What Architectural Inductive Bias Makes Diffusion Models Succeed? A Perspective from the Implicit Regularization of Gradient Descent

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Abstract

Diffusion models trained to predict noise or velocity achieve impressive generation quality, yet this success relies on locally connected architectures such as U-Nets or patch-based Transformers. A recent study reveals that switching to a fully connected network causes noise prediction to fail completely, while merely changing the prediction target to clean data restores success even with the same fully connected architecture. These findings expose a puzzling interchangeability between the prediction target and the connectivity pattern. This paper provides a unified explanation. We show that both clean-data prediction and sparse connectivity impose the same geometric constraint on gradient descent. They confine the weight updates to a low-dimensional subspace spanned by shared data patterns, preventing the network from memorizing individual training examples. Clean-data prediction achieves this constraint from the output side. An analytic skip connection allows the network to discard high-dimensional noise, and the internal representations collapse onto the data manifold. Sparse connectivity achieves the same constraint from the input side. Each filter receives only a small local patch whose vectors are concentrated, and every gradient update is a linear combination of these low-dimensional patch vectors. We formalize the architectural route through a stability analysis and prove that the capacity bound is governed by the geometry of the patch cloud. The critical quantity is the effective dimension of the vectors that compose the gradient, regardless of whether it is reduced by the prediction target or by the architecture.

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1. Introduction

Li & He (2025) recently demonstrated a stark asymmetry in diffusion model training. A fully connected network trained to predict noise or velocity cannot learn a two-dimensional distribution embedded in $\mathbb{R}^{512}$, while predicting clean data with the identical architecture succeeds. The authors attribute this gap to the manifold hypothesis. Clean data lies on a low-dimensional manifold and can be captured by limited capacity, while noise occupies the full ambient space and exceeds it.

Mainstream diffusion practice, however, routinely predicts noise or velocity and still obtains excellent results, using U-Nets (Ronneberger et al., 2015; Dhariwal & Nichol, 2021), DiT (Peebles & Xie, 2023), or SiT (Ma et al., 2024). These architectures share a structural signature absent in the fully connected baseline. Every processing unit sees only a spatially local, low-dimensional portion of the input, a small convolutional kernel or a small image (hidden represenation) patch. This suggests that architectural inductive bias can compensate for what the noise-prediction target does not provide. The situation raises a fundamental question. Why can two seemingly unrelated design dimensions, the prediction target and the connectivity pattern, substitute for each other?

This paper answers that both are different realizations of a single mechanism. They constrain the gradient updates of gradient descent to a low-dimensional subspace that reflects shared data structure. The mechanism follows directly from the back-propagation formula. For a weight matrix $\mathbf{W}^{(k)}$ at layer k, the gradient is

$$\nabla_{\mathbf{W}^{(k)}}\mathcal{L} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\delta}_i^{(k)} \mathbf{a}_i^{(k)\top}, \tag{1}$$

where $\mathbf{a}_i^{(k)}$ is the input activation and $\boldsymbol{\delta}_i^{(k)}$ the back-propagated error. The update direction is a linear combination of the activation vectors $\mathbf{a}_i^{(k)}$. If those activations span only a low-dimensional subspace, the optimizer cannot move weights along arbitrary ambient directions. It is forced to follow directions that recur across many training examples and thus capture population-level structure. Conversely, when the activations are approximately spherical

and high-dimensional, the network can dedicate a separate direction to each training sample. This is the dynamical signature of memorization.

We show that clean-data prediction and sparse connectivity both reduce the effective dimension of the vectors that appear in the gradient, albeit at different ends of the network. Clean-data prediction operates from the output side. The analytic relationship $\hat{\boldsymbol{\epsilon}} = \mathbf{z}_t/t - ((1-t)/t)f_\theta(\mathbf{z}_t, t)$ directly feeds the high-dimensional noise to the output, freeing the network f_θ from representing it. Consequently the residual stream can progressively discard the noise components and converge toward the low-dimensional clean data manifold. We verify that under x-prediction the effective dimension of hidden representations decreases with depth, producing low-dimensional activations in later layers and therefore low-rank gradients (Section 3).

Sparse connectivity operates from the input side. Every successful score estimator processes the image through local receptive fields of dimension $m \ll d$. A filter that sees only an m-dimensional patch receives gradient updates that are linear combinations of those patch vectors. Even when the full ambient data is isotropic Gaussian in $\mathbb{R}^d$, its projection onto a small m-dimensional subspace is highly concentrated. Therefore each filter's gradient is confined to the span of the concentrated patch cloud. The architecture itself imposes the low-dimensional constraint, and the prediction target can remain the high-dimensional noise (Section 4).

Thus the two design choices converge to the same effect. One reduces the dimension of the activations from the output end; the other reduces the effective input dimension from the input end. In both cases the optimizer is geometrically prevented from memorizing individual training points, and the network is compelled to learn features that are shared across the data. We formalize this geometric principle for the architectural route using the Below-Edge-of-Stability (BEoS) framework (Cohen et al., 2021; Mulayoff et al., 2021; Wu & Su, 2023; Qiao et al., 2024). The analysis yields a patch-geometry-weighted capacity bound (Theorem 5.3) and sharply contrasting generalization and memorization regimes (Theorems 5.4 and 5.5). Section 6 connects this principle to broader trends in vision backbone design and outlines diagnostic tools for large-scale models.

2. Related Work

Diffusion and flow generative modeling. Diffusion models (Ho et al., 2020; Song et al., 2021) frame generation as iterative denoising and have been extended through improved noise schedules (Karras et al., 2022; Lin et al., 2024), latent-space formulations (Rombach et al., 2022), few-step distillation (Song et al., 2023), and continuous-time flow matching and rectified flow (Lipman et al., 2023; Liu et al., 2023). Studies that hold the objective fixed and vary architecture or training details (Karras et al., 2022; Ma et al., 2024) establish that the denoiser is a primary performance driver, but do not identify which architectural properties are responsible.

Prediction parameterization. Li & He (2025) show that clean-data prediction can rescue a large-patch Transformer that fails under noise or velocity prediction, attributing the gap to the dimensionality of the prediction target. Our analysis gives a gradient-dynamic explanation for this observation via the BEoS framework.

Architectures for denoising. U-Net (Ronneberger et al., 2015) has been the canonical denoiser since early score-based models (Song & Ermon, 2020; Dhariwal & Nichol, 2021); patch-based Transformer denoisers followed as scalable alternatives (Peebles & Xie, 2023; Bao et al., 2023; Hatamizadeh et al., 2024; Ma et al., 2024). A shared structural feature is that the first processing stage acts on a local patch of the input rather than the full vector. Kamb & Ganguli (2025) show analytically that locality and equivariance together predict the mosaic structure of trained U-Net denoisers; Niedoba et al. (2025) approximate pretrained denoisers by patch-level aggregation; An et al. (2025) find that early-layer attention locality predicts generalization in DiTs. Our work provides an optimization-theoretic basis for these empirical observations.

Edge of stability and implicit regularization. A line of work studies generalization through solutions that gradient descent can stably reach (Mulayoff et al., 2021; Nacson et al., 2023; Wu & Su, 2023; Ding et al., 2024; Qiao et al., 2024). Liang et al. (2025; 2026) identify neural shattering (stable memorization under spherical data) as the key failure mode for FCNs and prove that BEoS-based generalization bounds become vacuous as inputs concentrate near a hypersphere. We extend their framework to sparsely connected networks; the extension to locality and weight sharing in the convolutional case is also developed in Liang et al. (2025; 2026).

Generalization theory for CNNs. The advantage of CNNs over FCNs has been studied from approximation (Mhaskar et al., 2017; Mao et al., 2021; Shi et al., 2025) and statistical (Li et al., 2021; Wang & Wu, 2023; Lahoti et al., 2024) perspectives. These results typically assume explicit regularization or structural properties of the target function. Our analysis requires neither; the implicit regularization comes entirely from the stability condition.

Patch-based representation learning. Early unsupervised image representation methods decomposed images into local patches and learned sparse dictionaries or manifolds from these primitives (Aharon et al., 2006; Peyré, 2009). Patch based representations have also shown success in un-

supervised learning (Paulin et al., 2017) and as preprocessing for image classification (Coates et al., 2011; Thiry et al., 2021; Brutzkus et al., 2022). These results highlight that patch space is highly structured, but it remains open how neural network training exploits such structure. Our work helps bridge this gap by analyzing generalization below the edge of stability.

Implicit bias of gradient descent. Many existing work on the implicit bias of gradient descent relies on strong assumptions on the data distributions (e.g., linearly separable data), simplified model architectures (e.g., linear activation) to make a gradient dynamics analysis tractable (Soudry et al., 2018; Gunasekar et al., 2018a;b), or weight initialization schemes that keep the model in the kernel regimes (Jacot et al., 2018; Arora et al., 2019). While CNNs were studied (Gunasekar et al., 2018b; Arora et al., 2019), the nature of the results are different from ours.

3. Predicting Clean Data Yields Low-Dimensional Internal Representations

We now test the first half of the geometric hypothesis. If the prediction target is chosen so that the network can shed high-dimensional noise from its representations, the activations in deeper layers should become low-dimensional, and gradient descent should be regularized. We examine this using a fully connected network trained on a two-dimensional spiral embedded in $\mathbb{R}^{256}$, varying only the output parameterization.

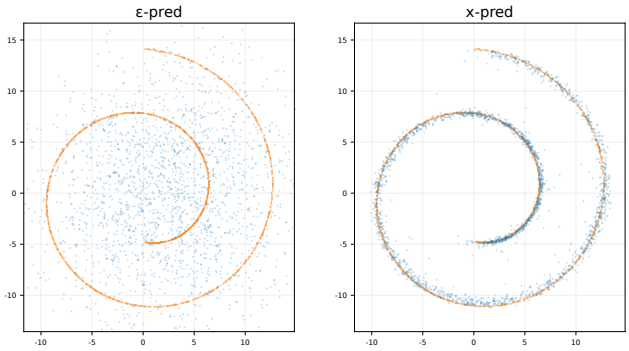
Setup. Diffusion models define a forward noising process and train a network to reverse it (Ho et al., 2020). For a clean sample $\boldsymbol{x}_0 \sim p_{\text{data}}$ and $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \boldsymbol{I})$, the corrupted input is $\boldsymbol{x}_t = \alpha_t \boldsymbol{x}_0 + \sigma_t \boldsymbol{\epsilon}$, and the noise-prediction objective is $\mathcal{L}_{\text{noise}}(\theta) = \mathbb{E}_{t,\boldsymbol{x}_0,\boldsymbol{\epsilon}}\left[\|\boldsymbol{\epsilon}_\theta(\boldsymbol{x}_t, t) - \boldsymbol{\epsilon}\|_2^2\right]$. Learning to predict $\boldsymbol{\epsilon}$ is equivalent to learning the score (Ho et al., 2020; Song et al., 2021); flow matching and rectified flow use velocity as the target with the same mathematical structure (Lipman et al., 2023; Liu et al., 2023). Throughout this section the objective is fixed; only the output parameterization changes.

3.1. The Output Skip Induced by Clean-Data Prediction

When a network predicts clean data $\boldsymbol{x}$ from a corrupted input $\boldsymbol{z}_t = (1-t)\boldsymbol{x} + t\boldsymbol{\epsilon}$, the corresponding noise predictor takes the form

$$\hat{\boldsymbol{\epsilon}} = \frac{\boldsymbol{z}_t - (1-t) f_\theta(\boldsymbol{z}_t, t)}{t} = \underbrace{\frac{1}{t}\boldsymbol{z}_t}_{\text{analytic skip}} - \frac{1-t}{t} f_\theta(\boldsymbol{z}_t, t). \tag{2}$$

The corrupted input $\boldsymbol{z}_t$ passes to the output through a fixed analytic function of t; the network branch only needs to approximate $\boldsymbol{x}$, which lies on a low-dimensional manifold. Under direct ϵ-prediction, no such decomposition exists. The full noise estimate must come from the network, and every layer must both build denoising features and preserve enough ambient information to decode the full noise direction at the output. The same observation holds for velocity prediction after the appropriate reparameterization, making x-prediction a general architectural principle.



(a) ϵ-prediction. *(b)* x-prediction.

Figure 1. **Same backbone, different output parameterization.** Two-dimensional spiral embedded in $\mathbb{R}^{256}$, projected back after sampling. ϵ-prediction spreads samples throughout the ambient space; x-prediction recovers the spiral.

3.2. Residual Stream Dimension Decreases with Depth

We verify this on a two-dimensional spiral embedded into $\mathbb{R}^{256}$ by a fixed orthogonal projection. Both models use the same 5-block residual FCN with pre-norm AdaLN-zero conditioning and zero-initialized output projection. The only change is the prediction target.

Figure 1 shows the sampling outcome. To understand why, we write the residual backbone schematically as

$$\boldsymbol{h}^0 = E\boldsymbol{z}_t, \quad \boldsymbol{h}^k = \boldsymbol{h}^0 + \sum_{\ell=1}^{k} B_\ell(\boldsymbol{h}^{\ell-1}, t), \quad f_\theta = R(\boldsymbol{h}^L),$$

and measure the effective dimension of each layer's hidden states: the number of principal components needed to explain 95% of the variance aggregated over the training distribution of t.

Under x-prediction, the effective dimension decreases with depth (Figure 2): the analytic skip in (2) frees the residual stream from carrying the high-dimensional noisy component, so each block B_ℓ only needs to improve the representation toward the low-dimensional $\boldsymbol{x}$. Under ϵ-prediction, even if early blocks extract a useful low-dimensional signal, later blocks are pressured to transform the stream back into a representation from which the full noise vector can be decoded; the effective dimension stays high throughout.

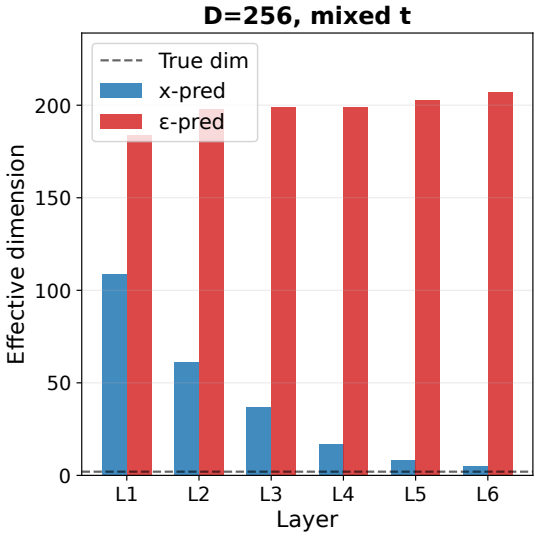


Figure 2. **Effective dimension of hidden representations across layers.** Under x-prediction the effective dimension decreases with depth as the residual stream converges toward the data manifold. Under ϵ-prediction it stays high throughout, because every layer must also carry information to decode the full ambient noise direction.

3.3. Low-Dimensional Activations Confine Gradient Updates

These activation patterns have a direct consequence for gradient descent. The gradient formula (1) shows that $\nabla_{\mathbf{W}^{(k)}}\mathcal{L}$ is a linear combination of the activation vectors $\mathbf{a}_i^{(k)}$. Under x-prediction, the activations in deeper layers lie in a low-dimensional subspace, so the gradient has low rank and can only update weights along those restricted directions. What gradient descent can find is determined by the available directions, and those directions are the representation geometry at each layer. Under ϵ-prediction the residual stream remains high-dimensional throughout, leaving the gradient unconstrained and the network free to memorize.

This is precisely the geometric regularization that the unified principle anticipates. The output-side analytic skip forces the network to discard dimensions orthogonal to the data manifold, which in turn limits the search space of the optimizer. Having verified this half of the principle, we now ask whether the same geometric confinement can be achieved from the input side, without changing the prediction target.

4. Sparse Connectivity Confines Gradient Updates to the Patch Subspace

The previous section demonstrated that collapsing the internal representations to a low-dimensional subspace regularizes gradient descent. But mainstream diffusion models

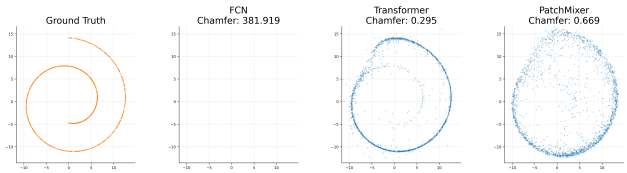


Figure 3. **Architecture determines noise prediction quality.** Same objective, same data, same training setup. FCN fails; patch-based models recover the spiral. Chamfer distance (lower is better) shown below each result.

predict noise, not clean data. They succeed by using architectures with sparse, local connectivity. We now show that such architectures impose the same geometric constraint from the opposite direction. They restrict the gradient of every filter to the span of low-dimensional local patches, regardless of the prediction target.

Consider a filter that processes an m-dimensional patch rather than the full d-dimensional input. Its weight gradient is a linear combination of the patch vectors that activate it. Formally, for a two-layer sparse network with weight sharing, the gradient with respect to filter $\mathbf{w}_k$ is

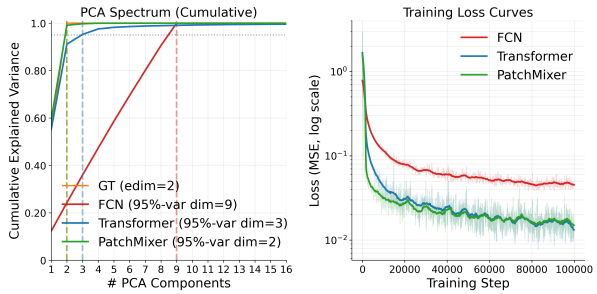
$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}_k} = \frac{v_k}{nJ} \sum_{i=1}^{n} \sum_{j=1}^{J} r_i \cdot \mathbb{1}\{\mathbf{w}_k^\top \pi_j(\mathbf{x}_i) > b_k\} \cdot \pi_j(\mathbf{x}_i), \tag{3}$$

where $\pi_j(\mathbf{x}_i)$ extracts the j-th m-dimensional patch. The update lies entirely in the span of the activated patch vectors $\pi_j(\mathbf{x}_i)$. If the cloud of training patches is concentrated in a low-dimensional region of $\mathbb{R}^m$, then the effective search space for $\mathbf{w}_k$ is drastically reduced, even when the ambient data is spherical in $\mathbb{R}^d$.

4.1. Same Objective, Different Architecture, Different Outcome

We hold the noise-prediction objective, noise schedule, optimizer, and sampling procedure fixed and vary only the denoiser architecture. The data is a two-dimensional spiral embedded into $\mathbb{R}^{256}$ by a fixed orthogonal projection $\boldsymbol{P} \in \mathbb{R}^{256\times 2}$. We compare an FCN, a patch Transformer (Dosovitskiy et al., 2021), and a PatchMixer (Tolstikhin et al., 2021). Both patch-based models partition the 256 ambient coordinates into 16 patches of size 16; the Transformer embeds each patch into a 96-dimensional token and processes the sequence with single-head self-attention; PatchMixer alternates token-mixing and channel-mixing ReLU layers.

Figures 3 and 4 show the outcome. The FCN fails to recover the spiral, spreading sample energy across many ambient dimensions and training more slowly. Patch-based models succeed, and generated samples concentrate in two or three principal components.



(a) PCA spectrum of generated samples.

(b) Training curves.

Figure 4. **Geometry and optimization under noise prediction. (a)** Ground truth is 2-dimensional. PatchMixer and Transformer reach 95% variance with 2–3 components; FCN needs 9. **(b)** FCN optimizes the same objective more slowly and plateaus at a higher value.

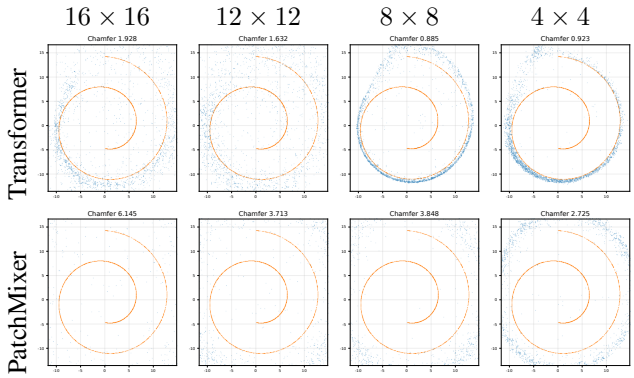


Figure 5. **Patch-size ablation for noise prediction.** A two-dimensional spiral embedded in a 32×32 ambient space; token count is fixed across all conditions. Smaller patches consistently improve sample quality; larger patches degrade toward the FCN failure mode.

4.2. The Local Dimension m Is the Critical Variable

To isolate the role of local input dimension, we embed the same spiral into a 32×32-dimensional ambient space and train Transformer and PatchMixer denoisers under four patch sizes: 4×4, 8×8, 12×12, and 16×16. In all conditions the number of patches is held constant by adjusting the stride, so the token count is identical across patch sizes and the only variable is the local input dimension m per neuron.

Figure 5 confirms the trend. Smaller patches produce samples that better concentrate on the ground-truth spiral, while larger patches (which expose each neuron to a higher-dimensional local view) produce more diffuse generations. The benefit of patch-based architectures comes from the local input dimension being sufficiently small, exactly as the gradient geometry principle predicts.

5. Patch Geometry Determines the Strength of Implicit Regularization

The patch-size experiment shows a continuous transition from success to failure as m increases, and the gradient formula (3) explains why. A small m concentrates the patch cloud and forces every filter gradient to be an average over many diverse patches. We now formalize this geometric intuition through the Below-Edge-of-Stability (BEoS) condition. Although the analysis is developed explicitly for patch-based architectures, the underlying logic applies equally to clean-data prediction. There, the low-dimensional span comes from collapsed activations instead of restricted receptive fields. In both cases the key quantity is the effective dimension of the vectors that constitute the gradient update.

5.1. Setup and the Gradient Factorization

Sparse connectivity pattern. Fix integers $d \geq 1$ and $1 \leq m \leq d$. A *sparse connectivity pattern* is a collection of coordinate subsets $\mathcal{S} = \{S_j\}_{j=1}^{J}$, each of size m, inducing projections $\pi_j : \mathbb{R}^d \to \mathbb{R}^m$. The vector $\boldsymbol{x}^{(S_j)} := \pi_j(\boldsymbol{x})$ is a *patch*. Given $\mathcal{S}$ and width $K \in \mathbb{N}$, a two-layer sparsely connected network with weight sharing (SCN) is

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sum_{k=1}^{K} v_k \left(\frac{1}{J} \sum_{j=1}^{J} \phi\big(\boldsymbol{w}_k^{\mathsf{T}} \pi_j(\boldsymbol{x}) - b_k\big) \right) + \beta, \quad (4)$$

where each unit $\phi(\boldsymbol{w}_k^{\mathsf{T}} \cdot - b_k)$ is a shared filter applied to every patch. When $m = d$ and $J = 1$, this reduces to a fully connected network.

The gradient structure. The gradient of the training loss $\mathcal{L}(\boldsymbol{\theta}) = \frac{1}{2n} \sum_i (f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)^2$ with respect to filter $\boldsymbol{w}_k$ is

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{w}_k} = \frac{v_k}{nJ} \sum_{i=1}^{n} \sum_{j=1}^{J} r_i \cdot \mathbb{1}\left\{\boldsymbol{w}_k^{\mathsf{T}} \pi_j(\boldsymbol{x}_i) > b_k\right\} \cdot \pi_j(\boldsymbol{x}_i), \quad (5)$$

where $r_i = f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i$ is the residual. Two roles separate cleanly. The gate-and-residual term aggregates the per-patch learning signal. The patch vectors $\pi_j(\boldsymbol{x}_i) \in \mathbb{R}^m$ then map this signal into parameter space: the update to $\boldsymbol{w}_k$ is a linear combination of exactly those m-dimensional patch vectors that activated the filter. The architecture therefore controls which directions gradient descent can effectively move in: if the patch cloud is confined to a low-dimensional subspace of $\mathbb{R}^m$, the gradient is confined there too, regardless of the ambient dimension d. For a fully connected network, $\pi_j(\boldsymbol{x}_i) = \boldsymbol{x}_i \in \mathbb{R}^d$, and the gradient can spread into any ambient direction.

From gradient dynamics to stability. Directly tracking gradient trajectories throughout training is analytically dif-

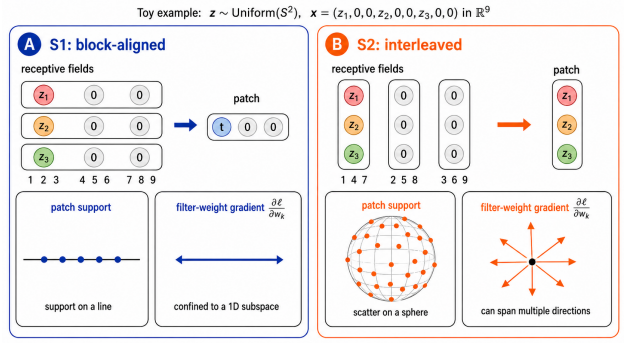


Figure 6. **Architecture × data determines gradient geometry.** The same spherical ambient data yields a one-dimensional patch cloud under $\mathcal{S}_1$ and a three-dimensional spherical cloud under $\mathcal{S}_2$. By (3), every filter gradient under $\mathcal{S}_1$ is confined to one dimension; under $\mathcal{S}_2$ it can span all three.

ficult. We instead use the below-edge-of-stability condition as a tractable proxy, as motivated in Section 3.

