DGD^2: A Linearly Convergent Distributed Algorithm For High-dimensional Statistical Recovery

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Abstract

We study linear regression from data distributed over a network of agents (with no master node) under high-dimensional scaling, which allows the ambient dimension to grow faster than the sample size. We propose a novel decentralization of the projected gradient algorithm whereby agents iteratively update their local estimates by a "double-mixing" mechanism, which suitably combines averages of iterates and gradients of neighbouring nodes. Under standard assumptions on the statistical model and network connectivity, the proposed method enjoys global *linear* convergence up to the *statistical precision* of the model. This improves on guarantees of (plain) DGD algorithms, whose iteration complexity grows undesirably with the ambient dimension. Our technical contribution is a novel convergence analysis that resembles (albeit different) algorithmic stability arguments extended to high-dimensions and distributed setting, which is of independent interest.

1 Introduction

Consider M-estimation over a network of m agents, modeled as an undirected graph with no server (termed *mesh* network). Each agent i owns a sample of n i.i.d observations, drawn from an unknown, common distribution on $\mathcal{Z} \subseteq \mathbb{R}^p$, and collected in the set S_i. Agents aim at minimizing the total empirical loss over $N = m \cdot n$ samples, resulting in the following empirical risk minimization (ERM):

$$\hat{\theta} \in \operatorname*{argmin}_{\theta \in \Omega : \mathcal{R}(\theta) \leq R} \mathcal{L}(\theta) \triangleq \frac{1}{m} \sum_{i=1}^{m} \mathcal{L}_i(\theta), \quad \text{with} \quad \mathcal{L}_i(\theta) \triangleq \frac{1}{2n} \sum_{j \in S_i} (x_j^\top \theta - y_j)^2, \tag{1}$$

where $\mathcal{L}_i(\theta)$ is the empirical loss of agent i, with $x_j \in \mathbb{R}^d$ being the vector of predictors and $y_j \in \mathbb{R}$ the associated response; $\mathcal{R}$ is a (convex) regularizer (with $R > 0$) controlling the complexity of the solution; and Ω is some (convex) subset of $\mathbb{R}^d$.

The ERM (1) is a surrogate of the minimization of the (strongly convex) population loss

$$\theta^\star = \operatorname*{argmin}_{\theta \in \mathbb{R}^d} \bar{\mathcal{L}}(\theta) \triangleq \frac{1}{2} \mathbb{E}_{(x,y) \sim \mathbb{P}} [(x^\top \theta - y)^2]. \tag{2}$$

We are interested in estimation problems that are underdetermined, meaning that the ambient dimension d exceeds (and grows faster than) the total sample size N. Therefore, we assume that $\theta^\star$ lies in a smaller subset of Ω or is well approximated by a member of it–the regularizer $\mathcal{R}$ in (1) enforces this constraint. Instances of such M-estimation problems include ℓ_1-constrained sparse linear models, low-rank matrix recovery via nuclear norm, and matrix regression with soft-rank constraints [2].

Our goal is to study statistical *and* computational guarantees of first-order distributed algorithms for (1). The benchmark is the centralized Projected Gradient Algorithm (PGA) [2]: under suitably

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restricted notions of strong convexity and smoothness of $\mathcal{L}$ (see Sec. 2)–which hold with high probability for a variety of statistical models–as well as conditions for statistical consistency–e.g., $s \log d/N = o(1)$ for s-sparsity–the iterates $\{\theta^t\}$ generated by the PGA (starting from θ^0) satisfy:

$$\|\theta^t - \hat{\theta}\|^2 \leq \lambda^t \, \|\theta^0 - \hat{\theta}\|^2 + o(\|\hat{\theta} - \theta^\star\|^2), \tag{3}$$

with $\lambda \in (0, 1)$. In words: the optimization error $\|\theta^t - \hat{\theta}\|^2$ decays *linearly* with rate λ, up to a tolerance of a smaller order than the statistical error of the model, $\|\hat{\theta} - \theta^\star\|^2$. Therefore every limit point of $\{\theta^t\}$ is within the statistical error from $\theta^\star$. This is the the best one can hope for statistically (ignoring lower order terms) and computationally (within first-order, non accelerated methods).

The PGA is not implementable on mesh networks: agents cannot compute locally the full gradient $\nabla\mathcal{L}$. This calls for the design of distributed algorithms. The following questions arise naturally:

(Q1) Can (3) be achieved by distributed algorithms? Of primary interest is the regime wherein the local sample size n is not sufficient for statistical consistency while the total one N is so.

(Q2) How do sample rate and convergence rate scale with the model parameters (d, n, N) and network parameters (connectivity, size m, and topology)?

As documented next, current literature does not provide a satisfactory answer to the above questions for the majority of distributed algorithms. In a nutshell, most of existing studies are of pure optimization type, they lack statistical analysis, and break down under high-dimensional scaling $d/N \to \infty$.

1.1 Related works

The ERM (1) is an instance of a (non strongly) convex optimization problem–several distributed algorithms are applicable. Early decentralizations of the PGA have been proposed in [18, 19]; when applied to (1), they reduce to the following distributed gradient descent (DGD):

$$\theta_i^{t+1} = \prod_{\Omega \cap \{\mathcal{R}(\theta) \leq R\}} \left(\sum_{j \in \mathcal{N}_i} \tilde{w}_{ij} \, \theta_j^t - \alpha \nabla \mathcal{L}_i(\theta_i^t) \right), \quad i = 1, \ldots, m, \tag{4}$$

where θ_i^t is the estimate from agent i of the common variable θ at iteration t; $\alpha \in (0, 1]$ is the stepsize; $\mathcal{N}_i$ is the set of neighbors of agent i; $\tilde{w}_{ij}$'s are suitably chosen nonnegative weights; and $\prod_{\Omega \cap \mathcal{R}(\theta) \leq R}(\bullet)$ is the Euclidean projection of its argument onto the convex set $\{\theta \in \Omega \, : \, \mathcal{R}(\theta) \leq R\}$. Subsequent works [33, 34, 7] extended convergence analyses of (4) to other classes of loss functions, and [5, 6, 24] proposed alternative forms of mixing of local and neighboring information.

When applied to the minimization of an average loss $f(\theta) = 1/m \sum_{i=1}^m f_i(\theta)$, convergence guarantees of the above DGD-like algorithms, including (4), can be roughly summarized as follows: (i) when f_i are smooth, strongly convex, DGD schemes using constant stepsize converge at linear rate, but only to a neighborhood of the minimizer of the average-loss f [33, 34]. Convergence to the exact minimizer is achieved employing diminishing stepsize rules, at the price of slower sublinear rate [34, 12]. These results are unsatisfactory when applied to the ERM (1). First, for fix d and N, they would predict sublinear convergence rate, as the loss $\mathcal{L}$ is convex but not strongly convex (recall $d > N$); this would suggest a negative answer to **(Q1)**–a conclusion that seems to be confuted by recent experiments [28, Fig. 1], showing instead linear convergence of DGD (4) up to a tolerance. Second, when d grows faster than N–the typical situation in high-dimension–these optimization studies break down. In fact, they all require *global smoothness* of the loss functions $\mathcal{L}_i$'s and $\mathcal{L}$, a property that no longer holds under the high-dimensional scaling $d/N \to \infty$: for commonly used designs of predictors x_i's, the Lipschitz constant of $\nabla\mathcal{L}$ grows indefinitely with d/N [30].

A statistical study of a DGD-like algorithm solving the LASSO problem (in the Lagrangian form) over mesh networks was recently proposed in [13]: For standard statistical models of predictors, the iterates generated by the algorithm enter an ε-neighborhood of a statistically optimal estimate after

$$\mathcal{O}\left(\frac{\kappa}{1-\rho} \cdot d \cdot m^2 \log m \cdot \log \frac{1}{\varepsilon} \right) \quad \text{iterations}, \tag{5}$$

where $\rho \in [0, 1)$ is a measure of the connectivity of the network and κ is the condition number of the data covariance matrix (population loss). This improves on classical optimization analyses, showing

that this instance of DGD achieves centralized statistical accuracy at a linear rate. However, this rate scales as $\mathcal{O}(d)$. In this paper, we prove that the DGD scheme (4) applied to (1) inherits the same undersirable scaling of the rate–see Appendix H. This would provide a negative answer to **(Q1)**.

A natural question is whether other distributed algorithms offer a more favorable rate scaling, in particular those employing some form of correction of the local gradient direction in the agents' updates. Examples include: primal-dual schemes [27, 11, 25, 26, 14] and gradient tracking methods [21, 17, 32, 8, 29]. However convergence of these algorithms in high-dimension has not been investigated. For fixed d and N, with $d > N$, existing analyses predict a pessimistic *sublinear* convergence rate of those methods applied to (1). Furthermore, with the exception of [28], they lack of any statistical analysis and, as those for DGD methods, they break down under high-dimension scaling, d/N growing. The very recent work [28] studied the LASSO problem over networks and proposed a decentralization of PGA based on gradient tracking, termed NetLASSO. Under suitable assumptions, the scheme is proved to reach a neighborhood of a statistically optimal estimate of an s-sparse parameter at linear rate. The analysis of NetLASSO is ad-hoc, as it builds on the specific dynamics of the algorithm–hence questions **(Q1)** and **(Q2)** remain open for other distributed algorithms. Furthermore, [28] seems to suggest that gradient-tracking is needed to achieve dimension independent (linear) convergence rates over mesh networks. In this paper we prove that gradient tracking is actually not necessary.

1.2 Major contributions

(i) Algorithm design: To cope with the 'speed-accuracy dilemma' of DGD in high-dimensions, we propose a new distributed algorithm whereby agents update their local estimate and optimization directions via a 'double-mixing' mechanism, aiming at enforcing consensus on *both* iterates and local gradients. We termed the algorithm DGD^2. At the high-level, our approach brings a new perspective to the design of distributed algorithms in high-dimension: the ultimate goal is not solving the ERM (1) *exactly* but reaching instead an estimate of $\theta^\star$ *within the statistical error* . This can significantly relax the algorithmic design. For instance, contrary to what one might infer from [28], gradient-tracking *is not needed* to efficiently achieve statistically optimal solutions, as our design and analysis prove.

(ii) Statistical guarantees: We identify deterministic conditions under which the iterates generated by DGD^2 converge at *linear* rate to a limit point that is within a fixed tolerance from the unknown parameter $\theta^\star$. When customized to statistical models–including sparse linear regression with ℓ_1 regularization, low-rank matrix recovery with nuclear norm, and matrix completion–our conditions hold with high-probability, and the tolerance becomes of the order of the statistical error $\|\hat{\theta} - \theta^\star\|$. For such models, DGD^2 enters an ε-neighborhood of a *statistically optimal* estimate of $\theta^\star$ after

$$\mathcal{O}\left(\kappa \cdot \log \frac{1}{\varepsilon}\right) \text{ iterations (gradient calls)} \quad \text{and} \quad \mathcal{O}\left(\frac{\kappa}{1-\rho} \cdot \log(m \cdot \kappa) \cdot \log \frac{1}{\varepsilon}\right) \text{ communications,} \tag{6}$$

for *any* $\rho \in [0, 1)$. The former matches the rate of PGA; the latter compares favorably with the complexity of DGD in (5), which instead degrades with d. Finally, (6) is showed to be invariant under high-dimensional scaling $d/N \to \infty$, as long as $\|\hat{\theta} - \theta^\star\| = \mathcal{O}(1)$. All this addresses **(Q1)** and **(Q2)**. Quite interestingly, it also shows that gradient-tracking as in [28] *is not needed in high-dimensions*.

(iii) New convergence analysis: We put forth a new analysis, inspired by the idea of algorithmic stability for stochastic optimization [4, 9]–note that that theory is *not* applicable in high-dimensions and distributed setting. Leveraging the more favorable landscape of the population risk, our approach contrasts the trajectory of DGD^2 to that of a "virtual" DGD^2 instance applied to a population (penalized) variant of the ERM, and establishes conditions for the two trajectories to stay within the desired tolerance while letting DGD^2 inherit fast linear converge of its population counterpart. This contrasts with existing works of distributed algorithms in high-dimension [13, 28], whose convergence is established on the empirical landscape, splitting thus the estimation error $\|\theta_i^t - \theta^\star\|$ between optimization $\|\theta_i^t - \hat{\theta}\|$ and statistical $\|\hat{\theta} - \theta^\star\|$ errors. This makes the analysis very algorithmic-dependent and hard to extend to other solution methods (see remark in Sec. 4.2 for technical details). On the contrary, our approach leads to simpler proofs, potentially applicable to other distributed algorithms than DGD^2 in high-dimensions (e.g., primal-dual). An example is given in Appendix H, where convergence of a variant of DGD in [13] applied to the constrained LASSO problem is provided. Our approach will be applied to other distributed algorithms in future works.

2 Setup and Background

We develop our analyses under the following assumptions.

Assumption 1. *The population risk $\bar{\mathcal{L}}$ is μ-strongly convex and L-smooth on $\mathbb{R}^d$, with $\mu, L \in (0, \infty)$. The condition number is $\kappa \triangleq L/\mu$.*

We consider regularizers $\mathcal{R}$ in ERM (1) that are *decomposable* [2]. Let $\mathcal{M} \subseteq \Omega$ be the so-called *parameter* subspace, capturing constraints from the model (e.g., vectors with a particular support or a subspace of low-rank matrices); and let $\bar{\mathcal{M}}^{\perp}$ be the *perturbation* subspace, representing the deviations from the model subspace $\mathcal{M}$. Decomposability is defined as follows.

Assumption 2 ([30])**.** ***(i)*** *$\mathcal{R} : \Omega \to \mathbb{R}$ is a norm, with $\Omega \subseteq \mathbb{R}^d$ convex. Furthermore, given a subspace pair $(\mathcal{M}, \bar{\mathcal{M}}^{\perp})$, such that $\mathcal{M} \subseteq \bar{\mathcal{M}}$:* ***(ii)*** *$\mathcal{R}$ is $(\mathcal{M}, \bar{\mathcal{M}}^{\perp})$-decomposable, that is, $\mathcal{R}(x+y) = \mathcal{R}(x) + \mathcal{R}(y)$, for all $x \in \mathcal{M}$ and $y \in \bar{\mathcal{M}}^{\perp}$; and* ***(iii)*** *When $\bar{\mathcal{M}} \neq \{0\}$, $\mathcal{R}$ is $\Psi(\bar{\mathcal{M}})$-Lipschitz over $\bar{\mathcal{M}}$ with respect to some norm $\|\bullet\|$: $\Psi(\bar{\mathcal{M}}) \triangleq \sup_{\theta \in \bar{\mathcal{M}} \setminus \{0\}} \mathcal{R}(\theta)/\|\theta\|$. If $\bar{\mathcal{M}} = \{0\}$, we set $\Psi(\{0\}) = 0$.*

Fast convergence of the PGA relies on the empirical risk $\mathcal{L}$ satisfying suitably restricted notions of strong convexity and smoothness [2]. Under Assumption 1, this can be enforced by controlling the deviation of the Hessian $\nabla^2 \mathcal{L}$ of $\mathcal{L}$ from the Hessian $\nabla^2 \bar{\mathcal{L}}$ of the population loss $\bar{\mathcal{L}}$, along certain directions, as stated below. Note that $\nabla^2 \bar{\mathcal{L}} \succ 0$ is the covariance matrix of the predictors x_i's while $\nabla^2 \mathcal{L} = (1/N) X^T X$, with $X = [x_1, x_2, \ldots, x_N]^\top$.

Assumption 3. *There exist $\gamma_{g,1}$, $\gamma_{g,2}, \tau_g > 0$ and a matrix $\Sigma' = \Sigma'^{\top} \succ 0$ such that*

$$\Delta^\top \big(\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L}\big) \Delta \leq \gamma_{g,1} \|\Delta\|_{\Sigma'}^2 + \tau_g \mathcal{R}^2(\Delta), \; \forall \Delta \in \Omega';$$

$$\Delta^\top \big(\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L}\big) \Delta \geq -\gamma_{g,2} \|\Delta\|_{\Sigma'}^2 - \tau_g \mathcal{R}^2(\Delta), \; \forall \Delta \in \Omega'.$$

Assumption 3 enforces a curvature property on $\nabla^2 \mathcal{L}$ along the directions $\Delta \in \Omega' \subseteq \mathbb{R}^d$ where $\mathcal{R}^2(\Delta)$ is sufficiently small. In the distributed setting, we need also a local counterpart of Assumption 3, associated with the agents' Hessian matrices $\nabla^2 \mathcal{L}_i = (1/n) X_i^\top X_i$, with $X_i = (x_j)_{j \in S_i}$.

Assumption 4. *There exist $\gamma_{\ell,1}, \gamma_{\ell,2}$ $\tau_\ell > 0$ and a matrix $\Sigma' = \Sigma'^{\top} \succ 0$ such that, for all i,*

$$\Delta^\top \big(\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L}_i\big) \Delta \leq \gamma_{\ell,1} \|\Delta\|_{\Sigma'}^2 + \tau_\ell \mathcal{R}^2(\Delta), \; \forall \Delta \in \Omega';$$

$$\Delta^\top \big(\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L}_i\big) \Delta \geq -\gamma_{\ell,2} \|\Delta\|_{\Sigma'}^2 - \tau_\ell \mathcal{R}^2(\Delta), \; \forall \Delta \in \Omega'.$$

The practical utility of Assumptions 3 and 4 is that they can be certified with high probability by a variety of random data generations. Here, we consider the following widely used statistical models, which cover a variety of estimation tasks, such as sparse linear models with ℓ_1 regularization, low-rank matrix recovery with nuclear norm, and matrix regression with soft-rank constraints [15, 2, 30].

Assumption 5. *The random predictors $x_i \in \mathbb{R}^d$ are i.i.d. and fulfill one of the following conditions:*

- *(i)* $x_i \sim \mathcal{N}(0, \Sigma)$ *and are i.i.d., for some $\Sigma \succ 0$;*
- *(ii)* x_i *are $(\tau^2, \Sigma)-$sub-Gaussian and i.i.d., with $\Sigma \succ 0$;*
- *(iii)* $x_i = e_j$ *where $j \sim$ uniform$[1, d]$ and i.i.d.*

Under Assumption 5 the curvature conditions in Assumptions 3 and 4 hold with high probability. The following lemma makes this formal under Assumption 5(i), and is a minor modification of [2, Prop. 1, supplementary]. Similar results under Assumptions 5(ii)-(iii) can be found in Appendix F.

Lemma 1. *Under Assumption 5(i), there exist universal constants $c_0, c_1, c_2 > 0$ such that, with probability at least $1 - c_0 \exp(-c_1 N + \log m)$ and $N \geq 10$, Assumptions 3 hold with parameters*

$$\Sigma' = \nabla^2 \bar{\mathcal{L}}, \;\; \gamma_{g,1} = 1/2, \;\; \gamma_{g,2} = 1, \;\; \tau_g = c_2 (\mathbb{E}[\mathcal{R}^*(x_i)])^2 / N;$$

and so does Assumption 4 with parameters

$$\Sigma' = \nabla^2 \bar{\mathcal{L}}, \;\; \gamma_{\ell,1} = 1, \;\; \gamma_{\ell,2} = 4m, \;\; \tau_\ell = c_2 \left(\mathbb{E}[\mathcal{R}^*(x_i)]\right)^2 / n,$$

where $\mathcal{R}^*$ is the dual norm of $\mathcal{R}$. When $\mathcal{R}(\cdot) = \|\cdot\|_1$, $(\mathbb{E}[\mathcal{R}^*(x_i)])^2 \leq 9(\max_j (\Sigma)_{jj}) \log(d)$.

Network setup: Agents are embedded in a communication network, modelled as an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where the vertices $\mathcal{V} = [m] \triangleq \{1, \ldots, m\}$ correspond to the agents and $\mathcal{E}$ is the set of edges of the graph; $(i, j) \in \mathcal{E}$ if and only if there is a communication link between agents i and j. The set of neighbors of each agent i is denoted by $\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\} \cup \{i\}$. We focus on distributed algorithms using gossip weight matrices in their communication steps, as below.

