theorem 6) *Suppose we initialize X with a Gaussian distribution with small enough variance that scales with α^2, and use gradient descent with a small enough constant step size to optimize the matrix factorization loss (3.1). Let X_t denote the factor matrix at the t-th iteration. Then with high probability over the initialization, there exists $T^{(0)} > 0$ such that we have*[1] [2]

$$\|X_tX_t^\top - M^*\|_F^2 \geq \left(\alpha^2/t\right)^2, \forall t \geq T^{(0)}. \tag{3.2}$$

Technical Insight: We find the root cause of the slow convergence is from the *redundant space* in $XX^\top$, which converges to 0 at much slower rate compared to the *signal space* which converges to M^* with a linear rate. To derive the lower bound, we construct a potential function and use some novel analyses of the updating rule to show that the potential function decreases slowly after a few rounds. See the precise theorem and more technical discussions in Apepndix C .

1. For clarity, in our informal theorems in Section 1, we only display the dependency on α and T, and ignore parameters such as dimension, condition number, and step size.
2. $T^{(0)}$ here and $T^{(1)}, T^{(2)}, T^{(3)}$ in theorems below represent the burn-in time to get to a neighborhood of an optimum, which can depend on initialization scale α, condition number, dimension, and step size.

4. Asymmetric and Symmetric Matrix Sensing with Asymmetric Parameterization

Next, we consider the more general asymmetric matrix sensing problem where the ground-truth $M^* \in \mathbb{R}^{n_1 \times n_2}$ is asymmetric matrix of rank r. For this setting, we must use the *asymmetric parameterization*. Specifically, we use $FG^\top$ to learn M^* where $F \in \mathbb{R}^{n_1 \times k}$ and $G \in \mathbb{R}^{n_2 \times k}$. Same as in the symmetric case, exact-parameterization means $k = r$ and over-parameterization means $k > r$. We still use gradient descent to optimize the L_2 loss for training:

$$L_{\mathrm{tr}}(F, G) = \frac{1}{2m} \sum_{i=1}^{m} \left(y_i - \langle A_i, FG^\top \rangle \right)^2, \tag{4.1}$$

and the performance metric is still: $L(F, G) = \frac{1}{2} \|FG^\top - M^*\|_F^2$.

Also note that even for the *symmetric matrix sensing problem* where M^* is positive semi-definite, one can still use *asymmetric parameterization*. Although doing so seems unnecessary at the first glance, we will soon see using asymmetric parameterization enjoys an *exponential gain*.

Contribution 2: Global Exact Convergence of Gradient Descent for Asymmetric Exact-Parameterization with a Linear Convergence Rate. Our second contribution is a global exact convergence result for randomly initialized gradient descent, and we show it enjoys a linear convergence rate.[3]

Theorem 2 (Informal version of Theorem 9) *In the exact-parameterization setting ($k = r$), suppose we initialize F and G using a Gaussian distribution with small enough variance α^2 and use gradient descent with a small enough constant step size to optimize the asymmetric matrix sensing loss (4.1). Let F_t and G_t denote the factor matrices at the t-the iteration. Then with high probability over the random initialization, there exists $T^{(1)} > 0$ such that we have*

$$\|F_t G_t^\top - M^*\|_F^2 = \exp\left(-\Omega\left(t\right)\right), \forall t \geq T^{(1)}. \tag{4.2}$$

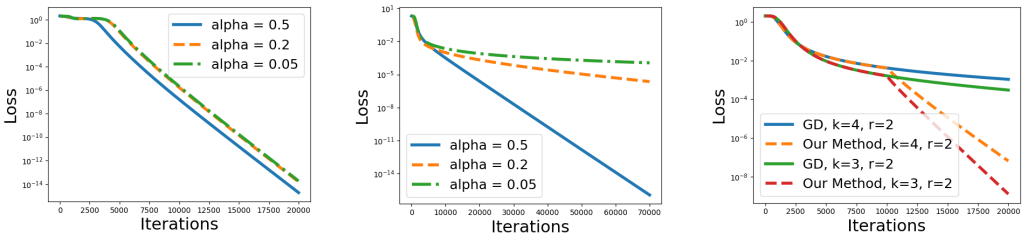
To our knowledge, this is the first global convergence result for randomly initialized GD. Prior result either requires initialization to be close to optimal or can only guarantee to find a point with error of similar scale as the initialization [29]. In contrast, our result only relies on random initialization and guarantees the error goes to 0 as t goes to infinity. Notably, this convergence rate is independent of α. See Figure 2(*a*)subfigure. Technically, unlike the symmetric case, here one cannot simply combine existing results because one needs an *exact balancing* of F and G to apply the local result in [23], and the result in [29] is not enough.

Contribution 3: Global Exact Convergence of Gradient Descent for Asymmetric Over-Parameterization with an Initialization-Dependent Linear Convergence Rate. Our next contribution is analogue theorem for the over-parameterization case with the caveat that the initialization scale α also appears in the convergence rate.

Theorem 3 (Informal Version of Theorem 8) *In the over-parameterization setting ($k > r$), suppose we initialize F and G using a Gaussian distribution with small enough variance α^2 and use gradient descent with a small enough constant step size to optimize the asymmetric matrix sensing loss (4.1). Let F_t and G_t denote the factor matrices at the t-the iteration. Then with high probability over the random initialization, there exists $T^{(2)} > 0$ such that we have*

$$\|F_t G_t^\top - M^*\|_F^2 = \exp\left(-\Omega\left(\alpha^2 t\right)\right), \forall t \geq T^{(2)}. \tag{4.3}$$

3. By exact convergence we mean the error goes to 0 as t goes to infinity in contrast to prior works which only guarantee to converge to a point with the error proportional to the initialization scale α.



(a) Exact-parameterized case (b) Over-parameterized case (c) Loss of our method

Figure 2: Curve of asymmetric matrix sensing. Figure 2(a)subfigure shows that the convergence rate is linear and independent on the initialization scale under the exact-parameterized case. Figure 2(b)subfigure shows that the convergence rate is linear and dependent on the initialization scale under the over-parameterized case. When the initialization scale is larger, the convergence speed is faster. Figure 2(c)subfigure shows the efficacy of our new method. See Appendix L for experimental details.

This is also the first global exact convergence result of randomly initialized gradient descent in the over-parameterized case. Recall that for the symmetric matrix sensing problem, even if M^* is positive semi-definite, one can still use an asymmetric parameterization $FG^\top$ to learn M^*, and Theorem 3 still holds. Comparing Theorem 3 and Theorem 1, we obtain a surprising corollary:

*For the **symmetric** matrix sensing problem, using **asymmetric** parameterization is **exponentially faster** than using symmetric parameterization.*

Also notice that different from Theorem 2, the convergence rate of Theorem 3 also depends on the initialization scale α which we require it to be small. Empirically we verify this dependency is necessary. See Figure 2(b)subfigure. We also study a special case in Section D.1 to show the dependency on the initialization scale is necessary in the worst case.

Technical Insight: Our key technical finding that gives the exponential acceleration is the *imbalance* of F and G. Denote the imbalance matrix $\Delta_t = F_t^\top F_t - G_t^\top G_t$. We show that the converge rate is linear when Δ_t is positive definite, and the rate depends on the minimum eigenvalue of Δ_t. We use imbalance initialization so that the minimum eigenvalue of Δ_0 is proportional to α, we can further show that the minimum eigenvalue Δ_t will not decrease too much, so the final convergence rate is linear. Furthermore, such a connection to α also inspires us to design a faster algorithm.

Contribution 4: A Simple Algorithm with Initialization-Independent Linear Convergence Rate for Asymetric Over-Parameterization. Our key idea is to *increase the degree of imbalance* when F and G are close to the optimum. We develop a new simple algorithm to accelerate GD. The algorithm only involves transforming the factor matrices F and G in one of iteration to intensify the degree of imbalance (cf. Equation (E.1)).

Theorem 4 (Informal Version of Theorem 10) *In the over-parameterization setting ($k > r$), suppose we initialize F and G using a Gaussian distribution with small enough variance α^2, gradient descent with a small enough constant step size, and the procedure described in Section E to optimize the asymmetric matrix sensing loss (4.1). Let F_t and G_t denote the factor matrices at the t-the iteration. Then with high probability over the random initialization, there exists $T^{(3)} > 0$ such that we have*

$$\|F_t G_t^\top - M^*\|_F^2 = \exp\left(-\Omega\left(t - T^{(3)}\right)\right), \forall t \geq T^{(3)}. \tag{4.4}$$

References

[1] Zeyuan Allen-Zhu, Yuanzhi Li, and Yingyu Liang. Learning and generalization in overparameterized neural networks, going beyond two layers. *Advances in neural information processing systems*, 32, 2019.

[2] Sanjeev Arora, Nadav Cohen, Wei Hu, and Yuping Luo. Implicit regularization in deep matrix factorization. *Advances in Neural Information Processing Systems*, 32, 2019.

[3] Emmanuel Candes and Benjamin Recht. Exact matrix completion via convex optimization. *Communications of the ACM*, 55(6):111–119, 2012.

[4] Emmanuel J Candes and Yaniv Plan. Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements. *IEEE Transactions on Information Theory*, 57(4):2342–2359, 2011.

[5] Chih-Fan Chen, Chia-Po Wei, and Yu-Chiang Frank Wang. Low-rank matrix recovery with structural incoherence for robust face recognition. In *2012 IEEE conference on computer vision and pattern recognition*, pages 2618–2625. IEEE, 2012.

[6] Lenaic Chizat, Edouard Oyallon, and Francis Bach. On lazy training in differentiable programming. *Advances in neural information processing systems*, 32, 2019.

[7] Mark A Davenport and Justin Romberg. An overview of low-rank matrix recovery from incomplete observations. *IEEE Journal of Selected Topics in Signal Processing*, 10(4):608–622, 2016.

[8] Simon S Du, Chi Jin, Jason D Lee, Michael I Jordan, Aarti Singh, and Barnabas Poczos. Gradient descent can take exponential time to escape saddle points. *Advances in neural information processing systems*, 30, 2017.

[9] Simon S Du, Wei Hu, and Jason D Lee. Algorithmic regularization in learning deep homogeneous models: Layers are automatically balanced. *Advances in neural information processing systems*, 31, 2018.

[10] Simon S Du, Xiyu Zhai, Barnabas Poczos, and Aarti Singh. Gradient descent provably optimizes over-parameterized neural networks. *arXiv preprint arXiv:1810.02054*, 2018.

[11] Raaz Dwivedi, Nhat Ho, Koulik Khamaru, Martin J Wainwright, Michael I Jordan, and Bin Yu. Singularity, misspecification and the convergence rate of em. 2020.

[12] Cong Fang, Jason Lee, Pengkun Yang, and Tong Zhang. Modeling from features: a mean-field framework for over-parameterized deep neural networks. In *Conference on learning theory*, pages 1887–1936. PMLR, 2021.

[13] Rong Ge, Chi Jin, and Yi Zheng. No spurious local minima in nonconvex low rank problems: A unified geometric analysis. In *International Conference on Machine Learning*, pages 1233–1242. PMLR, 2017.

[14] Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks. *Advances in neural information processing systems*, 31, 2018.

[15] Liwei Jiang, Yudong Chen, and Lijun Ding. Algorithmic regularization in model-free overparametrized asymmetric matrix factorization. *arXiv preprint arXiv:2203.02839*, 2022.

[16] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In *International conference on machine learning*, pages 1724–1732. PMLR, 2017.

[17] Jikai Jin, Zhiyuan Li, Kaifeng Lyu, Simon Shaolei Du, and Jason D Lee. Understanding incremental learning of gradient descent: A fine-grained analysis of matrix sensing. In *International Conference on Machine Learning*, pages 15200–15238. PMLR, 2023.

[18] Xingguo Li, Junwei Lu, Raman Arora, Jarvis Haupt, Han Liu, Zhaoran Wang, and Tuo Zhao. Symmetry, saddle points, and global optimization landscape of nonconvex matrix factorization. *IEEE Transactions on Information Theory*, 65(6):3489–3514, 2019.

[19] Yuanzhi Li, Tengyu Ma, and Hongyang Zhang. Algorithmic regularization in overparameterized matrix sensing and neural networks with quadratic activations. In *Conference On Learning Theory*, pages 2–47. PMLR, 2018.

[20] Zhiyuan Li, Yuping Luo, and Kaifeng Lyu. Towards resolving the implicit bias of gradient descent for matrix factorization: Greedy low-rank learning. *arXiv preprint arXiv:2012.09839*, 2020.

[21] Guangcan Liu, Zhouchen Lin, Shuicheng Yan, Ju Sun, Yong Yu, and Yi Ma. Robust recovery of subspace structures by low-rank representation. *IEEE transactions on pattern analysis and machine intelligence*, 35(1):171–184, 2012.

[22] Yiping Lu, Chao Ma, Yulong Lu, Jianfeng Lu, and Lexing Ying. A mean field analysis of deep resnet and beyond: Towards provably optimization via overparameterization from depth. In *International Conference on Machine Learning*, pages 6426–6436. PMLR, 2020.

[23] Cong Ma, Yuanxin Li, and Yuejie Chi. Beyond procrustes: Balancing-free gradient descent for asymmetric low-rank matrix sensing. *IEEE Transactions on Signal Processing*, 69:867–877, 2021.

[24] Phan-Minh Nguyen and Huy Tuan Pham. A rigorous framework for the mean field limit of multilayer neural networks. *arXiv preprint arXiv:2001.11443*, 2020.

[25] Yigang Peng, Jinli Suo, Qionghai Dai, and Wenli Xu. Reweighted low-rank matrix recovery and its application in image restoration. *IEEE transactions on cybernetics*, 44(12):2418–2430, 2014.

[26] Benjamin Recht, Maryam Fazel, and Pablo A Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM review*, 52(3):471–501, 2010.

[27] Frederieke Richert, Roman Worschech, and Bernd Rosenow. Soft mode in the dynamics of over-realizable online learning for soft committee machines. *Physical Review E*, 105(5): L052302, 2022.

[28] Itay Safran and Ohad Shamir. Spurious local minima are common in two-layer relu neural networks. In *International conference on machine learning*, pages 4433–4441. PMLR, 2018.

[29] Mahdi Soltanolkotabi, Dominik Stöger, and Changzhi Xie. Implicit balancing and regularization: Generalization and convergence guarantees for overparameterized asymmetric matrix sensing. *arXiv preprint arXiv:2303.14244*, 2023.

[30] Dominik Stöger and Mahdi Soltanolkotabi. Small random initialization is akin to spectral learning: Optimization and generalization guarantees for overparameterized low-rank matrix reconstruction. *Advances in Neural Information Processing Systems*, 34:23831–23843, 2021.

[31] Tian Tong, Cong Ma, and Yuejie Chi. Accelerating ill-conditioned low-rank matrix estimation via scaled gradient descent. *The Journal of Machine Learning Research*, 22(1):6639–6701, 2021.

[32] Stephen Tu, Ross Boczar, Max Simchowitz, Mahdi Soltanolkotabi, and Ben Recht. Low-rank solutions of linear matrix equations via procrustes flow. In *International Conference on Machine Learning*, pages 964–973. PMLR, 2016.

[33] Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge university press, 2018.

[34] Lingxiao Wang, Xiao Zhang, and Quanquan Gu. A unified computational and statistical framework for nonconvex low-rank matrix estimation. In *Artificial Intelligence and Statistics*, pages 981–990. PMLR, 2017.

[35] Colin Wei, Jason D Lee, Qiang Liu, and Tengyu Ma. Regularization matters: Generalization and optimization of neural nets vs their induced kernel. *Advances in Neural Information Processing Systems*, 32, 2019.

[36] Zhiyuan Weng and Xin Wang. Low-rank matrix completion for array signal processing. In *2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 2697–2700. IEEE, 2012.

[37] Fan Wu and Patrick Rebeschini. Implicit regularization in matrix sensing via mirror descent. *Advances in Neural Information Processing Systems*, 34:20558–20570, 2021.

[38] Yihong Wu and Harrison H Zhou. Randomly initialized em algorithm for two-component gaussian mixture achieves near optimality in o(n) iterations. *Mathematical Statistics and Learning*, 4(3), 2021.

[39] Weihang Xu and Simon Du. Over-parameterization exponentially slows down gradient descent for learning a single neuron. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 1155–1198. PMLR, 2023.

[40] Tian Ye and Simon S Du. Global convergence of gradient descent for asymmetric low-rank matrix factorization. *Advances in Neural Information Processing Systems*, 34:1429–1439, 2021.

[41] Gavin Zhang, Salar Fattahi, and Richard Y Zhang. Preconditioned gradient descent for over-parameterized nonconvex burer-monteiro factorization with global optimality certification. *J. Mach. Learn. Res.*, 24:163–1, 2023.

[42] Jialun Zhang, Salar Fattahi, and Richard Y Zhang. Preconditioned gradient descent for over-parameterized nonconvex matrix factorization. *Advances in Neural Information Processing Systems*, 34:5985–5996, 2021.

[43] Xiao Zhang, Lingxiao Wang, and Quanquan Gu. A unified framework for nonconvex low-rank plus sparse matrix recovery. In *International Conference on Artificial Intelligence and Statistics*, pages 1097–1107. PMLR, 2018.

[44] Xiao Zhang, Lingxiao Wang, Yaodong Yu, and Quanquan Gu. A primal-dual analysis of global optimality in nonconvex low-rank matrix recovery. In *International conference on machine learning*, pages 5862–5871. PMLR, 2018.

[45] Bo Zhao, Justin P Haldar, Cornelius Brinegar, and Zhi-Pei Liang. Low rank matrix recovery for real-time cardiac mri. In *2010 ieee international symposium on biomedical imaging: From nano to macro*, pages 996–999. IEEE, 2010.

[46] Tuo Zhao, Zhaoran Wang, and Han Liu. A nonconvex optimization framework for low rank matrix estimation. *Advances in Neural Information Processing Systems*, 28, 2015.

[47] Qinqing Zheng and John Lafferty. A convergent gradient descent algorithm for rank minimization and semidefinite programming from random linear measurements. *Advances in Neural Information Processing Systems*, 28, 2015.

[48] Zhihui Zhu, Qiuwei Li, Gongguo Tang, and Michael B Wakin. Global optimality in low-rank matrix optimization. *IEEE Transactions on Signal Processing*, 66(13):3614–3628, 2018.

[49] Zhihui Zhu, Qiuwei Li, Gongguo Tang, and Michael B Wakin. The global optimization geometry of low-rank matrix optimization. *IEEE Transactions on Information Theory*, 67(2): 1308–1331, 2021.

[50] Jiacheng Zhuo, Jeongyeol Kwon, Nhat Ho, and Constantine Caramanis. On the computational and statistical complexity of over-parameterized matrix sensing. *arXiv preprint arXiv:2102.02756*, 2021.

[51] Difan Zou, Yuan Cao, Dongruo Zhou, and Quanquan Gu. Gradient descent optimizes over-parameterized deep relu networks. *Machine learning*, 109:467–492, 2020.

Supplementary Material

Contents

1 Introduction **1**

2 Model and Prelimanaries **2**

3 Symmetric Matrix Sensing with Symmetric Parameterization **3**

4 Asymmetric and Symmetric Matrix Sensing with Asymmetric Parameterization **4**

A Related Work **12**

B Additional Preliminaries **13**

C Lower Bound of Symmetric Matrix Factorization **14**
C.1 Proof Sketch of Theorem 6 14

D Convergence of Asymmetric Matrix Sensing **15**
D.1 A Toy Example of Asymmetric Matrix Factorization 15
D.2 Theoretical Results for Asymmetric Matrix Sensing 16

E A Simple Fast convergence method **18**

F Proof of Theorem 6 **19**
F.1 Proof outline of Theorem 6 19
F.2 Phase 1 20
F.3 Phase 2 27
F.4 Phase 3: lower bound of convergence rate 29

G Proof of Theorem 7 **32**

H Proof of Theorem 8 **34**
H.1 Preliminaries 34
H.2 Proof Outline 35
H.3 Initial iterations 36
H.4 Phase 1: linear convergence phase. 38
H.5 Phase 2: Adjustment Phase. 41
H.6 Phase 3: local convergence 45

I Proof of Theorem 9 **55**

J Proof of Theorem 10 **57**
J.1 Proof Sketch of Theorem 10 57
J.2 Proof of Theorem 10 58

K Technical Lemma **62**
K.1 Proof of Lemma 11 62
K.2 Proof of Lemma 12 63

L Experiment Details **64**

Table 1: Comparison of previous representative work. The second row shows that the results hold for symmetric matrix factorization/sensing or asymmetric matrix factorization/sensing. The third row lists different types of initialization, where "Random" means the algorithm uses random initialization (typically Gaussian), "Local" indicates a requirement for initialization to be close to the optimal point. The fourth row "exact-cnvrg" represents whether the loss will go to zero when round T goes to infinity. The fifth row indicates whether the result applies to over-parameterization case or just the exact-parameterization case. The last row lists the convergence rate of algorithms with exact-convergence results.

	Is Symmetric	Init.	exact-cnvrg	k Range	Rate
[30]	Symmetric	Random	✗	$k \geq r$	N/A
[50]	Symmetric	Local	✓	$k \geq r$	$\mathcal{O}(1/T^2)$
[30]+[50]	Symmetric	Random	✓	$k \geq r$	$\mathcal{O}(1/T^2)$
[29]	Asymmetric	Random	✗	$k \geq r$	N/A
[32]	Both	Local	✓	$k = r$	$\exp(-\Omega(T))$
[23]	Asymmetric	Local	✓	$k = r$	$\exp(-\Omega(T))$
Theorem 2 (our paper)	Asymmetric	Random	✓	$k = r$	$\exp(-\Omega(T))$
Theorem 3 (our paper)	Asymmetric	Random	✓	$k > r$	$\exp(-\Omega(\alpha^2 T))$
Theorem 1 (our paper)	Symmetric	Random	✓	$k \geq r$	$\Omega(1/T^2)$

Appendix

Appendix A. Related Work

Slow Down Due to Over-parameterization Similar exponential slowdown phenomena caused by over-parameterization have been observed in other problems beyond matrix recovery, such as teacher-student neural network training [27, 39] and Expectation-Maximization algorithm on Gaussian mixture model [11, 38].

Matrix Sensing. Matrix sensing aims to recover the low-rank matrix based on measurements. In earlier years, previous papers [3, 21] propose convex optimization-based algorithms, which minimize the nuclear norm of a matrix, and another work [26] show that projected subgradient methods can recover the nuclear norm minimizer. The previous work [37] also proposes a mirror descent algorithm, which guarantees to converge to a nuclear norm minimizer. See [7] for a comprehensive review.

Factorization Approach. The factorization approach models the true matrix as the product of multiple matrices. For symmetric factorization, under the exact-parameterization setting $k = r$, papers [32, 47] shows the linear convergence of gradient descent when starting at a local point that is close to the optimal point. This initialization can be implemented by spectral initialization. However, when Paper [30] shows that with a small initialization, the gradient descent achieves a small error that is dependent on the initialization scale, rather than the *exact-convergence*. Paper [50] shows exact convergence with $\mathcal{O}(1/T^2)$ convergence rate in the overparamterization setting. Previous work [41, 42] have an intuitive discussion about why the gradient descent fails to achieve linear convergence, and provides a new preconditioner to make the gradient descent converge faster. However, there is no rigorous theoretical result to give a lower bound about the convergence rate.

In this project, we aim to provide a $\Omega(1/T^2)$ lower bound for the vanilla gradient descent for the over-parameterized symmetric matrix factorization problem.

In recent years, a flurry of works have studied the asymmetric factorization [29, 31, 40]. Many previous works [9, 13, 31, 32, 34, 40, 43, 44, 46] consider the exact-parameterization setting, where assume the user-specific rank k is exactly equal to true rank r. To be more specific, the papers [13, 32] adds a balance regularization term $\frac{1}{8}\|F^\top F - G^\top G\|_F^2$ to the loss function, to make sure that F and G are keeping balanced during the process of gradient descent. The paper [9] proves a global convergence result with a diminishing step size $\eta_t = \mathcal{O}(1/\sqrt{t})$. The paper [40] first proves the global convergence of the asymmetric matrix factorization problem in the exact-parameterization setting, with vanilla gradient descent and constant step size. The previous work [31] further consider the ill-conditioned situation when the condition number κ of $M^\star$ (i.e. $\kappa = \sigma_1(M^\star)/\sigma_r(M^\star)$) becomes large, and adds a scaled operator after the gradient. The authors show that the new algorithm can converge with a rate that is independent with κ, avoids bad performance in the ill-conditioned situation. Previous works [34, 43, 44, 46] consider either general problem formulations beyond the quadratic loss or more complicated algorithms beyond the gradient descent.

Although exact-parameterized asymmetric matrix factorization and matrix sensing problems have been explored intensively in the last decade, our understanding of the over-parameterization setting, i.e., $k > r$, remains limited. To our best knowledge, there are two very recent works [15, 29] consider the gradient descent for the asymmetric matrix sensing problem. Indeed, the recent work [15] considers the asymmetric matrix factorization setting, and proves that starting with a small initialization, the vanilla gradient descent sequentially recovers the principled component of the ground-truth matrix. Paper [29] proves the convergence of gradient descent in the asymmetric matrix sensing setting. However, both works prove that the gradient descent achieves a small error when stopped early, and the error depends on the initialization scale. However, whether the gradient descent can achieve *exact-convergence* remains open, and we solve this problem by extra techniques and analyses.

Landscape Analysis of Non-convex Low-rank Problems. There are some works establishing the general landscape analysis for the non-convex low-rank problems. The previous work [13] proposes a general framework for non-convex low-rank matrix problems using strict saddle property, and some following works [18, 48, 49] provide geometric analysis for solving general low-rank optimizations using the factorization approach. Note these paper on the global convergence requires injecting noise to the gradient descent dynamics [16] whereas we analyze the vanilla gradient descent. Injecting noise is required if one only uses the landscape analysis alone because there exists exponential lower bounds for standard GD [8]. One recent work [41] provides geometry landscape analysis of the over-parameterized non-convex Burer-Monteiro factorization using preconditioned gradient descent.

Appendix B. Additional Preliminaries

Diagonal Matrix Simplification. Since both the RIP and the loss are invariant to orthogonal transformation, we assume without loss generality that $M^\star = \Sigma$ in the rest of the paper for clarity, following prior work [15, 40]. For the same reason, we also assume $n_1 = n_2 = n$ to simplify notations, and our results can be easily extended to $n_1 \neq n_2$.

To make the problem approachable, we shall make the following standard assumption on $\mathcal{A}$: Restricted Isometry Property (RIP) [26].

Definition 5 (Restricted Isometry Property) *An operator $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ satisfies the Restricted Isometry Property of order r with constant $\delta > 0$ if for all matrices $M : \mathbb{R}^{n_1 \times n_2}$ with rank at most r, we have $(1-\delta)\|M\|_F^2 \le \|\mathcal{A}(M)\|^2 \le (1+\delta)\|M\|_F^2$.*

Symmetric Matrix Factorization. In this setting, we further assume $M^\star$ is symmetric and positive semidefinite, and $\mathcal{A}$ is the identity map. Since $M^\star$ admits a factorization $M^\star = F_\star F_\star^\top$ for some $F_\star \in \mathbb{R}^{n \times r}$, we can force the factor $F = G = X$ in (2.1) and the loss becomes $L(X) = \frac{1}{2}\|XX^\top - \Sigma\|_F^2$. Here, the factor $X \in \mathbb{R}^{n \times k}$. We shall focus on the over-parameterization setting, i.e., $k > r$ in the Section C . The gradient descent with a step size $\eta > 0$ becomes

$$X_{t+1} = X_t - 2\eta(X_t X_t^\top - M^*)X_t. \tag{B.1}$$

Appendix C. Lower Bound of Symmetric Matrix Factorization

We present a sublinear lower bound of the convergence rate of the gradient descent (B.1) for symmetric matrix factorization starting from a small random initialization. Our result supports the empirical observations that overparmetrization slows down gradient descent [41, 42, 50] and Figure 1.

Theorem 6 *Let $X_0 = \alpha \cdot \tilde{X}_0$, where each entry of $\tilde{X}_0$ is $\mathcal{N}(0, 1/k)$. For some universal constants $c_i, 1 \le i \le 7$, if the gradient descent method (B.1) starting at X_0 with the initial scale α, the search rank k, and the stepsize η satisfying that*

$$0 < \alpha \le \frac{c_1\sqrt{\sigma_1}}{\sqrt{n}\log(r\sqrt{n})}, \quad k \ge c_2\left((r\kappa)^2 \log(r\sqrt{\sigma_1}/\alpha)\right)^4, \quad \textit{and} \quad 0 < \eta \le \frac{c_3}{n^2\kappa\sigma_1}, \tag{C.1}$$

then with probability at least $1 - 2n^2\exp(-\sqrt{c_4 k}) - 2n\exp(-c_5 k/4)$, for all $t \ge T^{(0)} = \frac{c_6 \log(r\sqrt{\sigma_1})/\alpha}{\eta\sigma_r}$, we have

$$\|X_t X_t^\top - \Sigma\|_{\mathrm{F}}^2 \ge \left(\frac{c_7 \log(\sqrt{r\sigma_1}/\alpha)\alpha^2}{8\sigma_r \eta n t}\right)^2, \quad \forall t \ge T^{(0)}. \tag{C.2}$$

The proof of Theorem 1 demonstrates that, following a rapid convergence phase, the gradient descent eventually transitions to a *sublinear* convergence rate. Additionally, the ultimate error is influenced by the initialization scale α. Hence, opting for a smaller initialization scale leads the method to approach Σ at a linear rate, up to an error proportional to α^2, aligning with the findings in [30].

