Autoencoding Hyperbolic Representation for Adversarial Generation

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Abstract

With the recent advance of geometric deep learning, neural networks have been extensively used for data in non-Euclidean domains. In particular, hyperbolic neural networks have proved successful in processing hierarchical information of data. However, many hyperbolic neural networks are numerically unstable during training, which precludes using complex architectures. This crucial problem makes it difficult to build hyperbolic generative models for real and complex data. In this work, we propose a hyperbolic generative network in which we design novel architecture and layers to improve stability in training. Our proposed network contains a hyperbolic autoencoder (AE) that produces hyperbolic embedding for input data and a hyperbolic generative adversarial network (GAN) for generating the hyperbolic latent embedding of the AE from simple noise. Our generator inherits the decoder from the AE and the generator from the GAN. Our architecture fosters expressive and numerically stable representation in the hyperbolic space. Theoretically, we validate the training of GAN in the hyperbolic space, and prove stability of our hyperbolic layers used in the AE. Experiments show that our model is capable of generating tree-like graphs as well as complex molecular data with comparable structure-related performance.

1 Introduction

High-dimensional data often show an underlying geometric structure, which cannot be easily captured by neural networks designed for Euclidean spaces. Recently, there is intense interest in learning good representation for hierarchical data, for which the most natural underlying geometry is hyperbolic. A hyperbolic space is a Riemannian manifold with a constant negative curvature (Anderson, 2006). The exponential growth of the radius of the hyperbolic space provides high capacity, which makes it particularly suitable for modeling tree-like hierarchical structures. Hyperbolic representation has been successfully applied to, for instance, social network data in product recommendation (Wang et al., 2019), molecular data in drug discovery (Yu et al., 2020; Wu et al., 2021), and skeletal data in action recognition (Peng et al., 2020).

Many recent works (Ganea et al., 2018; Shimizu et al., 2021; Chen et al., 2021) have successfully designed hyperbolic neural operations. These operations have been used for generating samples in the hyperbolic space. For instance, several recent works (Nagano et al., 2019; Mathieu et al., 2019; Dai et al., 2021b) have built hyperbolic variational autoencoders (VAE) (Kingma & Welling, 2014). On the other hand, Lazcano et al. (2021) have generalized generative adversarial networks (GAN) (Goodfellow et al., 2014; Arjovsky et al., 2017) to the hyperbolic space. However, the above hyperbolic generative models are known to suffer

from gradient explosion when the networks are deep. In order to build hyperbolic networks that can generate real data, it is desired to have a framework that has both representation power and numerical stability.

In recent advancements, there has been a notable shift towards the manipulation of the latent space of autoencoders (AEs), including models like Latent Diffusion (Van Den Oord et al., 2017; Rombach et al., 2022), which has opened up a plethora of benefits across various domains. This manipulation has allowed for the latent representations to serve not just as a means for dimensionality reduction but as foundational elements in generative modeling. Researchers and practitioners have leveraged these representations for generating new, synthetic instances of data that retain the complexity and diversity of their training sets (Razavi et al., 2019; Esser et al., 2020). Beyond serving as a powerful tool for learning representations, the latent representation is increasingly utilized for its inherent smoothness (Rubanova et al., 2019) and numerically stable gradient properties (Li et al., 2020).

To leverage these insights, we design a novel stable hybrid model which learns complex structures and hyperbolic embeddings from data, and then generates examples by sampling from random noises in the hyperbolic space. Altogether, our model contains three parts: first, it uses a hyperbolic AE to learn the embedding of training data in the latent hyperbolic space; second, we use a hyperbolic GAN to learn the mapping from a wrapped normal noise to the latent distribution; third, we generate samples by sequentially applying the GAN's generator and the AE's decoder. The use of AE latent space in GAN was pioneered by Prykhodko et al. (2019) but our work considers the hyperbolic space and different neural operations. We name our model as Hyperbolic AE-GAN, or HAEGAN for short. The advantage of this architecture is twofold: it boasts expressivity by processing noise through both generator and decoder layers, and it allows for the AE's flexible design without compromising the GAN's sampling efficiency. In addition, HAEGAN avoids the complicated form of ELBO in hyperbolic VAE, which is one source of numerical instability. The main contributions of this paper are as follows:

- We design a hybrid framework for learning hyperbolic distributions with novel neural operations. To the best of our knowledge, similar architectures have not been explored in the hyperbolic space. Compared with other models, our model HAEGAN enjoys both expressivity and numerical stability.
- We theoretically validate the Wasserstein GAN formulation in the hyperbolic space, especially the way of sampling from the geodesic connecting a real sample and a generated sample.
- We design a novel concatenation operation in the hyperbolic space. We theoretically investigate its numerical stability and empirical comparisons with other hyperbolic networks.
- In the experiments part, we illustrate that HAEGAN has the capacity not only to faithfully generate synthetic hyperbolic data, but also to generate real data with sound quality. In particular, we show that HAEGAN achieves comparable performance in molecular generation, especially in metrics related to structural properties.

2 Background

2.1 Hyperbolic Geometry

Hyperbolic geometry is a special kind of Riemannian geometry with a constant negative curvature (Cannon et al., 1997; Anderson, 2006). To extract hyperbolic representations, it is necessary to choose a "model", or coordinate system, for the hyperbolic space. Popular choices include the Poincaré ball model and the Lorentz model, where the latter is found to be more numerically stable (Nickel & Kiela, 2018). We work with the Lorentz model $\mathbb{L}_K^n = (\mathcal{L}, \mathfrak{g})$ with a constant negative curvature K, which is an n-dimensional manifold $\mathcal{L}$ embedded in the $(n+1)$-dimensional Minkowski space, together with the Riemannian metric tensor $\mathfrak{g} = \mathrm{diag}([-1, \mathbf{1}_n^\top])$, where $\mathbf{1}_n$ denotes the n-dimensional vector whose entries are all 1's. Every point in $\mathbb{L}_K^n$ is represented by $\boldsymbol{x} = [x_t, \boldsymbol{x}_s^\top]^\top, x_t > 0, \boldsymbol{x}_s \in \mathbb{R}^n$, and satisfies $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathcal{L}} = 1/K$, where $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is the Lorentz inner product induced by $\mathfrak{g}^K$: $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{L}} \coloneqq \boldsymbol{x}^\top \mathfrak{g} \boldsymbol{y} = -x_t y_t + \boldsymbol{x}_s^\top \boldsymbol{y}_s,\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$. In the rest of the paper, we will refer to x_t as the "time component" and $\boldsymbol{x}_s$ as the "spatial component". Extensive details are provided in Appendix A.1.

Notation We use $d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})$ to denote the length of a geodesic ("distance" along the manifold) connecting $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$. For each point $\boldsymbol{x} \in \mathbb{L}_K^n$, the tangent space at $\boldsymbol{x}$ is denoted by $\mathcal{T}_{\boldsymbol{x}}\mathbb{L}_K^n$. The norm $\|\cdot\|_{\mathcal{L}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathcal{L}}}$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$ and $\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}}\mathbb{L}_K^n$, we use $\exp_{\boldsymbol{x}}^K(\boldsymbol{v})$ to denote the exponential map of $\boldsymbol{v}$ at $\boldsymbol{x}$; on the other hand, we use $\log_{\boldsymbol{x}}^K : \mathbb{L}_K^n \to \mathcal{T}_{\boldsymbol{x}}\mathbb{L}_K^n$ to denote the logarithmic map such that $\log_{\boldsymbol{x}}^K(\exp_{\boldsymbol{x}}^K(\boldsymbol{v})) = \boldsymbol{v}$. For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$, we use $\mathrm{PT}_{\boldsymbol{x}\to\boldsymbol{y}}^K$ to denote the parallel transport map which "transports" a vector from $\mathcal{T}_{\boldsymbol{x}}\mathbb{L}_K^n$ to $\mathcal{T}_{\boldsymbol{y}}\mathbb{L}_K^n$ along the geodesic from $\boldsymbol{x}$ to $\boldsymbol{y}$.

2.2 Hyperbolic Neural Operations

In this section, we introduce the fundamental linear layers designs in Hyperbolic space. Additional introductions of Graph Neural Network layers and Hyperbolic-Euclidean conversion layers can be found in the Appendix A.2.

Fully connected linear layers are the fundamental building block of Euclidean deep learning. In Euclidean space, each linear layer can be represented by a linear transformation add bias. Specifically, suppose the input is $\boldsymbol{x} \in \mathbb{R}^n$, an Euclidean linear layer is

$$\boldsymbol{y} = W\boldsymbol{x} + \boldsymbol{b} \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}^m$ is the output, $W \in \mathbb{R}^{m\times n}$ is learnable weight matrix, $\boldsymbol{b} \in \mathbb{R}^m$ is the learnable bias.

To define a linear layer in hyperbolic space, the crucial part is to define a hyperbolic linear transformation. However, if we have an $\boldsymbol{x} \in \mathbb{L}_K^n$ and use an arbitrary matrix $W \in \mathbb{R}^{(n+1)\times(m+1)}$ as weight matrix, the output $\boldsymbol{y}$ is not guaranteed in the $\mathbb{L}_K^m$. To constraint the output in the hyperbolic space, there are two lines of works, the tangent linear layers (Ganea et al., 2018; Shimizu et al., 2021; Chami et al., 2019) and fully hyperbolic linear layers (Chen et al., 2021).

Tangent Linear Layers The idea of this kind of linear layers is to perform linear transformation in the tangent space. Since the tangent space of any Riemannian manifold is an Euclidean subspace, it is possible to perform linear transformation in such space. Chami et al. (2019) defined the linear transformation of $\boldsymbol{x} \in \mathbb{L}_K^n$ to $\boldsymbol{y} \in \mathbb{L}_K^m$ by first logmap $\boldsymbol{x}$ to the tangent space of the origin, perform Euclidean linear transformation by matrix $W \in \mathbb{R}^{n\times M}$, and expmap it back to $\mathbb{L}_K^m$:

$$\boldsymbol{y} = W \otimes^K \boldsymbol{x} = \exp_{\boldsymbol{o}}^K\left(W \log_{\boldsymbol{o}}^K(\boldsymbol{x})\right) \tag{2}$$

However, cascading many exponential and logarithmic maps may lead to numerical instability (Chami et al., 2019; Chen et al., 2021). Thus, people are interested in developing hyperbolic linear layers without the use of exponential and logarithmic maps.

Fully Hyperbolic Linear Layers In Lorentz model, the constraint for the point $\boldsymbol{x} = \left[\begin{smallmatrix} x_t \\ \boldsymbol{x}_s \end{smallmatrix}\right] \in \mathbb{L}_K^n$ are

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathcal{L}} = -x_t^2 + |\boldsymbol{x}_s|^2 = 1/K. \tag{3}$$

where $|\cdot|$ is the Euclidean 2-norm. Thus, whatever the spacial component $\boldsymbol{x}_s$ changes, we could always calculate the $x_t = \sqrt{|\boldsymbol{x}_s|^2 - 1/K}$ to make it in the Lorentz space.

Using this idea, Chen et al. (2021) proposed the fully hyperbolic linear layer, which applies a genreal linear transformation to the spatial component of the input, and calculate the time component so that the whole vector lies in the hyperboloid. Specifically, it maps $\boldsymbol{x} \in \mathbb{L}_K^n$ to $\begin{bmatrix} \sqrt{\|W\boldsymbol{x}\|^2 - 1/K} \\ W\boldsymbol{x} \end{bmatrix}$, where $W \in \mathbb{R}^{n\times(m+1)}$. With additional activation, bias and normalization, a general expressive linear layer transforms an input $\boldsymbol{x} \in \mathbb{L}_K^n$ to

$$\boldsymbol{y} = \mathrm{HLinear}(\boldsymbol{x}) = \begin{bmatrix} \sqrt{\|h(\boldsymbol{x})\|^2 - 1/K} \\ h(\boldsymbol{x}) \end{bmatrix} \in \mathbb{L}_K^n. \tag{4}$$

Here,

$$h(\boldsymbol{x}) = \frac{\lambda\sigma\left(\boldsymbol{v}^\top \boldsymbol{x} + b'\right)}{\|W\tau(\boldsymbol{x}) + \boldsymbol{b}\|}(W\tau(\boldsymbol{x}) + \boldsymbol{b}), \tag{5}$$

where $\boldsymbol{v} \in \mathbb{R}^{n+1}$ and $W \in \mathbb{R}^{n\times(m+1)}$ are trainable weights, $\boldsymbol{b} \in \mathbb{R}^n$ and $b' \in \mathbb{R}$ are trainable biases, σ is the sigmoid function, τ is the activation function, and the trainable parameter $\lambda > 0$ scales the range. We may also write $h(\boldsymbol{x}; W, \boldsymbol{b})$ to emphasize its dependence on W and $\boldsymbol{b}$.

We remark that using Lorentz model comes with many advantage in numerical stability. We will demonstrate this in the next section, and in the experiments in §5.2.

3 Lorentz Concatenation

3.1 Motivation and Definition

Concatenation and split are essential operations in neural networks for feature combination, parallel computation, etc. Shimizu et al. (2021) proposed Poincaré β-concatenation and β-split in the Poincaré model. Specifically, they first use the logarithmic map to lift hyperbolic points to the tangent plane of the origin, then perform Euclidean concatenation and split in this tangent space, and finally apply β regularization and apply the exponential map to bring it back to the Poincaré ball.

Since we use the Lorentz model, the above operations are not useful and we need to define concatenation and split in the Lorentz space. One could define operations in the tangent space similarly to the Poincaré β-concatenation and β-split. More specifically, if we want to concatenate the input vectors $\{\boldsymbol{x}_i\}_{i=1}^N$ where each $\boldsymbol{x}_i \in \mathbb{L}_K^{n_i}$, we could follow a "Lorentz tangent concatenation": first lift each $\boldsymbol{x}_i$ to $\boldsymbol{v}_i = \log_{\boldsymbol{o}}^K(\boldsymbol{x}_i) = \left[\begin{smallmatrix} v_{i_t} \\ \boldsymbol{v}_{i_s} \end{smallmatrix}\right] \in \mathbb{R}^{n_i+1}$, and then perform the Euclidean concatenation to get $\boldsymbol{v} := \left[0, \boldsymbol{v}_{1_s}^\top, \ldots, \boldsymbol{v}_{N_s}^\top\right]^\top$. Finally, we would get $\boldsymbol{y} = \exp_{\boldsymbol{o}}^K(\boldsymbol{v})$ as a concatenated vector in the hyperbolic space. We denote $\boldsymbol{y} = \mathrm{HTCat}(\{\boldsymbol{x}_i\}_{i=1}^N)$. Similarly, we could perform the "Lorentz tangent split" on an input $\boldsymbol{x}_i \in \mathbb{L}_K^n$ with split sub-dimensions $\sum_{i=1}^N n_i = n$ to get $\boldsymbol{v} = \log_{\boldsymbol{o}}^K(\boldsymbol{x}) = \left[0, \boldsymbol{v}_{1_s}^\top \in \mathbb{R}^{n_1}, \ldots, \boldsymbol{v}_{N_s}^\top \in \mathbb{R}^{n_N}\right]^\top$, $\boldsymbol{v}_i = \left[\begin{smallmatrix} 0 \\ \boldsymbol{v}_{i_s} \end{smallmatrix}\right] \in \mathcal{T}_{\boldsymbol{o}}\mathbb{L}_K^{n_i}$, and the split vectors $\boldsymbol{y}_i = \exp_{\boldsymbol{o}}^K(\boldsymbol{v}_i)$ successively.

