MIN-MAX MULTI-OBJECTIVE BILEVEL OPTIMIZATION WITH APPLICATIONS IN ROBUST MACHINE LEARNING

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ABSTRACT

We consider a generic min-max multi-objective bilevel optimization problem with applications in robust machine learning such as representation learning and hyperparameter optimization. We design MORBiT, a novel single-loop gradient descent-ascent bilevel optimization algorithm, to solve the generic problem and present a novel analysis showing that MORBiT converges to the first-order stationary point at a rate of $\widetilde{\mathcal{O}}(n^{1/2}K^{-2/5})$ for a class of weakly convex problems with n objectives upon K iterations of the algorithm. Our analysis utilizes novel results to handle the non-smooth min-max multi-objective setup and to obtain a sublinear dependence in the number of objectives n. Experimental results on robust representation learning and robust hyperparameter optimization showcase (i) the advantages of considering the min-max multi-objective setup, and (ii) convergence properties of the proposed MORBiT.

1 INTRODUCTION

We begin by examining the classic bilevel optimization (BLO) problem as follows:

$$\min_{x\in\mathcal{X}\subseteq\mathbb{R}^{d_x}} f(x, y^\star(x)) \quad \text{subject to} \quad y^\star(x) \in \arg\min_{y\in\mathcal{Y}=\mathbb{R}^{d_y}} g(x, y) \tag{1}$$

where $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is the upper-level (UL) objective function and $g : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is the lower-level (LL) objective function. $\mathcal{X}$ and $\mathcal{Y}$, respectively, denote the domains for the UL and LL optimization variables x and y, incorporating any respective constraints. Equation 1 is called BLO because the UL objective f depends on both x and the solution $y^\star(x)$ of the LL objective g. BLO has been well-studied in the optimization literature (Bard, 2013; Dempe, 2002). Recently, BLO has found various applications in modern machine learning (Liu et al., 2021; Chen et al., 2022a), such as hyperparameter optimization (HPO) (Franceschi et al., 2018), reinforcement learning (Hong et al., 2020) (RL), multi-task learning (MTL) and representation learning (Arora et al., 2020).

In this work, we focus on a *robust* generalization of equation 1 to the multi-objective setting, where there are n different objective function pairs (f_i, g_i). Let $[n] \triangleq \{1, 2, \cdots, n\}$ and $f_i : \mathcal{X} \times \mathcal{Y}_i \to \mathbb{R}$, $g_i : \mathcal{X} \times \mathcal{Y}_i \to \mathbb{R}$ denote the ith UL and LL objectives respectively. We study the following problem:

$$\min_{x\in\mathcal{X}\subseteq\mathbb{R}^{d_x}} \max_{i\in[n]} f_i(x, y_i^\star(x)) \quad \text{subject to} \quad y_i^\star(x) \in \arg\min_{y_i\in\mathcal{Y}_i=\mathbb{R}^{d_{y_i}}} g_i(x, y_i), i \in [n]. \tag{2}$$

Here, the optimization variable x is shared across all objectives $f_i, g_i, i \in [n]$, while the variables $y_i, i \in [n]$ are only involved in their corresponding objectives f_i, g_i. The goal is to find a *robust solution* $x \in \mathcal{X}$ such that the worst-case across all objectives is minimized. This is a generic problem which reduces to equation 1 if we have a single objective pair, that is $n = 1$. Such a robust optimization problem is useful in various applications, and especially necessary in any safety-critical ones. For example, in decision optimization, the different objectives (f_i, g_i) can correspond to different "scenarios" (such as plans for different scenarios), with x being the shared decision variable and y_i's being scenario-specific decision variables. The goal of equation 2 is to find the robust shared decision x which provides robust performance across all the n considered scenarios, so that such a robust assignment of decision variables will generalize well on other scenarios. In machine learning, robust representation learning is important in object recognition and facial recognition where we desire robust worst-case performance across different groups of objects or different population

demographics. In RL applications with multiple agents (Busoniu et al., 2006; Li et al., 2019; Gronauer & Diepold, 2022), our robust formulation in equation 2 would generate a shared model of the world – the UL variable x – such that the worst-case utility, $\max_i f_i(x, y_i^\star(x))$, of the agent-specific optimal action – the LL variable $y_i^\star(x)$ – is optimized, ensuring robust performance across all agents.

An additional technical advantage of the general multi-objective problem in equation 2 is that it allows the objective-specific variables $y_i \in \mathcal{Y}_i$ to come from different domains, that is, $\mathcal{Y}_i \neq \mathcal{Y}_j, i, j \in [n]$; as stated in equation 2, this implies that the dimensionality d_{y_i} for the per-objective y_i need not be the same across all objectives.This allows for a larger class of problems where each objective can then have different number of objective specific variables but we still require a robust shared variable x. For example, in multi-agent RL, different agents can have different action spaces because they need to operate in different mediums (land, water, air, etc).

The main contributions of this work are highlighted as follows:

- (New algorithm design) We present a single loop **M**ulti-**O**bjective **R**obust **Bi**level **T**wo-timescale optimization algorithm, MORBiT, which uses (i) SGD for the unconstrained strongly convex LL problem, and (ii) projected SGD for the *constrained weakly convex* UL problem.
- (Theoretical convergence guarantees) We demonstrate that, under standard smoothness and regularity conditions, MORBiT with n objectives converges to a $\widetilde{\mathcal{O}}(n^{1/2}K^{-5/2})$-stationary point with K iterations, matching the best convergence rate for single-loop single-objective ($n = 1$) BLO algorithms with the constrained UL problem while using vanilla SGD for the LL problem, and providing a *sublinear* $n^{1/2}$*-dependence* on the number of objective pairs n.
- (Two sets of applications) We present two applications involving min-max multi-objective bilevel problems, robust representation learning and robust hyperparameter optimization (HPO), and demonstrate the effectiveness of our proposed algorithm MORBiT.

Paper Outline In the following section 2, we further discuss the different aspects of the problem in equation 2 and compare that to the problems and solutions considered in existing literature. We present our novel algorithm, MORBiT, and analyse its convergence properties in section 3, and empirically evaluate it in section 4. We conclude with future directions in section 5.

2 Problem and Related Work

We first discuss the different aspects of the robust multi-objective BLO problem with constrained UL in equation 2. While BLO is used in machine learning (Liu et al., 2021; Chen et al., 2022a), **multi-objective** BLO has not received much attention. In multi-task learning (MTL), the optimization problem is a multi-objective problem in nature, but is usually solved by summing the objectives and using a single-objective solver, that is, optimizing the objective $\sum_i f_i$. The robust **min-max** extension of MTL (Mehta et al., 2012; Collins et al., 2020) and RL (Li et al., 2019) have been shown to improve generalization performance, supporting the need for a more complex multi-objective optimization problem that replaces the objective $\sum_i f_i$ with the objective $\max_i f_i$. Similar

For SGD-based algorithms solving BLO, one critical aspect is whether the algorithm is **single-loop** (a single SGD step for both x and y in each iteration) or **double-loop** (the LL variable y taking multiple SGD steps between each step of the UL variable x). Double-loop algorithms can have faster empirical convergence, but are more computationally intensive and their performance is extremely sensitive to the step-sizes and termination criterion for LL iterations. Double-loop algorithms are not applicable when the (stochastic) gradients of the LL and UL problems are only provided sequentially, such as in logistics, motion planning and RL problems. Hence, we develop and analyse a single-loop algorithm.

A final aspect of BLO is the *constrained* UL *problem*. When the UL variable x corresponds to some decision variable in a decision optimization problem or a hyperparameter in HPO, we must consider a constrained form, $x \in \mathcal{X} \subset \mathbb{R}^{d_x}$. To capture a more general form of the bilevel problem, we focus on the constrained UL setup. In the remainder of this section, we will review existing literature on single-objective and multi-objective BLO and robust optimization, especially in the context of machine learning. Table 1 provides a snapshot of the properties of the problems and algorithms (with rigorous convergence analysis) studied in recent machine learning literature.

Table 1: The problem studied here relative to representative related work. If the studied problem is not a BLO, the notion of single-loop or constrained UL ($\mathcal{X} \subset \mathbb{R}^{d_x}$) is not applicable. The first row block lists general problems, the second block lists algorithms with analyses, and the final row is our MORBiT. †: This problem has been formulated both as a single-level and a BLO problem. □: The problem can be multi-objective but is treated as single-objective by summing the objectives. △: In the BLO form of adversarial learning, the UL is unconstrained but the LL is constrained.

Problem/Method	Bilevel	Multi-objective	Min-max	Single-loop	$\mathcal{X} \subset \mathbb{R}^{d_x}$
Distributionally Robust Learning	†	✗	✓	-	-
Adversarially Robust Learning	†	✗	✓	-	△
Multi-task Learning (MTL)	†	□	✗	-	-
Robust MTL (Mehta et al., 2012)	†	✓	✓	-	-
Meta-learning	†	□	✗	-	-
BSA (Ghadimi & Wang, 2018)	✓	✗	✗	✗	✓
HiBSA (Lu et al., 2020)	✗	✗	✓	✓	✓
GDA (Lin et al., 2020)	✗	✗	✓	✓	✗
TR-MAML (Collins et al., 2020)	✗	✓	✓	✓	✓
TTSA (Hong et al., 2020)	✓	✗	✗	✓	✓
StocBio (Ji et al., 2021)	✓	✗	✗	✗	✗
MRBO (Yang et al., 2021)	✓	✗	✗	✓	✗
VRBO (Yang et al., 2021)	✓	✗	✗	✗	✗
ALSET (Chen et al., 2021)	✓	✗	✗	✓	✗
STABLE (Chen et al., 2022b)	✓	✗	✗	✓	✓
MMB (Hu et al., 2022)	✓	✗	✓	✓	✗
MORBiT (Ours)	✓	✓	✓	✓	✓

Single-Objective BLO Lately, many new algorithms have been proposed to solve the single-objective BLO problem in equation 1. BSA (Ghadimi & Wang, 2018) proposed the first double-loop approach. StocBio (Ji et al., 2021) and VRBO (Yang et al., 2021) are double-loop schemes that improve upon the convergence rate of BSA but do not consider constrained UL problems. TTSA (Hong et al., 2020) is a single-loop algorithm that handles UL constraints. MRBO (Yang et al., 2021) and ALSET (Chen et al., 2021) are single-loop algorithms improving TTSA's convergence rate but do not consider UL constraints. STABLE (Chen et al., 2022b) improves upon TTSA by leveraging an additive *correction term* in the LL update step (beyond a basic SGD step) while still handling UL constraints. In contrast to the above single-objective bilevel setup, our formulation in equation 2 gives flexibility for inherently multi-objective problems in a robust manner to obtain stronger guarantees, ensuring convergence of each individual objective, rather than the average objective.

Multi-Objective BLO There has been a limited number of works analyzing multi-objective BLO schemes (Sinha et al., 2015; Deb & Sinha, 2009; Ji et al., 2017). All of these works analyze the multi-objective BLO problem from a game-theoretic point of view, using a vector-valued objective with the notion of Pareto optimality. In contrast, we are the first to study the multi-objective BLO problem from a traditional optimization perspective in terms of convergence properties and consider a min-max robust version of the multi-objective problem which produces *a single solution that ensures the convergence of each individual objective* instead of generating multiple Pareto-optimal solutions which trade-off the optimality of the different objectives.

Min-max Robust Optimization in Machine Learning Min-max optimization is commonly used to achieve robustness, such as in distributionally robust learning (DRL) and adversarially robust learning (ARL). In DRL, Duchi & Namkoong (2018) and Shalev-Shwartz & Wexler (2016) showed that a min-max loss improves generalization due to variance regularization. HiBSA (Lu et al., 2020) and GDA (Lin et al., 2020) compute quasi-Nash equilibria with convergence guarantees. Robust optimization is shown to have strong generalization for new tasks in multi-task learning (Mehta et al., 2012) and meta-learning (Collins et al., 2020). Note that the classic MAML (Finn et al., 2017) can be formulated as a BLO problem (Rajeswaran et al., 2019), and the precise problem analysed in Collins et al. (2020) is a single-level one as well. In fact, we consider the bilevel form of the TR-MAML problem as one of our applications for empirical evaluation. In ARL, the minimum is over the loss and the maximum is over the worst-case perturbation to inputs (Madry et al., 2017; Wang et al., 2019). In both DRL and ARL, the min-max objective is in the form $\min_x \max_y f(x, y)$ with a single-objective. In contrast, we study general robust multi-objective BLO where the UL objective is dependent on the LL solutions, and where the minimization is over the variable x shared across all objectives, and the maximization is over the multiple objectives, ensuring that each individual objective converges fast.

Closely related and concurrent work Given that our goals align with the properties of TTSA (Hong et al., 2020) – the single-loop nature and the ability to handle UL constraints – our proposed MORBiT is inspired by TTSA and can be viewed as a robust multi-objective version of it. Beyond this advancement, our contribution and novelty also lie in the convergence analysis of MORBiT, which significantly diverges from that of TTSA. After our MORBiT was developed and released, STABLE (Chen et al., 2022b) was recently presented as an improvement of TTSA, and we wish to explore similar improvements to MORBiT in future work. A very recent work (Hu et al., 2022) studies a problem that appears to be quite similar to equation 2, with common elements such as bilevel and min-max, and proposes a single-loop multi-block min-max bilevel (MMB) algorithm. However, there are significant differences: (i) Firstly, in their setup, they consider an extension of a min-max single level problem to a min-max BLO, and min-max is **not** meant to provide "robustness" among objectives. The applications in Hu et al. (2022) are restricted to problems such as multi-task AUC maximization instead of the common bilevel applications of representation learning and HPO. (ii) Also, Hu et al. (2022) do not consider a constrained UL problem. The problem in equation 2 is *not* a generalization of their problem – both our work and theirs are considering different setups with high-level commonalities. Table 1 shows how our setup compares to existing literature. To the best of our knowledge, *the precise problem in equation 2 has not been studied in ML literature*.

3 Algorithm and Analysis

In this section, we propose a simple single-loop algorithm MORBiT to solve equation 2, and establish a rigorous convergence rate and sample complexity for this algorithm. For the theoretical results, we defer the precise assumptions, statements and proofs to Appendix A and present the high-level theoretical results and critical novel proof steps here. In the sequel, we will always use the subscript $i \in [n]$ to denote the objective index and the superscript (k) to denote the iteration index, with $x^{(k)}$ denoting the kth iterate of the shared variable $x \in \mathcal{X} \subseteq \mathbb{R}^{d_x}$ and $y_i^{(k)}$ denoting the kth iterate of the ith-objective-specific variable $y_i \in \mathbb{R}^{d_{y_i}}$. We will also use the shorthand $\boldsymbol{y}$ to denote all the per-objective variables $[y_1, y_2, \ldots, y_n]$, with $\boldsymbol{y}^{(k)}$ denoting the kth iterate of all the per-objective variables $[y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}]$. Given our assumption that the LL objectives g_i are strongly convex, we define $y_i^\star(x) \triangleq \arg\min_{y_i \in \mathbb{R}^{d_{y_i}}} g_i(x, y)$, and use the shorthand $\ell_i(x) \triangleq f_i(x, y_i^\star(x))$.

3.1 MORBiT Algorithm

We begin with a standard reformulation of robust min-max problems (Duchi et al., 2008). We can rewrite the non-smooth min-max problem in equation 2 as

$$\min_{x \in \mathcal{X}} \max_{\lambda \in \Delta_n} \sum_{i=1}^{n} \lambda_i f_i(x, y_i^\star(x)) \quad \text{subject to} \quad y_i^\star(x) = \arg\min_{y_i \in \mathbb{R}^{d_{y_i}}} g_i(x, y_i), i \in [n] \tag{3}$$

where $\Delta_n \in \mathbb{R}_+^n$ is the n-simplex defined as $\Delta_n := \{\lambda \in \mathbb{R}_+^n \colon \lambda_i \geq 0, \forall i \in [n], \sum_{i \in [n]} \lambda_i = 1\}$. This problem is equivalent to the min-max problem in equation 2, but allows us to solve the problem with (projected) gradient based methods. The gradient for $y_i, i \in [n]$ is the straightforward $\nabla_{y_i} g_i(x, y_i)$ and we denote h_i as its stochastic estimate, with $\boldsymbol{h}$ as the shorthand for the per-objective stochastic gradient estimates $[h_1, h_2, \ldots, h_n]$. The gradient for the x-update is more involved because of the hierarchical structure of the BLO problem. Then, we consider the following weighted objectives utilizing the simplex variable $\lambda \in \Delta_n$ to define the necessary gradients:

$$F(x, \lambda) = \sum_{i \in [n]} \lambda_i \ell_i(x), \quad F(x, y, \lambda) = \sum_{i \in [n]} \lambda_i f_i(x, y_i). \tag{4}$$

Note that $F(x, \lambda)$ is the UL objective in equation 3, and the UL gradients can be defined as:

$$\nabla_x F(x, \lambda) = \sum_{i \in [n]} \lambda_i \nabla \ell_i(x), \quad \nabla_\lambda F(x, \lambda) = [\ell_1(x), \cdots, \ell_n(x)]^\top, \tag{5}$$

where $\nabla_x \ell_i(x)$ for any $i \in [n]$ can be defined as follows utilizing the strong convexity of the LL problem and implicit gradients (Gould et al., 2016):

$$\nabla \ell_i(x) = \nabla_x f_i(x, y_i^\star(x)) - \nabla^2_{xy_i} g_i(x, y_i^\star(x)) \left[\nabla^2_{y_i y_i} g_i(x, y_i^\star(x))\right]^{-1} \nabla_{y_i} f_i(x, y_i^\star(x)). \tag{6}$$

Note that in general, $y_i^\star(x)$ cannot be computed exactly. Following Ghadimi & Wang (2018), we use an approximation of $\nabla_x \ell_i(x)$ as a surrogate, denoted by $\overline{\nabla}_x f_i(x, y_i)$, by replacing $y_i^\star(x)$ in equation 6 with any $y_i \in \mathbb{R}^{d_{y_i}}$ as follows:

$$\overline{\nabla}_x f_i(x, y_i) = \nabla_x f_i(x, y_i) - \nabla^2_{xy_i} g_i(x, y_i) \left[\nabla^2_{y_i y_i} g_i(x, y_i)\right]^{-1} \nabla_{y_i} f_i(x, y_i). \tag{7}$$

Consequently, we define our approximate gradients for the UL variables x (and λ) as:

$$\overline{\nabla}_x F(x, \boldsymbol{y}, \lambda) = \sum_{i \in [n]} \lambda_i \overline{\nabla}_x f_i(x, y_i), \quad \overline{\nabla}_\lambda F(x, \boldsymbol{y}, \lambda) = [f_1(x, y_i), \cdots, f_n(x, y_n)]^\top. \tag{8}$$

We denote the (possibly biased) stochastic estimates of $\overline{\nabla}_x F(x, \boldsymbol{y}, \lambda)$ as h_x and $\overline{\nabla}_\lambda F(x, \boldsymbol{y}, \lambda)$ as h_λ.