C.1. Proof Sketch of Theorem 6

We provide a proof sketch of Theorem 6 in this section, deferring the details to §F.

The main intuition of Theorem 6 is that the last $n - r$ rows of X_t, corresponding to the space of 0 eigenvalues of Σ, converge to 0 at speed no faster than $\frac{1}{T^2}$. To make this intuition precise, for each $t \ge 0$, we let $X_t \in \mathbb{R}^{n \times k} = [x_1^t, \cdots, x_n^t]^\top$ where $x_i^t \in \mathbb{R}^k$. We let the potential function be $A_t = \sum_{i>r} \|x_i^t\|^2$. We aim to show the following two key inequalities,

$$\|x_i^T\|^2 \ge \alpha^2/8, \text{ for all } i > r, \tag{C.3a}$$

$$A_{t+1} \ge A_t(1 - \mathcal{O}(\eta A_t)), \text{ for all } t \ge T^{(0)}. \tag{C.3b}$$

Suppose (C.3) is true, then it implies that $A_t \geq \mathcal{O}\left(\frac{\alpha^2}{t^2}\right)$ for all $t \geq T^{(0)}$. Since $(X_t X_t^\top - \Sigma)_{ii} = \|x_i\|^2$, the lower bound (C.2) is established by noting that $\|X_t X_t^\top - \Sigma\|_F^2 \geq \max_{i>r} \|x_i^t\|^2 \geq A_t/n$. We leave the details in checking (C.3) to §F.

Appendix D. Convergence of Asymmetric Matrix Sensing

Here we investigate the dynamic of GD in the context of the asymmetric matrix sensing problem. Surprisingly, we demonstrate that the convergence rate of gradient descent for asymmetric matrix sensing problems is linear, so long as the initialization is *imbalanced*. However, this linear rate is contingent upon the chosen initialization scale.

D.1. A Toy Example of Asymmetric Matrix Factorization

We first use a toy example of asymmetric matrix factorization to demonstrate the behavior of GD. If we assume $\mathcal{A}$ is the identity map, and the loss and the GD update become

$$L(F, G) = \frac{1}{2}\|FG^\top - \Sigma\|_F^2. \tag{D.1}$$

$$F_{t+1} = F_t - \eta(F_t G_t^\top - \Sigma)G_t, \quad G_{t+1} = G_t - \eta(F_t G_t^\top - \Sigma)^\top F_t \tag{D.2}$$

The following theorem tightly characterizes the convergence rate for a toy example.

Theorem 7 *Consider the asymmetric matrix factorization (D.1), with $k = r + 1$. Choose $\eta \in [0, 1/6]$ and $\alpha \in [0, 1]$. Assume that the diagonal matrix $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$, where $\sigma_i = 1$ for $i \leq r$ and is 0 otherwise. Also assume that gradient descent (D.2) starts at F_0, G_0, where $(F_0)_{ii} = \alpha$ for $1 \leq i \leq k$, and $(G_0)_{ii} = \alpha$ for $1 \leq i \leq r$, $(G_0)_{ii} = \alpha/3$ for $i = r+1$, and all other entries of F_0 and G_0 are zero. Then, the iterate (F_t, G_t) of (D.2) satisfies that*

$$\frac{\alpha^4}{36}(1 - 4\eta\alpha^2)^{2t} \leq \|F_t G_t^\top - \Sigma\|_F^2 \leq 4n \cdot (1 - \eta\alpha^2/4)^{(t - T_1)}, \ \forall t \geq T_1.$$

where $T_1 = c_1 \log(1/\alpha)/\eta$, and c_1 is a universal constant.

The above initialization implicitly assumes that we know the singular vectors of Σ. Such an assumption greatly simplifies our presentations below. Note that we have a different initialization scale for F_t and G_t. As we shall see, such an *imbalance* is the key to establishing the convergence of $F_t G_t^\top$.

We introduce some notations before our proof. Denote the matrix of the first r row of F, G as $U, V \in \mathbb{R}^{r \times k}$ respectively, and the matrix of the last $n - r$ row of F, G as $J, K \in \mathbb{R}^{(n-r) \times k}$ respectively. Further denote the corresponding iterate of gradient descent as U_t, V_t, J_t, and K_t. The difference $F_t G_t^\top - \Sigma$ can be written in a block form as $F_t G_t^\top - \Sigma = \begin{pmatrix} U_t V_t^\top - \Sigma_r & J_t V_t^\top \\ U_t K_t^\top & J_t K_t^\top \end{pmatrix}$ where $\Sigma_r \in \mathbb{R}^{r \times r}$ is the identity matrix. Hence, we may bound $\|F_t G_t^\top - \Sigma\|$ by

$$\|J_t K_t^\top\| \leq \|F_t G_t^\top - \Sigma\| \leq \|U_t V_t^\top - \Sigma_r\| + \|J_t V_t^\top\| + \|U_t K_t^\top\| + \|J_t K_t^\top\|. \tag{D.3}$$

From (D.3), we shall upper bound $\|U_t V_t^\top - \Sigma_r\|$, $\|J_t V_t^\top\|$, $\|U_t K_t^\top\|$, and $\|J_t K_t^\top\|$, and lower bound $\|J_t K_t\|^\top$. Let us now prove Theorem 7.

Proof With our particular initialization and the formula of gradient descent (D.2), we have the following equality for all t:

$$U_t K_t^\top = 0, \quad J_t V_t^\top = 0, \quad U_t = V_t, \quad J_t = a_t A, \quad K_t = b_t A, \quad U_t = (\alpha_t I_r, 0), \tag{D.4a}$$

$$a_0 = \alpha, \quad b_0 = \alpha/3, \quad \alpha_0 = \alpha \tag{D.4b}$$

$$a_{t+1} = a_t - \eta a_t b_t^2, \tag{D.4c}$$

$$b_{t+1} = b_t - \eta a_t^2 b_t. \tag{D.4d}$$

$$\alpha_{t+1} = \alpha_t(1 + \eta - \eta \alpha_t^2), \tag{D.4e}$$

where $A \in \mathbb{R}^{(n-r)\times k}$ is the matrix that $(A)_{1k} = 1$ and other elements are all zero, and $a_t, b_t, \alpha_t \in \mathbb{R}$. We leave the detailed verification of (D.4) to §G. By considering (D.3) and (D.4), we see that we only need to keep track of three sequences a_t, b_t, α_t. In particular, we have the following inequalities for a_t, b_t, α_t for all $t \geq T_1$:

$$a_t \in \left[\frac{1}{2}\alpha, \alpha\right], \; b_t \in \left[\frac{\alpha}{3}(1 - 4\eta\alpha^2)^t, \; \frac{\alpha}{3}(1 - \frac{\eta\alpha^2}{4})^{t/2}\right], \text{ and } |\alpha_t - 1| \leq (1 - \eta/2)^{t - T_1}. \tag{D.5}$$

It is then easy to derive the upper and lower bounds. We leave the detail in checking (D.5) to §G. Our proof is complete. ■

Technical Insight. This proof of the toy case tells us why the imbalance initialization in the asymmetric matrix factorization helps us to break the $\Omega(1/T^2)$ convergence rate lower bound of the symmetric case. Since we initialize F_0 and G_0 with a *different scale*, this difference makes the norm of K converge to zero at a linear rate while keeping J larger than a constant. However, in the symmetric case, we have $a_t = b_t$, so they must both converge to zero when the loss goes to zero (as $\|F_t G_t^\top - \Sigma\| \geq a_t b_t$), leading to a sublinear convergence rate. In short, the imbalance property in the initialization causes faster convergence in the asymmetric case.

D.2. Theoretical Results for Asymmetric Matrix Sensing

Motivated by the toy case in Section D.1, the imbalance property is the key ingredient for a linear convergence rate. If we use a slightly imbalanced initialization $F_0 = \alpha \cdot \tilde{F}_0, G_0 = (\alpha/3) \cdot \tilde{G}_0$, where the elements of $\tilde{F}_0$ and $\tilde{G}_0$ are $\mathcal{N}(0, 1/n)$, we have $\|F_0^\top F_0 - G_0^\top G_0\| = \Omega(\alpha^2)$. Then, we can show that the imbalance property keeps true when the step size is small, and thus, the gradient descent (2.2) converges with a linear rate similar to the toy case.

Our result is built upon the recent work [29] in dealing with the initial phase. Define the following quantities α_0, η_0 according to [29, Theorem 1]:

$$\alpha_0 = \frac{c\sqrt{\sigma_1}}{k^5 \max\{2n, k\}^2} \cdot \left(\frac{\sqrt{k} - \sqrt{r-1}}{\kappa^2 \sqrt{\max\{2n, k\}}}\right)^{C\kappa}, \eta_0 = \frac{c}{k^5 \sigma_1 \log\left(\frac{2\sqrt{2\sigma_1}}{\alpha(\sqrt{k} - \sqrt{r-1}}\right)}, \tag{D.6}$$

where c and C are some numerical constants. Below, we show exact convergence results for both $k = r$ and $k > r$.

Theorem 8 *Consider the matrix sensing problem (4.1) and the gradient descent (2.2). For some numerical constants $c_i > 0$, $1 \le i \le 7$, if the search rank k satisfies $r < k < \frac{n}{8}$, the initial scale α and η satisfy*

$$\alpha \le \min\left\{c_1\kappa^{-2}\sqrt{\sigma_r}, \alpha_0\right\}, \quad \eta \le \min\left\{c_1\alpha^4/\sigma_1^3, \eta_0\right\}, \tag{D.7}$$

where α_0, η_0 are defined in (D.6), and the operator $\mathcal{A}$ has the RIP of order $2k+1$ with constant δ satisfying

$$\delta\sqrt{2k+1} \le \min\left\{c_1\kappa^{-6}\log^{-1}(\sqrt{\sigma_r}/(n\alpha)), \frac{c_1}{\kappa^3\sqrt{r}}, 1/128\right\}, \tag{D.8}$$

then the gradient descent (2.2) starting with $F_0 = \alpha \cdot \tilde{F}_0, G_0 = (\alpha/3) \cdot \tilde{G}_0$, where $\tilde{F}_0, \tilde{G}_0 \in \mathbb{R}^{n\times k}$ whose entries are i.i.d. $\mathcal{N}(0, 1/n)$, satisfies that

$$\|F_tG_t^\top - \Sigma\|_F^2 \le \frac{\sigma_r^4 n}{c_7\alpha^4\kappa^2}\left(1 - \frac{\eta\alpha^2}{8}\right)^{t/4}, \quad \forall t \ge T^{(1)}, \tag{D.9}$$

with probability at least $1-2e^{-c_2n}-c_3e^{-c_4k}-(c_5\upsilon)^{(k-r+1)}$, where $T^{(1)} = c_6\log(\sqrt{\sigma_r}/n\alpha\upsilon)/\eta\sigma_r)$ and $\upsilon \in [0,1]$ is an arbitrary parameter.

Next, we state our results on exact parametrization.

Theorem 9 *Consider the same setting as Theorem 8 except assuming $k = r$, then with probability at least $1 - 2e^{-c_2n} - c_3e^{-c_4k} - c_5\upsilon$, the gradient descent (2.2) achieves*

$$\|F_tG_t^\top - \Sigma\|_F^2 \le 2n\sigma_r \cdot \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^t, \quad \forall t \ge T^{(2)}, \tag{D.10}$$

where $T^{(2)} = c_7\log(\sqrt{\sigma_r}/n\alpha\upsilon)/\eta\sigma_r)$ for some numerical constant c_7.

Now we highlight two bullet points of Theorem 8 and 9.

Exact Convergence. The main difference between the above theorems and previous convergence results in [29] is that we prove the *exact convergence* property, i.e., the loss finally degenerates to zero when T tends to infinity (cf. Table 1). Moreover, we prove that the convergence rate of the gradient descent depends on the initialization scale α, which matches our empirical observations in Figure 4.

Discussions about Parameters. First, since we utilize the initial phase result in [29] to guarantee that the loss degenerates to a small scale, our parameters δ, α, and η should satisfy the requirement $\delta_0 = \mathcal{O}(\frac{1}{\kappa^3\sqrt{r}}), \alpha_0, \eta_0$ in [29]. We further require $\delta_{2k+1} = \widetilde{\mathcal{O}}(\kappa^{-6})$, $\alpha = \mathcal{O}(\kappa^{-2}\sqrt{\sigma_r})$, which are both polynomials of the conditional number κ. In addition, the step size η has the requirement $\eta = \mathcal{O}(\alpha^4/\sigma_1^3)$, which can be much smaller than the requirements $\eta = \widetilde{\mathcal{O}}(1/\kappa^5\sigma_1)$ in [29]. In Section E, we propose a novel algorithm that allows larger learning rate which is independent of α.

Technical insight Similar to the asymmetric matrix factorization case in the proof of Theorem 7, the main effort is in characterizing the behavior of $J_tK_t^\top$. In particular, the update rule of K_t is

$$K_{t+1} = K_t(1 - \eta F_t^\top F_t) + \eta E, \tag{D.11}$$

where E is some error matrix since $\mathcal{A}$ is not an identity. Because of our initialization, we know the following holds for $t = 0$ and $\Delta_t = F_t^\top F_t - G_t^\top G_t$,

$$c\alpha^2 I \preceq \Delta_t \preceq C\alpha^2 I. \tag{D.12}$$

for some numerical constant $c, C > 0$. Hence, we can show $\|K_t\|$ shrinks towards 0 so long as (D.11) is true, $E = 0$, and G_t is well-bounded. Indeed, we can prove (D.12) and G_t, J_t upper bounded for all $t \geq 0$ via a proper induction. We may then be tempted to conclude $J_t K_t^\top$ converges to 0. However, the actual analysis of the gradient descent (2.2) for matrix sensing is much more complicated due to the error E. It is now unclear whether $\|K_t\|$ will shrink under (D.12). To deal with it, we further consider the structure of E. We leave the details to §H.

Appendix E. A Simple Fast convergence method

As discussed in Section D, the fundamental reason that the convergence rate depends on the initialization scaling α is that the imlabace between F and G determines the convergence rate, but the imbalance between F and G remains at the initialization scale. This observation motivates us to do a straightforward additional step in one iteration to *intensify the imbalance*. Specifically, suppose at the T_0 iteration we have reached a neighborhood of an optimum that satisfies: $\|\mathcal{A}^*\mathcal{A}(\widetilde{F}_{T^{(3)}}\widetilde{G}_{T^{(3)}}^\top - \Sigma)\| \leq \gamma$ where the radius $\sigma_r^{1/4} \cdot \|F_{T^{(3)}} G_{T^{(3)}}^\top\|^{3/4}/8$ is chosen for some technical reasons (cf. Section J). Here we use $\widetilde{F}_t$ and $\widetilde{G}_t$ to denote the iterates before we make the change we describe below and F_t and G_t to denote the iterates after make the change.

Let the singular value decomposition of $\widetilde{F}_{T^{(3)}} = A\Sigma' B$ with the diagonal matrix $\Sigma' \in \mathbb{R}^{k\times k}$ and $\Sigma'_{ii} = \sigma'_i$, then let $\Sigma_{inv} \in \mathbb{R}^{k\times k}$ be a diagonal matrix and $(\Sigma_{inv})_{ii} = \beta/\sigma'_i$ for some small constant $\beta = O(\sigma_r)$, then we transform the matrix $F_{T^{(3)}}, G_{T^{(3)}}$ by

$$F_{T^{(3)}} = \widetilde{F}_{T^{(3)}} B^\top \Sigma_{inv}, G_{T^{(3)}} = \widetilde{G}_{T^{(3)}} B \Sigma_{inv}^{-1} \tag{E.1}$$

We can show that, when F and G have reached a local region of an optimum, their magnitude will have similar scale as M^*. Therefore, the step Equation (E.1) can create an imbalance between F and G as large the magnitude of M^*, which is significantly larger than the initial scaling α. The following theorem shows we can obtain a convergence rate independent of the initialization scaling α. The proof is deferred to §J.

Theorem 10 *With the same setting as Theorem 8, suppose that at the step $T^{(3)}$ we have $\|\mathcal{A}^*\mathcal{A}(\widetilde{F}_{T^{(3)}}\widetilde{G}_{T^{(3)}}^\top - \Sigma)\| \leq \gamma$ for some $\gamma > 0$, and we do one step as in Equation (E.1). Then with probability at least $1 - 2e^{-c_2 n} - c_3 e^{-c_4 k} - (c_5 \upsilon)^{(k-r+1)}$, we have for all $t > T^{(3)}$,*

$$\|F_t G_t^\top - \Sigma\|_F^2 \leq \frac{n\beta^{12}}{\sigma_1^4}\left(1 - \frac{\eta\beta^2}{2}\right)^{2(t-T^{(3)})},$$

so long as $0 < c_7\gamma^{1/6}\sigma_1^{1/3} \leq \beta \leq c_8\sigma_r$, and the step size satisfies $\eta \leq c_9\beta^2/\sigma_1^2$ from the iteration $T^{(3)} \leq c_{10}\log(\sqrt{\sigma_r}/n\alpha\upsilon)/\eta\sigma_r$ for some positive numerical constants $c_i, i = 1, \ldots, 10$.

Appendix F. Proof of Theorem 6

In this proof, we denote

$$X \in \mathbb{R}^{n\times k} = \begin{bmatrix} x_1^\top \\ x_2^\top \\ \cdots \\ x_n^\top \end{bmatrix}, \tag{F.1}$$

where $x_i \in \mathbb{R}^{k\times 1}$ is the transpose of the row vector. Since the updating rule can be written as

$$X_{t+1} = X_t - \eta(X_t X_t^\top - \Sigma)X_t,$$

where we choose η instead of 2η for the simplicity, which does not influence the subsequent proof. By substituting the equation (F.1), the updating rule can be written as

$$(x_i^{t+1})^\top = (1-\eta(\|x_i^t\|^2 - \sigma_i))x_i^\top - \sum_{j=1, j\neq i}^{n} \eta((x_i^t)^\top x_j^t (x_j^t)^\top)$$

where $\sigma_i = 0$ for $i > r$. Denote

$$\theta = \max_{j,k} \frac{(x_j^\top x_k)^2}{\|x_j\|^2\|x_k\|^2}$$

is the maximum angle between different vectors in $x_1, \cdots, x_n$. We start with the outline of the proof.

F.1. Proof outline of Theorem 6

Recall we want to establish the key inequalities (C.3). The updating rule (B.1) gives the following lower bound of x_i^{t+1} for $i > r$:

$$\|x_i^{t+1}\|^2 \geq \|x_i^t\|^2 \left(1 - 2\eta\theta_t^U \sum_{j\leq r} \|x_j^t\|^2 - 2\eta \sum_{j>r} \|x_j^t\|^2\right), \tag{F.2}$$

where the quantity $\theta_t^U = \max_{i,j:\min\{i,j\}\leq r} \theta_{ij,t}$ and the square cosine $\theta_{ij,t} = \cos^2 \angle(x_i, x_j)$. Thus, to establish the key inequalities (C.3), we need to control the quantity θ_t^U. Our analysis then consists of three phases. In the last phase, we show (C.3) holds and our proof is complete.

In the first phase, we show that $\|x_i^t\|^2$ for $i \leq r$ becomes large, while $\|x_i^t\|^2$ for $i > r$ still remains small yet bounded away from 0. In addition, the quantity $\theta_{ij,t}$ remains small. Phase 1 terminates when $\|x_i^t\|^2$ is larger than or equal to $\frac{3}{4}\sigma_i$.

After the first phase terminates, in the second and third phases, we show that θ_t^U converges to 0 linearly and the quantity $\theta_t^U \sigma_1 / \sum_{j>r} \|x_j^t\|^2$ converges to zero at a linear rate as well. We also keep track of the magnitude of $\|x_i^t\|^2$ and show $\|x_i^t\|$ stays close to σ_i for $i \leq r$, and $\|x_i^t\|^2 \leq 2\alpha^2$.

The second phase terminates once $\theta_t^U \leq \mathcal{O}(\sum_{j>r} \|x_j^t\|^2/\sigma_1)$ and we enter the last phase: the convergence behavior of $\sum_{j>r} \|x_j^t\|^2$. Note with $\theta_t^U \leq \mathcal{O}(\sum_{j>r} \|x_j^t\|^2/\sigma_1)$ and $\|x_i^t\|^2 \leq 2\sigma_r$ for $i \leq r$, we can prove (C.3b). The condition (C.3a) can be proven since the first two phases are quite short and the updating formula of x_i for $i > r$ shows $\|x_i\|^2$ cannot decrease too much.

F.2. Phase 1

In this phase, we show that $\|x_i^t\|^2$ for $i \le r$ becomes large, while $\|x_i^t\|^2$ for $i > r$ still remains small. In addition, the maximum angle between different column vectors remains small. Phase 1 terminates when $\|x_i^t\|^2$ is larger than a constant.

To be more specific, we have the following two lemmas. Lemma 11 states that the initial angle $\theta_0 = \mathcal{O}(\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa)^2)$ is small because the vectors in the high-dimensional space are nearly orthogonal.

Lemma 11 *For some constant c_4 and c, if $k \ge \frac{c^2}{16\log^4(r\sqrt{\sigma_1}/\alpha)(r\kappa)^4}$, with probability at least $1 - c_4 n^2 k \exp(-\sqrt{k})$, we have*

$$\theta_0 \le \frac{c}{\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa)^2} \tag{F.3}$$

Proof See §K.1 for proof. ■

Lemma 12 states that with the initialization scale α, the norm of randomized vector x_i^0 is $\Theta(\alpha^2)$.

Lemma 12 *With probability at least $1 - 2n\exp(-c_5 k/4)$, for some constant c, we have*

$$\|x_i^0\|^2 \in [\alpha^2/2, 2\alpha^2].$$

Proof See §K.2 for the proof. ■

Now we prove the following three conditions by induction.

Lemma 13 *There exists a constant C_1, such that $T_1 \le C_1(\log(\sqrt{\sigma_1}/n\alpha)/\eta\sigma_r)$ and then during the first T_1 rounds, with probability at least $1 - 2c_4 n^2 k\exp(-\sqrt{k}) - 2n\exp(-c_5 k/4)$ for some constant c_4 and c_5, the following four statements always hold*

$$\|x_i^t\|^2 \le 2\sigma_1 \tag{F.4}$$

$$\alpha^2/4 \le \|x_i^t\|^2 \le 2\alpha^2 \quad (i > r) \tag{F.5}$$

$$2\theta_0 \ge \theta_t \tag{F.6}$$

Also, if $\|x_i^t\|^2 \le 3\sigma_i/4$, we have

$$\|x_i^{t+1}\|^2 \ge (1 + \eta\sigma_r/4)\|x_i^t\|^2. \tag{F.7}$$

Moreover, at T_1 rounds, $\|x_i^{T_1}\|^2 \ge 3\sigma_i/4$, and Phase 1 terminates.

Proof

By Lemma 11 and Lemma 12, with probability at least $1-2c_4 n^2 k\exp(-\sqrt{k})-2n\exp(-c_5 k/4)$, we have $\|x_i^0\|^2 \in [\alpha^2/2, 2\alpha^2]$ for $i \in [n]$, and $\theta_0 \le \frac{c}{\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa)^2}$. Then assume that the three conditions hold for rounds before t, then at the $t+1$ round, we proof the four statements above one by one.

Proof of Eq.(F.5) For $i > r$, we have

$$(x_i^{t+1})^\top = (x_i^t)^\top - \eta \sum_{j=1}^n (x_i^t)^\top x_j^t (x_j^t)^\top$$

Then the updating rule of $\|x_i^t\|^2$ can be written as

$$\|(x_i^{t+1})\|_2^2 = \|x_i^t\|^2 - 2\eta \sum_{j=1}^n ((x_i^t)^\top x_j^t)^2 + \eta^2 (\sum_{j,k=1}^n (x_i^t)^\top x_j^t (x_j^t)^\top x_k^t (x_k^t)^\top x_i^t) \leq \|x_i^t\|^2. \tag{F.8}$$

The last inequality in (F.8) is because

$$(x_i^t)^\top x_j^t (x_j^t)^\top x_k^t (x_k^t)^\top (x_i^t) \leq (x_j^t)^\top x_k^t (((x_i^t)^\top x_j^t)^2 + ((x_k^t)^\top x_i^t)^2)/2 \tag{F.9}$$

$$\leq \sigma_1 ((x_i^t)^\top x_j^t)^2 + ((x_k^t)^\top x_i^t)^2), \tag{F.10}$$

and then

$$\begin{aligned} \eta^2 \sum_{j,k=1}^n (x_i^t)^\top x_j^t (x_j^t)^\top x_k^t (x_k^t)^\top (x_i^t) &\leq \eta^2 \sum_{j,k=1}^n \sigma_1 ((x_i^t)^\top x_j^t)^2 + ((x_k^t)^\top x_i^t)^2) \\ &= \eta^2 \cdot n\sigma_1 \sum_{j=1}^n ((x_i^t)^\top x_j^t)^2 \\ &\leq \eta \sum_{j=1}^n ((x_i^t)^\top x_j^t)^2. \end{aligned} \tag{F.11}$$

where the last inequality holds because $\eta \leq 1/n\sigma_1$. Thus the ℓ_2-norm of $x_i^\top$ does not increase, and the right side of Eq.(F.5) holds.

Also, we have

$$\begin{aligned} \|x_i^{t+1}\|^2 &\geq \|x_i^t\|^2 - 2\eta \sum_{j=1}^n ((x_i^t)^\top x_j^t)^2 + \eta^2 \left\| \sum_{j=1}^n (x_i^t)^\top x_j^t (x_j^t)^\top \right\|^2 \\ &\geq \|x_i^t\|^2 - \|x_i^t\|^2 \cdot 2\eta\theta_t \cdot \sum_{j \neq i}^n \|x_j^t\|^2 - 2\eta \|x_i\|^4 \end{aligned} \tag{F.12}$$

Equation (F.2) is because $\frac{((x_i^t)^\top x_j^t)^2}{\|x_i^t\|^2 \|x_j^t\|^2} = \theta_{ij,t} \leq \theta_t$. Now by (F.4) and (F.5), we can get

$$\sum_{j \neq i}^n \|x_j^t\|^2 \leq r \cdot 2\sigma_1 + (n-r) \cdot 2\alpha^2 \leq 2\sigma_1 + 2n\alpha^2$$

Hence we can further derive

$$\begin{aligned} \|x_i^{t+1}\|^2 &\geq \|x_i^t\|^2 \cdot \left(1 - 2\eta\theta_t (2r\sigma_1 + 2n\alpha^2) - 2\eta \cdot 2\alpha^2\right) \\ &\geq \|x_i^t\|^2 \cdot \left(1 - \eta(8\theta_t \sigma_1 + 4\alpha^2)\right), \end{aligned}$$

where the last inequality is because $\alpha \le \sqrt{r\sigma_1}/\sqrt{n}$. Thus by $(1-a)(1-b) \ge (1-a-b)$ for $a, b > 0$, we can get

$$\begin{aligned}
\|x_i^{T_1}\|^2 &\ge \|x_i^0\|^2 \cdot (1 - \eta(8\theta_t\sigma_1 + 4\alpha^2))^{T_1} \\
&\ge \frac{\alpha^2}{2} \cdot (1 - T_1\eta(8 \cdot (2\theta_0)\sigma_1 + 4\alpha^2)) && \text{(F.13)} \\
&\ge \frac{\alpha^2}{4} && \text{(F.14)}
\end{aligned}$$

Equation (F.13) holds by induction hypothesis (F.6), and the last inequality is because of our choice on T_1, α, and $\theta_0 \le O(\frac{1}{r\kappa\log(\sqrt{\sigma_1}/\alpha)})$ from the induction hypothesis. Hence, we complete the proof of Eq.(F.5).