Unfortunately, there are two problems with both the Lorentz tangent concatenation and the Lorentz tangent split. First, they are not "regularized", which means that the norm of the spatial component will increase after concatenation, and decrease after split. This will make the hidden embeddings numerically unstable. This problem could be partially solved by adding a hyperbolic linear layer after each concatenation and split, similarly to Ganea et al. (2018), so that we have a trainable scaling factor λ to regularize the norm of the output. The second and more important problem is that if we use the Lorentz tangent concatenation and split in a deep neural network, there would be too many exponential and logarithmic maps. It on one hand suffers from severe precision issue due to the inaccurate float representation (Yu & De Sa, 2019; 2021), and on the other hand easily suffers from gradient explosion. Moreover, the tangent space is chosen at $\boldsymbol{o}$. If the points to concatenate are not close to $\boldsymbol{o}$, their hyperbolic relation may not be captured very well. Therefore, we abandon the use of the tangent space and propose more direct and numerically stable operations, which we call the "Lorentz direct concatenation and split", defined as follows.

Given the input vectors $\{\boldsymbol{x}_i\}_{i=1}^N$ where each $\boldsymbol{x}_i \in \mathbb{L}_K^{n_i}$ and $M = \sum_{i=1}^N n_i$, the Lorentz direct concatenation of $\{\boldsymbol{x}_i\}_{i=1}^N$ is defined to be a vector $\boldsymbol{y} \in \mathbb{L}_k^M$ given by

$$\boldsymbol{y} = \mathrm{HCat}(\{\boldsymbol{x}_i\}_{i=1}^N) = \left[\sqrt{\sum_{i=1}^N x_{i_t}^2 + (N-1)/K}, \boldsymbol{x}_{1_s}^\top, \cdots, \boldsymbol{x}_{N_s}^\top\right]^\top. \tag{6}$$

Note that each $\boldsymbol{x}_{i_s}$ is the spatial component of $\boldsymbol{x}_i$. If we consider $\boldsymbol{x}_i \in \mathbb{L}_K^{n_i}$ as a point in $\mathbb{R}^{n_i+1}$, the projection of $\boldsymbol{x}_i$ onto the Euclidean subspace $\{0\}\times\mathbb{R}^n$, or the closest point there, is $\boldsymbol{x}_{i_s}$. The Lorentz direct concatenation can thus be considered as Euclidean concatenation of projections, where the Euclidean concatenated point is mapped back to $\mathbb{L}_K^M$ by the inverse map of the projection. We remark that this concatenation directly inherits from the Lorentz model.

We also define the Lorentz split for completeness, though our main focus is on concatenation: Given an input $\boldsymbol{x} \in \mathbb{L}_K^n$, the Lorentz direct split of $\boldsymbol{x}$, with sub-dimensions $n_1, \cdots, n_N$ where $\sum_{i=1}^N n_i = n$, will be

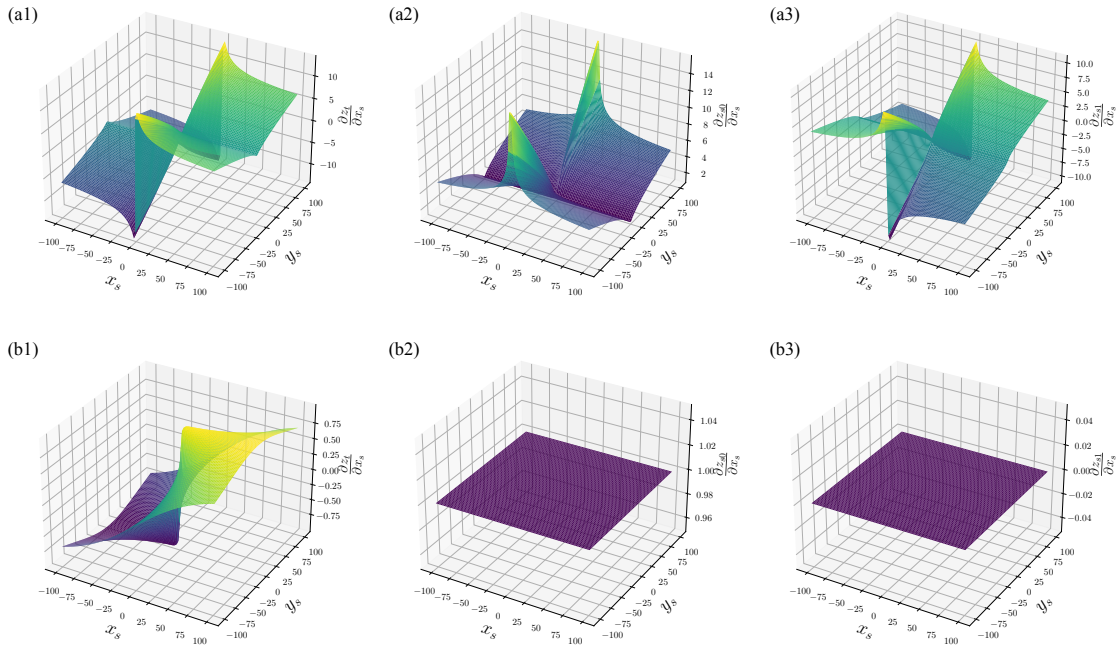

Figure 1: Illustration of the gradients of both concatenation methods (a) Lorentz tangent concatenation (b) Lorentz direct concatenation. (a1) & (b1) $\frac{\partial z_t}{\partial x_s}$; (a2) & (b2) $\frac{\partial z_{s0}}{\partial x_s}$; (a3) & (b3) $\frac{\partial z_{s1}}{\partial x_s}$.

$\{\boldsymbol{y}_i\}_{i=1}^N$, where each $\boldsymbol{y}_i \in \mathbb{L}_K^{n_i}$ is given by first splitting $\boldsymbol{x}$ as $\boldsymbol{x} = \left[x_t, \boldsymbol{y}_{1_s}^\top, \cdots, \boldsymbol{y}_{N_s}^\top\right]^\top$, and then calculating the corresponding time dimension as $\boldsymbol{y}_i = \begin{bmatrix} \sqrt{\|\boldsymbol{y}_{i_s}\|^2 - 1/K} \\ \boldsymbol{y}_{i_s} \end{bmatrix}$.

3.2 Advantage of Lorentz Direct Concatenation

Firstly, we state the following theoretical result regarding the exploding gradient of the Lorentz tangent concatenation.

Theorem 3.1. *Let $\{\boldsymbol{x}_i\}_{i=1}^N$, where $\boldsymbol{x}_i \in \mathbb{L}_K^{n_i}$, denote the input features. Let $\boldsymbol{y} = \mathrm{HCat}(\{\boldsymbol{x}_i\}_{i=1}^N)$ denote the output of the Lorentz direct concatenation and $\boldsymbol{z} = \mathrm{HTCat}(\{\boldsymbol{x}_i\}_{i=1}^N)$ denote the output of the Lorentz tangent concatenation. Fix $j \in \{1, \cdots, N\}$. The following results hold:*

1. *For any $\{\boldsymbol{x}_i\}_{i=1}^N$ and any entry y^* of $\boldsymbol{y}$, $\|\partial y^* / \partial \boldsymbol{x}_{j_s}|_{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N}\| \leq 1$.*
2. *For any $M > 0$, there exist $\{\boldsymbol{x}_i\}_{i=1}^N$ and an entry z^* of $\boldsymbol{z}$ for which $\|\partial z^* / \partial \boldsymbol{x}_{j_s}|_{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N}\| \geq M$.*

This theorem shows that while the Lorentz direct concatenation has bounded gradients, there is no control on the gradients of Lorentz tangent concatenation. The proof can be found in Appendix C.1. We provide a numerical validation in §3.2.1 and an empirical validation in §3.2.2. We also include additional analysis on the effect of the two concatenation methods to the hyperbolic distance in Appendix C.2.

3.2.1 Numerical Validation of Theorem 3.1

In this section, we numerically validate Theorem 3.1 by showing the gradients of the Lorentz tangent concatenation and the Lorentz direct concatenation under the following simple setting. We concatenate two 1-dimensional Lorentz vectors to obtain a 2-dimensional Lorentz vector.

Denote $\boldsymbol{x} = \left[\sqrt{x_s^2 + 1}, x_s\right]^\top \in \mathbb{L}_{-1}^1$ and $\boldsymbol{y} = \left[\sqrt{y_s^2 + 1}, y_s\right]^\top \in \mathbb{L}_{-1}^1$ to be the two input vectors under the Lorentz model with $K = -1$. For both the Lorentz tangent concatenation and the Lorentz direct

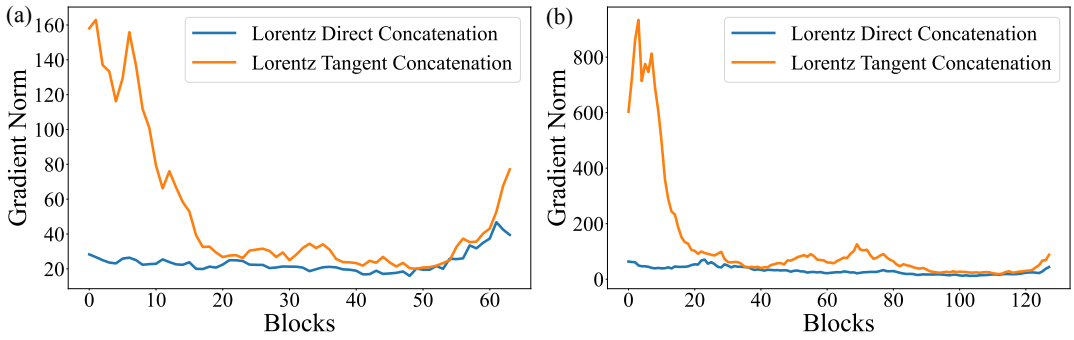

Figure 2: Average gradient norm of each block in training. (a) 64 blocks. (b) 128 blocks.

concatenation, let $\boldsymbol{z} = [z_t, z_{s0}, z_{s1}]^\top \in \mathbb{L}^2_{-1}$ denote the concatenated vector of $\boldsymbol{x}$ and $\boldsymbol{y}$. Numerically, we consider the range of $x_s, y_s \in [-100, 100]$ and calculate the gradients of each entry of $\boldsymbol{z}$ with respect to x_s, that is, $\left.\frac{\partial z_t}{\partial x_s}\right|_{x_s, y_s \in [-100,100]}$, $\left.\frac{\partial z_{s_0}}{\partial x_s}\right|_{x_s, y_s \in [-100,100]}$ and $\left.\frac{\partial z_{s_1}}{\partial x_s}\right|_{x_s, y_s \in [-100,100]}$. We plot them in Figure 1. We remark that due to symmetry, the graphs are the same for $\frac{\partial}{\partial y_s}$.

From Figure 1, we observe unbounded gradients in each component of the gradient of the Lorentz tangent concatenation. On the other hand, all the components of the gradient of the Lorentz direct concatenation has an absolute value bounded by 1. The results of this numerical experiment have validated the conclusion in Theorem 3.1.

3.2.2 Empirical Validation of Theorem 3.1

In this section, we design the following simple experiment to show the advantage of our Lorentz direct concatenation over the Lorentz tangent concatenation when they are used in simple neural networks. The hyperbolic neural network in this simple experiment consists of a cascading of L blocks, and the architecture of each block is illustrated in Figure 3. A d-dimensional input is fed into two different hyperbolic linear layers, whose outputs are then concatenated by the Lorentz direct concatenation and the Lorentz tangent concatenation, respectively. Then, the concatenated output further goes through another hyperbolic linear layer whose output is again d-dimensional. Specifically, for $l = 0, \cdots, L-1$,

$$\begin{aligned} \boldsymbol{h}_1^{(l)} &= \mathrm{Hlinear}_{d,d}(\boldsymbol{x}^{(l)}), \quad \boldsymbol{h}_2^{(l)} = \mathrm{Hlinear}_{d,d}(\boldsymbol{x}^{(l)}); \\ \boldsymbol{h}^{(l)} &= \mathrm{HCat}(\boldsymbol{h}_1^{(l)}, \boldsymbol{h}_2^{(l)}); \quad \boldsymbol{x}^{(l+1)} = \mathrm{Hlinear}_{2\times d,d}(\boldsymbol{h}^{(l)}). \end{aligned} \tag{7}$$

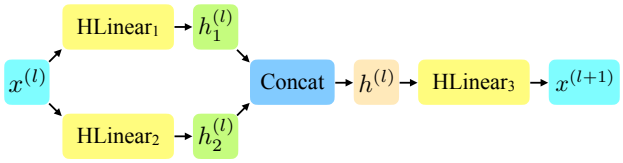

Figure 3: Illustration of the l-th block in the simple hyperbolic neural network.

In our test, we take $d = 64$. We sample input and output data from two wrapped normal distributions with different means (input: origin $\boldsymbol{o}$, output: $\mathrm{E2H}(\mathbf{1}_{64})$) and variances (input: $\mathrm{diag}(\mathbf{1}_{64})$, output: $3 \times \mathrm{diag}(\mathbf{1}_{64})$). Taking the input as $\boldsymbol{x}^{(0)}$, we fit $\boldsymbol{x}^{(L)}$ to the output data. We record the average gradient norm of the three hyperbolic linear layers in each block. The results for $L = 64$ blocks and $L = 128$ blocks are shown in Figure 2. Clearly, for the first 20 blocks, the Lorentz tangent concatenation leads to significantly larger gradient norms. This difference in norms is clearer when the network is deeper. The gradients from the Lorentz direct concatenation are much more stable.

4 Hyperbolic Auto-Encoder Generative Adversarial Networks

4.1 Architecture of HAEGAN

Although recent proposals of hyperbolic generative models have transferred VAE and GAN to the hyperbolic domain (Nagano et al., 2019; Mathieu et al., 2019; Dai et al., 2021b; Lazcano et al., 2021), they are known to suffer from numerical instability, which hinders building large-scale hyperbolic generative models. To this end, we design our HAEGAN model to contain both a hyperbolic AE and a hyperbolic GAN. First, we train a hyperbolic AE and use the encoder to embed the dataset into a latent hyperbolic space. Then, we use our hyperbolic GAN to learn the latent distribution of the embedded data. Finally, we sample hyperbolic embeddings using the generator and use the decoder to get samples in the original space. An illustration of HAEGAN is shown in Figure 4.

The overall structure of HAEGAN distributes learning into two architectures and thus reduces the scale of either standalone network. Nevertheless, both the GAN and the AE require carefully chosen structures in order to guarantee numerical stability. We discuss the hyperbolic GAN and hyperbolic AE in the next two subsections respectively.

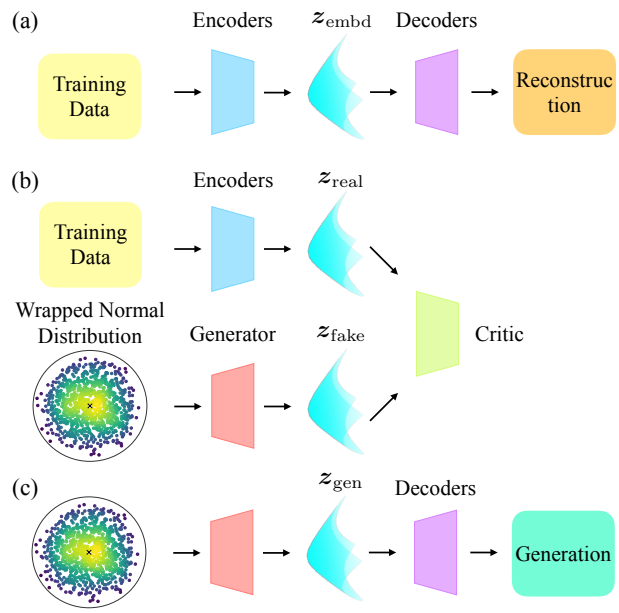

Figure 4: Overview of HAEGAN. (a) The hyperbolic AE. (b) The hyperbolic GAN for generating the latent embeddings. The encoders in (b) are identical to (a). (c) The process for sampling molecules. The generator in (c) is identical to (b) and the decoders in (c) are identical to (a).

