Size Lowerbounds for Deep Operator Networks

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Abstract

Deep Operator Networks are an increasingly popular paradigm for solving regression in infinite dimensions and hence solve families of PDEs in one shot. In this work, we aim to establish a first-of-its-kind data-dependent lowerbound on the size of DeepONets required for them to be able to reduce empirical error on noisy data. In particular, we show that for low training errors to be obtained on n data points it is necessary that the common output dimension of the branch and the trunk net be scaling as $\Omega\left(\sqrt{n}\right)$. This inspires our experiments with DeepONets solving the advection-diffusion PDE, where we demonstrate the possibility that at a fixed model size, to leverage increase in this common output dimension and get monotonic lowering of training error, the size of the training data might necessarily need to scale quadratically with it.

1 Introduction

Data-driven approaches to analyze, model, and optimize complex physical systems are becoming more popular as Machine Learning (ML) methodologies are gaining prominence. Dynamic behaviour of such systems is frequently characterized using systems of Partial Differential Equations (PDEs). A large body of literature exists for using analytical or computational techniques to solve these equations under a variety of situations, such as various domain geometries, input parameters, and initial and boundary conditions. Very often one wants to solve a "parametric" family of PDEs i.e have a mechanism of quickly obtaining new solutons to the PDE upon variation of some parameter in the PDE like say the viscosity in a fluid dynamics model. This is tantamount to obtaining a mapping between the space of possible parameters and the corresponding solutions to the PDE. The cost of doing this task with conventional tools such as finite element methods (Brenner & Carstensen, 2004) is enormous since distinct simulations must be run for each unique value of the parameter, be it domain geometry or some input or boundary value. Fortuitously, in recent times there have risen a host of ML methods under the umbrella of "operator learning" to achieve this with the promise of providing better speed-accuracy trade-offs than conventional methods, (Ray et al., 2023)

A reviewed in (Ray et al., 2023), we recognize that operator learning is itself a part of the larger program of rapidly increasing interest, "physics informed machine learning" (Karniadakis et al., 2021). This program encompasses all the techniques that are being developed to utilize machine learning methods, in particular neural networks for the numerical solution of dynamics of physical systems, oftenn described as differential equations. Notable methodologies that fall under this ambit are, Physics Inspired Neural Nets (Raissi & Karniadakis, 2018), DeepONet (Lu et al., 2019), Fourier Neural Operator (Li et al., 2020b), Wavelet Neural Operator (Tripura & Chakraborty, 2022) etc.

Physics-Informed Neural Networks (PINNs) have emerged as a notable approach when there is one specific PDE of interest that needs to be solved. To the best of our knowledge some of the earliest proposals of this were made in, (Dissanayake & Phan-Thien, 1994; Lagaris et al., 1998; 2000). The modern avatar of this idea and the naming of PINNs happened in (Raissi et al., 2019). This learning framework involves minimizing the residual of the underlying partial differential equation (PDE) within the class of neural networks. Notably, PINNs are by definition an unsupervised learning method and hence they can solve PDEs with no need for knowing any sample solutions. They have demonstrated significant efficacy and computational efficiency in approximating solutions to PDEs, as evidenced by (Raissi et al., 2018), (Lu

et al., 2021), (Mao et al., 2020), (Pang et al., 2019), (Yang et al., 2021), (Jagtap & Karniadakis, 2021), (Jagtap et al., 2020), (Bai et al., 2021), A detailed review of this field can be seen at, (Cuomo et al., 2022).

As opposed to the question being solved by PINNs, Deep Operator Networks train a pair of nets in tandem to learn a (possibly nonlinear) operator mapping between infinite-dimensional Banach spaces - which de-facto then becomes a way to solve a family of parameteric PDEs in "one-shot". Its shallow version was proposed in (Chen & Chen, 1995b) and more recently its deeper versions were investigated in (Lu et al., 2019) and its theoretical foundations laid in (Lanthaler et al., 2022a).

Till date numerous variants of DeepONet models (Park et al., 2023), (Liu & Cai, 2021), (Hadorn, 2022), (Almeida et al., 2022), (Lin et al., 2022), (Xu et al., 2022), (Tan & Chen, 2022), (Zhang et al., 2022), (Goswami et al., 2022) have been proposed and this training process takes place offline within a predetermined input space. As a result, the inference phase is rapid because no additional training is needed as long as the new conditions fall within the input space that was used during training.

Other such neural operators like FNO (Li et al., 2020b), WNO[(Tripura & Chakraborty, 2022) enable efficient and accurate solutions to complex mathematical problems, opening up new possibilities for scientific computing and data-driven modeling. They have shown promise in various scientific and engineering applications including physics simulations (Choubineh et al., 2023), (Gopakumar et al., 2023), (Li et al., 2022b), (Lehmann et al., 2023), (Li et al., 2022a), image processing (Johnny et al., 2022), (Tripura et al., 2023), and weather-modelling (Kurth et al., 2022), (Pathak et al., 2022).

A deep mystery with neural nets is the effect of their size on their performance. On one hand, we know from various experiments as well as theory that the asymptotically wide nets are significantly weaker than actual neural nets and they have very different training dynamics than what is true for practically relevant nets. But, it is also known that there are specific ranges of overparametrization at which the neural net performs better than at any lower size. Modern learning architectures exploit this possibility and they are almost always designed with a large number of training parameters than the size of the training set. It seems to be surprisingly easy to find overparametrized architectures which generalize well. This contradicts the traditional understanding of the trade-off between approximation and generalization, which suggests that the generalization error initially decreases but then increases due to overfitting as the number of parameters increases (forming a U-shaped curve). However, recent research has revealed a puzzling non-monotonic dependency on model size of the generalization error at the empirical risk minimum of neural networks. This curious pattern is referred to as the "double-descent" curve,(Belkin et al., 2019). Some of the current authors had pointed out (Gopalani & Mukherjee, 2021), that the nature of this double-descent curve might be milder (and hence the classical region exists for much large range of model sizes) for DeepONets - which is the focus of this current study.