Assumption 6. *$W = [w_{ij}]_{ij=1}^m$ satisfies: **(i)** $w_{ij} > 0$, if $(i, j) \in \mathcal{E}$; otherwise $w_{ij} = 0$; furthermore, $w_{ii} > 0$, for all $i \in [m]$; **(ii)** $W = W^\top$ and $W1 = 1$ (stochastic); and **(iii)** there holds $\rho \triangleq \|W - J\|_2 < 1$, where $J \triangleq 11^\top/m$.*

Assumption 6 is standard in the literature of distributed algorithms and is satisfied by several weight matrices; see, e.g., [16]. Note that $\rho < 1$ holds true by construction for connected graphs. Roughly speaking, ρ measures how fast the network mixes information; the smaller ρ, the faster the mixing.

3 Algorithm Design: DGD^2

We begin providing some rationale on the new design. Our goal is to cope with the speed-accuracy dilemma of DGD (4)–see (5)–while retaining linear convergence to statistically optimal solutions.

It is convenient to rewrite (4) in matrix/vector form. Define the "augmented" quantities:

$$\boldsymbol{\theta}^t \triangleq [(\theta_1^t)^\top, \ldots, (\theta_m^t)^\top]^\top, \quad \boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) \triangleq \frac{1}{m} \sum_{i=1}^m \mathcal{L}_i(\theta_i) \quad \text{and} \quad \tilde{\mathbf{W}} \triangleq \tilde{W} \otimes I_d, \tag{7}$$

with $\otimes$ denoting the Kronecker product. Then, DGD (4) can be rewritten as

$$\boldsymbol{\theta}^{t+1} = \prod_{\Omega \cap \mathcal{R}(\theta) \leq R} \left(\tilde{\mathbf{W}} \boldsymbol{\theta}^t - \alpha\, m\, \nabla \boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^t) \right), \tag{8}$$

where the projector operator is now intended acting block-wise on each d-dimensional block. It is well-known that (8) can be interpreted as the iterate dynamics of the projected gradient algorithm (with stepsize $m\,\alpha$) applied to the following optimization problem [33]: using $\boldsymbol{\theta} \triangleq [(\theta_1)^\top, \ldots, (\theta_m)^\top]^\top$,

$$\min_{\theta_i \in \Omega\,:\, \mathcal{R}(\theta_i) \leq R,\, \forall i \in [m]} \boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) + \frac{1}{2\, m\, \alpha} \|\boldsymbol{\theta}\|^2_{I - \tilde{\mathbf{W}}}, \tag{9}$$

The quadratic penalty in the objective aims at enforcing consensus among agents' variables θ_i's, as $\alpha \downarrow 0$. Our analysis (see Appendix H) shows that exact consensus is not needed; in fact, $\alpha = \mathcal{O}(d^{-1})$ is enough to drive the estimation error $(1/m)\|\boldsymbol{\theta}^t - 1 \otimes \theta^*\|^2$ within an ϵ- neighborhood of a *statistically optimal* solution of (2) in a number of communications steps as in (5). Numerical results confirm that $\alpha = \mathcal{O}(d^{-1})$ is necessary for statistical consistency.

This undesirable scaling of the rate (5) with d is a consequence of the restrictive condition $\alpha = \mathcal{O}(d^{-1})$. At a high level our goal is then clear: we need to ease such a constraint. Our idea is to share the consensus-achieving burden in (9) between α (the penalty term) and the first term $\boldsymbol{\mathcal{L}}(\boldsymbol{\theta})$. Notice that at consensus, $\boldsymbol{\theta} = \mathbf{J}\boldsymbol{\theta}$, and thus $\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) = \boldsymbol{\mathcal{L}}(\mathbf{J}\boldsymbol{\theta})$, where $\mathbf{J} \triangleq J \otimes I_d$. However, $\boldsymbol{\mathcal{L}}(\mathbf{J}\boldsymbol{\theta})$ (and its gradient) is not computable distributively, as agents cannot perform full averages in a single communication step, as subsumed by the operation $\mathbf{J}\boldsymbol{\theta}$. We propose then to approximate $\boldsymbol{\mathcal{L}}(\mathbf{J}\boldsymbol{\theta})$ by $\boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta})$, where $\mathbf{W} \triangleq W \otimes I_d$, for some gossip matrix W (Assumption 6). The gradient of $\boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta})$ is now additively separable across the agents and thus locally computable. We thus replace (9) with

$$\min_{\theta_i \in \Omega\,:\, \mathcal{R}(\theta_i) \leq R,\, \forall i \in [m]} \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}) + \frac{1}{2\, m\, \alpha} \|\boldsymbol{\theta}\|^2_{I - \tilde{\mathbf{W}}}. \tag{10}$$

Our new algorithm is obtained as projected gradient method applied to (10) (with stepsize $m\alpha$):

$$\boldsymbol{\theta}^{t+1} = \prod_{\Omega \cap \mathcal{R}(\theta) \leq R} \left(\tilde{\mathbf{W}} \boldsymbol{\theta}^t - \alpha\, m\, \mathbf{W} \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t) \right).$$

This algorithm is fully distributed. However, for arbitrary $\mathbf{W}$ and $\tilde{\mathbf{W}}$, it requires three communication exchanges per iteration. This number can be reduced to two by choosing $\tilde{\mathbf{W}} = \mathbf{W}^2$, yielding

$$\boldsymbol{\theta}^{t+1} = \prod_{\Omega \cap \mathcal{R}(\theta) \leq R} \left(\mathbf{W} \left(\mathbf{W} \boldsymbol{\theta}^t - \alpha\, m\, \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t) \right) \right). \tag{11}$$

The agent-level implementation is given in Algorithm 1. Two rounds of mixing (communications) are performed by each agent; hence, we term the algorithm DGD^2.

Algorithm 1 : DGD^2

Data: Any feasible θ_i^0, for all $i = 1, \dots, m$, $\alpha \in (0, 1]$;
Iterate $t = 1, 2, \dots$

[S.1]: Each agent i exchanges θ_i^t with $\mathcal{N}_i$ and updates

$$s_i^{t+1/2} = \sum_{j \in \mathcal{N}_i} w_{ij} \theta_j^t - \alpha \nabla \mathcal{L}_i \Big(\sum_{j \in \mathcal{N}_i} w_{ij} \theta_j^t \Big);$$

[S.2]: Each agent i exchanges $s_i^{t+1/2}$ with $\mathcal{N}_i$ and updates:

$$\theta_i^{t+1} = \prod_{\Omega \cap \mathcal{R}(\theta) \leq R} \left(\sum_{j \in \mathcal{N}_i} w_{ij} s_j^{t+1/2} \right).$$

4 Statistical-Computational Guarantees

This section studies convergence of DGD^2. Our first result (Theorem 1) establishes conditions under which the estimation error $(1/m) \sum_{i=1}^m \|\theta_i^t - \theta^\star\|_2^2$ shrinks at *linear* rate up to some tolerance Δ^2. In a number of subsequent corollaries, we show that our conditions hold with high probability for the statistical models in Assumption 5, and the tolerance Δ^2 is within minimax statistical rates.

We begin introducing some notation. The tolerance Δ^2 is composed of two terms, Δ^2_{stat} and Δ^2_{net}:

$$\Delta^2_{\text{stat}} \triangleq \underbrace{\frac{\Psi^2(\bar{\mathcal{M}})}{\mu^2} \mathcal{R}^*(\nabla \mathcal{L}(\theta^\star))^2}_{\text{Stat. error}} + \underbrace{\frac{\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))}{\mu} \mathcal{R}^*(\nabla \mathcal{L}(\theta^\star))}_{\text{Misspecification error}}$$

$$\Delta^2_{\text{net}} \triangleq \frac{\Psi^2(\bar{\mathcal{M}})}{\mu^2} m^3 \rho^2 \max_{j \in [m]} \left(\mathcal{R}^* \left(\nabla \mathcal{L}_j(\theta^\star) \right) \right)^2 + \frac{\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))}{\mu} m^{3/2} \rho \max_{j \in [m]} \mathcal{R}^* \left(\nabla \mathcal{L}_j(\theta^\star) \right),$$

where $\Pi_{\mathcal{M}^\perp}$ (resp. $\Pi_{\bar{\mathcal{M}}}$) denotes the Euclidean projection onto the orthogonal complement of the subspace $\mathcal{M}$ (resp. onto $\bar{\mathcal{M}}$).,Notice that Δ^2_{stat} matches the tolerance achievable by the PGA in the centralized setting [2] on star networks. The second term in Δ^2_{stat} is zero if $\theta^\star \in \mathcal{M}$ (as in sparse vector recovery). On the other hand, Δ^2_{net} is a network dependent error, and is the price for decentralization. Notice that the smaller ρ, the smaller Δ^2_{net}. The overall tolerance is defined as

$$\Delta^2 \triangleq \left(1 + \frac{\zeta(\bar{\mathcal{M}}, W)}{2L} \right) \left(\Delta^2_{\text{stat}} + \Delta^2_{\text{net}} \right) + \frac{\mathcal{R}\left(\Pi_{\mathcal{M}^\perp}(\theta^\star)\right)^2}{\mu \Psi^2(\bar{\mathcal{M}})} \zeta(\bar{\mathcal{M}}, W) \left(1 + \frac{\zeta(\bar{\mathcal{M}}, W)}{2L} \right).$$

where

$$\zeta(\bar{\mathcal{M}}, W) \triangleq \Psi^2(\bar{\mathcal{M}}) \cdot \tau_g + \Psi^2(\bar{\mathcal{M}}) \cdot \tau_\ell \cdot \rho \cdot m^{1.5}. \tag{12}$$

Notice that Δ^2 depends also on the parameters τ_g and τ_ℓ (see Assumptions 3 and 4) via $\zeta(\bar{\mathcal{M}}, W)$. The first term in (12) accounts for the lack of strong convexity and smoothness in a global sense of the empirical loss–the same dependence is observed in the centralized setting [2]–while the second term, depending on ρ, is a consequence of the distributed nature of the estimation.

We are now equipped to state the main convergence result.

Theorem 1. *Consider the ERM problem* (1) *and associated population minimization (2), under Assumption 1, and regularity conditions in Assumptions 3 and 4 such that:* $R = \mathcal{R}(\theta^\star)$,

$$\gamma_{g,1} \Sigma' + \frac{\mu}{2} I_d \preceq \nabla^2 \bar{\mathcal{L}} \quad \textit{and} \quad \Psi^2(\bar{\mathcal{M}}) \cdot \tau_g \leq C_0, \tag{13}$$

for some constant $C_0 > 0$. *Let* $\{(\theta_i^t)_{i=1}^m\}$ *be the sequence generated by DGD^2, with stepsize* $\alpha = (2L)^{-1}$ *and gossip matrix* W *satisfying Assumption 6 and such that*

$$\rho \leq \text{poly} \left(\kappa m \max(\gamma_{\ell_1}, \gamma_{\ell_2}) \frac{\tau_\ell}{\tau_g} \right)^{-1}. \tag{14}$$

Then, the estimation error $r^t \triangleq (1/m)\sum_{i=1}^m \|\theta_i^t - \theta^\star\|_2^2$ *satisfies: for some constant* $C_1 > 0$,

$$r^t \leq \left(\frac{1-(8\kappa)^{-1} + C_1 \cdot \big(\zeta(\bar{\mathcal{M}}, W)\big)/L}{1 - C_1 \cdot \big(\zeta(\bar{\mathcal{M}}, W)\big)/L}\right)^t r^0 \; + \; \mathcal{O}\left(\Delta^2\right). \tag{15}$$

The expression of the constants, a more general statement (Theorem 2), and its proof can be found in Appendix D. A sketch of the proof highlighting our new analysis is presented in Sec.4.2.

On the rate and tolerance: Theorem 1 guarantees linear convergence of the estimation error up to a tolerance $\mathcal{O}\left(\Delta^2\right)$. The contraction factor depends on the condition number κ of the population loss as well as the network connectivity ρ and the tolerance parameters τ_g and τ_ℓ via $\zeta(\bar{\mathcal{M}}, W)$–the latter being a consequence of the lack of strong convexity and smoothness of the empirical loss globally. Notice that, for sufficiently small $\zeta(\bar{\mathcal{M}}, W)$, a condition that holds for increasing sample size N and decreasing ρ, this rate is of the order of $1-(8\kappa)^{-1}$, which matches that of the PGA in the centralized setting. Referring to Δ^2, it will be shown in the examples to follow (see Sect. 4.1) that, for a variety of statistical models Δ^2 is of the order of the *centralized* statistical error, $\Delta^2 = O(\|\hat{\theta} - \theta^\star\|^2)$.

On the conditions (13) and (14): Conditions in (13) are the same as those required in the centralized setting [2]; they are satisfied by a variety of statistical models (as those considered in Sec. 4.1); in particular, $\Psi^2(\bar{\mathcal{M}}) \cdot \tau_g \leq C_0$ calls for a minimum number of samples N which, for the models in Sec. 4.1, is near (minimax) optimal. Notice that no extra conditions are imposed on the local samples size n, which can be as small as one. On the other hand, (14) is a consequence of the decentralization of the M-estimation; it calls for a sufficiently connected network–the larger m or the more ill-conditioned the population loss (larger κ), the smaller ρ. When the graph $\mathcal{G}$ is not part of the design and W is given, (14) can be enforced by employing multiple rounds of communications per gradient evaluations. For instance, given W satisfying Assumption 6, one can build a new matrix, $\bar{W} = W^K$, whose associated $\bar{\rho} = \|\bar{W} - J\| = \rho^K$ satisfies (14); it is sufficient to choose

$$K = \left\lceil (1-\rho)^{-1} \log\left(\frac{108\sigma_1(\Sigma')\gamma_\ell m^{1.5}}{\mu}\right)\right\rceil.$$

Algorithm 1 is then implemented using the weights $\bar{W}$, which results in K rounds of communications per iteration t, each one using the weights W. An ε-estimate of $\theta^\star$, with $\varepsilon \geq \mathcal{O}\left(\Delta^2\right)$, is then achived in $\mathcal{O}\left(\kappa \log 1/\varepsilon\right)$ and $\widetilde{\mathcal{O}}\left(\kappa(1-\rho)^{-1}\log 1/\varepsilon\right)$ numbers of iterations (gradient evaluations) and communications, respectively, where $\widetilde{\mathcal{O}}$ hides log-factors *independent* on the dimension d. If instead $\bar{W}$ is chosen according to Chebyshev polynomials [31], the dependence on ρ in the communication complexity can be improved to $(1-\rho)^{-1/2}$.

4.1 Statistical guarantees

We now develop some consequences of Theorem 1 for specific statistical models. Specifically, Corollary 1 below considers (hard and weak-)sparse linear regression with ℓ_1 regularization while Corollary 2 deals with a matrix completion problem. In Appendix C, we study sub-Gaussian linear regression (Corollary 3) and matrix regression with nuclear norm constraint (Corollary 4).

1) Sparse vector regression: We assume that $\theta^\star$ is sparse. To capture hard- and weak-sparsity we let $\|\theta^\star\|_q \leq R_q$, with $q \in [0,1]$. Note that $q = 0$ corresponds to vectors with support on a set of cardinality at most R_0 (hard-sparsity); the case $q \in (0,1]$ models approximate sparsity. We consider the following linear generation model over the network: $y_j = x_j^\top \theta^\star + w_j$, $j \in S_i$ and $i \in [m]$, where $w_j \sim \mathcal{N}(0, \sigma^2)$ is the measure noise and x_j satisfies Assumption 5(i). For convenience, we define $\eta_\Sigma \triangleq \max_i([\Sigma]_{ii})$. With this setup we have the following consequence of Theorem 1.

Corollary 1. *Consider the linear regression problem above via ERM (1), with* $\mathcal{R}(\cdot) = \|\cdot\|_1$, $R = \|\theta^\star\|_1$, *and* $\Omega \equiv \mathbb{R}^d$. *Suppose that* (13) *holds, which becomes:* $\Psi^2(\bar{\mathcal{M}}) \cdot \tau_g = C_1 R_q \eta_\Sigma \left(\log d/N\right)^{1-q/2} \leq C_0$, *for some constants* $C_0, C_1 > 0$, *and given* $q \in [0,1]$. *Let* $\{(\theta_i^t)_{i=1}^m\}$ *be the sequence generated by DGD*^{2} *under the conditions of Theorem 1, where (14) becomes* $\rho \leq C_2(\kappa\, m^{5/2})^{-1}$, *for some constant* $C_1 > 0$. *Then, with probability at least* $1 - c_0 \exp\left(-c_1 N + \log(m)\right)$, *for some* $c_0, c_1 > 0$ C_3, $C_4 > 0$, *the following holds:*

$$r_t^2 \leq \lambda^t r_0^2 + \mathcal{O}\left(\Delta^2\right) + o(\Delta^2),$$

where

$$\lambda = \frac{1-(8\,\kappa)^{-1} + C_3 \cdot \left(\Psi^2(\bar{\mathcal{M}}) \cdot \tau_g/L\right)}{1 - C_3 \cdot \left(\Psi^2(\bar{\mathcal{M}}) \cdot \tau_g/L\right)} \text{ and}$$

$$\Delta^2 = C_4 R_q \left(\frac{\log d}{N}\right)^{1-\frac{q}{2}} \left(\frac{\eta_\Sigma \sigma^2}{\mu^2} + \frac{\eta_\Sigma^{\frac{1}{2}}\sigma}{\mu}(1-\delta(q))\right),$$

and $\delta(q) = 1$ *when* $q = 0$*, and* 0 *otherwise.*

Note that Δ^2_{stat} above is (near) minimax optimal for either cases of $q = 0$ and $q \in (0, 1]$ [22]. Furthermore, the rate λ is *invariant* under high-dimensional scaling $d/N \to \infty$ and fixed m, as long as $\Delta^2_{\text{stat}} = \mathcal{O}(1)$, a condition necessary for statistical consistency. This matches guarantees of the PGA in the centralized setting [2]. Also, **(i)** it improves on communication complexity (5) of DGD (4) (applicable here when $q = 0$), which scales undesirably with d [13]; and **(ii)** compares favorably with [28], exhibiting same convergence rate order but sightly more favorable scaling of ρ with m ($m^{2.5}$ vs. m^8). This shows that gradient-tracking is not needed to match centralized statistical errors.

2) Matrix Completion: In this model, the observation y_j is a noisy version of a randomly selected entry $[\Theta^\star]_{a(j),b(j)}$ of the unknown matrix $\Theta^\star \in \mathbb{R}^{p\times p}$. We assume that $\|\Theta^\star\|_q \le R_q$, with $q \in [0, 1]$, where $\|\Theta^\star\|_q \triangleq \sum_{i=1}^p |\sigma_i(\Theta^\star)|^q$, and $\sigma_i(\Theta^\star)$ denotes the i-th singular value of $\Theta^\star$. Note that $q = 0$ corresponds to matrices $\Theta^\star$ with rank at most R_q while $q \in (0, 1]$ promotes matrices of all ranks but with relatively fast rate of decay of the singular values. Denoting by X_j the $p \times p$ matrix with $[X_j]_{a(j),b(j)} = 1$ and all the other entries zero, we can write: $y_j = \langle X_j, \Theta^\star\rangle + (\sigma/d)w_j$, $j \in S_i$ and $i \in [m]$, where $\langle\cdot,\cdot\rangle$ is the trace inner product; w_i are noise samples, assumed to be sub-exponential with parameter 1, i.i.d., and independent from X_i; $x_i \triangleq \text{vec}(X_i)$ satisfies Assumption 5(iii); $\sigma > 0$ is given, and $d = p^2$. This M-estimation can be written in the ERM form (1), with $\theta = \text{vec}(\Theta) \in \mathbb{R}^d$, with $\Theta \in \mathbb{R}^{p\times p}$ being the unknown matrix, and $x_i = \text{vec}(X_i)$. We choose the set $\Omega = \{\Theta \in \mathbb{R}^{p\times p} : \|\Theta\|_\infty \le \omega/p\}$ of candidate matrices with bounded element-wise ℓ_∞ norm, with $\omega \ge 1$. This eliminates matrices that concentrate too much their mass in a single position. With this setup, we can specialize Theorem 1 as follows.