Algorithm 1: MORBiT with learning rates α, β and γ for $x, \boldsymbol{y}, \lambda$ respectively

for $k = 1, 2, \cdots, K$ **do**
  $\boldsymbol{y}^{(k+1)} \leftarrow \boldsymbol{y}^{(k)} - \beta \boldsymbol{h}^{(k)}$
  $x^{(k+1)} \leftarrow \mathrm{proj}_{\mathcal{X}}(x^{(k)} - \alpha h_x^{(k)})$
  $\lambda^{(k+1)} \leftarrow \mathrm{proj}_{\Delta_n}(\lambda^{(k)} + \gamma h_\lambda^{(k)})$
end
Sample $\tau \sim \mathcal{U}(\{1, \cdots, K\})$
return $\bar{x} \leftarrow x^{(\tau)}, \bar{y}_i \leftarrow y_i^{(\tau-1)}, \bar{\lambda} \leftarrow \lambda^{(\tau)}$

Given the gradients and their stochastic estimates, we present our single-loop algorithm MORBiT in algorithm 1, where we utilize learning rates $\alpha, \beta, \gamma > 0$ for the UL variable x, LL variables $y_i, i \in [n]$ and the simplex variable λ respectively. The algorithm tracks three sets of variables $x^{(k)}$, $\boldsymbol{y}^{(k)} = [y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}]$ and $\lambda^{(k)}$ through a total of K iterations. The per-iterate gradient estimates $\boldsymbol{h}^{(k)}$ of the LL variables $\boldsymbol{y}$ is defined as the collection of the per-objective gradient estimate $h_i^{(k)}$ evaluated at $(x^{(k)}, y_i^{(k)})$ for all $i \in [n]$. The gradient estimates $h_x^{(k)}$ and $h_\lambda^{(k)}$ of the UL variables x and λ are evaluated at $(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)})$. We perform a standard gradient descent update for the objective specific variables $y_i, i \in [n]$ from $y_i^{(k)}$ to $y_i^{(k+1)}$. For the shared UL variable x, we perform a **projected gradient descent** to satisfy the UL constraints, where $\mathrm{proj}_{\mathcal{X}}(\cdot)$ denotes the projection operation onto the constrained set $\mathcal{X}$. We update the simplex variable λ via **projected gradient ascent**, where we project the variable back onto the n-simplex after a gradient ascent step with $\mathrm{proj}_{\Delta_n}(\cdot)$. Given the learning rates (α, β, γ), MORBiT is quite straightforward in terms of implementation. When $n = 1$, the problem reduces to single-objective BLO, $\lambda^{(k)} = 1$, and MORBiT reduces to TTSA (Hong et al., 2020).

3.2 Analysis

Given the single-loop MORBiT, we establish conditions under which MORBiT has finite-horizon convergence. The coupling of the stochastic errors due to the sampling process makes the convergence analysis of this three-sequence-based algorithm much more challenging than existing BLO algorithms.

Assumptions. We summarize the following typical assumptions (detailed in Appendix A.1) for all objective pairs $(f_i, g_i), i \in [n]$. Focusing on the smoothness and regularity properties of the objectives, we assume that (i) the LL objective g_i is strongly convex in y_i, twice-differentiable, and has sufficiently smooth first and second order gradients (Assumption 2 in Appendix A.1), (ii) the UL objective f_i has sufficiently smooth first order gradients, and (iii) the function $\ell_i(x) \triangleq f_i(x, y_i^\star(x))$ is weakly convex, bounded and has bounded first-order gradients (Assumption 1 in Appendix A.1). Regarding the quality of the gradient estimates $h_i^{(k)}, i \in [n]$, $h_x^{(k)}$ and $h_\lambda^{(k)}$, we assume that, for all $k > 0$, (i) $h_i^{(k)}$ is an unbiased estimate with bounded variance, (ii) $h_\lambda^{(k)}$ is an unbiased estimate, and (iii) $h_x^{(k)}$ has bounded variance, and can be a biased estimate of the $\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)})$ term defined in equation 8, but the bias norm at iteration k is bounded by $b_k \geq 0$, with $\{b_k, k \geq 0\}$ forming a non-increasing sequence. These gradient estimate quality assumptions are detailed in Assumption 3 in Appendix A.1. While the assumptions on $h_i^{(k)}$ and $h_\lambda^{(k)}$ are standard (Hong et al., 2020; Lu et al., 2022), the assumption on $h_x^{(k)}$ actually can be easily satisfied when a Hessian inverse approximation (HIA) based mini-batch sampling strategy is adopted, which can also avoid the matrix inversion by leveraging the Neumann series (Agarwal et al., 2017; Ghadimi & Wang, 2018; Hong et al., 2020).

Optimality and Stationarity of Solutions. To quantify the convergence properties of the solutions $\bar{x}, \bar{y}_i, i \in [n], \bar{\lambda}$ generated by MORBiT, we use the following optimality properties of the optimal solutions $x^\star, y_i^\star, i \in [n], \lambda^\star$ of the problem in equation 3. (i) The per-objective optimal LL variable $y_i^\star = y_i^\star(x^\star) = \arg\min_{y_i \in \mathbb{R}^{d_{y_i}}} g_i(x^\star, y_i)$; (ii) The optimal simplex variable $\lambda^\star$: $F(x^\star, \lambda^\star) =$

$\max_{\lambda \in \Delta_n} F(x^\star, \lambda)$. Given the constrained UL, the first-order stationarity condition is satisfied if $\langle \nabla_x F(x^\star, \lambda^\star), x - x^\star \rangle \geq 0 \forall x \in \mathcal{X}$. (iii) For establishing near-stationarity of UL variable x, the **proximal map** $\hat{x}(z) \in \mathcal{X}$, defined below,

$$\hat{x}(z) \triangleq \arg\min_{x \in \mathcal{X}} \frac{\rho}{2}\|x - z\|^2 + F(x, \lambda), \quad \rho > 0 \text{ is a fixed constant.} \tag{9}$$

is employed (Davis & Drusvyatskiy, 2018; Hong et al., 2020) to quantify the convergence for a constrained variable x in the stochastic setting. If $\|\hat{x}(x^{(k)}) - x^{(k)}\|^2$ is small, then, near-stationarity of $x^{(k)}$ is achieved at iteration k. Therefore, we need to bound $\|\hat{x}(\bar{x}) - \bar{x}\|$ to guarantee the convergence of the UL solution $\bar{x}$ returned by MORBiT. Given the convergence of the UL $\bar{x}$, we also need to bound $\|\bar{y}_i - y_i^\star(\bar{x})\|^2$ *for each* $i \in [n]$ *simultaneously* to quantify the convergence of the LL variables. Finally, the convergence of $\bar{\lambda}$ requires us to bound the difference between $F(\bar{x}, \bar{\lambda})$ and $\max_{\lambda \in \Delta_n} F(\bar{x}, \lambda)$.

Theoretical Convergence Rate. Now, we are ready to state our main theoretical result: a rigorous convergence rate for the solution returned by MORBiT (algorithm 1). We state an abbreviated version of the result, deferring details to Theorem 2 in Appendix A.2:

Theorem 1 (MORBiT convergence)**.** *Suppose that the previously stated assumptions holds and learning rates are set as* $\alpha = \mathcal{O}(K^{-3/5})$, $\beta = \mathcal{O}(K^{-2/5})$ *and* $\gamma = \mathcal{O}(n^{-1/2}K^{-3/5})$. *Then, if* $b_k^2 \leq \alpha$, *the solutions* $\bar{x}, \bar{y}_i, i \in [n], \bar{\lambda}$ *generated by algorithm 1 satisfy:*

$$\mathbb{E}[\|\hat{x}(\bar{x}) - \bar{x}\|^2] \leq \widetilde{\mathcal{O}}(\sqrt{n}K^{-2/5}), \tag{10a}$$

$$\mathbb{E}\left[\max_{i \in [n]} \|\bar{y}_i - y_i^\star(\bar{x})\|^2\right] \leq \widetilde{\mathcal{O}}(\sqrt{n}K^{-2/5}), \tag{10b}$$

$$\max_{\lambda} \mathbb{E}[F(\bar{x}, \lambda)] - \mathbb{E}[F(\bar{x}, \bar{\lambda})] \leq \widetilde{\mathcal{O}}(\sqrt{n}K^{-2/5}), \tag{10c}$$

with expectation over the stochastic gradient estimates and the random index τ *(algorithm 1, line 6).*

This result establishes the $\widetilde{\mathcal{O}}(n^{1/2}K^{-2/5})$-stationarity achieved by K iterations of MORBiT for both the UL and LL variables if all the assumptions are satisfied and the learning rates are selected appropriately. Note that, if the UL problem is unconstrained, that is $x \in X = \mathbb{R}^{d_x}$, the definition of the proximal map (equation 9) implies that $\mathbb{E}\|\nabla_x F(\bar{x}, \bar{\lambda})\| \leq \widetilde{\mathcal{O}}(n^{1/2}K^{-2/5})$, providing the convergence of $\bar{x}$ to a $\widetilde{\mathcal{O}}(n^{1/2}K^{-2/5})$-stationary point if the UL problem is unconstrained.

Comparison with Related Work. We would like to further highlight the differences between the convergence results of TTSA and MORBiT to highlight the major novelties in our analyses and theorem proving techniques. First, we consider a more general proximal map in equation 9 involving *a weighted sum of weakly convex functions* ℓ_i *instead of a single weakly convex function* in TTSA, requiring new construction of potential functions for establishing the convergence of the UL variable $\bar{x}$ in equation 10a. Secondly, even though TTSA provides a convergence rate for a single LL variable (equivalent to bounding $\mathbb{E}[\|\bar{y}_i - y_i^\star(\bar{x})\|^2]$ for a single $i \in [n]$), we provide a much stronger result for multiple LL optimization objectives, in the sense that *simultaneously establishing convergence for all* LL *variables in equation 10b* through measuring the convergence rate of $\mathbb{E}[\max_{i \in [n]} \|\bar{y}_i - y_i^\star(\bar{x})\|^2]$. This is especially challenging since a bounded $\mathbb{E}[\|\bar{y}_i - y_i^\star(\bar{x})\|]$ for each $i \in [n]$ does not directly imply a bounded $\mathbb{E}[\max_{i \in [n]} \|\bar{y}_i - y_i^\star(\bar{x})\|^2]$; in fact this can be generally unbounded. Finally, to satisfy the requirements of the min-max problem in equation 3, we have to additionally establish convergence for the simplex solution $\bar{\lambda}$ in equation 10c while TTSA does not have any such analysis.

Given the convergence rate, another related quantity of interest is the **sample complexity** which pertains to the number of queries to the stochastic gradient oracle required to achieve a desired level of stationarity. For example, for an iterative algorithm that converges to a $\mathcal{O}(K^{-\mu})$-stationary point with K iterations for some $\mu > 0$, requiring $\mathcal{O}(1)$ queries to the stochastic gradient oracle in each iteration, the sample complexity to find an ϵ-optimal solution is $\mathcal{O}(\epsilon^{-1/\mu})$. The number of stochastic gradient oracle queries required is directly related to the conditions in the gradient estimate quality assumptions (Assumption 3 in Appendix A.1 in our case). While the conditions on the per-iterate gradient estimates $h_i^{(k)}$ (for the per-objective LL variables) and $h_\lambda^{(k)}$ (for the simplex variable λ) both only require $\mathcal{O}(1)$ stochastic gradient oracle queries from each of the n objective pairs in each iteration, the condition $b_k^2 \leq \alpha$ on the non-increasing squared norm of the per-iterate bias of the gradient estimate $h_x^{(k)}$ (for the UL variable) require $\mathcal{O}(\log K)$ stochastic gradient oracle queries for

each of the n objective pairs leveraging the HIA sampling techniques in Ghadimi & Wang (2018) and Hong et al. (2020) using the Neumann series (Agarwal et al., 2017). This gives us the following sample complexity bound for MORBiT:

Corollary 1. *Under the conditions in Theorem 1,* MORBiT *converges to ϵ-(near)-stationarity with $\mathcal{O}(n^{5/4}\epsilon^{-5/2}\log(1/\epsilon))$ queries to the stochastic gradient oracle for each of the n objective pairs.*

Proof Sketch of Theorem 1. We now give a proof sketch of our main theorem, with constant terms abstracted away with $\mathcal{O}$ notation. In order to show equation 10a of Theorem 1 (convergence of x), we will derive a descent lemma comparing successive iterates $x^{(k)}$ and $x^{(k+1)}$. Descent lemmas often contain a quadratic term $\|x^{(k+1)} - x^{(k)}\|^2$, so it is natural that we must bound $\|h_x^{(k)}\|^2$. In Lemma 1, we bound the expected squared norm of the stochastic gradient estimate:

Lemma 1. *Under our regularity assumptions,* $\mathbb{E}[\|h_x^{(k)}\|^2] \leq \mathcal{O}\left(\sum_{i=1}^n \lambda_i^{(k)}\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2\right)$.

Turning to equation 10b of Theorem 1, we use a descent relation on $y_i^{(k)} - y_i^\star(x^{(k-1)})$. While ideally we would obtain a descent relation purely involving $y_i^{(k)}$ terms themselves, the intricate coupling of the x and y_i terms result in an extra $x^{(k-1)} - x^{(k)}$ term. The resulting relation is shown in Lemma 2. Here, we have that $c_1 = 1 - \frac{\mu_g\beta}{2}, c_2 = \frac{2}{\mu_g\beta} - 1$, where $\mu_g\beta < 1$.

Lemma 2. $\mathbb{E}[\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2] \leq \mathcal{O}\left((1-c_1)\mathbb{E}[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2] + c_2\mathbb{E}[\|x^{(k-1)} - x^{(k)}\|^2]\right)$.

From this lemma, intuitively, we know that $\mathbb{E}[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2]$ is decreasing as k increases, *as long as the* $\mathbb{E}[\|x^{(k-1)} - x^{(k)}\|^2]$*'s are not too large*. Therefore, it is important to have another descent relation that upper bounds this quantity, which we do next in Lemma 3. The lemma naturally involves the objective $F(x^{(k)}, \lambda^{(k)})$, which will telescope. Here, we have $c_3 = \frac{1}{4\alpha} - \frac{L_f}{2}, c_4 = 4\alpha L^2$. As $k \to \infty$, $\alpha \to 0$, and c_3 is positive.

Lemma 3. *Let* $\mathcal{L}^{(k)} \triangleq \mathbb{E}[F(x^{(k)}, \lambda^{(k)})] = \sum_{i=1}^n \lambda_i^{(k)}\mathbb{E}[\ell_i(x^{(k)})]$. *Then, the* $\mathcal{L}^{(k)}$ *satisfies:*

$$\mathcal{L}^{(k+1)} - \mathcal{L}^{(k)} \leq \mathcal{O}\left(-c_3\mathbb{E}[\|x^{(k+1)} - x^{(k)}\|^2] + c_4\max_{i\in[n]}\mathbb{E}[\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2] + \sqrt{n}\gamma + \alpha\right).$$

Following the intuition previously described, we then use Lemmas 2 and 3 to show that the $\mathbb{E}[\|x^{(k-1)} - x^{(k)}\|^2]$ terms are small enough and that the $y_i^{(k)}$ iterates converge in Lemma 4.

Lemma 4. $1/K\sum_{k=1}^K \max_{i\in[n]}\mathbb{E}\left[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2\right] \leq \mathcal{O}(\sqrt{n}K^{-2/5})$.

Finally, we can use the convergence of y_i to prove convergence of λ and x. Theorem 1 then follows. A more detailed proof plan can be found in Appendix A.3.

4 Experimental Results

In this section, we consider two applications where the min-max multi-objective bilevel formulation in equation 2 enhances robustness – multi-task representation learning and hyperparameter optimization. We will highlight the advantage of the min-max formulation and the convergence of MORBiT on these applications. We use PyTorch (Paszke et al., 2019), and implementation details are in Appendix C.