Proof of Eq.(F.7) For $i \le r$, if $\|x_i^t\|^2 \le 3\sigma_i/4$, by the updating rule,

$$\begin{aligned}
\|x_i^{t+1}\|_2^2 &\ge (1 - \eta(\|x_i^t\|^2 - \sigma_i))^2\|x_i^t\|^2 - 2\eta\sum_{j\neq i}^{n}((x_i^t)^\top x_j^t)^2 + \eta^2(\|x_i^t\|^2 - \sigma_i)\sum_{j\neq i}^{n}((x_i^t)^\top x_j^t)^2 && \text{(F.15)} \\
&\ge (1 - \eta(\|x_i^t\|^2 - \sigma_i))^2\|x_i^t\|^2 - 2\eta\sum_{j\neq i}^{n}((x_i^t)^\top x_j^t)^2 - \eta^2|\|x_i^t\|^2 - \sigma_i| \cdot \sum_{j\neq i}^{n}\|x_i^t\|^2\|x_j^t\|^2 \\
&\ge (1 - \eta(\|x_i^t\|^2 - \sigma_i))^2\|x_i^t\|^2 - 2\eta\sum_{j\neq i}^{n}((x_i^t)^\top x_j^t)^2 - 4\eta^2(n\sigma_1^2)\|x_i^t\|^2.
\end{aligned}$$

THe last inequality uses the fact that $|\|x_i^t\|^2 - \sigma_i| \le 2\sigma_1$ and $\|x_j^t\|^2 \le 2\sigma_1$. Then by $((x_i^t)^\top x_j^t)^2 \le \|x_i^t\|^2\|x_j^t\|^2 \cdot \theta$, we can further get

$$\begin{aligned}
\|x_i^{t+1}\|^2 &\ge \left(1 - 2\eta(\|x_i^t\|^2 - \sigma_i) - 2\eta\sum_{j\neq i}^{n}\|x_j^t\|^2\theta - 2\eta^2(n\sigma_1^2)\right)\|x_i^t\|^2 \\
&\ge (1 + \eta\sigma_i/2 - 2\eta^2(n\sigma_1^2) - \eta\sigma_r/16)\|x_i^t\|^2 && \text{(F.16)} \\
&\ge (1 + \sigma_i(\eta/2 - \eta/16 - \eta/16))\|x_i^t\|^2 && \text{(F.17)} \\
&\ge (1 + \eta\sigma_i/4)\|x_i^t\|^2.
\end{aligned}$$

The inequality (F.16) uses the fact $\theta \le 2\theta_0 \le \frac{1}{128\kappa r}$ and $\sum_{j\neq i}^{n}\|x_j\|^2 \le 2\sigma_1 r + 2n\alpha^2 \le 4\sigma_1 r \le \frac{\sigma_r}{32\theta}$. The inequality (F.17) uses the fact that $\eta \le \frac{1}{32n\sigma_1^2}$.

Proof of Eq.(F.4) If $\|x_i^t\|^2 \geq 3\sigma_i/4$, by the updating rule, we can get

$$
\begin{aligned}
|\|x_i^{t+1}\|_2^2 - \sigma_i| \leq & \left(1 - 2\eta\|x_i^t\|^2 + \eta^2(\|x_i^t\|^2 - \sigma_i)\|x_i^t\|^2 + \eta^2 \sum_{j\neq i}^{n} ((x_i^t)^\top x_j^t)^2\right) |\|x_i^t\|^2 - \sigma_i| \\
& + 2\eta \sum_{j\neq i}^{n} ((x_i^t)^\top x_j^t)^2 + \eta^2 \left(\sum_{j,k\neq i}^{n} ((x_i^t)^\top x_j^t (x_j^t)^\top x_k^t (x_k^t)^\top x_i^t)\right) \\
\leq & (1 - \eta\sigma_i)|\|x_i^t\|^2 - \sigma_i| + 3\,\eta \underbrace{\sum_{j\neq i}^{n} ((x_i^t)^\top x_j^t)^2}_{\text{(a)}} \qquad \text{(F.18)}
\end{aligned}
$$

The last inequality holds by Eq.(F.11) and

$$
\begin{aligned}
& 2\eta\|x_i^t\|^2 - \eta^2(\|x_i^t\|^2 - \sigma_i)\|x_i^t\|^2 - 2\eta^2 \sum_{j\neq i}^{n} ((x_i^t)^\top x_j^t)^2 && \text{(F.19)} \\
& \geq \frac{3\eta}{2}\sigma_i - \eta^2(2\sigma_1)\cdot 2\sigma_1 - 2\eta^2 n\sigma_1^2 && \text{(F.20)} \\
& \geq \eta\sigma_i, && \text{(F.21)}
\end{aligned}
$$

where (F.20) holds by $\|x_i^t\|^2 \geq \frac{3\sigma_i}{4}$, $\|x_i^t\|^2 \leq 2\sigma_1$ for all $i \in [n]$. The last inequality (F.21) holds by $\eta \leq C(\frac{1}{n\sigma_1\kappa})$ for small constant C. The first term of (F.18) represents the main converge part, and (a) represents the perturbation term. Now for the perturbation term (a), since $\alpha \leq \frac{1}{4\kappa n^2}$ and $\theta \leq 2\theta_0 \leq \frac{1}{20r\kappa^2} = \frac{\sigma_i^2}{20r\sigma_1^2}$, we can get

$$
\begin{aligned}
\text{(a)} = & \sum_{j\neq i, j\leq r} ((x_i^t)^\top x_j^t)^2 + \sum_{j\neq i, j>r} ((x_i^t)^\top x_j^t)^2 && \text{(F.22)} \\
\leq & (r\sigma_1 + 2n\alpha^2)\theta_t \cdot 2\sigma_1 && \text{(F.23)} \\
\leq & 2r\sigma_1 \cdot \theta_t \cdot 2\sigma_1 && \text{(F.24)} \\
= & 4r\sigma_1^2 \cdot \theta_t \\
\leq & \sigma_i^2/5, && \text{(F.25)}
\end{aligned}
$$

where (F.23) holds by (F.4) and (F.5). (F.24) holds by $\alpha = \mathcal{O}(\sqrt{r\sigma_1/n})$, and the last inequality (F.25) holds by θ is small, i.e. $\theta_t \leq 2\theta_0 = \mathcal{O}(1/r\kappa^2)$. Now it is easy to get that $(x_i^{t+1})^\top x_i^{t+1} \leq 2\sigma_i$ by

$$
|\|x_i^{t+1}\|^2 - \sigma_i| \leq (1 - \eta\sigma_i)(\|x_i^t\|^2 - \sigma_i) + \frac{3\eta\sigma_i^2}{5} \leq (1 - \eta\sigma_i)\sigma_i + \frac{3\eta\sigma_i^2}{5} \leq \sigma_i. \qquad \text{(F.26)}
$$

Hence we complete the proof of Eq.(F.4).

Proof of Eq.(F.6) Now we consider the change of θ. For $i \neq j$, denote

$$
\theta_{ij,t} = \frac{((x_i^t)^\top x_j^t)^2}{\|x_i\|^2\|x_j\|^2}
$$

Now we first calculate the $(x_i^{t+1})^\top x_j^{t+1}$ by the updating rule:

$$
\begin{aligned}
&(x_i^{t+1})^\top x_j^{t+1} \\
&= \underbrace{\left(1-\eta(\|x_i^t\|^2-\sigma_i)\right)\left(1-\eta(\|x_j^t\|^2-\sigma_j)\right)(x_i^t)^\top x_j^t}_{\text{A}} \underbrace{-\eta\|x_j^t\|^2(1-\eta(\|x_j^t\|^2-\sigma_j))(x_i^t)^\top x_j^t}_{\text{B}} \\
&\quad \underbrace{-\eta\|x_i^t\|^2(1-\eta(\|x_i^t\|^2-\sigma_j))(x_i^t)^\top x_j^t}_{\text{C}} + \underbrace{\eta^2\sum_{k,l\neq i,j}(x_i^t)^\top x_k^t(x_k^t)^\top x_l^t(x_l^t)^\top x_j^t}_{\text{D}} \\
&\quad \underbrace{-\eta(2-\eta(\|x_i^t\|^2-\sigma_i)-\eta(\|x_j^t\|^2-\sigma_j))\sum_{k\neq i,j}^{n}(x_i^t)^\top x_k^t(x_k^t)^\top x_j}_{\text{E}} \\
&\quad \underbrace{+\eta^2\sum_{k\neq i,j}x_i^\top x_j^t(x_j^t)^\top x_k^t(x_k^t)^\top x_j^t + \eta^2\sum_{k\neq i,j}(x_i^t)^\top x_k^t(x_k^t)^\top x_i^t(x_i^t)^\top x_j^t}_{\text{F}}.
\end{aligned}
$$

Now we bound A, B, C, D, E and F respectively. First, by $\|x_i^t\|^2 \le 2\sigma_1$ for any $i \in [m]$, we have

$$
\begin{aligned}
\text{A} &\le \left(1-\eta(\|x_i^t\|^2-\sigma_i)-\eta(\|x_j^t\|^2-\sigma_j)+\eta^2(\|x_i^t\|^2-\sigma_i)\left(\|x_j^t\|^2-\sigma_j)\right)\right)(x_i^t)^\top x_j^t \\
&\le \left(1-\eta\left(\|x_i^t\|^2+\|x_j^t\|^2-\sigma_i-\sigma_j\right)+\eta^2\cdot 4\sigma_1^2\right)(x_i^t)^\top x_j^t, \qquad \text{(F.27)}
\end{aligned}
$$

Now we bound term B. We have

$$
\begin{aligned}
\text{B}+\text{C} &= \left(-\eta(\|x_i^t\|^2+\|x_j^t\|^2)+\eta^2\left((\|x_j^t\|^2-\sigma_j)\|x_j^t\|^2+(\|x_i^t\|^2-\sigma_i)\|x_i^t\|^2\right)\right)(x_i^t)^\top x_j^t \\
&\le \left(-\eta(\|x_i^t\|^2+\|x_j^t\|^2)+\eta^2\cdot(8\sigma_1^2)\right)(x_i^t)^\top x_j^t. \qquad \text{(F.28)}
\end{aligned}
$$

Then, for D, by $\theta_t \le 1$, we have

$$
\begin{aligned}
\text{D} &= \eta^2\left(\sum_{k,l\neq i,j}\|x_k^t\|^2\|x_l^t\|^2\cdot\sqrt{\theta_{ik,t}\theta_{kl,t}\theta_{lj,t}/\theta_{ij,t}}\right)(x_i^t)^\top x_j^t \\
&\le \left(\eta^2\cdot n^2\cdot 4\sigma_1^2\cdot\theta_t/\sqrt{\theta_{ij,t}}\right)(x_i^t)^\top x_j^t. \qquad \text{(F.29)}
\end{aligned}
$$

For E, since we have

$$
\begin{aligned}
\text{E} &\le 2\eta\sum_{k\neq i,j}|(x_i^t)^\top x_k^t(x_k^t)^\top x_j^t| + 4\sigma_1\eta^2\sum_{k\neq i,j}|(x_i^t)^\top x_k^t(x_k^t)^\top x_j^t| \\
&\le \left(2\eta\sum_{k\neq i,j}\|x_k^t\|^2\cdot\sqrt{\theta_{ik,t}\theta_{kj,t}/\theta_{ij,t}} + 4\sigma_1\eta^2\sum_{k\neq i,j}\|x_k^t\|^2\cdot\sqrt{\theta_{ik,t}\theta_{kj,t}/\theta_{ij,t}}\right)(x_i^t)^\top x_j^t \\
&\le \left(2\eta\sum_{k\neq i,j}\|x_k^t\|^2\cdot\sqrt{\theta_{ik,t}\theta_{kj,t}/\theta_{ij,t}} + 4n\sigma_1\eta^2\cdot(2\sigma_1)\cdot\theta_t/\sqrt{\theta_{ij,t}}\right)(x_i^t)^\top x_j^t. \qquad \text{(F.30)}
\end{aligned}
$$

Lastly, for F, since $(x_j^t)^\top x_k^t (x_k^t)^\top x_j^t \le \|x_j^t\|^2 \|x_k^t\|^2 \le 4\sigma_1^2$, we have

$$\mathrm{F} \le \eta^2 8n\sigma_1^2 (x_i^t)^\top x_j^t. \tag{F.31}$$

Now combining (F.27), (F.28), (F.29), (F.30) and (F.31), we can get

$$(x_i^{t+1})^\top x_j^{t+1} \tag{F.32}$$

$$\le \left(1 - \eta(2\|x_i\|^2 + 2\|x_j\|^2 - \sigma_i - \sigma_j) + 2\eta \sum_{k\neq i,j} \|x_k\|^2 \cdot \sqrt{\theta_{ik,t}\theta_{kj,t}/\theta_{ij,t}} + 30n^2\sigma_1^2\eta^2\theta_t/\sqrt{\theta_{ij,t}})\right)(x_i^t)^\top x_j^t. \tag{F.33}$$

On the other hand, consider the change of $\|x_i^t\|^2$. By Eq.(F.15),

$$\begin{aligned}
\|x_i^{t+1}\|^2 &\ge (1 - \eta(\|x_i^t\|^2 - \sigma_i))^2 \|x_i^t\|^2 - 2\eta \sum_{j\neq i}^n ((x_i^t)^\top x_j^t)^2 + \eta^2(\|x_i^t\|^2 - \sigma_i) \sum_{j\neq i}^n ((x_i^t)^\top x_j^t)^2 \\
&\ge (1 - 2\eta(\|x_i^t\| - \sigma_i) - 2\eta \sum_{j\neq i}^n \|x_j^t\|^2 \theta_{ij,t} - 4\eta^2 n\theta_t\sigma_1^2)\|x_i^t\|^2 \\
&\ge (1 - 2\eta(\|x_i^t\| - \sigma_i) - 2\eta \sum_{k=1}^n \|x_j^t\|^2 \theta_{ij,t} - 4\eta^2 n\theta_t\sigma_1^2)\|x_i^t\|^2
\end{aligned}$$

Hence the norm of x_i^{t+1} and x_j^{t+1} can be lower bounded by

$$\begin{aligned}
&\|x_i^{t+1}\|^2 \|x_j^{t+1}\|^2 \\
&\ge \Big(1 - 2\eta(\|x_i^t\|^2 - \sigma_i) - 2\eta(\|x_j^t\|^2 - \sigma_j) - 2\eta \sum_{k\neq i,j} \|x_k\|^2(\theta_{ik,t} + \theta_{jk,t}) - 2\eta(\|x_j\|^2 + \|x_i\|^2)\theta_{ij,t} \\
&\qquad - 4\eta^2\theta_t n^2\sigma_1^2 + \sum_{l=i,j} 4\eta^2(\|x_l^t\|^2 - \sigma_l) \sum_{k=1}^n \|x_k^t\|^2 \theta_{ik,t} + \sum_{l=i,j} 2\eta(\|x_l^t\|^2 - \sigma_l)\eta^2 n^2\theta_t\sigma_1^2\Big)\|x_i^t\|^2\|x_j^t\|^2 \\
&\ge \Big(1 - 2\eta(\|x_i^t\|^2 - \sigma_i) - 2\eta(\|x_j^t\|^2 - \sigma_j) - 2\eta \sum_{k\neq i,j} \|x_k\|^2(\theta_{ik,t} + \theta_{jk,t}) - 2\eta(\|x_j\|^2 + \|x_i\|^2)\theta_{ij,t} \\
&\qquad - 4\eta^2\theta_t n^2\sigma_1^2 - 2\cdot 4\eta^2 \cdot (2\sigma_1)n \cdot (2\sigma_1)\theta_t - 2\cdot 4\eta\sigma_1 \cdot \eta^2 n^2\theta_t\sigma_1^2\Big)\|x_i^t\|^2\|x_j^t\|^2 \qquad \text{(F.34)} \\
&\ge \Big(1 - 2\eta(\|x_i^t\|^2 - \sigma_i) - 2\eta(\|x_j^t\|^2 - \sigma_j) - 2\eta \sum_{k\neq i,j} \|x_k\|^2(\theta_{ik,t} + \theta_{jk,t}) - 2\eta(\|x_j\|^2 + \|x_i\|^2)\theta_{ij,t} \\
&\qquad - 6\eta^2\theta_t n^2\sigma_1^2\Big)\|x_i^t\|^2\|x_j^t\|^2, \qquad \text{(F.35)}
\end{aligned}$$

where (F.35) holds by $n > 8k \ge 8$ and $2\eta(\|x_i^t\|^2 - \sigma_i) \le 4\eta\sigma_1 \le 1$. Then by (F.33) and (F.35), we have

$$\begin{aligned}
\theta_{ij,t+1} &= \theta_{ij,t} \cdot \frac{(x_i^{t+1})^\top x_j^{t+1}}{(x_i^t)^\top x_j^t} \cdot \frac{\|x_i^{t+1}\|^2\|x_j^{t+1}\|^2}{\|x_i^t\|^2\|x_j^t\|^2} \\
&\le \theta_{ij,t} \cdot \left(\frac{1 - A + B}{1 - A - C}\right) \qquad \text{(F.36)}
\end{aligned}$$

where

$$A = 2\eta(\|x_i^t\|^2 - \sigma_i + \|x_j^t\|^2 - \sigma_i)) \le 4\eta\sigma_1 \tag{F.37}$$

$$B = 2\eta\|x_k\|^2 \cdot \sqrt{\theta_{ik,t}\theta_{kj,t}/\theta_{ij,t}} + 30n^2\sigma_1^2\eta^2\theta_t/\sqrt{\theta_{ij,t}} \tag{F.38}$$

and

$$C = 2\eta \sum_{k\neq i,j} \|x_k\|^2(\theta_{ik,t} + \theta_{jk,t}) + 2\eta(\|x_j\|^2 + \|x_i\|^2)\theta_{ij,t} + 6\eta^2 n^2\theta_t\sigma_1^2 \tag{F.39}$$

$$\le \left(8\eta\sigma_1 + 2\eta(2n\alpha^2 + 2r\sigma_1) + 6\eta^2 n^2\sigma_1^2\right)\theta_t, \tag{F.40}$$

where the last inequality uses the fact that

$$\sum_{k\neq i,j} \|x_k^t\|^2 \le \sum_{k\le r} \|x_k^t\|^2 + \sum_{k>r} \|x_k^t\|^2 \le 2r\sigma_1 + 2n\alpha^2.$$

Hence we choose $\eta \le \frac{1}{1000n\sigma_1}$ to be sufficiently small so that $\max\{A, C\} \le 1/100$, then by $\frac{1-A+B}{1-A-C} \le 1 + 2B + 2C$ for $\max\{A, C\} \le 1/100$,

$$\begin{aligned}
&\theta_{ij,t} \cdot \left(\frac{1-A+B}{1-A-C}\right) \\
&\le \theta_{ij,t}(1 + 2B + 2C) \\
&\le \theta_{ij,t} + 4\eta \sum_{k\neq i,j} \|x_k\|^2 \cdot \sqrt{\theta_{ik,t}\theta_{kj,t}\theta_{ij,t}} + 60n^2\sigma_1^2\eta^2\theta_t\sqrt{\theta_{ij,t}} \\
&\qquad + \theta_t^2\Big(8\eta\sigma_1 + 2\eta(2n\alpha^2 + 2r\sigma_1) + 6\eta^2 n^2\sigma_1^2\Big) \\
&\le \theta_{ij,t} + 4\eta(2r\sigma_1 + 2n\alpha^2)\theta_t^{3/2} + 60n^2\sigma_1^2\eta^2\theta_t^{3/2} \\
&\qquad + \theta_t^2\Big(8\eta\sigma_1 + 2\eta(2n\alpha^2 + 2r\sigma_1) + 6\eta^2 n^2\sigma_1^2\Big) \\
&\le \theta_{ij,t} + 6\eta(2r\sigma_1 + 2n\alpha^2)\theta_t^{3/2} + 60n^2\sigma_1^2\eta^2\theta_t^{3/2} + 8\eta\sigma_1\theta_t^2 + 6n^2\eta^2\sigma_1^2\theta_t^2) \\
&\le \theta_{ij,t} + 98\eta \cdot (r\sigma_1\theta_t^{3/2})
\end{aligned}$$

The last inequality holds by $\alpha \le \sqrt{\sigma_1}/\sqrt{n}$, and $n^2\sigma_1\eta^2 \le \eta$ because $\eta \le \frac{1}{n^2\sigma_1}$.

Hence

$$\theta_{t+1} \le \theta_t + 98\eta(r\sigma_1)\theta_t^{3/2} \tag{F.41}$$

The Phase 1 terminates when $\|x_i^{T_1}\|^2 \ge \frac{3\sigma_i}{4}$. Since $\|x_i^0\|^2 \ge \alpha^2/2$ and

$$\|x_i^{t+1}\|^2 \ge (1 + \eta\sigma_i/4)\|x_i^t\|^2, \tag{F.42}$$

there is a constant C_3 such that $T_1 \le C_1(\log(\sqrt{\sigma_1}/\alpha)/\eta\sigma_i)$. Hence, before round T_1,

$$\theta_{T_1} \le \theta_0 + 98\eta T_1 \cdot r\sigma_1 \cdot (2\theta_0)^{3/2} \le \theta_0 + 98C_1 r\kappa(2\theta_0)^{3/2}\log(\sqrt{\sigma_1}/\alpha) \le 2\theta_0.$$

This is because

$$\theta_0 = \mathcal{O}((\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa))^2)$$

by Lemma 11 and choosing $k \ge c_2((r\kappa)^2\log(r\sqrt{\sigma_1/\alpha}))^4$ for large enough c_2 ■

F.3. Phase 2

Denote $\theta_t^U = \max_{\min\{i,j\}\le r} \theta_{ij,t}$. In this phase, we prove that θ_t^U is linear convergence, and the convergence rate of the loss is at least $\Omega(1/T^2)$. To be more specific, we will show that

$$\theta_{t+1}^U \le \theta_t^U \cdot (1 - \eta \cdot \sigma_r/4) \le \theta_t^U \tag{F.43}$$

$$\frac{\theta_{t+1}^U}{\sum_{i>r} \|x_i^{t+1}\|^2} \le \frac{\theta_t^U}{\sum_{i>r} \|x_i^t\|^2} \cdot \left(1 - \frac{\eta\sigma_r}{8}\right) \tag{F.44}$$

$$|\|x_i^t\|^2 - \sigma_i| \le \frac{1}{4}\sigma_i \ \ (i \le r) \tag{F.45}$$

$$\|x_i^t\|^2 \le 2\alpha^2 \ \ (i > r) \tag{F.46}$$

First, the condition (F.45) and (F.46) hold at round T_1. Then, if it holds before round t, consider round $t+1$, similar to Phase 1, condition (F.46) also holds. Now we prove Eq.(F.43), (F.44) and (F.45) one by one.

Proof of Eq.(F.45) For $i \le r$, if $\|x_i^t\|^2 \ge 3\sigma_i/4$, by Eq.(F.18)

$$|\|x_i^{t+1}\|_2^2 - \sigma_i| \le (1 - \eta\sigma_i)|\|x_i^t\|^2 - \sigma_i| + 3\eta \sum_{j\ne i}^{n} ((x_i^t)^\top x_j^t)^2 \tag{F.47}$$

Hence by (F.45) and (F.46), we can get

$$\begin{aligned}
\sum_{j\ne i}^{n} ((x_i^t)^\top x_j^t)^2 &\le \sum_{j\ne i, j\le r} ((x_i^t)^\top x_j^t)^2 + \sum_{j\ne i, j>r} ((x_i^t)^\top x_j^t)^2 \\
&\le (r\sigma_1 + 4n\sigma_1\alpha^2)\theta_t^U \\
&\le 2r\sigma_1\theta_t^U && \text{(F.48)} \\
&\le 2r\sigma_1\theta_{T_1}^U && \text{(F.49)} \\
&\le 2r\sigma_1 \cdot 2\theta_0 \le \sigma_i/20. && \text{(F.50)}
\end{aligned}$$

The inequality (F.48) is because $\alpha \le \frac{1}{4n\sigma_1}$, the inequality (F.49) holds by induction hypothesis (F.43), and the last inequality (F.50) is because of (F.6) and $\theta_0 \le \frac{1}{80r\kappa}$.

Hence if $|\|x_i^t\|^2 - \sigma_i| \le \sigma_i/4$, by combining (F.47) and (F.50), we have

$$|\|x_i^{t+1}\|^2 - \sigma_i| \le (1 - \eta\sigma_i)|\|x_i^t\| - \sigma_i| + 3\eta\sigma_i/20 \le \sigma_i/4.$$

Now it is easy to get that $|\|x_i^t\|^2 - \sigma_i| \le 0.25\sigma_i$ for $t \ge T_1$ by induction because of $|\|x_i^{T_1}\|^2 - \sigma_i| \le 0.25\sigma_i$. Thus we complete the proof of Eq.(F.45).

Proof of Eq.(F.43) First, we consider $i \le r, j \ne i \in [n]$ and $\theta_{ij,t} > \theta_t^U/2$, since (F.4) and (F.5) still holds with (F.45) and (F.46), similarly, we can still have equation (F.36), i.e.

$$\theta_{ij,t+1} = \theta_{ij,t} \cdot \left(\frac{1 - A - B}{1 - A - C}\right).$$

where

$$
\begin{aligned}
A &= 2\eta(\|x_i^t\|^2 - \sigma_i) + 2\eta(\|x_j^t\|^2 - \sigma_j) \geq -2\eta(2 \cdot (\sigma_i/4)) \geq -1/100. \\
B &= 2\eta(\|x_i^t\|^2 + \|x_j^t\|^2) - 2\eta \sum_{k \neq i,j} \|x_k\|^2 \cdot \sqrt{\theta_{ik,t}\theta_{kj,t}/\theta_{ij,t}} - 30n^2\eta^2\sigma_1^2\sqrt{\theta_t^U}/\sqrt{\theta_{ij,t}} \\
&\geq 2\eta(\|x_i^t\|^2 + \|x_j^t\|^2) - 4\eta \sum_{k \leq r} \|x_k\|^2\sqrt{\theta^U} - 4n\eta\alpha^2 - 40n^2\eta^2\sigma_1^2 && \text{(F.51)} \\
&\geq 2\eta \cdot \frac{3\sigma_i}{4} - 8\eta r\sigma_1\sqrt{2\theta_{T_0}} - 4n\eta\alpha^2 - 40n^2\eta^2\sigma_1^2 && \text{(F.52)} \\
&\geq \eta \cdot \sigma_r && \text{(F.53)}
\end{aligned}
$$

The inequality Eq.(F.51) holds by $\theta_{ij,t} > \theta_t^U/2$, the inequality (F.52) holds by (F.43), and (F.53) holds by

$$
\theta_{T_0} = \mathcal{O}\left(\frac{1}{r^2\kappa^2}\right), \quad \alpha = \mathcal{O}(\sqrt{\sigma_r/n}), \quad \eta = \mathcal{O}(1/n^2\kappa\sigma_1). \tag{F.54}
$$

The term C is defined and can be bounded by

$$
\begin{aligned}
C &= 2\eta \sum_{k \neq i,j} \|x_k\|^2(\theta_{ik,t} + \theta_{jk,t}) + 2\eta(\|x_i\|^2 + \|x_j\|^2)\theta_{ij,t} + 6\eta^2\theta_t n^2\sigma_1^2 \\
&\leq 4\eta \sum_{k \leq r} \|x_k\|^2\theta_t^U + 4\eta n\alpha^2\theta_t + 6\eta^2\theta_t n^2\sigma_1^2 \\
&\leq 8r\eta\sigma_1\theta_t^U + 4\eta n\alpha^2 + 6\eta^2 n^2\sigma_1^2 \\
&\leq 8r\eta\sigma_1\theta_{T_0} + 4\eta n\alpha^2 + 6\eta^2 n^2\sigma_1^2 && \text{(F.55)} \\
&\leq \eta \cdot \sigma_r/2. && \text{(F.56)}
\end{aligned}
$$

The inequality (F.55) holds by (F.43), and the inequality (F.56) holds by (F.54).