Definition 5.1 (Below Edge of Stability (Qiao et al., 2024, Definition 2.3))**.** A parameter $\boldsymbol{\theta}$ is *Below the Edge of Stability (BEoS)* if $\lambda_{\max}(\nabla^2 \mathcal{L}(\boldsymbol{\theta})) \leq 2/\eta$.

We study the BEoS class $\boldsymbol{\Theta}^{\mathcal{S}\mathrm{CN}}_{\mathrm{BEoS}}(\eta, \mathcal{D}) := \{\boldsymbol{\theta} \in \boldsymbol{\Theta}^{\mathcal{S}\mathrm{CN}} \mid \lambda_{\max}(\nabla^2 \mathcal{L}(\boldsymbol{\theta})) \leq 2/\eta\}$ and ask how the receptive field system $\mathcal{S}$ changes its effective capacity.

5.2. A Toy Example: Architecture Determines Gradient Geometry

Consider $z \sim \mathrm{Uniform}(\mathbb{S}^2)$ embedded into $\mathbb{R}^9$ by padding each coordinate with two zeros. The ambient data is spherical, so BEoS alone yields no useful constraint for an FCN (Liang et al., 2026). Equip a two-layer SCN with two receptive field systems:

- $\mathcal{S}_1$ **(block-aligned)**: groups 3 consecutive coordinates; every patch has the form $(t, 0, 0)$, a one-dimensional cloud.
- $\mathcal{S}_2$ **(interleaved)**: groups coordinates modulo the stride, recovering (z_1, z_2, z_3), a three-dimensional spherical cloud.

By (3), under $\mathcal{S}_1$ every filter gradient is a linear combination of $(t, 0, 0)$-vectors, confined to a one-dimensional subspace of $\mathbb{R}^3$. Under $\mathcal{S}_2$, activated patches span the full three-dimensional patch space. The two architectures produce fundamentally different gradient dynamics on the same ambient data. The sets of solutions that GD can stably reach therefore differ dramatically. The difference lies not in the function classes but in which directions each architecture allows gradient descent to move.

5.3. BEoS Implies a Patch-Geometry-Weighted Capacity Bound

The weight function $g_{\mathcal{D},\mathcal{S}}$ makes the connection between patch geometry and implicit regularization precise. Let $\boldsymbol{X}^{\mathcal{S}}_{\mathcal{D}}$ be drawn uniformly from the patch multiset $\{\pi_j(\boldsymbol{x}_i)\}_{(i,j)\in[n]\times[J]} \subset \mathbb{R}^m$. Define

$$g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) := \mathbb{E}\big[\phi(\boldsymbol{u}^\top \boldsymbol{X}^{\mathcal{S}}_{\mathcal{D}} - t)\big] \cdot \sqrt{\mathbb{P}(\boldsymbol{u}^\top \boldsymbol{X}^{\mathcal{S}}_{\mathcal{D}} > t)^2 + \big\|\mathbb{E}\big[\boldsymbol{X}^{\mathcal{S}}_{\mathcal{D}} \mathbb{1}\{\boldsymbol{u}^\top \boldsymbol{X}^{\mathcal{S}}_{\mathcal{D}} > t\}\big]\big\|_2^2}. \tag{6}$$

Intuitively, $g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t)$ evaluates a filter's activation boundary $(\boldsymbol{u}, t)$ against the patch multiset. The terms inside are exactly the ingredients of the gradient formula (3): the activation mass $\mathbb{P}(\boldsymbol{u}^\top \boldsymbol{X}^{\mathcal{S}}_{\mathcal{D}} > t)$ measures what fraction of training patches activate the filter, which is precisely what determines whether the filter's gradient in (3) reflects population structure or is driven by isolated residuals. A filter with large activation mass has its gradient shaped by many diverse training patches and is meaningfully constrained by BEoS. A filter that activates rarely has a gradient dominated by a handful of sample residuals; it can fit arbitrary labels without paying a stability cost.

Remark 5.2. When $m = d$ and $J = 1$ (fully connected network), $g_{\mathcal{D},\mathcal{S}}$ reduces to the ambient-space weight function of Liang et al. (2025). For spherical ambient data, this weight can be driven to zero by a boundary isolating a single point, which is precisely the neural shattering mechanism where gradient updates are driven by individual sample residuals. Sparse connectivity shifts the relevant geometry from $\mathbb{R}^d$ to the lower-dimensional patch space $\mathbb{R}^m$, and when $m \ll d$ this shift determines whether pointwise isolation is possible.

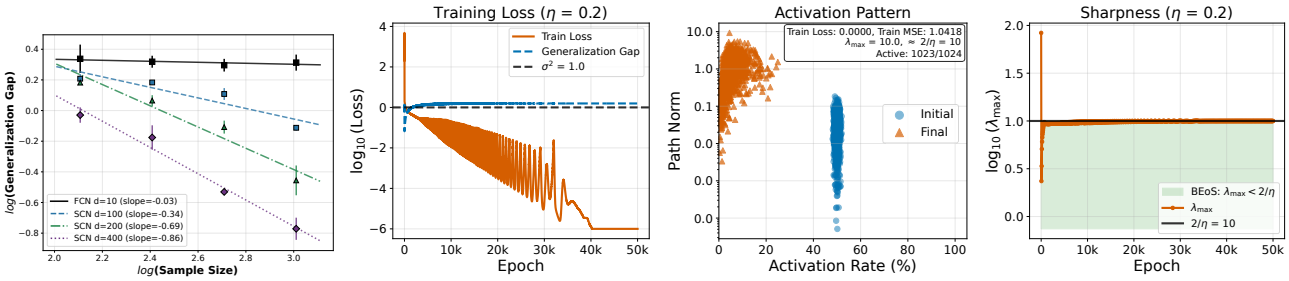
Theorem 5.3 (BEoS implies patch-geometry-weighted path norm bound)**.** *Fix $\mathcal{D}$ and $\mathcal{S}$. For any $\boldsymbol{\theta} \in \boldsymbol{\Theta}^{\mathcal{S}\mathrm{CN}}$,*

$$\sum_{k=1}^{K} |v_k| \, \|\boldsymbol{w}_k\| \, g_{\mathcal{D},\mathcal{S}}\Big(\tfrac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}, \tfrac{b_k}{\|\boldsymbol{w}_k\|}\Big) \leq \tfrac{1}{2}\Big(\lambda_{\max}(\nabla^2 \mathcal{L}(\boldsymbol{\theta})) + 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})} - 1\Big). \tag{7}$$

In particular, for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}^{\mathcal{S}\mathrm{CN}}_{\mathrm{BEoS}}(\eta, \mathcal{D})$,

$$\sum_{k=1}^{K} |v_k| \, \|\boldsymbol{w}_k\| \, g_{\mathcal{D},\mathcal{S}}\Big(\tfrac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}, \tfrac{b_k}{\|\boldsymbol{w}_k\|}\Big) \leq \tfrac{1}{\eta} - \tfrac{1}{2} + (R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})}. \tag{8}$$

The proof is in Appendix D. The architecture $\mathcal{S}$ enters entirely through $g_{\mathcal{D},\mathcal{S}}$, so different receptive field systems produce different penalty structures on the path norm. The weight function has a direct gradient-dynamics interpretation. A filter activated by many patches receives gradient



(a) GenGap scaling with n. *(b)* Stable interpolation. *(c)* Rare-activation filters. *(d)* Below the EoS.

Figure 7. **Numerical validation of both regimes. (a)** Generalization gap vs. n on a log–log scale. FCN slope ≈ -0.03 (nearly flat); SCN slopes grow more negative as d increases (slope $= -0.34,\ -0.69,\ -0.86$ at $d = 100, 200, 400$), consistent with the generalization regime in Theorem 5.4: a larger ambient dimension d makes patch projections more concentrated, strengthening the BEoS constraint. **(b)–(d)** Memorization regime (Theorem 5.5), where each input contains a unique unit-norm patch: the SCN interpolates noisy labels while remaining below the EoS ($\lambda_{\max} \approx 2/\eta = 10$), and many filters have large path norms but activate on only a tiny fraction of patches, exactly the gradient-level signature of filters driven by individual sample residuals rather than by population structure.

updates that average over population-level residuals, making $g_{\mathcal{D},\mathcal{S}}$ large and the stability constraint strong. A filter activated by only a few patches has its gradient driven by individual residuals, making $g_{\mathcal{D},\mathcal{S}}$ small and allowing memorization (Liang et al., 2026).

Returning to the toy example, under $\mathcal{S}_1$ the patch cloud is one-dimensional and concentrated, so $g_{\mathcal{D},\mathcal{S}}$ is bounded away from zero and the constraint is informative. Under $\mathcal{S}_2$ the patch cloud is spherical, a filter can activate on a single patch, $g_{\mathcal{D},\mathcal{S}} \approx 0$, and the constraint collapses. The next two theorems quantify these two regimes.

5.4. Concentrated vs. Spherical Patches: Two Contrasting Regimes

The next two results delineate when the patch-geometry constraint helps and when it does not.

The positive result (Theorem 5.4) identifies when the BEoS constraint is informative; the negative result (Theorem 5.5) identifies when it is not. Together they delineate the two regimes previewed by the toy example and the patch-size ablation.

Theorem 5.4 (Generalization under concentrated patches)**.** *Let $\boldsymbol{x} \sim \mathrm{Uniform}(\mathbb{S}^{d-1})$, $\mathcal{D}$ be n i.i.d. samples, $d > 3$, $1 \le m < \frac{d(d-3)}{d+3}$. For any $\boldsymbol{\theta} \in \boldsymbol{\Theta}^{\mathrm{SCN}}_{\mathrm{BEoS}}(\eta, \mathcal{D})$ with $\|f_{\boldsymbol{\theta}}\|_\infty \le M$, with probability $\ge 1 - 2\delta$,*

$$\mathrm{GenGap}(f_{\boldsymbol{\theta}}, \mathcal{D}) \lessapprox_d \mathrm{poly}\Big(\tfrac{1}{\eta}, J, M\Big) \cdot n^{-\frac{(d-m)(d+3)}{6d^2-2md+6d-6m}}. \tag{9}$$

When $m \ll d$, the m-dimensional patch projections concentrate near the origin of $\mathbb{R}^m$ (Proposition D.11), so any filter cutting through the dense interior must activate on many patches and its gradient reflects population structure. By contrast, Liang et al. (2026) prove that on the same spherical data a FCN can interpolate *any* labeling while satisfying BEoS. Sparse connectivity breaks this impossibility by shifting the relevant geometry to the concentrated patch cloud. As d grows with m fixed, patch projections concentrate more strongly, and the generalization guarantee *improves* with ambient dimension, a "blessing of dimensionality" confirmed in Figure 7(a).

Theorem 5.5 (Memorization compatible with stability)**.** *Suppose every nonzero-labeled training example contains a unit-norm patch appearing nowhere else in the patch multiset. Then there exists a network (4) with $K \le n$ that interpolates $\mathcal{D}$ and satisfies*

$$\lambda_{\max}\big(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}\big) \le 1 + \tfrac{D^2 + 2/J^2}{n}. \tag{10}$$

When each labeled example has a unique patch, one can construct a filter that activates only on that patch; its gradient (3) is driven by a single sample's residual (the patch-level counterpart of neural shattering) while the Hessian contribution remains small ($g_{\mathcal{D},\mathcal{S}} \approx 0$). The full construction is in Appendix F. Figure 7 validates both theorems: panel (a) shows the SCN generalization gap decaying faster with n as d increases (consistent with the blessing of dimensionality) while the FCN gap is flat; panels (b)–(d) confirm that in the memorization setting, the SCN interpolates arbitrary labels while remaining below the EoS, with many filters showing large path norms but low activation rates, the gradient signature of per-sample tuning.

6. Discussion

This paper began with a puzzle. Why can clean-data prediction rescue a fully connected denoiser that fails under noise prediction, while patch-based architectures succeed even without changing the target? Our analysis traces both phenomena to a single geometric principle. Gradient descent generalizes when its weight updates are confined to a

Table 1. The design trend of sparse connectivity in modern vision backbones. The key quantity in our framework is m, the dimension of the local vector each operator processes; this dimension shrinks consistently from ResNet to ConvNeXt and from ViT to Swin.

Backbone	**Core sparse connectivity**	**Dim of processed unit m**
ResNet-50 (He et al., 2016)	3×3 bottleneck convolution	$3\times3\times C_{\text{in}}$ (up to 512)
ResNeXt-50 (Xie et al., 2017)	3×3 grouped convolution	$3\times3\times C_{\text{in}}/g$ (up to g=32)
ConvNeXt-T (Liu et al., 2022)	7×7 depthwise convolution	49
ViT-B/16 (Dosovitskiy et al., 2021)	uniform Transformer	768 (all blocks)
Swin-T (Liu et al., 2021b)	hierarchical Transformer	$96\rightarrow192\rightarrow384\rightarrow768$

low-dimensional subspace spanned by shared data patterns. Memorization occurs when the updates can explore the full ambient space.

Clean-data prediction enforces this confinement by inserting an analytic skip at the output. The network is freed from representing high-dimensional noise, its deep activations collapse to the data manifold, and gradients become low-rank. Sparse connectivity enforces the same confinement by restricting each filter to an m-dimensional local patch. The patch vectors are concentrated, and every filter gradient is a linear combination of these low-dimensional vectors. In both cases the optimizer is geometrically prevented from dedicating separate directions to individual samples and is compelled to learn reusable features.

The key variable that emerges from our theory is the effective dimension of the vectors that appear in the gradient update. For clean-data prediction this dimension is the manifold dimension of the data; for patch architectures it is the local input dimension m. This variable has been a quiet driver of vision backbone evolution (Table 1). Convolutional architectures have steadily reduced per-operator receptive field size through bottlenecks, grouped convolutions, and depthwise convolutions. Vision Transformers have shifted from uniform global token spaces to hierarchical local ones. Our framework suggests these trends are not incidental. They systematically strengthen the implicit regularization of gradient descent.

The two routes explored in this paper are complementary. A natural next question is whether combining clean-data prediction with small-patch architectures yields further improvement, and how the output-side and input-side constraints interact. More broadly, our framework suggests concrete diagnostics that can be measured on large-scale models. The concentration of the patch cloud, the activation rates of individual filters, and the effective rank of the patch matrix all indicate whether gradient updates are confined to a low-dimensional subspace. These metrics offer a way to test whether the mechanism we have characterized in controlled settings extends to practical generative models.

The main limitation of our formal theory is its restriction to two-layer sparse networks. Extending the BEoS analysis to deep networks with residual connections, normalization, and attention remains an important open problem. Nevertheless, the underlying gradient-dynamics principle is architecture-agnostic. We expect it to hold qualitatively in far more complex settings, and to guide the design of future diffusion backbones.

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A. More Related Work

Geometric awareness in deep learning. Recent theoretical work has explored how architectural design incorporates geometric awareness to improve neural network training and approximation. Approximation theory research has demonstrated that CNN architectures overcome the curse of dimensionality by exploiting compositional structure in natural images (Mhaskar et al., 2017; Zhou, 2020; Poggio, 2022; Poggio & Fraser, 2024), with Mao et al. (2021) formally proving the superiority of CNNs over FCNs for learning certain composite functions. More recently, Vision Transformers (Dosovitskiy et al., 2021) have been analyzed through the same lens: Shi et al. (2025) extend these approximation guarantees to ViTs, showing they outperform FCNs for compositional functions, and Trockman & Kolter (2022) identify patch extraction, rather than attention, as the critical component, further reinforcing the strength of sparsely connected architectures. These results, however, either assume the labeling function satisfies a compositional structure or operate in settings with explicit regularization. Our work requires no assumptions on the labeling function and operates in the overparameterized regime.

Theoretical analysis of CNNs and separation from FCNs. The advantages of CNNs over FCNs have been established from both approximation-theoretic and statistical perspectives. Approximation theory shows that CNNs with appropriate sparse weights can achieve near-minimax optimal sample complexity (Oono & Suzuki, 2019) and adapt to intrinsic dimension when data lie on a low-dimensional manifold (Liu et al., 2021a), with Zhang et al. (2024) extending these results to overparameterized regimes under weight decay. However, these analyses rely on reducing CNNs to fully connected networks (Yarotsky, 2017) and therefore do not reveal the architectural insights specific to CNNs that we describe in this paper. Specifically, we do not require the data to be supported on a low-dimensional manifold to avoid the curse of dimensionality, nor do we require explicit regularization such as weight decay or a sparsity constraint. On the statistical side, a sample complexity separating CNNs, locally connected networks without weight sharing, and FCNs has been established (Li et al., 2021; Wang & Wu, 2023; Lahoti et al., 2024). Our results are for the same model family, but differ in that we consider an overparameterized regime without explicit regularization, which allows us to prove a stronger separation between FCNs and CNNs. Lastly, the significance of the input data distribution, rather than the labeling function, was not discovered in prior work.

Edge of stability, minima stability, and generalization by large stepsizes. Our approach builds upon a recent line of work that studies the set of solutions that gradient descent can visit (or converge to) via either Edge-of-stability observation or the minima stability theory (Ding et al., 2024; Mulayoff et al., 2021; Nacson et al., 2023; Wu & Su, 2023; Qiao et al., 2024; Liang et al., 2025; 2026). These approaches enable formal analysis of the generalization properties without having to analyze gradient dynamics. To the best of our knowledge, they all focused on feedforward neural networks, and we are the first to study CNNs and the impact of model architecture choices in the implicit regularization of large stepsizes.

B. Broader (Informal) Discussion on Network Architecture Design: Bridge our Theoretical Insights to Empirical Observations

This appendix expands on the architectural implications discussed in Section 6. The purpose is not to claim that the stability analysis in the main text provides a complete theory of modern deep vision architectures. Rather, the point is that stability gives a tractable lens through which one can see a more general object: the geometry of the local vectors exposed to gradient descent. This object appears explicitly in the shallow model analyzed in this paper, and it has close analogues in the patchification and local-operator design choices used in modern CNNs and Vision Transformers.

The patch matrix as the geometry seen by backpropagation. In the setting of this paper, consider the patch matrix

$$\mathbf{X} \in \mathbb{R}^{(nJ)\times m},$$

whose row indexed by (i,j) equals $\pi_j(\boldsymbol{x}_i)^\mathsf{T}$. Let $r_i := f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i$, and define the lifted residual vector $\bar{\mathbf{r}} \in \mathbb{R}^{nJ}$ by

$$(\bar{\mathbf{r}})_{(i,j)} := r_i.$$

Let $\mathbf{S}(\boldsymbol{\theta}) \in \{0,1\}^{(nJ)\times K}$ be the gating matrix

$$\mathbf{S}(\boldsymbol{\theta})_{(i,j),k} \;:=\; \mathbb{1}\left\{\boldsymbol{w}_k^\mathsf{T}\pi_j(\boldsymbol{x}_i) > b_k\right\},$$

and let $\boldsymbol{v} := (v_1,\ldots,v_K)^\mathsf{T}$. Stack first-layer weights as

$$\mathbf{W} := [\boldsymbol{w}_1^\mathsf{T};\ldots;\boldsymbol{w}_K^\mathsf{T}] \in \mathbb{R}^{K\times m}.$$

A direct calculation gives a backprop-aligned factorization of the gradient. Define the patch-level backprop signal

$$\mathbf{R}(\boldsymbol{\theta}) := \frac{1}{nJ}\,\bar{\mathbf{r}}\,\boldsymbol{v}^{\mathsf{T}} \in \mathbb{R}^{(nJ)\times K}, \quad \mathbf{G}(\boldsymbol{\theta}) := \mathbf{R}(\boldsymbol{\theta}) \odot \mathbf{S}(\boldsymbol{\theta}),$$

where $\odot$ denotes entrywise multiplication. Then

$$\nabla_{\mathbf{W}}\mathcal{L}(\boldsymbol{\theta}) \;=\; \mathbf{G}(\boldsymbol{\theta})^{\mathsf{T}}\mathbf{X}. \tag{11}$$

Equivalently, writing $\mathbf{V} := \mathrm{diag}(v_1, \ldots, v_K)$, one can express the same signal as

$$\mathbf{G}(\boldsymbol{\theta}) = \frac{1}{nJ}\,\mathrm{diag}(\bar{\mathbf{r}})\,\mathbf{S}(\boldsymbol{\theta})\,\mathbf{V}.$$

The factorization (11) separates two roles. The matrix $\mathbf{G}(\boldsymbol{\theta})$ aggregates the learning signal generated by residuals, readout weights, and gates. The patch matrix $\mathbf{X}$ then maps this signal into parameter updates. Thus the architecture does not only restrict the number of parameters, it determines the geometry through which backpropagated signals are converted into motion in parameter space. In this sense, $\mathbf{X}$ acts as a signal rectifier in the sense that data priors and receptive-field structure jointly shape the directions along which gradient descent can effectively move. For example, Figure 9a shows that, for CIFAR-10, 90% of the energy of the convolutional patch matrix is concentrated in only a few principal directions, suggesting a strong geometric constraint on the filter dynamics.

This viewpoint also clarifies why the stability condition in the main text is informative. The terms appearing in the definition of $g_{\mathcal{D},\mathcal{S}}$ are drawn from the same ingredients as (11): gate statistics induced by $\mathbf{S}(\boldsymbol{\theta})$ and geometric moments of the patch cloud encoded by $\mathbf{X}$. Theorem 5.4 can therefore be viewed as a static stability proxy for the backpropagation geometry in (11). It does not say that stability alone explains generalization. Indeed, Theorem 5.5 shows that stable solutions may still memorize when the patch cloud admits isolating half-spaces. If corresponding backpropagation direction may enable a filter move to a location such that it only activates on a rare, nearly isolated patch, then the network may fit labels while paying little stability cost. If the patch cloud is concentrated and resistant to such isolation, the same stability lens yields an effective regularity constraint.

Patch size as local-operator dimension in Vision Transformers. Vision Transformers make the role of patch geometry especially explicit. The original ViT converts an image into a sequence of non-overlapping patches and applies a shared patch embedding before global token mixing (Dosovitskiy et al., 2021). This patchification step is often introduced as an efficiency device, since larger patches shorten the token sequence and reduce the cost of attention. The factorization (11) suggests a second interpretation. Patchification also chooses the first local geometry seen by training. A larger patch presents each shared local operator with a higher-dimensional and more compressed local vector. A smaller patch reduces this local dimension and preserves finer spatial information, while increasing the number of tokens that later layers must mix.

Several empirical lines of work support the idea that patch size is a genuine training and generalization variable, not only an implementation detail. FlexiViT trains a single ViT over a range of patch sizes and demonstrates that patch size controls an accuracy–compute tradeoff at deployment time (Beyer et al., 2023). Self-supervised ViTs also reveal the importance of this axis: DINO identifies small patches as one ingredient behind the strong emergent properties of self-supervised ViTs (Caron et al., 2021). Early-convolution studies make the same point from the optimization side. The standard ViT patchify stem is a large-kernel, large-stride convolution, and replacing it by a short stack of small-stride convolutions significantly changes optimization stability and improves performance under common training recipes (Xiao et al., 2021). These results are consistent with the idea that the first patch matrix is not an innocuous preprocessing choice: it shapes the geometry of the signals that subsequent layers and backpropagation operate on.

Recent patchification scaling studies push this observation further. Wang et al. (Wang et al., 2025b) systematically reduce the patch size from the standard coarse-token regime toward pixel-level tokenization and report smooth gains across classification, semantic segmentation, object detection, and instance segmentation. Table 2 summarizes the results most relevant to our discussion. The most informative part is their sequence-length ablation. When coarse patch tokens are merely interpolated into a longer sequence, the gains remain small. When genuinely smaller patches are extracted from the image, the gains are much larger. This suggests that patch-size scaling changes the local visual primitives exposed to the model, not only the number of tokens. In the notation of this paper, reducing patch size changes the initial patch matrix itself: it lowers spatial compression in each local vector and changes the multiset of local rows through which the network forms

Table 2. Selected patchification-scaling results adapted from (Wang et al., 2025b). Patch size denotes the side length of the square image patch used to form one token. Panel A shows that reducing patch size improves ImageNet accuracy. Panel B separates true patch-size reduction from only increasing the sequence length by interpolating coarse tokens. Here we only list their results on DeiT-B (Touvron et al., 2021) and Adventurer-B (Wang et al., 2025a).