Representation Learning. In this setup, each objective pair corresponds to a learning "task" $i \in [n]$, with its own training/validation dataset pair $D_i^{\text{t}}, D_i^{\text{v}}$. We consider a shared representation network ϕ_x with ReLU nonlinearity (making the UL problem weakly convex) parameterized with $x \in \mathbb{R}^{d_x}$ and a per-task linear model w_{y_i} parameterized with $y_i \in \mathbb{R}^{d_{y_i}}$. Here we consider an unconstrained UL. Using $L(f, D)$ to denote the loss of a model f on data D, we consider the problem in equation 2 with

$$f_i(x, y_i) = L(w_{y_i} \circ \phi_x, D_i^{\text{v}}), \quad g_i(x, y_i) = L(w_{y_i} \circ \phi_x, D_i^{\text{t}}) + \rho\|w_{y_i}\|_2^2, \tag{11}$$

with $\rho > 0$ as a regularization penalty (ensuring that the LL problem is strongly convex). We first consider a multi-task setup with $n = 10$ binary classification tasks from the FashionMNIST dataset (Xiao et al., 2017). The goal is to learn a shared representation and per-task models so

that each of the tasks generalizes well. Usually, this problem is solved as a single-objective bilevel problem by minimizing $1/n \sum_i f_i$; we call this **min-avg**. We theoretically show that solving the min-max multi-objective bilevel problem in equation 2 provides a *tighter* generalization guarantee in Proposition 2 in Appendix B.

Here we demonstrate the same in figure 1 – we plot the worst-case UL objective (the validation loss) and the worst-case generalization loss across all tasks/objectives throughout the optimization trajectory, comparing the behaviour of the solution of min-avg problem to that of the min-max problem. The results indicate **solving the min-max problem** significantly reduces the worst-case validation loss and this also **results in a significant reduction of the worst-case generalization loss**, highlighting the utility of solving the min-max multi-objective bilevel problem in equation 2.

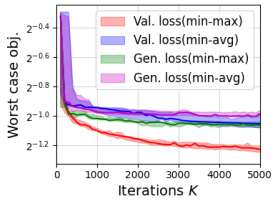

Figure 1: Quality of min-avg vs min-max MTL solution.

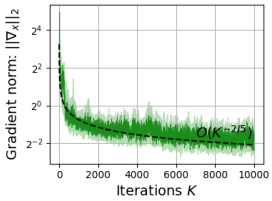

Figure 2: Cvg. of $\|\nabla_x\|^2$.

We also show the convergence of the UL variable for the above min-max problem in the form of the trajectory of gradient norm $\|\nabla_x\|^2$ in figure 2, comparing it to the theoretical $\widetilde{\mathcal{O}}(K^{-2/5})$ rate. The plot shows that **the empirical trajectory of the gradient norm closely tracks the theoretical rate**.

We also consider a bilevel extension of the robust meta-learning application (Collins et al., 2020) for a sinusoid regression task (a commonly used meta-learning application introduced by Finn et al. (2017)). [1] Here the goal of solving the problem in equation 2 with the objectives defined in equation 11 would be to learn a robust representation network such that we not only improve generalization on tasks seen during the optimization but also improve generalization for related unseen tasks.

We theoretically show that that solving the min-max multi-objective bilevel problem in equation 2 also provides a tighter generalization guarantee for the unseen tasks in Proposition 3 in Appendix B. Note that these results are similar in spirit to those of Mehta et al. (2012) and Collins et al. (2020) but our results are the first for a general bilevel setup. The results in figure 3 support this showing that solving the min-max problem not only improves the generalization on seen tasks, but significantly improves the generalization on unseen tasks when compared to solving the min-avg problem. These results are also consistent with the results for robust MTL in figure 1.

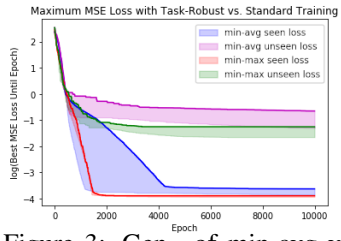

Figure 3: Gen. of min-avg vs min-max meta-learning solution.

Hyperparameter Optimization. In this setup, each objective pair again corresponds to a learning "task" $i \in [n]$, each with its own d dimensional training/validation dataset pair $D_i^{\text{t}}, D_i^{\text{v}}$. We consider a shared hyperparameter optimization problem for a kernel logistic regression (Zhu & Hastie, 2001) with K random Fourier features (RFFs) (Rahimi & Recht, 2007), where $x = \{x_\rho \in \mathbb{R}_+^K, x_\sigma \in \mathbb{R}_+^d\}$ are the regularization penalty and the bandwidth hyperparameters respectively, with ϕ_{x_σ} denoting the RFF. [2] The per-task linear model w_{y_i} on top of the RFF are parameterized with $y_i \in \mathbb{R}^K$. In this setup we have a weakly convex *constrained* UL *problem* (the hyperparameters need to be positive) and an unconstrained strongly convex LL problem. Again using $L(f, D)$ to denote the learning loss of a model f on a dataset D, we consider the problem in equation 2 with

$$f_i(x, y_i) = L(w_{y_i} \circ \phi_{x_\sigma}, D_i^{\text{v}}), \quad g_i(x, y_i) = L(w_{y_i} \circ \phi_{x_\sigma}, D_i^{\text{t}}) + \|x_\rho \odot w_{y_i}\|_2^2, \tag{12}$$

where $\odot$ denotes the elementwise vector multiplication, and we consider a weighted regression penalty.[3] We generate $n = 16$ binary classification tasks from the Letter dataset (Frey & Slate, 1991) and compare the generalization of the min-max solution of equation 2 to that of the min-avg.

[1] We use the formulation of Raghu et al. (2020) to separate the representation and the model parameters.

[2] For a d-dimension point p, the RFF $\phi_{x_\sigma}(x) = [\sin(W(x_\sigma \odot p))^\top, \cos(W(x_\sigma \odot p))^\top]^\top \in \mathbb{R}^K$, where $W \in \mathbb{R}^{K \times d}$ is a random normal matrix and the $\sin(\cdot)$ and $\cos(\cdot)$ are applied elementwise.

[3] The weighted regression penalty mitigates bias especially in the high-dimensional learning setting (Candes et al., 2008; Gasso et al., 2009; Šehić et al., 2022) which is common when using RFFs.

The results in figure 4 indicate that the solution of equation 2 provides a robust solution x (hyperparameters), significantly improving not only the worst-case validation loss but also the worst-case generalization loss for the supervised learning problems. This result highlights the advantage of solving the min-max problem in equation 2 and the ability of the single-loop MORBiT to handle a weakly convex constrained UL problem.

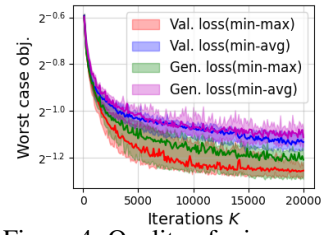

Figure 4: Quality of min-avg vs min-max HPO solution.

We study the effect of the number of objective pairs n on the convergence. We consider $n \in \{4, 16, 64\}$, increasing n with a factor of 4 (implying theoretically that the convergence should slow down with a factor of 2) to study how the convergence matches the $\sqrt{n}$-dependence in our theoretical result.

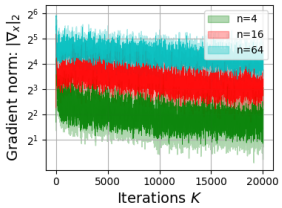

Figure 5: Effect of n on cvg.

In this case, we again consider the trajectory of the gradient norm $\|\nabla_x\|^2$ (as in figure 2). The results in figure 5 display such a behaviour – for a fixed number of outer iterations K, as the number of tasks is increased 4-fold, the gradient norm approximately increases 2-fold (note the $\log_2$-scale on the vertical axis). This validates our theoretical dependence on the number of objective pairs n.

We also study the effect of the batch size on the generalization performance of the min-max solution. In the previous experiments, we considered a batch size of 8 for both the UL and LL stochastic gradients. Here, we will consider batch sizes from $\{8, 32, 128\}$, using the same batch size for gradients of both levels and variables.

Note that, in this problem, each of the 16 learning tasks (and hence, objective pairs) has a training set size of around 900 samples (for the LL loss), with 300 samples each for the UL loss and for computing the generalization loss. Unlike figures 1 and 4, we only show the generalization loss (dropping the validation loss) in figure 6. The results indicate that increasing the batch size improves the stability and reduces the variance of the overall generalization. However, the convergence follows a similar trend for all batch sizes, and converges to a very similar level of generalization, supporting the $\widetilde{\mathcal{O}}(1)$ batch size requirement for convergence.

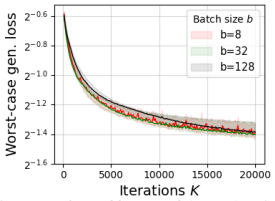

Figure 6: Effect of batch size.

Empirical conclusion

The empirical evaluations highlight that considering the more robust min-max problem in equation 2 does provide improved generalization in multiple applications (representation learning for MTL and meta-learning, and for hyperparameter optimization).

The results also highlight the validity of our theoretical convergence analysis both in terms of the number of iterations K and the number of objective pairs n.

5 Concluding Remarks

Motivated by the desiderata of robustness in bilevel learning applications, we study a new min-max multi-objective BLO framework (equation 2) that provides full flexibility and generality. We propose MORBiT (algorithm 1), a single-loop gradient descent-ascent based algorithm for finding an solution to our proposed min-max multi-objective framework. We establish its convergence rate (Theorem 1) and sample complexity (Corollary 1), demonstrating both the advantage of the min-max multi-objective BLO framework and the validity of our theoretical analyses on robust representation learning and hyperparameter optimization applications. We wish to explore further applications where robustness would be beneficial such as in RL, federated learning and domain generalization. On the theoretical side, we wish to develop single-loop algorithms with improved convergence rates (for example, exploring techniques in Chen et al. (2022b)) and double-loop algorithms with convergence guarantees for applications where a single-loop algorithm is not feasible (e.g., federated learning). Finally, we also wish to develop algorithms for large n (the number of objective pairs) or even $n \to \infty$ where MORBiT is not computationally feasible.

6 REPRODUCIBILITY STATEMENT

The formal definitions, assumptions, precise theorem statments, high level proof outline and detailed proofs for our main theoretical results are presented in Appendix A. We provide appropriate citations for the datasets used in our experiments and the experimental setup and details are presented in Appendix C.

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A CONVERGENCE ANALYSIS OF MORBiT

A.1 ASSUMPTIONS

First, we begin by listing the assumptions we make:

Assumption 1 (Regularity of the outer functions)**.** *For all $i \in [n]$, assume that outer functions $f_i(x, y)$ and $\ell_i(x) = f_i(x, y_i^\star(x))$ satisfy the following properties:*

- *For any $x \in \mathcal{X}$, $f_i(x, \cdot)$ is Lipschitz (w.r.t. y) with constant $G_f > 0$.*
- *For any $x \in \mathcal{X}$, $\nabla_x f_i(x, \cdot)$ and $\nabla_{y_i} f_i(x, \cdot)$ are Lipschitz continuous (w.r.t. y_i) with constants $L_{f_x} > 0$ and $L_{f_y} > 0$.*
- *For any $y_i \in \mathbb{R}^{d_{y_i}}$, $\nabla_{y_i} f_i(\cdot, y_i)$ is Lipschitz continuous (w.r.t. x) with constant $\bar{L}_{f_y} > 0$.*
- *For any $x \in \mathcal{X}, y_i \in \mathbb{R}^{d_{y_i}}$, we have $\|\nabla_{y_i} f_i(x, y_i)\| \leq C_{fy}$ for $C_{fy} > 0$.*
- *The function $\ell_i(\cdot)$ is μ_ℓ weakly convex (in x), so that for all $v, w \in \mathcal{X}$,*

$$\ell_i(w) \geq \ell_i(v) + \langle \nabla \ell_i(v), w - v \rangle - \mu_\ell \|w - v\|^2. \tag{13}$$

- *For all $x \in \mathcal{X}$, $\|\ell_i(x)\| \leq B_\ell$ for $B_\ell > 0$.*
- *For all $x \in \mathcal{X}$, $\|\nabla \ell_i(x)\| \leq C_\ell$, for $C_\ell > 0$.*

Assumption 2 (Regularity of the inner functions)**.** *Assume that inner functions $g_i(x, y_i), \forall i \in [n]$ satisfy:*

- *For any $x \in \mathcal{X}$ and $y_i \in \mathbb{R}^{d_{y_i}}$, $g_i(x, y_i)$ is twice continuously differentiable in (x, y_i).*
- *For any $x \in \mathcal{X}$, $\nabla_{y_i} g_i(x, \cdot)$ is Lipschitz continuous (w.r.t. y_i) with constant L_g.*
- *For any $x \in \mathcal{X}$, $g_i(x, \cdot)$ is μ_g-strongly convex in y_i, so that for all $v, w \in \mathbb{R}^{d_{y_i}}$,*

$$g_i(x, w) \geq g_i(x, v) + \langle \nabla_v g_i(x, v), w - v \rangle + \mu_g \|w - v\|^2. \tag{14}$$

- *For any $x \in \mathcal{X}$, $\nabla^2_{xy_i} g_i(x, \cdot)$ and $\nabla^2_{y_i} g_i(x, \cdot)$ are Lipschitz continuous (w.r.t. y_i) with constants $L_{gxy} > 0$ and $L_{gyy} > 0$, respectively.*
- *For any $x \in \mathcal{X}$ and $y_i \in \mathbb{R}^{d_{y_i}}$, we have $\|\nabla^2_{xy_i} g_i(x, y_i)\| \leq C_{gxy}$ for some $C_{gxy} > 0$.*
- *For any $y_i \in \mathbb{R}^{d_{y_i}}$, $\nabla^2_{xy_i} g_i(\cdot, y_i)$ and $\nabla^2_{y_i} g_i(\cdot, y_i)$ are Lipschitz continuous (w.r.t. x) with constants $\bar{L}_{gxy} > 0$ and $\bar{L}_{gyy} > 0$, respectively.*

From these assumptions, we can show a few additional regularity-type conditions. Since these conditions can also be found in Ghadimi & Wang (2018, Lemma 2.2) and Hong et al. (2020, Lemma 2), we state these results without proof.

Lemma 5 (Corollary of Assumptions)**.** *Under Assumptions 1 and 2 stated above, for all $x, x_1, x_2 \in \mathcal{X} \subseteq \mathbb{R}^{d_x}, y \in \mathbb{R}^{d_{y_i}}, i \in [n]$, we have*

$$\|\overline{\nabla}_x f_i(x, y) - \nabla \ell_i(x)\| \leq L \|y_i^\star(x) - y\|, \tag{15a}$$

$$\|y_i^\star(x_1) - y_i^\star(x_2)\| \leq G_y \|x_1 - x_2\|, \tag{15b}$$

$$\|\nabla \ell_i(x_1) - \nabla \ell_i(x_2)\| \leq L_f \|x_1 - x_2\|, \tag{15c}$$

where we define

$$L \triangleq L_{f_x} + \frac{L_{f_y} C_{gxy}}{\mu_g} + C_{f_y} \left(\frac{L_{gxy}}{\mu_g} + \frac{L_{gyy} C_{gxy}}{\mu_g^2} \right), \tag{16a}$$

$$L_f \triangleq L_{f_x} + \frac{(\bar{L}_{f_y} + L) C_{gxy}}{\mu_g} + C_{f_y} \left(\frac{\bar{L}_{gxy}}{\mu_g} + \frac{\bar{L}_{gyy} C_{gxy}}{\mu_g^2} \right), \tag{16b}$$

$$G_y \triangleq \frac{C_g}{\mu_g}. \tag{16c}$$

Assumption 3 (Quality of stochastic gradient estimates)**.** *For any iteration $k > 0$ and all $i \in [n]$, the gradient estimates $h_i^{(k)}$ for the* LL *variable y_i satisfy the following for some $\sigma_g > 0$ (Hong et al., 2020; Lu et al., 2022):*

$$\mathbb{E}[h_i^{(k)}] = \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)}), \tag{17}$$

$$\mathbb{E}[\|h_i^{(k)} - \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})\|^2] \leq \sigma_g^2(1 + \|\nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})\|^2). \tag{18}$$

For any iteration $k > 0$, the gradient estimate $h_\lambda^{(k)}$ for the simplex variable λ satisfies:

$$\mathbb{E}[h_\lambda^{(k)}] = \nabla_\lambda F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) = \left[f_1(x^{(k)}, y_1^{(k+1)}), \cdots, f_n(x^{(k)}, y_n^{(k+1)})\right]^\top. \tag{19}$$

For any $k \geq 0$ and a $\sigma_f > 0$, we assume that there exists a non-increasing sequence $\{b_k\}_{k\geq 0}$ such that

$$\mathbb{E}[h_x^{(k)}] = \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k, \|B_k\| \leq b_k, \tag{20}$$

$$\mathbb{E}[\|h_x^{(k)} - \mathbb{E}[h_x^{(k)}]\|^2] \leq \sigma_f^2. \tag{21}$$

A.2 Main Theorem and Remarks

We now state our main theorem, in full. Recall our notation from equation 9: for a fixed constant $\rho > 0$, we defined the proximal map to be

$$\hat{x}(z) \triangleq \arg\min_{x\in\mathcal{X}} \frac{\rho}{2}\|x - z\|^2 + F(x, \lambda). \tag{22}$$

We also define the Moreau envelope as

$$\Phi_{1/\rho}(z) \triangleq \min_x \frac{\rho}{2}\|x - z\|^2 + \sum_{i=1}^n \lambda_i \ell_i(x). \tag{23}$$