Then for $i \leq r, j \neq i \in [n]$ and $\theta_{ij,t} > \theta_t^U/2$, we can get

$$
\begin{aligned}
\theta_{ij,t+1} &\leq \theta_{ij,t} \cdot \left(\frac{1 - A - B}{1 - A - C}\right) \\
&\leq \theta_{ij,t} \cdot \left(\frac{2 - \eta \cdot \sigma_r}{2 - \eta \cdot \sigma_r/2}\right) \\
&\leq \theta_{ij,t} \cdot \left(\frac{1 - \eta \cdot \sigma_r/2}{1 - \eta \cdot \sigma_r/4}\right) \leq \theta_{ij,t} \cdot (1 - \eta \cdot \sigma_r/4) && \text{(F.57)}
\end{aligned}
$$

For $i \leq r, j \in [n]$ and $\theta_{ij,t} \leq \theta_t^U/2$, we have

$$
\begin{aligned}
B &\geq -2\eta \sum_{k \leq r} \|x_k\|^2\theta_t^U/\sqrt{\theta_{ij,t}} - 2\eta \sum_{k > r} \|x_k\|^2\sqrt{\theta_t^U}/\sqrt{\theta_{ij,t}} - 30n^2\eta^2\sigma_1^2\sqrt{\theta_t^U}/\sqrt{\theta_{ij,t}} && \text{(F.58)} \\
&\geq -4\eta r\sigma_1\theta_t^U/\sqrt{\theta_{ij,t}} - (4n\eta\alpha^2 + 30n^2\eta^2\sigma_1^2)\sqrt{\theta_t^U}/\sqrt{\theta_{ij,t}} && \text{(F.59)}
\end{aligned}
$$

$$
\begin{aligned}
\theta_{ij,t+1} &\le \theta_{ij,t} \cdot \left(\frac{1-A-B}{1-A-C}\right) \\
&\le \theta_{ij,t} \cdot (1-2B+2C) \\
&\le \theta_{ij,t} + 8\eta r\sigma_1\theta_t^U\sqrt{\theta_{ij,t}} + (4n\eta\alpha^2 + 30n^2\eta^2\sigma_1^2)\sqrt{\theta_t^U\theta_{ij,t}} + 2C\theta_{ij,t} \\
&\le \frac{\theta_t^U}{2} + 8\eta r\sigma_1\theta_t^U + (4n\eta\alpha^2 + 30n^2\eta^2\sigma_1^2)\theta_t^U + \eta\sigma_r\theta_t^U \\
&\le \frac{3\theta_t^U}{4}.
\end{aligned} \tag{F.60}
$$

The last inequality is because $8\eta r\sigma_1 + 4n\eta\alpha^2 + 30n^2\eta^2\sigma_1^2 + \eta\sigma_r \le \frac{1}{4}$ by $\eta \le \mathcal{O}(1/n\sigma_1)$ and $\eta \le \mathcal{O}(1/n\alpha^2)$. Hence by Eq.(F.57) and (F.60) and the fact that $\eta\sigma_r/4 \le 1/4$,

$$
\theta_{t+1}^U \le \theta_t^U \cdot \max\left\{\frac{3}{4}, 1-\eta\cdot\sigma_r/4\right\} = (1-\eta\cdot\sigma_r/4)\theta_t^U. \tag{F.61}
$$

Thus we complete the proof of Eq.(F.43)

Proof of Eq.(F.44) Also, for $i > r$, denote $\theta_{ii,t} = 1$, then

$$
\begin{aligned}
\|x_i^{t+1}\|^2 &= \|x_i^t\|^2 - 2\eta\sum_{j=1}^n((x_i^t)^\top x_j^t)^2 + \eta^2\left(\sum_{j,k=1}^n (x_i^t)^\top x_j^t (x_j^t)^\top\right)^2 \\
&\ge \|x_i^t\|^2(1-2\eta\sum_{j=1}^n\|x_j^t\|^2\theta_{ij,t}) \\
&\ge \|x_i\|^2(1-2\eta r\sigma_1\theta_t^U - 2\eta n\alpha^2) \\
&\ge \|x_i\|^2(1-\eta\cdot\sigma_r/8)
\end{aligned} \tag{F.62}
$$

The last inequality holds because

$$
\theta_t^U \le \theta_0 \le \mathcal{O}(1/r\kappa) \tag{F.63}
$$

$$
\alpha \le \sqrt{\sigma_r/n} \tag{F.64}
$$

Hence the term $\theta^U/\|x_i\|^2$ for $i > r$ is also linear convergence by

$$
\frac{\theta_{t+1}^U}{\sum_{i>r}\|x_i^{t+1}\|^2} \le \frac{\theta_t^U}{\sum_{i>r}\|x_i^t\|^2}\cdot\frac{1-\eta\cdot\sigma_r/4}{1-\eta\cdot\sigma_r/8} \le \frac{\theta_t^U}{\sum_{i>r}\|x_i^t\|^2}\cdot\left(1-\frac{\eta\sigma_r}{8}\right).
$$

Hence we complete the proof of Eq.(F.44).

F.4. Phase 3: lower bound of convergence rate

Now by (F.44), there are constants c_6 and c_7 such that, if we denote $T_2 = T_1 + c_7(\log(\sqrt{r\sigma_1}/\alpha)/\eta\sigma_r) = c_6(\log(\sqrt{r\sigma_1}/\alpha)/\eta\sigma_r)$, then we will have

$$
\theta_{T_2}^U < \sum_{i>r}\|x_i^{T_2}\|^2/r\sigma_1 \tag{F.65}
$$

because of the fact that $\theta_{T_1}^U / \sum_{i>r} \|x_i^{T_1}\|^2 \leq \frac{4}{n \cdot \alpha^2} \leq 4/\alpha^2$. Now after round T_2, consider $i > r$, we can have

$$\begin{aligned}
\|x_i^{t+1}\|^2 &\geq \|x_i^t\|^2 (1 - 2\eta \sum_{j=1}^n \|x_j^t\|^2 \theta_{ij,t}) \\
&\geq \|x_i^t\|^2 (1 - 2\eta r \sigma_1 \theta_t^U - 2\eta \sum_{j>r} \|x_j^t\|^2)
\end{aligned}$$

Hence by Eq.(F.62), we have

$$\sum_{j>r} \|x_j^{t+1}\|^2 \geq \left(\sum_{j>r} \|x_j^t\|^2 \right) \left(1 - 2\eta r \sigma_1 \theta_t^U - 2\eta \sum_{j>r} \|x_j^t\|^2 \right) \tag{F.66}$$

$$\geq \left(\sum_{j>r} \|x_j^t\|^2 \right) \left(1 - 4\eta \sum_{j>r} \|x_j^t\|^2 \right), \tag{F.67}$$

where the second inequality is derived from (F.65).

Hence we can show that $\sum_{j>r} \|x_j^t\|^2 = \Omega(1/T^2)$. In fact, suppose at round T_2, we denote $D_{T_2} = \sum_{j>r} \|x_j^{T_2}\|^2$, then by

$$\begin{aligned}
\|x_i^{t+1}\|^2 &\geq \|x_i^t\|^2 (1 - 2\eta \sum_{k=1}^n \|x_k^t\|^2 \theta_{ik,t})) \\
&\geq \|x_i^t\|^2 (1 - 2\eta r \sigma_1 \theta^U - 2\eta n \alpha^2)
\end{aligned}$$

we can get

$$\begin{aligned}
\|x_i^{T_2}\|^2 &\geq \|x_i^{T_1}\|^2 (1 - 2\eta r \sigma_1 \theta_{T_1}^U - 2\eta n \alpha^2)^{T_2 - T_1} \\
&\geq \|x_i^{T_1}\|^2 \cdot (1 - c_5 (\log(r\sqrt{\sigma_1}/\alpha)/\eta \sigma_r) \cdot \left(2\eta r \sigma_1 \theta_{T_1} + 2\eta n \alpha^2\right)) \\
&\geq \|x_i^{T_1}\|^2 \cdot (1 - c_5 \log(r\sqrt{\sigma_1}/\alpha) \cdot (4 r \kappa \theta_0 + 2n\alpha^2/\sigma_r)) \\
&\geq \frac{1}{2} \|x_i^{T_1}\|^2 \qquad \text{(F.68)} \\
&\geq \frac{\alpha^2}{8}
\end{aligned}$$

where the inequality (F.68) is because

$$\theta_0 \leq \mathcal{O}\left(\frac{1}{r\kappa \log(r\sqrt{\sigma_1}/\alpha)} \right) \tag{F.69}$$

$$\alpha^2 \leq \mathcal{O}\left(\frac{\sqrt{\sigma_r}}{n \log(r\sqrt{\sigma_1}/\alpha)} \right). \tag{F.70}$$

Hence

$$T_2 D_{T_2} \geq T_2 \cdot (n - r) \frac{\alpha^2}{8} \geq c_7 (\log(\sqrt{r\sigma_1}/\alpha)/\eta \sigma_r) \cdot \frac{\alpha^2}{8}. \tag{F.71}$$

by $n > r$. Define $D_{T_2+i+1} = D_{T_2+i}(1 - 4\eta D_{T_2+i})$, by Eq.(F.67), we have

$$D_{T_2+i} \le D_{T_2} = \sum_{i>r} \|x_i^{T_2}\|^2 \le 2n\alpha^2. \tag{F.72}$$

On the other hand, if $\eta(T_2 + i)D_{T_2+i} \le 1/8$, and then

$$\begin{aligned}
\eta(T_2 + i + 1)D_{T_2+i+1} &= \eta(T_2 + i + 1)D_{T_2+i}(1 - 4\eta D_{T_2+i}) \\
&= \eta(T_2 + i)D_{T_2+i} - (T_2 + i)4\eta^2 D_{T_2+i}^2 + \eta D_{T_2+i}(1 - 4\eta D_{T_2+i}) \\
&\ge \eta(T_2 + i)D_{T_2+i} - (T_2 + i)4\eta^2 D_{T_2+i}^2 + \eta D_{T_2+i}/2 \qquad \text{(F.73)} \\
&\ge \eta(T_2 + i)D_{T_2+i} - \eta D_{T_2+i}/2 + \eta D_{T_2+i}/2 \\
&\ge \eta(T_2 + i)D_{T_2+i},
\end{aligned}$$

where (F.73) holds by $\eta D_{T_2+i} \le 2n\eta\alpha^2 \le 1/8$.

If $\eta(T_2 + i)D_{T_2+i} > 1/8$, since $\eta D_{T_2+i} \le 1/8$, we have $\eta D_{T_2} \le 2n\eta\alpha^2 \le 1/8$.

$$\begin{aligned}
\eta(T_2 + i + 1)D_{T_2+i+1} &\ge \eta(T_2 + i)D_{T_2+i}(1 - 4\eta D_{T_2+i}) + \eta D_{T_2+i}(1 - 4\eta D_{T_2+i}) \\
&\ge \frac{1}{8} \cdot \frac{1}{2} + \eta D_{T_2+i} \cdot \frac{1}{2} \\
&\ge \frac{1}{16}.
\end{aligned}$$

Thus by the two inequalities above, at round $t \ge T_2$, we can have

$$\eta t D_t \ge \min\{\eta T_2 D_{T_2}, 1/16\}.$$

Now by (F.71),

$$\eta T_2 D_{T_2} \ge \frac{c_7 \log(\sqrt{r\sigma_1}/\alpha)\alpha^2}{8\sigma_r}, \tag{F.74}$$

then for any $t \ge T_2$, we have

$$\eta t D_t \ge \min\left\{\frac{c_7 \log(\sqrt{r\sigma_1}/\alpha)\alpha^2}{8\sigma_r}, 1/16\right\} \tag{F.75}$$

Now by choosing $\alpha = \widetilde{\mathcal{O}}(\sqrt{\sigma_r})$ so that $\frac{c_7 \log(\sqrt{r\sigma_1}/\alpha)\alpha^2}{8\sigma_r} \le 1/16$, we can derive

$$D_t \ge \frac{c_7 \log(\sqrt{r\sigma_1}/\alpha)\alpha^2}{8\sigma_r \eta t}. \tag{F.76}$$

Since for $j > r$, $(X_t X_t^\top - \Sigma)_{jj} = \|x_j^t\|^2$, we have $\|X_t X_t^\top - \Sigma\| \ge \max_{j>r} \|x_j^t\|^2$ and

$$\|X_t X_t^\top - \Sigma\|^2 \ge \left(\max_{j>r}\{\|x_j^t\|^2\}\right)^2 \ge D_t^2/n^2 \ge \left(\frac{c_7 \log(\sqrt{r\sigma_1}/\alpha)\alpha^2}{8\sigma_r \eta n t}\right)^2.$$

Appendix G. Proof of Theorem 7

Denote the matrix of the first r row of F, G as U, V respectively, and the matrix of the last $n-r$ row of F, G as J, K respectively. Hence $U, V \in \mathbb{R}^{r\times k}, J, K \in \mathbb{R}^{(n-r)\times k}$. In this case, the difference $F_t G_t^\top - \Sigma$ can be written in a block form as

$$F_t G_t^\top - \Sigma = \begin{pmatrix} U_t V_t^\top - \Sigma_r & J_t V_t^\top \\ U_t K_t^\top & J_t K_t^\top \end{pmatrix}, \tag{G.1}$$

where $\Sigma_r = I \in \mathbb{R}^{r\times r}$. Hence the loss can be bounded by

$$\|J_t K_t^\top\| \le \|F_t G_t^\top - \Sigma\| \le \|U_t V_t^\top - \Sigma_r\| + \|J_t V_t^\top\| + \|U_t K_t^\top\| + \|J_t K_t^\top\|. \tag{G.2}$$

The updating rule for (U, V, J, K) under gradient descent in (D.2) can be rewritten explicitly as

$$\begin{aligned} U_{t+1} &= U_t + \eta\Sigma_r V_t - \eta U_t(V_t^\top V_t + K_t^\top K_t) \\ V_{t+1} &= V_t + \eta\Sigma_r U_t - \eta V_t(U_t^\top U_t + J_t^\top J_t) \\ J_{t+1} &= J_t - \eta J_t(V_t^\top V_t + K_t^\top K_t) \\ K_{t+1} &= K_t - \eta K_t(U_t^\top U_t + J_t^\top J_t). \end{aligned}$$

Note that with our particular initialization, we have the following equality for all t:

$$U_t K_t^\top = 0, J_t V_t^\top = 0, \quad \text{and} \quad U_t = V_t. \tag{G.3}$$

Indeed, the conditions (G.3) are satisfied for $t = 0$. For $t+1$, we have

$$U_{t+1} = U_t + \eta(\Sigma_r - U_t V_t^\top)V_t = V_t + \eta(\Sigma_r - U_t V_t^\top)U_t = V_{t+1}, \;\; K_{t+1} = K_t - \eta K_t J_t^\top J_t$$
$$U_{t+1}K_{t+1}^\top = U_t K_t^\top + \eta(\Sigma_r - U_t V_t^\top)U_t K_t^\top - \eta V_t J_t^\top J_t K_t^\top - \eta^2(\Sigma_r - U_t V_t^\top)U_t J_t^\top J_t K_t^\top = 0$$

The last equality arises from the fact that $U_t K_t^\top = 0, J_t V_t^\top = 0$ and $U_t = V_t$. Similarly, we can get $J_{t+1}V_{t+1}^\top = 0$. Hence we can rewrite the updating rule of J_t and K_t as

$$J_{t+1} = J_t - \eta J_t K_t^\top K_t \tag{G.4}$$
$$K_{t+1} = K_t - \eta K_t J_t^\top J_t. \tag{G.5}$$

Let us now argue why the convergence rate can not be faster than $\Omega((1-6\eta\alpha^2)^t)$. Denote $A \in \mathbb{R}^{(n-r)\times k}$ as the matrix that $(A)_{1k} = 1$ and other elements are all zero. We have that $J_0 = \alpha A$ and $K_0 = (\alpha/3)\cdot A$. Combining this with Eq.(G.4) and Eq.(G.5), we have $J_t = a_t A, K_t = b_t A$, where

$$a_0 = \alpha, b_0 = \alpha/3, \tag{G.6a}$$
$$a_{t+1} = a_t - \eta a_t b_t^2, \tag{G.6b}$$
$$b_{t+1} = b_t - \eta a_t^2 b_t. \tag{G.6c}$$

It is immediate that $0 \le a_{t+1} \le a_t, 0 \le b_{t+1} \le b_t$, $\max\{a_t, b_t\} \le \alpha$ because of $\eta b_t^2 \le \eta b_0^2 = \eta\alpha^2 \le 1$ and similarly $\eta a_t^2 \le 1$. Now by $\eta\alpha^2 \le 1/4$,

$$\|J_{t+1}K_{t+1}^\top\| = a_{t+1}b_{t+1} = (1-\eta a_t^2)(1-\eta b_t^2)a_t b_t \ge (1-2\eta\alpha^2)^2 a_t b_t \ge (1-4\eta\alpha^2)a_t b_t. \tag{G.7}$$

By Eq.(G.2) that $\|F_tG_t^\top - \Sigma\| \geq \|J_tK_t^\top\|$, the convergence rate of $\|F_tG_t^\top - \Sigma\|$ can not be faster than $a_0b_0(1-4\eta\alpha^2)^t \geq \frac{\alpha^2}{3}(1-4\eta\alpha^2)^t$.

Next, we show why the convergence rate is exactly $\Theta((1-\Theta(\eta\alpha^2))^t)$ in this toy case. By Eq.(G.3), the loss $\|F_tG_t^\top - \Sigma\| \leq \|U_tU_t^\top - \Sigma_r\| + \|J_tK_t^\top\|$. First, we consider the norm $\|U_tU_t^\top - \Sigma_r\|$. Since in this toy case, $\Sigma_r = I_r$ and $U_t = V_t$ for all t, the updating rule of U_t can be written as

$$U_{t+1} = U_t - \eta(U_tU_t^\top - I)U_t \tag{G.8}$$

Note that $U_0 = (\alpha I_r, 0) \in \mathbb{R}^{r\times k}$. By induction, we can show that $U_t = (\alpha_t I_r, 0)$ and $\alpha_{t+1} = \alpha_t - \eta(\alpha_t^2 - 1)\alpha_t$ for all $t \geq 0$. If $\alpha_t \leq 1/2$, we have

$$\alpha_{t+1} = \alpha_t(1+\eta-\eta\alpha_t^2) \geq \alpha_t(1+\eta/2).$$

Then there exists a constant c_1 and $T_1 = c_1(\log(1/\alpha)/\eta)$ such that after T_1 rounds, we can get $\alpha_t \geq 1/2$. By the fact that $\alpha_{t+1} = \alpha_t(1+\eta(1-\alpha_t^2)) \leq \max\{\alpha_t, 2\}$ when $\eta < 1$, it is easy to show $\alpha_t \leq 2$ for all $t \geq 0$. Thus when $\eta < 1/6$, we can get $1-\eta(\alpha_t+1)\alpha_t > 0$ and then

$$\begin{aligned}|\alpha_{t+1} - 1| &= |(\alpha_t - 1) - \eta(\alpha_t - 1)(\alpha_t+1)\alpha_t| \\ &= |\alpha_t - 1|(1-\eta(\alpha_t+1)\alpha_t) \\ &\leq |\alpha_t - 1|(1-\eta/2).\end{aligned}$$

we know that $\|U_tU_t^\top - \Sigma_r\| = \alpha_t^2 - 1$ converges at a linear rate

$$\|U_tU_t^\top - \Sigma\| \leq (1-\eta/2)^{t-T_1} \overset{\text{(a)}}{\leq} (1-\eta\alpha^2/4)^{(t-T_1)/2}, \tag{G.9}$$

where (a) uses the fact that

$$1-\eta\alpha^2/4 \geq 1-\eta \geq (1-\eta/2)^2 \tag{G.10}$$

Hence we only need to show that $\|J_tK_t^\top\|$ converges at a relatively slower speed $\mathcal{O}((1-\Theta(\eta\alpha^2))^t)$. To do this, we prove the following statements by induction.

$$\alpha \geq a_t \geq \alpha/2, \;\; b_{t+1}^2 \leq b_t^2(1-\eta\alpha^2/4) \tag{G.11}$$

Using $b_0 = \alpha/3$, we see the above implies that $\|J_tK_t^\top\| = a_tb_t \leq \mathcal{O}((1-\Theta(\eta\alpha^2))^t)$.

Let us prove (G.11) via induction. It is trivial to show it holds at $t=0$ and the upper bound of a_t by (G.6). Suppose (G.11) holds for $t' \leq t$, then at round $t+1$, we have

$$b_{t+1}^2 = b_t^2(1-\eta a_t^2)^2 \leq b_t^2(1-\eta\alpha^2/4)^2 \leq b_t^2(1-\eta\alpha^2/4). \tag{G.12}$$

Using $a_{t+1} = a_t(1-\eta b_t^2)$, we have

$$a_{t+1} = a_0\prod_{i=1}^t(1-\eta b_i^2) \overset{(a)}{\geq} a_0\left(1-\eta\sum_{i=1}^t b_i^2\right) \overset{(b)}{\geq} \alpha\cdot\left(1-\eta\cdot\frac{\alpha^2}{9}\cdot\frac{4}{\eta\alpha^2}\right) \geq \alpha/2. \tag{G.13}$$

where the step (a) holds by recursively using $(1-a)(1-b) \geq (1-(a+b))$ for $a, b \in (0,1)$, and the step (b) is due to $b_i^2 \leq b_0^2\cdot(1-\eta\alpha^2/4)^t \leq \frac{\alpha^2}{9}\cdot(1-\frac{\eta\alpha^2}{4})^t$ and the sum formula for geometric series. Thus the induction is complete, and

$$\|J_tK_t^\top\| = a_tb_t \leq (\alpha^2/3)\cdot(1-\eta\alpha^2/4)^{t/2} \leq (1-\eta\alpha^2/4)^{t/2} \leq (1-\eta\alpha^2/4)^{(t-T_1)/2}. \tag{G.14}$$

Combining (G.9) and (G.14), with $\|A\|_2 \leq \|A\|_{\mathrm{F}} \leq \mathrm{rank}(A)\cdot\|A\|_2$, we complete the proof.

Appendix H. Proof of Theorem 8

We prove Theorem 8 in this section. We start with some preliminaries.

H.1. Preliminaries

In the following, we denote $\delta_{2k+1} = \sqrt{2k+1}\delta$. Also denote the matrix of the first r row of F, G as U, V respectively, and the matrix of the last $n-r$ row of F, G as J, K respectively. Hence $U, V \in \mathbb{R}^{r\times k}$, $J, K \in \mathbb{R}^{(n-r)\times k}$. We denote the corresponding iterates as U_t, V_t, J_t, and K_t.

Also, define $E(X) = \mathcal{A}^*\mathcal{A}(X) - X$. We also denote $\Gamma(X) = \mathcal{A}^*\mathcal{A}(X)$. By Lemma 17, we can show that $\|E(X)\| \le \delta_{2k+1} \cdot \|X\|$ for matrix X with rank less than $2k$ by Lemma 17. Decompose the error matrix $E(X)$ into four submatrices by

$$E(X) = \begin{pmatrix} E_1(X) & E_2(X) \\ E_3(X) & E_4(X) \end{pmatrix},$$

where $E_1(X) \in \mathbb{R}^{r\times r}, E_2(X) \in \mathbb{R}^{r\times(n-r)}, E_3(X) \in \mathbb{R}^{(n-r)\times r}, E_4(X) \in \mathbb{R}^{(n-r)\times(n-r)}$. Then the updating rule can be rewritten in this form:

$$U_{t+1} = U_t + \eta\Sigma V_t - \eta U_t(V_t^\top V_t + K_t^\top K_t) + \eta E_1(F_tG_t^\top - \Sigma)V_t + \eta E_2(F_tG_t^\top - \Sigma)K_t \quad \text{(H.1)}$$

$$V_{t+1} = V_t + \eta\Sigma U_t - \eta V_t(U_t^\top U_t + J_t^\top J_t) + \eta E_1^\top(F_tG_t^\top - \Sigma)U_t + \eta E_3^\top(F_tG_t^\top - \Sigma)J_t \quad \text{(H.2)}$$

$$J_{t+1} = J_t - \eta J_t(V_t^\top V_t + K_t^\top K_t) + \eta E_3(F_tG_t^\top - \Sigma)V_t + \eta E_4(F_tG_t^\top - \Sigma)K_t \quad \text{(H.3)}$$

$$K_{t+1} = K_t - \eta K_t(U_t^\top U_t + J_t^\top J_t) + \eta E_2^\top(F_tG_t^\top - \Sigma)U_t + \eta E_4^\top(F_tG_t^\top - \Sigma)J_t. \quad \text{(H.4)}$$

Since the submatrices' operator norm is less than the operator norm of the whole matrix, the matrices $E_i(F_tG_t^\top - \Sigma)$, $i = 1, \ldots, 4$ satisfy that

$$\|E_i(F_tG_t^\top - \Sigma)\| \le \|E(F_tG_t^\top - \Sigma)\| \le \delta_{2k+1}\|F_tG_t^\top - \Sigma\|, \quad i = 1, \ldots, 4.$$

Imbalance term An important property in analyzing the asymmetric matrix sensing problem is that $F^\top F - G^\top G = U^\top U + J^\top J - V^\top V - K^\top K$ remains almost unchanged when step size η is sufficiently small, i.e., the balance between two factors F and G are does not change much throughout the process. To be more specific, by

$$F_{t+1} = F_t - \eta(F_tG_t^\top - \Sigma)G_t - E(F_tG_t^\top - \Sigma)G_t \quad \text{(H.5)}$$

$$G_{t+1} = G_t - \eta(F_tG_t^\top - \Sigma)^\top F_t - (E(F_tG_t^\top - \Sigma))^\top F_t \quad \text{(H.6)}$$

we have

$$\left\|\left(F_{t+1}^\top F_{t+1} - G_{t+1}^\top G_{t+1}\right) - \left(F_t^\top F_t - G_t^\top G_t\right)\right\| \le 2\eta^2 \cdot \|F_tG_t^\top - \Sigma\|^2 \cdot \max\{\|F_t\|, \|G_t\|\}^2. \quad \text{(H.7)}$$

In fact, by the updating rule, we have

$$\begin{aligned} &F_{t+1}^\top F_{t+1} - G_{t+1}^\top G_{t+1} \\ &= F_t^\top F_t - G_t^\top G_t + \eta^2\Big(G_t^\top(F_tG_t^\top - \Sigma)^\top(F_tG_t^\top - \Sigma)G_t - F_t^\top(F_tG_t^\top - \Sigma)(F_tG_t^\top - \Sigma)^\top F_t\Big), \end{aligned}$$

so that

$$\begin{aligned}&\|F_{t+1}^\top F_{t+1} - G_{t+1}^\top G_{t+1} - (F_t^\top F_t - G_t^\top G_t)\| \\ \le& 2\eta^2 \|F_t\|^2 \|G_t\|^2 \|F_t G_t^\top - \Sigma\|^2 \\ \le& 2\eta^2 \cdot \|F_t G_t^\top - \Sigma\| \cdot \max\{\|F_t\|^2, \|G_t\|^2\}\end{aligned} \tag{H.8}$$

Thus, we will prove that, during the proof process, the following inequality holds with high probability during all $t \ge 0$:

$$2\alpha^2 I \ge U_t^\top U_t + J_t^\top J_t - V_t^\top V_t - K_t^\top K_t \ge \frac{\alpha^2}{8} I. \tag{H.9}$$

Next, we give the outline of our proof.

H.2. Proof Outline

In this subsection, we give our proof outline.

• Recall $\Delta_t = F_t^\top F_t - G_t^\top G_t = U_t^\top U_t + J_t^\top J_t - V_t^\top V_t - K_t^\top K_t$. In Section H.3, we show that with high probability, Δ_0 has the scale α, i.e., $C\alpha^2 I \ge \Delta_0 \ge c\alpha^2 I$, where $C > c$ are two constants. Then we apply the converge results in [29] to argue that the algorithm first converges to a local point. By the previous work [29], this converge phase takes at most $T_0 = \mathcal{O}((1/\eta\sigma_r \upsilon)\log(\sqrt{\sigma_1}/n\alpha))$ rounds.

• Then, in Section H.4 (Phase 1), we mainly show that $M_t = \max\{\|U_t V_t^\top - \Sigma\|, \|U_t K_t^\top\|, \|J_t V_t^\top\|\}$ converges linearly until it is smaller than

$$M_t \le \mathcal{O}(\sigma_1 \delta + \alpha^2)\|J_t K_t^\top\|. \tag{H.10}$$

This implies that the difference between estimated matrix $U_t V_t^\top$ and true matrix Σ, $\|U_t V_t^\top - \Sigma\|$, will be dominated by $\|J_t K_t^\top\|$. Moreover, during Phase 1 we can also show that Δ_t has the scale α. Phase 1 begins at T_0 rounds and terminates at T_1 rounds, and T_1 may tend to infinity, which implies that Phase 1 may not terminate. In this case, since M_t converges linearly and $M_t > \Omega(\sigma_1\delta + \alpha^2)\|J_t K_t^\top\|$, the loss also converges linearly. Note that, in the exact-parameterized case, i.e., $k = r$, we can prove that Phase 1 will not terminate since the stopping rule (H.10) is never satisfied as shown in Section I.

• The Section H.5 (Phase 2) mainly shows that, after Phase 1, the $\|U_t - V_t\|$ converges linearly until it achieves

$$\|U_t - V_t\| \le \mathcal{O}(\alpha^2/\sqrt{\sigma_1}) + \mathcal{O}(\delta_{2k+1}\|J_t K_t^\top\|/\sqrt{\sigma_1}).$$

Assume Phase 2 starts at round T_1 and terminates at round T_2. Then since we can prove that $\|U_t - V_t\|$ decreases from [4] $\mathcal{O}(\sigma_1)$ to $\Omega(\alpha^2)$, Phase 2 only takes a relatively small number of rounds, i.e. at most $T_2 - T_1 = \mathcal{O}(\log(\sqrt{\sigma_r}/\alpha)/\eta\sigma_r)$ rounds. We also show that M_t remains small in this phase.

• The Section H.6 (Phase 3) finally shows that the norm of K_t converges linearly, with a rate dependent on the initialization scale. As in Section D.2, the error matrix in matrix sensing brings additional challenges for the proof. We overcome this proof by further analyzing the convergence of (a) part of K_t that aligns with U_t, and (b) part of K_t that lies in the complement space of U_t. We also utilize that M_t and $\|U_t - V_t\|$ are small from the start of the phase and remain small. See Section H.6 for a detailed proof.