Corollary 2. *Consider the ERM (1) solving the matrix completion problem above, with* $\mathcal{R}(\Theta) = \|\Theta\|_1$ *and* $R = \|\Theta^\star\|_1$*. Suppose that* (13) *holds, which becomes:* $\left(p\log p/N\right)^{1-cq/2} \le C_0$*, for some constant* $C_0 > 0$ *and* $q \in (0, 1]$*. Let* $\{(\theta_i^t)_{i=1}^m\}$ *be the sequence generated by DGD*^{2} *under the conditions of Theorem 1, where* α *reduces to* $\alpha = 1$ *and (14) becomes* $\rho \le C_1 m^{-5}$*, for some constant* $C_1 > 0$*. Then, with probability at least 1 -* $c_0(m+1)\exp(-p\log(p))$*, for some* $c_0 > 0$*,* $C_2 > 0$ *there holds:*

$$r_t^2 \le (1-1/8)^t\, r_0^2 + \mathcal{O}\left(\Delta^2\right), \quad \textit{with} \quad \Delta^2 = C_2 R_q \left(p\log p/N\right)^{1-\frac{q}{2}} \left(\omega^{2-q} + \sigma^2\right).$$

Although quantitative aspects of the rates are different, Corollary 2 is analogous to Corollary 1: linear convergence is guaranteed up to a tolerance of the order of (near optimal) statistical minimax rates for the matrix completion problem under the soft-rank model [23]. The more demanding scaling of ρ with $m^{-5.5}$ (vs. $m^{-2.5}$) is due to the slower rate the empirical loss concentrates around the population for such statistical models.

4.2 Sketch of proof of Theorem 1

We highlight here the key novelty of our convergence analysis–see Appendix D for the full proof.

At the high level, our proof studies the dynamics of the estimation error $r^t = (1/m)\sum_{i=1}^m \|\theta_i^t - \theta^\star\|_2^2$ using as potential function the following *population* instance of (10), which inherits the curvature properties of $\bar{\mathcal{L}}$:

$$\bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}) \triangleq \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}) + \frac{1}{2\,m\,\alpha}\|\boldsymbol{\theta}\|^2_{I-\mathbf{W}}, \quad \text{with} \quad \bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}) \triangleq \frac{1}{m}\sum_{i=1}^m \bar{\mathcal{L}}(\theta_i). \tag{16}$$

Leveraging *global* strong-convexity and smoothness of $\bar{\mathcal{L}}$, we establish the following decay for r^{t+1} (see Lemma 2 in Appendix E):

$$r^{t+1} \le \lambda\, r^t + 2\alpha\langle \nabla\mathcal{L}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla\bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t), \mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle. \tag{17}$$

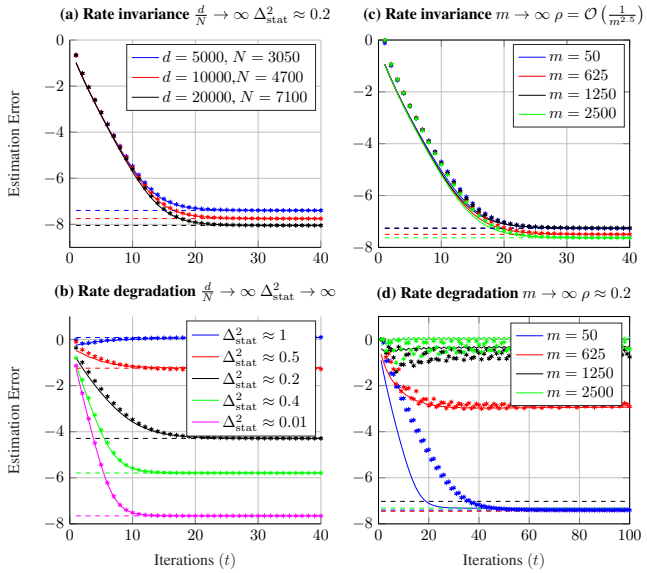

Figure 1: Estimation errors vs. iterations, for DGD^2 (solid lines) and NetLASSO (dotted lines); centralized statistical accuracy (horizontal dashed lines). **(a)** Rate invariance under $d/N \uparrow$, $m = 5$, and $\Delta^2_{\text{stat}} \approx 0.2$; **(b)** Growing Δ^2_{stat}, with $m = 5$ and W fixed. **(c)** Fixed d, and $m \uparrow$, with $\rho \downarrow 0$. **(d)** Same as **(c)** with ρ constant.

for some $\lambda \in (0, 1)$. The first term on the RHS is contractive, due to the strong convexity of $\bar{\boldsymbol{\mathcal{L}}}^{\alpha}$, while the second term captures the discrepancy between the gradient of the empirical loss and that of the proxy population loss along the direction $\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^{\star})$. The rest of the proof boils down to carefully control this error term. This is sensible; for instance, the approach used to study algorithmic stability [4, 9] is not satisfactory here: it would upper bound the aforementioned inner product generating in particular the term

$$\|\nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t)\|^2 = \mathcal{O}(d/n),$$

which grows indefinitely under $d/n \to \infty$, the typical scaling in high-dimensions.

This suggests that the discrepancy of the two gradients needs to be controlled along the direction $\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^{\star})$. This is the core of our analysis. Under Assumptions 3-4–limiting the deviation of the empirical loss $\mathcal{L}$ from the population loss $\bar{\mathcal{L}}$ along suitable directions–and proper choices of the regularization value R and sufficiently small network connectivity ρ, we prove that the inner product in (17) generates an error that preserves contraction (from the first term) while adding an overall tolerance, which can be made of the order of the statistical error.

Remark: Our proof differs from existing convergence studies of first-order methods in high-dimensions, both *centralized* [2] and distributed [13, 28]; they all use exclusively the *empirical* loss, decoupling the estimation error $\|\theta_i^t - \theta^{\star}\|$ in optimization $\|\theta_i^t - \hat{\theta}\|$ and statistical $\|\hat{\theta} - \theta^{\star}\|$ errors. While $\|\hat{\theta} - \theta^{\star}\|$ is known for several estimators in the centralized setting [30], this is no longer the case for distributed ERM problems, e.g., as those arising from the design of some DGD-like algorithms [13]–which deal with lifted, penalized formulations–or primal-dual methods, based on saddle-point reformulations. This makes the analyses using the above decomposition problem-dependent and hard to generalize to other formulations (algorithms). Our approach passing throughout the population loss bypasses this issue.

5 Numerical Results

We provide some numerical results validating our theoretical findings; more experiments (including on real data) can be found in Appendix B of the supporting material. All simulations are performed on a computer with an Intel i7-8650U CPU @ 1.9 GHz using 16 GB RAM running Windows 10.

We simulate the distributed s-sparse linear regression problem, as in Corollary 1. We set $q = 0$, $R_0 = s$, $\Sigma = I_d$, $\sigma^2 = 0.25$, and $R = \|\theta^{\star}\|_1$. The statistical error for this problem reads $\Delta^2_{\text{stat}} \triangleq$

$s \log d/N$. The communication network is generated using an Erdős-Rényi graph with m agents and link activation probability p. Unspecified values above are given for each particular experiment, as needed. DGD^2 is tuned according to the theory, $\alpha = 1/(2L) = 1/2$; we use same stepsize for NetLASSO. All curves are obtained using 10 repetitions via Montecarlo simulation.

(i) Dependence on Δ^2_{stat}. Fig. 1(a): We set $m = 5$, $p = 0.1$ to generate a base graph and $\bar{W}$; we let $W = \bar{W}^T$, with $T = 41$, to achieve a connectivity of $\rho = 0.023 \approx m^{-2.5}$, as required by Corollary 1. We plot the log-normalized estimation error $\log\left(\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 / \|\boldsymbol{\theta}^\star\|_2^2\right)$ versus the iteration, for different values of d and N as indicated in the legend and $s = \lceil \sqrt{d} \rceil$, so that d grows faster than N while keeping $\Delta^2_{\text{stat}} \approx 0.2$. Dashed-line curves correspond to the solution to (1) obtained by PGA while solid-and dotted-line curves refer to DGD^2 and NetLASSO, respectively. **Fig. 1(b):** We use the same fixed network W ($m = 5$) as in Fig. 1(a) and plot the same quantities, now parametrized on the statistical error $\Delta^2_{\text{stat}} \in \{1, 0.5, 0.2, 0.4, 0.01\}$. To do so, we fix $d = 20000$, $s = \lceil 2^{-1} \log(d) \rceil$ and vary N accordingly, i.e., $N \in \{50, 100, 250, 1250, 5000\}$. **Discussion:** Fig. 1(a)-(b) validates (14): under s, d/N growing, the convergence rate of DGD^2 is linear and remains invariant, as long as Δ^2_{stat} stays constant (Fig. 1.(a)) while algorithm's performance degrades when Δ^2_{stat} increases (Fig. 1(b)). Note that DGD^2 matches the rate of the centralized PGA and performs as good as NetLASSO without employing any gradient tracking.

(ii) Dependence on m and ρ. Fig. 1(c): We plot the same quantities as in Fig. 1(a), now parametrized on m, for fixed $d = 5000$, $s = \lceil \sqrt{d} \rceil$, and $N = 2500 = m \cdot n$ (this varying n). We choose $m \in \{50, 625, 1250, 2500\}$ and generate the underlying graph such that ρ varies to satisfy the required condition $\rho \lesssim m^{-2.5}$. For this, we set $p = 0.8$, resulting in a base graph with associated gossip $\bar{W}_m$; we then build $W_m = \bar{W}_m^T$, $T = 7$, and use such W_m in all the algorithms. **Fig. 1(d):** We use the same setup as in Fig. 1(c) with the difference that the underlying graphs are generated so that $\rho \approx 0.2$, for all m. To achieve this we set the link activation probabilities to $p \in \{0.87, 0.4, 0.22, 0.15\}$ for the associated values of m. **Discussion:** Fig. 1(c)-(d) demonstrates the necessity of a decreasing network connectivity ρ for increasing m and fixed N, as predicted by out theory. In fact, under $\rho \lesssim m^{-2.5}$, the convergence rate and statistical error remain roughly invariant, as m grows (Fig. 1(c)). On the other hand, all the algorithms break down if ρ does not decrease with m (Fig. 1(d)).

6 Concluding Remarks

We proposed DGD^2, a decentralized scheme solving high-dimensional quadratic M-estimation problems over mesh networks, which employs a double-mixing procedure averaging suitably iterates and local gradients. DGD^2 converges linearly to statistically optimal solutions, without requiring any condition on the local sample size (but centralized statistical consistency); differently from DGD-like schemes, the rate is independent on the ambient dimension (under standard assumptions on statistical consistency of the estimators), Quite interestingly our results show that, despite common wisdom, centralized statistical consistency can be achieved over networks without employing any explicit gradient tracking mechanism but instead just mixing local gradients at sufficiently fast rate. The analysis of other distributed algorithms for high-dimensional M-estimation problem remains a task of future investigations.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] The limitations of our work follow from the provided assumptions.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] See Supplementary Material
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [No]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 5
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [N/A]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] See Section 5
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   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
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5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
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   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

Appendix

This appendix contains additional theoretical and numerical results as well as the proofs of the results presented in the paper. The appendix is organized as follows.

Sec. A summarizes the notation used in the paper;

Sec. B contains additional numerical experiments;

Sec. C contains some consequences of Theorem 1 for (i) the ℓ_1 constrained sub-Gaussian regression problem under Assumption 5(ii) (Corollary 3); and (ii) and the nuclear norm constrained matrix regression problem under Assumption 5(ii) (Corollary 4). The proof of these corollaries along with Corollary 1 and Corollary 2 in the main text can be found in **Sec. G**;

Sec. D contains the main theorem of the paper (Theorem 2) along with its proof, which is a more general version of Theorem 1;

Sec. E contains the proof of some intermediate results stated in Sec. D and instrumental to prove Theorem 2;

Sec. F summarizes some concentration bounds regarding the discrepancy between empirical and population Hessian matrices, for the data models considered in Assumption 5, which are instrumental to show that Assumptions 3 and 4 hold with high probability; and

Sec. H studies statistical and computational guarantees of the DGD algorithm (4), applied to the ℓ_1-constrained sparse regression problem. This validates (5) (with a slightly better scaling with m) and complements results in [13] obtained for the DGD algorithm applied to a Lagrangian formulation of the LASSO problem over networks.

A Summary of the Main Notaton Adopted in the Paper

Problem size:

Symbols	Location	Description
d	Section 1	Problem dimension
n	(1)	Number of local samples
m	(1)	Number of agents
$N = n \cdot m$	Section 1	Total number of samples

Population and Empirical risk variants:

Symbols	Location	Description
$\mathcal{L}_i$	(1)	Local Empirical Loss
$\mathcal{L}$	(1)	Global Empirical Loss
$\mathcal{L}$	(2)	Population Loss
$\boldsymbol{\mathcal{L}}(\boldsymbol{\theta})$	(7)	Stacked empirical risk
$\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta})$	(16)	Staked population risk
$\bar{\boldsymbol{\mathcal{L}}}^{\alpha}(\boldsymbol{\theta})$	(16)	Augmented stacked population risk

Population curvature parameters:

Symbols	Location	Description
κ	(5), Assumption 1	Condition number
L, μ	Assumption 1	Lipschitz and strong convexity constants

Network quantities:

Symbols	Location	Description
ρ	(5), Assumption 6	Network connectivity
W, $\tilde{W}$	Assumption 6	Gossip matrices
$\mathbf{W}$, $\tilde{\mathbf{W}}$	Section 3	Stacked Gossip matrices
J	Section 3	$J = \frac{1}{m}\mathbf{1}\mathbf{1}_m^\top$
$\mathbf{J}$	Section 3	Staked fully connected network

Regularization quantities:

Symbols	Location	Description
$\mathcal{R}$	(1)	Norm constraint
R	(1)	Constraint radius
$\mathcal{M}$	Assumption 2	Model subspace
$\bar{\mathcal{M}}^\perp$	Assumption 2	Perturbation subspace
$\Psi^2(\mathcal{M})$	Assumption 2	$\mathcal{R} - \ell_2$ Lipschitz constant on $\mathcal{M}$
$\mathcal{R}^*$	Section 4	Dual norm to $\mathcal{R}$

Tolerances:

Symbols	Location	Description
$\gamma_{g,1},\ \gamma_{g,2},\ \gamma_{\ell,1},\ \gamma_{\ell,2}, \tau_g\Sigma'$	Assumption 3	Tolerances in global Hessian deviation
$\gamma_{\ell,1},\ \gamma_{\ell,2}, \tau_\ell$	Assumption 4	Tolerances in local Hessian deviations

Miscellanea:

Symbols	Location	Description
S_i	(1)	Indexes of samples corresponding to agent i
α	(4)	Step size

For any given matrix $A \in \mathbb{R}^{p\times p}$ we denote by

$$\|A\|_1 \triangleq \sum_{i,j=1}^{p} |a_{i,j}|, \qquad |||A|||_1 \triangleq \max_{1\le j\le p} \sum_{i=1}^{p} |a_{i,j}|,$$

$$\|A\|_F \triangleq \left(\sum_{i,j=1}^{p} |a_{i,j}|^2\right)^{1/2}, \qquad |||A|||_2 \triangleq \left(\lambda_{\max}(A^\top A)\right)^{1/2},$$

$$\|A\|_\infty \triangleq \max_{1\le i,j\le p} |a_{i,j}|, \qquad |||A|||_\infty \triangleq \max_{1\le j\le p} \left|\sum_{j=1}^{p} a_{i,j}\right|.$$

B Additional Numerical Experiments

This section contains additional numerical experiments on read and synthetic data.

1) Simulations on real data: We test DGD^2 on the data set `eyedata` in the `NormalBetaPrime` package [3]; $d = 200$ and $N = 120$. We generate a base graph with $m = 10$ and $p = 0.9$, yielding a gossip matrix $\bar{W}$. To achieve $\rho \approx m^{-2.5}$ we build $W = \bar{W}^7$. We split the data set into $N_{\text{test}} = 40$ and $N_{\text{train}} = 80$, and further split the train data evenly across 10 agents. Fig 1(e) plots the logarithm of the objective function value along the iterates generated by PGA (as benchmark), DGD^2 and NetLASSO. Observe that the distributed schemes converge linearly at the same rate as the centralized scheme. Fig 1(f) evaluates the prediction errors on the test set, as follows. Denote by y the output of the test set and by X the test covariates. Then, we build the predictors $y_i^t = X\theta_i^t$ $i \in [m]$, where θ_i^t is the estimate of $\theta^\star$ at agent i's side at iteration t based on the training set.

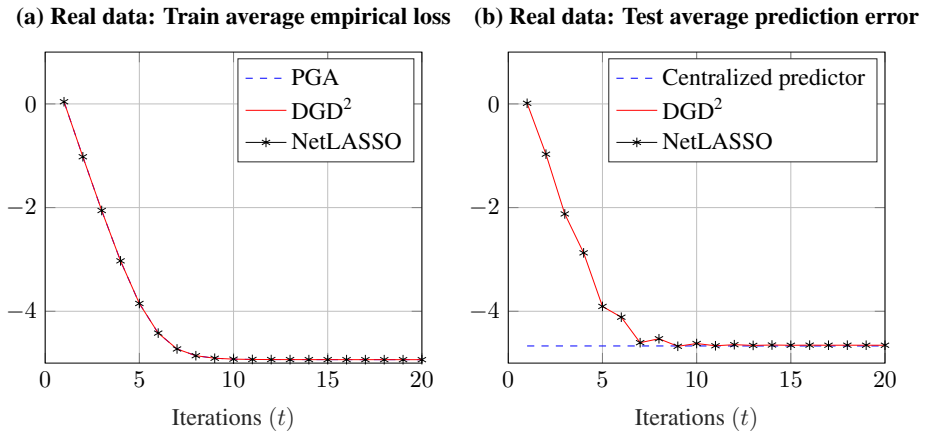

Figure 2: PGA, DGD^2 and NetLASSO on real data. **(a):** Rates (logarithm of objective function) vs. iterations; **(b):** Average prediction error on the test set (compared to centralized predictor on the test set).

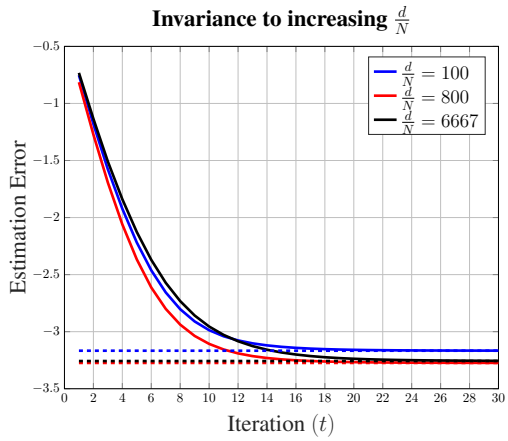

Figure 3: Estimation error generated by DGD^2 (solid line) vs. iterations, for different values of d/N and $s \log d/N = 0.0921$, $m = 5$; horizontal dashed line curves mark centralized statistical accuracy.

Fig 1(f) plots $\log\left((mN_{\text{test}})^{-1} \sum_{i=1}^{m} \|y - y_i^t\|_2^2\right)$ versus the iterations, for DGD^2 and NetLASSO. The dashed line-curve corresponds to $\log((N_{\text{test}}m)^{-1}\|y - \hat{y}\|_2^2)$ with $\hat{y} = X\hat{\theta}$, where $\hat{\theta}$ denotes the estimator obtained via PGA. As on synthetic data, both distributed schemes converge linearly up to the (estimated) statistical error (as our theory predicts), which matches centralized guarantees.