In addition, we use the notation $\Delta_{y_i}^{(k)} \triangleq \mathbb{E}[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2]$ and $i^{(k)} \triangleq \arg\max_{i\in[n]} \Delta_{y_i}^{(k)}$. Finally, we define $\tilde{\sigma}_f^2 = \sigma_f^2 + 3C_\ell^2$ (see Lemma 12). Now, we are ready to state the full version of our main theorem:

Theorem 2 (Convergence of MORBiT)**.** *Under Assumptions 1, 2 and 3, and the terms defined in Lemma 5, when step sizes are chosen as*

$$\alpha = \min\left(\frac{\mu_g}{16G_y L}\nu, \frac{K^{-3/5}}{4G_y L}\right), \tag{24}$$

$$\beta = \min\left(\nu, \frac{4K^{-2/5}}{\mu_g}\right), \tag{25}$$

$$\gamma = \frac{2K^{-3/5}}{B_\ell n^{1/2}}, \tag{26}$$

where $\nu = \min\left({}^{\mu_g}/_{L_g^2(1+\sigma_g^2)}, {}^{1}/_{\mu_g}\right)$, Algorithm 1 produces $\bar{x}, \bar{\lambda}, \bar{y}_i, i \in [n]$ satisfying:

$$\mathbb{E}\left[\max_{i\in[n]} \|\bar{y}_i - y_i^\star(\bar{x})\|^2\right] \leq A, \tag{27}$$

$$\max_\lambda \mathbb{E}[F(\bar{x}, \lambda)] - \mathbb{E}[F(\bar{x}, \bar{\lambda})] \leq \sqrt{2}B_\ell\sqrt{n}K^{-2/5} + G_f A, \tag{28}$$

$$\mathbb{E}[\|\hat{x}(\bar{x}) - \bar{x}\|^2] \leq \frac{16\Phi_{1/\rho}(x^{(0)})G_y L}{(-\mu_\ell + \rho)\rho K^{2/5}} + \frac{8(b_0^2 + L^2 A)}{(-\mu_\ell + \rho)^2} + \frac{2\alpha(\tilde{\sigma}_f^2 + 3b_0^2 + 3L^2 A)}{-\mu_\ell + \rho}, \tag{29}$$

where

$$A = \frac{\Delta_{y_{i^{(0)}}}^{(0)}/\mu_g}{K^{3/5}} + \frac{16\sigma_g^2/\mu_g^2}{K^{7/5}} + \frac{G_y/L}{K^{4/5}} + \frac{2\sqrt{n}B_\ell G_y/L}{K^{2/5}} + \frac{(b_0^2 + \frac{1}{2}\sigma_f^2)/(L^2)}{K^{2/5}} + \frac{16\sigma_g^2/\mu_g^2}{K^{2/5}}. \tag{30}$$

Connection of TTSA (Hong et al., 2020) As we generalize Hong et al. (2020), our proof follows a similar structure. In particular, Lemma 6 is a generalization of Hong et al. (2020, Equation (14)), and our Lemma 7 combines Hong et al. (2020, Lemma 3 and 4). Lemmas 8, 9, and 11 in our work parallel Hong et al. (2020, Lemmas 6, 5, and 7), respectively. Lemma 10 in our work deals with the maximization problem w.r.t. λ, so there is no analogue in Hong et al. (2020). However, it borrows techniques from Collins et al. (2020, Theorem 1).

We also discuss the convergence rate. Note that A in equation 30 is dominated by the fourth term, $\sqrt{n}/K^{2/5}$, so it is clear that equation 27 and equation 28 converge at a rate of $\mathcal{O}(\sqrt{n}K^{-2/5})$. We give special attention to equation 29). Apart from the ${}^{8b_0^2}/_{(-\mu_\ell+\rho)^2}$ term, we see that the RHS of equation 29 converges at a rate of $\mathcal{O}(\sqrt{n}K^{-2/5})$. To understand the convergence of this term, we turn to (20) from Assumption 3. As discussed in section 3.2, b_k can be made arbitrarily small by running more iterations of the subroutine for estimating $h_x^{(k)}$ (for example utilizing the HIA sampling scheme (Agarwal et al., 2017; Ghadimi & Wang, 2018; Hong et al., 2020)). Therefore, as long as we run enough iterations ($O(\log K)$ for HIA) such that $b_k^2 \leq \alpha$, (29) will also converge at a rate of $\mathcal{O}(\sqrt{n}K^{-2/5})$.

A.3 Proof Plan

Overall Roadmap: In what follows, we give a proof sketch of our main theorem, with constant terms abstracted away with $\mathcal{O}$ notation. c_1, c_2, c_3, c_4 are positive constants depending on L_f (defined in Lemma 5), μ_g (the LL objective convexity defined in Assumption 2) and the learning rates α, β (in algorithm 1).

In order to show equation 10a of Theorem 1 (the convergence of x), we will derive a descent lemma comparing successive iterates $x^{(k)}$ and $x^{(k+1)}$. Descent lemmas often contain a quadratic term $\|x^{(k+1)} - x^{(k)}\|^2$, so it is natural that we will have to bound $\|h_x^{(k)}\|^2$. As such, in Lemma 6, we bound the averaged squared norm of the stochastic gradient estimate, $\mathbb{E}[\|h_x^{(k)}\|^2]$:

Lemma 6. *Under our regularity assumptions, the average squared norm of $h_x^{(k)}$ can be bounded as follows, where the expectation is over the filtration $\mathcal{F}_i' \triangleq \{y_i^{(0)}, x^{(0)}, \cdots, y_i^{(k)}, x^{(k)}, y_i^{(k+1)}\}$:*

$$\mathbb{E}[\|h_x^{(k)}\|^2] \leq \mathcal{O}\left(\sum_{i=1}^{n} \lambda_i^{(k)} \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2\right). \tag{31}$$

Turning to equation 10a of Theorem 1, we'll use a descent relation on $y_i^{(k)} - y_i^\star(x^{(k-1)})$. While ideally we would obtain a descent relation purely involving $y_i^{((k))}$ terms themselves, the intricate coupling of the x and y terms result in an extra $x^{(k-1)} - x^{(k)}$ term. The resulting relation is shown in Lemma 7. Here, we have that $c_1 = 1 - \frac{\mu_g \beta}{2}, c_2 = \frac{2}{\mu_g \beta} - 1$, where $\mu_g \beta < 1$.

Lemma 7. *The distance between the algorithm's iterates $y_i^{(k)}$ and the true inner optimum $y_i^\star(x^{(k)})$ satisfies the following descent equation,*

$$\mathbb{E}[\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2] \leq \mathcal{O}((1-c_1)\mathbb{E}[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2] + c_2\mathbb{E}[\|x^{(k-1)} - x^{(k)}\|^2]). \tag{32}$$

From this lemma, intuitively, we know that $\mathbb{E}[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2]$ is decreasing as k increases, *as long as the $\mathbb{E}[\|x^{(k-1)} - x^{(k)}\|^2]$'s are not too large*. Therefore, it is important to have another descent relation that upper bounds this quantity, which we do next in Lemma 8. The lemma naturally involves the objective $\mathcal{L}^{(k)} = F(x^{(k)}, \lambda^{(k)})$, which will telescope. Here, we have $c_3 = \frac{1}{4\alpha} - \frac{L_f}{2}, c_4 = 4\alpha L^2$. As $k \to \infty$, $\alpha \to 0$, and c_3 is positive.

Lemma 8. *Let $\mathcal{L}^{(k)} \triangleq \mathbb{E}[F(x^{(k)}, \lambda^{(k)})] = \sum_{i=1}^{n} \lambda_i^{(k)} \mathbb{E}[\ell_i(x^{(k)})]$. Then, the $\mathcal{L}^{(k)}$ satisfies the descent equation*

$$\mathcal{L}^{(k+1)} - \mathcal{L}^{(k)} \leq \mathcal{O}\left(-c_3\mathbb{E}[\|x^{(k+1)} - x^{(k)}\|^2] + c_4 \max_{i\in[n]} \mathbb{E}[\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2] + \sqrt{n}\gamma + \alpha\right). \tag{33}$$

Following the intuition previously described, we then use Lemmas 7 and 8 to show that the $\mathbb{E}[\|x^{(k-1)} - x^{(k)}\|^2]$ terms are small enough and that the $y_i^{(k)}$ iterates converge:

Lemma 9 (Informal, see Appendix A.7 for precise statement)**.**

$$\frac{1}{K}\sum_{k=1}^{K}\max_{i\in[n]}\mathbb{E}\left[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2\right] \leq \mathcal{O}(\sqrt{n}K^{-2/5}). \tag{34}$$

Lemma 10 then leverages the convergence of $y_i, i \in [n]$ to bound the convergence of λ, and Lemma 11 shows the bound on x. By plugging in our step-sizes into Lemmas 9, 10, and 11, Theorem 1 directly follows.

Lemma 10. *For any $\lambda \in \Delta_n$, the iterates of Algorithm 1 satisfy*

$$\frac{1}{K}\mathbb{E}\left[\sum_{k=1}^{K} F(x^{(k)}, \lambda) - F(x^{(k)}, \lambda^{(k)})\right] \leq \mathcal{O}(\sqrt{n}K^{-2/5}). \tag{35}$$

Lemma 11. *The iterates of Algorithm 1 satisfy*

$$\frac{1}{K}\sum_{k=1}^{K}\mathbb{E}[\|\hat{x}(x^{(k)}) - x^{(k)}\|^2] \leq \mathcal{O}(\sqrt{n}K^{-2/5}). \tag{36}$$

A.4 Proof of Lemma 1 (Lemma 6)

Stating Lemma 1 more precisely:

Lemma 12. *Under Assumptions 1, 2 and 3, the average squared norm of $h_x^{(k)}$ can be bounded as follows, where $\tilde{\sigma}_f^2 = \sigma_f^2 + 3C_\ell^2$:*

$$\mathbb{E}[\|h_x^{(k)}\|^2] \leq \tilde{\sigma}_f^2 + 3b_k^2 + 3L^2\sum_{i=1}^{n}\lambda_i^{(k)}\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2, \tag{37}$$

where the expectation is over the filtration $\mathcal{F}_i' \triangleq \{y_i^{(0)}, x^{(0)}, \cdots, y_i^{(k)}, x^{(k)}, y_i^{(k+1)}\}$.

Note that here the expectation is over $\mathcal{F}_i' \triangleq \{y_i^{(0)}, x^{(0)}, \cdots, y_i^{(k)}, x^{(k)}, y_i^{(k+1)}\}$, so no expectation is needed in the last term.

Proof. We can derive the following:

$$\begin{aligned}
\mathbb{E}[\|h_x^{(k)}\|^2] &\overset{(1)}{=} \mathbb{E}[\|h_x^{(k)} - \mathbb{E}[h_x^{(k)}]\|^2] + \|\mathbb{E}[h_x^{(k)}]\|^2 && (38)\\
&\overset{(2)}{=} \mathbb{E}[\|h_x^{(k)} - \mathbb{E}[h_x^{(k)}]\|^2] + \|\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k\|^2 && (39)\\
&\overset{(3)}{\leq} \sigma_f^2 + \|\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k\|^2 && (40)\\
&\overset{(4)}{=} \sigma_f^2 + \left\|\sum_{i=1}^{n}\lambda_i^{(k)}\overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)}) + B_k\right\|^2 && (41)\\
&\overset{(5)}{\leq} \sigma_f^2 + 3b_k^2 + \frac{3}{2}\left\|\sum_{i=1}^{n}\lambda_i^{(k)}\overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)})\right\|^2. && (42)
\end{aligned}$$

(1) is true because

$$\begin{aligned}
\mathbb{E}[\|h_x^{(k)} - \mathbb{E}[h_x^{(k)}]\|^2] + \|\mathbb{E}[h_x^{(k)}]\|^2 &= \mathbb{E}[\|h_x^{(k)}\|^2] + \|\mathbb{E}[h_x^{(k)}]\|^2 - 2\mathbb{E}\langle h_x^{(k)}, \mathbb{E}[h_x^{(k)}]\rangle + \|\mathbb{E}[h_x^{(k)}]\|^2 && (43)\\
&= \mathbb{E}[\|h_x^{(k)}\|^2] + \|\mathbb{E}[h_x^{(k)}]\|^2 - \langle\mathbb{E}[h_x^{(k)}], \mathbb{E}[h_x^{(k)}]\rangle + \|\mathbb{E}[h_x^{(k)}]\|^2 && (44)\\
&= \mathbb{E}[\|h_x^{(k)}\|^2]. && (45)
\end{aligned}$$

(2) follows from (20), (3) follows from (21), (4) follows from definition of $\overline{\nabla}_x$, and (5) follows from Young's inequality, $\|a+b\|^2 \leq 3\|a\|^2 + \frac{3}{2}\|b\|^2$. Next, we bound the last term in (42). We start by using the fact that

$$\begin{aligned}
&\Big\|\sum_{i=1}^n \lambda_i^{(k)} \overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)})\Big\|^2 \\
&\overset{(1)}{\leq} 2\Big\|\sum_{i=1}^n \lambda_i^{(k)} (\overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)}) - \nabla \ell_i(x^{(k)}))\Big\|^2 + 2\Big\|\sum_{i=1}^n \lambda_i^{(k)} \nabla \ell_i(x^{(k)})\Big\|^2 \qquad (46)\\
&\overset{(2)}{\leq} 2\sum_{i=1}^n \lambda_i^{(k)} \|\overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)}) - \nabla \ell_i(x^{(k)})\|^2 + 2\sum_{i=1}^n \lambda_i^{(k)} \|\nabla \ell_i(x^{(k)})\|^2, \qquad (47)
\end{aligned}$$

where (1) follows from $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$, and (2) follows from $\Big\|\sum_{i=1}^n p_i a_i\Big\|^2 \leq \sum_{i=1}^n p_i \|a_i\|^2$. Next, we bound the first term in (47). From Lemma 5, we have

$$\|\overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)}) - \nabla \ell_i(x^{(k)})\|^2 \leq L\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2. \qquad (48)$$

Therefore, we can obtain

$$\begin{aligned}
\mathbb{E}[\|h_x^{(k)}\|^2] &\overset{(1)}{\leq} \sigma_f^2 + 3b_k^2 + 3L^2 \sum_{i=1}^n \lambda_i^{(k)} \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2 + 3\sum_{i=1}^n \lambda_i^{(k)} \|\nabla \ell_i(x^{(k)})\|^2 \qquad (49)\\
&\overset{(2)}{\leq} \tilde{\sigma}_f^2 + 3b_k^2 + 3L^2 \sum_{i=1}^n \lambda_i^{(k)} \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2, \qquad (50)
\end{aligned}$$

where (1) comes from plugging (48) into (47) and (47) into (42), (2) comes from the definition of $\tilde{\sigma}_f^2$ using $\|\nabla \ell_i(x^{(k)})\|^2 \leq C_\ell^2$ and $\lambda^{(k)} \in \Delta_n$. □

A.5 Proof of Lemma 2 (Lemma 7)

We state the precise version of Lemma 2 here:

Lemma 13. *Under Assumptions 1, 2, and 3, when $\beta^2(1+\sigma_g^2)L_g^2 \leq \beta\mu_g$ and $\mu_g\beta < 1$, the iterates $y_i^{(k)}$ satisfy the descent equation:*

$$\begin{aligned}
&\mathbb{E}[\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2] \qquad (51)\\
&\quad \leq \Big(1 - \frac{\mu_g\beta}{2}\Big)\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2 + \Big(\frac{2}{\mu_g\beta} - 1\Big)G_y^2\|x^{(k-1)} - x^{(k)}\|^2 + \beta^2\sigma_g^2.
\end{aligned}$$

Proof. For a particular (fixed) realization of the iterates $x^{(1)}, \cdots, x^{(k)}, y_i^{(1)}, \cdots, y_i^{(k)}$ for some $i \in [n]$, we have

$$\begin{aligned}
\mathbb{E}[\|h_i^{(k)}\|^2] &\overset{(1)}{=} \mathbb{E}[\|h_i^{(k)} - \mathbb{E}[h_i^{(k)}]\|^2] + \|\mathbb{E}[h_i^{(k)}]\|^2 \qquad (52)\\
&\overset{(2)}{=} \mathbb{E}[\|h_i^{(k)} - \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})\|^2] + \|\nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})\|^2 \qquad (53)\\
&\overset{(3)}{\leq} \sigma_g^2 + (1+\sigma_g^2)\|\nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})\|^2 \qquad (54)\\
&\overset{(4)}{\leq} \sigma_g^2 + (1+\sigma_g^2)\|\nabla_{y_i} g_i(x^{(k)}, y_i^{(k)}) - \nabla_{y_i} g_i(x^{(k)}, y_i^\star(x^{(k)}))\|^2 \qquad (55)\\
&\overset{(5)}{\leq} \sigma_g^2 + (1+\sigma_g^2)L_g^2\|y_i^{(k)} - y_i^\star(x^{(k)})\|^2, \qquad (56)
\end{aligned}$$

where (1) follows from algebra, (2) follows from $\mathbb{E}[h_i^{(k)}] = \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})$ in equation 17 in Assumption 3, (3) is from equation 18 in Assumption 3, (4) is from $\nabla_{y_i} g_i(x^{(k)}, y_i^\star(x^{(k)})) = 0$ due to the optimality of $y_i^\star(x^{(k)})$, and (5) is due to the L_g-Lipschitz continuity of $\nabla_{y_i} g_i(x, \cdot)$.