4.2 Hyperbolic GAN

4.2.1 Architecture of HGAN

We could use the hyperbolic linear layers in §2.2 to define a hyperbolic GAN whose generator and critic are in the hyperbolic space. The generator pushes forward a wrapped normal distribution $\mathcal{G}(\boldsymbol{o}, \boldsymbol{I})$ to a hyperbolic distribution via a cascading of l_{gen} hyperbolic linear layers. Specifically, we sample $\boldsymbol{z}^{(0)} \sim \mathcal{G}(\boldsymbol{o}, \boldsymbol{I})$ from the wrapped normal distribution (Nagano et al., 2019), and produce $\boldsymbol{z}_{\text{fake}}$ following

$$\begin{aligned} \boldsymbol{z}^{(l)} &= \text{HLinear}_{d_{l-1},d_l}(\boldsymbol{z}^{(l-1)}), \quad l = 1, \cdots, l_{\text{gen}}, \\ \boldsymbol{z}_{\text{fake}} &= \boldsymbol{z}^{(l_{\text{gen}})}. \end{aligned} \tag{8}$$

The critic aims to distinguish between fake data generated from the generator and real data. It contains a cascading of l_{dis} hyperbolic linear layers, and a centroid distance layer whose output is a score in $\mathbb{R}$. Specifically,

$$\begin{aligned} \boldsymbol{z}^{(l)} &= \text{HLinear}_{d_{l-1},d_l}(\boldsymbol{z}^{(l-1)}), \quad l = 1, \cdots, l_{\text{dis}}, \\ s &= \text{HCDist}_{d_{l_{\text{dis}}},1}(\boldsymbol{z}^{(l_{\text{dis}})}). \end{aligned} \tag{9}$$

4.2.2 Hyperbolic Wasserstein GAN and Gradient Penalty

We adopt the framework of Wasserstein GAN (Arjovsky et al., 2017), which aims to minimize the Wasserstein-1 (W_1) distance between the distribution pushed forward by the generator and the data distribution. Since $d_{\mathcal{L}}$ is a valid metric, the W_1 distance between two hyperbolic distribution $\mathbb{P}_r, \mathbb{P}_g$ defined on the Lorentz space is

$$W_1(\mathbb{P}_r, \mathbb{P}_g) = \inf_{\gamma \in \Pi(\mathbb{P}_r, \mathbb{P}_g)} \mathbb{E}_{(\boldsymbol{x},\boldsymbol{y})\sim\gamma}[d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})], \tag{10}$$

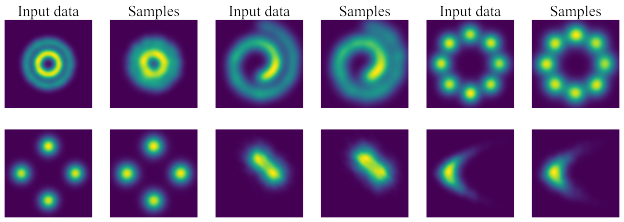

Figure 5: Input data and generated samples from the hyperbolic GAN. The hyperbolic data points are transformed to the tangent space of the origin by the logarithmic map.

where $\Pi(\mathbb{P}_r, \mathbb{P}_g)$ is the set of all joint distributions whose marginals are $\mathbb{P}_r$ and $\mathbb{P}_g$, respectively. By Kantorovich-Rubinstein duality (Villani, 2009), we have the following more tractable form of W_1 distance

$$W_1(\mathbb{P}_r, \mathbb{P}_g) = \sup_{\|D\|_L \leq 1} \mathbb{E}_{\boldsymbol{x} \sim \mathbb{P}_r}[D(\boldsymbol{x})] - \mathbb{E}_{\boldsymbol{x} \sim \mathbb{P}_g}[D(\boldsymbol{x})], \tag{11}$$

where the supremum is over all 1-Lipschitz functions $D : \mathbb{L}_K^n \to \mathbb{R}$, represented by the critic. To enforce the 1-Lipschitz constraint, we adopt a penalty term on the gradient following Gulrajani et al. (2017). The loss function is thus

$$L_{\text{WGAN}} = \mathop{\mathbb{E}}_{\tilde{\boldsymbol{x}} \sim \mathbb{P}_g}[D(\tilde{\boldsymbol{x}})] - \mathop{\mathbb{E}}_{\boldsymbol{x} \sim \mathbb{P}_r}[D(\boldsymbol{x})] + \lambda \mathop{\mathbb{E}}_{\hat{\boldsymbol{x}} \sim \mathbb{P}_{\hat{x}}} \left[\left(\|\nabla D(\hat{\boldsymbol{x}})\|_{\mathcal{L}} - 1 \right)^2 \right], \tag{12}$$

where D is the critic, $\nabla D(\hat{\boldsymbol{x}})$ is the Riemannian gradient of $D(\boldsymbol{x})$ at $\hat{\boldsymbol{x}}$, $\mathbb{P}_g$ is the generator distribution and $\mathbb{P}_r$ is the data distribution. Crucially $\mathbb{P}_{\hat{\boldsymbol{x}}}$ samples uniformly along the geodesic between pairs of points sampled from $\mathbb{P}_g$ and $\mathbb{P}_r$ instead of a linear interpolation. This manner of sampling is validated in the following proposition,

Proposition 4.1. *Let $\mathbb{P}_r$ and $\mathbb{P}_g$ be two distributions in $\mathbb{L}_K^n$ and f^* be an optimal solution of $\max_{\|f\|_L \leq 1} \mathbb{E}_{\boldsymbol{y} \sim \mathbb{P}_r}[f(\boldsymbol{y})] - \mathbb{E}_{\boldsymbol{x} \sim \mathbb{P}_g}[f(\boldsymbol{x})]$ where $\|\cdot\|_L$ is the Lipschitz norm. Let π be the optimal coupling between $\mathbb{P}_r$ and $\mathbb{P}_g$ that minimizes $W(\mathbb{P}_r, \mathbb{P}_g) = \inf_{\pi \in \Pi(\mathbb{P}_r, \mathbb{P}_g)} \mathbb{E}_{x,y \sim \pi}[d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})]$, where $\Pi(\mathbb{P}_r, \mathbb{P}_g)$ is the set of joint distributions $\pi(\boldsymbol{x}, \boldsymbol{y})$ whose marginals are $\mathbb{P}_r$ and $\mathbb{P}_g$, respectively. Let $\boldsymbol{x}_t = \gamma(t), 0 \leq t \leq 1$ be the geodesic between $\boldsymbol{x}$ and $\boldsymbol{y}$, such that $\gamma(0) = \boldsymbol{x}, \gamma(1) = \boldsymbol{y}, \gamma'(t) = \boldsymbol{v}_t \in \mathcal{T}\mathbb{L}_K^n, \|\boldsymbol{v}_t\|_{\mathcal{L}} = d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})$. If f^* is differentiable and $\pi(\boldsymbol{x} = \boldsymbol{y}) = 0$, then it holds that*

$$\mathbb{P}_{(\boldsymbol{x}, \boldsymbol{y}) \sim \pi} \left[\nabla f^*(\boldsymbol{x}_t) = \frac{\boldsymbol{v}_t}{d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})} \right] = 1. \tag{13}$$

The proof of this proposition can be found in Appendix D.1.

4.2.3 Toy Distribution Generation

As a proof-of-concept, we use a set of challenging toy 2D distributions explored by Rozen et al. (2021) to test the effectiveness of the hyperbolic WGAN-GP formulation. For our experiment, the training data are prepared in the following manner. We first sample 5,000 points from the toy 2D distributions and scale the coordinates to $[-1, 1]$. Then, we use the E2H operation (24) to map the points to the hyperbolic space. These points are treated as the input data of the hyperbolic GAN. Next, we use the hyperbolic GAN to learn the hyperbolic toy distributions. The generator and the critic both contain 3 layers of hyperbolic linear layers and 64 hidden dimensions at each layer. The input dimension for the generator is 128.

After we train the hyperbolic GAN, we sample from it and compare with the input data. Note that the input data and the generated samples are both in the hyperbolic space. To illustrate them, we map both the input data and the generated samples to the tangent space of the origin by applying the logarithmic map. We present the mapped input data and generated samples in Figure 5. Clearly, the hyperbolic GAN can faithfully represent the challenging toy distributions in the hyperbolic space.

4.3 Hyperbolic AE

HAEGAN enjoys expressivity and flexibility in choosing the AE for embedding the hyperbolic distribution. For many datasets with hierarchical structures that are not readily presented in the hyperbolic space, the hyperbolic AE can also learn to find their appropriate hyperbolic representations. In this case, the first layer of the hyperbolic AE is constructed by the hyperbolic embedding layer (Nagano et al., 2019) from Euclidean to the hyperbolic space.

The design of AE structure varies according to the type of data, which will be described in detail for each experimental task. Nevertheless, one common operation for building hyperbolic decoders for tree-structured data is message passing, which requires concatenating messages from different parts of a tree. We use our novel concatenation operation (§3) here to enforce stable training.

5 Experiments

In this section, we present two experiments utilizing HAEGAN: Random Tree Generation (§5.1) and *de novo* Molecular Generation (§5.2). Additionally, a proof-of-concept MNIST generation experiment is included in the Appendix E.

5.1 Random Tree Generation

Recent studies (Boguná et al., 2010; Krioukov et al., 2010; Sala et al., 2018; Sonthalia & Gilbert, 2020) have found that hyperbolic spaces are suitable for tree-like graphs. Thus, we perform an experiment in which we use HEAGAN to generate random trees. We compare the performance of HAEGAN with other hyperbolic models. In this experiment, the AE consists of a tree encoder and a tree decoder. The details of the model architecture is described in Appendix F.2.

5.1.1 Experimental Settings

Dataset Our dataset consists of 500 randomly generated trees. Each tree is created by converting a uniformly random Prüfer sequence (Prüfer, 1918). The number of nodes in each tree is uniformly sampled from $[20, 50]$. The dataset is randomly split into 400 for training and 100 for testing.

Baselines and Ablations We compare HAEGAN with the following baseline hyperbolic generation methods. HGAN, where we only use a hyperbolic Wasserstein GAN without the AE structure, where the tree decoder as the generator and a hyperbolic MLP as the critic. HVAE-w and HVAE-r, where we use the same AE but follow the ELBO loss function used by Mathieu et al. (2019) instead of having a GAN ("w" and "r" refer to using wrapped and Riemannian normal distributions, respectively). Although we mainly focus on hyperbolic methods, we also compare with the following Euclidean generation methods: GraphRNN (You et al., 2018b) and AEGAN. The latter has the same architecture as HAEGAN but all layers and operations are Euclidean.

As discussed in previous sections, our default choice in HAEGAN is to use Lorentz *direct* Concatenation and *fully* hyperbolic linear layers (Chen et al., 2021). We also consider the following ablations of HAEGAN: HAEGAN-H, where the fully hyperbolic linear layers are replaced with the tangent linear layers defined by Ganea et al. (2018); HAEGAN-β, where the concatenation in HAEGAN is replaced by β-concatenation (Shimizu et al., 2021); HAEGAN-T, where the concatenation is replaced by the Lorentz tangent concatenation.

Metrics We use the following metrics from You et al. (2018b) to evaluate the models: The Maximum Mean Discrepancy (MMD) of degree distribution (*Degree*), MMD of the orbit counts statistics distribution (*Orbit*); Average difference of orbit counts statistics (*Orbit*), Betweenness Centrality (*Betweenness*), Closeness Centrality (*Closeness*). All metrics are calculated between the test dataset and 100 samples generated from the models.

Table 1: Results of the tree generation experiments. "NaN" indicates NaN loss during training.

Model	Concat	HNN	MMD ($\times 10^{-3}$)		Average Difference			Time
			Degree	Orbit	Orbit	Betweenness	Closeness	(s/step)
HGAN	Direct	Fully	0.5660±0.0105	0.0560±0.0021	0.1315±0.0032	0.0271±0.0014	0.0229±0.0008	1.5472±0.0203
HVAE-w	Direct	Fully	NaN	NaN	NaN	NaN	NaN	1.6712±0.0257
HVAE-r	Direct	Fully	NaN	NaN	NaN	NaN	NaN	1.7394±0.0229
GraphRNN	N/A	N/A	2.6810±0.0352	0.1060±0.0041	0.1439±0.0035	0.0256±0.0013	0.0221±0.0010	0.0933±0.0012
AEGAN	N/A	N/A	1.3430±0.0204	0.0500±0.0020	0.1405±0.0037	0.0257±0.0012	0.0219±0.0007	1.1874±0.0186
HAEGAN-H	Direct	Tangent	0.7430±0.0126	0.0100±0.0006	0.1382±0.0030	0.0245±0.0011	0.0220±0.0008	1.8513±0.0234
HAEGAN-β	Beta	Fully	0.4700±0.0093	**0.0010±0.0015**	0.1299±0.0034	0.0261±0.0010	0.0224±0.0009	1.7529±0.0192
HAEGAN-T	Tangent	Fully	0.3140±0.0078	0.0520±0.0023	0.1316±0.0035	0.0242±0.0012	0.0219±0.0007	1.6385±0.0181
HAEGAN	Direct	Fully	**0.1560±0.0052**	0.0050±0.0004	**0.1233±0.0030**	**0.0237±0.0010**	**0.0217±0.0008**	1.3146±0.0157

5.1.2 Results

Table 1 presents results and runtime of all models. For all metrics, a smaller number implies a better result. Our default choice of HAEGAN (with direct concatenation and fully hyperbolic linear layers) performs the best across all metrics except the MMD of orbit counts statistics distribution, in which it just marginally falls behind the β-concatenation. In particular, the Lorentz direct concatenation generally performs better and more efficiently than Lorentz tangent concatenation and β-concatenation. Also, the fully hyperbolic linear layer is superior to the tangent linear layer in both effectiveness and efficiency. Our results also show the advantage of the overall framework compared with either a single GAN or VAE. On one hand, the performance of HAEGAN is much better than HGAN. On the other hand, we note that the hyperbolic VAE-based methods suffer from numerical instability for this simple dataset even when using fully hyperbolic linear layers and direct concatenation, possibly because of the complicated ELBO loss. Finally, we remark that it is clear from the results that the hyperbolic models are better at generating trees than the Euclidean ones.

5.2 *De Novo* Molecular Generation

It is a crucial task in machine learning to learn the structure of molecules, which has important applications in the discovery of drugs and proteins (Elton et al., 2019). Since molecules naturally show a graph structure, many recent works use graph neural networks to extract their information and accordingly train molecular generators (Simonovsky & Komodakis, 2018; De Cao & Kipf, 2018; Jin et al., 2018; 2019). In particular, Jin et al. (2018; 2019) proposed a bi-level representation of molecules where both a junction-tree skeleton and a molecular graph are used to represent the original molecular data. In this way, a molecule is represented in a hierarchical manner with a tree-structured scaffold. Given that hyperbolic spaces can well-embed such hierarchical and tree-structured data (Peng et al., 2021), we expect that HAEGAN can leverage the structural information. To validate its effectiveness, in this section, we test HAEGAN using molecular generative tasks, where the latent distribution is embedded in a hyperbolic manifold.

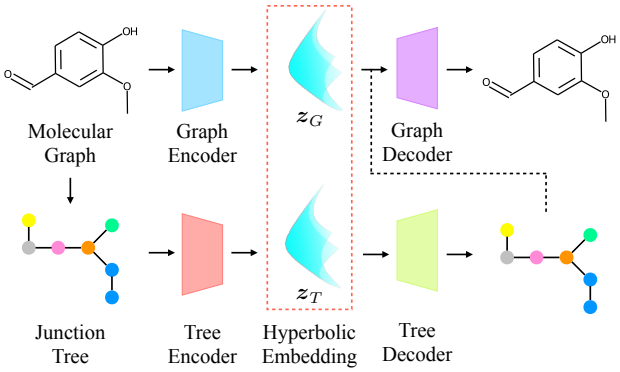

Figure 6: Illustration of the auto-encoder used in the HAEGAN for molecular generation. The input molecular graph is firstly coarsened into the junction tree. Then both of them are encoded using graph and tree encoders to their respective hyperbolic embeddings $\boldsymbol{z}_T$ and $\boldsymbol{z}_G$. To reconstruct the molecule, we first decode the junction tree from $\boldsymbol{z}_T$, and then reconstruct the molecular graph using the junction tree and $\boldsymbol{z}_G$.