2) Synthetic data: We provide some additional numerical results (on synthetic data) validating the rate invariance under the asymptotic scaling s, d, N growing, for fixed m and $s \log d/N$. Specifically, we simulate the distributed $s-$sparse linear regression problem as described in Section 5; we set $q = 0$, $R_0 = s = 1$, $\Sigma = I_d$, $\sigma_d^2 = 0.25$ and $R = \|\theta^\star\|_1$. We use the same communication network and number of communications as in Fig. 1(a), and set the step size $\alpha = \frac{1}{2}$. We vary $d = \{10^4, 10^5, 10^6\}$ and $n = \{20, 25, 30\}$, so that $\frac{s \log d}{N} \approx 0.0921$ for the three cases. We observe that, as predicted by the theory, the performance (convergence rate and achieved statistical error) of DGD^2 remain invariant as d/N grows.

C Additional Statistical Models

In this section we apply Theorem 1 to the ℓ_1-constrained regression problem (Sec. C.1) and the nuclear norm constrained matrix regression (Sec. C.2), under the data model satisfying Assumption 5(ii) and (i), respectively.

C.1 Sub-Gaussian linear regression

Let $\theta^\star$ be (either hard-or weak-)sparse, which we capture by assuming $\|\theta^\star\|_q \leq R_q$, with $q \in [0,1]$. Consider the network data generation model

$$y_j = x_j^\top \theta^\star + w_j, \quad j \in S_i \quad \text{and} \quad i \in [m],$$

where w_j is assumed to be $w_j \in \mathcal{N}(0, \sigma^2)$. Denote

$$X_{S_i} \triangleq [x_{j_1}^\top, \ldots, x_{j_m}^\top]^\top, \quad j_l \in S_i, \quad l \in [n], \quad \text{and} \quad i \in [m],$$

where X_{S_i} is assumed to be C-column normalized.

Corollary 3 (ℓ_1-constrained sub-Gaussian regression)**.** *Consider the ERM* (1) *solving the linear regression problem above, with* $\mathcal{R}(\cdot) = \|\cdot\|_1$, *and* $\Omega \equiv \mathbb{R}^d$. *Suppose that*

$$\Psi^2(\bar{\mathcal{M}}) \cdot \tau_g = C_1 R_q \max\left\{\frac{\tau^4}{\mu}, \mu\right\} \left(\frac{\log d}{N}\right)^{1-\frac{q}{2}} \leq C_0, \quad q \in [0,1],$$

for some constants $C_0, C_1 > 0$. *Let* $\{(\theta_i^t)_{i=1}^m\}$ *be the sequence generated by DGD*^{2} *under the conditions of Theorem 2, where* (14) *becomes*

$$\rho \leq \frac{C_2}{m^{5/2}},$$

for some constant $C_2 > 0$. *Then, with probability at least*

$$1 - 2\left(-c_0 N \min\left\{\frac{\mu^2}{\tau^4}, 1\right\} + \log(m+1)\right) - 2d\exp(-2\log(d)) - 2md\exp(-2\log(md)),$$

for some constant $c_0 > 0$, *there holds*

$$r_t^2 \leq \left(\frac{1 - \frac{1}{8\kappa} - C_3\frac{\Psi^2(\bar{\mathcal{M}})\cdot\tau_g}{L}}{1 - C_3\frac{\Psi^2(\bar{\mathcal{M}})\cdot\tau_g}{L}}\right)^t r_0^2 + \mathcal{O}\left(\Delta^2\right) + o(\Delta^2),$$

where $C_3,\ C_4 > 0$ *and*

$$\Delta^2 = C_4 R_q \left(\frac{\log d}{N}\right)^{1-\frac{q}{2}} \left(\frac{C\sigma^2}{\mu^2} + \frac{C^{1/2}\sigma}{\mu}(1-\delta(q))\right) \quad q \in [0,1].$$

Proof. See Appendix G. □

C.2 Gaussian Matrix Regression

Consider the observation model

$$y_j = \langle X_j, \Theta^\star \rangle + w_j, \quad j \in S_i \quad \text{and} \quad i \in [m],$$

where $X_j \in \mathbb{R}^{p\times p}$ is such that $\text{vec}(X_j)$ satistifes Assumption 5(i), and $w_j \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. and independent of X_j,. The unknown matrix $\Theta^\star \in \mathbb{R}^{p\times p}$ satisfies $\|\Theta^\star\|_q \leq R_q$, with $q \in [0,1]$, and $R_q > 0$.

Corollary 4 (Nuclear norm constrained Gaussian Matrix Regression)**.** *Consider the ERM* (1) *solving the matrix regression problem above with* $\Omega \equiv \mathbb{R}^{p\times p}$, $\mathcal{R}(\Theta) = \|\Theta\|_1$, *and* $R = \|\Theta^\star\|_1$. *Suppose that*

$$\Psi^2(\bar{\mathcal{M}}) \cdot \tau_g = C_1 R_q \eta_\Sigma \sigma^2 \left(\frac{p}{N}\right)^{1-\frac{q}{2}} \leq C_0, \quad q \in [0,1],$$

for some $C_0,\ C_1 > 0$. *Let* $\{(\theta_i^t)_{i=1}^m\}_{t\geq 0}$ *be the sequence generated by DGD*^{2} *under the conditions of Theorem 1, where* (14) *reduces to*

$$\rho \leq C_2 \frac{\kappa^{-1}}{m^2\sqrt{m}},$$

for some constant $C_2 > 0$. Then, with probability at least

$$1 - c_0 m \exp\left(-\frac{N}{2}\right) - \exp(-c_1 N) - (m+1)\exp\left(-c_2 p\right),$$

for some $c_0,\ c_1,\ c_2 > 0$, there holds

$$r_t^2 \leq \left(\frac{1 - \frac{\kappa^{-1}}{8} + C_3 \frac{\Psi^2(\bar{\mathcal{M}})\cdot\tau_g}{L}}{1 - C_3 \frac{\Psi^2(\bar{\mathcal{M}})\cdot\tau_g}{L}}\right) + \mathcal{O}(\Delta^2) + o(\Delta^2),$$

where $C_3,\ C_4 > 0$ and

$$\Delta^2 = C_4 R_q \left(\frac{p}{M}\right)^{1-\frac{q}{2}} \left(\frac{\sigma^2 \eta_\Sigma}{\mu^2} + \frac{\sigma \eta_\Sigma^{1/2}}{\mu}(1-\delta(q))\right), \quad q \in [0,1],$$

where $\delta(q) = 1$ for $q = 0$ and 0 otherwise.

Proof. See Appendix G. □

Corollaries 3 and 4 establish for the statistical models above the same type of statistical computational guarantees as Corollary 1 and Corollary 2.

D Theorem 1 and some Generalizations

In this section we introduce and prove a more general version of Theorem 1, which will follow as a special case. We begin introducing some notation. Denote by

$$\Delta_{\text{stat}}^2 \triangleq \frac{64\Psi^2(\bar{\mathcal{M}})}{\mu^2}\left(\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\right)^2 + \frac{24}{\mu}\mathcal{R}\left(\Pi_{\mathcal{M}^\perp}(\theta^\star)\right)\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right),$$

$$\Delta_{\text{net}}^2 \triangleq \frac{64\Psi^2(\bar{\mathcal{M}})}{m\mu} 3\sum_{i=1}^m \left(\sum_{j=1}^m (w_{i,j} - 1/m)\nabla\mathcal{L}_j(\theta^\star)\right)^2 + \frac{24}{m\mu}\mathcal{R}\left(\Pi_{\mathcal{M}^\perp}(\theta^\star)\right)\sum_{i=1}^m \mathcal{R}^*\left(\sum_{j=1}^m (w_{i,j} - 1/m)\nabla\mathcal{L}_j(\theta^\star)\right),$$

and the overall tolerance

$$\Delta^2 \triangleq (\Delta_{\text{stat}}^2 + \Delta_{\text{net}}^2)\left(1 + \frac{\alpha\zeta(\bar{\mathcal{M}}, W)}{1 - \alpha\zeta(\bar{\mathcal{M}}, W)}\right) + \frac{144}{\mu\Psi^2(\bar{\mathcal{M}})}\mathcal{R}^2\left(\Pi_{\mathcal{M}^\perp}(\theta^\star)\right)\zeta(\bar{\mathcal{M}}, W)\left(1 + \frac{\alpha\zeta(c\bar{M}, W)}{1 - \alpha\,\zeta(\bar{\mathcal{M}}, W)}\right),$$

where we recall that α is the stepsize and

$$\zeta(\bar{\mathcal{M}}, W) \triangleq 24\tau_\ell \Psi^2(\bar{\mathcal{M}})(\rho^2 m^3 + 2\rho m\sqrt{m}) + 24\tau_g \Psi^2(\bar{\mathcal{M}}).$$

Finally, for any matrix $V \in \mathbb{R}^{k\times k}$, we denote by $\sigma_i(V)$, $i \in [k]$, its i-th largest singular value.

We are now ready to provide our main result.

Theorem 2. *Consider the ERM problem* (1) *and the associated population minimization* (2) *under Assumptions 1-4, with parameters therein such that the following hold: $R = \mathcal{R}(\theta^\star)$,*

$$\gamma_{g,1}\Sigma' + \frac{\mu}{2}I_d \preceq \nabla^2\mathcal{L} \quad \textit{and} \quad \zeta(\bar{\mathcal{M}}, W) \leq \frac{5\mu}{36}.$$

Let $\{\boldsymbol{\theta}^t\}_{t\geq 0}$ be the sequence generated by DGD^2 with stepsize

$$\alpha \leq \frac{1}{L + \gamma_{g,2}\sigma_1(\Sigma')},$$

and gossip matrix W satisfying Assumption 6 and such that

$$\rho \leq \frac{\mu}{108 m^{1.5}\sigma_1(\Sigma')\max\{\gamma_{\ell,1}, \gamma_{\ell,2}\}}.$$

Then, the estimation error $r^t \triangleq \frac{1}{m}\sum_{i=1}^m \|\theta_i^t - \theta^\star\|_2^2$ satisfies

$$r_t^2 \leq \left(\frac{1 - \frac{\alpha\mu}{2} + \alpha\sigma_1(\Sigma')3\gamma_\ell(2m\sqrt{m}\rho + \rho^2 m^3) + \alpha\zeta(\bar{\mathcal{M}}, W)}{1 - \alpha\zeta(\bar{\mathcal{M}}, W)}\right)^t r_0^2 + \Delta^2.$$

D.1 Proof of Theorem 2

The key idea in proving the theorem is characterizing the dynamics of the estimation error based on the favorable landscape of the population risk while carefully controlling the error due to the mismatch between the empirical and population gradients.

We begin introducing some notation. Recall that DGD^2 can be interpreted as the gradient descent applied to the proxy problem

$$\min_{\theta_i \in \mathbb{B}_{\mathcal{R}}(\rho) \cap \Omega} \quad \boldsymbol{\mathcal{L}}^{\alpha}(\boldsymbol{\theta}) \triangleq \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}) + \frac{1}{2\alpha m}\|\boldsymbol{\theta}\|^2_{I-\mathbf{W}^2}. \tag{18}$$

The population counterpart of (18) is based on the following population augmented loss:

$$\bar{\boldsymbol{\mathcal{L}}}^{\alpha}(\boldsymbol{\theta}) \triangleq \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}) + \frac{1}{2\alpha m}\|\boldsymbol{\theta}\|^2_{I-\mathbf{W}^2}, \tag{19}$$

where

$$\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}) \triangleq \frac{1}{m}\sum_{i=1}^{m} \bar{\mathcal{L}}(\theta_i).$$

Consequently,

$$\nabla \bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}) = \frac{1}{m}[\nabla \bar{\mathcal{L}}(\theta_1)^\top, \dots, \nabla \bar{\mathcal{L}}(\theta_m)^\top]^\top \quad \text{and} \quad \nabla^2 \bar{\boldsymbol{\mathcal{L}}} \triangleq \frac{1}{m}\nabla^2 \bar{\mathcal{L}} \otimes I_m = \frac{1}{m}\Sigma \otimes I_m.$$

Notice that, by construction, $\bar{\boldsymbol{\mathcal{L}}}^{\alpha}(\boldsymbol{\theta})$ inherits much of the properties of the population risk $\bar{\mathcal{L}}$; in particular,

$$\bar{\boldsymbol{\mathcal{L}}}^{\alpha}(\boldsymbol{\theta}') - \bar{\boldsymbol{\mathcal{L}}}^{\alpha}(\boldsymbol{\theta}) - \langle \nabla \bar{\boldsymbol{\mathcal{L}}}^{\alpha}(\boldsymbol{\theta}), \boldsymbol{\theta}' - \boldsymbol{\theta}\rangle = \frac{1}{2}\|\mathbf{W}(\boldsymbol{\theta} - \boldsymbol{\theta}')\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\alpha m}\|\boldsymbol{\theta} - \boldsymbol{\theta}'\|^2_{I-\mathbf{W}^2}, \tag{20}$$

for all $\boldsymbol{\theta}$, $\boldsymbol{\theta}' \in \mathbb{R}^{md}$. Also, $\boldsymbol{\theta}^\star$ is the unique minimizer of $\bar{\boldsymbol{\mathcal{L}}}^{\alpha}$ as well.

Our analysis begins exploiting the positive curvature of $\bar{\mathcal{L}}$ in (20) to establish a decrease of the estimation error along the trajectory $\{\boldsymbol{\theta}^t\}$ of DGD^2 (11), as stated next.

Lemma 2. *Under Assumptions 1 and 6, the iterates $\{\boldsymbol{\theta}^t\}_{t\geq 0}$ generated by DGD^2 (11) for any $\alpha > 0$, satisfy*

$$\begin{aligned}\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 \leq\ & \|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star)\|^2_{\mathbf{I}-\alpha m \nabla^2 \bar{\boldsymbol{\mathcal{L}}}} - \alpha m \|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} \\ & - \|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|^2_{I-\alpha m \nabla^2 \bar{\boldsymbol{\mathcal{L}}}} - \|\boldsymbol{\theta}^{t+1}\|^2_{\mathbf{I}-\mathbf{W}^2} \\ & + 2\alpha m \langle \mathbf{W}\left(\nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)\right), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle.\end{aligned} \tag{21}$$

Proof. See Appendix E. □

Notice that the first term on the RHS of (21) captures the contraction properties of the gradient algorithm applied to the population loss while the inner product accounts for the trajectory mismatch between the empirical and population-based updates along sparse directions. The rest of the proof consists in suitably bounding this inner-product term.

Rewrite (21) as

$$\begin{aligned}&\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 \leq \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 - \alpha m \left(\|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}\right) \\ &- \left(\|\boldsymbol{\theta}^t\|^2_{I-\mathbf{W}^2} + \|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2}\right) + \alpha m \|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} - \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2_{\mathbf{W}^2} \\ &+ 2\alpha m \langle \mathbf{W}(\nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t\rangle.\end{aligned}$$

Using $\nabla^2 \bar{\boldsymbol{\mathcal{L}}} \preceq \frac{L}{m} I_{md}$ we have

$$\begin{aligned}&\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 \leq \|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 - \alpha m \left(\|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}\right) \\ &- \|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2} + (\alpha L - 1)\|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|_2^2 + 2\alpha m \langle \mathbf{W}\left(\nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)\right), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle.\end{aligned} \tag{22}$$

When $\alpha \leq 1/L$, the only detrimental term on the RHS of the above inequality is the inner product; we aim at controlling this term using Assumptions 3 and 4. To do so, we add and subtract therein

$$2\alpha m\langle \nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) - \nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star), \mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle,$$

so that

$$\begin{aligned}&2\alpha m\langle \mathbf{W}(\nabla\bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla\boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) + \nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle = \\ &2\alpha m\langle (\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star), \mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle,\end{aligned}$$

and consequently

$$\begin{aligned}&2\alpha m\langle \mathbf{W}\left(\nabla\bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla\boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)\right), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle \\ &= 2\alpha m\langle (\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star), \mathbf{J}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle && (23)\\ &\quad + 2\alpha m\langle (\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star), \mathbf{J}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle && (24)\\ &\quad + 2\alpha m\langle (\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star), (\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle && (25)\\ &\quad + 2\alpha m\langle (\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star), (\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle && (26)\\ &\quad + 2\alpha m\langle \mathbf{W}\left(\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) - \nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)\right), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle.\end{aligned}$$

The above factorization decomposes the quantities in the inner product in two contributions, those along the consensus directions $\mathbf{J}\boldsymbol{\theta}^{t+1}$ and $\mathbf{J}\boldsymbol{\theta}^{t+1}$, and those in the orthogonal space. Terms along the consensus directions are bounded employing Lemma 3 (see Sec. E) while those orthogonal to consensus, i.e. dependent on $(\mathbf{W} - \mathbf{J})$, are bounded using Lemma 4 (see Sec. E).

More specifically, denote by $\gamma_\ell \triangleq \max\{\gamma_{\ell,1}, \gamma_{\ell,2}\}$; invoking Lemma 3 and Lemma 4, we have

$$\begin{aligned}&2\alpha m\langle \mathbf{W}(\nabla\bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla\boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle \\ &\alpha\gamma_{g,1}\left(\|\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'} + \|\mathbf{J}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}\right) + \alpha\gamma_{g,2}\|\mathbf{J}\boldsymbol{\theta}^t - \mathbf{J}\boldsymbol{\theta}^{t+1}\|^2_{\boldsymbol{\Sigma}'} \\ &+ \gamma\left(6m\sqrt{m}\rho\sigma_1(\Sigma')\gamma_\ell + 3\rho^2m^3\sigma_1(\Sigma')\gamma_\ell\right)\left(\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 + \|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2\right) \\ &+ 3\alpha m\tau_g\left(\mathcal{R}^2(\theta^t_{\text{av}} - \theta^\star) + \mathcal{R}^2(\theta^{t+1}_{\text{av}} - \theta^\star)\right) \\ &+ \alpha 3\left(\rho^2m^3 + \rho m\sqrt{m}\right)\sum_{i=1}^m \tau_\ell\left(\mathcal{R}^2(\theta^t_i - \theta^\star) + \mathcal{R}^2(\theta^{t+1}_i - \theta^\star)\right) && (27)\\ &+ \alpha 3m\sqrt{m}\rho\sum_{i=1}^m \tau_\ell\left(\mathcal{R}^2(\theta^t_{\text{av}} - \theta^\star) + \mathcal{R}^2(\theta^{t+1}_{\text{av}} - \theta^\star)\right) \\ &+ 2\alpha m\langle \mathbf{W}(\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) - \nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle.\end{aligned}$$

Observe that because $\theta^\star$ is the solution to (19) $\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) = 0$.