Next, we can bound the difference between $y_i^{(k+1)}$ and $y_i^\star(x^{(k)})$ as the following, where again we assume that $x^{(1)}, \cdots, x^{(k)}, y^{(1)}, \cdots, y_i^{(k)}$ is fixed and the expectation is over the stochasticity of the gradient estimates:

$$
\begin{aligned}
&\mathbb{E}[\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2] \\
&\overset{(1)}{=} \mathbb{E}[\|y_i^{(k)} - \beta h_i^{(k)} - y_i^\star(x^{(k)})\|^2] && (57)\\
&\overset{(2)}{=} \|y_i^{(k)} - y_i^\star(x^{(k)})\|^2 + \beta^2\mathbb{E}[\|h_i^{(k)}\|]^2 - 2\beta\langle y_i^{(k)} - y_i^\star(x^{(k)}), \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})\rangle && (58)\\
&\overset{(3)}{\leq} (1-2\beta\mu_g)\|y_i^{(k)} - y_i^\star(x^{(k)})\|^2 + \beta^2\mathbb{E}[\|h_i^{(k)}\|^2] && (59)\\
&\overset{(4)}{\leq} (1-2\beta\mu_g)\|y_i^{(k)} - y_i^\star(x^{(k)})\|^2 + \beta^2\sigma_g^2 + \beta^2(1+\sigma_g^2)L_g^2\|y_i^{(k)} - y_i^\star(x^{(k)})\|^2 && (60)\\
&\overset{(5)}{\leq} (1-\beta\mu_g)\|y_i^{(k)} - y_i^\star(x^{(k)})\|^2 + \beta^2\sigma_g^2 && (61)\\
&\overset{(6)}{\leq} (1-\beta\mu_g)\left[(1+c)\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2 + \left(1+\frac{1}{c}\right)\|y_i^\star(x^{(k-1)}) - y_i^\star(x^{(k)})\|^2\right] + \beta^2\sigma_g^2 && (62)\\
&\overset{(7)}{\leq} (1-\beta\mu_g)\left[(1+c)\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2 + \left(1+\frac{1}{c}\right)G_y^2\|x^{(k-1)} - x^{(k)}\|^2\right] + \beta^2\sigma_g^2, && (63)
\end{aligned}
$$

where (1) is true by definition, and (2) holds by direct algebra and the unbiasedness assumption $\mathbb{E}[h_i^{(k)}] = \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)})$ in equation 17 in Assumption 3, (3) is from strong convexity, $\beta\left\langle \nabla_{y_i} g_i(x^{(k)}, y_i^{(k)}), y_i^{(k)} - y_i^\star(x^{(k)})\right\rangle \geq \beta\mu_g\|y_i^{(k)} - y_i^\star(x^{(k)})\|^2$, (4) is from equation 56, (5) is from the assumption $\beta^2(1+\sigma_g^2)L_g^2 \leq \beta\mu_g$, (6) is from the inequality $\|a+b\|^2 \leq (1+1/c)\|a\|^2 + (1+c)\|b\|^2$, and (7) is from the G_y-lipschitzness of $y_i^\star(\cdot)$ in Lemma 5.

Then, we choose $c = \frac{\mu_g\beta}{2(1-\mu_g\beta)}$, so that $(1-\beta\mu_g)(1+c) = 1-\beta\mu_g/2$ and $1+1/c = \frac{2}{\mu_g\beta} - 1$. We have $c > 0$ because $\mu_g\beta < 1$. Plugging these expressions into (63), we get

$$
\begin{aligned}
&\|y_i^{(k+1)} - y_i^\star(x^{(k)})\|^2 \\
&\quad\leq \left(1-\frac{\mu_g\beta}{2}\right)\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2 + \left(\frac{2}{\mu_g\beta} - 1\right)G_y^2\|x^{(k-1)} - x^{(k)}\|^2 + \beta^2\sigma_g^2,
\end{aligned} \tag{64}
$$

which completes the proof. □

A.6 Proof of Lemma 3 (Lemma 8)

We state the precise version of Lemma 3 here:

Lemma 14. *Let $\mathcal{L}^{(k)} \triangleq \mathbb{E}[F(x^{(k)}, \lambda^{(k)})] = \sum_{i=1}^n \lambda_i^{(k)}\mathbb{E}[\ell_i(x^{(k)})]$. Under Assumptions 1, 2 and 3, assume that the iterates $\{x^{(k)}, y_i^{(k)}, i \in [n], \lambda^{(k)}, \forall k\}$ are generated by* MORBiT*, then, $\mathcal{L}^{(k)}$ satisfies the descent equation*

$$
\begin{aligned}
&\mathcal{L}^{(k+1)} - \mathcal{L}^{(k)} \\
&\quad\leq 4\alpha L^2 \max_{i\in[n]} \Delta_{y_i}^{(k+1)} + \left(\frac{L_f}{2} - \frac{1}{4\alpha}\right)\mathbb{E}[\|x^{(k+1)} - x^{(k)}\|^2] + \gamma n B_\ell^2 + 4\alpha b_0^2 + 2\alpha\sigma_f^2.
\end{aligned} \tag{65}
$$

Proof. First, since ℓ_i is L_f-smooth, we know that for all $i \in [n]$,

$$
\ell_i(x^{(k+1)}) \leq \ell_i(x^{(k)}) + \langle x^{(k+1)} - x^{(k)}, \nabla\ell_i(x^{(k)})\rangle + \frac{L_f}{2}\|x^{(k+1)} - x^{(k)}\|^2. \tag{66}
$$

Taking $\lambda_i^{(k)}$ times the equation for i in (66) and summing, we can get

$$\begin{aligned}&\sum_{i=1}^n \lambda_i^{(k)} \ell_i(x^{(k+1)}) \\ &\quad \leq \sum_{i=1}^n \lambda_i^{(k)} \ell_i(x^{(k)}) + \left\langle x^{(k+1)} - x^{(k)}, \sum_{i=1}^n \lambda_i^{(k)} \nabla \ell_i(x^{(k)}) \right\rangle + \frac{L_f}{2}\|x^{(k+1)} - x^{(k)}\|^2.\end{aligned} \tag{67}$$

Therefore, we have

$$\begin{aligned}&\sum_{i=1}^n \lambda_i^{(k+1)} \ell_i(x^{(k+1)}) - \sum_{i=1}^n \lambda_i^{(k)} \ell_i(x^{(k)}) \\ &\quad \leq \underbrace{\sum_{i=1}^n \lambda_i^{(k+1)} \ell_i(x^{(k+1)}) - \sum_{i=1}^n \lambda_i^{(k)} \ell_i(x^{(k+1)})}_{\triangleq (A)} + \underbrace{\langle x^{(k+1)} - x^{(k)}, \sum_{i=1}^n \lambda_i^{(k)} \nabla \ell_i(x^{(k)}) \rangle}_{\triangleq (B)} \\ &\quad + \frac{L_f}{2}\|x^{(k+1)} - x^{(k)}\|^2.\end{aligned} \tag{68}$$

Next, we bound (A) and (B) respectively as follows. First, we upper bound term (A). First, from the non-expansiveness of projections and $\lambda^{(k+1)} = \mathrm{proj}_{\Delta_n}(\lambda^{(k)} + \gamma h_\lambda^{(k)})$, we have $\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq \|\gamma h_\lambda^{(k)}\|$. Since $\lambda^{(k+1)}, \lambda^{(k)} \in \Delta_n$, $\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq \sqrt{2}$. Therefore, we know that $\|\lambda^{(k+1)} - \lambda^{(k)}\| \leq \Lambda \triangleq \min\{\sqrt{2}, \gamma\|h_\lambda^{(k)}\|\}$. Based on these facts, we can have

$$(A) = \sum_{i=1}^n \lambda_i^{(k+1)} \ell_i(x^{(k+1)}) - \sum_{i=1}^n \lambda_i^{(k)} \ell_i(x^{(k+1)}) \tag{69}$$

$$\overset{(1)}{=} \sum_{i=1}^n \left(\lambda_i^{(k+1)} - \lambda_i^{(k)}\right) \ell_i(x^{(k+1)}) \tag{70}$$

$$\overset{(2)}{\leq} \left\|\lambda^{(k+1)} - \lambda^{(k)}\right\| \left\|\left[\ell_1(x^{(k+1)}), \ell_2(x^{(k+1)}), \cdots, \ell_n(x^{(k+1)})\right]^\top\right\| \tag{71}$$

$$\overset{(3)}{\leq} \Lambda B_\ell \sqrt{n} \quad \overset{(4)}{\leq} \sqrt{n} \min\{\sqrt{2}, \|\gamma h_\lambda^{(k)}\|\} B_\ell \quad \overset{(5)}{\leq} \sqrt{n}\gamma \|h_\lambda^{(k)}\| B_\ell, \tag{72}$$

where (1) is straightforward, (2) follows from Cauchy-Schwarz, (3) follows from the update rule for λ and the fact that $|\ell_i(\cdot)| \leq B_\ell$ from Assumption 1, (4) is from plugging in the definition of Λ, and (5) follows from $\gamma\|h_\lambda^k\| \leq \sqrt{2}$.

Then, we upper bound (B). First, from the non-expansiveness of projection and the update rule $x^{(k+1)} = \mathrm{proj}_{\mathcal{X}}(x^{(k)} - \alpha h_x^{(k)})$, we know that

$$\|x^{(k+1)} - x^{(k)} + \alpha h_x^{(k)}\|^2 \leq \| - \alpha h_x^{(k)}\|^2, \tag{73}$$

$$\Rightarrow \|x^{(k+1)} - x^{(k)}\|^2 + 2\alpha\langle x^{(k+1)} - x^{(k)}, h_x^{(k)}\rangle \leq 0, \tag{74}$$

$$\Rightarrow \frac{1}{2\alpha}\|x^{(k+1)} - x^{(k)}\|^2 + \langle x^{(k+1)} - x^{(k)}, h_x^{(k)}\rangle \leq 0. \tag{75}$$

Therefore, we can have

$$(B) = \left\langle \left(\sum_{i=1}^{n} \lambda_i^{(k)} \nabla \ell_i(x^{(k)}) \right), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle \overset{(1)}{=} \left\langle \nabla_x F(x^{(k)}, \lambda^{(k)}), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle \tag{76}$$

$$\overset{(2)}{=} \left\langle \left(\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k \right), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle + \left\langle \left(\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k \right), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle \tag{77}$$

$$\overset{(3)}{\leq} \left\langle \left(\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k \right), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle \tag{78}$$

$$+ \left\langle \left(\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k - h_x^{(k)} \right), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle - \frac{1}{2\alpha} \|x^{(k+1)} - x^{(k)}\|^2$$

$$\overset{(4)}{\leq} \frac{1}{2c} \|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k\|^2 + \frac{c}{2} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{1}{2d} \|\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k - h_x^{(k)}\|^2 + \frac{d}{2} \|x^{(k+1)} - x^{(k)}\|^2 - \frac{1}{2\alpha} \|x^{(k+1)} - x^{(k)}\|^2 \tag{79}$$

$$\overset{(5)}{\leq} \frac{1}{2c} \|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k\|^2 + \frac{c}{2} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\sigma_f^2}{2d} + \frac{d}{2} \|x^{(k+1)} - x^{(k)}\|^2 - \frac{1}{2\alpha} \|x^{(k+1)} - x^{(k)}\|^2 \tag{80}$$

$$\overset{(6)}{=} \frac{1}{2c} \|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k\|^2 + \left(\frac{c+d}{2} - \frac{1}{2\alpha} \right) \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\sigma_f^2}{2d}, \tag{81}$$

where (1) is by definition of $F(x^{(k)}, \lambda^{(k)})$, (2) is from adding and subtracting $\left\langle \left(\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) + B_k \right), \left(x^{(k+1)} - x^{(k)} \right) \right\rangle$, (3) is from adding (75) to the previous inequality, (4) is from applying the inequality $\langle a, b \rangle \leq \frac{1}{2c}\|a\|^2 + \frac{c}{2}\|b\|^2$ to both inner product terms, (5) is from equation 20 and equation 21 in Assumption 3, and (6) is from algebra.

Plugging in our expressions for (A) and (B) into (68), we get

$$\sum_{i=1}^{n} \lambda_i^{(k+1)} \ell_i(x^{(k+1)}) - \sum_{i=1}^{n} \lambda_i^{(k)} \ell_i(x^{(k)}) \leq \gamma \|h_\lambda^{(k)}\| \sqrt{n} B_\ell + \frac{1}{2c} \|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k\|^2 + \left(\frac{c+d+L_f}{2} - \frac{1}{2\alpha} \right) \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\sigma_f^2}{2d}. \tag{82}$$

Next, we work on bounding $\|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k\|^2$ in equation 81. Observe that

$$\begin{aligned}
&\|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - B_k\|^2 \\
&\quad \overset{(1)}{\leq} 2\|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)})\|^2 + 2\|B_k\|^2 && (83)\\
&\quad \overset{(2)}{\leq} 2\|\nabla_x F(x^{(k)}, \lambda^{(k)}) - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)})\|^2 + 2b_k^2 && (84)\\
&\quad \overset{(3)}{\leq} 2\left\|\sum_{i=1}^n \left[\lambda_i^{(k)}\left(\nabla \ell_i(x^{(k)}) - \overline{\nabla}_x f_i(x^{(k)}, y_i^{(k+1)})\right)\right]\right\|^2 + 2b_k^2 && (85)\\
&\quad \overset{(4)}{\leq} 2\left(\sum_{i=1}^n \lambda_i^{(k)} L\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|\right)^2 + 2b_k^2 && (86)\\
&\quad \overset{(5)}{\leq} 2L^2 \max_{i\in[n]} \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2 + 2b_k^2, && (87)
\end{aligned}$$

where (1) comes from $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$, (2) comes from $\|B_k\| \leq b_k$ in Assumption 3, (3) is from expanding the definitions of $\nabla_x F(\cdot,\cdot)$ and $\overline{\nabla}_x F(\cdot,\cdot,\cdot)$, (4) is from Lemma 5. Therefore, plugging in equation 87 into equation 82, using $c = d = \frac{1}{4\alpha}$, and taking expectation over the stochasticity of the gradient estimates, we get:

$$\begin{aligned}
&\sum_{i=1}^n \lambda_i^{(k+1)} \ell_i(x^{(k+1)}) - \sum_{i=1}^n \lambda_i^{(k)} \ell_i(x^{(k)}) \\
&\quad \leq \sqrt{n}\gamma \|h_\lambda^{(k)}\| B_\ell + 4\alpha L^2 \max_{i\in[n]} \mathbb{E}\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2 \\
&\qquad + 4\alpha b_k^2 + \left(\frac{L_f}{2} - \frac{1}{4\alpha}\right)\mathbb{E}[\|x^{(k+1)} - x^{(k)}\|^2] + 2\alpha\sigma_f^2. && (88)
\end{aligned}$$

Now, observe that the LHS looks like a telescoping sum. To make this more apparent, define $\mathcal{L}^{(k)} = \sum_{i=1}^n \lambda_i^{(k)} \mathbb{E}[\ell_i(x^{(k)})]$ and $\Delta_{y_i}^{(k)} = \mathbb{E}[\|y_i^{(k)} - y_i^\star(x^{(k-1)})\|^2]$. Therefore, with the assumption that $b_k \leq b_0$, we have

$$\begin{aligned}
&\mathcal{L}^{(k+1)} - \mathcal{L}^{(k)} \\
&\quad \leq 4\alpha L^2 \max_{i\in[n]} \Delta_{y_i}^{(k+1)} + \left(\frac{L_f}{2} - \frac{1}{4\alpha}\right)\mathbb{E}[\|x^{(k+1)} - x^{(k)}\|^2] + \sqrt{n}\gamma\|h_\lambda^{(k)}\| B_\ell + 4\alpha b_0^2 + 2\alpha\sigma_f^2 \\
&\quad \leq 4\alpha L^2 \max_{i\in[n]} \Delta_{y_i}^{(k+1)} + \left(\frac{L_f}{2} - \frac{1}{4\alpha}\right)\mathbb{E}[\|x^{(k+1)} - x^{(k)}\|^2] + \gamma n B_\ell^2 + 4\alpha b_0^2 + 2\alpha\sigma_f^2. && (89)
\end{aligned}$$

□

A.7 Proof of Lemma 9

We restate Lemma 9 in more general terms here:

Lemma 15. *Assume that $\Omega^{(k)}, \Theta^{(k)}, \Upsilon_i^{(k)}, \lambda_i, c_0, c_1, c_2, d_0, d_1, d_2$ are real numbers such that for all $0 \leq k \leq K-1$,*

$$\Omega^{(k+1)} \leq \Omega^{(k)} - c_0\Theta^{(k+1)} + c_1 \max_{i\in[n]} \Upsilon_i^{(k+1)} + c_2 \tag{90}$$

and also for all $1 \leq k \leq K, 1 \leq i \leq N$,

$$\Upsilon_i^{(k+1)} \leq (1-d_0)\Upsilon_i^{(k)} + d_1\Theta^{(k)} + d_2. \tag{91}$$

In addition, assume that $1 - d_0 > 0$*,* $d_0 - d_1 c_1 c_0^{-1} > 0$ *and* $c_0 - c_1 d_1 d_0^{-1}$*, and that* $\Upsilon_i^{(k)}, \Omega^{(k)} \geq 0$ *for all* $k, i \in [n]$*. Then, if* $i^{(0)} = \arg\max_{i \in [n]} \Upsilon_i^{(0)}$*, we have*

$$\frac{1}{K}\sum_{k=1}^{K} \max_{i\in[n]} \Upsilon_i^{(k)} \leq (d_0 - d_1 c_0^{-1} c_1)^{-1} \left(\frac{\Upsilon_{i^{(0)}}^{(0)} + d_1 \Theta^{(0)} + d_2 + d_1 c_0^{-1} \Omega^{(0)}}{K} + d_1 c_0^{-1} c_2 + d_2 \right). \tag{92}$$