Panel A: ImageNet classification top-1 accuracy

Model and input	16×16	8×8	4×4	2×2	1×1
DeiT-B, 64×64 input	68.2	76.9	80.1	80.8	81.3
DeiT-B, 128×128 input	78.1	81.0	82.3	82.9	–
Adventurer-B, 224×224 input	82.6	83.9	84.3	84.5	84.6

Panel B: Sequence-length ablation on ImageNet top-1 accuracy

Model and input	**Seq. length**	**Interpolated coarse tokens**	**True patch scaling**
DeiT-B, 128×128 input	256	78.2 (+0.1)	81.0 (+2.9)
DeiT-B, 128×128 input	1,024	78.2 (+0.1)	82.3 (+4.2)
Adventurer-B, 224×224 input	784	82.7 (+0.1)	83.9 (+1.3)
Adventurer-B, 224×224 input	3,136	82.8 (+0.2)	84.3 (+1.7)
Adventurer-B, 224×224 input	12,544	82.8 (+0.2)	84.5 (+1.9)

its updates. This aligns with the gradient factorization $\nabla_{\mathbf{W}}\mathcal{L} = \mathbf{G}^{\top}\mathbf{X}$: the relevant object is the geometry of the rows in the patch or token matrix, not only the nominal sequence length. Related pixel-token studies push the same question to the extreme by asking how far visual models can go when patchification is removed almost entirely (Nguyen et al., 2024).

Why small patches require hierarchy and mixing. Smaller patches improve the local geometry exposed to the model, but they also increase the burden of mixing information across locations. This tradeoff helps explain a major trend in vision-transformer architecture design. Hierarchical Transformers such as Swin do not keep a fixed high-dimensional token space throughout the network; they begin with finer local tokens, restrict early mixing to local windows, and then gradually merge tokens while increasing representation dimension (Liu et al., 2021b). PVTv2 follows a related logic through overlapping patch embeddings, linear-complexity attention, and convolutional feed-forward layers (Wang et al., 2022). CvT introduces convolutional token embeddings and convolutional projections inside Transformer blocks (Wu et al., 2021). Tokens-to-Token ViT replaces one-shot patchification by a progressive tokenization module that recursively aggregates local neighboring tokens (Yuan et al., 2021). These designs are not only remedies for a defective large-patch ViT. They can be read as different ways of managing the same geometric tradeoff: expose early layers to controlled local vectors, then recover expressivity through hierarchy, overlap, convolutional projection, or stage-wise mixing.

In the notation of (11), a deep network replaces the fixed raw patch matrix $\mathbf{X}$ by a sequence of representation matrices. At layer ℓ, one should imagine a matrix

$$\mathbf{X}_\ell = \left[\boldsymbol{z}_{\ell,j}(\boldsymbol{x}_i)^{\top}\right]_{(i,j)}$$

whose rows are local tokens or local feature vectors produced by the preceding layers. Backpropagation through a local operator at layer ℓ again has the schematic form

$$\nabla_{\mathbf{W}_\ell}\mathcal{L} \approx \mathbf{G}_\ell(\boldsymbol{\theta})^{\top}\mathbf{X}_\ell,$$

up to the additional Jacobian factors introduced by normalization, residual connections, attention, and nonlinear mixing. Thus the relevant geometry is no longer fixed by the raw data distribution. It evolves during training. Patch size, windowing, overlap, token merging, and convolutional stems can all be viewed as mechanisms for shaping the sequence

$$\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_L$$

of optimization-facing geometries.

Patch geometry in CNNs: from spatial-channel mixing to depthwise convolutional kernel. CNN architecture design provides a sequence of examples in which the vector seen by each local operator is progressively controlled. In a residual bottleneck block, the 3×3 spatial convolution acts after a channel-reducing 1×1 projection, so the spatial operator does not see the full channel dimension of the block (He et al., 2016). ResNeXt makes this control explicit through grouped convolutions: each group applies a 3×3 operator to only a fraction of the channels, reducing the local vector from

$3 \times 3 \times C_{\text{in}}$ to $3 \times 3 \times C_{\text{in}}/g$ (Xie et al., 2017). ConvNeXt pushes the same principle further by using depthwise spatial convolutions inside an inverted-bottleneck block, with expressive channel mixing supplied by pointwise layers (Liu et al., 2022). Thus, from ResNet to ResNeXt to ConvNeXt, the spatial operator is increasingly restricted to a structured local geometry, while expressivity is recovered through channel mixing and depth.

The kernel-size ablation in ConvNeXt gives a useful empirical instance of the trade-off suggested by our theory. Increasing the depthwise kernel from 3×3 to 7×7 improves ImageNet accuracy, while increasing it further to 9×9 or 11×11 yields little additional gain (Liu et al., 2022). In the language of this paper, enlarging the spatial kernel increases the context available to each local operator, reducing the bias induced by overly local processing. At the same time, a larger local vector exposes gradient descent to a richer patch geometry, which can weaken the regularizing effect of locality. The saturation around 7×7 is consistent with a bias–variance trade-off for the geometry seen by local operators: enough spatial support improves representation, while excessive support brings less benefit once the local computation is already sufficiently expressive.

RepLKNet develops this direction further by treating very large kernels as a separate architectural scaling dimension (Ding et al., 2022). It shows that replacing ConvNeXt's 7×7 kernels with much larger stage-wise kernels can further improve performance, especially on dense prediction tasks. The key point for our purposes is not only the larger spatial support, but the training-time parameterization used to make such kernels effective. RepLKNet does not train a bare large kernel. It introduces *small-kernel structural re-parameterization*: during training, a large kernel branch is paired with a small kernel branch, such as a 3×3 or 5×5 branch. After training, the small kernel is padded to the center of the large kernel and algebraically merged, so inference uses a single large kernel.

This mechanism has a direct interpretation in our gradient-geometry framework. Let $\boldsymbol{x}_L \in \mathbb{R}^{K^2}$ denote the large local patch and let $\boldsymbol{x}_s = \boldsymbol{C}\boldsymbol{x}_L \in \mathbb{R}^{s^2}$ be the centered small patch selected from it. Ignoring normalization for clarity, the training-time operator has the form

$$h(\boldsymbol{x}) = \boldsymbol{w}_L^\top \boldsymbol{x}_L + \boldsymbol{w}_s^\top \boldsymbol{x}_s = (\boldsymbol{w}_L + \boldsymbol{C}^\top \boldsymbol{w}_s)^\top \boldsymbol{x}_L, \qquad \boldsymbol{w}_{\text{eff}} = \boldsymbol{w}_L + \boldsymbol{C}^\top \boldsymbol{w}_s.$$

At inference time this is just a large kernel with weight $\boldsymbol{w}_{\text{eff}}$. During training, however, the two branches expose different local geometries to gradient descent. If r denotes the backpropagated scalar signal, then

$$\nabla_{\boldsymbol{w}_L} \mathcal{L} = r\boldsymbol{x}_L, \qquad \nabla_{\boldsymbol{w}_s} \mathcal{L} = r\boldsymbol{C}\boldsymbol{x}_L, \qquad \Delta \boldsymbol{w}_{\text{eff}} = -\eta r(\boldsymbol{I} + \boldsymbol{C}^\top \boldsymbol{C})\boldsymbol{x}_L,$$

up to the branch-dependent scaling introduced by normalization. The small branch therefore gives the centered local subspace an additional optimization path. It biases training toward small-scale local structure while the large branch captures broad spatial context.

This example sharpens the message of the patch-matrix factorization $\nabla_{\mathbf{W}} \mathcal{L} = \mathbf{G}^\top \mathbf{X}$. The geometry relevant to learning is not determined solely by the inference-time operator. RepLKNet uses an inference-time large kernel, but its training-time architecture exposes gradient descent to both a large-patch geometry and a centered small-patch geometry. Thus structural re-parameterization is a practical mechanism for modifying the optimization-facing patch geometry without changing the final operator class. Together with the ConvNeXt ablation, this supports the broader view that CNN design balances spatial context against implicit regularization induced by controlled local geometry.

C. Functional Analysis of Shallow ReLU Networks

C.1. Path-norm and Variation Semi-norm of ReLU Networks

This subsection collects several basic facts from (Parhi & Nowak, 2023) and (Siegel & Xu, 2023) that we will use later.

Definition C.1. Let $f_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sum_{k=1}^{K} v_k \, \phi(\boldsymbol{w}_k^\top \boldsymbol{x} - b_k) + \beta$ be a fully-connected two-layer neural network. The (unweighted) path-norm of $f_{\boldsymbol{\theta}}$ is defined to be

$$\|f_{\boldsymbol{\theta}}\|_{\text{path}} := \sum_{k=1}^{K} |v_k| \, \|\boldsymbol{w}_k\|_2 \,. \tag{12}$$

Dictionary representation of ReLU networks. Using the positive 1-homogeneity of ReLU, one may rescale each hidden unit while leaving the realized function unchanged:

$$v_k \, \phi(\boldsymbol{w}_k^\top \boldsymbol{x} - b_k) = a_k \, \phi(\boldsymbol{u}_k^\top \boldsymbol{x} - t_k), \quad \boldsymbol{u}_k := \frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|_2} \in \mathbb{S}^{d-1}, \; t_k := \frac{b_k}{\|\boldsymbol{w}_k\|_2}, \; a_k := v_k \, \|\boldsymbol{w}_k\|_2 \,.$$

Consequently, f_θ can be written in the normalized finite-sum form

$$f(\boldsymbol{x}) = \sum_{k=1}^{K'} a_k\, \phi(\boldsymbol{u}_k^\mathsf{T}\boldsymbol{x} - t_k) + \boldsymbol{c}^\mathsf{T}\boldsymbol{x} + c_0. \tag{13}$$

Define the (ReLU) ridge dictionary as $\mathscr{D}_\phi := \left\{\phi(\boldsymbol{u}^\mathsf{T}\cdot - t) :\ \boldsymbol{u} \in \mathbb{S}^{d-1},\ t \in \mathbb{R}\right\}$. We focus on the *overparameterized, width-agnostic* collection obtained by taking the *union over all finite widths*

$$\mathcal{F}_{\text{fin}} := \bigcup_{K\geq 1} \left\{ \sum_{k=1}^{K} a_k\, \phi(\boldsymbol{u}_k^\mathsf{T}\cdot - t_k) + \boldsymbol{c}^\mathsf{T}(\cdot) + c_0 \right\}, \tag{14}$$

and quantify complexity via the smallest path-norm among all realizations of f:

$$\|f\|_{\text{path,min}} := \inf\left\{\|f_{\boldsymbol{\theta}}\|_{\text{path}} :\ f_{\boldsymbol{\theta}} \equiv f \text{ of the form (13)}\right\}.$$

From finite sums to a *width-agnostic* integral representation. Rather than fixing a particular width K, it is convenient to work with a convex, measure-theoretic formulation that *captures the closure/convex hull of* (14). Concretely, let ν be a finite signed Radon measure on $\mathbb{S}^{d-1} \times [-R, R]$ and consider

$$f(\boldsymbol{x}) = \int_{\mathbb{S}^{d-1}\times[-R,R]} \phi(\boldsymbol{u}^\mathsf{T}\boldsymbol{x} - t)\ \mathrm{d}\nu(\boldsymbol{u}, t) + c^\mathsf{T}\boldsymbol{x} + c_0. \tag{15}$$

Every finite network of the form (13) corresponds to an *atomic* (hence sparse) measure $\nu = \sum_{k=1}^{K} a_k\, \delta_{(\boldsymbol{u}_k, t_k)}$, and conversely any such atomic ν yields a finite network. Therefore, (15) should be viewed as a *width-agnostic relaxation* aligned with (12), rather than as an assumption of an infinite-width limit.

Definition C.2. The (unweighted) variation (semi)norm

$$|f|_{\mathrm{V}} := \inf\left\{\|\nu\|_{\mathcal{M}} :\ f \text{ admits (13) for some } (\nu, c, c_0)\right\}, \tag{16}$$

where $\|\nu\|_{\mathcal{M}}$ is the total variation of ν.

For the compact region $\Omega = \mathbb{B}_R^d$, we define the bounded variation function class as

$$\mathrm{V}_C(\Omega) := \left\{ f : \Omega \to \mathbb{R} \mid f = \int_{\mathbb{S}^{d-1}\times[-R,R]} \phi(\boldsymbol{u}^\mathsf{T}\boldsymbol{x} - t)\, \mathrm{d}\nu(\boldsymbol{u}, t) + \boldsymbol{c}^\mathsf{T}\boldsymbol{x} + b,\ |f|_{\mathrm{V}} \leq C \right\}. \tag{17}$$

In particular, identifying (13) with the atomic measure $\nu = \sum_k a_k \delta_{(\boldsymbol{u}_k, t_k)}$ yields

$$|f|_{\mathrm{V}} \leq \sum_k |a_k| = \|f_{\boldsymbol{\theta}}\|_{\text{path}}, \quad \text{hence} \quad |f|_{\mathrm{V}} \leq \|f\|_{\text{path,min}}.$$

Moreover, the minimal total variation required to represent f coincides with the minimal path-norm over all finite decompositions:

$$\|f\|_{\text{path,min}} = |f|_{\mathrm{V}}. \tag{18}$$

Thus, (16) provides a *nonparametric* analogue of the path-norm: it encodes the same complexity notion while *not committing to a fixed width K*.

Remark C.3 ("Arbitrary width" $\neq$ "infinite width")**.** All statements here are about $\mathcal{F}_{\text{fin}}$ in (14), namely networks of *finite* (but unconstrained) width. The integral representation (15) is introduced as a convenient convexification/closure of this union for analysis and regularization; it *does not* posit an infinite-width limit. In particular, when training in the variational form with a total-variation penalty on ν, first-order optimality implies that optimal measures are sparse (i.e., have finite support), which corresponds exactly to *finite-width* networks. Therefore, our results hold for *arbitrary (yet finite) width*; the continuum measure serves only as a tool to characterize and control $\|f\|_{\text{path,min}}$.

C.2. The Metric Entropy of Variation Spaces

Metric entropy is a standard way to describe how "compact" a subset A is inside a metric space (X, ρ_X). We recall the notions of covering numbers and metric entropy.

Definition C.4 (Covering Number and Entropy)**.** Let A be a compact subset of a metric space (X, ρ_X). For $t > 0$, the *covering number* $N(A, t, \rho_X)$ is the smallest number of closed balls of radius t whose union contains A:

$$N(t, A, \rho_X) := \min \left\{ N \in \mathbb{N} : \exists\, x_1, \ldots, x_N \in X \text{ s.t. } A \subset \bigcup_{i=1}^{N} \mathbb{B}(x_i, t) \right\}, \tag{19}$$

where $\mathbb{B}(x_i, t) = \{y \in X : \rho_X(y, x_i) \leq t\}$. The *metric entropy* of A at scale t is then

$$H_t(A)_X := \log N(t, A, \rho_X). \tag{20}$$

Covering/entropy bounds for bounded-variation-type classes have been established in the literature. In what follows, we will use the estimate stated below.

Proposition C.5 (Parhi & Nowak 2023, Appendix D)**.** *The metric entropy of* $\mathrm{V}_C(\mathbb{B}_R^d)$ *(see Definition C.2) with respect to the* $L^\infty(\mathbb{B}_R^d)$*-distance* $\|\cdot\|_\infty$ *satisfies*

$$\log N(t, \mathrm{V}_C(\mathbb{B}_R^d), \|\cdot\|_\infty) \lessapprox_d \left(\frac{C}{t}\right)^{\frac{2d}{d+3}}. \tag{21}$$

where $\lessapprox_d$ *hides constants (which could depend on* d*) and logarithmic factors.*

C.3. Generalization Gap of Unweighted Variation Function Class

As a middle step towards bounding the generalization gap of the weighted variation function class, we bound the generalization gap of the unweighted variation function class using chaining and Gaussian complexity, together with the L^∞ metric entropy bound in Proposition C.5.

Proposition C.6 (Wainwright 2019, Chapter 13)**.** *Fix design points* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ *and denote the empirical norm*

$$\|f\|_n^2 := \frac{1}{n} \sum_{i=1}^{n} f(\boldsymbol{x}_i)^2.$$

Let $\mathcal{F}$ *be a class of real-valued functions on* $\{\boldsymbol{x}_i\}_{i=1}^n$*, and define*

$$\widehat{\mathcal{G}}_n(\mathcal{F}) := \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(\boldsymbol{x}_i), \qquad \varepsilon_1, \ldots, \varepsilon_n \overset{i.i.d.}{\sim} \mathcal{N}(0,1), \quad \mathcal{G}_n(\mathcal{F}) := \mathbb{E}\, \widehat{\mathcal{G}}_n(\mathcal{F}).$$

Then

$$\mathcal{G}_n(\mathcal{F}) \;\lesssim\; \frac{16}{\sqrt{n}} \int_0^{\mathrm{diam}(\mathcal{F}, \|\cdot\|_n)} \sqrt{\log N\big(t,\ \mathcal{F},\ \|\cdot\|_n\big)}\, \mathrm{d}t, \tag{22}$$

where $\mathrm{diam}(\mathcal{F}, \|\cdot\|_n) := \sup_{f_1, f_2 \in \mathcal{F}} \|f_1 - f_2\|_n$*. Moreover, with probability at least* $1 - \delta$*,*

$$\widehat{\mathcal{G}}_n(\mathcal{F}) \;\leq\; \mathcal{G}_n(\mathcal{F}) + \mathrm{diam}(\mathcal{F}, \|\cdot\|_n)\, \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}} \qquad (\delta > 0). \tag{23}$$

Lemma C.7. *Let* $\mathcal{F}_{M,C} = \{f \in \mathrm{V}_C(\mathbb{B}_R^d) \mid \|f\|_\infty \leq M\}$ *with* $M \geq D$*, and let* $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n \sim \mathcal{P}^{\otimes n}$ *where* $|Y| \leq D$ *a.s. Then with probability at least* $1 - \delta$*,*

$$\sup_{f \in \mathcal{F}_{M,C}} \left| R(f) - \widehat{R}_{\mathcal{D}}(f) \right| \;\lesssim_d\; C^{\frac{d}{d+3}}\, M^{\frac{d+6}{d+3}}\, n^{-\frac{1}{2}} \;+\; M^2 \left(\frac{\log(1/\delta)}{n}\right)^{\frac{1}{2}}. \tag{24}$$

Proof. Let $\ell_f(\boldsymbol{x}, y) := (y - f(\boldsymbol{x}))^2$ and $\mathcal{L}_{M,C} := \{\ell_f : f \in \mathcal{F}_{M,C}\}$. Since $|y| \le D \le M$ and $\|f\|_\infty \le M$, for any $f, g \in \mathcal{F}_{M,C}$ and any $(\boldsymbol{x}, y)$,

$$\big|\ell_f(\boldsymbol{x}, y) - \ell_g(\boldsymbol{x}, y)\big| = |f(\boldsymbol{x}) - g(\boldsymbol{x})|\,|f(\boldsymbol{x}) + g(\boldsymbol{x}) - 2y| \le 4M\,|f(\boldsymbol{x}) - g(\boldsymbol{x})|.$$

Therefore, for the empirical norm on $\{\boldsymbol{x}_i\}_{i=1}^n$,

$$\|\ell_f - \ell_g\|_n \le 4M\,\|f - g\|_n \le 4M\,\|f - g\|_\infty, \tag{25}$$

and $\operatorname{diam}(\mathcal{L}_{M,C}, \|\cdot\|_n) \le 8M^2$.

A standard Gaussian symmetrization/concentration argument (see, e.g., Wainwright 2019, Chapter 5) yields that with probability at least $1 - \delta$,

$$\sup_{f \in \mathcal{F}_{M,C}} \big|R(f) - \widehat{R}_{\mathcal{D}}(f)\big| \;\lesssim\; \widehat{\mathcal{G}}_n(\mathcal{L}_{M,C}) \;+\; M^2 \left(\frac{\log(1/\delta)}{n}\right)^{\frac{1}{2}}. \tag{26}$$

It remains to bound $\widehat{\mathcal{G}}_n(\mathcal{L}_{M,C})$ by chaining. By (25), any t-cover of $\mathcal{F}_{M,C}$ in $\|\cdot\|_n$ induces a $(4Mt)$-cover of $\mathcal{L}_{M,C}$ in $\|\cdot\|_n$, hence

$$\log N\big(t, \mathcal{L}_{M,C}, \|\cdot\|_n\big) \le \log N\Big(\frac{t}{4M}, \mathcal{F}_{M,C}, \|\cdot\|_n\Big) \le \log N\Big(\frac{t}{4M}, \mathrm{V}_C(\mathbb{B}_R^d), \|\cdot\|_\infty\Big),$$

where we used $\|h\|_n \le \|h\|_\infty$. Proposition C.5 then gives, up to logarithmic factors,

$$\log N\big(t, \mathcal{L}_{M,C}, \|\cdot\|_n\big) \;\lessapprox_d\; \left(\frac{MC}{t}\right)^{\frac{2d}{d+3}}.$$

Applying Proposition C.6 with $\mathcal{G} = \mathcal{L}_{M,C}$ and $\operatorname{diam}(\mathcal{L}_{M,C}, \|\cdot\|_n) \le 8M^2$ yields

$$\begin{aligned}
\widehat{\mathcal{G}}_n(\mathcal{L}_{M,C}) \;&\lesssim_d\; \frac{1}{\sqrt{n}} \int_0^{8M^2} \left(\frac{MC}{t}\right)^{\frac{d}{d+3}} \mathrm{d}t \;+\; M^2 \left(\frac{\log(1/\delta)}{n}\right)^{\frac{1}{2}} \\
&\asymp_d\; C^{\frac{d}{d+3}}\, M^{\frac{d+6}{d+3}}\, n^{-\frac{1}{2}} \;+\; M^2 \left(\frac{\log(1/\delta)}{n}\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining with (26) proves (24). □

D. Proof of Theorem 5.3

Let $\iota_j : \mathbb{R}^m \to \mathbb{R}^d$ be the dual embedding such that $\pi_j \circ \iota_j = \mathrm{id}_{\mathbb{R}^m}$. Then (4) is equivalent to a fully connected neural network in a form of

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) \;=\; \sum_{k=1}^{K} \frac{1}{J} \sum_{j=1}^{J} v_k\, \phi\big(\iota_j(\boldsymbol{w}_k)^\mathsf{T} \boldsymbol{x} - b_k\big) + \beta. \tag{27}$$

Note that for any fixed k, $\|\iota_j(\boldsymbol{w}_k)\|_2 = \|\boldsymbol{w}_k\|_2$ for all $j = 1, \cdots, J$. Therefore, the notion of path norm and variation norm (together with their weighted version) still make sense for the CNN model

$$\|f_{\boldsymbol{\theta}}\|_{\mathrm{path}} = \frac{1}{J} \sum_{k=1}^{K} \sum_{j=1}^{J} |v_k|\, \|\iota_j(\boldsymbol{w}_k)\|_2 = \sum_{k=1}^{K} |v_k|\, \|\boldsymbol{w}_k\|_2\,. \tag{28}$$

By direct computation, the Hessian matrix of the loss function is expressed as

$$\nabla_{\boldsymbol{\theta}}^2 \mathcal{L} = \underbrace{\frac{1}{n} \sum_{i=1}^{n} \nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i) \nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)^\mathsf{T}}_{\boldsymbol{T}_{\mathcal{D}}} + \underbrace{\frac{1}{n} \sum_{i=1}^{n} (f(\boldsymbol{x}_i) - y_i) \nabla_{\boldsymbol{\theta}}^2 f(\boldsymbol{x}_i)}_{\boldsymbol{R}_{\mathcal{D}}}\,. \tag{29}$$

Definition D.1. Given a dataset $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$ and a set of local receptive fields $\mathcal{S} = \{S_j\}_{j=1}^J$, we define a random vector $\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}}$ uniformly draw from $\{\pi_j(\boldsymbol{x}_i)\}_{(i,j)}^{n\times J} \subset \mathbb{R}^m$. For any $\boldsymbol{u} \in \mathbb{S}^{m-1}, t \in \mathbb{R}$, we define the weight function

$$g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) = \min\Big\{\tilde{g}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t),\ \tilde{g}_{\mathcal{D},\mathcal{S}}(-\boldsymbol{u}, -t)\Big\}. \tag{30}$$

where

$$g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) := \mathbb{E}\left[\phi\left(\boldsymbol{u}^\mathsf{T}\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}} - t\right)\right]\ \sqrt{\mathbb{P}\left(\boldsymbol{u}^\mathsf{T}\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}} > t\right)^2 + \left\|\mathbb{E}\left[\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}}\mathbb{1}\left\{\boldsymbol{u}^\mathsf{T}\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}} > t\right\}\right]\right\|_2^2}. \tag{31}$$

Proposition D.2. *Given a data set $\mathcal{D} = \{\boldsymbol{x}_i, y_i\}_{i=1}^n$ and a network model (4) with local receptive fields $\mathcal{S}$. Then we have*

$$\lambda_{\max}\left(\boldsymbol{T}_{\mathcal{D}}\right) \geq 1 + 2\sum_{k=1}^K |v_k|\,\|\boldsymbol{w}_k\|\ g_{\mathcal{D},\mathcal{S}}\left(\frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}, \frac{b_k}{\|\boldsymbol{w}_k\|}\right). \tag{32}$$

Proof. We write $\boldsymbol{T}_{\mathcal{D}}$ in terms of tangent features

$$\boldsymbol{T}_{\mathcal{D}} = \frac{1}{n}\sum_{i=1}^n \nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)\,\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)^\mathsf{T}\ =\ \frac{1}{n}\,\Phi\Phi^\mathsf{T}.$$