In order to relate $\mathcal{R}(\cdot)$ to $\|\cdot\|_2$, we can proceed as in [2, Lemma 1] and write: for any $a > 0$,

$$\mathcal{R}^2(\theta^t_i - \theta^\star) \leq 4(1+a)\Psi^2(\bar{\mathcal{M}})\|\theta - \theta^\star\|_2^2 + 4\left(1 + \frac{1}{a}\right)\mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star)), \quad \forall i \in [m]. \tag{28}$$

For convenience, we set $a = 1$ and define

$$\eta \triangleq 2\mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star)).$$

The analogous to (28) holds for θ^{t+1}_i, and θ^t_{av} and θ^{t+1}_{av} (the latter due to the convexity of the norm-ball). Consequently

$$\begin{aligned}&\mathcal{R}^2(\theta^t_{\text{av}} - \theta^\star) + \mathcal{R}^2(\theta^{t+1}_{\text{av}} - \theta^\star) \leq 8\Psi^2(\bar{\mathcal{M}})(\|\theta^t_{\text{av}} - \theta^\star\|_2^2 + \|\theta^{t+1}_{\text{av}} - \theta^\star\|_2^2) + 8\eta \\ &= \frac{8\Psi^2(\bar{\mathcal{M}})}{m}(\|\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 + \|\mathbf{J}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2) + 8\eta, && (29a)\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^m\left(\mathcal{R}^2(\theta^t_i - \theta^\star) + \mathcal{R}^2(\theta^{t+1}_i - \theta^\star)\right) \leq \sum_{i=1}^m 8\Psi^2(\bar{\mathcal{M}})\|\theta^t_i - \theta^\star\|_2^2 + \sum_{j=1}^m 8\Psi^2(\bar{\mathcal{M}})\|\theta^{t+1}_j - \theta^\star\|_2^2 \\ &+ 8m\eta = 8\Psi^2(\bar{\mathcal{M}})\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 + 8\Psi^2(\bar{\mathcal{M}})\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 + 8m\eta. && (29b)\end{aligned}$$

Combining (27) and (29) and rearranging terms, yields

$$
\begin{aligned}
&2\alpha m\langle \mathbf{W}(\nabla^2\bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t)-\nabla^2\boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)),\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle \le \alpha\gamma_{g,1}(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}+\|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'})\\
&+\alpha\gamma_{g,2}\|\mathbf{J}\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^{t+1}\|^2_{\boldsymbol{\Sigma}'}\\
&+\alpha\left(6m\sqrt{m}\rho\sigma_1(\Sigma')\gamma_\ell+3\rho^2m^4\sigma_1(\Sigma')\gamma_\ell\right)\left(\|\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|_2^2+\|\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2\right)\\
&+\alpha\left(24(\rho^2m^3+2\rho m\sqrt{m})\tau_\ell\Psi^2(\bar{\mathcal{M}})\right)\left(\|\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|_2^2+\|\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2\right)\\
&+24\alpha\tau_g\Psi^2(\bar{\mathcal{M}})\left(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|_2^2+\|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2\right)+\alpha\eta m\left(24\tau_g+12\tau_\ell(\rho^2m^3+2\rho m\sqrt{m})\right)\\
&+2\alpha\langle\mathbf{W}(\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)),\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle. \qquad (30)
\end{aligned}
$$

We now focus on the remaining inner product, which is the main contributor to the final tolerance. We separate the remaining inner product in two terms: a part that will correspond to the centralized achievable statistical accuracy and an additional one which will represent the cost of decentralization. Specifically,

$$
\begin{aligned}
2\alpha m\langle\mathbf{W}(-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)),\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle &= 2\alpha m\langle-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star),\mathbf{J}(\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star)\rangle\\
&+2\alpha m\langle(\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star)\rangle.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\langle-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star),\mathbf{J}(\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star)\rangle=\langle-\nabla\mathcal{L}(\theta^\star),\theta^{t+1}_{\text{av}}-\theta^\star\rangle\\
&\langle-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star)\rangle=-\frac{1}{m}\sum_{i=1}^m\left\langle\sum_{j=1}^m(w_{i,j}-1/m)\,\nabla\mathcal{L}_j(\theta^\star),\theta_i^{t+1}-\theta^\star\right\rangle.
\end{aligned}
$$

Therefore, using Hölder's inequality

$$
\begin{aligned}
2\alpha m\langle\mathbf{W}(-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)),\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle &\le 2\alpha m\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\mathcal{R}(\theta^{t+1}_{\text{av}}-\theta^\star)\\
&+2\alpha\sum_{i=1}^m\mathcal{R}^*\left(\sum_{j=1}^m(w_{i,j}-1/m)\nabla\mathcal{L}_j(\theta^\star)\right)\mathcal{R}(\theta_i^{t+1}-\theta^\star).
\end{aligned}
$$

From [2, Lemma 1] we deduce

$$
\mathcal{R}(\theta-\theta^\star)\le 2\Psi(\bar{\mathcal{M}})\|\theta-\theta^\star\|_2+2\mathcal{R}(\Pi^\perp_{\mathcal{M}}(\theta^\star)).
$$

Then, it holds

$$
\begin{aligned}
&2\alpha m\langle\mathbf{W}(-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)),\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle\le 4\alpha m\Psi(\bar{\mathcal{M}})\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\|\bar{\theta}^{t+1}-\theta^\star\|_2\\
&+4\alpha\Psi(\bar{\mathcal{M}})\sum_{i=1}^m\|\theta_i^{t+1}-\theta^\star\|\mathcal{R}^*\left(\sum_{j\in\mathcal{N}_i}(w_{i,j}-1/m)\nabla\mathcal{L}_j(\theta^\star)\right)\\
&+4\gamma\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\left(m\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)+\sum_{i=1}^m\mathcal{R}^*\left(\sum_{j\in\mathcal{N}_i}(w_{i,j}-1/m)\nabla\mathcal{L}_j(\theta^\star)\right)\right).
\end{aligned}
$$

Applying Young's inequality we have

$$
\begin{aligned}
&2\alpha m\langle\mathbf{W}(\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star)-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)),\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle\le\alpha m\frac{2}{\varepsilon}\Psi^2(\bar{\mathcal{M}})\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))^2+2\varepsilon\alpha\|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2\\
&+2\frac{\alpha}{\epsilon}\sum_{i=1}^m\left(\Psi(\bar{\mathcal{M}})\mathcal{R}^*\left(\left(\sum_{j\in\mathcal{N}_i}w_{i,j}-1/m\right)\nabla\mathcal{L}_j(\theta^\star)\right)\right)^2+2\alpha\epsilon\|\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2 \qquad (31)\\
&+4\alpha\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\left(m\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)+\sum_{i=1}^m\mathcal{R}^*\left(\left(\sum_{j\in\mathcal{N}_i}w_{i,j}-1/m\right)\nabla\mathcal{L}_j(\theta^\star)\right)\right),
\end{aligned}
$$

for any $\varepsilon,\ \epsilon>0$.

For convenience, let $R_t^2 \triangleq \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2$. Combining the bounds (22), (30) and (31) yields

$$
\begin{aligned}
R_{t+1}^2 &\leq \|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 - \alpha m \left(\|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star)\|_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}^2 - \frac{\gamma_{g,1}}{m} \|\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}'}^2 + \|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\|_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}^2 \right) \\
&+ 2\alpha\left(\varepsilon + \epsilon\right) R_{t+1}^2 + \alpha m \frac{\gamma_{g,1}}{m} \|\mathbf{J}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}'}^2 + (\gamma L - 1)\left\|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1})\right\|_2^2 \\
&- \|\boldsymbol{\theta}^{t+1}\|_{\mathbf{I}-\mathbf{W}^2}^2 + \alpha \left(24 \left(\rho^2 m^3 + \rho m \sqrt{m} + m\sqrt{m}\rho + \frac{\tau_g}{\tau_\ell} \right) \tau_\ell \Psi^2(\bar{\mathcal{M}}) \right) (R_t^2 + R_{t+1}^2) \\
&+ \alpha \left(6m\sqrt{m}\rho\sigma_1(\Sigma')\gamma_\ell + 3\rho^2 m^3 \sigma_1(\Sigma')\gamma_\ell \right) (R_t^2 + R_{t+1}^2) + \alpha\gamma_{g,2} \|\mathbf{J}\boldsymbol{\theta}^t - \mathbf{J}\boldsymbol{\theta}^{t+1}\|_{\boldsymbol{\Sigma}'}^2 \\
&+ 2\alpha m \Psi^2(\bar{\mathcal{M}}) \frac{\left(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\right)^2}{\varepsilon} + 2\alpha\Psi^2(\bar{\mathcal{M}}) \frac{\sum_{i=1}^m \left(\mathcal{R}^* \left(\sum_{j=1}^m (w_{i,j} - 1/m) \nabla\mathcal{L}_j(\theta^\star) \right) \right)^2}{\epsilon} \\
&+ \alpha\eta m \left(12\tau_\ell(\rho^2 m^3) \right) + \alpha\eta m \left(2\rho m\sqrt{m} + 24\tau_g \right) \\
&+ 4\alpha\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)) \left(m\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star)) + \sum_{i=1}^m \mathcal{R}^* \left(\left(\sum_{j=1}^m w_{i,j} - 1/m \right) \nabla\mathcal{L}_j(\theta^\star) \right) \right).
\end{aligned}
$$

Observe that

$$
\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}'}^2 = \|\mathbf{J}\mathbf{W}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}}^2 \leq \|\mathbf{W}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}}^2.
$$

Under $\nabla^2\bar{\mathcal{L}} - \gamma_{g,1}\Sigma' \succeq \frac{\mu}{2} I_d$, and using $\|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2 \leq \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2$ and $\mathbf{W}^2 \succeq \mathbf{J}$ along with the $\frac{\mu}{m}-$strong convex of $\bar{\boldsymbol{\mathcal{L}}}$, we obtain

$$
\begin{aligned}
R_{t+1}^2 &\leq \left(1 - \alpha\frac{\mu}{2}\right) R_t^2 - \alpha\frac{\mu}{2} \|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 - \|\boldsymbol{\theta}^{t+1}\|_{\mathbf{I}-\mathbf{W}^2}^2 \\
&+ \left(\alpha L + \alpha\gamma_{g,2}\sigma_1(\Sigma') - 1\right) \|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1})\|^2 \\
&+ 2\alpha(\varepsilon + \epsilon) R_{t+1}^2 + 2\alpha\Psi^2(\bar{\mathcal{M}}) \left(m \frac{(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star)))^2}{\varepsilon} + \frac{\sum_{i=1}^m \left(\sum_{j=1}^m (w_{i,j} - 1/m) \nabla\mathcal{L}_i(\theta^\star) \right)^2}{\epsilon} \right) \\
&+ \alpha \left(12(1+a) \left(\rho^2 m^3 + \rho m\sqrt{m} + m\sqrt{m}\rho + \frac{\tau_g}{\tau_\ell} \right) \tau_\ell \Psi^2(\bar{\mathcal{M}}) \right) (R_t^2 + R_{t+1}^2) \\
&+ \alpha \left(6m\sqrt{m}\rho\sigma_1(\Sigma')\gamma_\ell + 3\rho^2 m^3 \sigma_1(\Sigma')\gamma_\ell \right) (R_{t+1}^2 + R_t^2) + \alpha\eta m \left(24\tau_g + 12\tau_\ell(\rho^2 m^3 + 2\rho m\sqrt{m}) \right) \\
&+ 4\alpha\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)) \left(m\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star)) \right) + 4\alpha\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)) \sum_{i=1}^m \mathcal{R}^* \left(\left(\sum_{j=1}^m w_{i,j} - 1/m \right) \nabla\mathcal{L}_j(\theta^\star) \right).
\end{aligned}
$$

Under $\alpha \leq \frac{1}{L}$, which implies $\alpha\frac{\mu}{2} < 1$, and using $\|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2 \leq \|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2$, yields

$$
-\alpha\frac{\mu}{2} \|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2 - \|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_{\mathbf{I}-\mathbf{W}^2}^2 \leq -\alpha\frac{\mu}{2} \|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 = -\alpha\frac{\mu}{2} R_{t+1}^2.
$$

Choosing

$$
\begin{aligned}
&\varepsilon = \frac{3\mu}{16}, && \epsilon = \frac{\mu}{32}, \\
&3\gamma_\ell\sigma_1(\Sigma')m^{1.5}\rho(2 + \rho m^{1.5}) \leq \frac{\mu}{36}, && \alpha \leq \frac{1}{L + \gamma_{g,2}\sigma_1(\Sigma')},
\end{aligned} \tag{32}
$$

we have

$$R_{t+1}^2 \leq \left(1-\alpha\frac{\mu}{2}+\alpha 6m\sqrt{m}\rho\sigma_1(\Sigma')\gamma_\ell+3\alpha\rho^2m^3\sigma_1(\Sigma')\gamma_\ell\right)R_t^2+\alpha 24\left(\tau_g\Psi^2(\bar{\mathcal{M}})\right)\left(R_t^2+R_{t+1}^2\right)$$
$$+\alpha\left(24\left(\rho^2m^3+2\rho m\sqrt{m}\right)\tau_\ell\Psi^2(\bar{\mathcal{M}})\right)\left(R_t^2+R_{t+1}^2\right)+\alpha\eta m\left(24\tau_g+24\tau_\ell\rho m\sqrt{m}\right)$$
$$+32\alpha m\Psi^2(\bar{\mathcal{M}})\left(\frac{(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star)))^2}{3\mu}+\frac{\sum_{i=1}^m\left(\mathcal{R}^*(\sum_{j=1}^m(w_{i,j}-1/m)\nabla\mathcal{L}_j(\theta^\star))\right)^2}{m\mu}\right)$$
$$+4\alpha\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\left(m\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))+\sum_{i=1}^m\mathcal{R}^*\left(\left(\sum_{j=1}^m w_{i,j}-1/m\right)\nabla\mathcal{L}_j(\theta^\star)\right)\right)+\alpha\eta(12\tau_\ell\rho^2m^3).$$

If $1-\alpha\zeta>0$, which is implied by

$$\zeta \triangleq 24\tau_\ell\Psi^2(\bar{\mathcal{M}})\left(\rho^2m^3+2\rho m\sqrt{m}+\frac{\tau_g}{\tau_\ell}\right)<\frac{1}{\alpha}, \tag{33}$$

and one divides both sides by $1-\alpha\zeta$, yields

$$R_{t+1}^2 \leq \frac{\left(1-\alpha\frac{\mu}{2}+\alpha 3\sigma_1(\Sigma')\gamma_\ell(2m\sqrt{m}\rho+\rho^2m^3)+\alpha\zeta\right)}{1-\alpha\zeta}r_t^2$$
$$+\frac{24\alpha m\mathcal{R}^2\left(\Pi_{\mathcal{M}^\perp}(\theta^\star)\right))\left(2\tau_g+\tau_\ell(\rho^2m^3+2\rho m\sqrt{m})\right)}{1-\alpha\zeta}$$
$$+\frac{32\alpha m\Psi^2(\bar{\mathcal{M}})}{1-\alpha\zeta}\left(\frac{(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star)))^2}{3\mu}+\frac{\sum_{i=1}^m\left(\mathcal{R}^*\left(\sum_{j=1}^m(w_{i,j}-1/m)\nabla\mathcal{L}_j(\theta^\star)\right)\right)^2}{m\mu},\right)$$
$$+\frac{4\alpha m\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))}{1-\alpha\zeta}\left(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))+\frac{1}{m}\sum_{i=1}^m\mathcal{R}^*\left(\left(\sum_{j=1}^m w_{i,j}-1/m\right)\nabla\mathcal{L}_j(\theta^\star)\right)\right).$$

Contraction (up to some tolerance) is guaranteed under

$$1-\alpha\frac{\mu}{2}+\alpha 3\sigma_1(\Sigma')\gamma_\ell(2m\sqrt{m}\rho+\rho^2m^3)+\alpha\zeta<1-\alpha\zeta,$$

enforced via

$$\zeta<\frac{2}{9}\mu. \tag{34}$$

Observe that

$$\frac{1}{1-\alpha\zeta}=1+\frac{\alpha\zeta}{1-\alpha\zeta}.$$

Let

$$\lambda \triangleq \frac{1-\alpha\frac{\mu}{2}+\alpha\zeta+3\alpha\sigma_1(\Sigma')\gamma_\ell(2m\sqrt{m}\rho+\rho^2m^3)}{1-\alpha\zeta}.$$

Then, it holds

$$R_{t+1}^2 \leq \lambda R_t^2+\frac{32\alpha m\Psi^2(\bar{\mathcal{M}})}{\mu}\left(\frac{\left(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\right)^2}{3}+\frac{1}{m}\sum_{i=1}^m\left(\mathcal{R}^*\left(\sum_{j=1}^m(w_{i,j}-1/m)\nabla\mathcal{L}_j(\theta^\star)\right)\right)^2\right)\times$$
$$\times\left(1+\frac{\alpha\zeta}{1-\alpha\zeta}\right)+\left(1+\frac{\alpha\zeta}{1-\alpha\zeta}\right)\left(24\alpha m\mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star))(2\tau_g+\tau_\ell(\rho^2m^3+2\rho m\sqrt{m}))\right)$$
$$+\left(1+\frac{\alpha\zeta}{1-\alpha\zeta}\right)4\alpha\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\left(m\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))+\sum_{i=1}^m\mathcal{R}^*\left(\left(\sum_{j=1}^m w_{i,j}-1/m\right)\nabla\mathcal{L}_j(\theta^\star)\right)\right).$$

Telescoping, we obtain

$$R_{t+1}^2 \leq \lambda^{t+1} R_0^2 + \frac{32\alpha m \Psi^2(\bar{\mathcal{M}})}{\mu(1-\lambda)} \left(\frac{(\mathcal{R}^*(\nabla \mathcal{L}(\theta^\star)))^2}{3} + \frac{1}{m} \sum_{i=1}^m \left(\mathcal{R}^* \left(\sum_{j=1}^m (w_{i,j} - 1/m) \nabla \mathcal{L}_n^{(j)}(\theta^\star) \right) \right)^2 \right) \times$$

$$\times \left(1 + \frac{\alpha\zeta}{1-\alpha\zeta}\right) + (1-\lambda)^{-1} \left(1 + \frac{\alpha\zeta}{1-\alpha\zeta}\right) \left(24\alpha m \mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star))(2\tau_g + \tau_\ell(\rho^2 m^3 + 2\rho m\sqrt{m}))\right)$$

$$+ \left(1 + \frac{\alpha\zeta}{1-\alpha\zeta}\right) \frac{4}{1-\lambda} \mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)) \left(m\mathcal{R}^*(\nabla \mathcal{L}(\theta^\star)) + \sum_{i=1}^m \mathcal{R}^* \left(\left(\sum_{j=1}^m w_{i,j} - 1/m \right) \nabla \mathcal{L}_j(\theta^\star) \right) \right).$$

We further require

$$(1-\lambda) \geq \frac{\alpha\mu}{6},$$

implied by

$$\zeta + 3\sigma_1(\Sigma')\gamma_\ell(2m\sqrt{m}\rho + \rho^2 m^3) \leq \frac{\mu}{6}.$$

A sufficient condition for the above is

$$3\gamma_\ell \sigma_1(\Sigma') m^{1.5} \rho(2 + \rho\, m^{1.5}) \leq \frac{\mu}{36},$$
$$\zeta \leq \frac{5\mu}{36},$$

which is compatible with requirements (32),(33) and (34), and fulfilled whenever

$$\rho \leq \frac{\mu}{108\sigma_1(\Sigma')\gamma_\ell m^{1.5}}.$$

Dividing by m and observing that $R_t^2 = m r_t^2$ yields the desired result.

E Technical Lemmata

In this section we provide the proofs of some technical intermediate results, used in the proof of Theorem 2.

E.1 Proof of Lemma 2

We are interested in establishing an upper bound on the estimation error $\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2$ based on that at time t. Observe that

$$\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2 = \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2 + 2\langle \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t, \boldsymbol{\theta}^t - \boldsymbol{\theta}^\star \rangle + \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2. \tag{35}$$

We proceed to upper bound the latter two terms.