In our experiments, we design both a hyperbolic tree AE and a hyperbolic graph AE in our HAEGAN to embed the structural information of the atoms in each molecule, as illustrated in Figure 6. Specifically, our model takes a molecular graph as the input, passes the original graph to the graph encoder and feeds

the corresponding junction tree to the tree encoder, acquiring hyperbolic latent representations $\boldsymbol{z}_G$ of the graph, as well as $\boldsymbol{z}_T$ for the junction tree. Then, the junction tree decoder constructs a tree from $\boldsymbol{z}_T$ autoregressively. Finally, the graph decoder recovers the molecular graph using the generated junction tree and $\boldsymbol{z}_G$. The hyperbolic representation is supposed to better leverage the hierarchical structure from the junction-tree than the graph neural networks (Jin et al., 2018; 2019). We describe the detailed model architecture in Appendix F.3.

5.2.1 Experimental Settings

Dataset We train and test our model on the MOSES benchmarking platform (Polykovskiy et al., 2020), which is refined from the ZINC dataset (Sterling & Irwin, 2015) and contains about 1.58M training, 176k test, and 176k scaffold test molecules. The molecules in the scaffold test set have different Bemis-Murcko scaffolds (Bemis & Murcko, 1996), which represent the core structures of compounds, than both the training and test sets. They are used to determine whether a model can generate novel molecular scaffolds.

Baselines We compare our model with the following baselines: non-neural models including the Hidden Markov Model (HMM), the N-Gram generative model (NGram) and the combinatorial generator; neural methods including CharRNN (Segler et al., 2018), AAE (Kadurin et al., 2017a;b; Polykovskiy et al., 2018), VAE (Gómez-Bombarelli et al., 2018; Blaschke et al., 2018), JTVAE (Jin et al., 2018), LatentGAN (Prykhodko et al., 2019), GraphINVENT (Mercado et al., 2021), DiGress (Vignac et al., 2023), cMolGPT (Wang et al., 2023), Group SELFIES (Cheng et al., 2023). The benchmark results from HMM to LatentGAN are taken from Polykovskiy et al. (2020) ; the results from GraphINVENT to Group SELFIES are taken from respective papers.

Ablations On one hand, we consider a Euclidean counterpart of HAEGAN, named as AEGAN, to examine whether the hyperbolic setting indeed contributes. The architecture of AEGAN is the same as HAEGAN, except that the hyperbolic layers are replaced with Euclidean ones. On the other hand, we also report the following alternative hyperbolic methods: HVAE-w and HVAE-r, where we use the same tree and graph AE but follow the ELBO loss function used by Mathieu et al. (2019) instead of having a GAN ("w" and "r" refer to using wrapped and Riemannian normal distributions, respectively); HGAN, where we train an end-to-end hyperbolic WGAN with the graph and tree decoder as the generator, and the graph and tree encoder as the critic; HAEGAN-H, HAEGAN-β, and HAEGAN-T as introduced in §5.1.

Metrics We briefly describe how the models are evaluated. Detailed descriptions of the following metrics can be found in the MOSES benchmarking platform (Polykovskiy et al., 2020). We generate a set of 30,000 molecules, which we call the "generated set". We first report the *Validity* metrics, which indicates the percentage of valid molecules in the generated set. Then, we also evaluate the following structure-related metrics by comparing the generated set with the test set and the scaffold set: Fréchet ChemNet Distance (*FCD*), Similarity to a Nearest Neighbor (*SNN*) and the Scaffold similarity (*Scaf*). FCD is the difference in the distribution of the last layer of ChemNet. SNN is the average Tanimoto similarly (Tanimoto, 1958) between the generated molecule and its nearest neighbor in the reference set. Scaf is cosine distances between the scaffold frequency vectors (Bemis & Murcko, 1996) of the generated and reference sets. In particular, SNN compares the detailed structures while Scaf compares the skeleton structures. By considering them with both the test and the scaffold test sets, we measure both the structural similarity to training data and the capability of searching for novel structures.

5.2.2 Results

We report in Table 2 the performance of HAEGAN and the baselines. For each metric described above, we take the mean and standard deviation from three independent samples. We use bold font to highlight the best performing model in each criterion.

First of all, HAEGAN achieves perfect validity scores, which implies the hyperbolic embedding adopted by HAEGAN does not break the rule of molecule structures and does not induce mode collapse. Moreover, our model significantly outperforms the baseline models in the SNN metric. This means that the molecules

Table 2: Performance in Validity, FCD, SNN, and Scaf metrics. Reported (mean ± std) over three independent samples. HVAE-w, HVAE-r, HGAN, HAEGAN-H, HAEGAN-β, HAEGAN-T are not included in the table as they all produce "NaN".

Model	Validity (↑)	FCD (↓)		SNN (↑)		Scaf (↑)	
		Test	TestSF	Test	TestSF	Test	TestSF
Train	*1*	*0.008*	*0.476*	*0.6419*	*0.5859*	*0.9907*	*0*
HMM	0.076±0.032	24.466±2.5251	25.431±2.5599	0.388±0.011	0.380±0.011	0.207±0.048	0.049±0.018
NGram	0.238±0.003	5.507±0.1027	6.231±0.0966	0.521±0.001	0.499±0.001	0.530±0.016	0.098±0.014
Combinatorial	**1.0±0.0**	4.238±0.037	4.511±0.0274	0.451±0.001	0.439±0.001	0.445±0.006	0.087±0.003
CharRNN	0.975±0.026	**0.073±0.0247**	**0.520±0.0379**	0.602±0.021	0.565±0.014	0.924±0.006	0.110±0.008
AAE	0.937±0.034	0.556±0.2033	1.057±0.2375	0.608±0.004	0.568±0.005	0.902±0.038	0.079±0.009
VAE	0.977±0.001	0.099±0.0125	0.567±0.0338	0.626±0.001	0.578±0.001	0.939±0.002	0.059±0.010
JTVAE	**1.0±0.0**	0.297±0.0087	0.828±0.0117	0.548±0.008	0.519±0.007	0.896±0.004	0.101±0.011
LatentGAN	0.897±0.003	0.395±0.0234	0.938±0.0531	0.537±0.001	0.513±0.001	0.887±0.001	0.107±0.010
GraphINVENT	0.964	0.682	1.223	0.569	0.539	0.885	0.127
DiGress	0.857	NA	1.19	NA	0.52	NA	0.148
cMolGPT	0.988	NA	NA	0.619	0.578	NA	NA
Group SELFIES	**1.0±0.0**	0.179±0.0029	0.734±0.0109	0.605±0.004	0.559±0.003	**0.965±0.002**	0.061±0.065
HAEGAN (Ours)	**1.0±0.0**	0.216±0.0044	0.751±0.0021	**0.658±0.003**	**0.613±0.003**	0.897±0.003	**0.139±0.005**
AEGAN (Ours)	**1.0±0.0**	0.381±0.0082	0.869±0.0074	0.459±0.006	0.452±0.006	0.203±0.004	0.058±0.008

generated by our model have a closer similarity to the reference set. It implies that our model better captures the underlying structure of the molecules and our hyperbolic latent space is more suitable for embedding molecules than its Euclidean counterparts. Our model also achieves competitive performance in the Scaf metric when the reference set is the scaffold test set. Given the small FCD, this shows that our model is better at searching on the manifold of scaffolds and can generate examples with novel core structures.

Next, although AEGAN can also achieve very good performance in validity, uniqueness, and novelty, we notice the big margin HAEGAN has over AEGAN in the structure-related metrics. This suggests that working with the hyperbolic space is necessary in our approach and the hyperbolic space better represents structural information.

Lastly, the alternative hyperbolic models all suffer from numerical instability and training reports NaN. This is not surprising since hyperbolic neural operations are known to easily make training unstable, especially in deep and complex networks. The result reveals the stronger numerical stability of HAEGAN, which highlights the importance of (1) the overall framework of HAEGAN (v.s. HVAE and HGAN); (2) the fully hyperbolic layers (v.s. HAEGAN-H); (3) the Lorentz direct concatenation (v.s. HAEGAN-β, HAEGAN-T).

6 Conclusion and Limitations

We proposed HAEGAN, a hybrid generative framework capable of generating both synthetic hyperbolic data and real molecular data with comparable performance in structure-related metrics. It is an effective hyperbolic generative model that accommodates numerically stable deep architectures. This is attributed to the following: first, the overall hybrid framework; second, the fully hyperbolic operations performed in the Lorentz space; third, the direct concatenation. We expect that HAEGAN can be applied to broader scenarios due to the flexibility in designing the hyperbolic AE and the possibility of building deep models.

Despite the promising results, we point out two possible limitations of the current model. First, not all complex modules are directly compatible with HAEGAN. Indeed, if the gated recurrent units (GRU) were used in our molecular generation task, the complex structure of GRU would cause unstable training (we observed NaN in experiments) and that is why a hyperbolic linear layer is used instead. Nevertheless, we expect defining more efficient hyperbolic operations that incorporate recurrent operations may alleviate the problem and leave it to future work. Second, although the hyperbolic operations in HAEGAN do not require going back and forth between the hyperbolic and the tangent spaces, we need to use exponential maps when sampling from the wrapped normal distribution. We will also work on more efficient ways of sampling from the hyperbolic Gaussian.

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A Preliminaries

A.1 Hyperbolic Geometry

We describe some fundamental concepts in hyperbolic geometry related to this work.

The Lorentz Model The Lorentz model $\mathbb{L}_K^n = (\mathcal{L}, \mathfrak{g})$ of an n dimensional hyperbolic space with constant negative curvature K is an n-dimensional manifold $\mathcal{L}$ embedded in the $(n+1)$-dimensional Minkowski space, together with the Riemannian metric tensor $\mathfrak{g} = \mathrm{diag}([-1, \mathbf{1}_n^\top])$, where $\mathbf{1}_n$ denotes the n-dimensional vector whose entries are all 1's. Every point in $\mathbb{L}_K^n$ is represented by $\boldsymbol{x} = \begin{bmatrix} x_t \\ \boldsymbol{x}_s \end{bmatrix}, x_t > 0, \boldsymbol{x}_s \in \mathbb{R}^n$ and satisfies $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\mathcal{L}} = 1/K$, where $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is the Lorentz inner product induced by $\mathfrak{g}$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{L}} \coloneqq \boldsymbol{x}^\top \mathfrak{g} \boldsymbol{y} = -x_t y_t + \boldsymbol{x}_s^\top \boldsymbol{y}_s, \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n. \tag{14}$$

Geodesics and Distances Geodesics are shortest paths in a manifold, which generalize the notion of "straight lines" in Euclidean geometry. In particular, the length of a geodesic in $\mathbb{L}_K^n$ (the "distance") between $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$ is given by

$$d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{\sqrt{-K}} \cosh^{-1}(K \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{L}}). \tag{15}$$

Tangent Space For each point $\boldsymbol{x} \in \mathbb{L}_K^n$, the tangent space at $\boldsymbol{x}$ is $\mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n \coloneqq \{\boldsymbol{y} \in \mathbb{R}^{n+1} \mid \langle \boldsymbol{y}, \boldsymbol{x} \rangle_{\mathcal{L}} = 0\}$. It is a first order approximation of the hyperbolic manifold around a point $\boldsymbol{x}$ and is a subspace of $\mathbb{R}^{n+1}$. We denote $\|\boldsymbol{v}\|_{\mathcal{L}} = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\mathcal{L}}}$ as the norm of $\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n$.

Exponential and Logarithmic Maps The exponential and logarithmic maps are maps between hyperbolic spaces and their tangent spaces. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$ and $\boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n$, the exponential map $\exp_{\boldsymbol{x}}^K(\boldsymbol{v}) : \mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n \to \mathbb{L}_K^n$ maps tangent vectors to hyperbolic spaces by assigning $\boldsymbol{v}$ to the point $\exp_{\boldsymbol{x}}^K(\boldsymbol{v}) \coloneqq \gamma(1)$, where γ is the geodesic satisfying $\gamma(0) = \boldsymbol{x}$ and $\gamma'(0) = \boldsymbol{v}$. Specifically,

$$\exp_{\boldsymbol{x}}^K(\boldsymbol{v}) = \cosh(\phi)\boldsymbol{x} + \sinh(\phi)\frac{\boldsymbol{v}}{\phi}, \phi = \sqrt{-K}\|\boldsymbol{v}\|_{\mathcal{L}}. \tag{16}$$

The logarithmic map $\log_{\boldsymbol{x}}^K(\boldsymbol{y}) : \mathbb{L}_K^n \to \mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n$ is the inverse map that satisfies $\log_{\boldsymbol{x}}^K(\exp_{\boldsymbol{x}}^K(\boldsymbol{v})) = \boldsymbol{v}$. Specifically,

$$\log_{\boldsymbol{x}}^K(\boldsymbol{y}) = \frac{\cosh^{-1}(\psi)}{\sqrt{-K}} \frac{\boldsymbol{y} - \psi \boldsymbol{x}}{\|\boldsymbol{y} - \psi \boldsymbol{x}\|_{\mathcal{L}}}, \psi = K \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{L}}. \tag{17}$$

Parallel Transport For two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}_K^n$, the parallel transport from $\boldsymbol{x}$ to $\boldsymbol{y}$ defines a map $\mathrm{PT}_{\boldsymbol{x} \to \boldsymbol{y}}^K$, which "transports" a vector from $\mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n$ to $\mathcal{T}_{\boldsymbol{y}} \mathbb{L}_K^n$ along the geodesic from $\boldsymbol{x}$ to $\boldsymbol{y}$. Parallel transport preserves the metric, i.e. $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathcal{T}_{\boldsymbol{x}} \mathbb{L}_K^n, \left\langle \mathrm{PT}_{\boldsymbol{x} \to \boldsymbol{y}}^K(\boldsymbol{v}), \mathrm{PT}_{\boldsymbol{x} \to \boldsymbol{y}}^K(\boldsymbol{u}) \right\rangle_{\mathcal{L}} = \langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\mathcal{L}}$. In particular, the parallel transport in $\mathbb{L}_K^n$ is given by

$$\mathrm{PT}_{\boldsymbol{x} \to \boldsymbol{y}}^K(\boldsymbol{v}) = \frac{\langle \boldsymbol{y}, \boldsymbol{v} \rangle_{\mathcal{L}}}{-1/K - \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\mathcal{L}}}(\boldsymbol{x} + \boldsymbol{y}). \tag{18}$$

A.2 Additional Hyperbolic Neural Operations

A.2.1 Hyperbolic Aggregation Operations

In hyperbolic space, the "average" operation is not well-defined, i.e. the Euclidean mean does not guarantee the result is still in hyperbolic space. There are two generalizations of Euclidean mean in hyperbolic space, the Fréchet Mean and the Hyperbolic Centroid.

Fréchet Mean Fréchet Mean is defined as the direct generalization of Euclidean mean for any metric space. In hyperbolic space, it can be defined as:

$$\boldsymbol{\mu}_0^* = \underset{\boldsymbol{\mu}\in\mathbb{L}_K^n}{\arg\min} \sum_{i=1}^{N} \nu_i [d_{\mathcal{L}}^K(\boldsymbol{x}_i, \boldsymbol{\mu})]^2, \tag{19}$$

However, the Fréchet mean does not have a closed form solution in hyperbolic space. There are works for calculating the hyperbolic Fréchet mean iteratively (Lou et al., 2020; Gu et al., 2019), but they were quite inefficient and hard to pass gradients.