It is worth noting that this phenomenon has been observed in decision trees and random features and in various kinds of deep neural networks such as ResNets, CNNs, and Transformers (Nakkiran et al., 2021). Also, various theoretical approaches have been suggested towards deriving the double-descent risk curve, (Belkin et al., 2018a), (Belkin et al., 2018b), (Deng et al., 2022), (Kini & Thrampoulidis, 2020).

In recent times, many kinds of generalization bounds for neural nets have also been derived, like those based on Rademacher complexity (Sellke, 2023), (Golowich et al., 2018), (Bartlett et al., 2017) which are uniform convergence bounds independent of the trained predictor or results as in (Li et al., 2020a) and (Muthukumar & Sulam, 2023) which have developed data-dependent non-uniform bounds. These help explain how the generaliation error of deep neural nets might not explicitly scale with the size of the nets. Some of the current authors had previously shown (Gopalani et al., 2022) the first-of-its-kind Rademacher complexity bounds for DeepONets which does not explicitly scale with the width (and hence the number of trainable parameters) of the nets involved. Despite all these efforts, to the best of our knowledge, it has generally remained unclear as to how one might explain the necessity for overparameterization for good performance in any such neural system.

In light of this, a key advancement was made in, (Bubeck & Sellke, 2023). They showed, that with high probability over sampling n training data in d dimensions, if there has to exist a neural net f of depth D and

p parameters such that it has empirical squared-loss error below a measure of the noise in the labels then it must be true that, $\mathrm{Lip}(f) \geq \tilde{\Omega}\left(\sqrt{\frac{nd}{Dp}}\right)$. This can be interpreted as an indicator of why large models might be necessary to get low training error on real world data. Building on this work, we prove the following result (stated informally) for the specific instance of operator learning as we consider,

Theorem 1.1 (Informal Statement of Theorem 4.2)**.** *Suppose one considers a DeepONet function class at a fixed bound on the weights and the total number of parameters and both the branch and the trunk nets ending in a layer of sigmoid gates. Then with high probability over sampling a $n-$sized training data set, if this class has to have a predictor which can achieve empirical training error below a label noise dependent threshold, then necessarily the common output dimension of the branch and the trunk must be lower bounded as $\Omega\left(\sqrt{n}\right)$.*

And notably, the prefactors suppressed by Ω scale inversely with the bound on the weights and the size of the model.

Thus, to the best of our knowledge, our result here makes a first-of-its-kind progress with explaining the size requirement for DeepONets and in particular how that is related to the available size of the training data. Further, motivated by the above, we shall give experiments to demonstrate that at a fixed model size, for DeepONets to leverage an increase in the size of the common output dimension of branch and trunk, the size of the training data might need to be scaled quadratically with that.

The proof in (Bubeck & Sellke, 2023) critically uses the Lipschitzness condition of the predictors to leverage isoperimetry of the data distribution. And that raises a fundamental mismatch with the setup of operator learning - since DeepONets are not Lipschitz functions. Thus our work embarks on a program to look for an analogous insight as in (Bubeck & Sellke, 2023) that applies to DeepONets.

1.1 The Formal Setup of DeepONets

We recall the formal setup of DeepONet (Ryck & Mishra, 2022). Given $T > 0$ and $D \subset \mathbb{R}^d$ compact, consider functions $u : [0,T] \times D \to \mathbb{R}^m$, for $m \geq 1$, that solve the following time-dependent PDE,

$$\mathcal{L}_a(u)(t,x) = 0 \quad \text{and } u(x,0) = u_0 \quad \forall (t,x) \in [0,T] \times D$$

Let $\mathcal{H}$ be the function space of PDE solutions ofthe above. Define a function space $\mathcal{Y}$ s.t $u_0 \in \mathcal{Y} \subset L^2(D)$ be the space of initial conditions and $\mathcal{L}_a : \mathcal{H} \to L^2([0,T] \times D)$ is a differential operator that can depend on a parameter (function) $a \in \mathcal{Z} \subset L^2(D)$.

Corresponding to the above we have the solution operator $\mathcal{G} : \mathcal{X} \to L^2(\Omega) : f \mapsto u$, where $f \in \{u_0, a\}$ $\mathcal{X} \in \{\mathcal{Y}, \mathcal{Z}\}$, $f \in K$ where $K \subset C(D)$, with D compact domain in $\mathbb{R}^{d_1}$, and $\Omega = D$ or $\Omega = [0,T] \times D$.