From (20) it follows that

$$\frac{1}{2}\|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^t\|^2_{\mathbf{I}-\mathbf{W}^2} \leq \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^\star) - \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^t) - \langle \nabla \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^t), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t \rangle$$
$$- \langle \nabla \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^t), \boldsymbol{\theta}^\star - \boldsymbol{\theta}^{t+1} \rangle.$$

Furthermore, still from (20),

$$\frac{1}{2}\|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^t\|^2_{I-\mathbf{W}^2} \leq \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^\star) - \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^{t+1}) + \frac{1}{2}\|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}$$
$$+ \frac{1}{2\alpha m}\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2} - \langle \nabla \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^t), \boldsymbol{\theta}^\star - \boldsymbol{\theta}^{t+1} \rangle.$$

Observe that

$$\bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^\star) = \bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) \quad \text{and} \quad \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^{t+1}) = \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^{t+1}) + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2}.$$

Furthermore, since

$$\bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^{t+1}) \geq \bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star) + \langle \nabla \bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star), \mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star)\rangle + \frac{1}{2}\|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}$$

and $\langle \nabla \bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star), \mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle \geq 0$, it holds

$$\frac{1}{2}\|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^t\|^2_{I-\mathbf{W}^2} \leq -\frac{1}{2}\|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} - \frac{1}{2\alpha m}\|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2} \tag{36}$$

$$+ \frac{1}{2}\|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2} + \langle \nabla \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^t), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle. \tag{37}$$

By the optimality of $\boldsymbol{\theta}^{t+1}$,

$$\boldsymbol{\theta}^{t+1} = \underset{\theta_i \in \Omega\,:\, \mathcal{R}(\theta_i) \leq R, \forall i \in [m]}{\operatorname{argmin}} \left\{ \langle \nabla \boldsymbol{\mathcal{L}}^\alpha(\boldsymbol{\theta}^t), \boldsymbol{\theta} - \boldsymbol{\theta}^t\rangle + \frac{1}{2\alpha m}\|\boldsymbol{\theta} - \boldsymbol{\theta}^t\|^2 \right\},$$

it follows

$$\langle \nabla \bar{\boldsymbol{\mathcal{L}}}^\alpha(\boldsymbol{\theta}^t) - \mathbf{W}\left(\nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)\right) + \frac{1}{\alpha m}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t), \boldsymbol{\theta} - \boldsymbol{\theta}^{t+1}\rangle \geq 0, \tag{38}$$

for any feasible $\boldsymbol{\theta}$. Setting, in particular, $\boldsymbol{\theta} = \boldsymbol{\theta}^\star$ and combining (38) with (36), yield

$$\begin{aligned}
&\frac{1}{\alpha m}\langle \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t, \boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\rangle + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t\|_2^2 \leq -\frac{1}{2}\left(\|\mathbf{W}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}\right) \\
&- \frac{1}{2\alpha m}\left(\|\boldsymbol{\theta}^t\|^2_{I-\mathbf{W}^2} + \|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2}\right) + \frac{1}{2}\|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\alpha m}\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2} \\
&- \frac{1}{2\alpha m}\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t\|_2^2 + \langle \mathbf{W}\left(\nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \nabla \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)\right), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle.
\end{aligned}$$

Multiplying the above by $2\alpha m$ and combining with (35) yield

$$\begin{aligned}
&\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_2^2 \leq \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2 - \alpha m\left(\|\mathbf{W}(\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} + \|\mathbf{W}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}}\right) \\
&- \|\boldsymbol{\theta}^t\|^2_{I-\mathbf{W}^2} - \|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}^2} \\
&+ \alpha m\|\mathbf{W}(\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t)\|^2_{\nabla^2 \bar{\boldsymbol{\mathcal{L}}}} - \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2_{\mathbf{W}^2} \\
&+ 2\alpha m\langle \mathbf{W}\left(\nabla \bar{\boldsymbol{\mathcal{L}}}(\mathbf{W}\boldsymbol{\theta}^t) - \boldsymbol{\mathcal{L}}(\mathbf{W}\boldsymbol{\theta}^t)\right), \boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\rangle.
\end{aligned}$$

E.2 Bounds on inner product elements (23)–(26)

Lemma 3. *Under Assumption 1 and 3 it holds*

$$\langle (\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), \mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star\rangle \leq \frac{\gamma_{g,1}}{2m}\left(\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'} + \|\mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}\right) + \frac{\gamma_{g,2}}{2m}\|\mathbf{J}\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}'\|^2_{\boldsymbol{\Sigma}'} \tag{39}$$

$$+ \frac{3}{2}\tau_g\left(\mathcal{R}^2(\theta_{\text{av}} - \theta^\star) + \mathcal{R}^2(\theta'_{\text{av}} - \theta^\star)\right),$$

where $\boldsymbol{\Sigma}' \triangleq \Sigma' \otimes I_m$, $\theta_{\text{av}} \triangleq \frac{1}{m}\sum_{i=1}^m \theta_i$ *and analogousloy for* θ'_{av}.

Proof. Observe that the RHS of (39) can be expanded as

$$\begin{aligned}
&\langle (\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), \mathbf{J}(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle = \frac{1}{2}\langle \mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star, (\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star)\rangle \\
&+ \frac{1}{2}\langle \mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star, (\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle - \frac{1}{2}\langle \mathbf{J}\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}', (\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}')\rangle \\
&= \frac{1}{2}\langle (\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L})(\theta_{\text{av}} - \theta^\star), \theta_{\text{av}} - \theta^\star\rangle + \frac{1}{2}\langle (\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L})(\theta'_{\text{av}} - \theta^\star), \theta'_{\text{av}} - \theta^\star\rangle \\
&- \frac{1}{2}\langle (\nabla^2 \bar{\mathcal{L}} - \nabla^2 \mathcal{L})(\theta_{\text{av}} - \theta'_{\text{av}}, \theta_{\text{av}} - \theta'_{\text{av}}\rangle.
\end{aligned}$$

From Assumption 3 it follows that

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star),\mathbf{J}(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle \leq \frac{1}{2}\gamma_{g,1}\left(\|\theta_{\rm av}-\theta^\star\|^2_{\Sigma'}+\|\theta'_{\rm av}-\theta^\star\|^2_{\Sigma'}\right)$$
$$+\frac{1}{2}\tau_g\left(\mathcal{R}^2(\theta_{\rm av}-\theta^\star)+\mathcal{R}^2(\theta'_{\rm av}-\theta^\star)\right)+\frac{1}{2}\gamma_{g,2}\|\theta_{\rm av}-\theta'_{\rm av}\|^2_{\Sigma'}+\frac{1}{2}\tau_g\mathcal{R}^2(\theta_{\rm av}-\theta'_{\rm av}).$$

Then, using triangle and Young's inequality on $\mathcal{R}^2(\theta_{\rm av}-\theta'_{\rm av})$ we obtain

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star),\mathbf{J}(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle \leq \frac{\gamma_{g,1}}{2m}(\|\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}+\|\mathbf{J}\boldsymbol{\theta}'-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'})+\frac{\gamma_{g,2}}{2m}\|\mathbf{J}\boldsymbol{\theta}-\mathbf{J}\boldsymbol{\theta}'\|^2_{\boldsymbol{\Sigma}'}$$
$$+\frac{1}{2}\tau_g\left(\mathcal{R}^2(\theta_{\rm av}-\theta^\star)+\mathcal{R}^2(\theta'_{\rm av}-\theta^\star)\right).$$

□

Lemma 4. *Under Assumptions 1, 3 4, and 6, and given $\gamma_\ell \triangleq \max\{\gamma_{\ell,1},\gamma_{\ell,2}\}$, there holds*

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle \leq \frac{3\sqrt{m}\rho}{2}\gamma_\ell\left(\|\boldsymbol{\theta}-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}+\|\boldsymbol{\theta}'-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}\right) \quad (40)$$
$$+\frac{3\sqrt{m}\rho}{2}\sum_{j=1}^m\tau_\ell\left(\mathcal{R}^2(\theta_{\rm av}-\theta^\star)+\mathcal{R}^2(\theta'_j-\theta^\star)\right)$$

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}-\boldsymbol{\theta}^\star),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle \leq \frac{3\rho^2m^2\gamma_\ell}{2}\left(\|\boldsymbol{\theta}-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}+\|\boldsymbol{\theta}'-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}\right) \quad (41)$$
$$+\frac{3\rho^2m^2}{2}\sum_{i=1}^m\tau_\ell\left(\mathcal{R}^2(\theta_i-\theta^\star)+\mathcal{R}^2(\theta'_i-\theta^\star)\right)$$

where $\boldsymbol{\Sigma}' \triangleq \Sigma'\otimes I_m$ and $\theta_{\rm av}\triangleq\frac{1}{m}\sum_{i=1}^m\theta_i$ and analogously for $\theta'_{\rm av}$.

Proof. We write the first bound as

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle$$
$$=\frac{1}{m}\sum_{i=1}^m\langle(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}_i)(\theta_{\rm av}-\theta^\star),\sum_{j=1}^m(w_{i,j}-1/m)(\theta'_j-\theta^\star)\rangle.$$

Then,

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle$$
$$\leq\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^m|w_{i,j}-1/m||\langle(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}_i)(\theta_{\rm av}-\theta^\star),\theta'_j-\theta^\star\rangle|.$$

Using

$$|\langle v,Au\rangle|\leq\frac{1}{2}|\langle v,Av\rangle|+\frac{1}{2}|\langle u,Au\rangle|+\frac{1}{2}|\langle u-v,A(u-v)\rangle|,$$

we have

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star),(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star)\rangle$$
$$\leq\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^m\frac{|w_{i,j}-1/m|}{2}|\langle(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}_i)(\theta_{\rm av}-\theta^\star),\theta_{\rm av}-\theta^\star\rangle|$$
$$+\frac{1}{m}\sum_{i=1}^m\sum_{j=1}^m\frac{|w_{i,j}-1/m|}{2}\left(|\langle(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}_i)(\theta'_j-\theta^\star),\theta'_j-\theta^\star\rangle|+|\langle(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}_i)(\theta_{\rm av}-\theta'_j),\theta_{\rm av}-\theta'_j\rangle|\right).$$

Further $\|\|W - J\|\|_\infty \leq \rho\sqrt{m}$ [28] and therefore

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle \leq \frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m |\langle(\nabla^2\bar{\mathcal{L}} - \nabla^2\mathcal{L}_i)(\theta_{\rm av} - \theta^\star), \theta_{\rm av} - \theta^\star\rangle|$$

$$+ \frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m \left(|\langle(\nabla^2\bar{\mathcal{L}} - \nabla^2\mathcal{L}_i)(\theta'_j - \theta^\star), \theta'_j - \theta^\star\rangle| + |\langle(\nabla^2\bar{\mathcal{L}} - \nabla^2\mathcal{L}_i)(\theta'_j - \theta_{\rm av}), \theta'_j - \theta_{\rm av}\rangle|\right).$$

From Assumption 4 it follows that

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle \leq \frac{\rho\sqrt{m}}{2m}\left(\sum_{i=1}^m\sum_{j=1}^m \gamma_\ell\left(\|\theta_{\rm av} - \theta^\star\|^2_{\Sigma'} + \|\theta'_j - \theta^\star\|^2_{\Sigma'}\right)\right)$$

$$+ \frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m \gamma_\ell\|\theta_{\rm av} - \theta'_j\|^2_{\Sigma'} + \frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m \tau_\ell\left(\mathcal{R}^2(\theta_{\rm av} - \theta^\star) + \mathcal{R}^2(\theta'_j - \theta^\star) + \mathcal{R}^2(\theta'_j - \theta_{\rm av})\right).$$

Using the triangle and Young's inequalities on both $\|\theta_{\rm av} - \theta'_j\|^2_{\Sigma'}$ and $\mathcal{R}^2(\theta_{\rm av} - \theta'_j)$, $j \in [m]$, yields

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle \leq \frac{3\rho\sqrt{m}}{2}\gamma_\ell\left(\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'} + \|\boldsymbol{\theta}' - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}}\right)$$

$$+ \frac{3\rho\sqrt{m}}{2}\sum_{j=1}^m \tau_\ell\left(\mathcal{R}^2(\theta_{\rm av} - \theta^\star) + \mathcal{R}^2(\theta'_j - \theta^\star)\right).$$

The next bound can be written as

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{W} - \mathbf{J})(\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (\mathbf{W} - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle$$

$$= \frac{1}{m}\sum_{i=1}^m \left\langle \left(\nabla^2\bar{\mathcal{L}} - \nabla^2\mathcal{L}_i\right)\left(\sum_{j=1}^m (w_{i,j} - 1/m)(\theta_j - \theta^\star)\right), \left(\sum_{l=1}^m (w_{i,l} - 1/m)(\theta'_l - \theta^\star)\right)\right\rangle,$$

where following the same procedure as for the previous bound we obtain the desired result with the main difference that an additional network term and additional summation is required. □

F Empirical-Population Hessian Deviation Bounds

We provide here some technical lemmata that are necessary to establish that Assumptions 3 and 4 hold with high probability, for the considered data generation models. Specifically, Lemma 1 is used to establish Corollaries 1 and 1 while Lemma 5 and Lemma 6 permit to establish Corollaries 3 and 2, respectively. Since these lemmata are minor modifications of existing results, we omit their proofs.

Proof of Lemma 1

The global parameters $\gamma_{g,1}$ $\gamma_{g,2}$ τ_g and Σ follow from Proposition 1 in [1]. Establishing the local quantities $\gamma_{\ell,1}$, $\gamma_{\ell,2}$ and τ_ℓ follows the same steps as the proof of Proposition 5 in [28]. Observe that while the proof in [28] is done for the particular case in which $\mathcal{R}(\cdot) = \|\cdot\|_1$ it can be extended without significant differences to more general decomposable regularizers $\mathcal{R}$.

Sub-Gaussian data generation model (c.f. Assumption 5(ii))

For Corollary 3 to hold under the sub-Gaussian data generation model we require that Assumptions 3 and 4 hold. We establish that this is the case with overwhelming probability in the following Lemma.

Lemma 5 (Sub-Gaussian data model and $\mathcal{R}(\cdot) = \|\cdot\|_1$)**.** *Consider the problem* (1) *and assume that Assumptions 1 and 5(ii) hold. Denote by* $X = [x_1^\top; \ldots; x_N^\top]$ *and* $X_{S_i} = [x_{j_1}^\top; \ldots; x_{j_n}^\top]$, $j_l \in S_i$, $l \in [n]$, $i \in [m]$. *Then, there exists universal constants* $c_0, c_1, c_2 > 0$ *such that given that*

$$N \geq c_0 \log d \max\left\{1, \frac{\tau^4}{\mu^2}\right\}$$

there exist universal constants c_0 and c_1 such that with probability at least

$$1 - 2\exp\left(-c_1 N \min\left\{\frac{\mu^2}{\tau^4}, 1\right\} + \log(m+1)\right)$$

it holds

$$\left|\left\langle v, \left(\Sigma - \frac{X^\top X}{N}\right) v\right\rangle\right| \leq \frac{\mu}{2}\|v\|_2^2 + c_2 \frac{\log d}{N} \max\left\{\frac{\tau^4}{\mu}, \mu\right\} \|v\|_1^2, \, \forall v \in \mathbb{R}^d \tag{42}$$

$$\left|\left\langle v_i, \left(\Sigma - \frac{X_{S_i}^\top X_{S_i}}{n}\right) v_i\right\rangle\right| \leq \frac{m\mu}{2}\|v_i\|_2^2 + c_2 \frac{\log d}{n} \max\left\{\frac{\tau^4}{\mu}, \mu\right\} \|v_i\|_1^2, \, \forall v_i \in \mathbb{R}^d, \, i \in [m]. \tag{43}$$

Proof. The inequality (42) is a restatement of Corollary 1 in [15]. To obtain (43) similar steps as in [28] are taken on Lemma 12 in [15] followed by a union bound argument. □

Uniform data generation model (c.f. Assumption 5(iii))

For Corollary 2 to hold under the uniform data generation model we require that a variation of Assumptions 3 and 4 hold–see Lemma 8 and 9 in [1] for a discussion on this matter. We provide next a technical result required to establish a variant of Assumptions 3 and 4 tailored to this statistical model. We refer the reader to the proof of Corollary 2 (c.f. Sec. G) to see how this result is used.

Lemma 6. *Let*

$$X^{(i)} = d\, s^{(i)} e_{a(i)} e_{b(i)}^\top \in \mathbb{R}^{p \times p}, \quad i \in [N],$$

where $p^2 = d$, $s^{(i)} \in \{-1, 1\}$ uniformly, and are i.i.d. Further, $a(i), b(i) \sim \mathcal{U}\{1, p\}$ are i.i.d. and independent of $s^{(i)}$. For convenience, denote by

$$X \triangleq \begin{pmatrix} \mathrm{vec}(X^{(1)})^\top \\ \vdots \\ \mathrm{vec}(X^{(m)})^\top \end{pmatrix}, \qquad X_{S_i} \triangleq \begin{pmatrix} \mathrm{vec}(X^{(j_1)})^\top \\ \vdots \\ \mathrm{vec}(X^{j_m})^\top \end{pmatrix},$$

where $j_l \in S_i$, $l \in [n]$, and $i \in [m]$. Then, there exist universal constants $c_i > 0$, $i \in \{0, \ldots, 5\}$, such that, given

$$N \geq c_0 p \log p,$$

with probability at least

$$1 - \exp\left(-p\log(p)\right) - c_1 m \exp(-p\log(p)),$$

it holds for all $V \in \mathbb{R}^{p \times p}$ and $V_i \in \mathbb{R}^{p \times p}$, $i \in [m]$,

$$\left|\left\langle \mathrm{vec}(V), \left(\frac{XX^\top}{N} - I_{p^2}\right) \mathrm{vec}(V)\right\rangle\right| \leq c_2 p \|V\|_\infty \|V\|_1 \sqrt{\frac{p\log p}{N}} + c_3 \left(p\|V\|_\infty \sqrt{\frac{p\log p}{N}}\right)^2, \tag{44}$$

$$\left|\left\langle \mathrm{vec}(V_i), \left(\frac{X_{S_i} X_{S_i}^\top}{n} - I_{p^2}\right) \mathrm{vec}(V_i)\right\rangle\right| \leq c_4 pm \|V_i\|_\infty \|V\|_1 \sqrt{\frac{p\log p}{n}} + c_5 \left(pm\|V\|_\infty \sqrt{\frac{p\log p}{n}}\right)^2. \tag{45}$$

Proof. The statement in (44) restates Proposition 2 in [1] and (45) corresponds to following a very similar process as that in the proof of Proposition 5 in [28] but applied to Theorem 1 in [20] which is a restatement of (44). □

G Proofs of Corollaries 1-4

G.1 Proof of Corollary 1

We establish that for the given statistical model, the parameters are

$$\begin{aligned}
&\gamma_{g,1} = \frac{1}{2}, && \gamma_{g,2} = 1 \\
&\gamma_{\ell,1} = 4m - 1 && \gamma_{\ell,1} = 1, \\
&\tau_g = \frac{9\eta_\Sigma c_1 \log d}{N}, && \tau_\ell = 9\eta_\Sigma \frac{\log d}{n}, \\
&\Sigma' = \Sigma = \nabla^2 \bar{\mathcal{L}}, && \Psi^2(\bar{\mathcal{M}}) = \sqrt{s}, \\
&\mathcal{R}^*\left(\nabla \mathcal{L}(\theta^\star)\right) \leq \sqrt{3\eta_\Sigma \sigma^2 \frac{\log d}{N}}, && \max_{j\in[n]} \mathcal{R}^*\left(\nabla \mathcal{L}_i(\theta^\star)\right) = \sqrt{3m\eta_\Sigma \sigma^2 \frac{\log(md)}{N}},
\end{aligned}$$

and $\Pi_{\mathcal{M}^\perp}(\theta^\star) = 0$, for $q = 0$, where the suitable choice of sub-spaces is given in [2]. In the case $q \in (0, 1]$ if $\|\theta^\star\|_q \leq R_q$, for appropriate choice of $\bar{\mathcal{M}}$ [2]

$$\begin{aligned}
\Psi^2(\bar{\mathcal{M}}) &\leq \left(\frac{\log d}{N}\right)^{-\frac{q}{2}} R_q, \\
\|\Pi_{\mathcal{M}^\perp}(\theta^\star)\|_1 &\leq \left(\frac{\log d}{N}\right)^{\frac{1-q}{2}} R_q.
\end{aligned}$$

All statements except the bounds on the dual norms of the gradient(s) follow with high probability from Lemma 1. The bounds of the dual norms of the gradient(s) can be established by combining results in [30, Ch. 7] and the bound on the largest of the dual norms can be established via a minor alteration of the proof of [13, Th. 4]. Taking the union bound over all events guarantees that they all hold simultaneously with high probability. The result then follows by applying Theorem 2.