Consequently,

$$\lambda_{\max}\big(\boldsymbol{T}_{\mathcal{D}}\big)\ =\ \max_{u\in\mathbb{S}^{n-1}} \frac{1}{n}\,\|\Phi u\|^2\ \geq\ \frac{1}{n^2}\,\|\Phi\mathbf{1}\|^2. \tag{33}$$

For any point $\boldsymbol{x}$ and abbreviate its patch extraction on S_j by $\boldsymbol{x}^{(S_j)} := \pi_j(\boldsymbol{x})$. Define the ReLU gate

$$m_k^{(S_j)}(\boldsymbol{x}) := \mathbb{1}\left\{\boldsymbol{w}_k^\mathsf{T}\boldsymbol{x}^{(S_j)} > b_k\right\}.$$

For any sample $\boldsymbol{x}_i$, denote $m_{k,i}^{(S_j)} := m_k^{(S_j)}(\boldsymbol{x}_i)$. Then the partial derivatives are

$$\frac{\partial f(\boldsymbol{x}_i)}{\partial v_k} = \frac{1}{J}\sum_{j=1}^J m_{k,i}^{(S_j)}\cdot\big(\boldsymbol{w}_k^\mathsf{T}\boldsymbol{x}_i^{(S_j)} - b_k\big), \quad \frac{\partial f(\boldsymbol{x}_i)}{\partial \boldsymbol{w}_k} = \frac{1}{J}\sum_{j=1}^J m_{k,i}^{(S_j)}\cdot v_k\cdot\boldsymbol{x}_i^{(S_j)},$$

$$\frac{\partial f(\boldsymbol{x}_i)}{\partial b_k} = -\frac{1}{J}\sum_{j=1}^J m_{k,i}^{(S_j)}\cdot\ v_k, \quad \frac{\partial f(\boldsymbol{x}_i)}{\partial \beta} = 1.$$

Stacking these over i and plugging $u = \mathbf{1}/\sqrt{n}$ in (33), we get

$$
\begin{aligned}
\frac{1}{n^2}\|\Phi\mathbf{1}\|^2 &= 1 + \frac{1}{n^2}\sum_{k=1}^{K}\Bigg[(v_k)^2\Bigg(\Bigg\|\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\boldsymbol{x}_i^{(S_j)}\Bigg\|^2 + \Bigg(\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\Bigg)^2\Bigg) \\
&\qquad + \Bigg(\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J}\phi\left(\boldsymbol{w}_k^{\mathsf{T}}\boldsymbol{x}_i^{(S_j)} - b_k\right)\Bigg)^2\Bigg] \\
&= 1 + \frac{1}{n^2}\sum_{k=1}^{K}\Bigg[(v_k)^2\Bigg(\Bigg\|\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\boldsymbol{x}_i^{(S_j)}\Bigg\|^2 + \Bigg(\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\Bigg)^2\Bigg) \\
&\qquad + \|\boldsymbol{w}_k\|_2^2\Bigg(\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J}\phi\left(\boldsymbol{u}_k^{\mathsf{T}}\boldsymbol{x}_i^{(S_j)} - t_k\right)\Bigg)^2\Bigg] \qquad (34) \\
&\geq 1 + \frac{2}{n^2}\sum_{k=1}^{K}|v_k|\,\|\boldsymbol{w}_k\|_2\left(\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J}\phi\left(\boldsymbol{u}_k^{\mathsf{T}}\boldsymbol{x}_i^{(S_j)} - t_k\right)\right) \\
&\qquad \cdot\sqrt{\Bigg\|\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\boldsymbol{x}_i^{(S_j)}\Bigg\|^2 + \Bigg(\sum_{i=1}^{n}\frac{1}{J}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\Bigg)^2} \quad (\text{since } a^2+b^2 \geq 2ab) \\
&= 1 + 2\sum_{k=1}^{K}|v_k|\,\|\boldsymbol{w}_k\|_2\left(\frac{1}{nJ}\sum_{i=1}^{n}\sum_{j=1}^{J}\phi\left(\boldsymbol{u}_k^{\mathsf{T}}\boldsymbol{x}_i^{(S_j)} - t_k\right)\right) \\
&\qquad \cdot\sqrt{\Bigg\|\frac{1}{nJ}\sum_{i=1}^{n}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\boldsymbol{x}_i^{(S_j)}\Bigg\|^2 + \Bigg(\frac{1}{nJ}\sum_{i=1}^{n}\sum_{j=1}^{J} m_{k,i}^{(S_j)}\Bigg)^2} \\
&= 1 + 2\sum_{k=1}^{K}|v_k|\,\|\boldsymbol{w}_k\|_2\,\mathbb{E}\left[\phi\left(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}} - t\right)\right]\sqrt{\mathbb{P}\left(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}} > t\right)^2 + \left\|\mathbb{E}\left[\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}}\mathbb{1}\left\{\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}} > t\right\}\right]\right\|_2^2} \\
&= 1 + 2\sum_{k=1}^{K}|v_k|\,\|\boldsymbol{w}_k\|_2\,g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}_k, t_k), \qquad \boldsymbol{u}_k = \frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|_2}, \quad t_k = \frac{b_k}{\|\boldsymbol{w}_k\|_2}.
\end{aligned}
$$

Then combining (33) and the above inequality yields the claim. □

Lemma D.3. *Consider the model* (4)

$$
f(\boldsymbol{x}) \;=\; \sum_{k=1}^{K}\frac{v_k}{J}\sum_{j=1}^{J}\phi\left(\boldsymbol{w}_k^{\mathsf{T}}\pi_j(\boldsymbol{x}) - b_k\right) + \beta.
$$

where the input satisfies $\|\boldsymbol{x}\|_2 \leq R$, each patch extractor $\pi_j : \mathbb{R}^d \to \mathbb{R}^m$ is a coordinate projection, and $\phi(t) = \max\{0, t\}$. Let $\boldsymbol{\theta} = \left(\boldsymbol{w}_1^{\mathsf{T}}, \ldots, \boldsymbol{w}_K^{\mathsf{T}}, b_1, \ldots, b_K, v_1, \ldots, v_K, \beta\right)^{\mathsf{T}}$ collect all parameters. Assume $f_{\boldsymbol{\theta}}(\boldsymbol{x})$ is twice differentiable with respect to $\boldsymbol{\theta}$ at $\boldsymbol{x}$, i.e., for all k and $S_j \in \mathcal{S}$ we have $\boldsymbol{w}_k^{\mathsf{T}}\boldsymbol{x}^{(S_j)} \neq b_k$. Then for any perturbation vector $\boldsymbol{\omega}$ with $\|\boldsymbol{\omega}\|_2 = 1$, it holds that

$$
\left|\boldsymbol{\omega}^{\mathsf{T}}\,\nabla_{\boldsymbol{\theta}}^2 f_{\boldsymbol{\theta}}(\boldsymbol{x})\,\boldsymbol{\omega}\right| \;\leq\; 2\,(R+1).
$$

Proof. Write $\boldsymbol{\theta} = \left(\boldsymbol{w}_1^{\mathsf{T}}, \ldots, \boldsymbol{w}_K^{\mathsf{T}}, b_1, \ldots, b_K, v_1, \ldots, v_K, \beta\right)^{\mathsf{T}}$. The total number of parameters is $N = K \cdot m + K + K + 1 = K(m+2) + 1$.

Let the corresponding perturbation vector be

$$\boldsymbol{\omega} = \big(\boldsymbol{\alpha}_1^\mathsf{T}, \ldots, \boldsymbol{\alpha}_K^\mathsf{T},\, \delta_1, \ldots, \delta_K,\, \gamma_1, \ldots, \gamma_K,\, \iota\big)^\mathsf{T} \in \mathbb{R}^N,$$

where $\boldsymbol{\alpha}_k \in \mathbb{R}^m$ corresponds to $\boldsymbol{w}_k$, $\delta_k \in \mathbb{R}$ to b_k, $\gamma_k \in \mathbb{R}$ to v_k, and $\iota \in \mathbb{R}$ to β. The normalization constraint is

$$\|\boldsymbol{\omega}\|_2^2 = \sum_{k=1}^K \|\boldsymbol{\alpha}_k\|_2^2 + \sum_{k=1}^K \delta_k^2 + \sum_{k=1}^K \gamma_k^2 + \iota^2 = 1.$$

For the fixed input $\boldsymbol{x}$, set $\boldsymbol{x}^{(S_j)} := \pi_j(\boldsymbol{x}) \in \mathbb{R}^m$ and define the ReLU gate

$$m_k^{(S_j)} \;:=\; \mathbb{1}\big\{\boldsymbol{w}_k^\mathsf{T} \boldsymbol{x}^{(S_j)} > b_k\big\} \in \{0, 1\}.$$

By the twice-differentiability assumption, all gates are constant in a neighborhood of $\boldsymbol{\theta}$.

Within this gate-fixed region, $f_{\boldsymbol{\theta}}$ is affine in $(\boldsymbol{w}_k, b_k)$ once v_k is held fixed, and affine in v_k once $(\boldsymbol{w}_k, b_k)$ are held fixed. Therefore the only nonzero second partial derivatives inside the k-th neuron block are the mixed ones with v_k

$$\frac{\partial^2 f_{\boldsymbol{\theta}}}{\partial \boldsymbol{w}_k\, \partial v_k} = \frac{\partial}{\partial v_k}\left(\frac{1}{J}\sum_{S_j \in \mathcal{S}} v_k\, m_k^{(S_j)}\, \boldsymbol{x}^{(S_j)}\right) = \frac{1}{J}\sum_{j=1}^J m_k^{(S_j)}\, \boldsymbol{x}^{(S_j)} =: \boldsymbol{s}_k \in \mathbb{R}^m, \tag{35}$$

$$\frac{\partial^2 f_{\boldsymbol{\theta}}}{\partial b_k\, \partial v_k} = \frac{\partial}{\partial v_k}\left(\frac{1}{J}\sum_{j=1}^J v_k\, m_k^{(S_j)}\right) = -\frac{1}{J}\sum_{j=1}^J m_k^{(S_j)} =: t_k \in \mathbb{R}. \tag{36}$$

All other second derivatives inside the block vanish (as do any cross-neuron second derivatives and all those involving β). Hence, with the block variable $\theta_k := (\boldsymbol{w}_k^\mathsf{T},\, b_k,\, v_k)^\mathsf{T}$, we have

$$\nabla^2_{(\theta_k)} f_{\boldsymbol{\theta}}(\boldsymbol{x}) = \begin{pmatrix} \mathbf{0}_{m\times m} & \mathbf{0}_m & \boldsymbol{s}_k \\ \mathbf{0}_m^\mathsf{T} & 0 & t_k \\ \boldsymbol{s}_k^\mathsf{T} & t_k & 0 \end{pmatrix}. \tag{37}$$

The full quadratic form splits over neuron blocks:

$$\begin{aligned} \boldsymbol{\omega}^\mathsf{T}\, \nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x})\, \boldsymbol{\omega} &= \sum_{k=1}^K \begin{pmatrix} \boldsymbol{\alpha}_k^\mathsf{T} & \delta_k & \gamma_k \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \boldsymbol{s}_k \\ \mathbf{0}^\mathsf{T} & 0 & t_k \\ \boldsymbol{s}_k^\mathsf{T} & t_k & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}_k \\ \delta_k \\ \gamma_k \end{pmatrix} \\ &= \sum_{k=1}^K 2\,\gamma_k\big(\boldsymbol{\alpha}_k^\mathsf{T}\boldsymbol{s}_k + \delta_k\, t_k\big). \end{aligned} \tag{38}$$

Recall the definition of the notations (35) and (36),

$$|t_k| = \frac{1}{J}\sum_{j=1}^J m_k^{(S_j)} \le 1, \tag{39}$$

$$\|\boldsymbol{s}_k\|_2 = \left\|\frac{1}{J}\sum_{j=1}^J m_k^{(S_j)}\, \boldsymbol{x}^{(S_j)}\right\|_2 \le \frac{1}{J}\sum_{j=1}^J m_k^{(S_j)}\, \|\boldsymbol{x}^{(S_j)}\|_2 \le \frac{1}{J}\sum_{j=1}^J m_k^{(S_j)}\, R = R|t_k|, \tag{40}$$

because $\boldsymbol{x}^{(S_j)} = \pi_j(\boldsymbol{x})$ is a coordinate projection of $\boldsymbol{x}$, $\|\boldsymbol{x}^{(S_j)}\|_2 \le \|\boldsymbol{x}\|_2 \le R$.

Then for each k,

$$\begin{aligned} \big|2\,\gamma_k(\boldsymbol{\alpha}_k^\mathsf{T}\boldsymbol{s}_k + \delta_k\, t_k)\big| &\le 2\,|\gamma_k|\Big(\|\boldsymbol{\alpha}_k\|_2\, \|\boldsymbol{s}_k\|_2 + |\delta_k|\, t_k\Big) \quad \text{(Cauchy-Schwarz)} \\ (40) \Longrightarrow \quad &\le 2\,|\gamma_k|\Big(R\, t_k\, \|\boldsymbol{\alpha}_k\|_2 + |\delta_k|\, t_k\Big) \\ (39) \Longrightarrow \quad &\le 2\,|\gamma_k|\Big(R\, \|\boldsymbol{\alpha}_k\|_2 + |\delta_k|\Big), \end{aligned} \tag{41}$$

Summing over k for (41) and plug in (38), we have

$$\begin{aligned}\left|\boldsymbol{\omega}^{\mathsf{T}}\,\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x})\,\boldsymbol{\omega}\right| &\le 2\Big(R\sum_k |\gamma_k|\,\|\boldsymbol{\alpha}_k\|_2 + \sum_k |\gamma_k|\,|\delta_k|\Big)\\ \text{(Cauchy-Schwarz)} &\le 2\Big(R\sqrt{\textstyle\sum_k \gamma_k^2}\sqrt{\textstyle\sum_k \|\boldsymbol{\alpha}_k\|_2^2} + \sqrt{\textstyle\sum_k \gamma_k^2}\sqrt{\textstyle\sum_k \delta_k^2}\Big). \end{aligned} \tag{42}$$

Using the normalization $\sum_k \|\boldsymbol{\alpha}_k\|_2^2 + \sum_k \delta_k^2 + \sum_k \gamma_k^2 + \iota^2 = 1$, we have $\sqrt{\sum_k \gamma_k^2} \le 1$, $\sqrt{\sum_k \|\boldsymbol{\alpha}_k\|_2^2} \le 1$, and $\sqrt{\sum_k \delta_k^2} \le 1$. Thus $\left|\boldsymbol{\omega}^{\mathsf{T}}\,\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x})\,\boldsymbol{\omega}\right| \le 2\,(R+1)$. □

Theorem D.4 (Restate Theorem 5.3)**.** *Given a data set $\mathcal{D} = \{\boldsymbol{x}_i, y_i\}_{i=1}^n$ and a network model (4) with the set of local receptive fields $\mathcal{S}$, we have*

$$\sum_{k=1}^K |v_k|\,\|\boldsymbol{w}_k\|\,g_{\mathcal{D},\mathcal{S}}\left(\frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}, \frac{b_k}{\|\boldsymbol{w}_k\|}\right) \;\le\; \frac{1}{2}\left(\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta})) + 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})} - 1\right).$$

Proof. It suffices to prove the first assertion. Recall that by direct computation, the Hessian matrix of the loss function is expressed as

$$\nabla^2_{\boldsymbol{\theta}}\mathcal{L} = \underbrace{\frac{1}{n}\sum_{i=1}^n \nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)^{\mathsf{T}}}_{\boldsymbol{T}_{\mathcal{D}}} + \underbrace{\frac{1}{n}\sum_{i=1}^n (f(\boldsymbol{x}_i) - y_i)\nabla^2_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)}_{\boldsymbol{R}_{\mathcal{D}}}. \tag{43}$$

Let $\boldsymbol{\omega}$ be the unit eigenvector (i.e., $\|\boldsymbol{\omega}\|_2 = 1$) corresponding to the largest eigenvalue of the matrix $\boldsymbol{T}_{\mathcal{D}}$, the maximum eigenvalue of the Hessian matrix of the loss can be lower-bounded as follows:

$$\begin{aligned}\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta})) &\ge \boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta})\boldsymbol{\omega}\\ &\ge \underbrace{\lambda_{\max}(\boldsymbol{T}_{\mathcal{D}})}_{\text{(Term A)}} + \underbrace{\frac{1}{n}\sum_{i=1}^n (f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)\boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)\boldsymbol{\omega}}_{\text{(Term B)}}. \end{aligned} \tag{44}$$

According to Proposition D.2, Term A is lower bounded by

$$\lambda_{\max}\left(\boldsymbol{T}_{\mathcal{D}}\right) \ge 1 + 2\sum_{k=1}^K |v_k|\,\|\boldsymbol{w}_k\|\,g_{\mathcal{D},\mathcal{S}}\left(\frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}, \frac{b_k}{\|\boldsymbol{w}_k\|}\right), \tag{45}$$

For (Term B), an upper bound can be established using the training loss $\mathcal{L}(\boldsymbol{\theta})$ via the Cauchy-Schwarz inequality. This also employs a notable uniform upper bound for $|\boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_n)\boldsymbol{\omega}|$, as detailed in Lemma D.3:

$$|\text{(Term B)}| \le \sqrt{\frac{1}{n}\sum_{i=1}^n \left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i\right)^2}\cdot\sqrt{\frac{1}{n}\sum_{i=1}^n \left(\boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)\boldsymbol{\omega}\right)^2} \le 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})}. \tag{46}$$

Thus, we have

$$\lambda_{\max}\left(\boldsymbol{T}_{\mathcal{D}}\right) \le \lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta})) + |\text{(Term B)}| \le \lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta})) + 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})}. \tag{47}$$

Finally, we plug (45) into (47)

$$\sum_{k=1}^K |v_k|\,\|\boldsymbol{w}_k\|\,g_{\mathcal{D},\mathcal{S}}\left(\frac{\boldsymbol{w}_k}{\|\boldsymbol{w}_k\|}, \frac{b_k}{\|\boldsymbol{w}_k\|}\right) \;\le\; \frac{1}{2}\left(\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}(\boldsymbol{\theta})) + 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})} - 1\right).$$

□

D.1. Empirical Process for the Weight Function

We study uniform deviations of the empirical weight function $g_{\mathcal{D},\mathcal{S}}$ from its population counterpart $g_{\mathcal{P},\mathcal{S}}$ under i.i.d. sampling $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \sim \mathcal{P}_{\boldsymbol{X}}$. Although the collection $\{\pi_j(\boldsymbol{x}_i)\}_{i,j}$ is not i.i.d. across (i, j) in general, for each fixed j the patches $\{\pi_j(\boldsymbol{x}_i)\}_{i=1}^n$ are i.i.d. in $\mathbb{R}^m$.

Definition D.5. For any $S_j \in \mathcal{S}$, let $\boldsymbol{X}_{\mathcal{D}}^{(S_j)}$ be a random vector drawn uniformly at random from the training examples $\{\pi_j(\boldsymbol{x}_i)\}_{i=1}^n$ with patch extraction to the local receptive field S_j. For any $\boldsymbol{u} \in \mathbb{S}^{m-1}, t \in \mathbb{R}$, define the aggregated gate probability, ReLU margin, and gated first moment

$$\begin{aligned}
\bar{p}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) &:= \frac{1}{J} \sum_{j=1}^{J} \mathbb{P}\left(\boldsymbol{u}^\mathsf{T} \boldsymbol{X}_{\mathcal{D}}^{(S_j)} > t\right), \\
\bar{r}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) &:= \frac{1}{J} \sum_{j=1}^{J} \mathbb{E}\left[\phi\left(\boldsymbol{u}^\mathsf{T} \boldsymbol{X}_{\mathcal{D}}^{(S_j)} - t\right)\right], \\
\bar{\boldsymbol{a}}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) &:= \frac{1}{J} \sum_{j=1}^{J} \mathbb{E}\left[\boldsymbol{X}_{\mathcal{D}}^{(S_j)} \mathbb{1}\left\{\boldsymbol{u}^\mathsf{T} \boldsymbol{X}_{\mathcal{D}}^{(S_j)} > t\right\}\right] \in \mathbb{R}^m.
\end{aligned} \tag{48}$$

Based on these quantities, define the empirical weight function

$$g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) := \bar{r}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) \sqrt{\bar{p}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t)^2 + \left\|\bar{\boldsymbol{a}}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t)\right\|_2^2}. \tag{49}$$

For each $j \in [J]$ and $(\boldsymbol{u}, t) \in \mathbb{S}^{m-1} \times [-1, 1]$, define the population components

$$p_j(\boldsymbol{u}, t) := \mathbb{P}\big(\boldsymbol{u}^\mathsf{T} \pi_j(\boldsymbol{X}) > t\big), \quad r_j(\boldsymbol{u}, t) := \mathbb{E}\big[\phi(\boldsymbol{u}^\mathsf{T} \pi_j(\boldsymbol{X}) - t)\big], \quad \boldsymbol{a}_j(\boldsymbol{u}, t) := \mathbb{E}\big[\pi_j(\boldsymbol{X}) \mathbb{1}\{\boldsymbol{u}^\mathsf{T} \pi_j(\boldsymbol{X}) > t\}\big].$$

Let $\bar{p}_{\mathcal{P},\mathcal{S}}$, $\bar{r}_{\mathcal{P},\mathcal{S}}$, and $\bar{\boldsymbol{a}}_{\mathcal{P},\mathcal{S}}$ denote their averages over $j \in [J]$. Define

$$g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t) := \bar{r}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t) \sqrt{\bar{p}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t)^2 + \|\bar{\boldsymbol{a}}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t)\|_2^2}.$$

Lemma D.6 (Complexity of the component classes)**.** *Under the constraint $\|\boldsymbol{z}\|_2 \le 1$, the following classes are uniformly bounded VC-subgraph classes with VC-subgraph dimension $O(m)$:*

$$\mathcal{H} := \left\{\boldsymbol{z} \mapsto \mathbb{1}\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} > t\} : \boldsymbol{u} \in \mathbb{S}^{m-1},\, t \in [-1, 1]\right\},$$

$$\mathcal{F}_{\mathrm{ReLU}} := \left\{\boldsymbol{z} \mapsto \phi(\boldsymbol{u}^\mathsf{T} \boldsymbol{z} - t) : \boldsymbol{u} \in \mathbb{S}^{m-1},\, t \in [-1, 1]\right\},$$

and

$$\mathcal{A} := \left\{\boldsymbol{z} \mapsto \boldsymbol{v}^\mathsf{T} \boldsymbol{z} \mathbb{1}\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} > t\} : \boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{m-1},\, t \in [-1, 1]\right\}.$$

Moreover, $\mathcal{H}$ and $\mathcal{A}$ are bounded by 1, and $\mathcal{F}_{\mathrm{ReLU}}$ is bounded by 2.

Proof. The class $\mathcal{H}$ is the class of halfspace indicators in $\mathbb{R}^m$, hence has VC-dimension at most $m + 1$.

For $\mathcal{F}_{\mathrm{ReLU}}$, the subgraph of $\boldsymbol{z} \mapsto \phi(\boldsymbol{u}^\mathsf{T} \boldsymbol{z} - t)$ is

$$\{(\boldsymbol{z}, s) : s < \phi(\boldsymbol{u}^\mathsf{T} \boldsymbol{z} - t)\} = \big(\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} > t\} \cap \{s < \boldsymbol{u}^\mathsf{T} \boldsymbol{z} - t\}\big) \cup \big(\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} \le t\} \cap \{s < 0\}\big).$$

For $\mathcal{A}$, the subgraph of $\boldsymbol{z} \mapsto \boldsymbol{v}^\mathsf{T} \boldsymbol{z} \mathbb{1}\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} > t\}$ is

$$\{(\boldsymbol{z}, s) : s < \boldsymbol{v}^\mathsf{T} \boldsymbol{z} \mathbb{1}\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} > t\}\} = \big(\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} > t\} \cap \{s < \boldsymbol{v}^\mathsf{T} \boldsymbol{z}\}\big) \cup \big(\{\boldsymbol{u}^\mathsf{T} \boldsymbol{z} \le t\} \cap \{s < 0\}\big).$$

Both subgraph classes are constant-size Boolean combinations of halfspaces in $\mathbb{R}^{m+1}$. By the standard growth-function closure argument for finite Boolean combinations of VC classes, their VC dimensions are $O(m)$. Equivalently, their underlying real-valued function classes have pseudo-dimension $O(m)$, see (Anthony & Bartlett, 1999, Ch. 3 and Definition 11.2). The boundedness claims follow from $\|\boldsymbol{z}\|_2 \le 1$, $t \in [-1, 1]$, and $\|\boldsymbol{u}\|_2 = \|\boldsymbol{v}\|_2 = 1$. □

Theorem D.7 (Uniform deviation for $g_{\mathcal{D},\mathcal{S}}$)**.** *Assume $\|\pi_j(\boldsymbol{X})\|_2 \le 1$ almost surely for all $j \in [J]$. There exists a universal constant $C_{\mathrm{ep}} > 0$ such that for every $\delta \in (0,1)$, with probability at least $1-\delta$,*

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{m-1},\, t\in[-1,1]} \left|g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u},t) - g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u},t)\right| \le C_{\mathrm{ep}}\sqrt{\frac{m+\log(2J/\delta)}{n}}.$$

Proof. For each fixed $j \in [J]$, let P_j denote the law of $\boldsymbol{Z}_j := \pi_j(\boldsymbol{X}) \in \mathbb{R}^m$, and let $P_{n,j} := n^{-1}\sum_{i=1}^n \delta_{\pi_j(\boldsymbol{x}_i)}$. Define

$$\hat{p}_j(\boldsymbol{u},t) := P_{n,j}\mathbb{1}\{\boldsymbol{u}^\mathsf{T}\boldsymbol{z} > t\}, \qquad \hat{r}_j(\boldsymbol{u},t) := P_{n,j}\phi(\boldsymbol{u}^\mathsf{T}\boldsymbol{z} - t), \qquad \hat{\boldsymbol{a}}_j(\boldsymbol{u},t) := P_{n,j}\left[\boldsymbol{z}\mathbb{1}\{\boldsymbol{u}^\mathsf{T}\boldsymbol{z} > t\}\right].$$

By the VC-subgraph entropy bound and maximal inequality (van der Vaart & Wellner, 1996, Theorems 2.6.7 and 2.14.1), together with McDiarmid's inequality, any class $\mathcal{F}$ uniformly bounded by B with VC-subgraph dimension V satisfies

$$\sup_{f\in\mathcal{F}} |(P_n - P)f| \lesssim B\sqrt{\frac{V+\log(1/\eta)}{n}}$$

with probability at least $1-\eta$.