4. The upper bound $\mathcal{O}(\sigma_1)$ of $\|U_t - V_t\|$ is proved in the first two phases.

H.3. Initial iterations

We start our proof by first applying results in [29] and provide some additional proofs for our future use. From [29], the converge takes at most $T_0 = \mathcal{O}((1/\eta\sigma_r\upsilon)\log(\sqrt{\sigma_1}/n\alpha))$ rounds.

Let us state a few properties of the initial iterations using Lemma 18.

Initialization By our imbalance initialization $F_0 = \alpha \cdot \widetilde{F}_0, G_0 = (\alpha/3) \cdot \widetilde{G}_0$, and by random matrix theory about the singular value [33, Corollary 7.3.3 and 7.3.4], with probability at least $1 - 2\exp(-cn)$ for some constant c, if $n > 8k$, we can show that $[\sigma_{\min}(F_0), \sigma_{\max}(F_0))] \subseteq [\frac{\sqrt{3}\alpha}{2}, \frac{\sqrt{3}\alpha}{\sqrt{2}}]$, $[\sigma_{\min}(G_0), \sigma_{\max}(G_0)] \subseteq [\frac{\sqrt{3}\alpha}{6}, \frac{\alpha}{\sqrt{6}}]$ and

$$\frac{3\alpha^2}{2}I \geq F_0^\top F_0 - G_0^\top G_0 = U_0^\top U_0 + J_0^\top J_0 - V_0^\top V_0 - K_0^\top K_0 \geq \frac{\alpha^2}{2}I \tag{H.11}$$

As we will show later, we will prove the (H.9) during all phases by (H.7) and (H.11).

First, we show the following lemma, which is a subsequent corollary of the Lemma 18.

Lemma 14 *There exist parameters $\zeta_0, \delta_0, \alpha_0, \eta_0$ such that, if we choose $\alpha \leq \alpha_0$, $F_0 = \alpha \cdot \tilde{F}_0, G_0 = (\alpha/2) \cdot \tilde{G}_0$, where the elements of $\tilde{F}_0, \tilde{G}_0$ is $\mathcal{N}(0,1)$,[5] and suppose that the operator $\mathcal{A}$ defined in Eq.(1.1) satisfies the restricted isometry property of order $2r+1$ with constant $\delta \leq \delta_0$, then the gradient descent with step size $\eta \leq \eta_0$ will achieve*

$$\|F_t G_t^\top - \Sigma\| \leq \min\{\sigma_r/2, \alpha^{1/2} \cdot \sigma_1^{3/4}\} \tag{H.12}$$

within $T_0 = c_2(1/\eta\sigma_r)\log(\sqrt{\sigma_1}/n\alpha)$ rounds with probability at least $1 - \zeta_0$ and constant $c_2 \geq 1$, where $\zeta_0 = c_1\exp(-c_2 k) + \exp(-(k-r+1))$ is a small constant. Moreover, during $t \leq T_0$ rounds, we always have

$$\max\{\|F_t\|, \|G_t\|\} \leq 2\sqrt{\sigma_1} \tag{H.13}$$

$$\|U_t - V_t\| \leq 4\alpha + \frac{40\delta_{2k+1}\sigma_1^{3/2}}{\sigma_r} \tag{H.14}$$

$$\|J_t\| \leq \mathcal{O}\Big(2\alpha + \frac{\delta_{2k+1}\sigma_1^{3/2}\log(\sqrt{\sigma_1}/n\alpha)}{\sigma_r}\Big) \tag{H.15}$$

$$\frac{13\alpha^2}{8}I \geq \Delta_t \geq \frac{3\alpha^2}{8}I \tag{H.16}$$

Proof Since the initialization scale $\alpha \leq \mathcal{O}(\sqrt{\sigma_1})$, Eq.(H.13), Eq.(H.14), Eq.(H.15) and Eq.(H.16) hold for $t' = 0$. Assume that Eq.(H.12), Eq.(H.13), Eq.(H.14), Eq.(H.15) and Eq.(H.16) hold for $t' = t-1$.

Proof of Eq.(H.12) and Eq.(H.13)

First, by using the previous global convergence result Lemma 18, the Eq.(H.12) holds by $\alpha^{3/5}\sigma_1^{7/10} < \sigma_r/2$ because $\alpha \leq \mathcal{O}(\sigma_r^{5/3}/\sigma_1^{7/6}) = \mathcal{O}(\kappa^{7/6}\sqrt{\sigma_r})$. Also, by Lemma 18, Eq.(H.13) holds for all $t \in [T_0]$.

Proof of Eq.(H.16)

5. Note that in work [29], the initialization is $F_0 = \alpha \cdot \tilde{F}_0$ and $G_0 = \alpha \cdot \tilde{G}_0$, while Lemma 18 uses an imbalance initialization. It is easy to show that their results continue to hold with this imbalance initialization.

Recall $\Delta_t = U_t^\top U_t + J_t^\top J_t - V_t^\top V_t - K_t^\top K_t$, then for all $t \le T_0$, we have

$$\|\Delta_t - \Delta_0\| \le 2\eta^2 \cdot 25\sigma_1^2 \cdot T_0 \cdot 4\sigma_1 \le 2c_2 \log(\sqrt{\sigma_1}/n\alpha)(20\sigma_1^3\eta/\sigma_r) = 200c_2\eta\kappa\sigma_1^2 \log(\sqrt{\sigma_1}/n\alpha) \le \alpha^2/8.$$

The first inequality holds by Eq.(H.7) and $\|F_tG_t - \Sigma\| \le \|F_t\|\|G_t\| + \|\Sigma\| \le 5\sigma_1$. The last inequality uses the fact that $\eta = \mathcal{O}(\alpha^2/\kappa\sigma_1^2 \log(\sqrt{\sigma_1}/n\alpha))$. Thus at $t = T_0$, we have $\lambda_{\min}(\Delta_{T_0}) \ge \lambda_{\min}(\Delta_0) - \alpha^2/8 \ge \alpha^2/2 - \alpha^2/8 = 3\alpha^2/8$ and $\|\Delta_{T_0}\| \le \|\Delta_0\| + 3\alpha^2/2 + \alpha^2/8 = 13\alpha^2/8$.

Proof of Eq.(H.14)

Now we can prove that $\|U - V\|$ keeps small during the initialization part. In fact, by Eq.(H.1) and Eq.(H.2), we have

$$\begin{aligned}
&\|(U_{t+1} - V_{t+1})\| \\
&\le \|U_t - V_t\|\|I - \eta\Sigma - \eta(V_t^\top V_t + K_t^\top K_t))\| + \eta\|V_t\|\|U_t^\top U_t + J_t^\top J_t - V_t^\top V_t - K_t^\top K_t\| \\
&\qquad + 4\eta\delta_{2k+1}\|F_tG_t^\top - \Sigma\| \max\{\|U_t\|, \|V_t\|, \|J_t\|, \|K_t\|\} \\
&\le (1 - \eta\sigma_r)\|U_t - V_t\| + 2\eta\alpha^2 \cdot 2\sqrt{\sigma_1} + 4\eta\delta_{2k+1} \cdot (\|F_t\|\|G_t\| + \|\Sigma\|) \cdot 2\sqrt{\sigma_1} \\
&\le (1 - \eta\sigma_r)\|U_t - V_t\| + 2\eta\alpha^2 \cdot 2\sqrt{\sigma_1} + 40\eta\delta_{2k+1} \cdot \sigma_1^{3/2}.
\end{aligned}$$

The second inequality uses the inequality (H.9), while the third inequality holds by $\max\{\|F_t\|, \|G_t\|\} \le 2\sqrt{\sigma_1}$. Thus since $\alpha = \mathcal{O}(\delta_{2k+1}\sigma_1^{3/2}/\sigma_r)$, we can get $\|U_0 - V_0\| \le 4\alpha \le 4\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2}$. If $\|U_t - V_t\| \le 4\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2}$, we know that

$$\begin{aligned}
\|U_{t+1} - V_{t+1}\| &\le (1 - \eta\sigma_r)\left(4\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2}\right) + 4\eta\alpha^2\sqrt{\sigma_1} + 40\eta\delta_{2k+1} \cdot \sigma_1^{3/2} \\
&\le (1 - \eta\sigma_r)\left(4\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2}\right) + 4\eta\sigma_r\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2} \\
&\le 4\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2}.
\end{aligned}$$

Hence $\|U_t - V_t\| \le 4\alpha + \frac{40}{\sigma_r}\delta_{2k+1}\sigma_1^{3/2}$ for $t \le T_0$ by induction. The second inequality holds by $\alpha = \mathcal{O}(\sigma_r/\sqrt{\sigma_1})$

Proof of Eq.(H.15)

Now we prove that J_t and K_t are bounded for all $t \le T_0$. By Eq.(H.3) and $\max\{\|F_t\|, \|G_t\|\} \le 2\sqrt{\sigma_1}$, denote $C_2 = \max\{21c_2, 32\} \ge 32$, we have

$$\begin{aligned}
\|J_{T_0}\| &\le \|J_0\| + \eta \sum_{t=0}^{T_0-1} \max\{\|F_t\|, \|G_t\|\} \cdot 2\delta_{2k+1} \cdot (\|F_t\|\|G_t\| + \|\Sigma\|) \\
&\le \|J_0\| + \eta T_0 \cdot 20\sigma_1^{3/2} \cdot \delta_{2k+1} \\
&\le \|J_0\| + 20c_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r) \\
&\le 2\alpha + 20c_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r) \\
&= 2\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r).
\end{aligned}$$

Similarly, we can prove that $\|K_{T_0}\| \le 2\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r)$. We complete the proof of Eq.(H.15). ■

H.4. Phase 1: linear convergence phase.

In this subsection, we analyze the first phase: the linear convergence phase. This phase starts at round T_0, and we assume that this phase terminates at round T_1. In this phase, the loss will converge linearly, with the rate independent of the initialization scale. Note that T_1 may tend to infinity, since this phase may not terminate. For example, when $k = r$, we can prove that this phase will not terminate (§I), and thus leading a linear convergence rate that independent on the initialization scale. In this phase, we provide the following lemma, which shows some induction hypotheses during this phase.

Lemma 15 *Denote $M_t = \max\{\|U_tV_t^\top - \Sigma\|, \|U_tK_t^\top\|, \|J_tV_t^\top\|\}$. Suppose Phase 1 starts at T_0 and ends at the first time T_1 such that*

$$\eta\sigma_r^2 M_{t-1}/64\sigma_1 < (17\eta\sigma_1\delta_{2k+1} + \eta\alpha^2)\|J_{t-1}K_{t-1}^\top\| \tag{H.17}$$

During Phase 1 that $T_0 \le t \le T_1$, we have the following three induction hypotheses:

$$\max\{\|U_t\|, \|V_t\|\} \le 2\sqrt{\sigma_1} \tag{H.18}$$

$$\|U_tV_t^\top - \Sigma\| \le \sigma_r/2. \tag{H.19}$$

$$\max\{\|J_t\|, \|K_t\|\} \le 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\kappa^2\sqrt{\sigma_1}) \le \sqrt{\sigma_1} \tag{H.20}$$

$$\frac{7\alpha^2}{4}I \ge \Delta_t \ge \frac{\alpha^2}{4}I \tag{H.21}$$

The induction hypotheses hold for $t = T_0$ due to Lemma 14. Let us assume they hold for $t' < t$, and consider the round t. Let us first prove that the r-th singular value of U and V are lower bounded by poly$(\sigma_r, 1/\sigma_1)$ at round t, if Eq.(H.19) holds at round t. In fact,

$$2\sqrt{\sigma_1}\cdot\sigma_r(U) \ge \sigma_r(U)\sigma_1(V) \ge \sigma_r(UV^\top) \ge \sigma_r/2. \tag{H.22}$$

which means

$$\sigma_r(U) \ge \sigma_r/4\sqrt{\sigma_1}. \tag{H.23}$$

Similarly, $\sigma_r(V) \ge \sigma_r/4\sqrt{\sigma_1}$.

Proof of Eq.(H.19) First, since $\|U_{t-1}V_{t-1}^\top - \Sigma\| \le \sigma_r/2$, by Eq.(H.23), we can get

$$\min\{\sigma_r(U_{t-1}), \sigma_r(V_{t-1})\} \ge \frac{\sigma_r}{4\sqrt{\sigma_1}} \tag{H.24}$$

Define $M_t = \max\{\|U_tV_t^\top - \Sigma\|, \|U_tK_t^\top\|, \|J_tV_t^\top\|\}$. By the induction hypothesis,

$$\max\{\|U_{t-1}\|, \|V_{t-1}\|\} \le 2\sqrt{\sigma_1},$$
$$\max\{\|J_{t-1}\|, \|K_{t-1}\|\} \le 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\sigma_1^{3/2}/\sigma_r).$$

Then by the updating rule and $C_2 \ge 1$, we can get

$$\begin{aligned} U_tK_t = {} & (1-\eta U_{t-1}U_{t-1}^\top)U_{t-1}K_{t-1}(1-\eta K_{t-1}K_{t-1}^\top) + \eta(\Sigma - U_{t-1}V_{t-1}^\top)VK^\top \\ & + \eta U_{t-1}J_{t-1}^\top J_{t-1}K_{t-1}^\top + A_t, \end{aligned} \tag{H.25}$$

where A_t is the perturbation term that contains all $\mathcal{O}(E_i(FG^\top - \Sigma))$ terms and $\mathcal{O}(\eta^2)$ terms such that

$$\begin{aligned}
\|A_t\| &\le 4\eta\delta_{2k+1}\|F_tG_t^\top - \Sigma\| \max\{\|F_t\|^2, \|G_t\|^2\} + 8\eta^2\|F_tG_t^\top - \Sigma\|^2 \max\{\|F_t\|^2, \|G_t\|^2\} \\
&\quad + \eta^2 \max\{\|F_t\|^2, \|G_t\|^2\}^2 \cdot \|F_tG_t - \Sigma\| \\
&\le 4\eta\delta_{2k+1}\|F_tG_t^\top - \Sigma\| \max\{\|F_t\|^2, \|G_t\|^2\} + 8\eta^2\|F_tG_t^\top - \Sigma\| \cdot 5\sigma_1 \cdot 4\sigma_1 \\
&\quad + \eta^2 \cdot 16\sigma_1^2 \cdot \|F_tG_t - \Sigma\| \\
&\le 4\eta\delta_{2k+1}(3M_{t-1} + \|J_{t-1}K_{t-1}^\top\|)4\sigma_1 + \eta\alpha^2(3M_{t-1} + \|J_{t-1}K_{t-1}^\top\|)
\end{aligned}$$

Using the similar technique for $J_tV_t^\top$ and $U_tV_t^\top - \Sigma$, we can finally get

$$\begin{aligned}
M_t &\le \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right) M_{t-1} + 2\eta M_{t-1} \cdot 2\sqrt{\sigma_1} \cdot \max\{\|J_{t-1}\|, \|K_{t-1}\|\} && \text{(H.26)} \\
&\quad + 4\eta\delta_{2k+1}(3M_{t-1} + \|J_{t-1}K_{t-1}^\top\|) \cdot 4\sigma_1 + \eta\alpha^2(3M_{t-1} + \|J_{t-1}K_{t-1}^\top\|) && \text{(H.27)} \\
&\le \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right) M_{t-1} + 2\eta M_{t-1} \cdot 2\sqrt{\sigma_1} \cdot \left(\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\sigma_1^{3/2}/\sigma_r\right) && \text{(H.28)} \\
&\quad + 4\eta\delta_{2k+1}(3M_{t-1} + \|J_{t-1}K_{t-1}^\top\|) \cdot 4\sigma_1 + \eta\alpha^2(3M_{t-1} + \|J_{t-1}K_{t-1}^\top\|) && \text{(H.29)} \\
&\le \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right) M_{t-1} + \mathcal{O}\left(\eta\sqrt{\sigma_1} \cdot \left(\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\sigma_1^{3/2}/\sigma_r\right)\right) \cdot M_{t-1} && \text{(H.30)} \\
&\quad + (17\eta\sigma_1\delta_{2k+1} + \eta\alpha^2)\|J_{t-1}K_{t-1}^\top\| && \text{(H.31)} \\
&\le \left(1 - \frac{\eta\sigma_r^2}{32\sigma_1}\right) M_{t-1} + (17\eta\sigma_1\delta_{2k+1} + \eta\alpha^2)\|J_{t-1}K_{t-1}^\top\|. && \text{(H.32)}
\end{aligned}$$

The last inequality holds by $\delta_{2k+1} = \mathcal{O}(\sigma_r^3/\sigma_1^3 \log(\sqrt{\sigma_1}/n\alpha))$ and $\alpha = \mathcal{O}(\sigma_r^2/\sigma_1^{3/2}) = \mathcal{O}(\sqrt{\sigma_r}\kappa^{-3/2})$.

During Phase 1, we have

$$\eta\sigma_r^2 M_{t-1}/64\sigma_1 \ge (17\eta\sigma_1\delta_{2k+1} + \eta\alpha^2)\|J_{t-1}K_{t-1}^\top\|,$$

then

$$M_t \le \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right) M_{t-1}. \tag{H.33}$$

Hence $\|U_tV_t^\top - \Sigma\| \le M_t \le M_{T_0} \le \|F_{T_0}G_{T_0}^\top - \Sigma\| \le \delta_{2k+1}$.

Proof of Eq.(H.18) Now we bound the norm of U_t and V_t. First, note that

$$\|(U_t - V_t)\| \le (1 - \eta\sigma_r)\|U_{t-1} - V_{t-1}\| + \eta \cdot 2\alpha^2 \cdot 2\sqrt{\sigma_1} + 40\eta \cdot \delta_{2k+1} \cdot \sigma_1^{3/2}$$

Hence $\|U_t - V_t\| \le 4\alpha + 40\delta_{2k+1}\sigma_1^{3/2}/\sigma_r$ still holds using the same technique in the initialization part.

Thus by the induction hypothesis Eq.(H.19) and $\sigma_1 \geq \delta_{2k+1}$, we have

$$\begin{aligned}
2\sigma_1 \geq \sigma_1 + \delta_{2k+1} \geq \|\Sigma\| + \|U_t V_t^\top - \Sigma\| &\geq \|U_t V_t^\top\| = \|V_t V_t^\top + (U_t - V_t)V_t^\top\| \\
&\geq \|V_t V_t^\top\| - \|U_t - V_t\|\|V_t\| \\
&\geq \|V_t\|^2 - \|V_t\| \cdot \left(4\alpha + \frac{40\delta_{2k+1}\sigma_1^{3/2}}{\sigma_r}\right) \\
&\geq \|V_t\|^2 - \|V_t\|.
\end{aligned}$$

Then we can get $\|V_t\| \leq 2\sqrt{\sigma_1}$. Similarly, $\|U_t\| \leq 2\sqrt{\sigma_1}$.

Proof of Eq.(H.20) Since during Phase 1,

$$\|J_t K_t^\top\| \leq M_t \cdot \frac{\sigma_r^2}{64\sigma_1(17\sigma_1\delta_{2k+1} + \alpha^2)} \leq M_t \cdot \frac{1}{1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r},$$

by $\delta_{2k+1} < 1/128$ and Eq.(H.33),

$$\begin{aligned}
\|F_t G_t^\top - \Sigma\| \leq 4\max\{\|J_t K_t^\top\|, M_t\} &\leq 4M_t \cdot \max\left\{1, \frac{1}{1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r}\right\} \\
&\leq \|F_{T_0}G_{T_0} - \Sigma\| \left(1 - \eta\sigma_r^2/64\sigma_1\right)^{t-T_0} / (1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r).
\end{aligned} \tag{H.34}$$

Thus, the maximum norm of J_t, K_t can be bounded by

$$\begin{aligned}
\|J_t\| &\leq \|J_{T_0}\| + 2\eta \cdot 2\sqrt{\sigma_1}\delta_{2k+1} \cdot \sum_{t'=T_0}^{t-1} \|F_t G_t - \Sigma\| \\
&\leq 2\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r) + \frac{4\eta\sqrt{\sigma_1}\delta_{2k+1}}{1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r} \cdot \|F_{T_0}G_{T_0} - \Sigma\| \cdot \frac{64\sigma_1}{\eta\sigma_r^2} \\
&= 2\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r) + \frac{\sigma_1^{3/2}}{4\kappa^2\sigma_r^2} \cdot \|F_{T_0}G_{T_0} - \Sigma\| \\
&\leq 2\alpha + C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \sigma_1^{3/2}/\sigma_r) + \frac{\alpha^{1/2}\sigma_1^{9/4}}{4\kappa^2\sigma_r^2} \\
&\leq 2\sqrt{\alpha}\sigma_1^{1/4} + C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \kappa^2\sqrt{\sigma_1}) \\
&\leq 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \kappa^2\sqrt{\sigma_1}).
\end{aligned}$$

The last inequality uses the fact that $2\alpha + \frac{\sqrt{\alpha}\sigma_1^{1/4}}{4} \leq 2\sqrt{\alpha}\sigma_1^{1/4}$ by $\alpha = \mathcal{O}(\sqrt{\sigma_r})$. Similarly, $\|K_t\| \leq 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \kappa^2 \cdot \sqrt{\sigma_1})$. We complete the proof of Eq.(H.20).

Proof of Eq.(H.21) Last, for $t \in [T_0, T_1)$, we have

$$\begin{aligned}
\|\Delta_t - \Delta_{T_0}\| &\leq \sum_{t=T_0}^{T_1-1} 2(\eta^2 \cdot \|F_t G_t^\top - \Sigma\|^2 \cdot \max\{\|F_t\|, \|G_t\|\}^2) \\
&\leq 2\eta^2 \|F_{T_0} G_{T_0} - \Sigma\|^2 \sum_{t=T_0}^{\infty} \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right)^{2(t-T_0)} \cdot 4\sigma_1 \\
&\leq 2\eta^2 \cdot 25\sigma_1^2 \cdot \frac{16\sigma_1}{\eta\sigma_r^2} \cdot 4\sigma_1 \\
&\leq 3200\eta\kappa^2\sigma_1^2 \\
&\leq \alpha^2/8,
\end{aligned}$$

where the last inequality arises from the fact that $\eta = \mathcal{O}(\alpha^2/\kappa^2\sigma_1^2)$. By $\frac{3\alpha^2}{8} I \leq \Delta_{T_0} \leq \frac{13\alpha^2}{8} I$, we can have $\|\Delta_t\| \leq 13\alpha^2/8 + \alpha^2/8 \leq 7\alpha^2/4$ and $\lambda_{\min}(\Delta_t) \geq 3\alpha^2/8 - \alpha^2/8 = \alpha^2/4$. Hence the inequality Eq.(H.21) still holds during Phase 1. Moreover, by Eq.(H.34), during the Phase 1, for a round $t \geq 0$, we will have

$$\begin{aligned}
\|F_{t+T_0} G_{t+T_0}^\top - \Sigma\| &\leq \|F_{T_0} G_{T_0} - \Sigma\| \left(1 - \eta\sigma_r^2/64\sigma_1\right)^t / (1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r) \\
&\leq \|F_{T_0} G_{T_0} - \Sigma\| \left(1 - \eta\sigma_r^2/64\sigma_1\right)^t \cdot \frac{\sigma_r}{64\alpha^2\kappa} \\
&\leq \frac{\sigma_r}{2} \cdot \left(1 - \eta\sigma_r^2/64\sigma_1\right)^t \cdot \frac{\sigma_r}{64\alpha^2\kappa} \\
&= \frac{\sigma_r^2}{128\alpha^2\kappa} \left(1 - \eta\sigma_r^2/64\sigma_1\right)^t. \qquad \text{(H.35)}
\end{aligned}$$

The conclusion (H.35) always holds in Phase 1. Note that Phase 1 may not terminate, and then the loss is linear convergence. We assume that at round T_1, Phase 1 terminates, which implies that

$$\sigma_r^2 M_{T_1-1}/64\sigma_1 < (17\sigma_1\delta_{2k+1} + \alpha^2)\|J_{T_1-1} K_{T_1-1}^\top\|, \qquad \text{(H.36)}$$

and the algorithm goes to Phase 2.

H.5. Phase 2: Adjustment Phase.

In this phase, we prove $U - V$ will decrease exponentially. This phase terminates at the first time T_2 such that

$$\|U_{T_2-1} - V_{T_2-1}\| \leq \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}\|J_{T_2-1} K_{T_2-1}^\top\|}{\sigma_r}. \qquad \text{(H.37)}$$

By stopping rule (H.37), since $\|U_{T_1} - V_{T_1}\| \leq \mathcal{O}(\sigma_1)$, this phase will take at most $\mathcal{O}(\log(\sqrt{\sigma_r}/\alpha)/\eta\sigma_r)$ rounds, i.e.

$$T_2 - T_1 = \mathcal{O}(\log(\sqrt{\sigma_r}/\alpha)/\eta\sigma_r). \qquad \text{(H.38)}$$

We use the induction to show that all the following hypotheses hold during Phase 2.

$$\max\{\|F_{t-1}\|, \|G_{t-1}\} \leq 2\sqrt{\sigma_1} \tag{H.39}$$

$$M_t \leq (1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r)\|J_tK_t^\top\| \leq \|J_tK_t^\top\| \tag{H.40}$$

$$\max\{\|J_{t-1}\|, \|K_{t-1}\|\} \leq 2\sqrt{\alpha}\sigma_1^{1/4} + (2C_2 + 16C_3)\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\kappa^2\sqrt{\sigma_1}) \leq \sigma_r/4\sqrt{\sigma_1} \tag{H.41}$$

$$\|J_tK_t^\top\| \leq \left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right)\|J_{t-1}K_{t-1}^\top\| \tag{H.42}$$

$$\|U_t - V_t\| \leq (1 - \eta\sigma_r/2)\|U_{t-1} - V_{t-1}\| \tag{H.43}$$

$$\frac{3\alpha^2}{16}\cdot I \leq \Delta_t \leq \frac{29\alpha^2}{16}\cdot I. \tag{H.44}$$

Proof of (H.41) To prove this, we first assume that this adjustment phase will only take at most $C_3(\log(\alpha)/\eta\sigma_r)$ rounds. By the induction hypothesis for the previous rounds,

$$\begin{aligned}
\|J_t\| &\leq J_{T_1} + \sum_{i=T_1}^{t-1}\eta\delta_{2k+1}\cdot\|F_tG_t^\top - \Sigma\| \\
&\leq 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r) + \sum_{i=T_1}^{t-1}\eta\delta_{2k+1}\cdot\|F_iG_i^\top - \Sigma\| \\
&\leq 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r) + C_3(\log(\sqrt{\sigma_1}/n\alpha)/\eta\sigma_r)\cdot\eta\delta_{2k+1}\cdot 4\|J_{i-1}K_{i-1}^\top\| \\
&\leq 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r) + C_3(\log(\sqrt{\sigma_1}/n\alpha)/\eta\sigma_r)\cdot\eta\delta_{2k+1}16\sigma_1 \\
&\leq 2\sqrt{\alpha}\sigma_1^{1/4} + (2C_2 + 16C_3)\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r).
\end{aligned}$$

Similarly, due to the symmetry property, we can bound the $\|K_t\|$ using the same technique. Thus

$$\max\{\|J_t\|, \|K_t\|\} \leq 2\sqrt{\alpha}\sigma_1^{1/4} + (2C_2 + 16C_3)\log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r).$$

Proof of (H.40) First, we prove that during $t \in [T_1, T_2)$,

$$M_t \leq (1088\kappa^2\delta_{2k+1} + 64\alpha^2\kappa/\sigma_r)\|J_tK_t^\top\| \leq \|J_tK_t^\top\| \leq 4\alpha\kappa^4\sigma_1^{1/2} + \delta_{2k+1}\sigma_1. \tag{H.45}$$

in this phase.