The DeepONet architecture as shown in Figure 1 consists of two nets, the Branch Net, is a neural net denoted by $\mathcal{N}_\mathrm{B}$ that performs the mapping $\mathbb{R}^{d_1} \to \mathbb{R}^q$ - which in use will take as input a d_1 point discretization of a real valued function f as a vector, $\mathbf{f} = (f(x_1), f(x_2), ..., f(x_{d_1}))$ corresponding to some arbitrary choice of "sensor points" $\{x_j \mid 1 \leq j \leq m\} \subset D$. On the other hand, the Trunk Net, denoted by $\mathcal{N}_\mathrm{T}$, performs the mapping $\mathbb{R}^{d_2} \to \mathbb{R}^q$ which takes evaluation points at the domain of solution space of PDE. Then the final output is

$$\mathcal{G}_\theta\left(\underbrace{f\left(\left(x_1\right), f\left(x_2\right), \cdots, f\left(x_m\right)\right)}_{\mathbf{s}}\right)(\mathbf{p}) \coloneqq \langle \mathcal{N}_\mathrm{B}(\boldsymbol{f}), \mathcal{N}_\mathrm{T}(\boldsymbol{p}) \rangle$$

Given m, fixed sensor locations $\{x_j \mid j = 1, \ldots, m\} \subset D$ and the corresponding sensor values $\{f(x_j) \mid j = 1, \ldots, m\}$ as input, and input location $\mathbf{p} \in U$ where U compact domain in $\mathbb{R}^{d_2}$, the objective of a DeepONet is to approximate the value $\mathcal{G}(f)(\mathbf{p})$ by $\mathcal{G}_\theta\left(f\left(x_1\right), f\left(x_2\right), \cdots, f\left(x_m\right)\right)(\mathbf{p}))$. where $([f\left(x_1\right), f\left(x_2\right), \cdots, f\left(x_m\right)]$ are discrete representations of f.

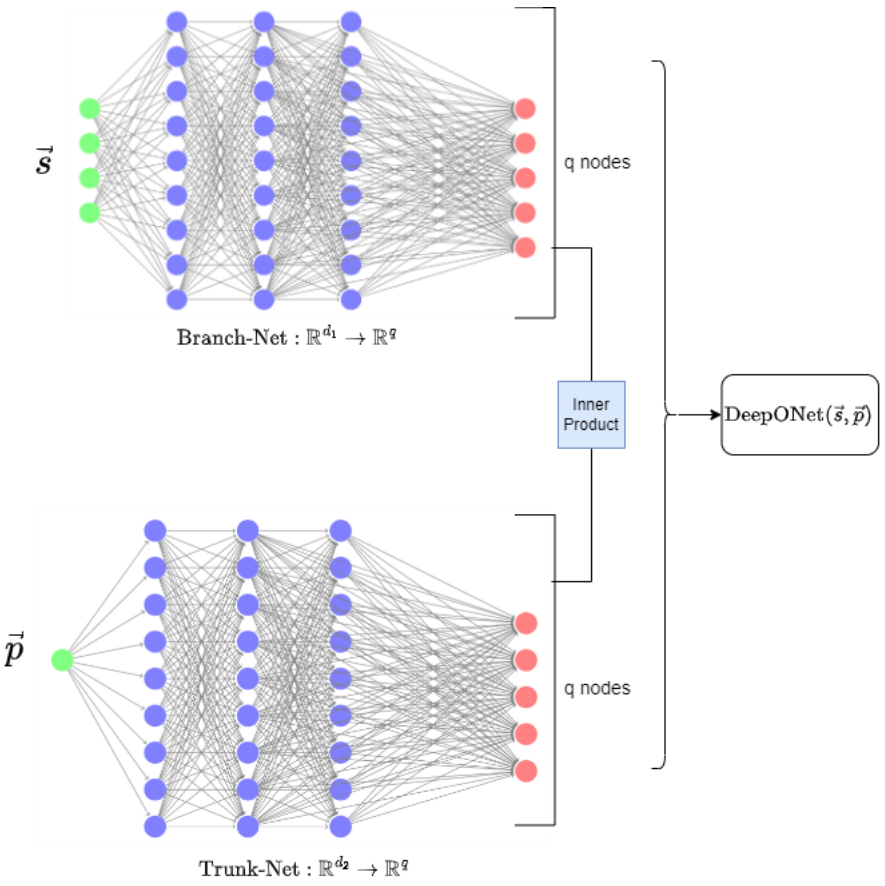

Figure 1: A Sketch of the DeepONet Architeture

Review of the Universal Approximation Property of DeepONets An universal approximation theorem for shallow DeepONets was established in (Chen & Chen, 1995a). A more general version of it was established in (Lanthaler et al., 2022b) which we shall now briefly review.

Consider two compact domains, $D \subset \mathbb{R}^d$ and $U \subset \mathbb{R}^n$, and two compact subsets of infinite dimensional Banach spaces, $K_1 \subset C(D)$ and $K_2 \subset C(U)$, where $C(D)$ represents the collection of all continuous functions defined on the domain D and similarly for $C(U)$. We then define a (possibly nonlinear) continuous operator $\mathcal{G} : K_1 \to K_2$.

Theorem 1.2. (Restatement of a key result from (Lanthaler et al., 2022b) on Generalised Universal Approximation for Operators). *Let $\mu \in \mathcal{P}(C(D))$ be a probability measure on $C(D)$. Assume that the mapping $\mathcal{G} : C(D) \to L^2(U)$ is Borel measurable and satisfies $\mathcal{G} \in L^2(\mu)$. Then, for any positive value ε, there exists an operator $\tilde{\mathcal{G}} : C(D) \to L^2(U)$, such that*

$$\|\mathcal{G} - \tilde{\mathcal{G}}\|_{L^2(\mu)} = \left(\int_{C(D)} \|\mathcal{G}(u) - \tilde{\mathcal{G}}(\mathbf{u})\|^2_{L^2(U)} d\mu(u) \right)^{1/2} < \varepsilon$$

In other words, $\tilde{\mathcal{G}}$ can approximate the original operator $\mathcal{G}$ arbitrarily close in the $L^2(\mu)$-norm with respect to the measure μ. The above approximation guarantee between DeepONets ($\tilde{\mathcal{G}}$) and solution operators of differential equations ($\mathcal{G}$) clearly motivates the use of DeepONets for solving differential equations.