Applying this bound to $\mathcal{H}$ and $\mathcal{F}_{\mathrm{ReLU}}$ from Lemma D.6, for every fixed j and $\eta \in (0,1)$, with probability at least $1-\eta$,

$$\sup_{\boldsymbol{u},t} |\hat{p}_j(\boldsymbol{u},t) - p_j(\boldsymbol{u},t)| \lesssim \sqrt{\frac{m+\log(1/\eta)}{n}}, \qquad \sup_{\boldsymbol{u},t} |\hat{r}_j(\boldsymbol{u},t) - r_j(\boldsymbol{u},t)| \lesssim \sqrt{\frac{m+\log(1/\eta)}{n}},$$

where all suprema are over $\boldsymbol{u} \in \mathbb{S}^{m-1}$ and $t \in [-1,1]$.

For the vector moment, use duality:

$$\begin{aligned}\sup_{\boldsymbol{u},t} \|\hat{\boldsymbol{a}}_j(\boldsymbol{u},t) - \boldsymbol{a}_j(\boldsymbol{u},t)\|_2 &= \sup_{\boldsymbol{u},t}\sup_{\boldsymbol{v}\in\mathbb{S}^{m-1}} \left|\boldsymbol{v}^\mathsf{T}\left(\hat{\boldsymbol{a}}_j(\boldsymbol{u},t) - \boldsymbol{a}_j(\boldsymbol{u},t)\right)\right| \\ &= \sup_{\boldsymbol{u},\boldsymbol{v},t} \left|(P_{n,j} - P_j)\left[\boldsymbol{v}^\mathsf{T}\boldsymbol{z}\mathbb{1}\{\boldsymbol{u}^\mathsf{T}\boldsymbol{z} > t\}\right]\right|.\end{aligned}$$

The last supremum is over the class $\mathcal{A}$ from Lemma D.6. Hence, with probability at least $1-\eta$,

$$\sup_{\boldsymbol{u},t} \|\hat{\boldsymbol{a}}_j(\boldsymbol{u},t) - \boldsymbol{a}_j(\boldsymbol{u},t)\|_2 \lesssim \sqrt{\frac{m+\log(1/\eta)}{n}}.$$

Taking $\eta = \delta/(3J)$ and applying a union bound over the three component bounds and over $j \in [J]$, we obtain that, with probability at least $1-\delta$, simultaneously for all $j \in [J]$,

$$\sup_{\boldsymbol{u},t} |\hat{p}_j(\boldsymbol{u},t) - p_j(\boldsymbol{u},t)| \le \varepsilon, \qquad \sup_{\boldsymbol{u},t} |\hat{r}_j(\boldsymbol{u},t) - r_j(\boldsymbol{u},t)| \le \varepsilon, \qquad \sup_{\boldsymbol{u},t} \|\hat{\boldsymbol{a}}_j(\boldsymbol{u},t) - \boldsymbol{a}_j(\boldsymbol{u},t)\|_2 \le \varepsilon,$$

where all suprema are over $\boldsymbol{u} \in \mathbb{S}^{m-1}$ and $t \in [-1,1]$, and $\varepsilon := C_0\sqrt{(m+\log(3J/\delta))/n}$ for a universal constant $C_0 > 0$. Averaging over j does not increase the deviations. Therefore, uniformly over $(\boldsymbol{u},t) \in \mathbb{S}^{m-1}\times[-1,1]$,

$$|\bar{p}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u},t) - \bar{p}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u},t)| \le \varepsilon, \qquad |\bar{r}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u},t) - \bar{r}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u},t)| \le \varepsilon, \qquad \|\bar{\boldsymbol{a}}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u},t) - \bar{\boldsymbol{a}}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u},t)\|_2 \le \varepsilon.$$

It remains to pass from the components to g. For both empirical and population quantities,

$$0 \le \bar{p} \le 1, \qquad 0 \le \bar{r} \le 2, \qquad \|\bar{\boldsymbol{a}}\|_2 \le \bar{p} \le 1.$$

Define $F(r,p,\boldsymbol{a}) := r\sqrt{p^2 + \|\boldsymbol{a}\|_2^2}$. On this domain,

$$|F(r,p,\boldsymbol{a}) - F(r',p',\boldsymbol{a}')| \le \sqrt{2}\,|r-r'| + 2\sqrt{(p-p')^2 + \|\boldsymbol{a}-\boldsymbol{a}'\|_2^2}.$$

Thus, uniformly over $(\boldsymbol{u},t) \in \mathbb{S}^{m-1}\times[-1,1]$,

$$\left|g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u},t) - g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u},t)\right| \lesssim \varepsilon \lesssim \sqrt{\frac{m+\log(2J/\delta)}{n}},$$

after adjusting the universal constant. This proves the theorem. □

D.2. The Computation of the Population Weight Functions for Uniform Distributions on Spheres

Assume that $\mathcal{P}_{\boldsymbol{X}}$ is a uniform distribution on $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Here we compute the population version of the components in the weighted function $g_{\mathcal{D},\mathcal{S}}$ defined in Definition D.5.

Let $d \geq 2$, $m < d$. Draw $\boldsymbol{X} \sim \mathrm{Uniform}(\mathbb{S}^{d-1}) \subset \mathbb{R}^d$, set $\boldsymbol{Z} = \mathrm{proj}(\boldsymbol{X}) \in \mathbb{R}^m$ by projecting the first m coordinates, and fix any unit $\boldsymbol{u} \in \mathbb{R}^m$. Define

$$Z := \boldsymbol{u}^\top \boldsymbol{Z} \in [-1, 1]. \tag{50}$$

By rotational invariance in $\mathbb{R}^d$, Z has the onecoordinate marginal of $\mathrm{Uniform}(\mathbb{S}^{d-1})$, hence its density is

$$f_d(z) = c_d\,(1-z^2)^{\frac{d-3}{2}} \mathbb{1}\{-1 \leq z \leq 1\}, \qquad c_d := \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)}. \tag{51}$$

All bounds below are independent of m and $\boldsymbol{u}$, depending only on d.

For shorthand, write $\alpha := \frac{d-3}{2} \geq -\frac{1}{2}$ and observe

$$(1-z^2)^\alpha = \big[(1-z)(1+z)\big]^\alpha, \qquad z \in [-1, 1], \tag{52}$$

together with the elementary bounds $1 \leq 1+z \leq 2$ for $z \in [0, 1]$.

Lemma D.8 ((Tail of Z))**.** *For all $t \in [0, 1)$,*

$$\frac{c_d}{\alpha+1}\,(1-t)^{\alpha+1} \;\leq\; \mathbb{P}(Z > t) \;\leq\; \frac{c_d\,2^\alpha}{\alpha+1}\,(1-t)^{\alpha+1}. \tag{53}$$

Consequently $\mathbb{P}(Z > t) \asymp (1-t)^{\frac{d-1}{2}}$ as $t \to 1^-$.

Proof. Using (51) and (52),

$$\mathbb{P}(Z > t) = \int_t^1 c_d\,(1-z^2)^\alpha\,dz = c_d \int_t^1 \big[(1-z)(1+z)\big]^\alpha\,dz. \tag{54}$$

Since $1 \leq 1+z \leq 2$ for $z \in [t, 1]$, we have

$$c_d \int_t^1 (1-z)^\alpha\,dz \;\leq\; \mathbb{P}(Z > t) \;\leq\; c_d\,2^\alpha \int_t^1 (1-z)^\alpha\,dz. \tag{55}$$

Evaluating $\int_t^1 (1-z)^\alpha\,dz = \frac{(1-t)^{\alpha+1}}{\alpha+1}$ (valid for $\alpha > -1$; here $\alpha \geq -\frac{1}{2}$), we obtain (53). □

Lemma D.9. *For all $t \in [0, 1)$,*

$$\frac{c_d}{(\alpha+1)(\alpha+2)}\,(1-t)^{\alpha+2} \;\leq\; \mathbb{E}\big[\phi(Z-t)\big] \;\leq\; \frac{c_d\,2^\alpha}{(\alpha+1)(\alpha+2)}\,(1-t)^{\alpha+2}. \tag{56}$$

Consequently $\mathbb{E}[\phi(Z-t)] \asymp (1-t)^{\frac{d+1}{2}}$ as $t \uparrow 1$.

Proof. By definition and (51),

$$\mathbb{E}[\phi(Z-t)] = \int_t^1 (z-t) f_d(z)\,dz = c_d \int_t^1 (z-t)\big[(1-z)(1+z)\big]^\alpha\,dz. \tag{57}$$

Bounding $1 \leq 1+z \leq 2$ for $z \in [t, 1]$ yields

$$c_d \int_t^1 (z-t)(1-z)^\alpha\,dz \;\leq\; \mathbb{E}[\phi(Z-t)] \;\leq\; c_d\,2^\alpha \int_t^1 (z-t)(1-z)^\alpha\,dz. \tag{58}$$

Substitute $y = 1-z$ to compute the shared integral:

$$\int_t^1 (z-t)(1-z)^\alpha\,dz = \int_0^{1-t} \big[(1-t)-y\big]\,y^\alpha\,dy = \frac{(1-t)^{\alpha+2}}{(\alpha+1)(\alpha+2)}, \tag{59}$$

valid for $\alpha > -1$. This gives (56). □

Proposition D.10. *There exist absolute constants (depending only on d)*

$$c_L(d) := \frac{c_d^2}{\big(\alpha+1\big)^2\big(\alpha+2\big)}, \qquad c_U(d) := \frac{c_d^2\, 2^{\,2\alpha}}{\big(\alpha+1\big)^2\big(\alpha+2\big)}, \tag{60}$$

such that for all $t \in [0,1)$,

$$c_L(d)\,(1-t)^{2\alpha+3} \;\le\; \mathbb{P}(Z>t)\cdot\mathbb{E}\big[\phi(Z-t)\big] \;\le\; c_U(d)\,(1-t)^{2\alpha+3}. \tag{61}$$

Equivalently, since $\alpha = \frac{d-3}{2}$,

$$\mathbb{P}(Z>t)\cdot\mathbb{E}\big[\phi(Z-t)\big] \asymp (1-t)^d \qquad \text{as } t\uparrow 1, \tag{62}$$

with explicit

$$c_L(d) = \frac{c_d^2}{\big(\frac{d-1}{2}\big)^2\big(\frac{d+1}{2}\big)}, \qquad c_U(d) = \frac{c_d^2\, 2^{\,d-3}}{\big(\frac{d-1}{2}\big)^2\big(\frac{d+1}{2}\big)}, \qquad c_d = \frac{\Gamma\big(\frac{d}{2}\big)}{\sqrt{\pi}\,\Gamma\big(\frac{d-1}{2}\big)}. \tag{63}$$

Proof. The inequality can be deduced by combining (53) and (56). □

Proposition D.11 (Boundary tail for projected radius)**.** *Fix integers $d \ge 2$ and $1 \le m < d$. Let $\boldsymbol{X} \sim \mathrm{Uniform}(\mathbb{S}^{d-1}) \subset \mathbb{R}^d$ and $\boldsymbol{Z} = \mathrm{proj}(\boldsymbol{X}) \in \mathbb{R}^m$. Then there exist universal constants*

$$c_L(d,m),\, c_U(d,m) \in (0,\infty)$$

depending only on d, m (and independent of the choice of projection and of t) such that for all $t \in (0, \frac{1}{4}]$,

$$c_L(d,m)\; t^{\frac{d-m}{2}} \;\le\; \mathbb{P}\big(\|\boldsymbol{Z}\| > 1-t\big) \;\le\; c_U(d,m)\; t^{\frac{d-m}{2}}. \tag{64}$$

In particular,

$$\mathbb{P}\big(\|\boldsymbol{Z}\| > 1-t\big) \asymp t^{\frac{d-m}{2}} \qquad \text{as } t\downarrow 0, \tag{65}$$

with one admissible choice

$$c_L(d,m) = \frac{2^{-|\frac{m}{2}-1|}}{\frac{d-m}{2}\, B\big(\frac{m}{2},\frac{d-m}{2}\big)}, \qquad c_U(d,m) = \frac{2^{\,|\frac{m}{2}-1|+\frac{d-m}{2}}}{\frac{d-m}{2}\, B\big(\frac{m}{2},\frac{d-m}{2}\big)}, \tag{66}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ denotes the Beta function.

Proof. Write $R := \|\boldsymbol{Z}\| \in [0,1]$ and $S := R^2$. A standard Gaussian-ratio representation gives

$$S \sim \mathrm{Beta}(a,b)\,, \qquad a := \frac{m}{2}, \quad b := \frac{d-m}{2}, \tag{67}$$

with density

$$f_S(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}\,\mathbf{1}_{\{0<x<1\}}. \tag{68}$$

For $t \in (0,1)$,

$$\mathbb{P}(R>1-t) = \mathbb{P}\big(S>(1-t)^2\big) = \frac{1}{B(a,b)}\int_{(1-t)^2}^{1} x^{a-1}(1-x)^{b-1}\,dx. \tag{69}$$

Set $y := 1-(1-t)^2 = 2t-t^2$. Then $y \in (0,2t)$ and for $t \in (0,\frac{1}{4}]$ we have $y \le \frac{1}{2}$ and $(1-y) \ge \frac{1}{2}$. Since $x \in [1-y,1]$, the elementary bound

$$2^{-|a-1|} \;\le\; x^{a-1} \;\le\; 2^{|a-1|} \qquad \text{for all } x \in \big[\tfrac{1}{2},1\big] \tag{70}$$

holds deterministically (because $\ln x \in [-\ln 2, 0]$ and $x^{a-1} = e^{(a-1)\ln x}$). Using (70) in (69) and integrating the $(1-x)^{b-1}$ part exactly gives

$$\frac{2^{-|a-1|}}{B(a,b)}\int_{1-y}^{1}(1-x)^{b-1}\,dx \;\le\; \mathbb{P}(R>1-t) \;\le\; \frac{2^{|a-1|}}{B(a,b)}\int_{1-y}^{1}(1-x)^{b-1}\,dx. \tag{71}$$

Since $\int_{1-y}^{1}(1-x)^{b-1}\,dx = \frac{y^b}{b}$, we obtain

$$\frac{2^{-|a-1|}}{b\,B(a,b)}\,y^b \;\le\; \mathbb{P}(R>1-t) \;\le\; \frac{2^{|a-1|}}{b\,B(a,b)}\,y^b. \tag{72}$$

Finally, because $t \le y \le 2t$, we have $t^b \le y^b \le (2t)^b$, and (72) yields

$$\frac{2^{-|a-1|}}{b\,B(a,b)}\,t^b \;\le\; \mathbb{P}(R>1-t) \;\le\; \frac{2^{|a-1|+b}}{b\,B(a,b)}\,t^b, \tag{73}$$

which is exactly (64) with the explicit constants in (66). This proves $\mathbb{P}(\|\boldsymbol{Z}\|>1-t) \asymp t^b = t^{(d-m)/2}$ as $t \downarrow 0$. □

E. Proof of Theorem 5.4

Theorem E.1 (Detailed version of Theorem 5.4)**.** *Suppose $\mathcal{P}$ is a joint distribution of $(\boldsymbol{x}, y)$. Assume that the marginal distribution of $\boldsymbol{x}$ is $\mathrm{Uniform}(\mathbb{S}^{d-1})$ and the marginal distribution of y is supported on $[-D, D]$ for some $D > 0$. Fix a dataset $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$, where each data point is drawn i.i.d. from $\mathcal{P}$. Let $M \ge D$. If $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\mathrm{BEoS}}^{\mathcal{S}_{\mathrm{CN}}}(\eta, \mathcal{D})$ and $\|f_{\boldsymbol{\theta}}\|_\infty \le M$, then, with probability $\ge 1-2\delta$, we have that for the plug-in risk estimator $\widehat{\mathcal{R}}_{\mathcal{D}}(f) := \frac{1}{n}\sum_{i=1}^n (f(\boldsymbol{x}_i)-y_i)^2$,*

$$\begin{aligned}&\left|\mathop{\mathbb{E}}_{(\boldsymbol{x},y)\sim\mathcal{P}}\left[(f_{\boldsymbol{\theta}}(\boldsymbol{x})-y)^2\right] - \widehat{\mathcal{R}}(f_{\boldsymbol{\theta}})\right| \\ &\lesssim_d JM^2\,\varepsilon^{\frac{d-m}{2}} + \left(A\,\varepsilon^{-d}\right)^{\frac{d}{d+3}} M^{\frac{d+6}{d+3}}\,n^{-\frac{1}{2}},\end{aligned} \tag{74}$$

for any $\varepsilon \in (0,1)$ such that $\varepsilon^d \gtrsim d^2\sqrt{\frac{m+\log(J/\delta)}{n}}$[1]*. Here $A = \frac{1}{\eta} - \frac{1}{2} + 4M$, and $\lesssim_d$ hides constants depending only on d and logarithmic factors in n and (J/δ).*

Proof. For $\varepsilon \in (0,1)$, we define

$$\mathsf{I}_\varepsilon^{\mathrm{all}} := \left\{\boldsymbol{x} \in \mathbb{S}^{d-1} : \max_{S_j \in \mathcal{S}} \|\boldsymbol{x}^{(S_j)}\| \le 1-\varepsilon\right\}, \tag{75}$$

$$\mathsf{O}_\varepsilon^{\mathrm{any}} := \left\{\boldsymbol{x} \in \mathbb{S}^{d-1} : \exists\, S_j \in \mathcal{S},\ \|\boldsymbol{x}^{(S_j)}\| > 1-\varepsilon\right\}. \tag{76}$$

Then $\mathbb{S}^{d-1} = \mathsf{I}_\varepsilon^{\mathrm{all}} \cup \mathsf{O}_\varepsilon^{\mathrm{any}}$ (disjoint). According to the law of total expectation, the population risk is decomposed into

$$\mathop{\mathbb{E}}_{(\boldsymbol{x},y)\sim\mathcal{P}}\left[(f(\boldsymbol{x})-y)^2\right] = \mathbb{P}(\boldsymbol{x} \in \mathsf{O}_\varepsilon^{\mathrm{any}}) \cdot \mathbb{E}_{\mathsf{O}}\left[(f(\boldsymbol{x})-y)^2\right] + \mathbb{P}(\boldsymbol{x} \in \mathsf{I}_\varepsilon^{\mathrm{all}}) \cdot \mathbb{E}_{\mathsf{I}}\left[(f(\boldsymbol{x})-y)^2\right], \tag{77}$$

where $\mathbb{E}_{\mathsf{O}}$ means that $\{\boldsymbol{x}, y\}$ is a new sample from the data distribution conditioned on $\boldsymbol{x} \in \mathsf{O}_\varepsilon^{\mathrm{any}}$ and $\mathbb{E}_{\mathsf{I}}$ means that $(\boldsymbol{x}, y)$ is a new sample from the data distribution conditioned on $\boldsymbol{x} \in \mathsf{I}_\varepsilon^{\mathrm{all}}$.

Similarly, we also have this decomposition for empirical risk

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n (f(\boldsymbol{x}_i)-y_i)^2 &= \frac{1}{n}\left(\sum_{i\in I}(f(\boldsymbol{x}_i)-y_i)^2 + \sum_{i\in O}(f(\boldsymbol{x}_i)-y_i)^2\right) \\ &= \frac{n_I}{n}\frac{1}{n_I}\sum_{i\in I}(f(\boldsymbol{x}_i)-y_i)^2 + \frac{n_O}{n}\frac{1}{n_O}\sum_{i\in O}(f(\boldsymbol{x}_i)-y_i)^2,\end{aligned} \tag{78}$$

where I is the set of data points with $\boldsymbol{x}_i \in \mathsf{I}^{\mathrm{all}}$ and O is the set of data points with $\boldsymbol{x}_i \in \mathsf{O}_\varepsilon^{\mathrm{any}}$. Then the generalization gap

[1] We only need $\mathrm{poly}(d)$ samples to make the feasible choice of ε non-vacuous. Here we hide the universal constant.

can be decomposed into

$$|R(f) - \hat{R}_{\mathcal{D}}(f)| \le \mathbb{P}(\boldsymbol{x} \in \mathsf{O}_{\varepsilon}^{\text{any}}) \cdot \mathbb{E}_{\mathsf{O}}\left[(f_{\boldsymbol{\theta}}(\boldsymbol{x}) - y)^2\right] + \frac{n_O}{n}\frac{1}{n_O}\sum_{i\in O}(f(\boldsymbol{x}_i) - y_i)^2 \tag{79}$$

$$+ \left|\mathbb{P}(\boldsymbol{x} \in \mathsf{I}_{\varepsilon}^{\text{all}}) - \frac{n_I}{n}\right| \frac{1}{n_I}\sum_{i\in I}(f(\boldsymbol{x}_i) - y_i)^2 \tag{80}$$

$$+ \mathbb{P}(\boldsymbol{x} \in \mathsf{I}_{\varepsilon}^{\text{all}}) \cdot \left|\mathbb{E}_{\mathsf{I}}\left[(f(\boldsymbol{x}) - y)^2\right] - \frac{1}{n_I}\sum_{i\in I}(f(\boldsymbol{x}_i) - y_i)^2\right|. \tag{81}$$

Using the property that the marginal distribution of $\boldsymbol{x}$ is $\mathrm{Uniform}(\mathbb{S}^{d-1})$ and its concentration property (Proposition D.11 + union bound),

$$\mathbb{P}\big(\mathsf{O}_{\varepsilon}^{\text{any}}\big) = \mathbb{P}\Big(\bigcup_{S_j\in\mathcal{S}}\{\|\boldsymbol{x}^{(S_j)}\| > 1-\varepsilon\}\Big) \lesssim_d J \cdot \varepsilon^{\frac{d-m}{2}}. \tag{82}$$

The Hoeffding inequality guarantees that, with probability at least $1 - \delta/3$,

$$\frac{n_O}{n} \le J \cdot \varepsilon^{\frac{d-m}{2}} + \sqrt{\frac{\log(6/\delta)}{n}} \tag{83}$$

Therefore, we may conclude that

$$(79) \lessapprox_d JM^2\varepsilon^{\frac{d-m}{2}}, \tag{84}$$

where $\lessapprox_d$ hides the constants that could depend on d and logarithmic factors of $1/\delta$.

For the term (80), with probability $1 - \delta/3$

$$\begin{cases} \left|\mathbb{P}(\boldsymbol{x} \in \mathsf{I}_{\varepsilon}^{\text{all}}) - \frac{n_I}{n}\right| & \lesssim \sqrt{\frac{\varepsilon^{(d-m)/2}\log(6J/\delta)}{n}}, \\ \frac{1}{n_I}\sum_{i\in I}(f(\boldsymbol{x}_i) - y_i)^2 & \lesssim M^2 \end{cases} \tag{85}$$

so we may also conclude that

$$(80) \lesssim M^2\sqrt{\frac{\varepsilon^{(d-m)/2}\log(6J/\delta)}{n}} \le M^2\sqrt{\frac{\log(6J/\delta)}{n}} \tag{86}$$

For the part of the interior (81), the scalar $\mathbb{P}(\boldsymbol{x} \in \mathbb{I}_{\varepsilon}^{d})$ is less than 1 with high-probability. Therefore, we just need to deal with the term

$$\mathbb{E}_{\mathbb{I}}\left[(f(\boldsymbol{x}) - y)^2\right] - \frac{1}{n_I}\sum_{i\in I}(f(\boldsymbol{x}_i) - y_i)^2. \tag{87}$$

Define the projected core index set $\mathcal{C}_{\varepsilon} := \{(\boldsymbol{u}, t) \in \mathbb{S}^{m-1} \times [-1, 1] : |t| \le 1 - \varepsilon\}$. For any $S_j \in \mathcal{S}$, $\boldsymbol{X} \sim \mathrm{Uniform}(\mathbb{S}^{d-1})$ and $\boldsymbol{X}^{(S_j)} = \pi_j(\boldsymbol{X})$,

- (53) shows that $\mathbb{P}(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}^{(S_j)} > t) \asymp (1-t)^{\frac{d-1}{2}}$, and
- (56) shows that $\mathbb{E}[\phi(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}^{(S_j)} - t)] \asymp (1-t)^{\frac{d+1}{2}}$.