G.2 Proof of Corollary 2

Consider the problem of recovering $\Theta^\star \in \mathbb{R}^{p\times p}$ from

$$\tilde{y}_i = {\Theta^\star}_{r(i),c(i)} + \frac{\sigma}{d}\tilde{w}_i,$$

where $d = p^2$, and $r(i)$ and $c(i)$ are such that $\left(\mathrm{vec}(\Theta^\star)\right)_{v(i)} = {\Theta^\star}_{r(i),c(i)}$, and where $v(i) \sim \mathcal{U}\{1, d\}$, and are i.i.d. Further $\tilde{w}_i$ is zero mean, symmetric, i.i.d. and independent of $v(i)$. The problem above is statistically equivalent to recovering $\Theta^\star \in \mathbb{R}^p$ from [20]

$$y_i = \langle X_i, \Theta^\star\rangle + \sigma w_i,$$

where

$$X_i = p s_i e_{r(i)} e_{c(i)}^\top,$$

where w_i follows the same distribution as $\tilde{w}_i$, and $s_i \in \{-1, 1\}$ uniformly and i.i.d. Thus, the problem formulation corresponds to

$$\min_{\Theta:\|\Theta\|_\infty \leq \frac{\alpha}{p}, \|\Theta\|_1 \leq \rho} \underbrace{\frac{1}{2N}\sum_{i=1}^N \left(\langle X_i, \Theta^\star\rangle - y_i^2\right)}_{\triangleq \mathcal{L}(\Theta)} = \underbrace{\frac{1}{2N}\sum_{i=1}^N \left(\mathrm{vec}(X_i)^\top \mathrm{vec}(\Theta^\star) - y_i\right)^2}_{\triangleq \mathcal{L}(\theta)}.$$

The population risk is given by

$$\|\Sigma^{1/2}(\mathrm{vec}(\Theta) - \mathrm{vec}(\Theta^\star))\|_2^2.$$

Observe that by setting $\mathcal{R}(\cdot) = \|\cdot\|_1$ we have that $\mathcal{R}^*(\cdot) = |||\cdot|||_2$. Under the Assumption that $\tilde{w}_i$ is sub-exponential with parameter one we establish bounds

$$|||\nabla \mathcal{L}(\Theta^\star)|||_2 = \frac{\left|\left|\left|\sum_{i=1}^N X_i \sigma w_i\right|\right|\right|_2}{N},$$

$$|||\nabla \mathcal{L}_i(\Theta^\star)|||_2 = \frac{\left|\left|\left|\sum_{j\in S_i} X_j \sigma w_j\right|\right|\right|_2}{n}.$$

Using Lemma 7 in [20] and using the union bound it follows that for some $c_1,\ c_2,\ c_3 > 0$

with probability at least

$$1 - c_2(m+1)\exp\left(-c_3 \log(p)\right),$$

it holds that

$$\mathcal{R}^*\left(\nabla \mathcal{L}(\Theta^\star)\right) \leq c_1 \sigma \sqrt{\frac{p\log p}{N}},$$

$$\mathcal{R}^*\left(\sum_{j=1}^m (w_{i,j} - 1/m)\nabla \mathcal{L}_j(\Theta^\star)\right)$$

$$\leq \sum_{j=1}^m |w_{i,j} - 1/m| \mathcal{R}^*\left(\nabla \mathcal{L}_j(\theta^\star)\right) \leq \sum_{j=1}^m |||W - J|||_\infty m c_1 \sigma \sqrt{\frac{p\log p}{N}} \leq m^2\sqrt{m}\rho c_1 \sigma \sqrt{\frac{p\log p}{N}}.$$

Let $X \triangleq \left(\mathrm{vec}(X_1), \ldots, \mathrm{vec}(X_N)\right)^\top$. Then, using Lemma 6 we have that for any pair $\Theta^{(1)} \in \mathbb{R}^{p\times p}$ and $\Theta^{(2)} \in \mathbb{R}^{p\times p}$ such that $\|\Theta_i\|_\infty \leq \frac{\omega}{p}$ $i \in [2]$ it holds that for some $c_4,\ c_5 > 0$

$$\left|\left\langle \mathrm{vec}(\Theta^{(1)} - \Theta^{(2)}), \left(\frac{X^\top X}{N} - I_{p^2}\right)\mathrm{vec}(\Theta^{(1)} - \Theta^{(2)})\right\rangle\right| \tag{46}$$

$$\leq 2\omega c_4 \|\Theta^{(1)} - \Theta^{(2)}\|_1 \sqrt{\frac{p\log p}{N}} + 4\omega^2 c_5 \frac{p\log p}{N}. \tag{47}$$

We proceed now to obtain the counterpart of Lemma 3 in the setting of Corollary 2. For convenience, denote by $\boldsymbol{\theta} \in \mathbb{R}^{mp^2}$ such that $\theta_i = \mathrm{vec}(\Theta_i)$ and $\boldsymbol{\theta} = [\theta_1^\top, \ldots, \theta_m^\top]^\top$, and analogously for $\boldsymbol{\theta}'$. Assume all Θ_i and Θ_i' are feasible. Following the same steps as in the proof of Lemma 3, one obtains

$$\langle \mathbf{J}(\boldsymbol{\theta} - \boldsymbol{\theta}^\star), \left(\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \bar{\boldsymbol{\mathcal{L}}}\right)(\mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle = \frac{1}{2}\left\langle \theta_{\mathrm{av}} - \theta^\star, \left(\frac{X^\top X}{N} - I_{p^2}\right)(\theta_{\mathrm{av}} - \theta^\star)\right\rangle$$

$$+ \frac{1}{2}\left\langle \theta_{\mathrm{av}}' - \theta^\star, \left(\frac{X^\top X}{N} - I_{p^2}\right)(\theta_{\mathrm{av}}' - \theta^\star)\right\rangle - \frac{1}{2}\left\langle \theta_{\mathrm{av}} - \theta_{\mathrm{av}}', \left(\frac{X^\top X}{N} - I_{p^2}\right)(\theta_{\mathrm{av}} - \theta_{\mathrm{av}}')\right\rangle$$

$$\leq \sqrt{\frac{p\log p}{N}}\omega c_4 \left(\|\Theta_{\mathrm{av}}' - \Theta^\star\|_1 + \|\Theta_{\mathrm{av}} - \Theta^\star\|_1 + \|\Theta_{\mathrm{av}}' - \Theta_{\mathrm{av}}\|_1\right) + 12\omega^2 c_5 \frac{p\log p}{N},$$

where we have used (46) to obtain the last inequality, $\theta_{\mathrm{av}} \triangleq \frac{1}{m}\sum_{i=1}^m \theta_i$, $\Theta_{\mathrm{av}} \triangleq \frac{1}{m}\sum_{i=1}^m \Theta_i$ and analogously for θ_{av}' and Θ_{av}'. Because Θ and Θ' are feasible, so are Θ_{av} and Θ_{av}'. By using the triangle inequality on $\|\Theta_{\mathrm{av}} - \Theta_{\mathrm{av}}'\|_1$, and invoking Lemma 1 in [2] we obtain

$$\langle \mathbf{J}(\boldsymbol{\theta} - \boldsymbol{\theta}^\star), \left(\nabla^2 \bar{\boldsymbol{\mathcal{L}}} - \nabla^2 \boldsymbol{\mathcal{L}}\right)(\mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle \leq \omega c_4 \sqrt{\frac{p\log p}{N}} 12 \|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1$$

$$+ 4\Psi(\bar{\mathcal{M}})\sqrt{\frac{p\log p}{N}}\omega c_4 \left(\|\theta_{\mathrm{av}}' - \theta^\star\|_2 + \|\theta_{\mathrm{av}} - \theta^\star\|_2\right) + 12\omega^2 c_5 \frac{p\log p}{N}$$

$$\leq \frac{1}{6}\left(\|\theta_{\mathrm{av}} - \theta^\star\|_2^2 + \|\theta_{\mathrm{av}}' - \theta^\star\|_2^2\right) + 12\omega c_4 \sqrt{\frac{p\log p}{N}} \|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1$$

$$+ 12\omega^2 c_5 \frac{p\log p}{N} + 24\omega^2 c_4^2 \Psi^2(\bar{\mathcal{M}}) \frac{p\log p}{N} = \frac{1}{6m}\left(\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|_2^2 + \|\mathbf{J}\boldsymbol{\theta}' - \boldsymbol{\theta}^\star\|_2^2\right)$$

$$+ \omega^2 \frac{p\log p}{N}\left(12c_5 + 24c_4^2\Psi^2(\bar{\mathcal{M}})\right) + 12\omega c_4 \sqrt{\frac{p\log p}{N}} \|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1.$$

Following the same steps as in the proof of Lemma 4 we obtain

$$\langle(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star),(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star)\rangle \leq$$
$$\frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m\left|\langle(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L})(\theta_{\text{av}}-\theta^\star),\theta_{\text{av}}-\theta^\star\rangle\right|$$
$$+\frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m\left(\left|\left\langle\left(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}\right)(\theta_j'-\theta^\star),\theta_j'-\theta^\star\right\rangle\right|+\left|\left\langle\left(\nabla^2\bar{\mathcal{L}}-\nabla^2\mathcal{L}_i\right)(\theta_j'-\theta_{\text{av}}),\theta_j'-\theta_{\text{av}}\right\rangle\right|\right)$$
$$\leq\frac{\rho\sqrt{m}}{2m}\sum_{i=1}^m\sum_{j=1}^m\left(c_6pm\|\Theta_{\text{av}}-\Theta^\star\|_\infty\|\Theta_{\text{av}}-\Theta^\star\|_1\sqrt{\frac{p\log p}{n}}+\left(c_7p^2m^2\|\Theta_{\text{av}}-\Theta^\star\|_\infty^2\frac{p\log p}{n}\right)\right)$$
$$+\frac{\rho\sqrt{m}}{2}\sum_{i=1}^m\sum_{j=1}^m\left(c_6pm\|\Theta_j'-\Theta^\star\|_\infty\|\Theta_j'-\Theta^\star\|_1\sqrt{\frac{p\log p}{n}}+\left(c_7p^2m^2\|\Theta_j'-\Theta^\star\|_\infty^2\frac{p\log p}{n}\right)\right)$$
$$+\frac{\rho\sqrt{m}}{2}\sum_{i=1}^m\sum_{j=1}^m\left(c_6pm\|\Theta_{\text{av}}-\Theta_j'\|_\infty\|\Theta_{\text{av}}-\Theta_j'\|_1\sqrt{\frac{p\log p}{n}}+\left(c_7p^2m^2\|\Theta_{\text{av}}-\Theta_j'\|_\infty^2\frac{p\log p}{n}\right)\right),$$

where the last inequality follows from Lemma 6. Observe that since Θ_{av}, $\Theta^\star$ and Θ_j' are all feasible, it holds that the infinity-norm terms can be upper bounded by $\frac{2\omega}{p}$. Further, we use the triangle inequality on $\|\Theta_{\text{av}}-\Theta_j'\|_1$ and obtain

$$\langle(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star),(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star)\rangle \leq$$
$$c_6\rho\sqrt{m}\sum_{i=1}^m\sum_{j=1}^m\left(2\omega\|\Theta_{\text{av}}-\Theta^\star\|_1+2\omega m\|\Theta_j'-\Theta^\star\|_1\right)\sqrt{\frac{p\log p}{n}}+\frac{3\rho\sqrt{m}}{2}\left(\omega^2 4c_7m^3\frac{p\log p}{n}\right).$$

Using feasibility of Θ_{av} and Θ_j' and in [2, Lemma 1], we can write

$$\begin{aligned}&\langle(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star),(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star)\rangle\\&\leq c_6\rho\sqrt{m}4\Psi(\bar{\mathcal{M}})\omega m\sqrt{\frac{p\log p}{n}}\sum_{j=1}^m\left(\|\theta_{\text{av}}-\theta^\star\|_2+\|\theta_j'-\theta^\star\|_2\right)\\&+c_7\rho\sqrt{m}4\omega m^2\sqrt{\frac{p\log p}{n}}\|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1+\frac{3\rho\omega^2\sqrt{m}m^3}{2}4c_7\frac{p\log p}{n}.\end{aligned}$$

Using Young's inequality yields

$$\langle(\mathbf{W}-\mathbf{J})(\boldsymbol{\theta}'-\boldsymbol{\theta}^\star),(\nabla^2\bar{\boldsymbol{\mathcal{L}}}-\nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star)\rangle\leq c_5\frac{\rho\sqrt{m}m}{2}\sum_{j=1}^m\left(\|\theta_{\text{av}}-\theta^\star\|_2^2+\|\theta_j'-\theta^\star\|_2^2\right)$$
$$+c_6\frac{\rho\sqrt{m}m^2}{2}16\Psi^2(\bar{\mathcal{M}})\omega^2\frac{p\log p}{n}+4c_6\omega\sqrt{m}m^3\rho\sqrt{\frac{p\log p}{n}}\|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1+12c_7\rho\sqrt{m}m^3\omega^2\frac{p\log p}{n}$$
$$=c_6\frac{\rho\sqrt{m}m}{2}\left(\|\mathbf{J}\boldsymbol{\theta}-\boldsymbol{\theta}^\star\|_2^2+\|\boldsymbol{\theta}'-\boldsymbol{\theta}^\star\|_2^2\right)+4c_6\omega m^3\sqrt{m}\rho\sqrt{\frac{p\log p}{n}}\|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1$$
$$+\left(c_6 8\Psi^2(\bar{\mathcal{M}})+12mc_6\right)\rho\omega^2\sqrt{m}m^2\frac{p\log p}{n}.$$

It remains to find bounds on $\Psi^2(\bar{\mathcal{M}})$ and $\|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1$. In the setting of the corollary, we can use [2] and write

$$\Psi^2(\mathcal{M})\leq R_q\omega^{-q}\left(\frac{p\log p}{N}\right)^{-\frac{q}{2}},$$
$$\|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1\leq R_q\omega^{1-q}\left(\frac{p\log p}{N}\right)^{\frac{1-q}{2}}.$$

By deriving similar bounds for the remaining quantities in Lemma 4 and following the same steps as in the proof of Theorem 2 while treating the extra tolerance in the same way as the misspecification error in the proof of Theorem 2, the desired result follows, that is,

$$r_t^2 \leq \lambda^t r_0^2 + \mathcal{O}\left(\left(\frac{p\log p}{N}\right)^{1-\frac{q}{2}} R_q\left(\sigma^2\omega^{-q} + \omega^{2-q} + \sigma\omega^{1-q}\right)\left(1+\rho m^5\right)\right).$$

Observe that in this particular case $\zeta(\bar{\mathcal{M}}, W)$ (c.f. Theorem 2) is equal to zero and that the dominating term is the latter since $\omega \geq 1$.

G.3 Proof of Corollary 3

Under the column normalized assumption, following the same steps as in the proof of Corollary 1, it follows that

$$\mathbb{P}\left(\max_{i\in[m]} \frac{\left\|X_{S_i}^\top w_{S_i}\right\|_\infty}{n} \leq C\sigma\sqrt{\frac{2m\log(md)}{N}}, \frac{\|X^\top w\|_\infty}{N} \leq C\sigma\sqrt{\frac{2\log d}{N}}\right)$$
$$\geq 1 - 2d\exp(-2\log(d) - 2md\exp(-2\log(dm))).$$

From Lemma 5 it follows that Assumptions 3 and 4 hold with high probability. Finally bounds for $\Psi^2(\bar{\mathcal{M}})$ and $\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))$ as in the proof of Corollary 1 apply. Thus, taking the overall union bound and invoking Theorem 1 yields the desired result.

G.4 Proof of Corollary 4

Observe that the empirical risk minimization problem can be written as

$$\min_{\Theta:\|\Theta\|_1\leq\bar{\rho}} \underbrace{\frac{1}{2N}\sum_{i=1}^N \left(\langle X_i, \Theta^\star\rangle - y_i\right)^2}_{\triangleq\mathcal{L}(\Theta)} = \underbrace{\frac{1}{2N}\sum_{i=1}^N \left(\mathrm{vec}(X_i)^\top \mathrm{vec}(\Theta^\star) - y_i\right)^2}_{\triangleq\mathcal{L}(\theta)}.$$

The population risk is given by

$$\bar{\mathcal{L}}(\theta) \triangleq \|\Sigma^{1/2}(\theta - \theta^\star)\|_2^2,$$

where $\theta^\star = \mathrm{vec}(\Theta^\star)$. Then, from Lemma 1 it follows that Assumptions 3 and 4 hold with the probability given in Lemma 1. Observe that $\mathbb{E}[\mathcal{R}^*\left(\mathrm{vec}(x_i)\right)] \leq c\eta_\Sigma \frac{p}{n}$, which establishes the terms accompanying $\mathcal{R}(\cdot) = \|\cdot\|_1$.

In this case, the statement in Lemma 1 in [2] can be written as

$$\|\Theta - \Theta^\star\|_1 \leq 2\Psi(\bar{\mathcal{M}})\|\Theta - \Theta^\star\|_F + 2\|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1.$$

As in [2], for $q = 0$ and $q \in (0, 1]$, we have (see [2] for the appropriate choices of $\mathcal{M}$, $\bar{\mathcal{M}}$).

$$\Psi^2(\bar{\mathcal{M}}) = r = R_0, \qquad \Psi^2(\bar{\mathcal{M}}) \leq R_q \frac{p}{N}^{-\frac{q}{2}},$$

$$\|\Pi_{\mathcal{M}^\perp}\left(\Theta^\star\right)\|_1 = 0, \qquad \|\Pi_{\mathcal{M}^\perp}(\Theta^\star)\|_1 \leq R_q\left(\frac{p}{N}\right)^{\frac{1-q}{2}},$$

where r denotes the rank of $\Theta^\star$, the left column corresponds to $q = 0$ and the right column to $q \in (0, 1]$.

It remains to establish upper bounds on

$$\frac{\|\sum_{i=1}^m X_i w_i\|_2}{N}, \qquad \frac{\|\sum_{j\in S_i}^m X_j w_j\|_2}{n}.$$

Following the proof of [30, Cor. 10.10] it can be shown that, if

$$N \geq c_0 p,$$

it holds

$$\mathbb{P}\left(\cap_{i\in[m]}\left(\frac{\|\sum_{j\in S_i} X_j w_j\|_2}{N} \leq c_1\sigma\eta_\Sigma^{1/2} m\sqrt{\frac{p}{N}}\right) \cap \left(\frac{\|\sum_{i=1}^m X_i w_i\|_2}{N} \leq c_1\sigma\eta_\Sigma^{1/2}\sqrt{\frac{p}{N}}\right)\right)$$
$$\geq 1 - (m+1)\left(\exp\left(-c_2 p\right)\right),$$

where c_0, c_1, $c_2 > 0$ are for universal constants. Invoking Lemma 1 and using the union bound yield the desired result.