2 Related Works

(Lanthaler et al., 2022b) have defined the DeepONet approximation error as follows,

$$\widehat{\mathcal{E}} = \left(\int_{C(D)} \int_U |\mathcal{G}(u)(y) - \mathcal{N}(u)(y)|^2 \, \mathrm{d}y \, \mathrm{d}\mu(u) \right)^{1/2},$$

where the DeepONet approximates the underlying operator $\mathcal{G} : C(D) \to C(U)$ and μ being as defined previously. To the best of our knowledge, the following is the only DeepONet size lowerbound proven previously,

Theorem 2.1. *Let $\mu \in \mathcal{P}\left(L^2(\mathbb{T})\right)$. Let $u \mapsto \mathcal{G}(u)$ denote the operator, mapping initial data $u(x)$ to the solution at time $t = \pi/2$,for the Burgers' PDE (Hon & Mao, 1998). Then there exists a universal constant $C > 0$ (depending only on μ, but independent of the neural network architecture), such that the DeepONet approximation error $\widehat{\mathcal{E}}$ is,*

$$\widehat{\mathcal{E}} \geqslant \frac{C}{\sqrt{p}}$$

where, p is the size of the trunk net.

Firstly, from above it does not seem possible to infer any direct connection between the net's architecture size required for any specified level of performance and training data size that is available to use. And that is a key connection that is being established in our work. *Secondly,* it is not obvious as to how one can infer any constraint on the branch net's size from the above - while our bound jointly constraints both the nets' architecture. *Thirdly,* the above-mentioned lower bound theorem is specific to Burger's PDE, while our theorem is PDE-independent.

Organization Starting in the next section we shall give the formal setup of our theory. In Section 4 we shall give the full statement of our theorem, in Section 5 we shall state all the intermediate lemmas that we need. In Section 6 we give the proof of our main theorem and in Section 7 we give the proofs of all the lemmas that are needed. Motivated by the theoretical results, in Section 8 we give an experimental demonstration revealing a property of DeepONets about how much training data is required to leverage any increase in the common output dimension of the branch and the trunk. We conclude in Section 9 delineating some open questions.

3 Our Setup

In this section we will give all the definitions about the training data and the function spaces that we shall need to state our main results.

Definition 1. Training Datasets
$(y_i, (s_i, p_i))$ be i.i.d. sampled input-output pairs and $y_i \in [-B, B]$, $\forall i$ and we define the conditional random variable $g(\boldsymbol{s}_i, \boldsymbol{p}_i) \coloneqq \mathbb{E}[y \mid (\boldsymbol{s}_i, \boldsymbol{p}_i)]$

Definition 2. Branch Functions & Trunk Functions

$$\mathcal{B} \coloneqq \{B_{\boldsymbol{w}} \text{ a function with } \leq d_B \text{ parameters } \mid B_{\boldsymbol{w}} : \mathbb{R}^{d_1} \to \mathbb{R}^q,\ \mathrm{Lip}(B_{\boldsymbol{w}}) \leq L_B \ \&\ \|\boldsymbol{w}\|_2 \leq W_B \ \&\ \|B_{\boldsymbol{w}}\|_\infty \leq \mathcal{C}\}$$

$$\mathcal{T} \coloneqq \{T_{\boldsymbol{w}} \text{ a function with } \leq d_T \text{ parameters} \mid T_{\boldsymbol{w}} : \mathbb{R}^{d_2} \to \mathbb{R}^q,\ \mathrm{Lip}(T_{\boldsymbol{w}}) \leq L_T \ \&\ \|\boldsymbol{w}\|_2 \leq W_T \ \&\ \|T_{\boldsymbol{w}}\|_\infty \leq \mathcal{C}\}$$

The functions in the set $\mathcal{B}$ shall be called the "Branch Functions" and the functions in the set $\mathcal{T}$ would be called the "Trunk Functions".

The bound of $\mathcal{C}$ in the above definitions abstracts out the model of the branch and the trunk functions being nets having a layer of bounded activation functions in their output layer - while they can have any other activation (like ReLU) in the previous layers.

Definition 3. DeepONets

$$\mathcal{H} \coloneqq \{h_{\boldsymbol{w}_b, \boldsymbol{w}_t} = h_{(\boldsymbol{w}_b, \boldsymbol{w}_t)} \mid \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \ni (\boldsymbol{s}, \boldsymbol{p}) \mapsto \boldsymbol{h}_{\boldsymbol{w}_b, \boldsymbol{w}_t}(\boldsymbol{s}, \boldsymbol{p}) \coloneqq \langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), \mathbf{T}_{\boldsymbol{w}_t}(\boldsymbol{p}) \rangle \in \mathbb{R}, B_{\boldsymbol{w}_b} \in \mathcal{B} \ \&\ \mathbf{T}_{\boldsymbol{w}_t} \in \mathcal{T}\}$$

Further, note that $\forall \theta > 0\ \exists$ a "θ-cover" of this function space $\mathcal{H}_\theta$ such that, $\forall h_{\boldsymbol{w}_b, \boldsymbol{w}_t} \in \mathcal{H}$, $\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}}, \boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta$ s.t $\left\|\boldsymbol{w}_b - \boldsymbol{w}_{b,\frac{\theta}{2}}\right\| \leq \frac{\theta}{2}$ and $\left\|\boldsymbol{w}_t - \boldsymbol{w}_{t,\frac{\theta}{2}}\right\| \leq \frac{\theta}{2}$ and $\boldsymbol{w}_{b,\frac{\theta}{2}}$ and $\boldsymbol{w}_{t,\frac{\theta}{2}}$ being elements of the $\frac{\theta}{2}$ covering space of the set of branch and trunk weights respectively.