Therefore we have

$$\left(\frac{1}{J}\sum_{j=1}^{J}\mathbb{P}(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}^{(S_j)} > t)\right)\left(\frac{1}{J}\sum_{j=1}^{J}\mathbb{E}[\phi(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}^{(S_j)} - t)]\right) \asymp (1-t)^d, \qquad t \uparrow 1. \tag{88}$$

According to (49) and the definition $g = \min\{\tilde{g}(\boldsymbol{u}, t), \tilde{g}(-\boldsymbol{u}, -t)\}$, we have the pointwise lower bound

$$g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t) \;\ge\; \min\Big\{\bar{r}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t)\,\bar{p}_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t),\; \bar{r}_{\mathcal{P},\mathcal{S}}(-\boldsymbol{u}, -t)\,\bar{p}_{\mathcal{P},\mathcal{S}}(-\boldsymbol{u}, -t)\Big\},$$

since $\sqrt{\bar{p}^2 + \|\bar{\boldsymbol{a}}\|_2^2} \geq \bar{p}$.

Under $\boldsymbol{X} \sim \mathrm{Uniform}(\mathbb{S}^{d-1})$, by rotational invariance, for each fixed j and unit $\boldsymbol{u}$ the quantities $\mathbb{P}(\boldsymbol{u}^\top \boldsymbol{X}^{(S_j)} > t)$ and $\mathbb{E}[\phi(\boldsymbol{u}^\top \boldsymbol{X}^{(S_j)} - t)]$ depend on t only through the scalar marginal Z in (51). Moreover both are non-increasing in t. Hence the product

$$h(t) := \mathbb{P}(Z > t) \cdot \mathbb{E}[\phi(Z - t)]$$

is non-increasing for $t \in [0, 1)$, and for any $t \in [-1, 1]$ we have

$$g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t) \;\gtrsim_d\; h(|t|).$$

Therefore, for the core $\mathcal{C}_\varepsilon = \{(\boldsymbol{u}, t) : |t| \leq 1 - \varepsilon\}$,

$$g_{\mathcal{P},\mathcal{S},\min}(\varepsilon) := \inf_{(\boldsymbol{u},t)\in\mathcal{C}_\varepsilon} g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t) \;\gtrsim_d\; h(1 - \varepsilon).$$

By Proposition D.10, $h(1 - \varepsilon) \asymp \varepsilon^d$, hence there exists $c_g(d) > 0$ such that

$$g_{\mathcal{P},\mathcal{S},\min}(\varepsilon) \;\geq\; c_g(d)\, \varepsilon^d. \tag{89}$$

On the simultaneous core $\mathsf{I}_\varepsilon^{\mathrm{all}}$, for *every* local receptive field S_j and any unit $\boldsymbol{u} \in \mathbb{R}^m$,

$$\|\boldsymbol{x}^{(S_j)}\| \leq 1 - \varepsilon \;\Rightarrow\; \begin{cases} t \geq 1 - \varepsilon \;\Rightarrow\; \phi(\boldsymbol{u}^\top \boldsymbol{x}^{(S_j)} - t) = 0, \\ t \leq -1 + \varepsilon \;\Rightarrow\; \phi(\boldsymbol{u}^\top \boldsymbol{x}^{(S_j)} - t) = \boldsymbol{u}^\top \boldsymbol{x}^{(S_j)} - t \text{ (affine)}. \end{cases} \tag{90}$$

Therefore all *large-offset* units ($|t| \geq 1 - \varepsilon$) are affine on $\mathsf{I}_\varepsilon^{\mathrm{all}}$ *simultaneously across all views* and can be absorbed into a global affine term. The remaining *core-offset* units with $|t| \leq 1 - \varepsilon$ are controlled by the $g_{\mathcal{S}}$-weighted variation. Using $|f_\theta|_{\mathrm{V}_g} \leq A$ and (89),

$$\Big|f_\theta\Big|_{\mathrm{V}(\mathsf{I}_\varepsilon^{\mathrm{all}})} \leq \frac{|f_\theta|_{\mathrm{V}_g}}{g_{\mathcal{P},\mathcal{S},\min}(\varepsilon)} \;\lesssim\; A\, \varepsilon^{-d}. \tag{91}$$

Therefore, we may leverage the generalization bounds for the unweighted path-norm constraint (see Lemma C.7) to deduce that with probability at least $1 - \delta/3$,

$$(81) \lessapprox_d (A\, \varepsilon^{-d})^{\frac{d}{d+3}}\, M^{\frac{d+6}{d+3}}\, n^{-\frac{1}{2}}, \tag{92}$$

Combining (84), (86) and (92), we obtain

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_{\mathrm{BEoS}}(\eta,\mathcal{D})} \mathrm{GenGap}(f_{\boldsymbol{\theta}}; \mathcal{D}) \;\lessapprox_d\; JM^2 \varepsilon^{\frac{d-m}{2}} + (A\varepsilon^{-d})^{\frac{d}{d+3}}\, M^{\frac{d+6}{d+3}}\, n^{-\frac{1}{2}}. \tag{93}$$

According to standard empirical process theory (see Theorem D.7), for any $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\sup_{\substack{\boldsymbol{u}\in\mathbb{S}^{m-1} \\ t\in[-1,1]}} \big|g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) - g_{\mathcal{P},\mathcal{S}}(\boldsymbol{u}, t)\big| \leq C_{\mathrm{ep}} \sqrt{\frac{m + \log(2J/\delta)}{n}} =: \zeta_n. \tag{94}$$

Consequently, for any $\varepsilon \in (0, 1)$,

$$g_{\mathcal{D},\mathcal{S},\min}(\varepsilon) \;:=\; \inf_{\substack{\boldsymbol{u}\in\mathbb{S}^{m-1} \\ |t|\leq 1-\varepsilon}} g_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) \;\geq\; g_{\mathcal{P},\mathcal{S},\min}(\varepsilon) - \zeta_n, \tag{95}$$

where $g_{\mathcal{P},\mathcal{S},\min}(\varepsilon)$ is the corresponding population quantity.

For $\mathcal{P}_{\boldsymbol{X}} = \mathrm{Uniform}(\mathbb{S}^{d-1})$, Proposition D.10 (see (64)) gives the explicit lower bound

$$g_{\mathcal{P},\mathcal{S},\min}(\varepsilon) \;\geq\; c_L(d)\, \varepsilon^d. \tag{96}$$

Therefore, a sufficient **validity condition** ensuring $g_{\mathcal{D},\mathcal{S},\min}(\varepsilon) \geq 2\zeta_n$ is

$$\varepsilon^d \;\geq\; \frac{2C_{\mathrm{ep}}}{c_L(d)}\sqrt{\frac{m+\log(2J/\delta)}{n}}. \tag{97}$$

The constant $C_{\mathrm{ep}} > 0$ is universal (it does not depend on d, m, n, J), and can be taken explicitly from any standard VC-/pseudo-dimension uniform convergence inequality; see, e.g., Mohri et al. (2018, Chapter 3) or Haussler (1992). Moreover, the constant $c_L(d)$ defined in Proposition D.10 satisfies

$$c_L(d) \;\gtrsim\; \frac{1}{d^2} \qquad (d \geq 3), \tag{98}$$

which follows from standard bounds on Gamma-function ratios (e.g. Gautschi's inequality; see dlm, §5.6(i)). Hence the admissible range for ε is nonempty (e.g. $\varepsilon \in [\varepsilon_{\min}, 1)$) as soon as $n \gtrsim \mathrm{poly}(d)\,(m+\log(2J/\delta))$. □

Corollary E.2. *Under the same conditions as Theorem 5.4, assume $d > 3$ and $1 \leq m < \frac{d(d-3)}{d+3}$. Let*

$$Q \coloneqq 3d^2 + 3d - md - 3m \qquad \textit{and} \qquad \varepsilon_{\min} \coloneqq \left(c_d\, d^2\sqrt{\frac{m+\log(J/\delta)}{n}}\right)^{1/d}.$$

Define the optimal choice

$$\varepsilon^\star \;\asymp\; A^{\frac{2d}{Q}}\, J^{-\frac{2(d+3)}{Q}}\, M^{-\frac{2d}{Q}}\, n^{-\frac{d+3}{D}}, \tag{99}$$

and choose the feasible/truncated value

$$\varepsilon^\dagger \coloneqq \max\{\varepsilon_{\min},\, \varepsilon^\star\}.$$

Then, with probability at least $1-\delta$,

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}^{\mathrm{SCN}}_{\mathrm{BEoS}}(\eta,\mathcal{D})} \left|\mathbb{E}\big[(f_{\boldsymbol{\theta}}(\boldsymbol{x})-y)^2\big] - \widehat{\mathcal{R}}_{\mathcal{D}}(f_{\boldsymbol{\theta}})\right| \;\lesssim_d\; JM^2(\varepsilon^\dagger)^{\frac{d-m}{2}} + \Big(A(\varepsilon^\dagger)^{-d}\Big)^{\frac{d}{d+3}} M^{\frac{d+6}{d+3}} n^{-1/2}. \tag{100}$$

Moreover, in the regime where $\varepsilon^\star \geq \varepsilon_{\min}$ (e.g., for n sufficiently large, treating A, J, M as constants), plugging $\varepsilon = \varepsilon^\star$ into (9) yields the optimized rate

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}^{\mathrm{SCN}}_{\mathrm{BEoS}}(\eta,\mathcal{D})} \left|\mathbb{E}\big[(f_{\boldsymbol{\theta}}(\boldsymbol{x})-y)^2\big] - \widehat{\mathcal{R}}_{\mathcal{D}}(f_{\boldsymbol{\theta}})\right| \;\lesssim_d\; A^{\alpha_A}\, J^{\alpha_J}\, M^{\alpha_M}\, n^{-\alpha_n}, \tag{101}$$

where

$$\alpha_A = \frac{d(d-m)}{Q}, \qquad \alpha_J = \frac{2d^2}{Q}, \qquad \alpha_M = \frac{4d^2}{Q} + \frac{(d+6)(d-m)(d+3)}{(d+3)Q}, \qquad \alpha_n = \frac{(d-m)(d+3)}{2Q}.$$

In particular, if m is fixed and $d \to \infty$, then

$$\alpha_n \to \frac{1}{6}, \qquad \alpha_A \to \frac{1}{3}, \qquad \alpha_J \to \frac{2}{3}, \qquad \alpha_M \to \frac{5}{3}.$$

Proof. The optimal choice of ε minimizing the RHS of (93) is

$$\varepsilon^* \asymp A^{\frac{2d}{Q}} J^{-\frac{2(2d+3)}{Q}} M^{-\frac{2d(d+3)}{(2d+3)Q}} n^{-\frac{d+3}{Q}}, \tag{102}$$

where

$$Q = (d-m)(d+3) + 2d^2 = 3d^2 + 3d - md - 3m.$$

Plugging ε^* into (93) yields

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}_{\mathrm{BEoS}}(\eta,\mathcal{D})} \mathrm{GenGap}(f_{\boldsymbol{\theta}};\mathcal{D}) \;\lessapprox_d\; A^{\frac{2d}{Q}}\, J^{-\frac{2(d+3)}{Q}} M^{-\frac{2d}{Q}} n^{-\frac{d+3}{D}} \lessapprox_{A,M,J,d} O\Big(n^{-\frac{(d-m)(d+3)}{2\left((d-m)(d+3)+2d^2\right)}}\Big). \tag{103}$$

In particular, when m is fixed and $d \to \infty$,

$$\frac{(d-m)(d+3)}{2\big((d-m)(d+3)+2d^2\big)} \;\longrightarrow\; \frac{1}{6}.$$

Therefore, this rate does not suffer from the curse of dimensionality when m is fixed.

Finally, we verify the validity of the plug-in choice ε^*. The validity condition (97) requires $\varepsilon^d \gtrsim_d n^{-1/2}$ (up to logarithmic factors). Since $(\varepsilon^*)^d \asymp n^{-d(d+3)/Q}$ (treating A, J, M as constants), plugging (102) into (97) gives

$$n^{-\frac{d(d+3)}{Q}} \gtrsim_d n^{-1/2}.$$

This requires

$$\frac{1}{2} > \frac{d(d+3)}{Q} = \frac{d(d+3)}{(d-m)(d+3)+2d^2},$$

which is equivalent to

$$(d-m)(d+3)+2d^2 > 2d(d+3) \iff d^2-3d > m(d+3) \iff m < \frac{d(d-3)}{d+3}.$$

□

F. Proof of Theorem 5.5

Let $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$ be a dataset with $\boldsymbol{x}_i \in \mathbb{R}^d$, and let $\mathcal{S} = \{\pi_j : \mathbb{R}^d \to \mathbb{R}^m\}_{j=1}^J$ be a collection of coordinate projections (patch extractors). For each $j \in [J]$, abbreviate the extracted patch by $\boldsymbol{x}^{(S_j)} := \pi_j(\boldsymbol{x})$, and for each sample write $\boldsymbol{x}_i^{(S_j)} := \pi_j(\boldsymbol{x}_i)$.

Assume labels are uniformly bounded: $|y_i| \le D$ for all $i \in [n]$. Define $I_{\neq 0} := \{i \in [n] : y_i \neq 0\}$.

Consider width-K two-layer sparsely connected ReLU models with Global Average Pooling (GAP),

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) \;=\; \sum_{k=1}^{K} \frac{v_k}{J} \sum_{j=1}^{J} \phi\Big(\boldsymbol{w}_k^{\mathsf{T}} \boldsymbol{x}^{(S_j)} - b_k\Big) + \beta, \qquad \phi(t) = \max\{t, 0\}, \tag{104}$$

where $\boldsymbol{w}_k \in \mathbb{R}^m$, $b_k \in \mathbb{R}$ are the shared filter weights and bias, $v_k \in \mathbb{R}$ is the output weight, and $\beta \in \mathbb{R}$. We write $\boldsymbol{\theta} = \{(\boldsymbol{w}_k, b_k, v_k)\}_{k=1}^K \cup \{\beta\}$.

Theorem F.1 (Flat interpolation with width $\le n$)**.** *Assume that $\big\|\boldsymbol{x}_i^{(S_j)}\big\|_2 \le 1$ for all $i \in [n]$, $j \in [J]$, and there exists a map $\tau : I_{\neq 0} \to [J]$ that assigns $\boldsymbol{p}_i := \boldsymbol{x}_i^{(S_{\tau(i)})}$ such that $\|\boldsymbol{p}_i\|_2 = 1$ and $\boldsymbol{p}_i \neq \boldsymbol{x}_\ell^{(S_j)}$ for all $(\ell, j) \neq (i, \tau(i))$. There exists a width $K \le n$ network of the form (4) that interpolates the dataset and whose Hessian operator norm satisfies*

$$\lambda_{\max}\big(\nabla^2_{\boldsymbol{\theta}} \mathcal{L}\big) \;\le\; 1 + \frac{D^2 + 2/J^2}{n}. \tag{105}$$

Construction F.2. Let $K := |I_{\neq 0}| \le n$ and index the hidden units by $k \in I_{\neq 0}$. For each $k \in I_{\neq 0}$ define the anchor patch $\boldsymbol{p}_k := \boldsymbol{x}_k^{(S_{\tau(k)})}$ and set

$$\rho_k := \max_{(\ell,j)\neq(k,\tau(k))} \big(\boldsymbol{x}_\ell^{(S_j)}\big)^{\mathsf{T}} \boldsymbol{p}_k \;<\; 1, \qquad b_k \in (\rho_k, 1), \qquad \boldsymbol{w}_k := \boldsymbol{p}_k. \tag{106}$$

Set the output bias and output weights

$$\beta := 0, \qquad v_k := \frac{J\, y_k}{1 - b_k}, \qquad k \in I_{\neq 0}. \tag{107}$$

(Justification of $\rho_k < 1$.) According to the assumption, for any $(\ell, j) \neq (k, \tau(k))$,

$$\big(\boldsymbol{x}_\ell^{(S_j)}\big)^\mathsf{T} \boldsymbol{p}_k \leq \big\|\boldsymbol{x}_\ell^{(S_j)}\big\|_2 \, \|\boldsymbol{p}_k\|_2 \leq 1.$$

If equality held then necessarily $\|\boldsymbol{x}_\ell^{(S_j)}\|_2 = 1$ and $\boldsymbol{x}_\ell^{(S_j)} = \boldsymbol{p}_k$, contradicting to the assumption. Hence $\rho_k < 1$.

By (106), for any sample index i and any patch index j,

$$\boldsymbol{w}_k^\mathsf{T} \boldsymbol{x}_i^{(S_j)} - b_k = \begin{cases} 1 - b_k \ > \ 0, & (i,j) = (k, \tau(k)), \\ \leq \rho_k - b_k \ < \ 0, & (i,j) \neq (k, \tau(k)). \end{cases} \tag{108}$$

Thus neuron k is activated on exactly one patch in the entire collection $\{\boldsymbol{x}_i^{(S_j)}\}_{i,j}$, namely $\boldsymbol{x}_k^{(S_{\tau(k)})}$, and inactivated on all other patches. Using (108) and (107), at $\boldsymbol{x}_k$ we have

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}_k) = \frac{v_k}{J}\, \phi(1 - b_k) = \frac{v_k}{J}\,(1 - b_k) = y_k, \qquad k \in I_{\neq 0},$$

and for $i \notin I_{\neq 0}$ all constructed units are inactivated on all patches of $\boldsymbol{x}_i$, hence $f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) = \beta = 0 = y_i$. Therefore, $f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) = y_i$ for all $i \in [n]$.

For each constructed unit, define

$$\tilde{v}_k := \operatorname{sign}(v_k) \in \{\pm 1\}, \qquad \tilde{\boldsymbol{w}}_k := |v_k|\, \boldsymbol{w}_k, \qquad \tilde{b}_k := |v_k|\, b_k. \tag{109}$$

Then for any input $\boldsymbol{x}$,

$$\frac{\tilde{v}_k}{J} \sum_{j=1}^{J} \phi\Big(\tilde{\boldsymbol{w}}_k^\mathsf{T} \boldsymbol{x}^{(S_j)} - \tilde{b}_k\Big) = \frac{v_k}{J} \sum_{j=1}^{J} \phi\Big(\boldsymbol{w}_k^\mathsf{T} \boldsymbol{x}^{(S_j)} - b_k\Big), \tag{110}$$

so interpolation is preserved. Moreover, the activation pattern on the full patch collection is unchanged because (108) has strict inequalities and $|v_k| > 0$. At the unique activated patch $(k, \tau(k))$, the (post-rescaling) pre-activation is

$$\tilde{z}_k := \tilde{\boldsymbol{w}}_k^\mathsf{T} \boldsymbol{x}_k^{(S_{\tau(k)})} - \tilde{b}_k = |v_k|\,(1 - b_k) = J|y_k| \ > \ 0, \qquad |\tilde{v}_k| = 1. \tag{111}$$

In what follows we work with the reparameterized network and drop tildes for readability, implicitly assuming $|v_k| = 1$ for all $k \in I_{\neq 0}$ and

$$z_k := \boldsymbol{w}_k^\mathsf{T} \boldsymbol{x}_k^{(S_{\tau(k)})} - b_k = J|y_k|. \tag{112}$$

Proposition F.3. *Let $\boldsymbol{\theta}$ be the model in Construction F.2 after the reparameterization (109). Then*

$$\lambda_{\max}\big(\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}\big) \leq 1 + \frac{D^2 + 2/J^2}{n}.$$

Proof. By direct computation, the Hessian $\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}$ is given by

$$\nabla_{\boldsymbol{\theta}}^2 \mathcal{L} = \frac{1}{n} \sum_{i=1}^{n} \nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) \nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)^\mathsf{T} + \frac{1}{n} \sum_{i=1}^{n} \big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i\big) \nabla_{\boldsymbol{\theta}}^2 f_{\boldsymbol{\theta}}(\boldsymbol{x}_i). \tag{113}$$

Since the model interpolates $f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) = y_i$ for all i, we have

$$\nabla_{\boldsymbol{\theta}}^2 \mathcal{L} = \frac{1}{n} \sum_{i=1}^{n} \nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) \nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)^\mathsf{T}. \tag{114}$$

Denote the tangent features matrix by

$$\Phi = [\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_1), \nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_2), \cdots, \nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_n)]. \tag{115}$$

Then (114) can be written as $\nabla^2_{\boldsymbol{\theta}} \mathcal{L} = \Phi\Phi^\mathsf{T}/n$, and thus

$$\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}} \mathcal{L}) = \max_{\boldsymbol{\gamma} \in \mathbb{S}^{(m+2)K}} \frac{1}{n} \|\Phi^\mathsf{T} \boldsymbol{\gamma}\|^2 = \max_{\boldsymbol{u} \in \mathbb{S}^{n-1}} \frac{1}{n} \|\Phi \boldsymbol{u}\|^2. \tag{116}$$

For the gate $m_k^{(S_j)}(\boldsymbol{x}) \coloneqq \mathbb{1}\left\{\boldsymbol{w}_k^\mathsf{T} \boldsymbol{x}^{(S_j)} > b_k\right\}$, let $m_{k,i}^{(S_j)} \coloneqq m_k^{(S_j)}(\boldsymbol{x}_i)$. From direct computation,

$$\begin{aligned}
\frac{\partial f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)}{\partial v_k} &= \frac{1}{J} \sum_{j=1}^{J} m_{k,i}^{(S_j)} \cdot \left(\boldsymbol{w}_k^\mathsf{T} \boldsymbol{x}_i^{(S_j)} - b_k\right), & \frac{\partial f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)}{\partial \boldsymbol{w}_k} &= \frac{1}{J} \sum_{j=1}^{J} m_{k,i}^{(S_j)} \cdot v_k \cdot \boldsymbol{x}_i^{(S_j)}, \\
\frac{\partial f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)}{\partial b_k} &= -\frac{1}{J} \sum_{j=1}^{J} m_{k,i}^{(S_j)} \cdot v_k, & \frac{\partial f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)}{\partial \beta} &= 1.
\end{aligned} \tag{117}$$

By the one-to-one activation property (108), each sample $\boldsymbol{x}_i$ with $i \in I_{\neq 0}$ activates exactly one unit (the unit $k = i$) on exactly one patch $S_{\tau(i)}$, and samples with $i \notin I_{\neq 0}$ activate none. Hence the sample-wise gradient $\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$ has support only on the parameter triplet $(\boldsymbol{w}_i, b_i, v_i, \beta)$ when $i \in I_{\neq 0}$, and is zero on $(\boldsymbol{w}_k, b_k, v_k)$ for all k when $i \notin I_{\neq 0}$. Writing the nonzero gradient block explicitly (recall $|v_i| = 1$ after reparameterization and (112)),

$$\begin{aligned}
\nabla_{(\boldsymbol{w}_i, b_i, v_i, \beta)} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) &= \begin{pmatrix} \nabla_{(\boldsymbol{w}_i, b_i, v_i)} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) \\ 1 \end{pmatrix}, \\
\nabla_{(\boldsymbol{w}_i, b_i, v_i)} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) &= \begin{cases} \begin{pmatrix} \frac{v_i}{J} \boldsymbol{x}_i^{(S_{\tau(i)})} \\ -\frac{v_i}{J} \\ \frac{1}{J}\left(\boldsymbol{w}_i^\mathsf{T} \boldsymbol{x}_i^{(S_{\tau(i)})} - b_i\right) \end{pmatrix} = \begin{pmatrix} \frac{v_i}{J} \boldsymbol{x}_i^{(S_{\tau(i)})} \\ -\frac{v_i}{J} \\ |y_i| \end{pmatrix}, & (i \in I_{\neq 0}), \\ \mathbf{0}, & (i \notin I_{\neq 0}). \end{cases}
\end{aligned} \tag{118}$$

Let $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{S}^{n-1}$ and plug (118) into (116). As in the fully connected case, after a row permutation (grouping neuron parameters) we obtain

$$\begin{aligned}
\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}} \mathcal{L}) &= \max_{\boldsymbol{u} \in \mathbb{S}^{n-1}} \frac{1}{n} \|\Phi \boldsymbol{u}\|^2 \\
&= \frac{1}{n} \max_{\boldsymbol{u} \in \mathbb{S}^{n-1}} \sum_{i=1}^{n} u_i^2 \left\|\nabla_{(\boldsymbol{w}_i, b_i, v_i)} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)\right\|_2^2 + \left(\sum_{i=1}^{n} u_i\right)^2 && (119) \\
&= \frac{1}{n} \max_{\boldsymbol{u} \in \mathbb{S}^{n-1}} \sum_{i \in I_{\neq 0}} u_i^2 \left(\frac{1}{J^2} \left\|\boldsymbol{x}_i^{(S_{\tau(i)})}\right\|_2^2 + \frac{1}{J^2} + y_i^2\right) + \left(\sum_{i=1}^{n} u_i\right)^2. && (120)
\end{aligned}$$

According to the assumption, $\|\boldsymbol{x}_i^{(S_{\tau(i)})}\|_2 = 1$ for all $i \in I_{\neq 0}$, and by bounded labels $y_i^2 \leq D^2$ for all i. Thus

$$\begin{aligned}
\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}} \mathcal{L}) &\leq \frac{1}{n} \left(\max_{i \in [n]} \left(\frac{1}{J^2} \left\|\boldsymbol{x}_i^{(S_{\tau(i)})}\right\|_2^2 + \frac{1}{J^2} + y_i^2\right) + \max_{\boldsymbol{u} \in \mathbb{S}^{n-1}} \left(\sum_{i=1}^{n} u_i\right)^2\right) \\
&\leq \frac{1}{n} \left(\frac{2}{J^2} + D^2 + n\right) = 1 + \frac{D^2 + 2/J^2}{n}.
\end{aligned}$$

If we remove the output bias term β from the parameters, then the last term $(\sum_i u_i)^2$ in (119) is removed. □

G. Extending the Framework beyond Weight Sharing and Global Average Pooling

The main text analyzes the minimal convolutional model (4) with weight sharing and global average pooling (GAP). These two design choices simplify the exposition but are not prerequisites for the stability-induced regularization mechanism. To demonstrate that *sparse connectivity alone* is the essential ingredient, this appendix analyzes a variant that retains the sparse receptive-field pattern of the main-text SCN but removes both weight sharing and global pooling.