Hyperbolic Centroid The notion of a centroid is extended to $\mathbb{L}_K^n$ by Law et al. (2019), defined to be the point $\boldsymbol{\mu}^*$ that minimizes a weighted sum of the squared Lorentzian distance:

$$\boldsymbol{\mu}^* = \underset{\boldsymbol{\mu}\in\mathbb{L}_K^n}{\arg\min} \sum_{i=1}^{N} \nu_i d_{\mathcal{L}}^2(\boldsymbol{x}_i, \boldsymbol{\mu}), \tag{20}$$

where $\{\boldsymbol{x}_i\}_{i=1}^N$ is the set of points to aggregate, $\boldsymbol{\nu}$ is the weight vector whose entries satisfy $\nu_i \geq 0$, $\sum_i \nu_i > 0$, $i = 1, \cdots, N$. The squared Lorentzian distance is defined as

$$d_{\mathcal{L}}^2(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_{\mathcal{L}}^2 = 2/K - 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle_{\mathcal{L}} \tag{21}$$

It satisfies all the axioms of a distance metric except the triangle inequality. Thus, it is a Fréchet mean of a pseudo-hyperbolic space. Fortunately, this centroid has a closed form solution given by

$$\mathrm{HCent}(\{\boldsymbol{x}_i\}_{i=1}^N, \boldsymbol{\nu}) = \boldsymbol{\mu}^* = \frac{\sum_{i=1}^N \nu_i \boldsymbol{x}_i}{\sqrt{-K}\left|\|\sum_{i=1}^N \nu_i \boldsymbol{x}_i\|_{\mathcal{L}}\right|}. \tag{22}$$

A.2.2 Hyperbolic Graph Neural Network

Given the hyperbolic linear layers and hyperbolic aggregation, it is straightforward to generalize the Euclidean Graph Convolutional Networks (Kipf & Welling, 2017) to hyperbolic space (Chami et al., 2019; Chen et al., 2021):

$$\boldsymbol{x}_v^{(l)} = \mathrm{HGCN}(\boldsymbol{X}^{(l-1)})_v = \mathrm{HCent}(\{\mathrm{HLinear}_{d_{l-1},d_l}(\boldsymbol{x}_u^{(l-1)}) \mid u \in N(v)\}, \mathbf{1}) \tag{23}$$

where $\boldsymbol{x}_v^{(l)}$ is the feature of node v in layer l, d_l denotes the dimensionality of layer l, and $N(v)$ is the set of neighbor points of node v.

Since hyperbolic space is better at embedding hierarchal data, graphs with Gromov hyperbolicity δ (Sonthalia & Gilbert, 2020) similar to that of a hyperboloid can we naturally embedded in hyperbolic space.

A.2.3 Hyperbolic-Euclidean Conversion Layers

Sometimes, it is necessary to find a learnable way to convert data between hyperbolic and Euclidean space. In this section, we introduce some learnable layers to convert between them.

Euclidean to Hyperbolic It is possible that a dataset is originally represented as Euclidean, albeit having a hierarchical structure. In this case, the most obvious way of processing is to view the data in the tangent space and use the exponential or logarithmic maps to transform it to hyperbolic space. In order to map $\boldsymbol{t} \in \mathbb{R}^n$ to the hyperbolic space $\mathbb{L}_K^m$, Nagano et al. (2019) add a zero padding to the front of $\boldsymbol{t}$ to make it a vector in $\mathcal{T}_{\boldsymbol{o}}\mathbb{L}_K^m$, and then apply the exponential map. This *Euclidean to Hyperbolic* (E2H) operation was originally used for sampling, but can also be generally used to map from the Euclidean to the hyperbolic spaces. Specifically,

$$\boldsymbol{y} = \mathrm{E2H}_m(\boldsymbol{t}) = \exp_{\boldsymbol{o}}^K\left(\left[\begin{smallmatrix}0\\ \boldsymbol{t}\end{smallmatrix}\right]\right), \tag{24}$$

where $\boldsymbol{o} = \left[\sqrt{-1/K}, 0, \ldots, 0\right]^\top$ is the hyperbolic origin.

For better expressivity, especially when the input $\boldsymbol{x} \in \mathbb{R}^n$ is one-hot, one can first map the input to a hidden embedding $\boldsymbol{h} = W\boldsymbol{x} \in \mathbb{R}^m$ with a trainable embedding matrix $W \in \mathbb{R}^{n \times m}$. Then, it is mapped to hyperbolic space by the E2H operation defined as above. That is,

$$\boldsymbol{y} = \mathrm{HEmbed}_{n,m}(\boldsymbol{x}) = \mathrm{E2H}\left(W\boldsymbol{x}\right). \tag{25}$$

This layer was previously used by Nagano et al. (2019) for word embedding.

Hyperbolic to Euclidean Some machine learning tasks requires Euclidean output (such as classification), thus it is necessary to construct a layer to transform Hyperbolic features to Euclidean. It is possible to directly use logarithmic map to convert hyperbolic features to the tangent space and view that as Euclidean features. However, it is hard to pass gradients through logarithmic map. Another way to extract Euclidean features is by calculating distances, since distances are intrinsically Euclidean.

The Hyperbolic Centroid Distance Layer (Liu et al., 2019) is a numerically stable way of converting Hyperbolic features to Euclidean. It maps points from $\mathbb{L}_K^n$ to $\mathbb{R}^m$. Given an input $\boldsymbol{x} \in \mathbb{L}_K^n$, it first initializes m trainable centroids $\{\boldsymbol{c}_i\}_{i=1}^m \subset \mathbb{L}_K^n$, then produces a vector of distances

$$\boldsymbol{y} = \mathrm{HCDist}_{n,m}(\boldsymbol{x}) = [d_{\mathcal{L}}^K(\boldsymbol{x}, \boldsymbol{c}_1) \cdots d_{\mathcal{L}}^K(\boldsymbol{x}, \boldsymbol{c}_m)]^\top, \tag{26}$$

B Related Works

Machine Learning in Hyperbolic Spaces A central topic in machine learning is to find methods and architectures that incorporate the geometric structure of data (Bronstein et al., 2021). Due to the data representation capacity of the hyperbolic space, many machine learning methods have been designed for hyperbolic data. Such methods include hyperbolic dimensionality reduction (Chami et al., 2021) and kernel hyperbolic methods (Fang et al., 2021). Besides these works, deep neural networks have also been proposed in the hyperbolic domain. One of the earliest such models was the Hyperbolic Neural Network (Ganea et al., 2018) which works with the Poincaré ball model of the hyperbolic space. This was recently refined in the Hyperbolic Neural Network ++ (Shimizu et al., 2021). Another popular choice is to use the Lorentz model of the hyperbolic space (Chen et al., 2021; Yang et al., 2022). Our model also uses Lorentz space for numerical stability.

Hyperbolic Graph Neural Networks Graph neural networks (GNNs) are successful models for learning representations of graph data. Recent studies (Boguná et al., 2010; Krioukov et al., 2010; Sala et al., 2018; Sonthalia & Gilbert, 2020) have found that hyperbolic spaces are suitable for tree-like graphs and a variety of hyperbolic GNNs (Chami et al., 2019; Liu et al., 2019; Bachmann et al., 2020; Dai et al., 2021a; Chen et al., 2021) have been proposed. In particular, Chami et al. (2019); Liu et al. (2019); Bachmann et al. (2020) all performed message passing, the fundamental operation in GNNs, in the tangent space of the hyperbolic space. On the other hand, Dai et al. (2021a); Chen et al. (2021) designed fully hyperbolic operations so that message passing can be done completely in the hyperbolic space. Some recent works address special GNNs. For instance, Sun et al. (2021) applied a hyperbolic time embedding to temporal GNN, while Zhang et al. (2021) designed a hyperbolic graph attention network. We also notice the recent survey on hyperbolic GNNs by Yang et al. (2022).

Hyperbolic Generative Models Generative neural networks in the Euclidean domain cannot embed information of the hyperbolic geometry. To address that, Nagano et al. (2019) designs a wrapped normal distribution in the hyperbolic space which enables taking gradient and uses it as the latent distribution of a variational autoencoder (VAE). Mathieu et al. (2019) considers both the wrapped normal distribution and maximum entropy normal distribution and uses them to construct an VAE on the Poincaré space. Dai et al. (2021b) also builds a VAE in the Poincaré space, which uses the primal-dual formulation of the KullbackLeibler (KL) divergence. Cho et al. (2024) proposes a pseudo-Gaussian manifold normal distribution for the latent space of VAE. Other than using VAE, other generation frameworks are also adapted to the hyperbolic space. For instance, Bose et al. (2020) lifts normalizing flows on the tangent plane of the hyperbolic space for generation; while Wen et al. (2024); Fu et al. (2024) adapts diffusion models to the hyperbolic space. Lazcano et al. (2021) uses hyperbolic linear layers in GAN for image generation. Despite the importance of the GAN framework, we are unaware of other hyperbolic GAN models. Our proposed model contains a hyperbolic encoder-decoder to learn the graph-to-graph mapping, as well as a hyperbolic GAN for generating latent embeddings, whose generator uses the wrapped normal distribution as input.

Molecular Generation State-of-the-art methods for molecular generation usually treat molecules as abstract graphs whose nodes represent atoms and edges represent chemical bonds. Early methods for molecular graph generations usually generate adjacency matrices via simple multilayer perceptrons (Simonovsky & Komodakis, 2018; De Cao & Kipf, 2018). Jin et al. (2018; 2019) proposed to treat a molecule as a multiresolution representation, with a junction-tree scaffold, whose nodes represent valid molecular substructures. Other molecular graph generation methods include You et al. (2018a); Shi* et al. (2020); Mercado et al. (2021); Vignac et al. (2023); Cheng et al. (2023). Methods that work with SMILES (Simplified Molecular Input Line Entry System) notations instead of graphs include Segler et al. (2018); Gómez-Bombarelli et al. (2018); Blaschke et al. (2018); Kadurin et al. (2017a;b); Polykovskiy et al. (2018); Prykhodko et al. (2019); Wang et al. (2023). In particular, Prykhodko et al. (2019) proposed an AE-GAN architecture which is similar to ours, though they did not use the graph structure of the molecules. Since the hyperbolic space is promising for tree-like structures, hyperbolic GNNs have also been recently used for molecular generation (Liu et al., 2019; Dai et al., 2021a).

C Additional Analysis of Concatenation

C.1 Proof of Theorem 3.1

Proof. 1. First, consider the case where $y*$ is one of the spatial components of $\boldsymbol{y}$. According to (6), y^* is a copy of an entry in $\boldsymbol{x}_{i_s}$ for some $i \in \{1, \cdots, N\}$. Therefore, if $i \neq j$, $\partial y^*/\partial \boldsymbol{x}_{j_s}$ is a zero vector; if $i = j$, $\partial y^*/\partial \boldsymbol{x}_{j_s}$ a one-hot vector. In both cases, $\|\partial y^*/\partial \boldsymbol{x}_{j_s}\| \leq 1$.

Next, consider the case where $y^* = y_t$ is the time component of $\boldsymbol{y}$. According to (6), $\partial y^*/\partial \boldsymbol{x}_{j_s} = \dfrac{\boldsymbol{x}_{j_s}}{\sqrt{\sum_{i=1}^N \|\boldsymbol{x}_{i_s}\|^2 - \frac{1}{K}}}$ and thus $\left\|\dfrac{\partial y^*}{\partial \boldsymbol{x}_{j_s}}\right\| = \dfrac{\|\boldsymbol{x}_{j_s}\|}{\sqrt{\sum_{i=1}^N \|\boldsymbol{x}_{i_s}\|^2 - \frac{1}{K}}} \leq 1$ since the curvature $K < 0$.

We conclude that $\|\partial y^*/\partial \boldsymbol{x}_{j_s}\| \leq 1$ for any entry y^* in $\boldsymbol{y}$.

2. Write $\boldsymbol{z} = \left[z_t, \boldsymbol{z}_{1_s}^\top, \cdots, \boldsymbol{z}_{N_s}^\top\right]^\top$. According to the definition of the Lorentz tangent concatenation as well as formulas (16) and (17), for $l \in \{1, \cdots, N\}$, the p-th entry of the vector $\boldsymbol{z}_{l_s}$ can be written in the following concrete form:

$$\begin{aligned} z_{l_s p} &= \frac{\sinh(\Delta)}{\Delta} \frac{\cosh^{-1}\left(-K\sqrt{\|\boldsymbol{x}_{l_s}\|^2 - \frac{1}{K}}\right)}{\sqrt{-K}} \frac{x_{l_s p}}{\|\boldsymbol{x}_{l_s}\|} \\ &= \frac{\sinh(\Delta)}{\Delta} \frac{\sinh^{-1}\left(\sqrt{K^2\|\boldsymbol{x}_{l_s}\|^2 - K - 1}\right)}{\sqrt{-K}} \frac{x_{l_s p}}{\|\boldsymbol{x}_{l_s}\|}, \end{aligned} \tag{27}$$

where

$$\Delta = \sqrt{-K}\left(\sum_{i=1}^N \left(\sinh^{-1}\left(\sqrt{K^2\|\boldsymbol{x}_{i_s}\|^2 - K - 1}\right)\right)^2\right)^{1/2}. \tag{28}$$

Let $j \neq l$. Differentiating $z_{l_s p}$ with respect to $x_{j_s k}$, the k-th entry of $\boldsymbol{x}_{j_s}$, yields

$$\begin{aligned} \frac{\partial z_{l_s p}}{\partial x_{j_s k}} &= C_l\left(\frac{\cosh(\Delta)}{\Delta^2} - \frac{\sinh(\Delta)}{\Delta^3}\right) \cdot \sinh^{-1}\left(\sqrt{K^2\|\boldsymbol{x}_{j_s}\|^2 - K - 1}\right) \cdot \\ &\quad \frac{1}{\sqrt{\|\boldsymbol{x}_{j_s}\|^2 - \frac{1}{K}}} \frac{x_{j_s k}}{\sqrt{\|\boldsymbol{x}_{j_s}\|^2 - \frac{1+K}{K^2}}}, \end{aligned} \tag{29}$$

where

$$C_l = \sinh^{-1}\left(\sqrt{K^2\|\boldsymbol{x}_{l_s}\|^2 - K - 1}\right) \frac{x_{l_s p}}{\|\boldsymbol{x}_{l_s}\|} \tag{30}$$

does not depend on $\boldsymbol{x}_{j_s}$.

Arbitrarily take fixed $\boldsymbol{x}_{i_s}$ for all $i \neq l$ with particularly $x_{j_s k} > 0$. Also arbitrarily take fixed $x_{l_s q}$ for $q \neq p$. We claim that $\dfrac{\partial z_{l_s p}}{\partial x_{j_s k}} \to \infty$ as $x_{l_s q} \to \infty$.

To prove the claim, first note that

$$\begin{aligned} \frac{\partial C_l}{\partial x_{l_s p}} &= \frac{x_{l_s p}^2}{\|\boldsymbol{x}_{l_s}\|^2 \sqrt{\|\boldsymbol{x}_{j_s}\|^2 - \frac{1}{K}}\sqrt{\|\boldsymbol{x}_{j_s}\|^2 - \frac{1+K}{K^2}}} + \\ &\quad \left(1 - \frac{x_{l_s p}^2}{\|\boldsymbol{x}_{l_s}^2\|}\right) \frac{\sinh^{-1}\left(\sqrt{K^2\|\boldsymbol{x}_{l_s}\|^2 - K - 1}\right)}{\|\boldsymbol{x}_{l_s}\|}. \end{aligned} \tag{31}$$

Since $x_{l_s p}^2 \leq \|\boldsymbol{x}_{l_s}\|^2$ and $\frac{\sinh^{-1}\left(\sqrt{K^2\|\boldsymbol{x}_{l_s}\|^2 - K - 1}\right)}{\|\boldsymbol{x}_{l_s}\|} > 0$, the second term in (31) is positive. Consequently, $\partial C_l/\partial x_{l_s p} > 0$ and thus C_l is a positive term that increases with $x_{l_s p}$. Moreover,

$\sinh^{-1}\left(\sqrt{K^2\left\|\boldsymbol{x}_{j_s}\right\|^2-K-1}\right)\frac{1}{\sqrt{\left\|\boldsymbol{x}_{j_s}\right\|^2-1/K}}\frac{x_{j_s k}}{\sqrt{\left\|\boldsymbol{x}_{j_s}\right\|^2-\frac{1+K}{K^2}}}$ in (29) is a fixed positive term that does not depend on $x_{l_s p}$. Hence, we only need to show that $\frac{\cosh(\Delta)}{\Delta^2}-\frac{\sinh(\Delta)}{\Delta^3}\to\infty$ as $x_{l_s p}\to\infty$. Since $\Delta\to\infty$ as $x_{l_s p}\to\infty$, this reduces to proving $f(t)=\frac{\cosh(t)}{t^2}-\frac{\sinh(t)}{t^3}\to\infty$ as $t\to\infty$, which is an immediate result once we write explicitly that

$$f(t)=\frac{1}{2}\left(\frac{e^{-t}}{t^3}+\frac{e^{-t}}{t^2}+\frac{(t-1)e^t}{t^3}\right). \tag{32}$$

We conclude that for any $M>0$, there exist $\{\boldsymbol{x}_i\}_{i=1}^N$ and an entry z^* of $\boldsymbol{z}$, i.e. $z_{l_s p}$ above, for which all entries of $\partial z^*/\partial\boldsymbol{x}_{j_s}$ are no less than M. Thus it also holds that $\|\partial z^*/\partial\boldsymbol{x}_{j_s}|_{\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N}\|\geq M$. □

C.2 Effect on Hyperbolic Distances

In this section, we perform additional analysis of Lorentz direct concatenation and Lorentz tangent concatenation, particularly their effect on hyperbolic distances.