It's easy to see how the above definition of $\mathcal{H}$ includes functions representable by the architecture given in Figure 1. Now we recall the following result about neural nets from (Bubeck & Sellke, 2023).

Lemma 3.1. *Let $f_{\boldsymbol{w}}$ be a neural network of depth D, mapping into $\mathbb{R}$ with the vector of parameters being $\boldsymbol{w} \in \mathbb{R}^p$ and all the parameters being bounded in magnitude by W i.e the set of neural networks parametrized by $\boldsymbol{w} \in [-W, W]^p$. Let Q be the maximum number of matrix or bias terms that are tied to a single parameter w_a for some $a \in [p]$. Corresponding to it we define, $B(w) \coloneqq \prod_{j \in [D]} \max\left(\|\boldsymbol{W}_j\|_{op}, 1\right)$, where $\boldsymbol{W}_j$ is the matrix in the j^{th}−layer of the net.*

Let $\boldsymbol{x} \in \mathbb{R}^d$ such that $\|\boldsymbol{x}\| \leq R$, and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{R}^p$ such that $B(\boldsymbol{w}_1), B(\boldsymbol{w}_2) \leq \bar{B}$. Then one has

$$|f_{\boldsymbol{w}_1}(\boldsymbol{x}) - f_{\boldsymbol{w}_2}(\boldsymbol{x})| \leq \bar{B}^2 Q R \sqrt{p} \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|.$$

Moreover for any $\boldsymbol{w} \in [-W, W]^p$ with $W \geq 1$, one has, $B(\boldsymbol{w}) \leq (W\sqrt{pQ})^D$.

In light of the above, we define J as follows,

Definition 4 (Defining J). *Given any two valid weight vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ for a "branch function" B we assume to have the following inequality for some fixed $J > 0$,*

$$\sup_{\boldsymbol{s}} \|B_{\boldsymbol{w}_1}(\boldsymbol{s}) - B_{\boldsymbol{w}_2}(\boldsymbol{s})\|_\infty \leq J \cdot \|\boldsymbol{w}_1 - \boldsymbol{w}_2\|$$

And similarly for the trunk functions.

One can see that the above inequality is easy to satisfy if the space of inputs to the branch or the trunk is bounded. Thus invocation of this inequality implicitly constraints the support of the data distribution.

4 Main Theorem

In the setup of the definitions given above, now we can state our main result as follows,

Theorem 4.1. *$\forall \delta \in (0,1)$ and an arbitrary positive constant Q and $\forall \theta \leq \frac{Q}{q^2}$, if we are to ensure that with probability at least $1-\delta$ with respect to the sampling of the data $\{(y_i,(\boldsymbol{s}_i,\boldsymbol{p}_i)) \mid i=1,\ldots,n\}$, $n \geq \frac{288\cdot B^2}{\theta^2}\cdot\log\frac{4}{1-\delta}$, $\exists h_{\boldsymbol{w}_b,\boldsymbol{w}_t} \in \mathcal{H}$ s.t*

$$\frac{1}{n}\sum_{i=1}^{n}\left(y_i - h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right)^2 \leq \sigma^2 - \mathcal{Q}\left(1+\mathcal{C}\cdot J\cdot\left(B+2\cdot\mathcal{C}^2\right)\right)$$

then,

$$q \geq \frac{\theta}{\sqrt{32}\cdot B\cdot\mathcal{C}^2}\cdot\sqrt{\frac{n}{\log\left(1+\frac{\left(2^{2(d_B+d_T)}\cdot\left(W_B\sqrt{d_B}\right)^{d_B}\cdot\left(W_T\sqrt{d_T}\right)^{d_T}\right)}{\theta^{(d_B+d_T)}}\right)+\log_e\frac{4}{1-\delta}}} \tag{1}$$

where $\sigma^2 := \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(y_i - g\left(\boldsymbol{s}_i,\boldsymbol{p}_i\right)\right)^2\right]$ and $g(\boldsymbol{s},\boldsymbol{p}) = \mathbb{E}\left[y \mid (\boldsymbol{s},\boldsymbol{p})\right]$.

The proof of the above can be seen in Section 6. For further insight we now specialize our Theorem 4.1 to using $\mathcal{C}=1$ – which then encompasses the case that we shall do experiments with that of having DeepONets whose branch and trunk nets end in a sigmoid gate.

Theorem 4.2. (Lowerbounds for DeepONets Whose Branch and Trunk End in Sigmoid Gates) *Let $\mathcal{C}=1$. Then $\forall \delta \in (0,1)$ and an arbitrary positive constant Q and $\forall \theta \leq \frac{Q}{q^2}$, if we are to ensure that with probability at least $1-\delta$ with respect to the sampling of the data $\{(y_i,(\boldsymbol{s}_i,\boldsymbol{p}_i)) \mid i=1,\ldots,n\}$, $n \geq \frac{288\cdot B^2}{\theta^2}\cdot\log\frac{4}{1-\delta}$, $\exists h_{\boldsymbol{w}_b,\boldsymbol{w}_t} \in \mathcal{H}$ s.t,*

$$\frac{1}{n}\sum_{i=1}^{n}\left(y_i - h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right)^2 \leq \sigma^2 - \mathcal{Q}\left(1+J\cdot\left(B+2\right)\right)$$

then,

$$q \geq \frac{\theta}{\sqrt{32}\cdot B}\cdot\frac{\sqrt{n}}{\sqrt{\log\left(1+\left(\frac{4}{\theta}\right)^{d_B+d_T}+\left(W_B\sqrt{d_B}\right)^{d_B}\cdot\left(W_T\sqrt{d_T}\right)^{d_T}\right)+\log_e\frac{4}{1-\delta}}}$$

where $\sigma^2 := \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(y_i - g\left(\boldsymbol{s}_i,\boldsymbol{p}_i\right)\right)^2\right]$ and $g(\boldsymbol{s},\boldsymbol{p}) = \mathbb{E}\left[y \mid (\boldsymbol{s},\boldsymbol{p})\right]$.