H DGD Algorithm (4) for Gaussian ℓ_1-sparse regression

Recall that the DGD-CTA iterates can be written as

$$\boldsymbol{\theta}^{t+1} = \prod_{\Omega \cap \mathcal{R}(\theta) \leq R} \left(\mathbf{W}\boldsymbol{\theta}^t - \alpha m \nabla \boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^t) \right). \tag{48}$$

We establish convergence of DGD-CTA for the particular case of ℓ_1 constrained Gaussian linear regression. We begin by introducing some notation. Denote by

$$\Delta_{\text{stat}}^2 \triangleq \frac{128\Psi^2(\bar{\mathcal{M}})}{\mu^2} \left(\mathcal{R}^*(\nabla \mathcal{L}(\theta^\star))\right)^2 + \frac{16}{\mu} \mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)) \mathcal{R}(\nabla \mathcal{L}(\theta^\star)),$$

$$\Delta_{\text{net}}^2 \triangleq \frac{16m\alpha}{(1-\rho)\mu} \|\nabla \mathcal{L}(\boldsymbol{\theta}^\star)\|_2^2,$$

and the tolerance

$$\Delta^2 \triangleq \left(\Delta_{\text{stat}}^2 + \Delta_{\text{net}}^2\right) \left(1 + \frac{\alpha\zeta(\bar{\mathcal{M}})}{1-\zeta(\bar{\mathcal{M}})}\right) + \frac{144}{\mu\Psi^2(\bar{\mathcal{M}})} \mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star)) \zeta(\bar{\mathcal{M}}) \left(1 + \frac{\alpha\zeta(\bar{\mathcal{M}})}{1-\alpha\zeta(\bar{\mathcal{M}})}\right),$$

where

$$\zeta(\bar{\mathcal{M}}) \triangleq 9\Psi^2(\bar{\mathcal{M}})\tau_g.$$

Theorem 3. *Consider the ERM problem* (1) *and the associated population minimization* (2)*, under Assumptions 1-3, with* $\gamma_{g,1} = \frac{1}{2}$ *and* $\gamma_{g,2} = 1$*, and*

$$\Sigma' \preceq \nabla^2 \mathcal{L} \quad \textit{and} \quad \zeta(\bar{\mathcal{M}}) \leq \frac{\mu}{16}.$$

Consider the DGD-CTA algorithm in (48)*, with weight matrix* $W \succ 0$ *satisfying Assumption 6 and stepsize*

$$\alpha \leq \min\left\{\frac{\sigma_m(W)}{2L}, \frac{1-\rho}{18\eta + 16L}\right\},$$

where

$$\eta \geq \max_{i \in [m]} \|\nabla^2 \bar{\mathcal{L}} - \nabla^2 \bar{\mathcal{L}}_i\|_2. \tag{49}$$

Then, the iterates of DGD-CTA (48) *satisfy*

$$r_t^2 \leq \left(\frac{1 - \alpha\frac{\mu}{4} + \alpha\zeta(\bar{\mathcal{M}})}{1-\alpha\zeta(\bar{\mathcal{M}})}\right)^t r_0^2 + \Delta^2.$$

Contrasting Theorem 2 and Theorem 3, the following comments are in order. First, in Theorem 2 Δ_{net}^2 can be made of the same order of Δ_{stat}^2 leveraging the network, mixing local information at sufficiently fast rate. On the other hand, in the setting of Theorem 3, even if the network is fully connected, i.e., $\rho = 0$,

$$\Delta_{\text{net}}^2 = \frac{16m\alpha}{\mu} \|\nabla \mathcal{L}(\boldsymbol{\theta}^\star)\|_2^2.$$

For the Gaussian linear regression problem, it holds with overwhelming probability that $\|\nabla \mathcal{L}(\boldsymbol{\theta}^\star)\|_2^2 = \mathcal{O}(d/n)$. This implies that, for Δ_{net}^2 to be of the same order as Δ_{stat}^2, the stepsize α must be chosen as $\alpha = \mathcal{O}(d^{-1})$.

H.1 Proof of Theorem 3 and auxiliary results

We begin establishing a variant of Lemma 2 for the proxy

$$\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}) + \frac{1}{2\alpha m} \|\boldsymbol{\theta}\|_{I-\mathbf{W}}^2,$$

which can be done following the same procedure as in the proof of Lemma 2 and yields

$$\begin{aligned}
&\|\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2 \le \|\boldsymbol{\theta}^{t}-\boldsymbol{\theta}^\star\|^2 - \alpha m\left(\|\boldsymbol{\theta}^{t}-\boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \|\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}\right)\\
&-\left(\|\boldsymbol{\theta}^{t}\|^2_{I-\mathbf{W}} + \|\boldsymbol{\theta}^{t+1}\|^2_{I-\mathbf{W}}\right) + \alpha m\|\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^t\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} - \|\boldsymbol{\theta}^{t}-\boldsymbol{\theta}^{t+1}\|^2_{\mathbf{W}}\\
&+2\alpha m\langle \nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^t)-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^t), \boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^t\rangle.
\end{aligned}$$

We split the inner product similarly to (23) -(26). Under Assumption 3 and (49) we have from Lemmas 3 and 7 that

$$\begin{aligned}
&2\alpha m\langle \nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^t)-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^t), (\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star)\rangle \le (\alpha\gamma_{g,1}+\alpha\epsilon\gamma_{g,1})\left(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'} + \|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}\right)\\
&+m\alpha\epsilon\left(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}\right) + \alpha\gamma_{g,2}\|\mathbf{J}\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^{t+1}\|^2_{\boldsymbol{\Sigma}'}\\
&+\left(1+\frac{1}{\epsilon}\right)\alpha\eta\left(\|\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^t\|_2^2 + \|\boldsymbol{\theta}^{t+1}-\mathbf{J}\boldsymbol{\theta}^{t+1}\|_2^2\right) + \frac{\alpha m}{\epsilon}\left(\|\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \|\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^t\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}\right)\\
&+\alpha m\tau_g(1+\epsilon)\left(\mathcal{R}^2(\theta^t_{\text{av}}-\theta^\star) + \mathcal{R}^2(\theta^{t+1}_{\text{av}}-\theta^\star)\right) - 2\alpha m\langle\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star), \boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle.
\end{aligned}$$

Using [2, Lemma 1], we have

$$\$\mathcal{R}^2(\theta^t_{\text{av}}-\theta^\star) \le 4(1+a)\Psi^2(\bar{\mathcal{M}})\|\theta^t_{\text{av}}-\theta^\star\|_2^2 + 4\left(1+\frac{1}{a}\right)\mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star)),$$

and analogously for θ^{t+1}_{av}, implying that

$$\begin{aligned}
&\alpha m\tau_g\left(1+\epsilon\right)\left(\mathcal{R}^2(\theta^t_{\text{av}}-\theta^\star) + \mathcal{R}^2(\theta^{t+1}_{\text{av}}-\theta^\star)\right) \le\\
&\alpha m\tau_g\left(1+\epsilon\right)4(1+a)\Psi^2(\bar{\mathcal{M}})\left(\|\theta^t_{\text{av}}-\theta^\star\|_2^2 + \|\theta^{t+1}_{\text{av}}-\theta^\star\|_2^2\right)\\
&+8\alpha m\tau_g(1+\epsilon)\delta,
\end{aligned}$$

where for convenience we set $\delta \triangleq \left(1+\frac{1}{a}\right)\mathcal{R}^2(\Pi_{\mathcal{M}^\perp}(\theta^\star))$. Further, using Hölder's inequality and Young's inequaity, for any $b_1, b_2 > 0$ it holds that

$$\begin{aligned}
&-2\alpha m\langle\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star), \boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\rangle = -2\alpha m\langle\nabla\mathcal{L}(\theta^\star), \theta^{t+1}_{\text{av}}-\theta^\star\rangle - 2\alpha m\langle\nabla\boldsymbol{\mathcal{L}}(\theta^\star), (I-\mathbf{J})\boldsymbol{\theta}^{t+1}\rangle\\
&\le 2\alpha m\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\mathcal{R}(\theta^{t+1}_{\text{av}}-\theta^\star) + 2\alpha m\|(I-\mathbf{J})\boldsymbol{\theta}^{t+1}\|_2\|\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)\|_2\\
&\le 4\alpha m\Psi(\bar{\mathcal{M}})\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\|\theta^{t+1}_{\text{av}}-\theta^\star\|_2\\
&+2\alpha m\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\mathcal{R}\left(\Pi_{\mathcal{M}^\perp}(\theta^\star)\right) + b_2\|(I-\mathbf{J})\boldsymbol{\theta}^{t+1}\|_2^2 + \frac{m^2\alpha^2}{b_2}\|\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)\|_2^2 \le\\
&+2\alpha b_1 m\|\theta^{t+1}_{\text{av}}-\theta^\star\|_2^2 + \frac{2\alpha m}{b_1}\Psi^2(\bar{\mathcal{M}})\left(\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\right)^2 + 2m\alpha\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\\
&+b_2\|(I-\mathbf{J})\boldsymbol{\theta}^{t+1}\|_2^2 + \frac{m^2\alpha^2}{b_2}\|\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)\|_2^2.
\end{aligned}$$

Combining yields

$$\begin{aligned}
&2\alpha m\langle \nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^t)-\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^t), (\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star)\rangle \le (\alpha\gamma_{g,1}+\alpha\epsilon\gamma_{g,1})\left(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'} + \|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}\right)\\
&+m\alpha\epsilon\left(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}\right) + \alpha\gamma_{g,2}\|\mathbf{J}\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^{t+1}\|^2_{\boldsymbol{\Sigma}'}\\
&+\left(1+\frac{1}{\epsilon}\right)\alpha\eta\left(\|\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^t\|_2^2 + \|\boldsymbol{\theta}^{t+1}-\mathbf{J}\boldsymbol{\theta}^{t+1}\|_2^2\right) + \frac{\alpha m}{\epsilon}\left(\|\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \|\boldsymbol{\theta}^t-\mathbf{J}\boldsymbol{\theta}^t\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}\right)\\
&+\alpha\tau_g(1+\epsilon)4(1+a)\Psi^2(\bar{\mathcal{M}})(\|\mathbf{J}\boldsymbol{\theta}^t-\boldsymbol{\theta}^\star\|_2^2 + \|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2) + 8\alpha m\tau_g(1+\epsilon)\delta\\
&+2\alpha b_1\|\mathbf{J}\boldsymbol{\theta}^{t+1}-\boldsymbol{\theta}^\star\|_2^2 + b_2\|(I-\mathbf{J})\boldsymbol{\theta}^{t+1}\|_2^2 + \frac{m^2\alpha^2}{b_2}\|\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star)\|_2^2\\
&+\frac{2\alpha m}{b_1}\Psi^2(\bar{\mathcal{M}})\left(\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\right)^2 + 2m\alpha\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)).
\end{aligned}$$

For convenience, denote by $R_t^2 \triangleq \|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_2^2$ and $C_t^2 \triangleq \|\boldsymbol{\theta}\|_{I-\mathbf{J}}^2$. Then, using the above to upper bound R_{t+1}^2 yields

$$
\begin{aligned}
&R_{t+1}^2 \leq R_t^2 - \alpha m \left(\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}^2 + \|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}^2\right) - (1-\rho)\left(C_{t+1}^2 + C_t^2\right) \\
&(\alpha\gamma_{g,1} + \alpha\epsilon\gamma_{g,1})\left(\|\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}'}^2 + \|\mathbf{J}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_{\boldsymbol{\Sigma}'}^2\right) + m\alpha\epsilon\left(\|\mathbf{J}\boldsymbol{\theta}^t - \boldsymbol{\theta}^\star\|_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}^2 + \|\mathbf{J}\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^\star\|_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}^2\right) \\
&+ \alpha\tau_g(1+\epsilon)4(1+a)\Psi^2(\bar{\mathcal{M}})(R_t^2 + R_{t+1}^2) + 2\alpha b_1 R_{t+1}^2 + \left(1 + \frac{1}{\epsilon}\right)\alpha\eta(C_t^2 + C_{t+1}^2) + \frac{\alpha L}{\epsilon}(C_t^2 + C_{t+1}^2) \\
&+ b_2 C_{t+1}^2 + \frac{m^2\alpha^2}{b_2}\|\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star)\|_2^2 + 8\alpha m\tau_g(1+\epsilon)\delta + \frac{2\alpha m}{b_1}\Psi^2(\bar{\mathcal{M}})\left(\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\right)^2 \\
&+ 2m\alpha\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)) + \alpha\gamma_{g,2}\|\mathbf{J}\boldsymbol{\theta}^t - \mathbf{J}\boldsymbol{\theta}^{t+1}\|_{\boldsymbol{\Sigma}'}^2 \\
&+ \alpha m\|\boldsymbol{\theta}^{t+1} - \boldsymbol{\theta}^t\|_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}^2 - \sigma_m(W)\|\boldsymbol{\theta}^t - \boldsymbol{\theta}^{t+1}\|^2.
\end{aligned}
$$

If $\Sigma' \preceq \nabla^2\bar{\mathcal{L}}$ Setting $b_1 = \frac{\mu}{8}$, $\epsilon = \frac{1}{8}$ $b_2 = \frac{1-\rho}{2}$, for $\alpha \leq \frac{\sigma_m(W)}{2L}$ we have

$$
\begin{aligned}
&R_{t+1}^2 \leq R_t^2 - \frac{\alpha\mu}{4}R_t^2 + \frac{36}{8}\alpha(1+a)\tau_g\Psi^2(\bar{\mathcal{M}})(R_t^2 + R_{t+1}^2) - \frac{(1-\rho)}{2}(C_{t+1}^2 + C_t^2) \\
&+ (9\alpha\eta + 8\alpha L)\left(C_t^2 + C_{t+1}^2\right) + \frac{m^2\alpha^2}{b_2}\|\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star)\|_2^2 + 9\alpha m\tau_g\delta + 16\alpha m\frac{\Psi^2(\bar{\mathcal{M}})}{\mu}\left(\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\right)^2 \\
&+ 2m\alpha\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star)).
\end{aligned}
$$

For the terms in C_t^2 and C_{t+1}^2 to vanish from the RHS of the above inequality we require that

$$
\alpha \leq \min\left\{\frac{\sigma_m(W)}{2L}, \frac{1-\rho}{2(9\eta + 8L)}\right\}.
$$

Then, for convenience, define

$$
\zeta \triangleq \frac{36}{8}(1+a)\tau_g\Psi^2(\bar{\mathcal{M}}).
$$

Since $1 - \alpha\zeta > 0$, dividing both sides by $1 - \alpha\zeta$, we obtain

$$
\begin{aligned}
R_{t+1}^2 \leq\ & \underbrace{\left(1 - \frac{\alpha\mu}{4} + \alpha\zeta\right)(1-\alpha\zeta)^{-1}}_{\triangleq\lambda} R_t^2 + \\
&(1-\alpha\zeta)^{-1}\left(\frac{m^2\alpha^2}{b_2}\|\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star)\|_2^2 + 9\alpha m\tau_g\delta\right) \\
&+ (1-\alpha\zeta)^{-1}\left(16\alpha m\frac{\Psi^2(\bar{\mathcal{M}})}{\mu}\left(\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\right)^2 + 2m\alpha\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star))\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\right).
\end{aligned}
$$

Observe that if

$$
\zeta \leq \frac{\mu}{16},
$$

λ is at most $1 - \frac{\gamma\mu}{8}$. Therefore, by telescoping

$$
\begin{aligned}
R_{t+1}^2 \leq\ & \lambda^{t+1}R_0^2 + (1-\alpha\zeta)^{-1}\left(\frac{16m^2\alpha}{1-\rho}\|\nabla\bar{\boldsymbol{\mathcal{L}}}(\boldsymbol{\theta}^\star)\|_2^2 + \frac{128m}{\mu^2}\Psi^2(\bar{\mathcal{M}})(\mathcal{R}^*(\nabla\mathcal{L}(\theta^\star)))^2\right) \\
&+ (1-\alpha\zeta)^{-1}(1-\lambda)^{-1}\alpha\left(9m\tau_g\delta + 2m\alpha\mathcal{R}^*\left(\nabla\mathcal{L}(\theta^\star)\right)\mathcal{R}(\Pi_{\mathcal{M}^\perp}(\theta^\star))\right).
\end{aligned}
$$

Setting $a = 1$ and using

$$
(1-\alpha\zeta)^{-1} = 1 + \frac{\alpha\zeta}{1-\alpha\zeta},
$$

yield the desired result.

The following lemma contains results analogous to those in Lemma 4, which are required for the proof of DGD-CTA.

Lemma 7. *Under Assumptions 1, 3, and* (49)*, and for any* $\epsilon > 0$*, it holds*

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (I - \mathbf{J})\boldsymbol{\theta}'\rangle \leq \frac{\epsilon\gamma_{g,1}}{2m}\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'} + \frac{\epsilon\tau_g}{2}\mathcal{R}^2(\theta_{\text{av}} - \theta^\star) + \frac{\epsilon}{2}\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}$$
$$+ \frac{\eta}{2\epsilon m}\|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|_2^2 + \frac{1}{2\epsilon}\|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}},$$

and

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}), \boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\rangle \leq \frac{\eta}{2m}\left(\|\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}\|_2^2 + \|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|_2^2\right).$$

Proof. We start with

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (I - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle = -\langle\nabla^2\boldsymbol{\mathcal{L}}(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (I - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle$$
$$\leq \frac{\epsilon}{2}\|(\nabla^2\boldsymbol{\mathcal{L}})^{1/2}(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star)\|_2^2 + \frac{1}{2\epsilon}\|(\nabla^2\boldsymbol{\mathcal{L}})^{1/2}(\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}')\|_2^2$$

for any $\epsilon > 0$. Then,

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (I - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle \leq \frac{\epsilon}{2}\langle(\nabla^2\boldsymbol{\mathcal{L}} - \nabla^2\bar{\boldsymbol{\mathcal{L}}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star)\rangle$$
$$+ \frac{\epsilon}{2}\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \frac{1}{2\epsilon}\langle\nabla^2\bar{\boldsymbol{\mathcal{L}}}(I - \mathbf{J})\boldsymbol{\theta}', (I - \mathbf{J})\boldsymbol{\theta}'\rangle + \frac{1}{2\epsilon}\|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}.$$

Under Assumption 3 and (49) it follows that

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star), (I - \mathbf{J})(\boldsymbol{\theta}' - \boldsymbol{\theta}^\star)\rangle \leq \frac{\epsilon\gamma_{g,1}}{2m}\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\boldsymbol{\Sigma}'}$$
$$+ \frac{\epsilon\tau_g}{2}\mathcal{R}^2(\theta_{\text{av}} - \theta^\star) + \frac{\epsilon}{2}\|\mathbf{J}\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}} + \frac{\eta}{2\epsilon m}\|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|_2^2 + \frac{1}{2\epsilon}\|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|^2_{\nabla^2\bar{\boldsymbol{\mathcal{L}}}}.$$

Finally, from (49), we conclude

$$\langle(\nabla^2\bar{\boldsymbol{\mathcal{L}}} - \nabla^2\boldsymbol{\mathcal{L}})(\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}), (\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}')\rangle \leq \frac{\eta}{2m}\|\boldsymbol{\theta} - \mathbf{J}\boldsymbol{\theta}\|_2^2 + \frac{\eta}{2m}\|\boldsymbol{\theta}' - \mathbf{J}\boldsymbol{\theta}'\|_2^2.$$

□

In order to establish the convergence of DGD-CTA when used to solve the Gaussian ℓ_1-constrained regression problem (c.f. Corollary 1) we require that (49) holds with high probability, which guaranteed by the following lemma.

Lemma 8. *Under the data model described above, for all* $i \in [m]$ *it holds that*

$$\left\|\frac{X_{S_i}^\top X_{S_i}}{n} - \Sigma\right\|_2 \leq 2L\left(3\sqrt{\frac{d}{n}} + 4\frac{d}{n}\right)$$

with probability at least

$$1 - 2\exp(-d/2 + \log(m)).$$

Proof. See Example 6.3 in [30] for the single agent statement. Then apply the union bound. □

Using the above lemma in conjunction with the union bound applied and the statements in Lemma 1 the requirements for convergence of DGD CTA are met. However, observe that η in (49) corresponds to

$$\eta = 2L\left(3\sqrt{\frac{d}{n}} + 4\frac{d}{n}\right).$$

Further, the ball Δ^2_{net} scales as

$$m\alpha\|\nabla\boldsymbol{\mathcal{L}}(\boldsymbol{\theta}^\star)\|_2^2 = \frac{\alpha}{m}\sum_{i=1}^m\|\nabla\mathcal{L}_i(\theta^\star)\|_2^2,$$

which corresponds to the rate of standard linear regression with random designs which with high probability scales as $\mathcal{O}\left(\alpha\frac{d}{n}\right)$ [10].