Distances to Origin First, we study the hyperbolic distances to the hyperbolic origin for both concatenation methods. Suppose we have $\boldsymbol{x}\in\mathbb{L}_K^n$ and $\boldsymbol{y}\in\mathbb{L}_K^m$. Let $\boldsymbol{z}=\mathrm{HCat}(\boldsymbol{x},\boldsymbol{y})\in\mathbb{L}_K^{n+m-1}$ and $\boldsymbol{z}'=\mathrm{HTCat}(\boldsymbol{x},\boldsymbol{y})\in\mathbb{L}_K^{n+m-1}$ be their hyperbolic direct concatenation and hyperbolic tangent concatenation, respectively. We compare the difference between $d_{\mathcal{L}}(\boldsymbol{z},\boldsymbol{o})$ and $d_{\mathcal{L}}(\boldsymbol{z}',\boldsymbol{o})$ as follows. Note that the distance between an arbitrary point $\boldsymbol{x}\in\mathbb{L}_K^n$ and the origin only depend on the time component:

$$d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{o})=\frac{1}{\sqrt{-K}}\cosh^{-1}(K\langle\boldsymbol{x},\boldsymbol{o}\rangle_{\mathcal{L}}) \tag{33}$$

$$=\frac{1}{\sqrt{-K}}\cosh^{-1}(-Kx_t). \tag{34}$$

Hence, the distance information is completely contained the time component. After the concatenation, the time component is $\sqrt{x_t^2+y_t^2+1/K}$. Consequently, for Lorentz direct concatenation, the distance is

$$d_{\mathcal{L}}(\boldsymbol{z},\boldsymbol{o})=\frac{1}{\sqrt{-K}}\cosh^{-1}\left(-K\sqrt{x_t^2+y_t^2+1/K}\right). \tag{35}$$

For Lorentz tangent concatenation, since both the logarithmic and exponential maps reserve the distances, one has

$$\begin{aligned}d_{\mathcal{L}}(\boldsymbol{z}',\boldsymbol{o})&=\sqrt{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{o})^2+d_{\mathcal{L}}(\boldsymbol{y},\boldsymbol{o})^2}\\&=\sqrt{\frac{1}{-K}\left(\cosh^{-2}(-Kx_t)+\cosh^{-2}(-Ky_t)\right)}.\end{aligned} \tag{36}$$

Although the hyperbolic distance $d_{\mathcal{L}}(\boldsymbol{z},o)$ is not the squared sum of $d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{o})$ and $d_{\mathcal{L}}(\boldsymbol{y},\boldsymbol{o})$, $d_{\mathcal{L}}(\boldsymbol{z},\boldsymbol{o})$ is larger than each of $d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{o})$ and $d_{\mathcal{L}}(\boldsymbol{y},\boldsymbol{o})$. On the other hand, after concatenation, $d_{\mathcal{L}}^2(\boldsymbol{z}',\boldsymbol{o})=d_{\mathcal{L}}^2(\boldsymbol{x},\boldsymbol{o})+d_{\mathcal{L}}^2(\boldsymbol{y},\boldsymbol{o})$. This relation agrees with the Euclidean concatenation. However, norm-preservation is not why concatenation works in the Euclidean domain. Therefore, we don't consider this as an advantage of the Lorentz tangent concatenation. The Lorentz direct concatenation is more efficient and stable, and no information is lost during concatenation. Therefore, it is still preferred as a neural layer.

Relative Distances More importantly, we study how concatenation changes the relative distances, which is closely related to stability. Specifically, we perform the following experiments. Given $\boldsymbol{x},\boldsymbol{y},\boldsymbol{c}\in\mathbb{L}_K^n$, let $\boldsymbol{x}_c=\mathrm{HCat}(\boldsymbol{x},\boldsymbol{c})$ be the direct-concatenated version of $\boldsymbol{x}$ and $\boldsymbol{c}$, while $\boldsymbol{y}_c=\mathrm{HCat}(\boldsymbol{y},\boldsymbol{c})$ be the direct-concatenated version of $\boldsymbol{y}$ and $\boldsymbol{c}$. Similarly we denote $\boldsymbol{x}_c'=\mathrm{HTCat}(\boldsymbol{x},\boldsymbol{c})$ and $\boldsymbol{y}_c'=\mathrm{HTCat}(\boldsymbol{y},\boldsymbol{c})$. Since the same vector $\boldsymbol{c}$ is attached to $\boldsymbol{x}$ and $\boldsymbol{y}$, we naturally hope $d_{\mathcal{L}}(\boldsymbol{x}_c,\boldsymbol{y}_c)$ and $d_{\mathcal{L}}(\boldsymbol{x}_c',\boldsymbol{y}_c')$ do not deviate much from $d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})$.

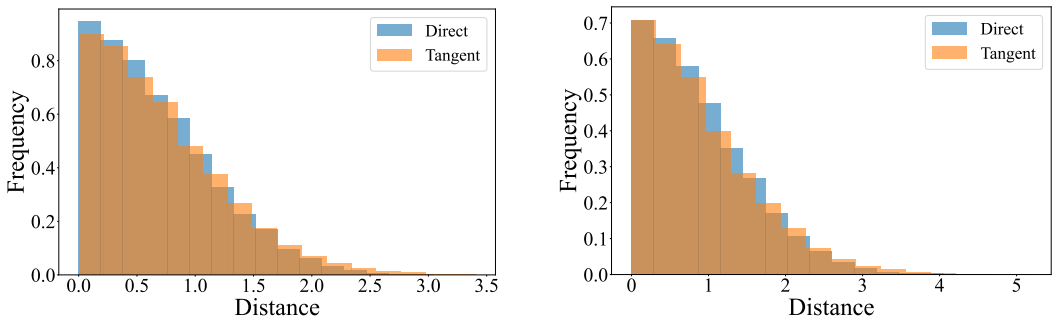

Figure 7: Difference between concatenated distances and original distances with $n = 3$. Left: spatial normal. Right: wrapped normal.

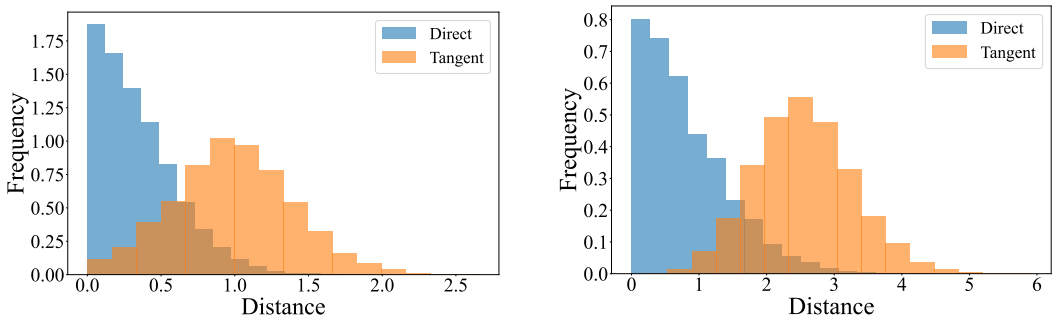

Figure 8: Difference between concatenated distances and original distances with $n = 16$. Left: spatial normal. Right: wrapped normal.

We describe our experiments as follows. Take $K = -1$. We randomly sample three points independently from $\mathbb{L}_K^n$ as $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{c}$ respectively. We have two scenarios for sampling the points: (1) "spatial normal": the points are sampled so that their spatial components follow the standard normal distribution; (2) "wrapped normal": the points are sampled from the wrapped normal distribution with unit variance. In each scenario, for $n \in \{3, 16, 64\}$, we do the experiments for 10,000 times. We report the distances $|d_{\mathcal{L}}(\boldsymbol{x}_c, \boldsymbol{y}_c) - d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})|$ and $|d_{\mathcal{L}}(\boldsymbol{x}'_c, \boldsymbol{y}'_c) - d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})|$ in Figures 7–9, as well as their differences in Figure 10.

Our experiments clearly show that, especially for large dimensions, the distance between $d_{\mathcal{L}}(\boldsymbol{x}_c, \boldsymbol{y}_c)$ and $d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})$ is smaller than the distance between $d_{\mathcal{L}}(\boldsymbol{x}'_c, \boldsymbol{y}'_c)$ and $d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})$. In particular, in many cases, $|d_{\mathcal{L}}(\boldsymbol{x}_c, \boldsymbol{y}_c) - d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})|$ is around zero. On the other hand, $|d_{\mathcal{L}}(\boldsymbol{x}'_c, \boldsymbol{y}'_c) - d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})|$ tend to be large when $n = 16, 64$, especially when samples follow the wrapped normal distribution. From this result, the Lorentz Direct Concatenation should be preferred to the Lorentz Tangent Concatenation. In particular, the significant expansion of distance when concatenating with the same vector, in the case $n = 64$, may be one cause of numerical instability.

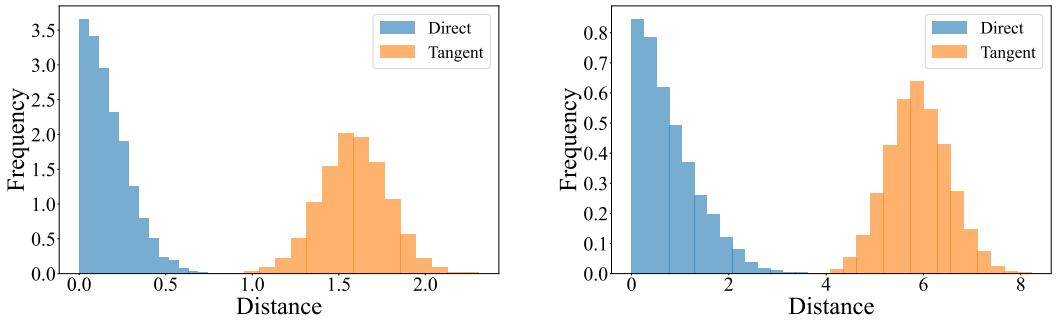

Figure 9: Difference between concatenated distances and original distances with $n = 64$. Left: spatial normal. Right: wrapped normal.

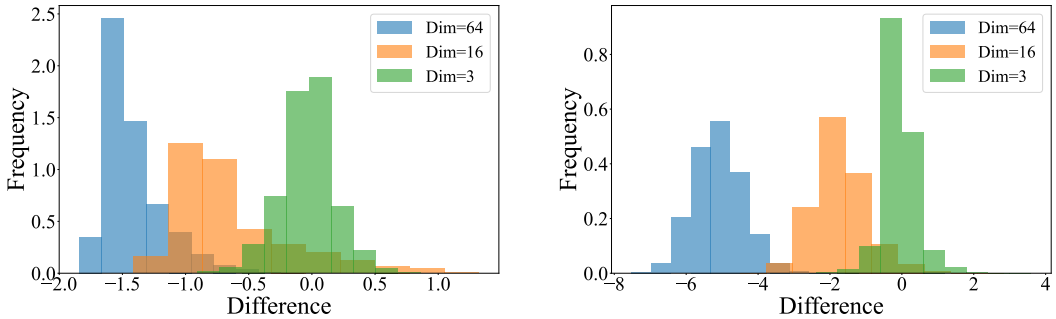

Figure 10: Illustration of $|d_{\mathcal{L}}(\boldsymbol{x}_c, \boldsymbol{y}_c) - d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})| - |d_{\mathcal{L}}(\boldsymbol{x}'_c, \boldsymbol{y}'_c) - d_{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y})|$. Left: spatial normal. Right: wrapped normal.

D Additional Details in Hyperbolic GAN

D.1 Proof of Proposition 4.1

Proof. For the optimal solution f^*, we have

$$\mathbb{P}_{(\boldsymbol{x},\boldsymbol{y})\sim\pi}\left(f^*(\boldsymbol{y}) - f^*(\boldsymbol{x}) = d_{\mathcal{L}}(\boldsymbol{y},\boldsymbol{x})\right) = 1. \tag{37}$$

Let $\psi(t) = f^*(\boldsymbol{x}_t) - f^*(\boldsymbol{x})$, $0 \leq t, t' \leq 1$. Following Gulrajani et al. (2017), it is clear that ψ is $d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})$-Lipschitz, and $f^*(\boldsymbol{x}_t) - f^*(\boldsymbol{x}) = \psi(t) = td_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})$, $f^*(\boldsymbol{x}_t) = f^*(\boldsymbol{x}) + td_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y}) = f^*(\boldsymbol{x}) + t\|\boldsymbol{v}_t\|_{\mathcal{L}}$.

Let $\boldsymbol{u}_t = \frac{\boldsymbol{v}_t}{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})} \in \mathcal{T}\mathbb{L}_K^n$ be the unit speed directional vector of the geodesic at point $\boldsymbol{x}_t$. Let $\alpha : [-1,1] \to \mathbb{L}_K^n$ be a differentiable curve with $\alpha(0) = \boldsymbol{x}_t$ and $\alpha'(0) = \boldsymbol{u}_t$. Note that $\gamma'(t) = d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})\alpha'(0)$. Therefore,

$$\lim_{h\to 0}\alpha(h) = \lim_{h\to 0}\gamma\left(t + \frac{h}{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})}\right) = \lim_{h\to 0}\boldsymbol{x}_{t+\frac{h}{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})}}. \tag{38}$$

The directional derivative can be thus calculated as

$$\begin{aligned}
&\nabla_{\boldsymbol{u}_t} f^*(\boldsymbol{x}_t) \\
&= \frac{d}{d\tau} f^*(\alpha(\tau))\Big|_{\tau=0} = \lim_{h\to 0}\frac{f^*(\alpha(h)) - f^*(\alpha(0))}{h} \\
&= \lim_{h\to 0}\frac{f^*\left(\boldsymbol{x}_{t+\frac{h}{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})}}\right) - f^*(\boldsymbol{x}_t)}{h} \\
&= \lim_{h\to 0}\frac{f^*(\boldsymbol{x}) + (t + \frac{h}{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})})d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y}) - f^*(\boldsymbol{x}) - td_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})}{h} \\
&= \lim_{h\to 0}\frac{h}{h} = 1.
\end{aligned} \tag{39}$$

Since f^* is 1-Lipschitz, we have $\|\nabla f^*(\boldsymbol{x}_t)\|_{\mathcal{L}} \leq 1$. This implies

$$\begin{aligned}
1 &\geq \|\nabla f^*(\boldsymbol{x})\|_{\mathcal{L}}^2 \\
&= \langle \boldsymbol{u}_t, \nabla f^*(\boldsymbol{x}_t)\rangle_{\mathcal{L}}^2 + \|\nabla f^*(\boldsymbol{x}_t) - \langle \boldsymbol{u}_t, \nabla f^*(\boldsymbol{x}_t)\rangle \boldsymbol{u}_t\|_{\mathcal{L}}^2 \\
&= |\nabla_{\boldsymbol{u}_t} f^*(\boldsymbol{x}_t)|^2 + \|\nabla f^*(\boldsymbol{x}_t) - \boldsymbol{u}_t\nabla_{\boldsymbol{u}_t} f^*(\boldsymbol{x}_t)\|_{\mathcal{L}}^2 \\
&= 1 + \|\nabla f^*(\boldsymbol{x}_t) - \boldsymbol{u}_t\|_{\mathcal{L}}^2 \geq 1.
\end{aligned} \tag{40}$$

Therefore, we have $1 = 1 + \|\nabla f^*(\boldsymbol{x}_t) - \boldsymbol{u}_t\|_{\mathcal{L}}^2$, $\nabla f^*(\boldsymbol{x}_t) = \boldsymbol{u}_t$. This yields $\nabla f^*(\boldsymbol{x}_t) = \frac{\boldsymbol{v}_t}{d_{\mathcal{L}}(\boldsymbol{x},\boldsymbol{y})}$.