To interpret the above theorem consider a sequence of DeepONet training being done for fixed training data (and hence a fixed n) and on different architectures having the same weight bound and the same number of parameters - but allowing for variations in q, the common output dimension of the branch and the trunk functions. Now we can see how the above theorem reveals a largeness requirement for DeepONets - that if there has to exist an architecture which can get the training error below a certain label-noise dependent threshold then necessarily the branch/trunk output dimension q has to be $\Omega(\sqrt{\text{training-data-size}})$.

Later, in Section 8, we shall conduct an experimental study motivated by the above and reveal something more than what the above theorem guarantees. We will see that over a sequence of training being done on

different DeepONet architectures (and a fixed PDE) having nearly the same number of parameters, one can get monotonic improvement in performance upon increasing training data size n if it is accompanied by an increase in q s.t $\frac{q}{\sqrt{n}}$ is nearly constant. We also show that a slightly smaller rate of growth for n for the same sequence of qs would break this monotonicity. Thus it reveals a "scaling law" for DeepONets - which is not yet within the ambit of our theoretical analysis.

5 Lemmas Towards Proving Theorem 4.1

Lemma 5.1. *Recall from Definition 2, that d_B and d_T are the total number of parameters in any function in the sets $\mathcal{B}$ and $\mathcal{T}$ respectively. Let $\mathcal{W}_{\mathcal{B}} \subseteq \mathbb{R}^{d_B}$, $\mathcal{W}_{\mathcal{T}} \subseteq \mathbb{R}^{d_T}$ and $\mathcal{W}_{\mathcal{H}} = \mathcal{W}_{\mathcal{B}} \times \mathcal{W}_{\mathcal{T}}$ denote the sets of allowed weights of $\mathcal{B}$, $\mathcal{T}$, and $\mathcal{H}$ (Definition 3), respectively. Then the following three bounds hold for any $\theta > 0$,*

$$N(\theta, \mathcal{W}_{\mathcal{B}}) \leq \left(\frac{2W_B\sqrt{d_B}}{\theta}\right)^{d_B} \qquad N(\theta, \mathcal{W}_{\mathcal{T}}) \leq \left(\frac{2W_T\sqrt{d_T}}{\theta}\right)^{d_T}$$

$$N(\theta, \mathcal{W}_{\mathcal{H}}) \leq N(\theta/2, \mathcal{W}_{\mathcal{B}}) \cdot N(\theta/2, \mathcal{W}_{\mathcal{T}})$$

In above for any space X with Euclidean metric, we have denoted as $N(\theta, X)$ the covering number of it at scale θ.

The proof of the above Lemma is given in Section 7.1

Lemma 5.2. *We recall the definition of $\mathcal{H}$ from Definition 3, B as given in Defintion 1 & J from Definiton 4, Then, $\forall \theta > 0$ we have,*

$$\hat{\mathcal{R}}(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}}, \boldsymbol{w}_{t,\frac{\theta}{2}})}) \leq \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b, \boldsymbol{w}_t)}) + q\mathcal{C}J\theta \cdot \left(B + 2q\mathcal{C}^2\right)$$

and $\boldsymbol{w}_{b,\frac{\theta}{2}}$ and $\boldsymbol{w}_{t,\frac{\theta}{2}}$ be s.t. $\left\|\boldsymbol{w}_b - \boldsymbol{w}_{b,\frac{\theta}{2}}\right\| \leq \frac{\theta}{2}$ and $\left\|\boldsymbol{w}_t - \boldsymbol{w}_{t,\frac{\theta}{2}}\right\| \leq \frac{\theta}{2}$ and for any h and any training data of the form as given in Theorem 4.1, $\hat{R}(h) := \frac{1}{n}\sum_{i=1}^{n}(y_i - h(\boldsymbol{s}_i, \boldsymbol{p}_i))^2$

Thus we see that it is quantifiable as to how much is the increment in the empirical risk when for a given training data a DeepONet is replaced by another with weights within a distance of θ from the original - and that this increment is parametric in θ. The proof of the above lemma is given in Section 7.2.

Lemma 5.3. *We recall the definition of $\mathcal{H}_\theta$ from Definition 3; d_B, d_T, W_B, W_T, $\mathcal{C}$ & q from Defintion 2 and B as given in Defintion 1. Then $\forall \theta > 0$, and for $z_i := y_i - g(\boldsymbol{s}_i, \boldsymbol{p}_i)$;*

$$\begin{aligned}
&\mathbb{P}\left(\exists\ h_{(\boldsymbol{w}_{b,\frac{\theta}{2}}, \boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \frac{1}{n}\sum_{i=1}^{n} h_{(\boldsymbol{w}_{b,\frac{\theta}{2}}, \boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i)\, z_i \geq \frac{\theta}{4}\right) \\
&\leq \frac{2^{2(d_B+d_T)+1}}{\theta^{(d_B+d_T)}} \cdot \left(W_B\sqrt{d_B}\right)^{d_B} \cdot \left(W_T\sqrt{d_T}\right)^{d_T} \cdot \exp\left(-\frac{2n\theta^2}{8^4 \cdot (B \cdot q\mathcal{C}^2)^2}\right) \\
&+ 2\exp\left(\frac{-n\theta^2}{8^3 \cdot B^2 \cdot q^2 \cdot \mathcal{C}^4}\right)
\end{aligned}$$