Proof. First, let $i^{(k)} \triangleq \arg\max_{i\in[n]} \Upsilon_i^{(k)}$, so that $\Upsilon_{i^{(k)}}^{(k)} = \max_{i\in[n]} \Upsilon_i^{(k)}$. Summing (90) from $k = 0, 1, \cdots, K-1$, we get:

$$c_0 \sum_{k=1}^{K} \Theta^{(k)} \leq \Omega^{(0)} - \Omega^{(k)} + c_1 \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} + c_2 K. \tag{93}$$

Next, we apply (91) for $i = i^{(k+1)}$. Noting that $1 - d_0 > 0$ and $\Upsilon_{i^{(k+1)}}^{(k)} \leq \Upsilon_{i^{(k)}}^{(k)}$ by definition of $i^{(k)}$, we have

$$\Upsilon_{i^{(k+1)}}^{(k+1)} \leq (1 - d_0)\Upsilon_{i^{(k+1)}}^{(k)} + d_1 \Theta^{(k)} + d_2 \tag{94}$$

$$\leq (1 - d_0)\Upsilon_{i^{(k)}}^{(k)} + d_1 \Theta^{(k)} + d_2. \tag{95}$$

Then summing for $k = 1$ to K, we get

$$d_0 \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} \leq \Upsilon_{i^{(1)}}^{(1)} - \Upsilon_{i^{(K+1)}}^{(K+1)} + d_1 \sum_{k=1}^{K} \Theta^{(k)} + d_2 K. \tag{96}$$

Now, we have

$$d_0 \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} \overset{(1)}{\leq} \Upsilon_{i^{(1)}}^{(1)} - \Upsilon_{i^{(K+1)}}^{(K+1)} + d_1 c_0^{-1} \left(\Omega^{(0)} - \Omega^{(k)} + c_1 \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} + c_2 K \right) + d_2 K \tag{97}$$

$$\overset{(2)}{\leq} \Upsilon_{i^{(1)}}^{(1)} + d_1 c_0^{-1} \left(\Omega^{(0)} + c_1 \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} + c_2 K \right) + d_2 K \tag{98}$$

$$\overset{(3)}{\leq} \Upsilon_{i^{(1)}}^{(1)} + d_1 c_0^{-1} \Omega^{(0)} + d_1 c_0^{-1} c_1 \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} + d_1 c_0^{-1} c_2 K + d_2 K, \tag{99}$$

where (1) holds from plugging (93) into (96), (2) is true because $\Upsilon, \Omega \geq 0$, and (3) follows from the distributive property. We can rewrite this equation as

$$(d_0 - d_1 c_0^{-1} c_1) \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} \leq \Upsilon_{i^{(1)}}^{(1)} + d_1 c_0^{-1} \Omega^{(0)} + d_1 c_0^{-1} c_2 K + d_2 K \tag{100}$$

$$\Rightarrow \frac{1}{K} \sum_{k=1}^{K} \Upsilon_{i^{(k)}}^{(k)} \leq (d_0 - d_1 c_0^{-1} c_1)^{-1} \left(\frac{\Upsilon_{i^{(1)}}^{(1)} + d_1 c_0^{-1} \Omega^{(0)}}{K} + d_1 c_0^{-1} c_2 + d_2 \right). \tag{101}$$

By plugging in $\Upsilon_{i^{(1)}}^{(1)} = \Upsilon_{i^{(0)}}^{(0)} + d_1 \Theta^{(0)} + d_2$ into (101), we get the statement of the lemma. □

Plugging the following values into Lemma 15 and utilizing Lemmas 13 and 14 and the learning rates α, β, γ from Theorem 2, we get the result in Lemma 9 in precise terms:

$$\begin{aligned}
&\Omega^{(k)} = \mathcal{L}^{(k)}, && \Theta^{(k)} = \mathbb{E}[\|x^{(k)} - x^{(k-1)}\|^2], && \Upsilon_i^{(k)} = \Delta_{y_i}^{(k)}, \\
&c_0 = \frac{1}{4\alpha} - \frac{L_f}{2}, && c_1 = 4\alpha L^2, && c_2 = \gamma n B_\ell^2 + 4\alpha b_0^2 + 2\alpha\sigma_f^2, \\
&d_0 = \mu_g \beta / 2, && d_1 = \left(\frac{2}{\mu_g \beta} - 1 \right) G_y^2, && d_2 = \beta^2 \sigma_g^2.
\end{aligned} \tag{102}$$

Next, recall that our step sizes were

$$\alpha = \min\left(\frac{\mu_g}{16G_yL}\nu, \frac{K^{-3/5}}{4G_yL}\right), \beta = \min\left(\nu, \frac{4K^{-2/5}}{\mu_g}\right), \gamma = \frac{2K^{-3/5}}{B_\ell n^{1/2}}, \tag{103}$$

where $\nu = \min(\frac{\mu_g}{L_g^2(1+\sigma_g^2)}, \frac{1}{\mu_g})$. Note that the choice of ν was motivated by the conditions of Lemma 5. First, observe that $1 - d_0 > 0$ is true because we chose $\beta < 2/\mu_g$. Now, observe that $\frac{\alpha}{\beta} \leq \frac{\mu_g}{16G_yL}$. Finally, will next show that $d_0 - d_1c_1(c_0)^{-1} > 0$ and $c_0 - c_1d_1(d_0)^{-1} > 0$, completing the set of conditions in Lemma 5. By direct algebraic manipulation, we have

$$\begin{aligned}
d_0 - d_1c_1(c_0)^{-1} &= \frac{\mu_g\beta}{2} - \frac{\left(\frac{2}{\mu_g\beta} - 1\right)G_y^2 \cdot 4L^2\alpha}{\left(\frac{1}{4\alpha} - \frac{L_f}{2}\right)} \geq \frac{\mu_g\beta}{2} - \frac{\frac{2}{\mu_g\beta} \cdot G_y^2 \cdot 4L^2\alpha}{\left(\frac{1}{4\alpha} - \frac{L_f}{2}\right)} \\
&\geq \frac{\mu_g\beta}{2} - \frac{8L^2G_y^2\alpha}{\beta\mu_g\left(\frac{1}{4\alpha} - \frac{L_f}{2}\right)} \geq \frac{\mu_g\beta}{2} - \frac{64L^2G_y^2}{\mu_g^2} \cdot \frac{\alpha^2}{\beta^2} \cdot \mu_g\beta \\
&\geq \frac{\mu_g\beta}{2} - \frac{64L^2G_y^2}{\mu_g^2} \cdot \frac{\mu_g^2}{256G_y^2L^2} \cdot \mu_g\beta = \frac{\mu_g\beta}{2} - \frac{\mu_g\beta}{4} = \frac{\mu_g\beta}{4}.
\end{aligned} \tag{104}$$

Similarly, we also have

$$\begin{aligned}
&c_0 - c_1d_1(d_0)^{-1} \\
&= \left(\frac{1}{4\alpha} - \frac{L_f}{2}\right) - \frac{4L^2\alpha \cdot \left(\frac{2}{\mu_g\beta} - 1\right)G_y^2}{\frac{\mu_g\beta}{2}} = \left(\frac{1}{4\alpha} - \frac{L_f}{2}\right) - \frac{8L^2\alpha\left(\frac{2}{\mu_g\beta} - 1\right)G_y^2}{\mu_g\beta} \\
&\geq \left(\frac{1}{4\alpha} - \frac{L_f}{2}\right) - \frac{8L^2\alpha\left(\frac{2}{\mu_g\beta}\right)G_y^2}{\mu_g\beta} \geq \left(\frac{1}{4\alpha} - \frac{L_f}{2}\right) - \frac{16L^2\alpha G_y^2}{\mu_g^2} \cdot \frac{\alpha^2}{\beta^2} \cdot \frac{1}{\alpha} \\
&\geq \frac{1}{8\alpha} - \frac{16L^2\alpha G_y^2}{\mu_g^2} \cdot \frac{\mu_g^2}{256G_y^2L^2} \cdot \frac{1}{\alpha} = \frac{1}{8\alpha} - \frac{1}{16\alpha} = \frac{1}{16\alpha}.
\end{aligned} \tag{105}$$

Now, we can bound the optimality of y by bounding the maximum difference $\Delta_{y_i}^{(k)}$:

$$\frac{1}{K}\sum_{k=1}^{K}\max_{i\in[n]}\Delta_{y_i}^{(k)}$$

$$\overset{(1)}{\leq}(d_0-d_1c_0^{-1}c_1)^{-1}\left(\frac{\Upsilon_{i^{(0)}}^{(0)}+d_1\Theta^{(0)}+d_2+d_1c_0^{-1}\Omega^{(0)}}{K}+d_1c_0^{-1}c_2+d_2\right) \tag{106}$$

$$\overset{(2)}{=}\frac{4}{\mu_g\beta}\left(\frac{\Upsilon_{i^{(0)}}^{(0)}+d_1\Theta^{(0)}+d_2+d_1c_0^{-1}\Omega^{(0)}}{K}+d_1c_0^{-1}c_2+d_2\right) \tag{107}$$

$$\overset{(3)}{\leq}\frac{4}{\mu_g\beta}\left(\frac{\Delta_{y_{i^{(0)}}}^{(0)}+\beta^2\sigma_g^2+\frac{2G_y^2}{\mu_g\beta}(8\alpha)\mathcal{L}^{(0)}}{K}\right.$$
$$\left.+\frac{2G_y^2}{\mu_g\beta}(8\alpha)(n\gamma B_\ell^2+4\alpha b_0^2+2\alpha\sigma_f^2)+\beta^2\sigma_g^2\right) \tag{108}$$

$$\overset{(4)}{=}\frac{\frac{4\Delta_{y_{i^{(0)}}}^{(0)}}{\mu_g\beta}+4\beta\frac{\sigma_g^2}{\mu_g}+\frac{64G_y^2\alpha}{\mu_g^2\beta^2}}{K}+\frac{64G_y^2\alpha}{\mu_g^2\beta^2}\left(n\gamma B_\ell+4\alpha b_0^2+2\alpha\sigma_f^2\right)+\frac{4\beta\sigma_g^2}{\mu_g} \tag{109}$$

$$\overset{(5)}{=}\frac{4\Delta_{y_{i^{(0)}}}^{(0)}}{\mu_g}\frac{1}{\beta K}+\frac{4\sigma_g^2}{\mu_g}\frac{\beta}{K}+\frac{64G_y^2}{\mu_g^2}\frac{\alpha}{\beta^2K}+\frac{64G_y^2}{\mu_g^2}\gamma B_\ell^2\frac{n\alpha}{\beta^2}$$
$$+\frac{64G_y^2(4b_0^2+2\sigma_f^2)}{\mu_g^2}\frac{\alpha^2}{\beta^2}+\frac{4\sigma_g^2}{\mu_g}\beta \tag{110}$$

$$\overset{(6)}{\leq}\frac{\Delta_{y_{i^{(0)}}}^{(0)}/\mu_g}{K^{3/5}}+\frac{16\sigma_g^2/\mu_g^2}{K^{7/5}}+\frac{G_y/L}{K^{4/5}}+\frac{2\sqrt{n}B_\ell G_y/L}{K^{2/5}}+\frac{(b_0^2+\frac{1}{2}\sigma_f^2)/(L^2)}{K^{2/5}}+\frac{16\sigma_g^2/\mu_g^2}{K^{2/5}}. \tag{111}$$

Here, (1) follows directly from plugging $\Upsilon_i^{(k)}$ from (102) into Lemma 4, (2) follows from (104), (3) comes from plugging in the rest of (102), (4) is direct algebraic manipulation, (5) separates the step sizes and n, K factors from the rest of the constants, and (6) applies the definition of the step sizes. This gives us the $\mathcal{O}(\sqrt{n}K^{-2/5})$ bound in Lemma 9.

A.8 Proof of Lemma 10

We present a precise form of Lemma 10 here:

Lemma 16. *For any $\lambda\in\Delta_n$, under Assumptions 1, 2, and 3, assume that the iterates $\{x^{(k)}, y_i^{(k)}, i\in[n]\lambda^{(k)}, \forall k\}$ generated by MORBiT, then we have*

$$\frac{1}{K}\mathbb{E}\left[\sum_{k=1}^{K}F(x^{(k)},\lambda)-F(x^{(k)},\lambda^{(k)})\right]$$
$$\leq\frac{1}{\sqrt{2}}\left(B_\ell\sqrt{n}K^{-3/5}+B_\ell\sqrt{n}K^{-2/5}\right)+\frac{G_f}{K}\sum_{k=1}^{K}\max_{i\in[n]}\Delta_{y_i}^{(k)}. \tag{112}$$

Proof. Recall that we defined

$$F(x,\lambda):=\sum_{i=1}^{n}\lambda_i f_i(x,y_i^\star(x))=\sum_{i=1}^{n}\lambda_i\ell_i(x). \tag{113}$$

For a fixed realization of $x^{(1)}, \cdots, x^{(k)}, y_i^{(1)}, \cdots, y_i^{(k)}, i \in [n]$, we have

$$
\begin{aligned}
&F(x^{(k)}, \lambda) - F(x^{(k)}, \lambda^{(k)}) \\
&\overset{(1)}{=} \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)}) f_i(x^{(k)}, y_i^\star(x^{(k)})) && (114)\\
&\overset{(2)}{=} \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)})(f_i(x^{(k)}, y_i^\star(x^{(k)})) - f_i(x^{(k)}, y_i^{(k+1)}) + f_i(x^{(k)}, y_i^{(k+1)}))) && (115)\\
&\overset{(3)}{=} \left\langle \left(\lambda - \lambda^{(k)}\right), \left[f_1(x^{(k)}, y_1^{(k+1)}), \cdots, f_n(x^{(k)}, y_n^{(k+1)})\right]^\top \right\rangle \\
&\qquad + \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)})(f_i(x^{(k)}, y_i^\star(x^{(k)})) - f_i(x^{(k)}, y_i^{(k+1)})) && (116)\\
&\overset{(4)}{=} \left\langle \left(\lambda - \lambda^{(k)}\right), h_\lambda^{(k)} \right\rangle + \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)})(f_i(x^{(k)}, y_i^\star(x^{(k)})) - f_i(x^{(k)}, y_i^{(k+1)})) && (117)\\
&\overset{(5)}{=} \frac{\|\lambda - \lambda^{(k)}\|^2 + \gamma^2 \|h_\lambda^{(k)}\|^2 - \|\lambda - \lambda^{(k)} - \gamma h_\lambda^{(k)}\|^2}{2\gamma} \\
&\qquad + \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)})(f_i(x^{(k)}, y_i^\star(x^{(k)})) - f_i(x^{(k)}, y_i^{(k+1)})) && (118)\\
&\overset{(6)}{\leq} \frac{\|\lambda - \lambda^{(k)}\|^2 + \gamma^2 \|h_\lambda^{(k)}\|^2 - \|\lambda - \lambda^{(k+1)}\|^2}{2\gamma} \\
&\qquad + \sum_{i=1}^{n} (\lambda - \lambda^{(k)})_i (f_i(x^{(k)}, y_i^\star(x^{(k)})) - f_i(x^{(k)}, y_i^{(k+1)})) && (119)\\
&\overset{(7)}{\leq} \frac{\|\lambda - \lambda^{(k)}\|^2 + \gamma^2 \|h_\lambda^{(k)}\|^2 - \|\lambda - \lambda^{(k+1)}\|^2}{2\gamma} + G_f \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)})(y_i^\star(x^{(k)}) - y_i^{(k+1)}), && (120)
\end{aligned}
$$

where (1) comes from the definition of F, (2) follows from adding and subtracting $(\lambda_i - \lambda_i^{(k)}) f_i(x^{(k)}, y_i^{(k+1)})$ terms, (3) follows from splitting the preceding sum and writing the first term as a dot product, (4) follows from definition of $h_\lambda^{(k)}$, (5) uses $\langle a, b\rangle = \frac{\|a\|^2 + \|b\|^2 - \|a-b\|^2}{2}$, (6) follows from the update $\lambda^{(k+1)} = \mathrm{proj}_{\Delta_n}(\lambda^{(k)} - \gamma h_\lambda^{(k)})$ and the projection property, and (7) follows from Lipschitzness of f.