□

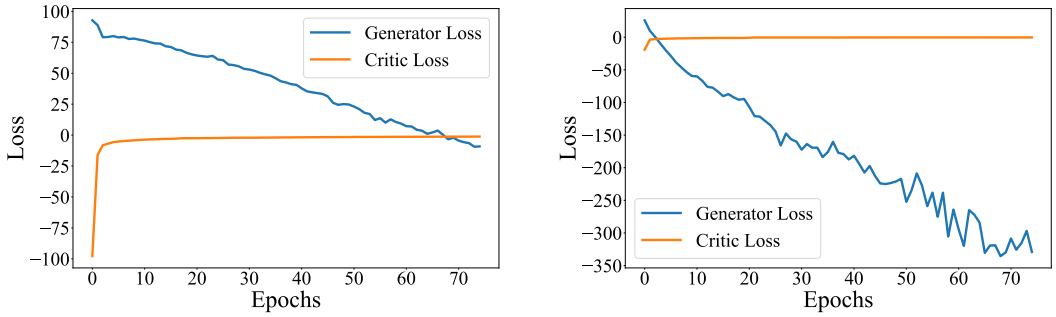

Figure 11: The training loss of generator and critic in the MNIST generation task. Left: hyperbolic Wasserstein GAN. Right: Euclidean Wasserstein GAN.

Figure 12: MNIST samples generated from HAEGAN.

E MNIST Generation Experiment

We train a HAEGAN with the MNIST dataset (LeCun et al., 2010). This serves as an important sanity check given the promise of using hyperbolic models for vision tasks (Mathieu et al., 2019; Nagano et al., 2019; Bose et al., 2020). We obverse faithfully generated examples.

E.1 Experimental Settings

Architecture In the HAEGAN for generating MNIST, the encoder of the AE consists of three convolutional layers, followed by an E2H layer and three hyperbolic linear layers, while the decoder consists of three hyperbolic linear layers, a logarithmic map to the Euclidean space, and three deconvolutional layers. We remark that in our implementation, we use the LipSwish activation function (Chen et al., 2019) in the HLinear layers for its smooth second order gradients and better Lipschitz property.

Dataset The MNIST (LeCun et al., 2010) (Modified National Institute of Standards and Technology) dataset is a widely used and well-known database of handwritten digits. It contains 60,000 training images and 10,000 test images of handwritten digits, 0 through 9. The images in the MNIST database are 28x28 pixels in size and are grayscale images.

Training We describe the training procedures as follows. Firstly, we normalize the MNIST dataset and train the AE by minimizing the reconstruction loss. Secondly, we use the encoder to embed the MNIST in hyperbolic space and train the hyperbolic GAN with the hyperbolic embedding. Finally, we sample a hyperbolic embedding using the generator and use it to produce an image by applying the decoder.

The training curves of the hyperbolic GAN of HAEGAN in the MNIST generation task are shown in Figure 11, which we compare with an Euclidean Wasserstein GAN. The critic loss includes the gradient penalty

Table 3: Quantitative comparison between HAEGAN and other methods in the MNIST generation task. Log Likelihood (±std) for different embedding dimensions is reported. 5000 samples to estimate the log-likelihood across all models. ⋆ indicates the results are replicated with ours setting with the original source code. NaN indicates numerically unstable settings.

Model	Dimensionality			
	2	5	10	20
$\mathcal{N}$-VAE (Mathieu et al., 2019)	-144.50±0.40	-114.70±0.10	-100.20±0.10	-97.60±0.10
$\mathcal{P}$-VAE (Wrapped) (Mathieu et al., 2019)	-143.80±0.60	-114.70±0.10	-100.00±0.10	-97.10±0.10
$\mathcal{P}$-VAE (Riemannian) (Mathieu et al., 2019)	-142.50±0.40	-114.10±0.20	-99.70±0.10	-97.00±0.10
Vanilla VAE (Nagano et al., 2019) ⋆	-140.12±0.53	-114.03±0.47	-99.52±0.36	-97.54±0.28
Hyperbolic VAE (Nagano et al., 2019) ⋆	-139.05±0.57	-114.25±0.46	-99.42±0.35	-97.32±0.29
$\mathcal{N}$-VAE (Bose et al., 2020) ⋆	-139.42±0.97	-115.63±0.24	-100.02±0.27	-97.53±0.36
$\mathbb{H}$-VAE (Bose et al., 2020) ⋆	NaN	-113.72±0.91	-99.82±0.23	-97.25±0.18
$\mathcal{N}$C (Bose et al., 2020) ⋆	-139.28±0.45	-115.24±0.65	-98.73±0.38	-97.05±0.27
$\mathcal{T}$C (Bose et al., 2020) ⋆	NaN	-112.55±0.29	-99.35±0.28	-97.12±0.22
$\mathcal{W}\mathbb{H}$C (Bose et al., 2020) ⋆	-137.23±1.97	-112.83±0.54	-99.45±0.26	-97.03±0.17
AEGAN (Ours)	-140.13±0.31	-110.62±0.41	-97.93±0.31	-94.19±0.53
HAEGAN (Ours)	-138.54±0.25	-110.17±0.52	-98.03±0.35	-96.62±0.61

Table 4: Quantitative comparison between HAEGAN and HGAN (Lazcano et al., 2021) in the MNIST generation task. Fréchet inception distance (±std) is reported. Results for (Lazcano et al., 2021) are taken directly from the paper.

Model	FID
HGAN (Lazcano et al., 2021)	54.95
HWGAN (Lazcano et al., 2021)	12.50
HCGAN (Lazcano et al., 2021)	12.43
HAEGAN (Ours)	8.01±0.22

term. We observe that the trend of loss in the hyperbolic GAN is similar to the Euclidean one (both the generator and critic) and no instability from the hyperbolic model.

E.2 Results

We present some generated samples in Figure 12. In Table 3 and Table 4, we report the quantitative results for the MNIST generation task. First, we compare the negative log-likelihood (NLL) results between our method and hyperbolic VAEs (Mathieu et al., 2019; Nagano et al., 2019; Bose et al., 2020). NLL is estimated with 5000 samples across all baselines and our model. The results of Mathieu et al. (2019) is directly taken from the paper, and the results of Nagano et al. (2019); Bose et al. (2020) is reproduced by us to ensure fair comparison. Then, we also calculate the FID and compare it with HGAN (Lazcano et al., 2021). Our NLL results are comparable with the hyperbolic VAEs while FID is slightly better than HGAN.

F Additional Details on Experiments

F.1 Experiment Details for MNIST Generation

We describe the detailed architecture for the MNIST Generation experiment.

F.1.1 Architecture Details of Auto-Encoder

Encoder

- Input: MNIST image with dimension (28×28)
- Convolutional Neural Network Encoder
  - Convolutional layer
    - Input channel: 1
    - Output channel: 8
    - Kernel Size: 3
    - Stride: 2
    - Padding: 1
  - LipSwish
  - Convolutional layer
    - Input channel: 8
    - Output channel: 16
    - Kernel Size: 3
    - Stride: 2
    - Padding: 1
  - Batch normalization layer (16)
  - LipSwish
  - Convolutional layer
    - Input channel: 16
    - Output channel: 32
    - Kernel Size: 3
    - Stride: 2
    - Padding: 0
  - Batch normalization layer (32)
  - LipSwish
  - Convolutional layer
    - Input channel: 32
    - Output channel: 64
    - Kernel Size: 3
    - Stride: 2
    - Padding: 0
- Map to hyperbolic space: $\mathbb{R}^{64} \to \mathbb{L}_K^{64}$
- Hyperbolic linear layers:
  - Input dimension: 64
  - Hidden dimension: 64
  - Depth: 3
  - Output dimension: 64
- Output: hyperbolic embeddings in $\mathbb{L}_K^{64}$

Decoder

- Input: hyperbolic embeddings in $\mathbb{L}_K^{64}$
- Hyperbolic linear layers:
  - Input dimension: 64
  - Hidden dimension: 64
  - Depth: 3
  - Output dimension: 64
- Map to Euclidean space: $\mathbb{L}_K^{64} \to \mathbb{R}^{64}$
- Transposed Convolutional Neural Network Decoder
  - Transposed Convolutional layer
    - Input channel: 64
    - Output channel: 64
    - Kernel Size: 3
    - Stride: 1
    - Padding: 0
    - Output Padding: 0
  - Batch normalization layer (64)
  - LipSwish
  - Transposed Convolutional layer
    - Input channel: 64
    - Output channel: 32
    - Kernel Size: 3
    - Stride: 2
    - Padding: 0
    - Output Padding: 0
  - Batch normalization layer (32)
  - LipSwish
  - Transposed Convolutional layer
    - Input channel: 32
    - Output channel: 16
    - Kernel Size: 3
    - Stride: 2
    - Padding: 1
    - Output Padding: 1
  - Batch normalization layer (16)
  - LipSwish
  - Transposed Convolutional layer
    - Input channel: 16
    - Output channel: 1
    - Kernel Size: 3
    - Stride: 2
    - Padding: 1
    - Output Padding: 1
- Output: Reconstructed MNIST image with dimension (28×28)

Hyperparameters

- Manifold curvature: $K = -1.0$
- For all hyperbolic linear layers:
  - Dropout: 0.0
  - Use bias: True
- Optimizer: Riemannian Adam ($\beta_1 = 0.9, \beta_2 = 0.9$)
- Learning Rate: 1e-4
- Batch size: 32
- Number of epochs: 20

F.1.2 Architecture Details of Hyperbolic Generative Adversarial Network

Generator

- Input: points in $\mathbb{L}_K^{128}$ sampled from $\mathcal{G}(\boldsymbol{o}, \mathrm{diag}(\mathbf{1}_{128}))$
- Hyperbolic linear layers:
  - Input dimension: 128
  - Hidden dimension: 64
  - Depth: 3
  - Output dimension: 64
- Output: points in $\mathbb{L}_K^{64}$

Critic

- Input: points in $\mathbb{L}_K^{64}$
- Hyperbolic linear layers:
  - Input dimension: 3
  - Hidden dimension: 64
  - Depth: 3
  - Output dimension: 64
- Hyperbolic centroid distance layer: $\mathbb{L}_K^{64} \to \mathbb{R}$
- Output: score in $\mathbb{R}$

Hyperparameters

- Manifold curvature: $K = -1.0$
- Gradient penalty coefficient: $\lambda = 10$
- For all hyperbolic linear layers:
  - Dropout: 0.0
  - Use bias: True
- Optimizer: Riemannian Adam ($\beta_1 = 0, \beta_2 = 0.9$)

- Learning Rate: 1e-4
- Batch size: 64
- Number of epochs: 20
- Gradient penalty λ: 10

F.2 Experiment Details for Random Tree Generation

We describe the detailed architecture and settings for the tree generation experiments.

F.2.1 Model Architecture

Tree Encoder The tree encoder for the random tree T contains the following layers. First, each node feature $\boldsymbol{x}_v$ (which is the zero vector in the random trees) is mapped to the hyperbolic space via

$$\boldsymbol{x}_v^{(0)} = \mathrm{E2H}_{d_{T_0}}(\boldsymbol{x}_v). \tag{41}$$

Next, the hyperbolic feature is passed to a hyperbolic GCN with l_T layers

$$\boldsymbol{x}^{(l)} = \mathrm{HGCN}(\boldsymbol{x}^{(l-1)}), \quad l = 1, \cdots, l_T. \tag{42}$$

Finally, we take the centroid of the embeddings of all vertices to get the hyperbolic embedding $\boldsymbol{z}_G$ of the entire tree,

$$\boldsymbol{z}_T = \mathrm{HCent}(\boldsymbol{x}^{(l_T)}). \tag{43}$$

Tree Decoder We adopted the tree decoder from Jin et al. (2018; 2019). The tree $T = (V_T, E_T)$ is generated using a tree recurrent neural network in a top-down and node-by-node fashion. The generation process resembles a depth-first traversal over the tree T. Staring from the root, at each time step t, the model makes a decision whether to continue generating a child node or backtracking to its parent node. If it decides to generate a new node, it will further predict the cluster label of the child node. It makes these decision based on the messages passed from the neighboring node. We remark that we do not use the gated recurrent unit (GRU) for message passing. The complex structure of GRU would make the training process numerically unstable for our hyperbolic neural network (we observed NaN in our experiments). We simply replace it with a hyperbolic linear layer.

Message Passing Let $\tilde{E} = \{(i_1, j_1), \ldots, (i_m, j_m)\}$ denote the collection of the edges visited in a depth-first traversal over T, where $m = 2|E_T|$. We store a hyperbolic message $\boldsymbol{h}_{i_t,j_t}$ for each edge in $\tilde{E}$. Let $\tilde{E}_t$ be the set of the first t edges in $\tilde{E}$. Suppose at time step t, the model visits node i_t and it visits node j_t at the next time step. The message $\boldsymbol{h}_{i_t,j_t}$ is updated using the node feature $\boldsymbol{x}_{i_t}$ and inward messages $\boldsymbol{h}_{k,i_t}$. We first use hyperbolic centroid to gather the inward messages to produce

$$\boldsymbol{z}_{\mathrm{nei}} = \mathrm{HCent}(\mathrm{HLinear}_{d_T,d_T}(\{\boldsymbol{h}_{k,i_t}\}_{(k,i_t)\in\tilde{E},k\neq j_t})), \tag{44}$$

and then map the tree node features to the hyperbolic space to produce

$$\boldsymbol{z}_{\mathrm{cur}} = \mathrm{HEmbed}_{d_{T_0},d_T}(\boldsymbol{x}_{i_t}). \tag{45}$$

Finally, we combine them using the Lorentz Direct Concatenation and pass them through a hyperbolic linear layer to get the message

$$\boldsymbol{h}_{i_t,j_t} = \mathrm{HLinear}_{2\times d_T,d_T}\left(\mathrm{HCat}(\{\boldsymbol{z}_{\mathrm{cur}}, \boldsymbol{z}_{\mathrm{nei}}\})\right). \tag{46}$$

Topological Prediction At each time step t, the model makes a binary decision on whether to generate a child node, using tree embedding $\boldsymbol{z}_T$, node feature $\boldsymbol{x}_{i_t}$, and inward messages $\boldsymbol{h}_{k,i_t}$ using the following layers successively:

$$\begin{aligned}
\boldsymbol{z}_{\mathrm{nei}} &= \mathrm{HCent}(\mathrm{HLinear}_{d_T,d_T}(\{\boldsymbol{h}_{k,i_t}\}_{(k,i_t)\in\tilde{E}})),\\
\boldsymbol{z}_{\mathrm{cur}} &= \mathrm{HEmbed}_{d_{T_0},d_T}(\boldsymbol{x}_{i_t}),\\
\boldsymbol{z}_{\mathrm{all}} &= \mathrm{HLinear}_{3\times d_T,d_T}\left(\mathrm{HCat}(\{\boldsymbol{z}_{\mathrm{cur}}, \boldsymbol{z}_{\mathrm{nei}}, \boldsymbol{z}_T\})\right),\\
\boldsymbol{p}_t &= \mathrm{Softmax}(\mathrm{HCDist}_{d_T,2}(\boldsymbol{z}_{\mathrm{all}})).
\end{aligned} \tag{47}$$

Label Prediction If a child node j_t is generated, we use the tree embedding $\boldsymbol{z}_T$ and the outward message $\boldsymbol{h}_{i_t,j_t}$ to predict its label. We apply the following two layers successively:

$$\begin{aligned} \boldsymbol{z}_{\text{all}} &= \text{HLinear}_{2\times d_T, d_T}\left(\text{HCat}(\{\boldsymbol{h}_{i_t,j_t}, \boldsymbol{z}_T\})\right), \\ \boldsymbol{q}_t &= \text{Softmax}(\text{HCDist}_{d_T, d_{T_0}}(\boldsymbol{z}_{\text{all}})). \end{aligned} \tag{48}$$

The output $\boldsymbol{q}_t$ is a distribution over the label vocabulary. When j_t is a root node, its parent i_t is dummy and the message is padded with the origin of the hyperbolic space $\boldsymbol{h}_{i_t,j_t} = \boldsymbol{o}$.