The proof of the above lemma is given in Section 7.3

Lemma 5.4. *We continue in the same setup as in the previous lemma and further recall the definition of σ as in Theorem 4.1. Then $\forall \theta > 0$*

$$\mathbb{P}\left(\exists\ h_{\boldsymbol{w}_b, \boldsymbol{w}_t} \in \mathcal{H} \mid \frac{1}{n}\sum_{i=1}^{n}(y_i - h_{\boldsymbol{w}_b, \boldsymbol{w}_t}(\boldsymbol{s}_i, \boldsymbol{p}_i))^2 \leq \sigma^2 - \theta\right) \leq 2\exp\left(-\frac{n\theta^2}{288B^2}\right) + \mathbb{P}\left(\exists\ h_{\boldsymbol{w}_b, \boldsymbol{w}_t} \in \mathcal{H} \mid \frac{1}{n}\sum_{i=1}^{n} h(\boldsymbol{s}_i, \boldsymbol{p}_i)\, z_i \geqslant \frac{\theta}{4}\right)$$

The above lemma reveals an intimate connection between the empirical error of DeepONets and the correlation of its output with label noise. The proof of the above lemma is given in Section 7.4

6 Proof of the (Main)Theorem 4.1

A careful study of the proof of Lemma 5.4 would reveal that it can as well be invoked on $\mathcal{H}_\theta$. And doing so we get,

$$
\begin{aligned}
&\mathbb{P}\left(\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \frac{1}{n}\sum_{i=1}^{n}\left(y_i - h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right) \leq \sigma^2 - \theta\right) \\
&\leq 2\exp\left(-\frac{n\theta^2}{288\cdot B^2}\right) + \mathbb{P}\left(\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \frac{1}{n}\sum_{i=1}^{n} h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\, z_i \geqslant \frac{\theta}{4}\right) \\
&\text{Using Lemma 5.3,} \\
&\leq 2\exp\left(-\frac{n\theta^2}{288\cdot B^2}\right) + \frac{2^{2(d_B+d_T)+1}}{\theta^{(d_B+d_T)}}\cdot\left(W_B\sqrt{d_B}\right)^{d_B}\cdot\left(W_T\sqrt{d_T}\right)^{d_T}\cdot\exp\left(-\frac{2n\theta^2}{8^4\cdot(B\cdot q\mathcal{C}^2)^2}\right) \\
&+ 2\exp\left(-\frac{n\theta^2}{8^3\cdot(B\cdot q\cdot\mathcal{C}^2)^2}\right)
\end{aligned}
\tag{2}
$$

Invoking $\theta \leq \frac{\mathcal{Q}}{q^2}$ as assumed in the theorem and using Lemma 5.2 and recalling that the $q \geq 1$ we have,

$$
\hat{\mathcal{R}}(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}) \leq \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}) + \mathcal{C}\cdot J\cdot\mathcal{Q}\cdot\left(B + 2\cdot\mathcal{C}^2\right) \tag{3}
$$

With respect to random sampling of the training data we define two events $\mathbf{E}_1$ (corresponding to the function class $\mathcal{H}$) and $\mathbf{E}_2$ (corresponding to the θ–cover of $\mathcal{H}$),

$$
\mathbf{E}_1 \coloneqq \left\{\exists h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)} \in \mathcal{H} \mid \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}) \leq \sigma^2 - \theta - \mathcal{C}\cdot J\cdot\mathcal{Q}\cdot\left(B + 2\cdot\mathcal{C}^2\right)\right\}
$$

$$
\mathbf{E}_2 \coloneqq \left\{\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \hat{\mathcal{R}}(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}) \leq \sigma^2 - \theta\right\}
$$

Thus if $\mathbf{E}_1$ is true, we can invoke the above inequality to get,

$$
\begin{aligned}
\hat{\mathcal{R}}(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}) &\leq \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}) + \mathcal{C}\cdot J\cdot\mathcal{Q}\cdot\left(B + 2\cdot\mathcal{C}^2\right) \\
&\leq \sigma^2 - \theta - \mathcal{C}\cdot J\cdot\mathcal{Q}\cdot\left(B + 2\cdot\mathcal{C}^2\right) + \mathcal{C}\cdot J\cdot\mathcal{Q}\cdot\left(B + 2\cdot\mathcal{C}^2\right) \leq \sigma^2 - \theta
\end{aligned}
$$

Thus we observe that, $\mathbf{E}_1 \implies \mathbf{E}_2$ and thus $\mathbb{P}(\mathbf{E}_1) \leq \mathbb{P}(\mathbf{E}_2)$ and noting that $\theta \leq Q$, we can invoke equation 2 to get,

$$
\begin{aligned}
&\mathbb{P}\left(\exists h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)} \in \mathcal{H} \mid \underbrace{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right)^2}_{\hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)})} \leq \sigma^2 - \mathcal{Q}\left(1 + \mathcal{C}\cdot J\cdot\left(B + 2\cdot\mathcal{C}^2\right)\right)\right) \\
&\leq \mathbb{P}\left(\exists h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)} \in \mathcal{H} \mid \underbrace{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right)^2}_{\hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)})} \leq \sigma^2 - \theta - \mathcal{C}\cdot J\cdot\mathcal{Q}\cdot\left(B + 2\cdot\mathcal{C}^2\right)\right) \\
&\leq 2\exp\left(-\frac{n\theta^2}{288\cdot B^2}\right) + \frac{2^{2(d_B+d_T)+1}}{\theta^{(d_B+d_T)}}\cdot\left(W_B\sqrt{d_B}\right)^{d_B}\cdot\left(W_T\sqrt{d_T}\right)^{d_T}\cdot\exp\left(-\frac{2n\theta^2}{8^4\cdot(B\cdot q\mathcal{C}^2)^2}\right) \qquad (4) \\
&+ 2\exp\left(\frac{-n\theta^2}{8^3\cdot B^2\cdot q^2\cdot\mathcal{C}^4}\right)
\end{aligned}
$$