G.1. Sparsely connected network without weight sharing

Setup and notation. Fix the collection of local receptive fields $\mathcal{S} = \{S_j\}_{j=1}^{J}$ (each $S_j \subset [d]$ with $|S_j| = m$) and the corresponding coordinate projections $\pi_j : \mathbb{R}^d \to \mathbb{R}^m$.

Definition G.1 (SCN without weight sharing)**.** A *sparsely connected network without weight sharing*, with receptive fields $\mathcal{S}$ and width K, is a function of the form

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) \;=\; \sum_{k=1}^{K}\sum_{j=1}^{J} v_{k,j}\,\phi\big(\boldsymbol{w}_{k,j}^{\mathsf{T}}\pi_j(\boldsymbol{x}) - b_{k,j}\big) \;+\; \beta, \tag{121}$$

where $\boldsymbol{w}_{k,j} \in \mathbb{R}^m$, $b_{k,j} \in \mathbb{R}$, $v_{k,j} \in \mathbb{R}$ are *independent* parameters for each neuron index $k \in [K]$ and location index $j \in [J]$, and $\beta \in \mathbb{R}$ is a scalar bias. We write $\boldsymbol{\theta} = \{(\boldsymbol{w}_{k,j}, b_{k,j}, v_{k,j})\}_{k\in[K],j\in[J]} \cup \{\beta\}$ for the full parameter vector.

The key structural property shared with the main-text SCN (4) is *sparse connectivity*: neuron (k, j) receives input only from the m-dimensional patch $\pi_j(\boldsymbol{x})$. The architecture (121) differs from (4) in two respects: (i) filter weights $\boldsymbol{w}_{k,j}$ and biases $b_{k,j}$ are *not* shared across locations j, and (ii) output weights $v_{k,j}$ may vary freely with j, so there is no global average pooling.

Data and loss. As in the main text, the dataset is $\mathcal{D} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^{n}$ with $\boldsymbol{x}_i \in B_R^d$ and $y_i \in [-D, D]$, and the training objective is $\mathcal{L}(\boldsymbol{\theta}) := \frac{1}{2n}\sum_{i=1}^{n}(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)^2$.

Location-specific weight function. Because neurons at different locations j now operate with independent parameters on different patch distributions, the relevant weight function is defined *per location*. For each $j \in [J]$, let $\boldsymbol{X}_{\mathcal{D}}^{(j)}$ be a random vector drawn uniformly from the training patches at position j, i.e. from $\{\pi_j(\boldsymbol{x}_i)\}_{i=1}^{n} \subset \mathbb{R}^m$. Define

$$g_{\mathcal{D},\mathcal{S}}^{(j)}(\boldsymbol{u}, t) \;:=\; \mathbb{E}\Big[\phi\big(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{(j)} - t\big)\Big]\,\sqrt{\mathbb{P}\big(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{(j)} > t\big)^2 + \Big\|\mathbb{E}\Big[\boldsymbol{X}_{\mathcal{D}}^{(j)}\,\mathbb{1}\Big\{\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{(j)} > t\Big\}\Big]\Big\|_2^2}. \tag{122}$$

This is the direct analogue of the global weight function $g_{\mathcal{D},\mathcal{S}}$ in Definition D.1, specialized to a single location.

G.2. From the BEoS condition to a location-wise weighted path norm

We establish the counterpart of Theorem 5.3 for the unshared architecture (121). The proof follows the same strategy (lower-bounding $\lambda_{\max}(\boldsymbol{T}_{\mathcal{D}})$ via the all-ones direction) but now each neuron (k, j) contributes independently, which allows the bound to decompose over locations.

Lemma G.2 (Hessian residual bound for the unshared SCN)**.** *For the architecture* (121) *and any* $\boldsymbol{x}$ *with* $\|\boldsymbol{x}\|_2 \le R$,

$$\big\|\nabla_{\boldsymbol{\theta}}^2 f_{\boldsymbol{\theta}}(\boldsymbol{x})\big\|_{\mathrm{op}} \;\le\; 2(R+1).$$

Proof. The proof of Lemma D.3 bounds $|\boldsymbol{\omega}^{\mathsf{T}}\nabla_{\boldsymbol{\theta}}^2 f(\boldsymbol{x})\,\boldsymbol{\omega}|$ for a unit perturbation $\boldsymbol{\omega}$ by examining each neuron independently: the only nonzero second derivatives within neuron (k, j) are the mixed partials between $(\boldsymbol{w}_{k,j}, b_{k,j})$ and $v_{k,j}$, exactly as in (35)–(36). Since different neurons occupy orthogonal parameter blocks, their contributions to $\boldsymbol{\omega}^{\mathsf{T}}\nabla_{\boldsymbol{\theta}}^2 f(\boldsymbol{x})\,\boldsymbol{\omega}$ sum without interaction. Each block contributes at most $2|\gamma_{k,j}|(R\,\|\boldsymbol{\alpha}_{k,j}\|_2 + |\delta_{k,j}|)$ (cf. (41)), where $(\boldsymbol{\alpha}_{k,j}, \delta_{k,j}, \gamma_{k,j})$ is the perturbation restricted to neuron (k, j). Summing over (k, j) and applying Cauchy–Schwarz with the normalization $\|\boldsymbol{\omega}\|_2 = 1$ yields the same bound $2(R+1)$. □

Proposition G.3 (Tangent-feature lower bound for the unshared SCN)**.** *Let* $\boldsymbol{T}_{\mathcal{D}} = \frac{1}{n}\sum_{i=1}^{n}\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)\,\nabla_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)^{\mathsf{T}}$. *Then*

$$\lambda_{\max}(\boldsymbol{T}_{\mathcal{D}}) \;\ge\; 1 \;+\; 2\sum_{k=1}^{K}\sum_{j=1}^{J}|v_{k,j}|\;\|\boldsymbol{w}_{k,j}\|\;g_{\mathcal{D},\mathcal{S}}^{(j)}\!\left(\frac{\boldsymbol{w}_{k,j}}{\|\boldsymbol{w}_{k,j}\|},\,\frac{b_{k,j}}{\|\boldsymbol{w}_{k,j}\|}\right). \tag{123}$$

Proof. Write the gate indicators $m_{k,j,i} := \mathbb{1}\left\{\boldsymbol{w}_{k,j}^{\mathsf{T}}\pi_j(\boldsymbol{x}_i) > b_{k,j}\right\}$, the normalized direction $\boldsymbol{u}_{k,j} := \boldsymbol{w}_{k,j}/\left\|\boldsymbol{w}_{k,j}\right\|_2$, and the normalized threshold $t_{k,j} := b_{k,j}/\left\|\boldsymbol{w}_{k,j}\right\|_2$. The partial derivatives of $f_{\boldsymbol{\theta}}$ at sample $\boldsymbol{x}_i$ are

$$\frac{\partial f}{\partial \boldsymbol{w}_{k,j}}(\boldsymbol{x}_i) = v_{k,j}\, m_{k,j,i}\, \pi_j(\boldsymbol{x}_i), \qquad \frac{\partial f}{\partial b_{k,j}}(\boldsymbol{x}_i) = -v_{k,j}\, m_{k,j,i}, \tag{124}$$

$$\frac{\partial f}{\partial v_{k,j}}(\boldsymbol{x}_i) = \phi\big(\boldsymbol{w}_{k,j}^{\mathsf{T}}\pi_j(\boldsymbol{x}_i) - b_{k,j}\big), \qquad \frac{\partial f}{\partial \beta}(\boldsymbol{x}_i) = 1. \tag{125}$$

Since each neuron (k,j) operates on the patch $\pi_j(\boldsymbol{x}_i)$ with *independent* parameters $(\boldsymbol{w}_{k,j}, b_{k,j}, v_{k,j})$, the gradient blocks are orthogonal across different (k,j) pairs.

Denoting the tangent-feature matrix $\Phi = [\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_1), \ldots, \nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_n)] \in \mathbb{R}^{p\times n}$, we use the standard lower bound (cf. (33))

$$\lambda_{\max}(\boldsymbol{T}_{\mathcal{D}}) = \frac{1}{n}\max_{\|\boldsymbol{u}\|=1}\|\Phi\boldsymbol{u}\|^2 \;\geq\; \frac{1}{n^2}\|\Phi\,\mathbf{1}\|^2 = \frac{1}{n^2}\Big\|\sum_{i=1}^{n}\nabla_{\boldsymbol{\theta}} f(\boldsymbol{x}_i)\Big\|^2. \tag{126}$$

Expanding the squared norm and using the orthogonality of neuron blocks gives

$$\frac{1}{n^2}\|\Phi\,\mathbf{1}\|^2 = 1 + \sum_{k=1}^{K}\sum_{j=1}^{J}\Bigg[\underbrace{v_{k,j}^2\left(\Big\|\frac{1}{n}\sum_i m_{k,j,i}\,\pi_j(\boldsymbol{x}_i)\Big\|^2 + \Big(\frac{1}{n}\sum_i m_{k,j,i}\Big)^2\right)}_{=:\; P_{k,j}} + \underbrace{\|\boldsymbol{w}_{k,j}\|_2^2\left(\frac{1}{n}\sum_i \phi\big(\boldsymbol{u}_{k,j}^{\mathsf{T}}\pi_j(\boldsymbol{x}_i) - t_{k,j}\big)\right)^2}_{=:\; Q_{k,j}}\Bigg]. \tag{127}$$

Here the "1" comes from $\left(\frac{1}{n}\sum_i \partial f/\partial\beta\right)^2 = 1$. The term $P_{k,j}$ collects the squared sums of the $\boldsymbol{w}_{k,j}$- and $b_{k,j}$-gradients (both proportional to $v_{k,j}$), while $Q_{k,j}$ comes from the $v_{k,j}$-gradient (proportional to $\|\boldsymbol{w}_{k,j}\|_2$). Crucially, the double sum decomposes over (k,j) because the parameters are not shared across locations.

We apply $a^2 + b^2 \geq 2ab$ to each (k,j) independently (cf. the analogous step in the proof of Proposition D.2):

$$P_{k,j} + Q_{k,j} \;\geq\; 2\,|v_{k,j}|\;\|\boldsymbol{w}_{k,j}\|_2\;\underbrace{\left(\frac{1}{n}\sum_i \phi\big(\boldsymbol{u}_{k,j}^{\mathsf{T}}\pi_j(\boldsymbol{x}_i) - t_{k,j}\big)\right)\sqrt{\Big(\frac{1}{n}\sum_i m_{k,j,i}\Big)^2 + \Big\|\frac{1}{n}\sum_i m_{k,j,i}\,\pi_j(\boldsymbol{x}_i)\Big\|^2}}_{=\; g_{\mathcal{D},\mathcal{S}}^{(j)}(\boldsymbol{u}_{k,j},\, t_{k,j})}. \tag{128}$$

To verify the identification with (122), note that the empirical averages over $i \in [n]$ are precisely the expectation, probability, and conditional first moment under the uniform draw $\boldsymbol{X}_{\mathcal{D}}^{(j)} \sim \mathrm{Uniform}(\{\pi_j(\boldsymbol{x}_i)\}_{i=1}^n)$:

$$\frac{1}{n}\sum_{i=1}^{n}\phi\big(\boldsymbol{u}^{\mathsf{T}}\pi_j(\boldsymbol{x}_i) - t\big) = \mathbb{E}\big[\phi(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{(j)} - t)\big], \qquad \frac{1}{n}\sum_{i=1}^{n} m_{k,j,i} = \mathbb{P}\big(\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{(j)} > t\big),$$

$$\frac{1}{n}\sum_{i=1}^{n} m_{k,j,i}\,\pi_j(\boldsymbol{x}_i) = \mathbb{E}\big[\boldsymbol{X}_{\mathcal{D}}^{(j)}\,\mathbb{1}\left\{\boldsymbol{u}^{\mathsf{T}}\boldsymbol{X}_{\mathcal{D}}^{(j)} > t\right\}\big].$$

Summing (128) over (k,j) and combining with (126)–(127) yields (123). □

Remark G.4 (Why the per-neuron decomposition succeeds here)**.** In the GAP model (4), the output weight v_k is shared uniformly across all locations via the $1/J$ average. This forces $\nabla_{\boldsymbol{w}_k} f = \frac{v_k}{J}\sum_j m_{k,i}^{(S_j)}\,\boldsymbol{x}_i^{(S_j)}$, in which contributions from different locations are combined with a *common* coefficient v_k/J. The common coefficient allows v_k^2 to be factored out, and the resulting $\frac{1}{n^2}\|\Phi\mathbf{1}\|^2$ factors through the *global* patch distribution $\boldsymbol{X}_{\mathcal{D}}^{\mathcal{S}}$, producing the global weight function $g_{\mathcal{D},\mathcal{S}}$ of Proposition D.2.

In the unshared architecture (121), each neuron (k,j) has its own output weight $v_{k,j}$, but since $\boldsymbol{w}_{k,j}$ is also independent across j, the gradient blocks are orthogonal and the squared norm decomposes cleanly into per-(k,j) terms (127). This per-neuron decomposition makes the AM–GM step (128) straightforward.

An intermediate architecture (shared filters $\boldsymbol{w}_k$ with location-dependent output weights $v_{k,j}$) would couple locations through $\nabla_{\boldsymbol{w}_k} f = \sum_j v_{k,j}\, m_{k,j,i}\, \pi_j(\boldsymbol{x}_i)$, in which contributions from different locations can cancel when $v_{k,j}$ have varying signs. This coupling prevents both the per-location decomposition used above and the common-coefficient factorization available under GAP. Analyzing such architectures would require different proof techniques.

Theorem G.5 (BEoS constraint for the unshared SCN)**.** *For the architecture* (121)*, if* $\boldsymbol{\theta} \in \boldsymbol{\Theta}^{\mathcal{S}\mathrm{CN}}_{\mathrm{BEoS}}(\eta, \mathcal{D})$*, i.e.,* $\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}) \le 2/\eta$*, then*

$$\sum_{k=1}^{K}\sum_{j=1}^{J} |v_{k,j}|\, \|\boldsymbol{w}_{k,j}\|\; g^{(j)}_{\mathcal{D},\mathcal{S}}\!\left(\frac{\boldsymbol{w}_{k,j}}{\|\boldsymbol{w}_{k,j}\|}, \frac{b_{k,j}}{\|\boldsymbol{w}_{k,j}\|}\right) \;\le\; \frac{1}{\eta} - \frac{1}{2} + (R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})}. \tag{129}$$

Proof. Decompose the Hessian as $\nabla^2_{\boldsymbol{\theta}}\mathcal{L} = \boldsymbol{T}_{\mathcal{D}} + \boldsymbol{R}_{\mathcal{D}}$ (cf. (29)) with

$$\boldsymbol{R}_{\mathcal{D}} = \frac{1}{n}\sum_{i=1}^{n} (f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)\, \nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i).$$

Let $\boldsymbol{\omega}$ be the unit eigenvector of $\boldsymbol{T}_{\mathcal{D}}$ corresponding to $\lambda_{\max}(\boldsymbol{T}_{\mathcal{D}})$. Then (cf. (44))

$$\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}) \;\ge\; \boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}}\mathcal{L}\,\boldsymbol{\omega} \;=\; \lambda_{\max}(\boldsymbol{T}_{\mathcal{D}}) + \underbrace{\frac{1}{n}\sum_{i=1}^{n}(f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)\,\boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)\,\boldsymbol{\omega}}_{=:\ \Delta}.$$

By the Cauchy–Schwarz inequality and Lemma G.2,

$$|\Delta| \;\le\; \sqrt{\frac{1}{n}\sum_i (f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)^2} \cdot \sqrt{\frac{1}{n}\sum_i \big(\boldsymbol{\omega}^{\mathsf{T}}\nabla^2_{\boldsymbol{\theta}} f_{\boldsymbol{\theta}}(\boldsymbol{x}_i)\,\boldsymbol{\omega}\big)^2} \;\le\; 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})}.$$

Therefore,

$$\lambda_{\max}(\boldsymbol{T}_{\mathcal{D}}) \;\le\; \lambda_{\max}(\nabla^2_{\boldsymbol{\theta}}\mathcal{L}) + |\Delta| \;\le\; \frac{2}{\eta} + 2(R+1)\sqrt{2\mathcal{L}(\boldsymbol{\theta})}.$$

Inserting the lower bound from Proposition G.3 and rearranging yields (129). □

G.3. Generalization analysis

We now derive the generalization guarantee for the unshared architecture (121), following the same interior/exterior decomposition as in Appendix E.

Interior/exterior decomposition. For $\varepsilon \in (0,1)$, define

$$\mathsf{I}^{\mathrm{all}}_{\varepsilon} := \Big\{\boldsymbol{x} \in \mathbb{S}^{d-1} : \max_{j\in[J]} \|\pi_j(\boldsymbol{x})\| \le 1-\varepsilon\Big\}, \qquad \mathsf{O}^{\mathrm{any}}_{\varepsilon} := \Big\{\boldsymbol{x} \in \mathbb{S}^{d-1} : \exists\, j \in [J],\ \|\pi_j(\boldsymbol{x})\| > 1-\varepsilon\Big\}.$$

By a union bound and Proposition D.11, $\mathbb{P}(\mathsf{O}^{\mathrm{any}}_{\varepsilon}) \lesssim J\,\varepsilon^{(d-m)/2}$.

Lower bound on $g^{(j)}_{\mathcal{D},\mathcal{S}}$ over the interior. On $\mathsf{I}^{\mathrm{all}}_{\varepsilon}$, every patch satisfies $\|\pi_j(\boldsymbol{x})\| \le 1-\varepsilon$ for all j. When the marginal of $\boldsymbol{x}$ is $\mathrm{Uniform}(\mathbb{S}^{d-1})$, each projected patch $\pi_j(\boldsymbol{x}) \in \mathbb{R}^m$ inherits the same rotational-invariance structure analyzed in Appendix D. By the same computation as in Proposition D.10 (see (61)), the population weight function at each location satisfies

$$g^{(j)}_{\mathcal{P}}(\boldsymbol{u}, t) \;\ge\; c_g(d)\,\varepsilon^d \qquad \text{for all } \boldsymbol{u} \in \mathbb{S}^{m-1},\ |t| \le 1-\varepsilon,$$

with a constant $c_g(d) > 0$ depending only on d. The uniform deviation bound of Theorem D.7, applied separately at each location j (the patches $\{\pi_j(\boldsymbol{x}_i)\}_{i=1}^n$ are i.i.d. for fixed j), gives

$$\sup_{\boldsymbol{u}\in\mathbb{S}^{m-1},\,|t|\le 1} \big|g^{(j)}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) - g^{(j)}_{\mathcal{P}}(\boldsymbol{u}, t)\big| \;\le\; C_{\mathrm{ep}}\sqrt{\frac{m + \log(2J/\delta)}{n}} \;=:\; \zeta_n,$$

with probability at least $1-\delta$, simultaneously for all $j \in [J]$ (via a union bound absorbing J into the logarithm). Hence, for any ε satisfying $\varepsilon^d \geq 2\zeta_n / c_g(d)$, we have

$$g^{(j)}_{\mathcal{D},\mathcal{S}}(\boldsymbol{u}, t) \;\geq\; \tfrac{1}{2}\, c_g(d)\, \varepsilon^d \qquad \text{for all } j \in [J],\; \boldsymbol{u} \in \mathbb{S}^{m-1},\; |t| \leq 1-\varepsilon. \tag{130}$$

From the BEoS constraint to an unweighted path norm. On the interior $\mathsf{I}^{\text{all}}_{\varepsilon}$, every neuron with $|t_{k,j}| \geq 1-\varepsilon$ is either identically zero or fully affine on all training patches (cf. (90)), and can be absorbed into the affine component. For the remaining neurons with $|t_{k,j}| \leq 1-\varepsilon$, inserting the lower bound (130) into the BEoS constraint (129) and setting $A := \frac{1}{\eta} - \frac{1}{2} + 4M$ yields

$$\sum_{\substack{k,j:\\ |t_{k,j}| \leq 1-\varepsilon}} |v_{k,j}|\, \|\boldsymbol{w}_{k,j}\| \;\leq\; \frac{A}{\frac{1}{2} c_g(d)\, \varepsilon^d} \;\asymp\; A\, \varepsilon^{-d}. \tag{131}$$

Controlling the generalization gap. The function class on the interior set $\mathsf{I}^{\text{all}}_{\varepsilon}$ has variation norm bounded by $A\varepsilon^{-d}$ (equation (131)). The generalization gap decomposes as in (79)–(81):

- *Exterior contribution.* Since $\|f_{\boldsymbol{\theta}}\|_\infty \leq M$ and $|y| \leq D \leq M$, the exterior contributes at most $O(JM^2 \varepsilon^{(d-m)/2})$ (cf. (84)).
- *Interior contribution.* By Lemma C.7 (metric entropy of variation spaces), the interior generalization gap satisfies

$$\left| \mathbb{E}_{\mathsf{I}}\big[(f_{\boldsymbol{\theta}} - y)^2\big] - \frac{1}{n_I} \sum_{i \in I} (f_{\boldsymbol{\theta}}(\boldsymbol{x}_i) - y_i)^2 \right| \;\lessapprox_d\; (A\varepsilon^{-d})^{\frac{d}{d+3}}\, M^{\frac{d+6}{d+3}}\, n^{-1/2}$$

  (cf. (92)).
- *Concentration of the interior fraction.* By Hoeffding's inequality, $|\mathbb{P}(\mathsf{I}^{\text{all}}_{\varepsilon}) - n_I/n| \lesssim \sqrt{\log(1/\delta)/n}$ with high probability, contributing a lower-order term $O(M^2 \sqrt{\log(1/\delta)/n})$ (cf. (86)).

Combining these three terms gives the following result.

Theorem G.6 (Generalization for the unshared SCN)**.** *Assume $\boldsymbol{x} \sim \mathrm{Uniform}(\mathbb{S}^{d-1})$ and $|y| \leq D$. Let $M \geq D$, $\|f_{\boldsymbol{\theta}}\|_\infty \leq M$, and suppose $\boldsymbol{\theta}$ satisfies the BEoS condition $\lambda_{\max}(\nabla^2_{\boldsymbol{\theta}} \mathcal{L}) \leq 2/\eta$. Assume $d > 3$ and $1 \leq m < \frac{d(d-3)}{d+3}$. Then, with probability at least $1-2\delta$,*

$$\mathrm{GenGap}(f_{\boldsymbol{\theta}}; \mathcal{D}) \;\lessapprox_d\; JM^2\, \varepsilon^{\frac{d-m}{2}} + \big(A\, \varepsilon^{-d}\big)^{\frac{d}{d+3}} M^{\frac{d+6}{d+3}} n^{-1/2}, \tag{132}$$

for any $\varepsilon \in (0,1)$ satisfying $\varepsilon^d \gtrsim d^2 \sqrt{(m + \log(J/\delta))/n}$, where $A = \frac{1}{\eta} - \frac{1}{2} + 4M$.

Proof. The proof follows the decomposition and estimates described above. The exterior bound uses Proposition D.11 and Hoeffding's inequality (see (84)). The interior bound uses the path-norm estimate (131) together with the metric entropy of variation spaces (Lemma C.7; see (92)). The concentration term $\left|\mathbb{P}(\mathsf{I}^{\text{all}}_{\varepsilon}) - n_I/n\right|$ is controlled as in (86) and is of smaller order. □

Optimizing over ε as in Corollary E.2 yields the rate exponent

$$-\frac{(d-m)(d+3)}{2\big((d-m)(d+3) + 2d^2\big)},$$

which tends to $-\frac{1}{6}$ for fixed m and $d \to \infty$, identical to the rate obtained for the weight-shared SCN in Theorem 5.4.