Training The topological and label prediction have two induced losses. Suppose $\hat{\boldsymbol{p}}_t, \hat{\boldsymbol{q}}_t$ are the the ground truth topological and label values, obtained by doing depth-first traversal on the real junction tree. The decoder minimizes the following cross-entropy loss:

$$L_{\text{topo}} = \sum_{t=1}^{m} L_{\text{cross}}(\hat{\boldsymbol{p}}_t, \boldsymbol{p}_t),\ L_{\text{label}} = \sum_{t=1}^{m} L_{\text{cross}}(\hat{\boldsymbol{q}}_t, \boldsymbol{q}_t),\ L_{\text{total}} = L_{\text{topo}} + \beta L_{\text{label}}, \tag{49}$$

where L_{cross} is the cross-entropy loss. During the training phase, we use the teacher forcing strategy: after the predictions at each time step, we replace them with the ground truth. This allows the model to learn from the correct history information.

F.2.2 Architecture Details

Tree Encoder

- Input: tree
- Hyperbolic GCN layers:
  - Input dimension: 1
  - Hidden dimension: 32
  - Depth: 2
  - Output dimension: 32
- Hyperbolic centroid on all vertices
- Output: tree embedding in $\mathbb{L}_K^{32}$

Tree Decoder

- Input: tree embedding in $\mathbb{L}_K^{32}$
- Message passing RNN:
  - Input: node feature of current tree node, inward messages
  - Hyperbolic linear layer on inward messages: $\mathbb{L}_K^{32} \to \mathbb{L}_K^{32}$
  - Hyperbolic centroid on inward messages
  - Hyperbolic linear layer: $\mathbb{L}_K^{32} \to \mathbb{L}_K^{32}$
  - Output dimension: 32
- Topological Prediction:
  - Input: tree embedding, inward messages
  - Hyperbolic linear layer on inward messages: $\mathbb{L}_K^{32} \to \mathbb{L}_K^{32}$
  - Hyperbolic centroid on inward messages
  - Lorentz Direct concatenation on inward message and tree embedding: $\mathbb{L}_K^{32} \to \mathbb{L}_K^{64}$
  - Hyperbolic linear layer: $\mathbb{L}_K^{64} \to \mathbb{L}_K^{32}$

  - Hyperbolic centroid distance layer: $\mathbb{L}_K^{32} \to \mathbb{R}^2$
  - Softmax on output
  - Output dimension: 2

- Output: tree

Hyperparameters

- Manifold curvature: $K = -1.0$
- For all hyperbolic linear layers:
  - Dropout: 0.0
  - Use bias: True
- Optimizer: Riemannian Adam ($\beta_1 = 0.0, \beta_2 = 0.999$)
- Learning rate: 5e-3
- Learning rate scheduler: StepLR (step = 20000, $\gamma = 0.5$)
- Batch size: 32
- Number of epochs: 20

F.2.3 Architecture Details of Hyperbolic Generative Adversarial Network

Generator

- Input: points sampled from wrapped normal distribution $\mathcal{G}(\boldsymbol{o}, \mathrm{diag}(\mathbf{1}_{16}))$ in $\mathbb{L}_K^{16}$
- Hyperbolic linear layers for tree embedding:
  - Input dimension: 16
  - Hidden dimension: 32
  - Depth: 2
  - Output dimension: 32
- Output: tree embedding in $\mathbb{L}_K^{32}$

Critic

- Input: tree embedding in $\mathbb{L}_K^{32}$
- Hyperbolic linear layers for tree embedding:
  - Input dimension: 32
  - Hidden dimension: 32
  - Depth: 2
  - Output dimension: 32
- Hyperbolic centroid distance layer: $\mathbb{L}_K^{32} \to \mathbb{R}$
- Output: score in $\mathbb{R}$

Hyperparameters

- Manifold curvature: $K = -1.0$
- Gradient penalty coefficient: $\lambda = 10$
- For all hyperbolic linear layers:
  - Dropout: 0.1
  - Use bias: True
- Optimizer: Riemannian Adam ($\beta_1 = 0, \beta_2 = 0.9$)
- Learning Rate: 1e-4
- Batch size: 64
- Number of epochs: 20
- Gradient penalty λ: 10

F.3 Experiment Details for Molecular Generation

We describe the detailed architecture and settings for the molecular generation experiments.

F.3.1 Model Architecture

Notation We denote a molecular graph as $G = (V_G, E_G)$, where V_G is the set of nodes (atoms) and E_G is the set of edges (bonds). Each node (atom) $v \in V_G$ has a node feature $\boldsymbol{x}_v$ describing its atom type and properties. The molecular graph is decomposed into a junction tree $T = (V_T, E_T)$ where V_T is the set of atom clusters. We use u, v, w to represent graph nodes and i, j, k to represent tree nodes, respectively. The dimensions of the node features of the graph $\boldsymbol{x}_v$ and the tree $\boldsymbol{x}_i$ are denoted by d_{G_0} and d_{T_0}, respectively. The hidden dimensions of graph and tree embeddings are d_G, d_T, respectively.

Graph and Tree Encoder The design of the graph encoder is the same as the encoder in the random tree generation experiment (§F.2.1). The graph encoder maps a graph to a latent hyperbolic feature $\boldsymbol{z}_G$:

$$\begin{aligned} \boldsymbol{x}_v^{(0)} &= \mathrm{E2H}_{d_{G_0}}(\boldsymbol{x}_v), \\ \boldsymbol{x}^{(l)} &= \mathrm{HGCN}(\boldsymbol{x}^{(l-1)}), \quad l = 1, \cdots, l_G, \\ \boldsymbol{z}_G &= \mathrm{HCent}(\boldsymbol{x}^{(l_G)}). \end{aligned} \tag{50}$$

The tree encoder is similar with the graph encoder, it encodes the junction tree to hyperbolic embedding $\boldsymbol{z}_T$ with a hyperbolic GCN of depth l_T. The only difference is that its input feature $\boldsymbol{x}_i$'s are one-hot vectors representing the atom clusters in the cluster vocabulary. We need to use a hyperbolic embedding layer as the first layer of the network accordingly.

$$\begin{aligned} \boldsymbol{x}_i^{(0)} &= \mathrm{HEmbed}_{d_{T_0}, d_T}(\boldsymbol{x}_i), \\ \boldsymbol{x}^{(l)} &= \mathrm{HGCN}(\boldsymbol{x}^{(l-1)}), \quad l = 1, \cdots, l_T, \\ \boldsymbol{z}_T &= \mathrm{HCent}(\boldsymbol{x}^{(l_T)}). \end{aligned} \tag{51}$$

Tree Decoder Similar to Jin et al. (2018; 2019), we generate a junction tree $T = (V_T, E_T)$ using a tree recurrent neural network in a top-down and node-by-node fashion. Its architecture is the same as the tree decoder in the random tree generation experiment (§F.2.1) and we will not elaborate here.

Graph Decoder The graph decoder assembles a molecular graph given a junction tree $\hat{T} = (\hat{V}, \hat{E})$ and graph embedding $\boldsymbol{z}_G$. Let $\mathcal{G}_i$ be the set of possible candidate subgraphs around tree node i, i.e. the different ways of attaching neighboring clusters to cluster i. We want to design a scoring function for each candidate subgraph $G_j^{(i)} \in \mathcal{G}_i$. To this end, we first use the hyperbolic GCN and hyperbolic centroid to acquire the hyperbolic embedding $\boldsymbol{z}_{G_j^{(i)}}$ of each subgraph $G_j^{(i)}$. Specifically,

$$\begin{aligned} \boldsymbol{x}_v^{(0)} &= \mathrm{E2H}_{d_{G_0}}(\boldsymbol{x}_v), \\ \boldsymbol{x}^{(l)} &= \mathrm{HGCN}(\boldsymbol{x}^{(l-1)}), \quad l = 1, \cdots, l_G, \\ \boldsymbol{z}_{G_j^{(i)}} &= \mathrm{HCent}(\boldsymbol{x}^{(l_G)}). \end{aligned} \tag{52}$$

Then, the embedding of the subgraph is combined with the embedding of the molecular graph $\boldsymbol{z}_G$ by the Lorentz Direct Concatenation to produce z_{all}, and gathered by a hyperbolic centroid to acquire a score

$$\begin{aligned} \boldsymbol{z}_{\text{all}} &= \mathrm{Hlinear}_{2\times d_G, d_G}(\mathrm{HCat}(\{\boldsymbol{z}_{G_j^{(i)}}, \boldsymbol{z}_G\})), \\ s_j^{(i)} &= \mathrm{HCDist}_{d_G,1}(\boldsymbol{z}_{\text{all}}) \in \mathbb{R}. \end{aligned} \tag{53}$$

Training We define the loss for the graph decoder to be the sum of the cross-entropy losses in each $\mathcal{G}_i$. Specifically, suppose the correct subgraph is $G_c^{(i)}$,

$$L_{\text{assm}} = \sum_i \left(s_c^{(i)} - \log \sum_{G_j^{(i)} \in \mathcal{G}_i} \exp(s_j^{(i)}) \right). \tag{54}$$

Similar to the tree decoder, we also use teacher forcing when training the graph decoder. We also utilize the LipSwish activation function (Chen et al., 2019) throughout the hyperbolic linear layers for stable and smooth second-order gradients.

F.3.2 Architecture Details of Hyperbolic Junction Tree Encoder-Decoder

Graph Encoder

- Input: graph node features in $\mathbb{R}^{35}$
- Map features to hyperbolic space: $\mathbb{R}^{35} \to \mathbb{L}_K^{35}$
- Hyperbolic GCN layers:
  - Input dimension: 35
  - Hidden dimension: 256
  - Depth: 4
  - Output dimension: 256
- Hyperbolic centroid on all vertices
- Output: graph embedding in $\mathbb{L}_K^{256}$

Tree Encoder

- Input: junction tree features in $\mathbb{R}^{828}$
- Hyperbolic embedding layer: $\mathbb{R}^{828} \to \mathbb{L}_K^{256}$
- Hyperbolic GCN layers:

  - Input dimension: 256
  - Hidden dimension: 256
  - Depth: 4
  - Output dimension: 256
- Hyperbolic centroid on all vertices
- Output: tree embedding in $\mathbb{L}_K^{256}$

Tree Decoder

- Input: tree embedding in $\mathbb{L}_K^{256}$
- Message passing RNN:
  - Input: node feature of current tree node, inward messages
  - Hyperbolic linear layer on inward messages: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{256}$
  - Hyperbolic centroid on inward messages
  - Hyperbolic embedding layer on node feature: $\mathbb{R}^{828} \to \mathbb{L}_K^{256}$
  - Lorentz Direct concatenation on node feature and inward message: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{512}$
  - Hyperbolic linear layer: $\mathbb{L}_K^{512} \to \mathbb{L}_K^{256}$
  - Output dimension: 256
- Topological Prediction:
  - Input: tree embedding, node feature of current tree node, inward messages
  - Hyperbolic linear layer on inward messages: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{256}$
  - Hyperbolic centroid on inward messages
  - Hyperbolic embedding layer on tree feature: $\mathbb{R}^{828} \to \mathbb{L}_K^{256}$
  - Lorentz Direct concatenation on node feature, inward message, and tree embedding: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{768}$
  - Hyperbolic linear layer: $\mathbb{L}_K^{768} \to \mathbb{L}_K^{256}$
  - Hyperbolic centroid distance layer: $\mathbb{L}_K^{256} \to \mathbb{R}^2$
  - Softmax on output
  - Output dimension: 2
- Label Prediction:
  - Input: tree embedding, outward messages
  - Lorentz Direct concatenation on outward message, and tree feature: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{512}$
  - Hyperbolic linear layer: $\mathbb{L}_K^{512} \to \mathbb{L}_K^{256}$
  - Hyperbolic centroid distance layer: $\mathbb{L}_K^{256} \to \mathbb{R}^{828}$
  - Softmax on output
  - Output dimension: 828
- Output: junction tree

Graph Decoder

- Input: junction tree, tree message, and graph embedding
- Construction candidate subgraphs
- Hyperbolic graph convolution layers on all subgraphs:

  - Input dimension: 256
  - Hidden dimension: 256
  - Depth: 4
  - Output dimension: 256
- Hyperbolic centroid on vertices of all subgraphs
- Lorentz Direct concatenation on subgraph embedding and graph embedding: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{512}$
- Hyperbolic linear layer: $\mathbb{L}_K^{512} \to \mathbb{L}_K^{256}$
- Hyperbolic centroid distance layer: $\mathbb{L}_K^{256} \to \mathbb{R}$
- Use subgraph score to construct molecular graph
- Output: molecular graph

Hyperparameters

- Manifold curvature: $K = -1.0$
- For all hyperbolic linear layers:
  - Dropout: 0.0
  - Use bias: True
- Optimizer: Riemannian Adam ($\beta_1 = 0.0, \beta_2 = 0.999$)
- Learning rate: 5e-4
- Learning rate scheduler: StepLR (step = 20000, $\gamma = 0.5$)
- Batch size: 32
- Number of epochs: 20

F.3.3 Architecture Details of Hyperbolic Generative Adversarial Network

Generator

- Input: points sampled from wrapped normal distribution $\mathcal{G}(\boldsymbol{o}, \mathrm{diag}(\mathbf{1}_{128}))$ in $\mathbb{L}_K^{128}$
- Hyperbolic linear layers for graph embedding:
  - Input dimension: 128
  - Hidden dimension: 256
  - Depth: 3
  - Output dimension: 256
- Hyperbolic linear layers for tree embedding:
  - Input dimension: 128
  - Hidden dimension: 256
  - Depth: 3
  - Output dimension: 256
- Output: graph embedding and tree embedding in $\mathbb{L}_K^{128}$

Critic

- Input: graph embedding and tree embedding in $\mathbb{L}_K^{128}$
- Hyperbolic linear layers for graph embedding:
  - Input dimension: 256
  - Hidden dimension: 256
  - Depth: 2
  - Output dimension: 256
- Hyperbolic linear layers for tree embedding:
  - Input dimension: 256
  - Hidden dimension: 256
  - Depth: 2
  - Output dimension: 256
- Lorentz Direct concatenation on graph embedding and tree embedding: $\mathbb{L}_K^{256} \to \mathbb{L}_K^{512}$
- Hyperbolic linear layer: $\mathbb{L}_K^{512} \to \mathbb{L}_K^{256}$
- Hyperbolic centroid distance layer: $\mathbb{L}_K^{256} \to \mathbb{R}$
- Output: score in $\mathbb{R}$

Hyperparameters

- Manifold curvature: $K = -1.0$
- Gradient penalty coefficient: $\lambda = 10$
- For all hyperbolic linear layers:
  - Dropout: 0.1
  - Use bias: True
- Optimizer: Riemannian Adam ($\beta_1 = 0, \beta_2 = 0.9$)
- Learning Rate: 1e-4
- Batch size: 64
- Number of epochs: 20
- Gradient penalty λ: 10