Then by $\delta_{2k+1} \leq \mathcal{O}(1/\log(\sqrt{\sigma_1}/n\alpha)\kappa^2)$ and $\alpha \leq \mathcal{O}(\sigma_r/\sqrt{\sigma_1})$, choosing sufficiently small coefficient, we can have

$$\begin{aligned}
J_tK_t^\top = &(I - \eta J_{t-1}J_{t-1}^\top)J_{t-1}K_{t-1}^\top(I - \eta K_{t-1}K_{t-1}^\top) + \eta^2 J_{t-1}J_{t-1}^\top J_{t-1}K_{t-1}^\top K_{t-1}K_{t-1}^\top \\
&- \eta J_{t-1}V_{t-1}^\top V_{t-1}K_{t-1}^\top - \eta J_tU_t^\top U_tK_t^\top + C_{t-1},
\end{aligned} \tag{H.46}$$

where C_t represents the relatively small perturbation term, which contains terms of $\mathcal{O}(\delta)$ and $\mathcal{O}(\eta^2)$. By (H.39), we can easily get

$$C_{t-1} \geq -\left(4\eta\delta_{2k+1}\cdot\|F_{t-1}G_{t-1}^\top - \Sigma\|\cdot 4\sigma_1\right) \tag{H.47}$$

Thus combining (H.46) and (H.47), we have

$$
\begin{aligned}
&\|J_t K_t^\top\| \\
&\geq \|I - \eta J_{t-1} J_{t-1}^\top\| \|I - \eta K_{t-1} K_{t-1}^\top\| \|J_{t-1} K_{t-1}^\top\| - 4\eta M_{t-1} \cdot 4\sigma_1 \\
&\quad - 4\eta \delta_{2k+1} \|J_{t-1} K_{t-1}\| \cdot 2\sigma_1 - \eta^2 64 \sigma_1^3 \\
&\geq \left(1 - 2\eta \max\{\|J_{t-1}\|, \|K_{t-1}\|\}^2 - 16 \cdot 1088 \eta \kappa^2 \delta_{2k+1} \sigma_1 - 1024 \eta \alpha^2 \kappa^2 - 8\eta \delta_{2k+1} \cdot \sigma_1\right) \|J_{t-1} K_{t-1}^\top\| \\
&\geq \left(1 - \frac{\eta \sigma_r^2}{128 \sigma_1}\right) \|J_{t-1} K_{t-1}^\top\|.
\end{aligned}
$$

The second inequality is because $M_{t-1} \leq (1088\kappa^2 \delta_{2k+1} + 64\alpha^2 \kappa/\sigma_r)\|J_{t-1} K_{t-1}^\top\|$, and the last inequality holds by Eq.(H.41) and

$$
\delta_{2k+1} = \mathcal{O}(\kappa^{-4}), \alpha = \mathcal{O}(\kappa^{-3/2}\sqrt{\sigma_r}) \tag{H.48}
$$

Then, note that by Eq.(H.32), we have

$$
M_t \leq \left(1 - \frac{\eta \sigma_r^2}{32 \sigma_1}\right) M_{t-1} + (17\eta \sigma_1 \delta_{2k+1} + \eta \alpha^2) \|J_{t-1} K_{t-1}^\top\|.
$$

Then by $M_{t-1} \leq (1088\kappa^2 \delta_{2k+1} + 64\alpha^2 \kappa/\sigma_r) \cdot \|J_{t-1} K_{t-1}^\top\|$ and denote $L = 17\sigma_1 \delta_{2k+1} + \alpha^2$, we have

$$
\begin{aligned}
M_t &\leq \left(1 - \frac{\eta \sigma_r^2}{32 \sigma_1}\right) M_{t-1} + (17\eta \sigma_1 \delta_{2k+1} + \eta \alpha^2) \|J_{t-1} K_{t-1}^\top\| \\
&\leq \left(1 - \frac{\eta \sigma_r^2}{32 \sigma_1}\right) \cdot (1088\kappa^2 \delta_{2k+1} + 64\alpha^2 \kappa/\sigma_r) \|J_{t-1} K_{t-1}^\top\| + \eta L \|J_{t-1} K_{t-1}^\top\| \\
&= \left(1 - \frac{\eta \sigma_r^2}{32 \sigma_1}\right) \cdot \frac{64 L \kappa}{\sigma_r} \|J_{t-1} K_{t-1}^\top\| + \eta L \|J_{t-1} K_{t-1}^\top\| \\
&\leq \left(\frac{64 L \kappa}{\sigma_r} - 2\eta L\right) \|J_{t-1} K_{t-1}^\top\| \\
&\leq \left(\frac{64 L \kappa}{\sigma_r} - 2\eta L\right) \Big/ \left(1 - \frac{\eta \sigma_r^2}{128 \sigma_1}\right) \|J_t K_t^\top\| \\
&\leq \frac{64 L \kappa}{\sigma_r} \|J_t K_t^\top\|.
\end{aligned}
$$

Hence

$$
M_t \leq \frac{64 L \kappa}{\sigma_r} \|J_t K_t^\top\| \leq \|J_t K_t^\top\|
$$

for all t in Phase 2. The last inequality is because $\delta_{2k+1} = \mathcal{O}(1/\kappa^2 \log(\sqrt{\sigma_1}/n\alpha))$. Moreover, by $\delta_{2k+1} \leq \mathcal{O}(1/\kappa^2 \log(\sqrt{\sigma_1}/n\alpha)^2)$ and $(a+b)^2 \leq 2a^2 + 2b^2$ we have

$$
\|J_t K_t^\top\| \leq \|J_t\| \|K_t\| \leq \left(2\sqrt{\alpha} \sigma_1^{1/4} + (2C_2 + 16C_3) \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \kappa^2 \sqrt{\sigma_1})\right)^2 \tag{H.49}
$$

$$
\leq 4\alpha \kappa^4 \sigma_1^{1/2} + \delta_{2k+1} \sigma_1. \tag{H.50}
$$

We complete the proof of Eq.(H.40).

Proof of Eq.(H.42) Moreover, by the updating rule of J_t and K_t, (H.46) and (H.47) we have

$$\|J_t K_t^\top\| \tag{H.51}$$
$$\leq \|(I - \eta J_{t-1}J_{t-1}^\top)J_{t-1}K_{t-1}^T(I - \eta K_{t-1}K_{t-1}^\top)\| + \|\eta^2(J_{t-1}J_{t-1}^\top)J_{t-1}K_{t-1}^T(K_{t-1}K_{t-1}^\top)\| \tag{H.52}$$
$$+ 4\eta M_{t-1}\cdot 4\sigma_1 + 4\eta\delta_{2k+1}\|J_{t-1}K_{t-1}^\top\|\cdot 2\sigma_1 \tag{H.53}$$
$$\leq \|J_{t-1}K_{t-1}^\top\| + \eta^2(\sqrt{\sigma_1}/2)^4\|J_{t-1}K_{t-1}^\top\| + 4\eta\frac{64L\kappa}{\sigma_r}\|J_{t-1}K_{t-1}^\top\|\cdot 4\sigma_1 + 8\eta\sigma_1\delta_{2k+1}\|J_{t-1}K_{t-1}^\top\| \tag{H.54}$$
$$= \|J_{t-1}K_{t-1}^\top\|\cdot\left(1 + \eta^2\sigma_1^2/16 + 1024L\kappa^2 + 8\sigma_1\delta_{2k+1}\right). \tag{H.55}$$

The last inequality uses the fact that $\|J_{t-1}\| \leq \sqrt{\sigma_1}/2$, $\|K_{t-1}\| \leq \sqrt{\sigma_1}/2$ and $M_{t-1} \leq \frac{64L\kappa}{\sigma_r}\|J_{t-1}K_{t-1}^\top\|$. Now by the fact that $L = 17\sigma_1\delta_{2k+1} + \alpha^2 = \mathcal{O}(\frac{\sigma_r^2}{\sigma_1\kappa^2})$, we can choose small constant so that

$$\eta^2\sigma_1^2/16 \leq \frac{\sigma_r^2}{384\sigma_1}, \quad 1024L\kappa^2 \leq \frac{\sigma_r^2}{384\sigma_1}, \quad 8\sigma_1\delta_{2k+1} \leq \frac{\sigma_r^2}{384\sigma_1}. \tag{H.56}$$

Thus we can have

$$\|J_t K_t^\top\| \leq \|J_{t-1}K_{t-1}^\top\|\cdot\left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right). \tag{H.57}$$

We complete the proof of (H.42)

Proof of (H.43) Hence, similar to Phase 1, by $\|U_t V_t^\top - \Sigma\| \leq M_t \leq 4\alpha\kappa^4\sigma_1^{1/2} + \delta_{2k+1}\sigma_1$ and $\|U_t - V_t\| \leq \|U_{T_1} - V_{T_1}\| \leq 4\alpha + \frac{40\delta\sigma_1^{3/2}}{\sigma_r}$, we can show that

$$\max\{\|U_t\|, \|V_t\|\} \leq 2\sqrt{\sigma_1}$$

Also, consider

$$\begin{aligned}
&U_t - V_t \\
&= (I - \eta\Sigma - V_t^\top V_t - K_t^\top K_t)(U_{t-1} - V_{t-1}) - \eta V_t\Delta_t \\
&\quad + \eta\cdot\left(E_1(F_{t-1}G_{t-1}^\top - \Sigma)V_{t-1} + E_2(F_{t-1}G_{t-1}^\top - \Sigma)K_{t-1}\right) \\
&\quad - \eta\cdot\left(E_1^\top(F_{t-1}G_{t-1}^\top - \Sigma)U_{t-1} + E_3^\top(F_{t-1}G_{t-1}^\top - \Sigma)J_{t-1}\right).
\end{aligned}$$

Hence by the RIP property and $\Delta_{t-1} \leq 2\alpha^2 I$ ((H.44)), we can get

$$\begin{aligned}
\|(U_t - V_t)\| &\leq (1 - \eta\sigma_r)\|U_{t-1} - V_{t-1}\| + 2\eta\alpha^2\cdot 2\sqrt{\sigma_1} + 4\eta\delta_{2k+1}\cdot 2\sqrt{\sigma_1}\cdot\|F_{t-1}G_{t-1}^\top - \Sigma\| \\
&\leq (1 - \eta\sigma_r)\|U_{t-1} - V_{t-1}\| + 2\eta\alpha^2\cdot 2\sqrt{\sigma_1} + 8\eta\delta_{2k+1}\cdot\sqrt{\sigma_1}\cdot 4\|J_{t-1}K_{t-1}^\top\| \\
&\leq (1 - \eta\sigma_r)\|U_{t-1} - V_{t-1}\| + 2\eta\alpha^2\cdot 2\sqrt{\sigma_1} + 32\eta\delta_{2k+1}\cdot\sqrt{\sigma_1}\cdot\|J_{t-1}K_{t-1}^\top\|
\end{aligned}$$

Since

$$\|U_{t-1} - V_{t-1}\| \geq \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}\|J_{t-1}K_{t-1}^\top\|}{\sigma_r}.$$

for all t in Phase 2, we can have

$$\|U_t - V_t\| \le (1 - \eta\sigma_r/2)\|U_{t-1} - V_{t-1}\|$$

during Phase 2.

Moreover, since Phase 2 terminates at round T_2, such that

$$\|U_{T_2-1} - V_{T_2-1}\| \le \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}\|J_{T_2-1}K_{T_2-1}^\top\|}{\sigma_r},$$

it takes at most

$$C_3 \log(\sqrt{\sigma_r}/\alpha)/\eta\sigma_r = t_2^* \tag{H.58}$$

rounds for some constant C_3 because (a) (H.43), (b) and $U_t - V_t$ decreases from $\|U_{T_1} - V_{T_1}\| \le 4\sqrt{\sigma_1}$ to at most $\|U_{T_2} - V_{T_2}\| = \Omega(\alpha^2\sqrt{\sigma_1}/\sigma_r)$. Also, the changement of Δ_t can be bounded by

$$\begin{aligned}
\|\Delta_t - \Delta_{T_1}\| &\le \sum_{t=T_1}^{T_2-1} 2(\eta^2 \cdot \|F_t G_t^\top - \Sigma\|^2 \cdot 4\sigma_1) \\
&\le 2(\eta^2) \cdot 100\sigma_1^3 \cdot (T_2 - T_1) \\
&\le 2(\eta^2) \cdot 100\sigma_1^3 \cdot C_3 \log(\sqrt{\sigma_1}/n\alpha)(1/\eta\sigma_r) \\
&\le 10C_3 \log(\sqrt{\sigma_1}/n\alpha)(\eta\kappa\sigma_1^2) \\
&\le \alpha^2/16.
\end{aligned}$$

The last inequality holds by choosing $\eta \le \alpha^2/160C_3\kappa\sigma_1^2$. Then $\lambda_{\min}(\Delta_t) \ge \lambda_{\min}\Delta_{T_1} - \alpha^2/16 \ge \alpha^2/4 - \alpha^2/16 = 3\alpha^2/16$ and $\|\Delta_t\| \le \|\Delta_{T_1}\| + \alpha^2/16 \le 7\alpha^2/4 + \alpha^2/16 \le 29\alpha^2/16$. Hence inequality (H.9) still holds during Phase 2.

H.6. Phase 3: local convergence

In this phase, we show that the norm of K_t will decrease at a linear rate. Denote the SVD of U_t as $U_t = A_t\Sigma_t W_t$, where $\Sigma_t \in \mathbb{R}^{r\times r}$, $W_t \in \mathbb{R}^{r\times k}$, and define $W_{t,\perp} \in \mathbb{R}^{(k-r)\times k}$ is the complement of W_t.

We use the induction to show that all the following hypotheses hold during Phase 3.

$$\max\{\|J_t\|, \|K_t\|\} \le \mathcal{O}(2\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \kappa^2\sqrt{\sigma_1}) \le \sqrt{\sigma_1}/2 \tag{H.59}$$

$$M_t \le \frac{64L\kappa}{\sigma_r}\|J_t K_t^\top\| \le \|J_t K_t^\top\| \tag{H.60}$$

$$\|J_t K_t^\top\| \le \left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right)\|J_{t-1}K_{t-1}^\top\| \tag{H.61}$$

$$\|U_t - V_t\| \le \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}\|J_t K_t^\top\|}{\sigma_r} \tag{H.62}$$

$$\frac{\alpha^2}{8} \cdot I \le \Delta_t \le 2\alpha^2 I \tag{H.63}$$

$$\|K_t\| \le 2\|K_t W_{t,\perp}^\top\| \tag{H.64}$$

$$\|K_{t+1}W_{t+1,\perp}^\top\| \le \|K_t W_{t,\perp}^\top\| \cdot \left(1 - \frac{\eta\alpha^2}{8}\right). \tag{H.65}$$

Assume the hypotheses above hold before round t, then at round t, by the same argument in Phase 1 and 2, the inequalities (H.60) and (H.62) still holds, then $\max\{\|U_t\|, \|V_t\|\} \le 2\sqrt{\sigma_1}$ and $\min\{\sigma_r(U), \sigma_r(V)\} \ge \sigma_r/4\sqrt{\sigma_1}$.

Last, we should prove the induction hypotheses (H.59) , (H.63), (H.64) and (H.65).

Proof of Eq.(H.61) Similar to the proof of (H.42) in Phase 2, we can derive (H.61) again.

Proof of Eq.(H.64) First, to prove (H.64), note that we can get

$$\begin{aligned} M_t \ge \|U_t K_t\| = \|A_t \Sigma_t W_t K_t^\top\| &= \|\Sigma_t W_t K_t^\top\| \\ &\ge \sigma_r(U) \cdot \|K_t W_t^\top\| \ge \frac{\|K_t W_t^\top\| \sigma_r}{4\sqrt{\sigma_1}} \ge \frac{\|K_t W_t^\top\| \sqrt{\sigma_r}}{4\sqrt{\kappa}}. \end{aligned}$$

Hence

$$\|K_t W_t^\top\| \le 4\sqrt{\kappa} M/\sqrt{\sigma_r} \le \frac{64\sigma_1 L\sqrt{\kappa}}{\sigma_r^{5/2}} \|J_t K_t^\top\| \le \frac{32 L \kappa^{3/2}}{\sigma_r^{3/2}} \|K_t\| \cdot \sqrt{\sigma_1} \le \frac{32 L \kappa^2}{\sigma_r} \|K_t\|. \tag{H.66}$$

Thus

$$\begin{aligned} \|K_t\| &\le \|K_t W_{t,\perp}^\top\| + \|K_t W_t^\top\| \\ &\le \|K_t W_{t,\perp}^\top\| + \frac{64 L\kappa}{\sigma_r} \|K_t\| \\ &\le \|K_t W_{t,\perp}^\top\| + \frac{1}{2} \|K_t\|. \end{aligned}$$

The last inequality uses the fact that $\delta_{2k+1} = \mathcal{O}(\sigma_r^3/\sigma_1^3)$ Hence $\|K_t W_{t,\perp}^\top\| \ge \|K_t\|/2$, and (H.64) holds during Phase 3.

Proof of Eq.(H.63) To prove the (H.63), by the induction hypothesis of Eq.(H.65), note that

$$\begin{aligned} \|\Delta_t - \Delta_{T_2}\| &\le 2\eta^2 \cdot \sum_{t'=T_2}^{t-1} \|F_{t'} G_{t'}^\top - \Sigma\|^2 4\sigma_1 \\ &\le 2\eta^2 \sum_{t'=T_2}^{t-1} 16\sigma_1 \|J_{t'} K_{t'}^\top\|^2 \\ &\le 64\sigma_1 \eta^2 \cdot \sum_{t'=T_2}^{\infty} \|J_{t'}\|^2 \|K_{t'} W_{t',\perp}^\top\|^2 \\ &\le 64\sigma_1 \cdot \eta^2 \left(\sigma_1 \cdot \|K_{T_2} W_{T_2,\perp}^\top\|^2 \cdot \frac{8}{\eta\alpha^2} \right) \\ &\le \frac{512\eta\sigma_1^2}{\alpha^2} \cdot \|K_{T_2}\|^2 \\ &\le \frac{128\eta\sigma_1^2}{\alpha^2} \cdot \sigma_1 \\ &\le \alpha^2/16. \end{aligned} \tag{H.67}$$

The Eq.(H.67) holds by the sum of geometric series. The last inequality holds by $\eta \le \mathcal{O}(\alpha^4/\sigma_1^3)$ Then we have

$$\|\Delta_t\| \le \|\Delta_{T_2}\| + \|\Delta_t - \Delta_{T_2}\| \le \frac{29\alpha^2}{16} + \frac{\alpha^2}{16} \le 2\alpha^2.$$
$$\lambda_{\min}(\Delta_t) \ge \lambda_{\min}(\Delta_{T_2}) - \|\Delta_t - \Delta_{T_2}\| \ge \frac{3\alpha^2}{16} - \frac{\alpha^2}{16} = \frac{\alpha^2}{8}.$$

Hence (H.63) holds during Phase 3.

Proof of Eq.(H.59) To prove the (H.59), note that

$$\|K_t\| \le 2\|K_t W_{t,\perp}^\top\| \le 2\|K_{T_2} W_{T_2,\perp}^\top\| \le 2\|K_{T_2}\| \le \mathcal{O}(\delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha)\cdot \sigma_1^{3/2}/\sigma_r). \quad \text{(H.68)}$$

On the other hand, by $\Delta_t \le 2\alpha^2 I$, we have

$$W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top - W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top - W_{t,\perp} V_t^\top V_t W_{t,\perp}^\top \le 2\alpha^2 \cdot I.$$

Hence, denote $L_t = \|J_t K_t^\top\| \le \sigma_1/4$,

$$\begin{aligned}
&W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top \le 2\alpha^2 I + W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top + W_{t,\perp} V_t^\top V_t W_{t,\perp}^\top \\
&= 2\alpha^2 I + W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top + W_{t,\perp}(V_t - U_t)^\top (V_t - U_t) W_{t,\perp}^\top \\
&\le 2\alpha^2 I + W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top + \left(\frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r}\right)^2 \cdot I \\
&= W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top + \left(2\alpha + \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r}\right)^2 I. \qquad \text{(H.69)}
\end{aligned}$$

Also, by inequality (H.69), we have

$$\begin{aligned}
\|J_t W_{t,\perp}^\top\| - \|K_t W_{t,\perp}^\top\| &\le \frac{\|J_t W_{t,\perp}^\top\|^2 - \|K_t W_{t,\perp}^\top\|^2}{\|J_t W_{t,\perp}^\top\| + \|K_t W_{t,\perp}^\top\|} \\
&\le \frac{\left(2\alpha + \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r}\right)^2}{2\|K_t W_{t,\perp}^\top\| + \|J_t W_{t,\perp}^\top\| - \|K_t W_{t,\perp}^\top\|} \\
&\le \frac{\left(2\alpha + \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r}\right)^2}{\|J_t W_{t,\perp}^\top\| - \|K_t W_{t,\perp}^\top\|}
\end{aligned}$$

Thus by $L_t \le \sigma_1/4$, we can get

$$\begin{aligned}
\|J_t W_{t,\perp}^\top\| &\le \|K_t W_{t,\perp}^\top\| + 2\alpha + \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r} \\
&\le \|K_{T_2}\| + 2\alpha + \frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r} \\
&\le \mathcal{O}(2\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha)\kappa^2\sqrt{\sigma_1}).
\end{aligned}$$

The second inequality holds by $\|K_t W_{t,\perp}^\top\| \le \|K_{T_2} W_{T_2,\perp}^\top\| \le \|K_{T_2}\|$. On the other hand, note that

$$
\begin{aligned}
\|J_t\| &\le \|J_t W_t^\top\| + \|J_t W_{t,\perp}^\top\| \\
&\le \|J_t U_t^\top\|/\sigma_r(U) + \|J_t W_{t,\perp}^\top\| \\
&\le \|J_t V_t\|/\sigma_r(U) + \|J_t(U_t - V_t)\|/\sigma_r(U) + \|J_t W_{t,\perp}^\top\| \\
&\le M_t/\sigma_r(U) + \|J_t\|\|(U_t - V_t)\|/\sigma_r(U) + \|J_t W_{t,\perp}^\top\| \\
&\le \frac{64L\kappa}{\sigma_r}\|J_t\|\|K_t\| \cdot \frac{4\sqrt{\sigma_1}}{\sigma_r} + \|J_t\|\frac{8\alpha^2\sqrt{\sigma_1} + 64\delta_{2k+1}\sqrt{\sigma_1}\|J_t K_t^\top\|}{\sigma_r} \cdot \frac{4\sqrt{\sigma_1}}{\sigma_r} + \|J_t W_{t,\perp}^\top\| \\
&\le \left(\frac{64\sigma_1^{3/2}L}{\sigma_r^3} \cdot \sqrt{\sigma_1} + \frac{32\alpha^2\sigma_1 + 256\delta_{2k+1}\sigma_1 \cdot \sigma_1}{\sigma_r^2}\right)\|J_t\| + \|J_t W_{t,\perp}^\top\| \\
&\le \frac{1}{2}\|J_t\| + \|J_t W_{t,\perp}^\top\|. \qquad \text{(H.70)}
\end{aligned}
$$

The last inequality holds because

$$
\delta_{2k+1} = \mathcal{O}(\kappa^{-4}\log^{-1}(\sqrt{\sigma_1}/n\alpha)), \quad \alpha \le \mathcal{O}(\sigma_r/\sqrt{\sigma_1})
$$

Hence by the inequality (H.70), we can get

$$
\|J_t\| \le 2\|J_t W_{t,\perp}^\top\| = \mathcal{O}(2\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \kappa^2\sqrt{\sigma_1}). \qquad \text{(H.71)}
$$

Thus (H.59) holds during Phase 3.

Proof of Eq.(H.65) Now we prove the inequality (H.65). We consider the changement of K_t. We have

$$
K_{t+1} = K_t(I - U_t^\top U_t - J_t^\top J_t) + E_3(F_t G_t^\top - \Sigma)U_t + E_4(F_t G_t^\top - \Sigma)J_t
$$

Now consider $K_{t+1}W_{t,\perp}^\top$, we can get

$$
\begin{aligned}
K_{t+1}W_{t,\perp}^\top &= K_t(I - \eta W_t^\top \Sigma^2 W_t - J_t^\top J_t)W_{t,\perp}^\top + \eta E_3(F_t G_t^\top - \Sigma)U_t W_{t,\perp}^\top + \eta E_4(F_t G_t^\top - \Sigma)J_t W_{t,\perp}^\top \\
&= K_t W_{t,\perp}^\top - \eta K_t J_t^\top J_t W_{t,\perp}^\top + \eta E_4(F_t G_t^\top - \Sigma)J_t W_{t,\perp}^\top \\
&= K_t W_{t,\perp}^\top - \eta K_t W_{t,\perp}^\top W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top - \eta K_t W_t^\top W_t J_t^\top J_t W_{t,\perp}^\top + \eta E_4(F_t G_t^\top - \Sigma)J_t W_{t,\perp}^\top
\end{aligned}
$$

Hence by the Eq.(H.66),

$$
\begin{aligned}
\|K_{t+1}W_{t,\perp}^\top\| &\le \|K_t W_{t,\perp}^\top(I - \eta W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top)\| + \frac{64\eta L\kappa^{3/2}}{\sigma_r^{3/2}}\|J_t K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\|\|J_t\| + 4\eta\delta_{2k+1}M_t\|J_t W_{t,\perp}^\top\| \\
&\le \|K_t W_{t,\perp}^\top(I - \eta W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top)\| + \frac{64\eta L\kappa^{3/2}}{\sigma_r^{3/2}}\|J_t K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\|\|J_t\| \\
&\quad + \frac{16\sigma_1\eta L}{\sigma_r^2}\|J_t K_t^\top\|\|J_t W_{t,\perp}^\top\| \\
&\le \|K_t W_{t,\perp}^\top(I - \eta W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top)\| + \frac{80\eta L\kappa^2}{\sigma_r}\|J_t K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\|
\end{aligned}
$$

The second inequality uses the fact that $\delta_{2k+1} \leq 1/16$ and (H.66). The last inequality uses the fact that $\|J_t\| \leq \sqrt{\sigma_1}$. Note that $\lambda_{\min}(\Delta_t) \geq \alpha^2/8 \cdot I$, then multiply the $W_{t,\perp}^\top$, we can get

$$W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top - W_{t,\perp} V_t^\top V_t W_{t,\perp}^\top - W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top \geq \frac{\alpha^2}{8} \cdot I.$$

Hence

$$W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top - W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top \geq \frac{\alpha^2}{8} \cdot I.$$

Thus, define $\phi_t = W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top - W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top$, then we can get

$$\begin{aligned}
\|K_{t+1} W_{t,\perp}^\top\| &\leq \|K_t W_{t,\perp}^\top (I - W_{t,\perp} J_t^\top J_t W_{t,\perp}^\top)\| + \frac{80 L \kappa^2}{\sigma_r} \|J_t K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\| \\
&\leq \|K_t W_{t,\perp}^\top (I - W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top - \eta \phi_t)\| + \frac{80 L \kappa^2}{\sigma_r} \|J_t K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\|
\end{aligned}$$

Define loss $L_t = \|J_t K_t^\top\|$. Note that

$$\begin{aligned}
L_t &= \|J_t K_t^\top\| \\
&= \|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top + J_t W_t^\top W_t K_t^\top\| \\
&\leq \|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| + \|J_t W_t^\top W_t K_t^\top\| \\
&\leq \|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| + \sqrt{\sigma_1} \cdot \frac{64 L \kappa^{3/2}}{\sigma_r^{3/2}} \|J_t K_t^\top\| \qquad \text{(H.72)} \\
&\leq \|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| + \frac{L_t}{2}.
\end{aligned}$$

The Eq.(H.72) holds by Eq.(H.66) and $\|W_t^\top\| = 1$, and the last inequality holds by $\delta_{2k+1} = \mathcal{O}(\kappa^4)$.