Therefore, applying the telescoping sum by adding the preceding inequality over $k = 1, 2, \ldots, K$, and taking expectation, we get:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{K} F(x^{(k)}, \lambda) - F(x^{(k)}, \lambda^{(k)})\right] \\
&\overset{(1)}{\leq} \frac{\gamma}{2} \sum_{k=1}^{K} \mathbb{E}\|h_\lambda^{(k)}\|^2 + \frac{\mathbb{E}[\|\lambda - \lambda^{(1)}\|^2]}{2\gamma} + G_f \sum_{k=1}^{K} \sum_{i=1}^{n} (\lambda_i - \lambda_i^{(k)}) \Delta_{y_i}^{(k)} && (121)\\
&\overset{(2)}{\leq} \frac{\gamma}{2} \sum_{k=1}^{K} \mathbb{E}\|h_\lambda^{(k)}\|^2 + \frac{1}{\gamma} + G_f \sum_{k=1}^{K} \max_{i \in [n]} \Delta_{y_i}^{(k)} && (122)\\
&\overset{(3)}{\leq} \frac{nKB^2\gamma}{2} + \frac{1}{\gamma} + G_f \sum_{k=1}^{K} \max_{i \in [n]} \Delta_{y_i}^{(k)} && (123)\\
&\overset{(4)}{\leq} \frac{\sqrt{2}}{2} \left(B_\ell \sqrt{n} K^{2/5} + B_\ell \sqrt{n} K^{3/5}\right) + G_f \sum_{k=1}^{K} \max_{i \in [n]} \Delta_{y_i}^{(k)} && (124)
\end{aligned}
$$

where (1) follows directly from (120) and the telescoping sum, (2) follows from $\lambda \in \Delta_n$ and $\|\lambda - \lambda^{(1)}\|^2 \leq 2$, (3) follows from $\|h_\lambda^{(k)}\|^2 \leq nB^2$, and (4) follows from selecting $\gamma = \frac{\sqrt{2}}{B_\ell \sqrt{n} K^{3/5}}$. □

Therefore, we obtain

$$\begin{aligned} &\frac{1}{K}\mathbb{E}\left[\sum_{k=1}^{K} F(x^{(k)}, \lambda) - F(x^{(k)}, \lambda^{(k)})\right] \\ &\leq \frac{1}{\sqrt{2}}\left(B_\ell \sqrt{n} K^{-3/5} + B_\ell \sqrt{n} K^{-2/5}\right) + G_f \mathcal{O}(\sqrt{n} K^{-2/5}). \end{aligned} \tag{125}$$

A.9 Proof of Lemma 11

We state Lemma 11 here in precise terms:

Lemma 17. *Under Assumptions 1, 2, and 3 with the iterates* $\{x^{(k)}, y_i^{(k)}, i \in [n], \lambda^{(k)}, \forall k\}$ *generated by* MORBiT*, then we have*

$$\begin{aligned} &\frac{1}{K}\sum_{k=1}^{K} \mathbb{E}[\|\hat{x}(x^{(k)}) - x^{(k)}\|^2] \\ &\leq \frac{4}{-\mu_\ell + \rho}\left(\frac{\Phi_{1/\rho}(x_0)}{\rho \alpha K} + \frac{2}{-\mu_\ell + \rho} b_0^2 + \right. \\ &\qquad \left. \left(\frac{2L^2}{-\mu_\ell + \rho} + \frac{3L^2 \alpha}{2}\right)\left(\frac{1}{K}\sum_{k=1}^{K} \max_{i \in [n]} \Delta_{y_i}^{(k)}\right) + \frac{\alpha}{2}\left(\tilde{\sigma}_f^2 + 3b_0^2\right)\right). \end{aligned} \tag{126}$$

Proof. Recall that we defined the Moreau envelope and proximal map as follows:

$$\Phi_{1/\rho}(z) \triangleq \min_x \frac{\rho}{2}\|x - z\|^2 + \sum_{i=1}^{n} \lambda_i \ell_i(x), \qquad \hat{x}(z) \triangleq \arg\min_x \frac{\rho}{2}\|x - z\|^2 + \sum_{i=1}^{n} \lambda_i \ell_i(x). \tag{127}$$

Therefore, we have

$$\Phi_{1/\rho}(x^{(k+1)}) \overset{(1)}{=} \sum_i \lambda_i \ell_i(\hat{x}(x^{(k+1)})) + \frac{\rho}{2}\|x^{(k+1)} - \hat{x}(x^{(k+1)})\|^2 \tag{128}$$

$$\overset{(2)}{\leq} \sum_i \lambda_i \ell_i(\hat{x}(x^{(k)})) + \frac{\rho}{2}\|x^{(k+1)} - \hat{x}(x^{(k)})\|^2 \tag{129}$$

$$\overset{(3)}{=} \sum_i \lambda_i \ell_i(\hat{x}(x^{(k)})) + \frac{\rho}{2}\|x^{(k+1)} - x^{(k)} + x^{(k)} - \hat{x}(x^{(k)})\|^2 \tag{130}$$

$$\begin{aligned} &\overset{(4)}{=} \sum_i \lambda_i \ell_i(\hat{x}(x^{(k)})) + \frac{\rho}{2}\|x^{(k+1)} - x^{(k)}\|^2 + \frac{\rho}{2}\|x^{(k)} - \hat{x}(x^{(k)})\|^2 \\ &\qquad + \rho\left\langle \left(x^{(k+1)} - x^{(k)}\right), \left(x^{(k)} - \hat{x}(x^{(k)})\right)\right\rangle \end{aligned} \tag{131}$$

where (1) is by definition of the proximal map, (2) comes from the optimality of the Moreau envelope, (3) is by adding and subtracting $x^{(k)}$, and (4) is from expanding out $\|a+b\|^2$ into $\|a\|^2 + \|b\|^2 + 2\langle a, b\rangle$.

Next, from the optimality condition of the update $x^{(k+1)} = \text{proj}_{\Delta_n}(x^{(k)} - \alpha h_x^{(k)})$, we have

$$\langle x^{(k+1)} - \hat{x}(x^{(k)}), x^{(k+1)} - x^{(k)} + \alpha h_x^{(k)} \rangle \leq 0 \tag{132}$$

$$\overset{(1)}{\Rightarrow} \langle x^{(k+1)} - x^{(k)} + x^{(k)} - \hat{x}(x^{(k)}) x^{(k+1)} - x^{(k)} + \alpha h_x^{(k)} \rangle \leq 0 \tag{133}$$

$$\begin{aligned} &\overset{(2)}{\Rightarrow} \langle x^{(k+1)} - x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \\ &\quad \leq -\|x^{(k+1)} - x^{(k)}\|^2 - \alpha \langle h_x^{(k)}, x^{(k+1)} - x^{(k)} \rangle - \alpha \langle h_x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \end{aligned} \tag{134}$$

$$\begin{aligned} &\overset{(3)}{\Rightarrow} \rho \langle x^{(k+1)} - x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \\ &\quad \leq -\rho \|x^{(k+1)} - x^{(k)}\|^2 - \rho\alpha \langle h_x^{(k)}, x^{(k+1)} - x^{(k)} \rangle - \rho\alpha \langle h_x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \end{aligned} \tag{135}$$

$$\begin{aligned} &\overset{(4)}{\Rightarrow} \rho \langle x^{(k+1)} - x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \\ &\quad \leq -\rho \|x^{(k+1)} - x^{(k)}\|^2 + \rho \langle \alpha h_x^{(k)}, x^{(k)} - x^{(k+1)} \rangle + \rho\alpha \langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \end{aligned} \tag{136}$$

$$\begin{aligned} &\overset{(5)}{\Rightarrow} \rho \langle x^{(k+1)} - x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \\ &\quad \leq -\rho \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\rho}{2} \left(\|\alpha h_x^{(k)}\|^2 + \|x^{(k)} - x^{(k+1)}\|^2 \right) \\ &\qquad + \rho\alpha \langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \end{aligned} \tag{137}$$

$$\begin{aligned} &\overset{(6)}{\Rightarrow} \rho \langle x^{(k+1)} - x^{(k)}, x^{(k)} - \hat{x}(x^{(k)}) \rangle \\ &\quad \leq -\frac{\rho}{2} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\rho\alpha^2}{2} \|h_x^{(k)}\|^2 + \rho\alpha \langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \end{aligned} \tag{138}$$

where (1) is by adding and subtracting $x^{(k)}$, (2) is from distributing the inner product $\langle (x^{(k+1)} - x^{(k)}) + (x^{(k)} - \hat{x}(x^{(k)})), (x^{(k+1)} - x^{(k)}) + \alpha h_x^{(k)} \rangle$, (3) is from multiplying both sides by ρ, (4) is from simple algebra, (5) is from rewriting $\langle a, b \rangle \leq \frac{\|a\|^2 + \|b\|^2}{2}$, and (6) is from combining terms. Therefore, substituting (138) into (131), we get

$$\begin{aligned} &\Phi_{1/\rho}(x^{(k+1)}) \\ &\quad \leq \sum_i \lambda_i \ell_i(\hat{x}(x^{(k)})) + \frac{\rho}{2} \|x^{(k)} - \hat{x}(x^{(k)})\|^2 + \frac{\rho\alpha^2}{2} \|h_x^{(k)}\|^2 + \rho\alpha \langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \end{aligned} \tag{139}$$

$$= \Phi_{1/\rho}(x^{(k)}) + \frac{\rho\alpha^2}{2} \|h_x^{(k)}\|^2 + \rho\alpha \langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \tag{140}$$

where the second equality is by definition of the Moreau envelope. Now, we bound the last term in (140):

$$\begin{aligned} &\langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \\ &\overset{(1)}{=} \left\langle \hat{x}(x^{(k)}) - x^{(k)}, h_x^{(k)} - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) \right\rangle \\ &\qquad + \left\langle \hat{x}(x^{(k)}) - x^{(k)}, \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - \nabla_x F(x^{(k)}, \lambda^{(k)}) \right\rangle \\ &\qquad + \left\langle \hat{x}(x^{(k)}) - x^{(k)}, \nabla_x F(x^{(k)}, \lambda^{(k)}) \right\rangle \end{aligned} \tag{141}$$

$$\begin{aligned} &\overset{(2)}{=} \underbrace{\langle \hat{x}(x^{(k)}) - x^{(k)}, B_k \rangle}_{(A)} + \underbrace{\langle \hat{x}(x^{(k)}) - x^{(k)}, \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - \nabla_x F(x^{(k)}, \lambda^{(k)}) \rangle}_{(B)} \\ &\quad + \underbrace{\langle \hat{x}(x^{(k)}) - x^{(k)}, \nabla_x F(x^{(k)}, \lambda^{(k)}) \rangle}_{(C)} \end{aligned} \tag{142}$$

where (1) follows from adding and subtracting $\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}), \nabla_x F(x^{(k)}, \lambda^{(k)})$ terms and (2) is from splitting the inner product and applying $h_x^{(k)} - \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) = B_k$ from equation 20 in Assumption 3. To bound (A) and (B), we simply apply $\langle a, b \rangle \leq \frac{c}{4}\|a\|^2 + \frac{1}{c}\|b\|^2$ to

both inner products:

$$(A) = \langle \hat{x}(x^{(k)}) - x^{(k)}, B_k \rangle \leq \frac{c}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 + \frac{1}{c}b_k^2 \tag{143}$$

$$\begin{aligned}(B) &= \left\langle \left(\hat{x}(x^{(k)}) - x^{(k)}\right), \overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - \nabla_x F(x^{(k)}, \lambda^{(k)}) \right\rangle \\ &\leq \frac{1}{c}\|\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - \nabla_x F(x^{(k)}, \lambda^{(k)})\|^2 + \frac{c}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2. \end{aligned} \tag{144}$$

We proceed to bound (C). First, from weak convexity of μ_ℓ, we have that for all $i \in [n]$,

$$\ell_i(\hat{x}(x^{(k)})) \geq \ell_i(x^{(k)}) + \langle \nabla \ell(x^{(k)}), \hat{x}(x^{(k)}) - x^{(k)} \rangle - \frac{\mu_\ell}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2. \tag{145}$$

Taking λ_i times the ith of these equations, we get

$$\sum_{i=1}^n \lambda_i \ell_i(\hat{x}(x^{(k)})) \geq \sum_{i=1}^n \lambda_i \ell_i(x^{(k)}) + \langle \sum_{i=1}^n \nabla \ell_i(x^{(k)}), \hat{x}(x^{(k)}) - x^{(k)} \rangle - \frac{\mu_\ell}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2. \tag{146}$$

By definition of the Moreau envelope, we also have

$$\sum_{i=1}^n \lambda_i \ell_i(x^{(k)}) \geq \sum_{i=1}^n \lambda_i \ell_i(\hat{x}(x^{(k)})) + \frac{\rho}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2. \tag{147}$$

Adding (146) and (147), we have

$$\frac{\mu_\ell - \rho}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \geq \langle \nabla_x F(x^{(k)}, \lambda^{(k)}), \hat{x}(x^{(k)}) - x^{(k)} \rangle. \tag{148}$$

If we let $c = \frac{-\mu_\ell + \rho}{2}$ in (143) and (144), we can rewrite (142) as

$$\begin{aligned}&\langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle \\ &\leq \frac{c}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 + \frac{1}{c}b_k^2 + \frac{1}{c}\|\overline{\nabla}_x F(x^{(k)}, \boldsymbol{y}^{(k+1)}, \lambda^{(k)}) - \nabla_x F(x^{(k)}, \lambda^{(k)})\|^2 \\ &\quad - \frac{-\mu_\ell + \rho}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \end{aligned} \tag{149}$$

$$\begin{aligned}&\leq \frac{c}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 + \frac{1}{c}b_k^2 + \frac{L^2}{c}\left(\sum_{i=1}^n \lambda_i \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2\right) \\ &\quad - \frac{-\mu_\ell + \rho}{2}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \end{aligned} \tag{150}$$

$$\leq \frac{2}{-\mu_\ell + \rho}b_k^2 + \frac{2L^2}{-\mu_\ell + \rho}\left(\sum_{i=1}^n \lambda_i \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2\right) - \frac{-\mu_\ell + \rho}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2. \tag{151}$$

Taking the full expectation $\mathcal{F}_i \triangleq \{y_i^{(0)}, x^{(0)}, \cdots, y_i^{(k)}, x^{(k)}\}$, we have

$$\begin{aligned}&\mathbb{E}[\langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)} \rangle] \\ &\leq \frac{2}{-\mu_\ell + \rho}b_k^2 + \frac{2L^2}{-\mu_\ell + \rho}\mathbb{E}\left[\sum_{i=1}^n \lambda_i \|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2\right] - \frac{-\mu_\ell + \rho}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \end{aligned} \tag{152}$$

$$= \frac{2}{-\mu_\ell + \rho}b_k^2 + \frac{2L^2}{-\mu_\ell + \rho}\left[\sum_{i=1}^n \lambda_i \mathbb{E}[\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2]\right] - \frac{-\mu_\ell + \rho}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \tag{153}$$

$$\leq \frac{2}{-\mu_\ell + \rho}b_k^2 + \frac{2L^2}{-\mu_\ell + \rho}\left(\max_{i \in [n]} \mathbb{E}[\|y_i^\star(x^{(k)}) - y_i^{(k+1)}\|^2]\right) - \frac{-\mu_\ell + \rho}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \tag{154}$$

$$= \frac{2}{-\mu_\ell + \rho}b_k^2 + \frac{2L^2}{-\mu_\ell + \rho}\left(\max_{i \in [n]} \Delta_{y_i}^{(k)}\right) - \frac{-\mu_\ell + \rho}{4}\|\hat{x}(x^{(k)}) - x^{(k)}\|^2 \tag{155}$$

Therefore, rewriting everything into (146) and taking the full expectation over $\mathcal{F}_i$, we have (recall that the definition of $\Delta_{y_i}^{(k)}$ includes an expectation):

$$
\begin{aligned}
&\mathbb{E}[\Phi_{1/\rho}(x^{(k+1)})] \\
&\overset{(1)}{\leq} \mathbb{E}[\Phi_{1/\rho}(x^{(k)})] + \frac{\rho\alpha^2}{2}\mathbb{E}[\|h_x^{(k)}\|^2] + \rho\alpha\mathbb{E}[\langle h_x^{(k)}, \hat{x}(x^{(k)}) - x^{(k)}\rangle] \qquad (156)\\
&\overset{(2)}{=} \mathbb{E}[\Phi_{1/\rho}(x^{(k)})] + \frac{2\rho\alpha}{-\mu_\ell+\rho}b_k^2 + \frac{2L^2\rho\alpha}{-\mu_\ell+\rho}\left(\max_{i\in[n]}\Delta_{y_i}^{(k)}\right) - \frac{\rho\alpha(-\mu_\ell+\rho)}{4}\mathbb{E}[\|\hat{x}(x^{(k)}) - x^{(k)}\|^2] \\
&\quad + \frac{\rho\alpha^2}{2}\left(\tilde{\sigma}_f^2 + 3b_k^2 + 3L^2\max_{i\in[n]}\Delta_{y_i}^{(k)}\right) \qquad (157)\\
&\overset{(3)}{=} \mathbb{E}[\Phi_{1/\rho}(x^{(k)})] + \frac{2\rho\alpha}{-\mu_\ell+\rho}b_k^2 + \left(\frac{2L^2\rho\alpha}{-\mu_\ell+\rho} + \frac{3L^2\rho\alpha^2}{2}\right)\left(\max_{i\in[n]}\Delta_{y_i}^{(k)}\right) \\
&\quad + \frac{\rho\alpha(\mu_\ell-\rho)}{4}\mathbb{E}[\|\hat{x}(x^{(k)}) - x^{(k)}\|^2] + \frac{\rho\alpha^2}{2}(\tilde{\sigma}_f^2 + 3b_k^2), \qquad (158)
\end{aligned}
$$

where (1) is a copy of (140) and (2) is from (158), plugging in $\|h_x^{(k)}\|^2$ from Lemma 1, and doing the same expectation calculation from (152) to (155). Finally, (3) is combining terms via algebra. Summing up from $k = 0, 1, \cdots, K-1$, we get the following bound:

$$
\begin{aligned}
&\frac{1}{K}\sum_{k=1}^{K}\mathbb{E}[\|\hat{x}(x^{(k)}) - x^{(k)}\|^2] \\
&\leq \frac{4}{-\mu_\ell+\rho}\left(\frac{\Phi_{1/\rho}(x_0)}{\rho\alpha K} + \frac{2}{-\mu_\ell+\rho}b_0^2\right. \\
&\qquad\qquad \left. + \left(\frac{2L^2}{-\mu_\ell+\rho} + \frac{3L^2\alpha}{2}\right)\left(\frac{1}{K}\sum_{k=1}^{K}\max_{i\in[n]}\Delta_{y_i}^{(k)}\right) + \frac{\alpha}{2}\left(\tilde{\sigma}_f^2 + 3b_0^2\right)\right) \qquad (159)\\
&= \frac{4}{-\mu_\ell+\rho}\left(\underbrace{\frac{\Phi_{1/\rho}(x_0)}{\rho\alpha K}}_{K^{-2/5}} + \frac{2}{-\mu_\ell+\rho}b_0^2\right. \\
&\qquad\qquad + \underbrace{\frac{2L^2}{-\mu_\ell+\rho}\left(\frac{1}{K}\sum_{k=1}^{K}\max_{i\in[n]}\Delta_{y_i}^{(k)}\right)}_{\sqrt{n}K^{-2/5}} \\
&\qquad\qquad \left. + \underbrace{\frac{\alpha}{2}\left(\tilde{\sigma}_f^2 + 3b_0^2 + \frac{3L^2}{K}\sum_{k=1}^{K}\max_{i\in[n]}\Delta_{y_i}^{(k)}\right)}_{K^{-3/5}+\sqrt{n}K^{-1}}\right). \qquad (160)
\end{aligned}
$$