Conclusion. The only architectural prerequisite for the stability-induced implicit regularization is the *sparse connectivity pattern*: each hidden neuron receives input from a small subset of the ambient coordinates. Weight sharing and GAP change how the patch geometry is aggregated. In the GAP model, the shared filter and common output coefficient couple

all locations and yield a single global patch-multiset weight $g_{\mathcal{D},\mathcal{S}}$. In the unshared model, the same mechanism appears as a sum of location-wise weights $g^{(j)}_{\mathcal{D},\mathcal{S}}$. Thus weight sharing and GAP are not necessary for stability-induced regularization. They determine whether the patch-geometry penalty is global and pooled or location-wise. This conclusion substantiates the design principle articulated in Section 6: the effective vector dimension seen by each local operator governs the strength of implicit regularization, regardless of whether parameters are shared across locations.

H. The Role of Weight Sharing

The analysis in Section 5 shows that sparse connectivity strengthens implicit regularization by shifting the geometric input to gradient descent. But the SCN architecture bundles two distinct inductive biases together: sparse fan-in and weight sharing. Does sparse fan-in alone produce the effect, or is weight sharing essential?

To test this, we introduce a third architecture identical to the SCN except that the filters are no longer shared across locations. This *sparsely connected network without weight sharing* (SCN-noWS) has independent parameters per patch position:

$$f_{\boldsymbol{\theta}}(\boldsymbol{x}) = \sum_{k=1}^{K}\sum_{j=1}^{J} v_{k,j}\,\phi(\boldsymbol{w}_{k,j}^{\top}\pi_j(\boldsymbol{x}) - b_{k,j}) + \beta.$$

Under weight sharing, a filter is driven by patches from all locations. To memorize a sample, the filter must isolate a patch that is distinguishable from every other patch in the entire multiset. Without weight sharing, a location-specific neuron only needs to find a patch that stands out within its own location. In typical data such as images, recurring local structures make cross-location isolation far harder than within-location isolation. Thus, weight sharing raises the threshold for memorization, pushing the network toward features that are common across the patch multiset.

The following experiment tests this hypothesis in a controlled denoising setting. **Data.** We construct D-dimensional inputs composed of J equal-sized disjoint patches. Within the patch multiset, most patches are pure Gaussian noise, but a small fraction carry signal. Concretely, fix two signal centers $\boldsymbol{v}_+, \boldsymbol{v}_- \in \mathbb{S}^{m-1}$. For each sample, a label $y \in \{\boldsymbol{v}_+, \boldsymbol{v}_-\}$ is drawn uniformly, and a random location $j^\star$ is selected. The patch at $j^\star$ is set to y plus a small Gaussian perturbation; all other patches are independent Gaussian noise with matched total energy. This creates a sharp cluster structure: the vast majority of patches form a dense noise cluster around the origin, while two small signal clusters lie near $\boldsymbol{v}_+$ and $\boldsymbol{v}_-$. Spatially, the signal is extremely sparse (one informative patch per sample).

Setup. We train three two-layer ReLU networks of identical width and learning rate: a fully connected network (FCN), an SCN-noWS, and an SCN. The task is regression: predict the label from the noisy input. The training objective is standard denoising MSE, without explicit regularization.

Results. Figure 8 shows training and test loss. All three architectures fit the training data to near-zero loss. However, only the SCN generalizes to held-out samples; the FCN and SCN-noWS both overfit. This is exactly what the memorization-threshold argument predicts. In the SCN-noWS, a neuron at location j needs only to find a signal patch that is distinguishable from other patches at the *same* location; with high probability, the one signal patch in a training sample appears as a local outlier. The SCN, by contrast, forces each filter to respond to patches across all locations. A filter that fires on a signal patch at one location will also fire on the many noise patches elsewhere. This drives up its activation mass and the associated stability penalty, pushing the network away from sample-specific memorization and toward filters tuned to the recurring signal clusters.

This experiment reveals a deeper property of weight sharing in denoising settings. Because the added noise is independent across coordinates, patches at different locations are perturbed by independent noise realizations. A filter restricted to a single location cannot separate signal from noise using cross-location statistics; it is forced to rely on the local noise realization. Weight sharing lifts this restriction: a shared filter sees many independent noise realizations of the same underlying patch structure, and its aggregated gradient suppresses the noise while amplifying the consistent signal directions. Thus weight sharing acts as an implicit denoising mechanism at the level of gradient dynamics.

Remark H.1 (Weight sharing as a geometric form of translation invariance)**.** Translation invariance is usually invoked to explain parameter efficiency and equivariance. Our analysis suggests a complementary geometric perspective. By coupling each filter to all spatial locations, weight sharing effectively redefines the "unit of isolation" for memorization. A training sample can be memorized without sharing if it has *any* distinguishable patch at *any* location. With sharing, merely local distinctiveness is not enough; the patch must be globally distinctive across the entire multiset. For natural images, where

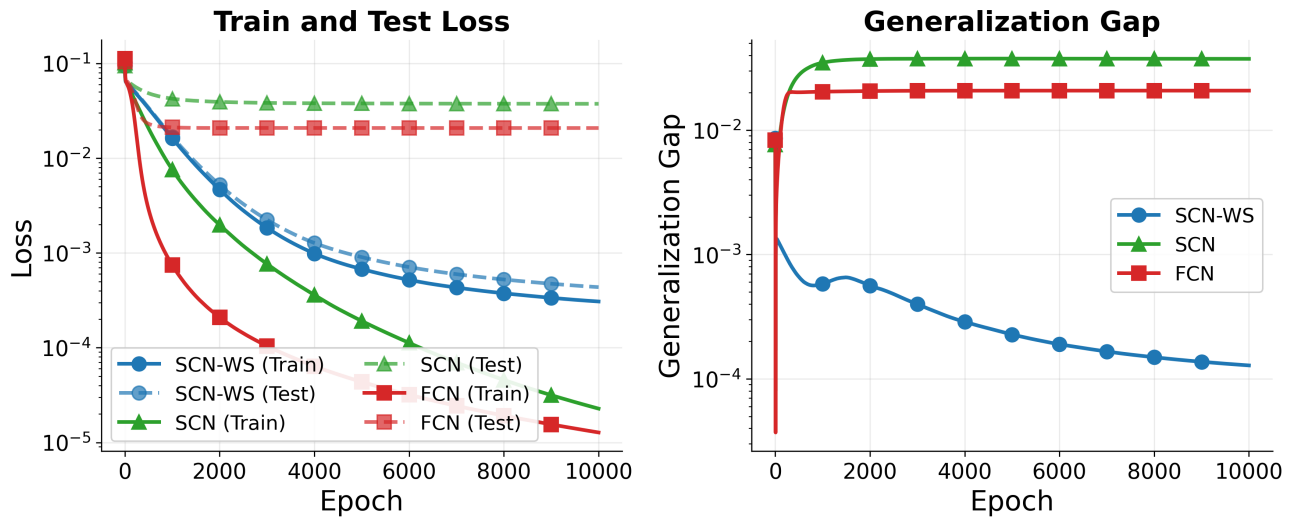
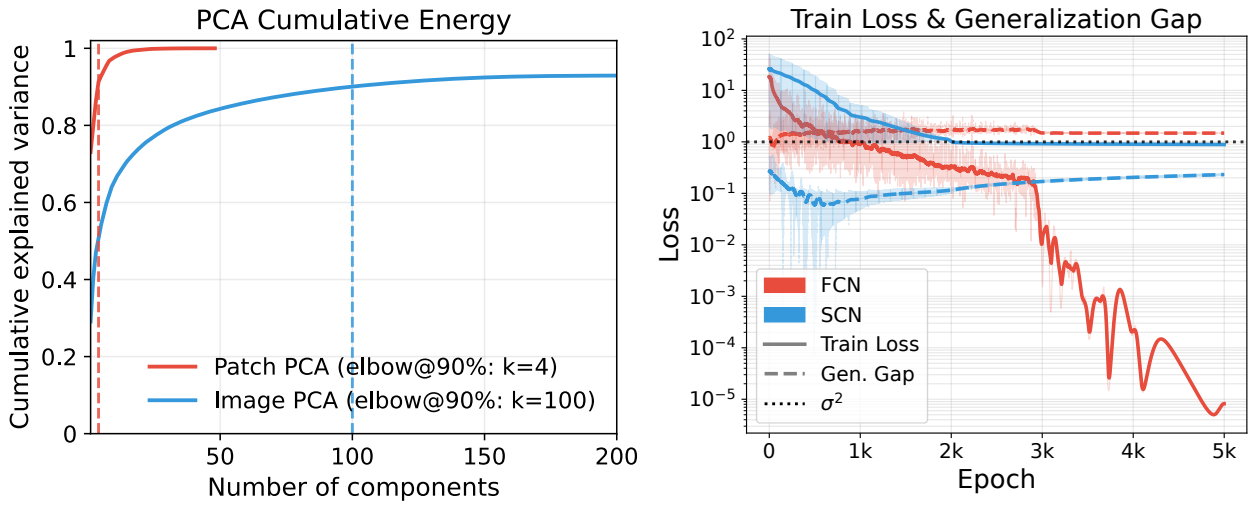


Figure 8. Training loss (solid) and test loss (dashed) for the three architectures on the clustered-patch denoising task. Only the SCN resists overfitting.

patches repeat across space, this condition is much harder to satisfy. Weight sharing thus funnels gradient descent away from sample-specific memorization and toward generalizable features, providing a form of implicit regularization rooted in the geometry of the patch multiset rather than in parameter counts.

I. Patch Geometry of Natural Images

We now examine which side of the dichotomy established in Figure 6 real image data falls on. We extract 3×3 convolutional patches (stride 1, no padding) from the CIFAR-10 training set, sample 10^7 patches. Figure 9a compares the patch cloud with the cloud of full images. The patch cloud is strongly concentrated: 90% of its variance lives in just 4 directions of $\mathbb{R}^{27}$, while the full images in $\mathbb{R}^{3072}$ need over 100 directions. Recalling the gradient view (3), such a low-dimensional patch cloud implies that the gradients themselves are constrained to a low-rank subspace, much like the one-dimensional $\mathcal{S}_1$ example in Figure 6. Moreover, Figure 9b indicates that SCN indeed benefit from the convolutional patch geometry.



(a) Patch vs Image.

(b) Regression on CIFAR-10.

Figure 9. **(a)** PCA explained-variance ratio for the patch cloud ($\mathbb{R}^{27}$, solid) and the ambient image cloud ($\mathbb{R}^{3072}$, dashed). Patches concentrate 90% variance in 3 principal directions; images need over 100. **(b)** The SCN stabilizes near the noise floor; the FCN interpolates.

The takeaway message is that local receptive-field design can affect implicit regularization, not only feature extraction. Even under the same ambient data distribution, different receptive-field systems can induce very different patch geometries, and hence can change the balance between expressivity and regularization. In particular, the size of patch/local receptive field size in the hidden layers is a concrete perspective.

J. Experimental Details

We adopt the random-design nonparametric regression setting.

The setting of nonparametric regression. The non-parametric regression with noisy labels on the random design is one typical setting in this program. Suppose $\{\boldsymbol{x}_i\}_{i=1}^n$ are i.i.d. sampled from a distribution $\mathcal{P}_{\boldsymbol{X}}$ supported on $\mathbb{B}_R^d$ and $y_i = f_{true}(\boldsymbol{x}_i) + \xi_i$ for $i \in [n]$, where $f_{true} \colon \mathbb{R}^d \to \mathbb{R}$ is the ground-true function and $\{\xi_i\}_{i=1}^n$ are i.i.d. Gaussian noises $\mathcal{N}(0, \sigma^2)$. In this setting, the goal of the regression task is to find a predictor f to minimize the mean squared error (MSE):

$$\mathrm{MSE}(f) = \frac{1}{n}\sum_{i=1}^{n}(f(\boldsymbol{x}_i) - f_{true}(\boldsymbol{x}_i))^2. \tag{133}$$

The population level of (133) is known as *excess risk*,

$$\mathrm{Excess}(f) \coloneqq \mathbb{E}_{\boldsymbol{x}\sim\mathcal{P}_{\boldsymbol{X}}}[(f(\boldsymbol{x}) - f_{true}(\boldsymbol{x}))^2], \tag{134}$$

which is also called the *estimation error* under $L^2(\mathcal{P}_{\boldsymbol{X}})$.

In this setting, the population risk (under squared loss) of a predictor f decomposes as

$$\mathcal{R}(f) = \mathbb{E}_{(\boldsymbol{x},y)\sim\mathcal{P}}\big[(f(\boldsymbol{x}) - y)^2\big] = \mathrm{Excess}(f) + \sigma^2, \tag{135}$$

The additive term σ^2 is the irreducible error contributed by the label noise, and it is achieved by the Bayes predictor $f^*(\boldsymbol{x}) = \mathbb{E}[y \mid \boldsymbol{x}] = f_{true}(\boldsymbol{x})$, namely $R(f^*) = \sigma^2$. Consequently, $\mathcal{R}(f) - \mathcal{R}(f^*) = \mathbb{E}_{\boldsymbol{x}\sim\mathcal{P}_{\boldsymbol{X}}}[(f(\boldsymbol{x}) - f_{true}(\boldsymbol{x}))^2]$, so controlling the excess risk is equivalent to controlling the population regret in squared-loss regression.

Moreover, the generalization gap in this random-design regression model admits the following equivalent form

$$\mathrm{GenGap}(f, \mathcal{D}) = \Big|\mathrm{Excess}(f) + \sigma^2 - \widehat{\mathcal{R}}_{\mathcal{D}}(f)\Big|. \tag{136}$$

This identity makes explicit how the generalization gap compares the *population* performance (excess risk plus irreducible noise) against the *training* performance measured by the empirical squared loss.

J.1. Experimental details for the interpolation below the edge of stability

We use synthetic data to empirically validate our claim that SCN generalizes well on spherical data when $m \ll d$, and the rate does not deteriorate as the ambient dimension increases.

Data generation: Fix a collection of receptive fields $\mathcal{S} = \{S_j\}_{j=1}^J$ with patch size m. In all experiments we use disjoint coordinate patches $S_j = \{(j-1)m+1, \ldots, jm\}$ so that $J = \lfloor d/m \rfloor$. We first sample a ground-truth predictor $f_{true} \in \boldsymbol{\Theta}^{\mathcal{S}\mathrm{CN}}$ from the architecture in (4) with a moderate width $K_{true} = 20$ (fixed across all n), and generate training inputs $\{\boldsymbol{x}_i\}_{i=1}^n$ i.i.d. from $\mathrm{Uniform}(\mathbb{S}^{d-1})$. The labels are generated by $y_i = f_{true}(\boldsymbol{x}_i) + \xi_i$, where $\xi_i \sim \mathcal{N}(0, \sigma^2)$ are i.i.d Gaussian noise. We also generate an independent test set $\{\tilde{\boldsymbol{x}}_r\}_{r=1}^N \sim \mathrm{Uniform}(\mathbb{S}^{d-1})$ (with $N \gg n$) for Monte Carlo evaluation of generalization gap.

Architecture: We compare two overparameterized two-layer ReLU models trained by full-batch GD: (i) **SCN** (two-layer sparsely connected ReLU network in (4)), and (ii) **FCN** obtained by taking $m = d$ and $J = 1$ in (4). Unless otherwise stated, both models use width $K = 1024$, which places the training in an overparameterized regime for the sample sizes we consider. Both use Kaiming-normal initialization for hidden weights and zero initialization for all biases.

Metrics: For each trained predictor f, we report (i) **train loss** $\widehat{\mathcal{R}}_{\mathcal{D}}(f)$ This quantity measures how well the predictor fit the training set; (ii) **estimated generalization gap**, $\widehat{\mathrm{GenGap}}(f, \mathcal{D})$, which is the main object predicted by our theory (Theorem 5.4). (iii) **estimated excess risk** $\widehat{\mathrm{Excess}}(f) = \frac{1}{N}\sum_{r=1}^{N}\big(f(\tilde{\boldsymbol{x}}_r) - f_{\mathrm{true}}(\tilde{\boldsymbol{x}}_r)\big)^2$, the Monte Carlo estimator of

$\mathrm{Excess}(f)$ defined in Eq (134); (iv) **Hessian Sharpness** $\lambda_{\max}(\nabla^2 \mathcal{L}(\theta_t))$, computed every 500 epochs via PyHessian (Yao et al., 2020) to verify the trajectory satisfies BEoS;

Optimization All models are trained with full-batch gradient descent (no momentum, no weight decay) on the squared loss $\mathcal{L}(\theta)$, using learning rate $\eta = 0.2$ for 30000 epochs; the corresponding BEoS threshold is $2/\eta = 10$.

Sweep and Scaling For SCN, we sweep (Figure 7) ambient dimension $d \in \{100, 200, 400\}$; for FCN, we fix $d = 10$ as a baseline where fully connected networks are expected to perform well. Sample sizes range over $n \in \{128, 256, 512, 1024\}$. Results averaged over 5 seeds. To estimate the sample-size scaling, we sweep $n \in \{128, 256, 512, 1024\}$ and repeat the full pipeline over multiple random seeds. We plot $\log \widehat{\mathrm{GenGap}}(f_{\hat{\boldsymbol{\theta}}}, \mathcal{D})$ versus $\log n$ and report the least-squares fitted slope (**SCN**: $d = 100, 200, 400$; **FCN**: $d = 10$).

J.2. Experimental details for Figure 7

Similar setup and metrics as the spherical-data setup above in J.1, with two changes. First, each patch is sampled independently from the unit sphere and concatenated: $\boldsymbol{x} = [\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(J)}]$ with $\boldsymbol{x}^{(j)} \sim \mathrm{Uniform}(\mathbb{S}^{m-1})$, instead of the whole-vector sampling $\boldsymbol{x} \sim \mathrm{Uniform}(\mathbb{S}^{d-1})$. Second, we fix $m = 10$, $J = 8$ (so $d = 80$), $K_{\mathrm{true}} = 50$, $n = 512$, $\sigma = 1$, and train at width $K = 1024$ for 50,000 epochs at $\eta = 0.2$ with gradient clipping. Additional experiments with similar setup in Fig 10, 11 and Table 3.

J.3. Experimental details for Figure 9a

We evaluate four patchification schemes on CIFAR-10: (i) *Conv Patches*, 4×4 patches with stride 2 (baseline); (ii) *Random Patches*, 4×4 contiguous crops at random spatial locations (same patch content as Conv but irregular grid); (iii) *ShufflePixel*, conv-style patches taken after a global pixel permutation (negative control); and (iv) *Full Image Space*, the flattened 3072-D image. For each scheme we draw 2,000,000 reference and 50,000 query patches and report two geometry metrics: (a) the PCA *effective rank*, taken as the elbow of the cumulative explained variance at 90% (lower values indicate more structured distributions); and (b) the area under the half-space (Tukey) depth concentration curve $\Psi(T) = \Pr[\hat{d}(\boldsymbol{z}) \geq T]$ for $T \in [0, 0.5]$ (higher values indicate distributions that are harder to shatter), computed via the random-projection approximation of Liang et al. (2026) with 512 uniformly random unit directions and 4096 histogram bins per direction. Full Image Space uses randomized PCA for the spectrum and 10000 query points.

J.4. Experimental details for Figure 9b

We repeat the above setup in J.1 with two changes. First, inputs are CIFAR-10 images $\boldsymbol{x} \in \mathbb{R}^{3072}$, and SCN uses patches ($3 \times 3 \times 3$, stride 1; $L = 900$, $m = 27$). Second, targets are generated by a *frozen* ground-truth SCN f_{true} of width $K_{\text{true}} = 20$ with the same patchification, $y_i = f_{\mathrm{true}}(\boldsymbol{x}_i) + \xi_i$, $\xi_i \sim \mathcal{N}(0, 1)$; excess risk is reported as the MSE against f_{true} on the standard CIFAR-10 test split ($N = 10000$). We train SCN and FCN, both at width $K = 1024$, on $n_{\text{train}} = 1024$ images for 5000 epochs with $\eta = 0.2$ and gradient clipping. Both networks are overparameterized in width ($K \gg n_{\text{train}}$), and SCN's training loss reaches the noise floor σ^2, ruling out underfitting as the explanation for its small excess risk.

Table 3. Generalization gap and excess risk for SCN on spherical data (corresponding to Figure 7a). As ambient dimension d increases with fixed patch size m, both metrics decreaseconfirming the "blessing of dimensionality".

d	J	Gen Gap	Excess Risk
100	10	1.283 ± 0.114	0.613 ± 0.101
200	20	0.780 ± 0.076	0.352 ± 0.047
400	40	0.295 ± 0.004	0.147 ± 0.010

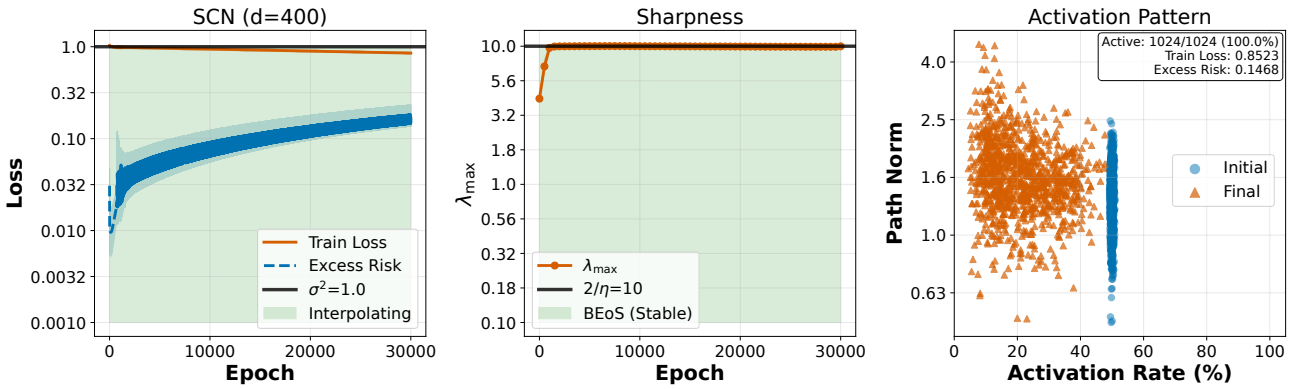


Figure 10. **SCN generalizes on high-dimensional spherical data.(Left)** With $d = 400$ and fixed patch size $m \ll d$, train loss plateaus near σ^2 while excess risk decreases to 0.15, confirming generalization rather than memorization. **(Middle)** Sharpness saturates at BEoS ($\lambda_{\max} \approx 2/\eta$). **(Right)** Neurons spread across moderate activation rates, unlike the sparse isolation in flat interpolation (Figure 7). This validates Theorem 5.5: when $m \ll d$, stability-induced regularization prevents overfitting. Results averaged over 5 seeds with $\eta = 0.2$, trained for 30k epochs.

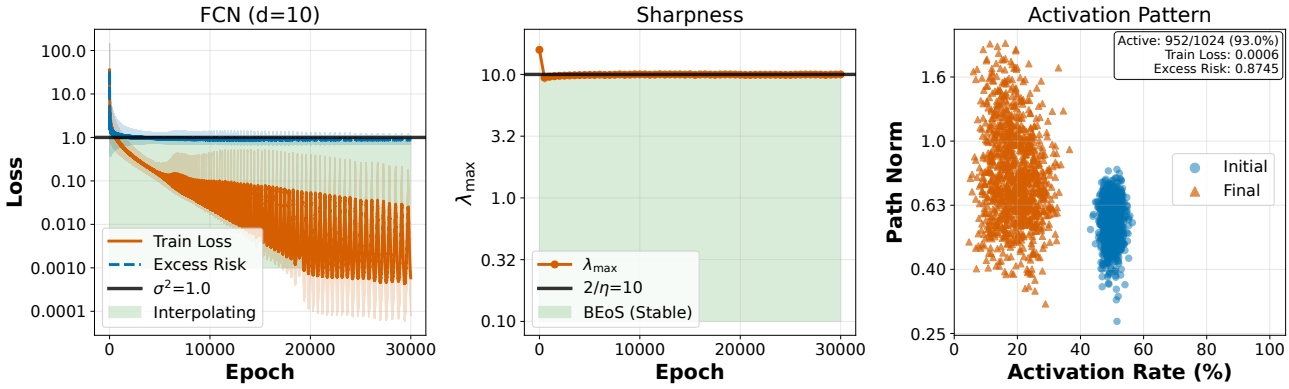


Figure 11. **FCN satisfies BEoS but still memorizes (Left)** FCN ($d = 10$) interpolates noisy labels (train loss $\to 0$) while excess risk remains $\approx \sigma^2$. **(Middle)** Sharpness saturates at BEoS. **(Right)** Activation pattern after training. Despite satisfying BEoS, FCN fails to generalizeconfirming that on spherical data, stability constraints alone are insufficient without convolutional structure. Averaged over 5 seeds; with similar setting as Figure 10