Hence

$$\|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| \geq L_t/2. \qquad \text{(H.73)}$$

Similarly,

$$\|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| \leq 2 L_t \qquad \text{(H.74)}$$

Then

$$\|K_{t+1} W_{t,\perp}^\top\| \leq \|K_t W_{t,\perp}^\top (I - \eta W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top - \eta \phi_t)\| + \frac{160 \eta L \kappa^2}{\sigma_r} \|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\|.$$

If $\|J_t W_{t,\perp}^\top\| \leq 10 \kappa \alpha$, we can get

$$
\begin{aligned}
\|K_{t+1}W_{t,\perp}^\top\| &\le \|K_tW_{t,\perp}^\top(I-\eta W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top-\eta\phi_t)\| + \frac{160\eta L\kappa^2}{\sigma_r}\|J_tW_{t,\perp}^\top W_{t,\perp}K_t^\top\|\cdot\|J_tW_{t,\perp}^\top\| \\
&\le \|K_tW_{t,\perp}^\top\|\|(I-\eta W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top-\eta\phi_t)\| + \frac{160\eta L\kappa^2}{\sigma_r}\|J_tW_{t,\perp}^\top W_{t,\perp}K_t^\top\|\cdot\|J_tW_{t,\perp}^\top\| \\
&\le \|K_tW_{t,\perp}^\top\|\left(1-\frac{\eta\alpha^2}{8}\right) + \frac{160\eta L\kappa^2}{\sigma_r}\|J_tW_{t,\perp}^\top\|\|W_{t,\perp}K_t^\top\|\cdot\|J_tW_{t,\perp}^\top\| \\
&\le \|K_tW_{t,\perp}^\top\|\cdot\left(1-\frac{\eta\alpha^2}{8}\right) + \frac{160\eta L\kappa^2}{\sigma_r}100\kappa^2\alpha^2\|K_tW_{t,\perp}^\top\| \\
&\le \|K_tW_{t,\perp}^\top\|\cdot\left(1-\frac{\eta\alpha^2}{16}\right) \qquad \text{(H.75)} \\
&\le \|K_tW_{t,\perp}^\top\|\cdot\left(1-\frac{\eta\|J_tW_{t,\perp}^\top\|}{1600\kappa^2}\right) \qquad \text{(H.76)}
\end{aligned}
$$

by choosing $\delta_{2k+1} \le \mathcal{O}(\kappa^{-5})$. Now if $\|J_tW_{t,\perp}^\top\| \ge 10\kappa\alpha$,

$$
\begin{aligned}
W_{t,\perp}J_t^\top J_tW_{t,\perp}^\top - W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top - W_{t,\perp}V_t^\top V_tW_{t,\perp}^\top &\le 2\alpha^2\cdot I \\
W_{t,\perp}J_t^\top J_tW_{t,\perp}^\top - W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top &\le 2\alpha^2\cdot I + W_{t,\perp}(U_t-V_t)^\top(U_t-V_t)W_{t,\perp}^\top
\end{aligned}
$$

Hence

If $\|J_tW_{t,\perp}^\top\| \ge 10\kappa\alpha$, then

$$
\begin{aligned}
\|J_tW_{t,\perp}\|^2 &= \|W_{t,\perp}J_t^\top J_tW_{t,\perp}^\top\| \\
&\le \|W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top\| + \left(2\alpha + \frac{8\alpha^2\sqrt{\sigma_1}+64\delta_{2k+1}\sqrt{\sigma_1}L_t}{\sigma_r}\right)^2 \\
&\le \|W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top\| + \left(2\alpha + \frac{8\alpha^2\sqrt{\sigma_1}+64\delta_{2k+1}\sqrt{\sigma_1}\|J_tW_{t,\perp}^\top\|\cdot\sqrt{\sigma_1}}{\sigma_r}\right)^2 \\
&\le \|W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top\| + (10\alpha + 64\delta_{2k+1}\kappa\|J_tW_{t,\perp}^\top\|)^2 \\
&\le \|W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top\| + (1/10\kappa + 64\delta_{2k+1}\kappa)\cdot\|J_tW_{t,\perp}^\top\|^2 \\
&\le \|W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top\| + (1/2)\cdot\|J_tW_{t,\perp}^\top\|^2.
\end{aligned}
$$

Thus $\|K_tW_{t,\perp}^\top\| \ge \|J_tW_{t,\perp}^\top\|/\sqrt{2} \ge \|J_tW_{t,\perp}^\top\|/2$.

$$
\begin{aligned}
\|K_{t+1}W_{t,\perp}^\top\| &\le \|K_tW_{t,\perp}^\top(I-\eta W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top-\eta\phi_t)\| + \frac{160\eta L\kappa^2}{\sigma_r}\|J_tW_{t,\perp}^\top W_{t,\perp}K_t^\top\|\cdot\|J_tW_{t,\perp}^\top\| \\
&\le \|K_tW_{t,\perp}^\top\|\|(I-\eta W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top-\eta\phi_t)\| + \frac{160\eta L\kappa^2}{\sigma_r}\|J_tW_{t,\perp}^\top W_{t,\perp}K_t^\top\|\cdot\|J_tW_{t,\perp}^\top\|
\end{aligned}
$$

Then if we denote $K' = K_tW_{t,\perp}^\top$, then we know $\|K'(1-\eta(K')^\top K')\| \le (1-\eta\frac{\sigma_1^2(K')}{2})\|K'\|$. Let $K' = A'\Sigma'W'$

$$
\begin{aligned}
\|K'(1-\eta(K')^\top K')\| &= \|A'\Sigma'W'(I-\eta(W')^\top(\Sigma')^2W')\| \\
&= \|\Sigma'(I-\eta(\Sigma')^2)\|
\end{aligned}
$$

Let $\Sigma'_{ii} = \zeta_i$ for $i \le r$, then $\Sigma'(I-\eta(\Sigma')^2)_{ii} = \zeta_i - \eta\zeta_i^3$, then by the fact that $\zeta_1 = \sigma_1(K_t W_{t,\perp}^\top) \le 1$, we can have $\zeta_1 - \eta\zeta_1^3 = \max_{1\le i\le r} \zeta_i - \eta\zeta_i^3$ and then

$$\|\Sigma(I-\eta\Sigma^2)\| = (1-\eta\|K'\|^2)\|K'\|.$$

Hence

$$\begin{aligned}
\|K_{t+1}W_{t,\perp}^\top\| &\le \|K_t W_{t,\perp}^\top (I - \eta W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top - \eta\phi_t)\| + \frac{160\eta L\kappa^2}{\sigma_r}\|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\| \\
&\le \|K_t W_{t,\perp}^\top (I - \eta W_{t,\perp} K_t^\top K_t W_{t,\perp}^\top)\| + \frac{160\eta L\kappa^2}{\sigma_r}\|J_t W_{t,\perp}^\top W_{t,\perp} K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\| \\
&\le \|K_t W_{t,\perp}^\top\| \left(1 - \eta\frac{\|K_t W_{t,\perp}^\top\|^2}{2}\right) + \frac{160\eta L\kappa^2}{\sigma_r}\|J_t W_{t,\perp}^\top\|\|W_{t,\perp} K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\| \\
&\le \|K_t W_{t,\perp}^\top\| \left(1 - \eta\frac{\|J_t W_{t,\perp}^\top\|^2}{8}\right) + \frac{160\eta L\kappa^2}{\sigma_r}\|J_t W_{t,\perp}^\top\|\|W_{t,\perp} K_t^\top\| \cdot \|J_t W_{t,\perp}^\top\| \\
&\le \|K_t W_{t,\perp}^\top\| \left(1 - \eta\frac{\|J_t W_{t,\perp}^\top\|^2}{16}\right) \qquad \text{(H.77)} \\
&\le \|K_t W_{t,\perp}^\top\| \left(1 - 4\eta\kappa^2\alpha^2\right). \qquad \text{(H.78)}
\end{aligned}$$

The fifth inequality is because $\delta_{2k+1} = O(\kappa^{-4})$. Thus for all cases, by Eq.(H.75), (H.76), (H.78) and (H.77), we have

$$\begin{aligned}
\|K_{t+1}W_{t,\perp}^\top\| &\le \|K_t W_{t,\perp}^\top\| \cdot \min\left\{\left(1 - \frac{\eta\alpha^2}{4}\right), \left(1 - \frac{\eta\|J_t W_{t,\perp}^\top\|^2}{1600\kappa^2}\right)\right\} \qquad \text{(H.79)} \\
&\le \|K_t W_{t,\perp}^\top\| \cdot \left(1 - \frac{\eta\alpha^2}{8}\right) \cdot \left(1 - \frac{\eta\|J_t W_{t,\perp}^\top\|^2}{3200\kappa^2}\right), \qquad \text{(H.80)}
\end{aligned}$$

where we use the inequality $\max\{a,b\} \le \sqrt{ab}$. Now we prove the following claim:

$$\|K_{t+1}W_{t+1,\perp}^\top\| \le \|K_{t+1}W_{t,\perp}^\top\| \cdot \left(1 + \mathcal{O}(\eta\delta_{2k+1}\|J_t W_{t,\perp}^\top\|^2/\sigma_r^{3/2})\right). \qquad \text{(H.81)}$$

First consider the situation that $\|J_t W_{t,\perp}^\top\| \le 10\kappa\alpha$. We start at these two equalities:

$$\begin{aligned}
K_{t+1} &= K_{t+1}W_{t,\perp}^\top W_{t,\perp} + K_{t+1}W_t^\top W_t \\
K_{t+1} &= K_{t+1}W_{t+1,\perp}^\top W_{t+1,\perp} + K_{t+1}W_{t+1}^\top W_{t+1}.
\end{aligned}$$

Thus we have

$$K_{t+1}W_{t,\perp}^\top W_{t,\perp} W_{t+1,\perp}^\top + K_{t+1}W_t^\top W_t W_{t+1,\perp}^\top = K_{t+1}W_{t+1,\perp}^\top$$

Consider

$$
\begin{aligned}
\|W_t W_{t+1,\perp}^\top\| &= \|W_{t+1,\perp} W_t^\top\| \\
&= \|W_{t+1,\perp} U_t^\top (U_t U_t^\top)^{-1/2}\| \\
&= \|W_{t+1,\perp} U_t^\top\| \|(U_t U_t^\top)^{-1/2}\| \\
&\le \|W_{t+1,\perp}\| \|U_{t+1} - U_t\| \cdot \sigma_r(U)^{-1} \\
&\le \frac{4\sqrt{\sigma_1}}{\sigma_r} \cdot \eta \cdot (2\sqrt{\sigma_1} \cdot M_t + 2\delta_{2k+1} \cdot (L_t + 3M_t) \cdot 2\sqrt{\sigma_1}) \\
&\le \frac{4\sqrt{\sigma_1}}{\sigma_r} \cdot \eta \cdot (3\sqrt{\sigma_1} \cdot M_t + 2\delta_{2k+1} \cdot L_t) \\
&\le \frac{4\sqrt{\sigma_1}}{\sigma_r} \cdot \eta (\frac{48L\kappa\sqrt{\sigma_1}}{\sigma_r} \|J_t K_t^\top\| + 2\sqrt{\sigma_1}\delta_{2k+1} \cdot L_t) \\
&\le C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_t K_t^\top\|.
\end{aligned}
$$

for some constant C. Also, note that $\|F_t G_t^\top - \Sigma\| \le L_t + 3M_t \le 4L_t$,

$$
\begin{aligned}
\|K_{t+1} W_t^\top\| &= \|(K_{t+1} - K_t) W_t^\top\| + \|K_t W_t^\top\| \\
&\le \|\eta K_t (U_t^\top U_t + J_t^\top J_t) W_t^\top\| + \eta\delta_{2k+1} \cdot (4L_t) \cdot 2\sqrt{\sigma_1} + \|K_t W_t^\top\| \\
&\le \|\eta K_t J_t^\top J_t W_t^\top\| + 8\sqrt{\sigma_1}\eta\delta_{2k+1} \cdot L_t + \frac{64L\kappa^{3/2}}{\sigma_r^{3/2}} L_t \\
&\le \eta L_t \|J_t W_t^\top\| + 8\sqrt{\sigma_1}\eta\delta_{2k+1} \cdot L_t + \frac{64L\kappa^{3/2}}{\sigma_r^{3/2}} L_t \\
&\le L_t \cdot (\eta \cdot \sqrt{\sigma_1} + 8\sqrt{\sigma_1}\eta\delta_{2k+1} + \frac{64L\kappa^{3/2}}{\sigma_r^{3/2}}) \\
&\le \frac{1}{4\sqrt{\sigma_1}} L_t \\
&\le \frac{1}{4}\|K_t\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|K_{t+1} W_{t,\perp}^\top\| &\ge \|K_t W_{t,\perp}^\top\| - \|(K_{t+1} - K_t) W_{t,\perp}^\top\| \\
&\ge \frac{1}{2}\|K_t\| - \eta\|K_t (U_t^\top U_t + J_t^\top J_t) W_t^\top\| - 8\sqrt{\sigma_1}\eta\delta_{2k+1} \cdot L_t \\
&\ge \frac{1}{2}\|K_t\| - \eta L_t \|J_t W_t^\top\| - 8\sqrt{\sigma_1}\eta\delta_{2k+1} \cdot L_t \\
&\ge \|K_t\|(\frac{1}{2} - \eta\|J_t\| \cdot \|J_t W_t^\top\| - 8\sqrt{\sigma_1}\eta\delta_{2k+1} \cdot \|J_t\|) \\
&\ge \|K_t\|(\frac{1}{2} - \eta\sigma_1 - 8\eta\delta_{2k+1}\sigma_1) \\
&\ge \frac{1}{4}\|K_t\| \\
&\ge \|K_{t+1} W_t^\top\|
\end{aligned}
$$

Here we use the fact that $\eta \le 1/\sigma_1$, $\delta_{2k+1} \le 1/32$ and $\|J_t\| \le \sqrt{\sigma_1}$. Hence we have

$$\begin{aligned}
\|K_{t+1}W_{t+1,\perp}^\top\| &\le \|K_{t+1}W_{t,\perp}^\top\|\|W_{t,\perp}W_{t+1,\perp}^\top\| + \|K_{t+1}W_t^\top\|\|W_tW_{t+1,\perp}^\top\| \\
&\le \|K_{t+1}W_{t,\perp}^\top\| + \|K_{t+1}W_{t,\perp}^\top\| \cdot C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)L_t \\
&\le \left(1 + C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)L_t\right)\|K_{t+1}W_{t,\perp}^\top\| \\
&\le \left(1 + 2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top W_{t,\perp}K_t^\top\|\right)\|K_{t+1}W_{t,\perp}^\top\| \\
&\le \left(1 + 2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top\|\|W_{t,\perp}K_t^\top\|\right)\|K_{t+1}W_{t,\perp}^\top\|
\end{aligned}$$

The inequality on the fourth line is because Eq.(H.73).

Note that

$$W_{t,\perp}J_t^\top J_tW_{t,\perp}^\top - W_{t,\perp}K_t^\top K_tW_{t,\perp}^\top \ge \frac{\alpha^2}{8} \cdot I.$$

Thus $\|K_tW_{t,\perp}^\top\| \le \|J_tW_{t,\perp}^\top\|$ and

$$\begin{aligned}
\|K_{t+1}W_{t+1,\perp}^\top\| &\le \left(1 + 2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top\|\|W_{t,\perp}K_t^\top\|\right)\|K_{t+1}W_{t,\perp}^\top\| \\
&\le \left(1 + 2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top\|^2\right)\|K_{t+1}W_{t,\perp}^\top\| \qquad \text{(H.82)}
\end{aligned}$$

By inequalities (H.80) and (H.82), we can get

$$\|K_{t+1}W_{t+1,\perp}^\top\| \qquad \text{(H.83)}$$

$$\le \left(1 + 2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top\|^2\right)\|K_{t+1}W_{t,\perp}^\top\| \qquad \text{(H.84)}$$

$$\le \left(1 + 2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top\|^2\right) \cdot \left(1 - \frac{\eta\alpha^2}{8}\right) \cdot \left(1 - \frac{\eta\|J_tW_{t,\perp}^\top\|^2}{3200\kappa^2}\right)\|K_tW_{t,\perp}^\top\| \qquad \text{(H.85)}$$

$$\le \left(1 - \frac{\eta\alpha^2}{8}\right)\|K_tW_{t,\perp}^\top\|. \qquad \text{(H.86)}$$

The last inequality is because

$$2C\eta(\delta_{2k+1}\kappa^4 + \alpha^2\kappa^2/\sigma_r)\|J_tW_{t,\perp}^\top\|^2 \le \frac{\eta\|J_tW_{t,\perp}^\top\|^2}{3200\kappa^2}$$

by choosing

$$\delta_{2k+1} = \mathcal{O}(\kappa^{-6}) \qquad \text{(H.87)}$$

and

$$\alpha = \mathcal{O}(\kappa^{-2} \cdot \sqrt{\sigma_r}). \qquad \text{(H.88)}$$

Thus we can prove $\|K_tW_{t,\perp}^\top\|$ decreases at a linear rate.

Now we have completed all the proofs of the induction hypotheses. Hence,

$$\begin{aligned}
\|F_t G_t^\top - \Sigma\| &\le 2\|J_t K_t^\top\| \\
&\le 4\|K_t^\top\| \cdot \sqrt{\sigma_1} \\
&\le 4\|K_t W_{t,\perp}^\top\| \sqrt{\sigma_1} \\
&\le 4\|K_t W_{T_2,\perp}^\top\| \cdot \sqrt{\sigma_1} \left(1 - \frac{\eta\alpha^2}{8}\right)^{t-T_2} \\
&\le 4\|K_{T_2}\| \cdot \sqrt{\sigma_1} \left(1 - \frac{\eta\alpha^2}{8}\right)^{t-T_2} \\
&\le 2\sigma_1 \left(1 - \frac{\eta\alpha^2}{8}\right)^{t-T_2}
\end{aligned} \tag{H.89}$$

Now combining three phases (H.35), (H.58) and (H.89), if we denote $t_2^* + T_0 = T' = \widetilde{\mathcal{O}}(1/\eta\sigma_r)$, then for any round $T \ge 4T'$, Phase 1 and Phase 3 will take totally at least $T - T'$ rounds. Now we consider two situations.

Situation 1: Phase 1 takes at least $\frac{3(T-T')}{4}$ rounds. Then by (H.35), suppose Phase 1 starts at T_0 rounds and terminates at T_1 rounds, we will have

$$\begin{aligned}
\|F_{T_1} G_{T_1}^\top - \Sigma\| &\le \frac{\sigma_r^2}{128\alpha^2\kappa} \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^{T_1 - T_0} \\
&\le \frac{\sigma_r^2}{128\alpha^2\kappa} \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^{T/2}.
\end{aligned} \tag{H.90}$$

The last inequality uses the fact that $T \ge 4T'$ and

$$T_1 - T_0 \ge \frac{3(T - T')}{4} \ge T/2$$

Then by (H.42), (H.40), (H.60) and (H.61), we know that

$$\begin{aligned}
\|F_T G_T^\top - \Sigma\| &\le 4\|J_T K_T^\top\| \\
&\le 4\|J_{T_1} K_{T_1}^\top - \Sigma\| \cdot \left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right)^{T - T_1} \\
&\le 4\|F_{T_1} G_{T_1}^\top - \Sigma\| \cdot \left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right)^{T - T_1} \\
&\le 4\|F_{T_1} G_{T_1}^\top - \Sigma\| \cdot \left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right)^{T/2}
\end{aligned} \tag{H.91}$$

The last inequality uses the fact that $T_1 - T_0 \geq \frac{3(T-T')}{4} \geq \frac{T}{2}$, which implies that $\frac{T}{2} \geq T - T_1$ Then combining with (H.90), we can get

$$\begin{aligned}
\|F_T G_T^\top - \Sigma\| &\leq \frac{\sigma_r^2}{128\alpha^2\kappa}\left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^{T/2} \cdot \left(1 + \frac{\eta\sigma_r^2}{128\sigma_1}\right)^{T/2} \\
&\leq \frac{\sigma_r^2}{128\alpha^2\kappa}\left(1 - \frac{\eta\sigma_r^2}{128\sigma_1}\right)^{T/2} \qquad \text{(H.92)} \\
&\leq \frac{\sigma_r^2}{128\alpha^2\kappa}\left(1 - \frac{\eta\alpha^2}{8}\right)^{T/2}. \qquad \text{(H.93)}
\end{aligned}$$

(H.92) uses the basic inequality $(1-2x)(1+x) \leq (1-x)$, and (H.93) uses the fact that $\alpha = \mathcal{O}(\kappa^{-2}\sqrt{\sigma_r}) = \mathcal{O}(\sqrt{\kappa\sigma_r})$.

Situation 2: Phase 3 takes at least $\frac{T-T'}{4}$ rounds. Then by (H.89), suppose Phase 3 starts at round T_2, we have

$$\begin{aligned}
\|F_T G_T^\top - \Sigma\| &\leq 2\sigma_1\left(1 - \frac{\eta\alpha^2}{8}\right)^{t-T_2} \\
&\leq 2\sigma_1\left(1 - \frac{\eta\alpha^2}{8}\right)^{(T-T')/4} \\
&\leq \frac{\sigma_r^2}{128\alpha^2\kappa}\left(1 - \frac{\eta\alpha^2}{8}\right)^{T/8}. \qquad \text{(H.94)}
\end{aligned}$$

The last inequality uses the fact that $\alpha = \mathcal{O}(\kappa^{-2}\sqrt{\sigma_r}) = \mathcal{O}(\kappa^{-1}\sqrt{\sigma_r})$ and $\frac{T-T'}{4} \geq \frac{T-T/4}{4} \geq T/8$. Thus by $\|F_T G_T^\top - \Sigma\|^2 \leq n \cdot \|F_T G_T^\top - \Sigma\|^2$, we complete the proof by choosing $4T' = T^{(1)}$ and $c_7 = 1/128^2$.

Appendix I. Proof of Theorem 9

By the convergence result in [29], the following three conditions hold for $t = T_0$.

$$\max\{\|J_t\|, \|K_t\|\} \leq \mathcal{O}\left(2\alpha + \frac{\delta_{2k+1}\sigma_1^{3/2}\log(\sqrt{\sigma_1}/n\alpha)}{\sigma_r}\right) \qquad \text{(I.1)}$$

$$\max\{\|U_t\|, \|V_t\|\} \leq 2\sqrt{\sigma_1} \qquad \text{(I.2)}$$

and

$$\|F_t G_t^T - \Sigma\| \leq \alpha^{1/2}\sigma_1^{3/4} \leq \sigma_r/2. \qquad \text{(I.3)}$$

Then we define $M_t = \max\{\|U_t V_t^\top - \Sigma\|, \|U_t K_t^\top\|, \|J_t V_t^\top\|\}$, by the same techniques in Section H.4, if we have

$$\sigma_r^2 M_{t-1}/64\sigma_1 \geq (17\sigma_1\delta_{2k+1} + \alpha^2)\|J_{t-1}K_{t-1}^\top\|, \qquad \text{(I.4)}$$

we can prove that

$$M_t \leq \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right) M_{t-1}. \tag{I.5}$$

and

$$\max\{\|J_t\|, \|K_t\|\} \leq 2\sqrt{\alpha}\sigma_1^{1/4} + 2C_2 \log(\sqrt{\sigma_1}/n\alpha)(\delta_{2k+1} \cdot \kappa^2\sqrt{\sigma_1}) \leq \sqrt{\sigma_1} \tag{I.6}$$

$$\|F_t G_t^\top - \Sigma\| \leq \sigma_r/2 \tag{I.7}$$

$$\max\{\|U_t\|, \|V_t\|\} \leq 2\sqrt{\sigma_1}. \tag{I.8}$$

Now note that

$$\|U_{t-1}K_{t-1}^\top\| \geq \lambda_{\min}(U_{t-1}) \cdot \|K_{t-1}^\top\| = \sigma_r(U_{t-1}) \cdot \|K_{t-1}^\top\| \geq \frac{\sigma_r}{4\sqrt{\sigma_1}} \cdot \|K_{t-1}\|, \tag{I.9}$$

Now since $\delta_{2k+1} = \mathcal{O}(\kappa^{-3})$ and $\alpha = \mathcal{O}(\kappa^{-1}\sqrt{\sigma_r})$ are small parameters, we can derive the M_t's lower bound by

$$\begin{aligned}
M_{t-1} &\geq \|U_{t-1}K_{t-1}^\top\| \\
&\geq \frac{\sigma_r}{4\sqrt{\sigma_1}} \cdot \|K_{t-1}\| \\
&\geq \frac{\sigma_r}{4\sqrt{\sigma_1}}\|K_{t-1}\| \cdot \frac{\|J_{t-1}\|}{\sqrt{\sigma_1}} && \text{(I.10)} \\
&\geq 64\sigma_1 \cdot \frac{17\sigma_1\delta_{2k+1} + \alpha^2}{\sigma_r^2}\|J_{t-1}K_{t-1}^\top\|. && \text{(I.11)}
\end{aligned}$$

Hence (I.4) always holds for $t \geq T_0$, and then by (I.5), we will have

$$\begin{aligned}
M_t &\leq \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right)^{t-T_0} M_{T_0} && \text{(I.12)} \\
&\leq \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right)^{t-T_0} \|F_{T_0}G_{T_0}^T\| && \text{(I.13)} \\
&\leq \frac{\sigma_r}{2} \cdot \left(1 - \frac{\eta\sigma_r^2}{16\sigma_1}\right)^{t-T_0}. && \text{(I.14)}
\end{aligned}$$

Thus we can bound the loss by

$$\begin{aligned}
\|F_t G_t^\top - \Sigma\| &\leq \|U_t V_t^\top - \Sigma\| + \|J_t V_t^\top\| + \|U_t K_t^\top\| + \|J_t K_t^\top\| \\
&\leq 3M_t + \|J_t K_t^\top\| \\
&\leq 3M_t + \mathcal{O}(2\alpha + \delta_{2k+1}\kappa\sqrt{\sigma_1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \frac{4\sqrt{\sigma_1}}{\sigma_r} M_t \\
&\leq 4M_t && \text{(I.15)} \\
&\leq 2\sigma_r \cdot \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^{t-T_0}.
\end{aligned}$$

where Eq.(I.15) uses the fact that $\delta_{2k+1} \leq \mathcal{O}(\kappa^{-2}\log^{-1}(\sqrt{\sigma_1}/n\alpha))$ and $\alpha \leq \mathcal{O}(\sigma_r/\sqrt{\sigma_1})$. Now we can choose $T^{(2)} = 2T_0$, and then by $t - T_0 \geq t/2$ for all $t \geq T^{(2)}$, we have

$$\|F_tG_t^T - \Sigma\|_F^2 \leq n\|F_tG_t^T - \Sigma\|^2 \leq 2n\sigma_r \cdot \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^{t-T_0} \leq 2n\sigma_r \cdot \left(1 - \frac{\eta\sigma_r^2}{64\sigma_1}\right)^{t/2}. \quad \text{(I.16)}$$

We complete the proof.

Appendix J. Proof of Theorem 10

During the proof of Theorem 10, we assume β satisfy that

$$\max\{c_7\gamma^{1/6}\sigma_1^{1/3}, c\delta_{2k+1}^{1/6}\kappa^{1/6}\sigma_1^{5/12}\} \leq \beta \leq c_8\sqrt{\sigma_r} \quad \text{(J.1)}$$

for some large constants c_7, c and small constant c_8. In particular, this requirement means that $\gamma \leq \sigma_r/4$. Then, since $\|\mathcal{A}^*\mathcal{A}(\tilde{F}_{T^{(3)}}\tilde{G}_{T^{(3)}}^\top - \Sigma)\| \geq \frac{1}{2}\|\tilde{F}_{T^{(3)}}\tilde{G}_{T^{(3)}}^\top - \Sigma\|$ by RIP property and $\delta_{2k+1} \leq 1/2$, we can further derive $\|F_{T^{(3)}}G_{T^{(3)}}^\top - \Sigma\| = \|\tilde{F}_{T^{(3)}}\tilde{G}_{T^{(3)}}^\top - \Sigma\| \leq \sigma_r/2$.

To guarantee (J.1), we can use choose γ to be small enough, i.e., $\gamma \ll \sigma_1\kappa^{-2}$, so that (J.1) holds easily. In the following, we denote $\delta_{2k+1} = \sqrt{2k+1}\delta$.

J.1. Proof Sketch of Theorem 10

First, suppose we modify the matrix $\widetilde{F}_{T^{(3)}}, \widetilde{G}_{T^{(3)}}$ to $F_{T^{(3)}}$ and $G_{T^{(3)}}$ at $t = T^{(3)}$, then $\|F_{T^{(3)}}\|^2 = \lambda_{\max}((F_{T^{(3)}})^\top F_{T^{(3)}}) = \beta^2$ and $\|U_{T^{(3)}}\|^2 \leq \beta^2$. Also, by $\|\widetilde{F}_{T^{(3)}}\| \leq 2\sqrt{\sigma_1}$, we can get that $\|G_{T^{(3)}}\| \leq \|\widetilde{G}_{T^{(3)}}\| \cdot \frac{\|\widetilde{F}_{T^{(3)}}\|}{\beta} \leq \|\widetilde{G}_{T^{(3)}}\| \cdot \frac{2\sqrt{\sigma_1}}{\beta}$ is still bounded. Similarly, $\|V_{T^{(3)}}\| \leq \|\widetilde{V}_{T^{(3)}}\| \cdot \frac{2\sqrt{\sigma_1}}{\beta}$ and $\|K_{T^{(3)}}\| \leq \|\widetilde{K}_{T^{(3)}}\| \cdot \frac{2\sqrt{\sigma_1}}{\beta}$ is still bounded. With these conditions, define $S_t = \max\{\|U_tK_t^\top\|, \|J_tK_t^\top\|\}$ and $P_t = \max\{\|J_tV_t^\top\|, \|U_tV_t^\top - \Sigma\|\}$. For $\|K_{t+1}\|$, since we can prove $\lambda_{\min}(F_t^\top F_t) \geq \beta^2/2$ for all $t \geq T^{(3)}$ using induction, with the updating rule, we can bound $\|K_{t+1}|$ as the following

$$\|K_{t+1}\| \leq \|K_t\|\|1 - \eta F_t^\top F_t\| + 2\eta\delta_{2k+1} \cdot \|F_tG_t^\top - \Sigma\| \max\{\|U_t\|, \|J_t\|\} \quad \text{(J.2)}$$

$$\leq \|K_t\| \cdot \left(1 - \frac{\eta\beta^2}{2}\right) + \left(4\eta\delta_{2k+1}\beta \cdot P_t + 4\beta^2\eta\delta_{2k+1}\|K_t\|\right). \quad \text{(J.3)}$$

The first term of (J.3) ensures the linear convergence, and the second term represents the perturbation term. To control the perturbation term, for P_t, with more calculation (see details in the rest of the section), we have

$$P_{t+1} \leq \left(1 - \eta\sigma_r^2/8\beta^2\right)P_t + \eta\|K_t\| \cdot \widetilde{\mathcal{O}}\left(\left(\delta_{2k+1}\sigma_1 + \sqrt{\alpha\sigma_1^{7/4}}\right)/\beta\right). \quad \text{(J.4)}$$

The last inequality uses the fact that $S_t \leq \|K_t\| \cdot \max\{\|U_t\|, \|J_t\|\} \leq \|K_t\| \cdot \|F_t\| \leq \sqrt{2}\beta \cdot \|K_t\|$.