□

B Generalization Bounds

In addition to convergence, we also show the generalization abilities of the bilevel optimizer. The theorem in this section is inspired by Collins et al. (2020, Theorem 4), but our results hold for the fully general min-max multi-objective BLO setup while Collins et al. (2020) study a min-max multi-objective single-level problem. Assume that for a learning task i, we observe m_i batches of train/test data, $D_{i,j}^{\text{t}}$ and $D_{i,j}^{\text{v}}, j \in [m_i]$. Assume that each train and test batch has K and J input-output pairs, respectively, so that sets $D_{i,j}^{\text{t}}, D_{i,j}^{\text{v}}$ are drawn from a common distribution $\mathcal{D}_i$. Also, let $y_i^\star(x; D_i^{\text{t}}) = \arg\min_{y_i} \frac{1}{m_i}\sum_{j'=1}^{m_i} g_i(x, y_i, D_{i,j'}^{\text{t}})$ be the value of y_i that minimizes the empirical inner loss on some dataset D_i^{t}. For outer and inner objectives f_i, g_i, we consider the following

function class $\mathcal{F}_i$, where $x \in \mathcal{X}$ is the set of optimization parameters introduced in equation 2 and $(D_i^{\text{t}}, D_i^{\text{v}})$ are any train and test datasets sampled from $\mathcal{D}_i$:

$$\mathcal{F}_i = \left\{ f_i\left(x, \hat{y}_i^\star(x; D_i^{\text{t}}), D_i^{\text{v}}\right), x \in \mathcal{X} \right\}.$$

We use f and g to denote the empirical function values evaluated at the points in D_i^{t} and D_i^{v}, and so the empirical Rademacher complexity of $\mathcal{F}_i$ on m_i samples $\mathbf{D}_i \triangleq \{(D_{i,j}^{\text{t}}, D_{i,j}^{\text{v}}\}_{j=1}^{m_i} \sim (\mathcal{D}_i)^{m_i}$ is

$$\mathcal{R}_{m_i}^i(\mathcal{F}_i) = E_{\mathbf{D}_i}\mathbb{E}_{\epsilon_j}\left[\sup_{x\in\mathcal{X}} \frac{1}{m_i}\sum_{j=1}^{m_i} \epsilon_j f_i\left(x, y_i^\star(x; D_{i,j}^{\text{t}}); D_{i,j}^{\text{v}}\right)\right],$$

where ϵ_js are Rademacher random variables ($\pm 1/2$ with equal probability). The empirical loss for fixed samples $D_{i,j}^{\text{t}}$ and $D_{i,j}^{\text{v}}$,

$$\hat{F}_i(x) \triangleq \frac{1}{m_i}\sum_{j=1}^{m_i} f_i\left(x, y_i^\star(x; D_{i,j}^{\text{t}}); D_{i,j}^{\text{v}}\right).$$

Similarly, define $F_i(x) \triangleq \mathbb{E}_{\mathbf{D}_i}[\hat{F}_i(x)]$. First, from classical generalization results such as Shalev-Shwartz & Ben-David (2014, Theorem 26.5) or Mohri et al. (2018, Theorem 3.3), we directly conclude the following proposition, which bounds the true loss of the classifier as a function of the empirical loss.

Proposition 1. *Assume the regularity assumptions considered in Appendix 3, specifically that the function ℓ is B_ℓ-bounded. Then, with probability at least $1-\delta$, we have*

$$F_i(x) \leq \hat{F}_i(x) + 2\mathcal{R}_{m_i}^i(\mathcal{F}_i) + B_\ell\sqrt{\frac{\log 1/\delta}{2m_i}}. \tag{161}$$

This extends to the following in a straightforward manner, providing a guarantee for the worst-case generalization for any learning task i:

Proposition 2. *Assume the regularity assumptions given in Section 5, specifically that the function ℓ is B_ℓ-bounded. Then, with probability at least $1-\delta$, we have*

$$\max_{i\in[n]} F_i(x) \leq \max_{i\in[n]} \hat{F}_i(x) + 2\mathcal{R}_m(\mathcal{F}) + B_\ell\sqrt{\frac{\log n/\delta}{2m}}. \tag{162}$$

Here we assume that $\mathcal{F}_i = \mathcal{F}$ and $m_i = m$ for all $i \in [n]$ and hence $\mathcal{R}_{m_i}^i(\mathcal{F}_i) = \mathcal{R}_m(\mathcal{F})$ for all $i \in [n]$.

Next, we proceed to bound the generalization on unseen tasks. Consider a new task with distribution $\mathcal{D}_{n+1} = \sum_{i=1}^n a_i\mathcal{D}_i$, for some $a \in \Delta_n$, meaning that the distribution of the new task is anywhere in the convex hull of the distribution of the old tasks. We make this assumption because if the new task is very dissimilar to the existing tasks, there is no reason to expect good generalization in the first place. We then show the following proposition:

Proposition 3. *For all $x \in \mathcal{X}$, with probability at least $1-\delta$, we have*

$$F_{n+1}(x) \leq \max_{p\in\Delta_n} p_i\hat{F}_i(x) + 2\sum_{i=1}^n a_i R_{m_i}^i(\mathcal{F}_i) + \sum_{i=1}^n a_i B_\ell\sqrt{\frac{\log(n/\delta)}{2m_i}}. \tag{163}$$

Notice that while Proposition 3 holds true for all x, the tighest upper bound is found when x minimizes $\max_{p\in\Delta_n} p_i\hat{F}_i(x)$, which is precisely when $x = x^\star$, the optimal solution to problem we study in equation 2. This highlights another advantage of our formulation over TTSA: when we use the solution obtained by the single averaged objective $\sum_i f_i$ in equation 1, $x^\star_{\text{min-avg}}$, we will have a looser upper bound for $F_{n+1}(x^\star_{\text{min-avg}})$ compared to $F_{n+1}(x^\star)$, showing that our formulation gives tighter robust (or worst case) generalization guarantees and this behaviour has been demonstrated empirically in section 4.

Proof. (of Proposition 3) First, by definition, for all x, we have that

$$F_{n+1}(x) = \mathbb{E}_{(\mathcal{D}^{\text{train}}_{n+1,j}, \mathcal{D}^{\text{test}}_{n+1,j}) \sim \mathcal{D}_{n+1}} \left[\hat{f}_i \left(x, \arg\min_{y_i} \sum_{j'=1}^{m_i} \hat{g}_i(x, y_i, D^{\text{train}}_{i,j'}), D^{\text{test}}_{i,j} \right) \right] \tag{164}$$

$$= \sum_{i=1}^{n} a_i \mathbb{E}_{(\mathcal{D}^{\text{train}}_{n+1,j}, \mathcal{D}^{\text{test}}_{n+1,j}) \sim \mathcal{D}_i} \left[\hat{f}_i \left(x, \arg\min_{y_i} \sum_{j'=1}^{m_i} \hat{g}_i(x, y_i, D^{\text{train}}_{i,j'}), D^{\text{test}}_{i,j} \right) \right] \tag{165}$$

$$= \sum_{i=1}^{n} a_i F_i(x). \tag{166}$$

Therefore, for all x, using a union bound over $1 \leq i \leq n$, we get that with probability at least $1 - n\delta'$, we have

$$F_{n+1}(x) = \sum_{i=1}^{n} a_i F_i(x) \leq \sum_{i=1}^{n} a_i \hat{F}_i(x) + 2 \sum_{i=1}^{n} a_i R^i_{m_i}(\mathcal{F}) + \sum_{i=1}^{n} a_i B \sqrt{\frac{\log 1/\delta'}{2m_i}}. \tag{167}$$

Letting $\delta = n\delta'$ in (167), we have

$$F_{n+1}(x) \leq \max_{p \in \Delta_n} p_i \hat{F}_i(x) + 2 \sum_{i=1}^{n} a_i R^i_{m_i}(\mathcal{F}) + \sum_{i=1}^{n} a_i B_\ell \sqrt{\frac{\log(n/\delta)}{2m_i}} \tag{168}$$

Therefore, plugging in $x^\star$ gives us

$$F_{n+1}(x^\star) \leq \min_{x \in \mathcal{X}} \max_{p \in \Delta_n} p_i \hat{F}_i(x) + 2 \sum_{i=1}^{n} a_i R^i_{m_i}(\mathcal{F}) + \sum_{i=1}^{n} a_i B_\ell \sqrt{\frac{\log(n/\delta)}{2m_i}}. \tag{169}$$

□

C Implementation details and Compute Resources

We perform our experiments in Python 3.7.10 and PyTorch 1.8.1 with Intel(R) Core(TM) i5-8265U CPU @ 1.60GHz.

C.1 Sinusoid Regression Task

We consider the sinusoid regression experiment (Finn et al., 2017), a multi-task representation learning problem where each task $\mathcal{T}_i$ is a regression problem $y = t_i(x) = a_i \sin(x - \phi_i)$. We uniformly sample the amplitude $a_i \in [0.1, 5]$, frequency and phase $\phi_i \in [0, \pi]$ for each task. We use $n = 3$ training tasks and 3 testing tasks, with 2 "easy tasks" ($a_i \in [0.1, 1.05]$) and one "hard tasks" ($a_i \in [4.95, 5]$) for each set. We use easy and hard tasks following the setup in (Collins et al., 2020). During training, for each task i, the learner is given samples (x, y), $x \in [-5, 5]$. The goal is to learn a function approximating t_i as best as possible in the mean squared error sense.

As described in Section 2, we use a neural network divided into two pieces, i.e., an embedding network and a task-specific network. The embedding network $f : \mathbb{R} \to \mathbb{R}^{10}$ consists of two hidden ReLU layers of size 80 and a final fully connected layer of size 10. Each task-specific network $g_i : \mathbb{R}^{10} \to \mathbb{R}, i \in [n]$ is a one-layer linear layer. Therefore, the loss on an input $x \in \mathbb{R}$ and $y = t_i(x)$ for task i is $(g_i(f(x)) - y)^2$, and the true loss of the network with parameters f, g_i are $\ell_i(f; g_i) = \mathbb{E}_{(x,y)}[(g_i(f(x)) - y)^2]$. The embedding network is as follows:

Training: At each iteration, we first perform the inner loop optimization step (meta-training) by sampling 10 shots from each of the tasks in order to update each of the task-specific network weights. We use just 1 inner loop step. Pseudocode for the inner loop is shown below in PyTorch-style:

```
for task_id in task_list:
```

Input ($\mathbb{R}$)
Linear FC Layer (output in $\mathbb{R}^{80}$)
ReLU
Linear FC Layer (output in $\mathbb{R}^{80}$)
ReLU
Linear FC Layer (output in $\mathbb{R}^{10}$)

```
xs, ys = sample_batch(task_id, n_shots)
embedding = embedding_network(xs)
for _ in range(n_inner):
    head_optimizers[task_id].zero_grad()
    total_loss = get_loss(task_id, xs, ys)
    total_loss.backward()
    head_optimizers[task_id].step()
```

For each outer loop optimization, we run a meta-validation batch again containing 10 shots from each of the tasks. We then take an outer-loop step, optimizing the embedding weights using the results of the meta-validation batch. The meta-validation batch is sampled in the exact same way as the meta-training batch shown above.

Regularization: First, as in (Ji et al., 2020), we add weight regualarization during inner loop training of the form $\epsilon_w \sum_{w \in \mathcal{W}} \|w\|$, where $\mathcal{W}$ denotes the set of weight parameters, where $\epsilon_w = 0.01$. In PyTorch, this is expressed as

```
l2_reg_constant = 0.01
for p in heads[task].parameters():
    l2_reg += p.norm(2)
total_loss += l2_reg * l2_reg_constant
```

Next, for the inner λ updates, we add a regularization term to the overall loss, $-\epsilon_\lambda \sum_{i=1}^{n} (\lambda_i - \frac{1}{n})^2$, where $\epsilon_\lambda = 3$, which pulls the λs closer to uniform. In PyTorch, the λ update is expressed as

```
task_gradient = task_losses[i]
reg_gradient = -mu_lambda * (lambdas[task] - 1/n)
lambdas[task] += (task_update + reg_update) * gamma
```

Parameters: For the Task-Robust version of the algorithm, we use $\alpha = 0.007, \beta = 0.005, \gamma = 0.003$. For the standard version of the algorithm, we use $\alpha = 0.007, \beta = 0.011, \gamma = 0.003$.

Loss curves: To approximate the true loss for measurement purposes, we use 100 equally-spaced samples from $[-5, 5]$. After each iteration, we calculated the maximum loss among all the tasks. In 3, we show the minimum of these maximum losses up until each epoch.

Results with more tasks: Finally, we show another figure similar to Figure 1, but with 20 training tasks and 20 test tasks. It can be observed that both the task-robust training loss and the task-robust testing loss greatly outperform their respective standard losses.

C.2 Nonlinear Representation Learning

We consider binary classification tasks generated from the FashionMNIST data set where we select 8 "easy" tasks (lowest log loss ~ 0.3 from independent training) and 2 "hard" tasks (lowest loss ~ 0.45 from independent training). We learn a shared representation network that maps the 784 dimensional (vectorized 28×28 images) to a 100 dimensional space. Each tasks then learns a binary classifier on top of this representation. The task specific objective g_i for task i corresponds to the cross-entropy loss on the training set, while the upper level objective f_i corresponds to the loss of the $y_i^\star(x)$ with the learned representation x on a validation set. We also maintain a heldout test set which we use to evaluate the generalization of the learned representation and per-task models.

For our data, we had $x \in \mathbb{R}^{784 \times 100}$ and $y \in \mathbb{R}^{100 \times 2}$. We used step sizes $\alpha = 0.01, \beta = 0.01$, and $\gamma = 0.3$. We used batch sizes of 8 and 128 to compute g_i for each inner step and f_i for each outer

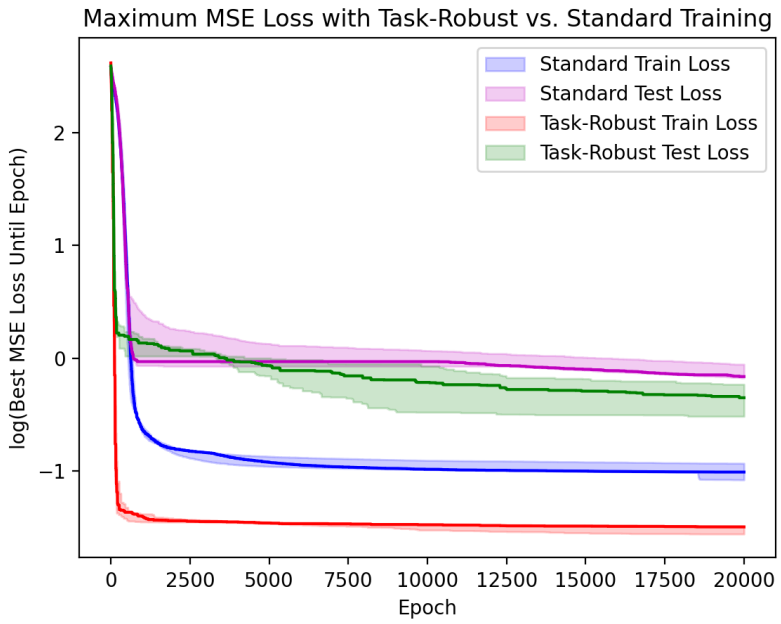

Figure 7: Comparison of standard training and robust training using 20 tasks

iteration, respectively. In addition, we included ℓ_2-regularization of y with regularization penalty 0.0005. We used vanilla SGD with a learning rate scheduler (ReduceLROnPlateau), invoked every 100 outer iterations, with patience of 10. Each optimization was executed for 10000 outer iterations. The results are generated by aggregation over runs with 10 different seeds.

C.3 Hyperparameter optimization

In this application, we use learning rates $\alpha = 0.0001$, $\beta = 0.001$, $\gamma = 0.001$ and 20000 outer iterations. We use a batch size of 8 for both the inner and outer steps for each $i \in [16]$ for the initial experiment in figure 4. The optimizer was vanilla SGD with a learning rate scheduler (ReduceLROnPlateau), invoked every 100 outer iterations, with patience of 30. The results are generated by aggregating over 10 runs with different seeds. For the other HPO experiments, the number of tasks n and the batch sizes are discussed in the main text.