Combining (J.4) and (J.3), we can show that $P_t + \sqrt{\sigma_1}\|K_t\|$ converges at a linear rate $(1 - \mathcal{O}(\eta\beta^2))$, since the second term of Eq. (J.4) and Eq.(J.3) contain δ_{2k+1} or α, which is relatively small and can be canceled by the first term. Hence $\|F_tG_t^\top - \Sigma\| \leq 2P_t + 2S_t \leq 2P_t + \sqrt{2}\beta\|K_t\|$ converges at a linear rate.

J.2. Proof of Theorem 10

At time $t \geq T^{(3)}$, we have $\sigma_{\min}(U_{T^{(3)}}V_{T^{(3)}}) \geq \sigma_{\min}(\Sigma) - \|U_{T^{(3)}}V_{T^{(3)}}^\top - \Sigma\| \geq \sigma_r - \alpha^{1/2} \cdot \sigma_1^{3/4} \geq \sigma_r/2$. The last inequality holds because $\alpha = O(\kappa^{-3/2} \cdot \sqrt{\sigma_r})$. Then given that $\|F_{T^{(3)}}\|^2 = \lambda_{\max}((F_{T^{(3)}})^\top F_{T^{(3)}}) = \beta^2$, we have $\|U_{T^{(3)}}\|^2 \leq \beta^2$. Hence by $\sigma_1(U) \cdot \sigma_r(V) \geq \sigma_r(UV^\top)$, we have

$$\sigma_r(V_{T^{(3)}}) \geq \frac{\sigma_r(U_{T^{(3)}}V_{T^{(3)}})}{\sigma_1(U_{T^{(3)}})} \geq \frac{\sigma_r}{2\beta}.$$

Also, by $\sigma_1' = \|\widetilde{F}_{T^{(3)}}\| \leq 2\sqrt{\sigma_1}$, we can get

$$\|G_{T^{(3)}}\| \leq \|\widetilde{G}_{T^{(3)}}\|\|B\Sigma_{inv}^{-1}\| \leq \|\widetilde{G}_{T^{(3)}}\| \cdot \frac{\sigma_1'}{\beta} \leq \|\widetilde{G}_{T^{(3)}}\| \cdot \frac{2\sqrt{\sigma_1}}{\beta}.$$

Similarly, $\|V_{T^{(3)}}\| \leq \|\widetilde{V}_{T^{(3)}}\| \cdot \frac{2\sqrt{\sigma_1}}{\beta}$ and $\|K_{T^{(3)}}\| \leq \|\widetilde{K}_{T^{(3)}}\| \cdot \frac{2\sqrt{\sigma_1}}{\beta}$.

Denote $S_t = \max\{\|U_tK_t^\top\|, \|J_tK_t^\top\|\}, P_t = \max\{\|J_tV_t^\top\|, \|U_tV_t^\top - \Sigma\|\}$. Now we prove the following statements by induction:

$$P_{t+1} \leq \left(1 - \frac{\eta\sigma_r^2}{8\beta^2}\right) P_t + \eta S_t \cdot \mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^2 + \sqrt{\alpha}\sigma_1^{7/4}}{\beta^2}\right) \tag{J.5}$$

$$\|F_{t+1}G_{t+1}^\top - \Sigma\| \leq \frac{\beta^6}{\sigma_1^2}\left(1 - \frac{\eta\beta^2}{2}\right)^{t+1-T^{(3)}} \leq \sigma_r/2 \tag{J.6}$$

$$\max\{\|F_{t+1}\|, \|G_{t+1}\|\} \leq 4\sigma_1/\beta \tag{J.7}$$

$$\frac{\beta^2}{2}I \leq F_{t+1}^\top F_{t+1} \leq 2\beta^2 I \tag{J.8}$$

$$\|K_t\| \leq \mathcal{O}(2\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \kappa^2\sqrt{\sigma_1}) \cdot \frac{2\sqrt{\sigma_1}}{\beta} \tag{J.9}$$

Proof of Eq.(J.5) First, since $\|F_t\|^2 = \lambda_{\max}((F_t)^\top F_t) \leq 2\beta^2$, we have $\|U_t\|^2 \leq 2\beta^2$. Then because $\sigma_{\min}(U_tV_t) \geq \sigma_{\min}(\Sigma) - \|U_tV_t^\top - \Sigma\| \geq \sigma_r/2$, by $\sigma_1(U) \cdot \sigma_r(V) \geq \sigma_r(UV^\top)$, we have

$$\sigma_r(V_t) \geq \frac{\sigma_r(U_tV_t)}{\sigma_1(U_t)} \geq \frac{\sigma_r}{2\beta}.$$

we write down the updating rule as

$$\begin{aligned}
&U_{t+1}V_{t+1}^\top - \Sigma \\
&= (1 - \eta U_tU_t^\top)(U_tV_t^\top - \Sigma)(1 - \eta V_tV_t^\top) - \eta U_tK_t^\top K_tV_t^\top - \eta U_tJ_t^\top J_tV_t^\top + B_t
\end{aligned}$$

where B_t contains the $\mathcal{O}(\eta^2)$ terms and $\mathcal{O}(E_i(F_tG_t^\top - \Sigma))$ terms

$$\|B_t\| \leq 4\eta\delta_{2k+1}(F_tG_t^\top - \Sigma)\max\{\|F_t\|^2, \|G_t\|^2\} + \mathcal{O}(\eta^2\|F_tG_t^\top - \Sigma\|^2\max\{\|F_t\|^2, \|G_t\|^2\})$$

Hence we have

$$\|U_{t+1}V_{t+1}^\top - \Sigma\| \tag{J.10}$$

$$\leq (1 - \frac{\eta\sigma_r^2}{4\beta^2})\|U_tV_t^\top - \Sigma\| + \eta\|U_tK_t^\top\|\|K_tV_t^\top\| + \eta\|J_tV_t^\top\|\|J_t^\top U_t^\top\| + \|B_t\| \tag{J.11}$$

$$\leq (1 - \frac{\eta\sigma_r^2}{4\beta^2})P_t + \eta S_t\|K_t\|\|V_t\| + \eta P_t\|J_t\|\|U_t\| + \|B_t\| \tag{J.12}$$

$$\leq (1 - \frac{\eta\sigma_r^2}{4\beta^2})P_t + \eta S_t \cdot \frac{4\sigma_1}{\beta^2} \cdot \mathcal{O}\left(2\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \kappa^2\sqrt{\sigma_1}\right) \cdot 2\sqrt{\sigma_1} + \eta P_t\beta \cdot \beta \tag{J.13}$$

$$+ 4\eta\delta_{2k+1} \cdot 2(P_t + S_t) \cdot 4\sigma_1 \cdot \frac{4\sigma_1}{\beta^2} + \mathcal{O}(\eta^2(P_t + S_t)^2 \cdot 4\sigma_1 \cdot \frac{4\sigma_1}{\beta^2}) \tag{J.14}$$

$$\leq \left(1 - \frac{\eta\sigma_r^2}{8\beta^2}\right)P_t + \eta S_t \cdot \mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^2 + \sqrt{\alpha}\sigma_1^{7/4}}{\beta^2}\right) \tag{J.15}$$

The last inequality uses the fact that

$$\beta^2 = \mathcal{O}(\sigma_r^{1/2}) \tag{J.16}$$

$$\delta_{2k+1} = \mathcal{O}(\kappa^{-2}) \tag{J.17}$$

$$P_t + S_t \leq 2\|F_tG_t^\top - \Sigma\| \leq \mathcal{O}(\sigma_1^2/\beta^2) \leq 1/\eta. \tag{J.18}$$

Similarly, we have

$$\begin{aligned}
&\|J_{t+1}V_{t+1}^\top\| \\
&\leq \left(1 - \eta J^\top J\right) JV^\top(1 - \eta V^\top V) - \eta JK^\top KV^\top - \eta JU^\top(UV^\top - \Sigma) + C_t
\end{aligned}$$

where C_t satisfies that

$$\begin{aligned}
\|C_t\| &\leq 4\eta\delta_{2k+1}(F_tG_t^\top - \Sigma)\max\{\|F_t\|^2, \|G_t\|^2\} + \mathcal{O}(\eta^2\|F_tG_t^\top - \Sigma\|\max\{\|F_t\|^2, \|G_t\|^2\}) \\
&\leq 4\eta\delta_{2k+1} \cdot 2(P_t + S_t) \cdot \frac{16\sigma_1^2}{\beta^2} + \mathcal{O}(\eta^2(P_t + S_t) \cdot \sigma_1 \cdot \frac{\sigma_1}{\beta^2}).
\end{aligned}$$

Thus similar to Eq.(J.15), we have

$$\|J_{t+1}V_{t+1}^\top\| \leq \left(1 - \frac{\eta\sigma_r^2}{8\beta^2}\right)P_t + \eta S_t \cdot \mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^2 + \sqrt{\alpha}\sigma_1^{7/4}}{\beta^2}\right).$$

Hence we have

$$P_{t+1} \leq \left(1 - \frac{\eta\sigma_r^2}{8\beta^2}\right)P_t + \eta S_t \cdot \mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^2 + \sqrt{\alpha}\sigma_1^{7/4}}{\beta^2}\right).$$

Proof of Eq.(J.6) We have $S_t \le \|K_t\| \cdot \max\{\|U_t\|, \|J_t\|\} \le \|K_t\| \cdot \|F_t\| \le \sqrt{2}\beta \cdot \|K_t\|$. So the inequality above can be rewritten as

$$P_{t+1} \le \left(1 - \frac{\eta\sigma_r^2}{8\beta^2}\right) P_t + \eta\sqrt{2}\beta \cdot \|K_t\| \cdot \mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^2 + \sqrt{\alpha}\sigma_1^{7/4}}{\beta^2}\right) \tag{J.19}$$

$$= \left(1 - \frac{\eta\sigma_r^2}{8\beta^2}\right) P_t + \eta\|K_t\| \cdot \mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^2 + \sqrt{\alpha}\sigma_1^{7/4}}{\beta}\right) \tag{J.20}$$

Also, for K_{t+1}, we have

$$\|K_{t+1}\| = \|K_t\|\|(1 - \eta F_t^\top F_t)\| + 2\delta_{2k+1} \cdot \|F_t G_t^\top - \Sigma\| \max\{\|U_t\|, \|J_t\|\} \tag{J.21}$$

$$\le \|K_t\|(1 - \frac{\eta\beta^2}{2}) + 2\eta\delta_{2k+1} \cdot (P_t + S_t) \cdot \sqrt{2}\beta \tag{J.22}$$

$$\le \|K_t\|(1 - \frac{\eta\beta^2}{2}) + 2\eta\delta_{2k+1} \cdot P_t \cdot \sqrt{2}\beta + 2\eta\delta_{2k+1} \cdot \sqrt{2}\beta\|K_t\| \cdot \sqrt{2}\beta \tag{J.23}$$

$$= \|K_t\|(1 - \frac{\eta\beta^2}{2}) + 4\eta\delta_{2k+1} \cdot \beta P_t + 4\beta^2\eta\delta_{2k+1} \cdot \|K_t\| \tag{J.24}$$

Thus we can get

$$\begin{aligned}
&P_{t+1} + \sqrt{\sigma_1}\|K_{t+1}\| \\
&\le \max\{1 - \frac{\eta\sigma_r^2}{8\beta^2}, 1 - \frac{\eta\beta^2}{2}\}(P_t + \|K_t\|) \\
&\quad + \eta \max\left\{\mathcal{O}\left(\frac{\log(\sqrt{\sigma_1}/n\alpha)\delta_{2k+1}\kappa^2\sigma_1^{3/2} + \sqrt{\alpha}\sigma_1^{5/4}}{c}\right) + 4\beta^2\delta_{2k+1}, 4\beta\sqrt{\sigma_1}\delta_{2k+1}\right\} \\
&\qquad \cdot (P_t + \sqrt{\sigma_1}\|K_t\|) \\
&\le (1 - \frac{\eta\beta^2}{4})(P_t + \sqrt{\sigma_1}\|K_t\|).
\end{aligned}$$

The last inequality uses the fact that $\beta \le \mathcal{O}(\sigma_r^{1/2})$ and

$$\delta_{2k+1} \le \mathcal{O}(\beta/\sqrt{\sigma_1}\log(\sqrt{\sigma_1}/n\alpha)). \tag{J.25}$$

Hence

$$\|K_t\| \le (P_{T^{(3)}}/\sqrt{\sigma_1} + \|K_{T^{(3)}}\|) \cdot \left(1 - \frac{\eta\beta^2}{2}\right)^{t - T^{(3)}} \tag{J.26}$$

$$\le \|K_{T^{(3)}}\| + \|F_t G_t^\top - \Sigma\|/\sqrt{\sigma_1} \tag{J.27}$$

$$\le \mathcal{O}(\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \kappa^2\sqrt{\sigma_1}) + \alpha^{1/2} \cdot \sigma_1^{1/4} \tag{J.28}$$

$$= \mathcal{O}(\sqrt{\alpha}\sigma_1^{1/4} + \delta_{2k+1}\log(\sqrt{\sigma_1}/n\alpha) \cdot \kappa^2\sqrt{\sigma_1}) \tag{J.29}$$

Hence $P_t + \sqrt{\sigma_1}\|K_t\|$ is linear convergence. Hence by $\beta \le \sqrt{\sigma_1}$,

$$\begin{aligned}
\|F_{t+1}G_{t+1}^\top - \Sigma\| &\le 2P_{t+1} + 2S_{t+1} && \text{(J.30)}\\
&\le 2P_{t+1} + \sqrt{2}\beta\|K_{t+1}\| && \text{(J.31)}\\
&\le (2+\sqrt{2}\beta/\sqrt{\sigma_1})(P_{t+1} + \sqrt{\sigma_1}\|K_{t+1}\|) && \text{(J.32)}\\
&\le 4(P_{T^{(3)}} + \sqrt{\sigma_1}\|K_{T^{(3)}}\|)\cdot\left(1-\frac{\eta\beta^2}{2}\right)^{t+1-T^{(3)}} && \text{(J.33)}
\end{aligned}$$

Last, note that by $\beta \ge c_7(\gamma^{1/6}\sigma_1^{1/3})$ and $\beta \ge c\delta_{2k+1}^{1/6}\kappa^{1/6}\sigma_1^{5/12}\log(\sqrt{\sigma_1}/n\alpha)^{1/6}$, by choosing for some constants c_7 and c, by choosing large c' and $c_7 = 2^6$, we can get

$$\gamma \le \frac{\beta^6}{2\sigma_1^2}, \quad \sqrt{\sigma_1}\cdot\mathcal{O}(\log(\sqrt{\sigma_1}/n\sqrt{\alpha})\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r)\cdot(2\sqrt{\sigma_1}/\beta) \le \frac{\beta^6}{2\sigma_1^2}$$

and

$$P_{T^{(3)}} + \sqrt{\sigma_1}\|K_{T^{(3)}}\| \le \gamma + \sqrt{\sigma_1}\cdot\mathcal{O}(\log(\sqrt{\sigma_1}/n\sqrt{\alpha})\delta_{2k+1}\cdot\sigma_1^{3/2}/\sigma_r)\cdot(2\sqrt{\sigma_1}/\beta) \le \beta^6/\sigma_1^2$$

we have

$$\|F_{t+1}G_{t+1}^\top - \Sigma\| \le \left(\frac{\beta^6}{\sigma_1^2}\right)\left(1-\frac{\eta\beta^2}{2}\right)^{t+1-T^{(3)}} \tag{J.34}$$

Proof of Eq.(J.7) Note that we have $\max\{\|F_{T^{(3)}}\|, \|G_{T^{(3)}}\|\} \le 4\sqrt{\sigma_1}\cdot\sqrt{\sigma_1}/\beta = 4\sigma_1/\beta$. Now suppose $\max\{\|F_{t'}\|, \|G_{t'}\|\} \le 4\sqrt{\sigma_1}\cdot\sqrt{\sigma_1}/\beta = 4\sigma_1/\beta$ for all $t' \in [T^{(3)}, t]$, then the changement of F_{t+1} and G_{t+1} can be bounded by

$$\|F_{t+1} - F_{T^{(3)}}\| \le \eta\sum_{t'=T^{(3)}}^{t} 2\|F_{t'}G_{t'} - \Sigma\|\|G_{t'}\| \le \eta\cdot 2\cdot\left(\frac{\beta^6}{\sigma_1^2} + \frac{\sigma_r}{2}\right)\cdot\frac{2}{\eta\beta^2}\frac{4\sigma_1}{\beta} \le \frac{16\beta^3}{\sigma_1} + \frac{8\sigma_1^2}{\beta^3}$$

$$\|G_t - G_{T^{(3)}}\| \le \eta\sum_{t'=T^{(3)}}^{t-1} 2\|F_{t'}G_{t'} - \Sigma\|\|F_{t'}\| \le \frac{16\beta^3}{\sigma_1} + \frac{8\sigma_1^2}{\beta^3}$$

Then by the fact that $\beta \le \mathcal{O}(\sigma_1^{-1/2})$, we can show that

$$\|F_{t+1}\| \le \|F_{T^{(3)}}\| + \|F_{t+1} - F_{T^{(3)}}\| \le \frac{2\sigma_1}{\beta} + \frac{16\beta^3}{\sigma_1} + \frac{8\sigma_1^2}{\beta^3} \le \frac{4\sigma_1}{\beta},$$

$$\|G_{t+1}\| \le \|G_{T^{(3)}}\| + \|G_{t+1} - G_{T^{(3)}}\| \le \frac{2\sigma_1}{\beta} + \frac{16c^3}{\sigma_1} + \frac{8\sigma_1^2}{\beta^3} \le \frac{4\sigma_1}{\beta}.$$

Proof of Eq.(J.8) Moreover, we have

$$\begin{aligned}
\sigma_k(F_{t+1}) &\ge \sigma_k(F_{T^{(3)}}) - \sigma_{\max}(F_{t+1} - F_{T^{(3)}})\\
&= \sigma_k(F_{T^{(3)}}) - \|F_{t+1} - F_{T^{(3)}}\|\\
&\ge \beta - \frac{16\beta^3}{\sigma_1}\\
&\ge \beta/\sqrt{2},
\end{aligned}$$

and

$$\|F_t\| \leq \|F_{T^{(3)}}\| + \|F_t - F_{T^{(3)}}\| \leq \beta + \frac{16\beta^3}{\sigma_1} \leq \sqrt{2}\beta.$$

The last inequality is because $\beta \leq \mathcal{O}(\sigma_1^{-1/2})$. Hence since $F_{t+1} \in \mathbb{R}^{n\times k}$, we have

$$\frac{\beta^2}{2} I \leq F_{t+1}^\top F_{t+1} \leq 2\beta^2 I \tag{J.35}$$

Thus, we complete the proof.

Appendix K. Technical Lemma

K.1. Proof of Lemma 11

Proof We only need to prove with high probability,

$$\max_{i,j\in[n]} \cos^2 \theta_{x_j,x_k} \leq \frac{c}{\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa)^2}. \tag{K.1}$$

In fact, since $\cos^2 \theta_{x_j,x_k} = \sin^2(\frac{\pi}{2} - \theta_{x_j,x_k}) \leq (\pi/2 - \theta_{x_j,x_k})^2$, we have

$$\mathbb{P}\left[|\pi/2 - \theta_{x_j,x_k}| > \mathcal{O}\left(\frac{\sqrt{c}}{\log(r\sqrt{\sigma_1}/\alpha)r\kappa}\right)\right] \geq \mathbb{P}\left[\cos^2 \theta_{x_j,x_k} > \mathcal{O}\left(\frac{c}{\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa)^2}\right)\right]. \tag{K.2}$$

Moreover, for any $m > 0$, by Lemma 16,

$$\mathbb{P}\left[|\pi/2 - \theta_{x_j,x_k}| > m\right] \leq \mathcal{O}\left(\frac{(\sin(\frac{\pi}{2} - m))^{k-2}}{1/\sqrt{k-2}}\right) = \mathcal{O}\left(\sqrt{k-2}(\cos m)^{k-2}\right) \tag{K.3}$$

$$\leq \mathcal{O}\left(\sqrt{k}(1 - m^2/4)^{k-2}\right) \tag{K.4}$$

$$\leq \mathcal{O}\left(\sqrt{k}\exp\left(-\frac{4k}{m^2}\right)\right). \tag{K.5}$$

The second inequality uses the fact that $\cos x \leq 1 - x^2/4$. Then if we choose

$$m = \frac{\sqrt{c}}{\log(r\sqrt{\sigma_1}/\alpha)r\kappa}$$

and let $k \geq 16/m^4 = \frac{16\log^4(r\sqrt{\sigma_1}/\alpha)(r\kappa)^4}{c^2}$, we can have

$$\mathbb{P}\left[\cos^2 \theta_{x_j,x_k} > m^2\right] \leq \mathbb{P}\left[|\pi/2 - \theta_{x_j,x_k}| > m\right] \tag{K.6}$$

$$\leq \mathcal{O}\left(k\exp\left(-\frac{m^2 k}{4}\right)\right) \tag{K.7}$$

$$\leq \mathcal{O}\left(k\exp\left(-\sqrt{k}\right)\right) \tag{K.8}$$

Thus by taking the union bound over $j, k \in [n]$, there is a constant c_2 such that, with probability at least $1 - c_4 n^2 k \exp(-\sqrt{k})$, we have

$$\theta_0 \leq \frac{c}{\log^2(r\sqrt{\sigma_1}/\alpha)(r\kappa)^2}. \tag{K.9}$$

■

K.2. Proof of Lemma 12

Proof Since $x_i = \alpha/\sqrt{k} \cdot \tilde{x}_i$, where each element in $\tilde{x}_i$ is sampled from $\mathcal{N}(0,1)$. By Theorem 3.1 in work [33], there is a constant c such that

$$\mathbb{P}\left[|\|\tilde{x}_i^0\|_2^2 - k| \geq t\right] \leq 2\exp(-ct) \tag{K.10}$$

Hence choosing $t = (1 - \frac{1}{\sqrt{2}})k$, we have

$$\mathbb{P}[\|\tilde{x}_i^0\|_2^2 \in [k/\sqrt{2}, \sqrt{2}k]] \leq \mathbb{P}[|\|\tilde{x}_i^0\|_2^2 - k| \geq t] \leq 2\exp(-ct) \leq 2\exp(-ck/4)$$

Hence

$$\mathbb{P}\Big[\|x_i^0\|^2 \in [\alpha^2/2, 2\alpha^2]\Big] = \mathbb{P}\Big[\|\tilde{x}_i^0\|^2 \in [k/\sqrt{2}, \sqrt{2}k]\Big] \leq 2\exp(-ck/4). \tag{K.11}$$

By taking the union bound over $i \in [n]$, we complete the proof. ■

Lemma 16 *Assume x, y are two random vectors in $\mathbb{R}^n$, then define θ as the angle between x, y, we have*

$$\mathbb{P}\Big(\Big|\theta - \frac{\pi}{2}\Big| \leq m\Big) \leq \frac{3\pi\sqrt{n-2}(\sin(\pi/2 - m))^{n-2}}{4\sqrt{2}}. \tag{K.12}$$

Proof Note that $\theta \in [0, \pi]$, and the CDF of θ is

$$f(\theta) = \frac{\Gamma(n/2)\sin^{n-2}(\theta)}{\sqrt{\pi}\Gamma(\frac{n-1}{2})} \tag{K.13}$$

Then we have

$$\mathbb{P}\Big(\Big|\theta - \frac{\pi}{2}\Big| > m\Big) = 1 - \frac{\int_{\pi/2-m}^{\pi/2+m} \sin^{n-2}\theta d\theta}{\int_0^{\pi} \sin^{n-2}\theta d\theta} = \frac{\int_0^{\pi/2-m} \sin^{n-2}\theta d\theta}{\int_0^{\pi/2} \sin^{n-2}\theta d\theta} \tag{K.14}$$

$$\leq \frac{(\pi/2) \cdot \sin^{n-2}(\pi/2 - m)}{\int_0^{\pi/2} \cos^{n-2}\theta d\theta} \tag{K.15}$$

$$\leq \frac{(\pi/2 \cdot (\pi/2 - m)^{n-2})}{\int_0^{\sqrt{2}} (1 - t^2/2)^{n-2} dt} \tag{K.16}$$

$$\leq \frac{(\pi/2) \cdot (\pi/2 - m)^{n-2}}{\frac{2\sqrt{2}}{3\sqrt{n-2}}} \tag{K.17}$$

$$= \frac{3\pi\sqrt{n-2}(\sin(\pi/2 - m))^{n-2}}{4\sqrt{2}}. \tag{K.18}$$

■

Lemma 17 (Lemma 7.3 (1) in [30]) *Let $\mathcal{A}$ is a linear measurement operator that satisfies the RIP property of order $2k+1$ with constant δ, then we have for all matrices with rank no mora than $2k$*

$$\|(I - \mathcal{A}^*\mathcal{A})(X)\| \leq \sqrt{2k} \cdot \delta \|X\|. \tag{K.19}$$

Lemma 18 ([29]) *There exist parameters ζ_0, $\delta_0, \alpha_0, \eta_0$ such that, if we choose $\alpha \leq \alpha_0$, $F_0 = \alpha \cdot \tilde{F}_0, G_0 = (\alpha/3) \cdot \tilde{G}_0$, where the elements of $\tilde{F}_0, \tilde{G}_0$ is $\mathcal{N}(0, 1/n)$,[6] and suppose that the operator $\mathcal{A}$ defined in Eq.(1.1) satisfies the restricted isometry property of order $2r+1$ with constant $\delta \leq \delta_0$, then the gradient descent with step size $\eta \leq \eta_0$ will achieve*

$$\|F_t G_t^\top - \Sigma\| \leq \alpha^{3/5} \cdot \sigma_1^{7/10} \tag{K.20}$$

within $T = \widetilde{\mathcal{O}}(1/\eta\sigma_r)$ rounds with probability at least $1 - \zeta_0$, where $\zeta_0 = c_1 \exp(-c_2 k) + (c_3 \upsilon)^{k-r+1}$ is a small constant. Moreover, during T rounds, we always have

$$\max\{\|F_t\|, \|G_t\|\} \leq 2\sqrt{\sigma_1}. \tag{K.21}$$

The parameters α_0, δ_0 and η_0 are selected by

$$\alpha_0 = \mathcal{O}\left(\frac{\sqrt{\sigma_1}}{k^5 \max\{2n, k\}^2}\right) \cdot \left(\frac{\sqrt{k} - \sqrt{r-1}}{\kappa^2 \sqrt{\max\{2n, k\}}}\right)^{C\kappa} \tag{K.22}$$

$$\delta_0 \leq \mathcal{O}\left(\frac{1}{\kappa^3 \sqrt{r}}\right) \tag{K.23}$$

$$\eta \leq \mathcal{O}\left(\frac{1}{k^5 \sigma_1} \cdot \frac{1}{\log\left(\frac{2\sqrt{2\sigma_1}}{\upsilon\alpha(\sqrt{k}-\sqrt{r-1}}\right)}\right) \tag{K.24}$$

Appendix L. Experiment Details

In this section, we provide experimental results to corroborate our theoretical observations.

Symmetric Lower Bound In the first experiment, we choose $n = 50, r = 2$, three different $k = 5, 3, 2$ and learning rate $\eta = 0.01$ for the symmetric matrix factorization problem. The results are shown in Figure 1, which matches our $\Omega(1/T^2)$ lower bound result in Theorem 6 for the over-parameterized setting, and previous linear convergence results for exact-parameterized setting.

Asymmetric Matrix Sensing In the second experiment, we choose configuration $n = 50, k = 4, r = 2$, sample number $m = 700 \approx nk^2$ and learning rate $\eta = 0.2$ for the asymmetric matrix sensing problem. To demonstrate the direct relationship between convergence speed and initialization scale, we conducted multiple trials employing distinct initialization scales $\alpha = 0.5, 0.2, 0.05$. The experimental results in Figure 4 offer compelling evidence supporting three key findings:

- The loss exhibits a linear convergence pattern.
- A larger value of α results in faster convergence under the over-parameterization setting
- The convergence rate is not dependent on the initialization scale under the exact-parameterization setting.

6. Note that in [29], the initialization is $F_0 = \alpha \cdot \tilde{F}_0$ and $G_0 = \alpha \cdot \tilde{G}_0$, while Lemma 18 uses a slightly imbalance initialization. It is easy to show that their techniques also hold with this imbalance initialization.

These observations highlight the influence of the initialization scale on the algorithm's performance.

In the last experiment, we run our new method with the same n and r but two different $k = 3, 4$. Unlike the vanilla gradient descent, at the midway point of the episode, we applied a transformation to the matrices F_t and G_t as specified by Eq. (E.1). As illustrated in Figure 2(c)subfigure, it is evident that the rate of loss reduction accelerates after the halfway mark. This compelling observation serves as empirical evidence attesting to the efficacy of our algorithm.