Hence if the required probability has to be at least $1-\delta$, its necessary that we have,

$$
(1-\delta) \leq 2\exp\left(-\frac{n\theta^2}{288\cdot B^2}\right) + \frac{2^{2(d_B+d_T)+1}}{\theta^{(d_B+d_T)}}\cdot\left(W_B\sqrt{d_B}\right)^{d_B}\cdot\left(W_T\sqrt{d_T}\right)^{d_T}\cdot\exp\left(-\frac{2n\theta^2}{8^4\cdot(B\cdot q\mathcal{C}^2)^2}\right) + 2\exp\left(\frac{-n\theta^2}{8^3\cdot(B\cdot q\mathcal{C}^2)^2}\right)
$$

Rearranging the above we can read a necessary condition for the above to be,

$$
q \geq \frac{\sqrt{2}\cdot\theta}{8^2\cdot B\cdot\mathcal{C}^2}\cdot\sqrt{\frac{n}{\log\left(1 + \frac{\left(2^{2(d_B+d_T)}\cdot\left(W_B\sqrt{d_B}\right)^{d_B}\cdot\left(W_T\sqrt{d_T}\right)^{d_T}\right)}{\theta^{(d_B+d_T)}}\right) + \log_e\frac{1}{\left(\frac{1-\delta}{2} - \exp\left(-\frac{n\theta^2}{288\cdot B^2}\right)\right)}}} \qquad (5)
$$

Now invoking the largeness assumption on the size of the training data, as given in the preamble of the theorem, we arrive at the largeness requirement on q as stated in the theorem.

7 Proofs of the Lemmas

7.1 Proof of Lemma 5.1

Proof. The first two equations are standard results, Example 27.1 of (Shalev-Shwartz & Ben-David, 2014)

Further define $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|_2$. Then, let $S \subset \mathbb{R}^{d_B}$ be a witness for $N(\theta/2, \mathcal{W}_\mathcal{B})$, that is, for all $\boldsymbol{w}_b \in \mathcal{W}_\mathcal{B}$, there is some $s \in S$ such that $d(\boldsymbol{w}_b, s) \leq \theta/2$. Similarly, let $P \subset \mathbb{R}^{d_T}$ be a witness for $N(\theta/2, \mathcal{W}_\mathcal{T})$. Then for all $\boldsymbol{w}_b \in \mathcal{W}_\mathcal{B}$, $\boldsymbol{w}_t \in \mathcal{W}_\mathcal{T}$, there exist a corresponding cover point $s \in S$ and $p \in P$. And since $(\boldsymbol{w}_b, \boldsymbol{w}_t) \in \mathcal{W}_\mathcal{H}$:

$$
\begin{aligned}
d((\boldsymbol{w}_b,\boldsymbol{w}_t),(s,p)) &\leq d((\boldsymbol{w}_b,\boldsymbol{w}_t),(s,\boldsymbol{w}_t)) + d((s,\boldsymbol{w}_t),(s,p)) \quad \text{(by triangle inequality)} \\
&= d(\boldsymbol{w}_b, s) + d(\boldsymbol{w}_t, p) \quad \text{(under } d \sim l_2\text{-norm)} \\
&\leq \theta \quad \text{(by definition of } S \text{ and } P)
\end{aligned}
$$

Hence, $S \times T$ is an θ-cover of $\mathcal{W}_\mathcal{H}$.

□

7.2 Proof of Lemma 5.2

Proof. Given an $\theta > 0$ and a $h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)} \in \mathcal{H}$, let $\boldsymbol{w}_{b,\frac{\theta}{2}}$ and $\boldsymbol{w}_{t,\frac{\theta}{2}}$ be s.t. $\left\|\boldsymbol{w}_b - \boldsymbol{w}_{b,\frac{\theta}{2}}\right\| \leq \frac{\theta}{2}$ and $\left\|\boldsymbol{w}_t - \boldsymbol{w}_{t,\frac{\theta}{2}}\right\| \leq \frac{\theta}{2}$. Then from the definition of J in Definition 4, the following inequalities hold,

$$\sup_{\boldsymbol{s}} \left\| B_{\boldsymbol{w}_b}(\boldsymbol{s}) - B_{\boldsymbol{w}_{b,\frac{\theta}{2}}}(\boldsymbol{s}) \right\|_\infty \leq J.\frac{\theta}{2} \text{ and } \sup_{\boldsymbol{p}} \left\| T_{\boldsymbol{w}_t}(\boldsymbol{p}) - T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\|_\infty \leq J.\frac{\theta}{2}$$

Further, we can simplify as follows, for any valid $(\boldsymbol{s},\boldsymbol{p})$ input to the function $h_{\boldsymbol{w}_b,\boldsymbol{w}_t} = \langle B_{\boldsymbol{w}_b}, T_{\boldsymbol{w}_t} \rangle$ and similarly for $h_{\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}}}$.

$$\begin{aligned}
&\left| \langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), T_{\boldsymbol{w}_t}(\boldsymbol{p}) \rangle - \left\langle B_{\boldsymbol{w}_{b,\frac{\theta}{2}}}(\boldsymbol{s}), T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\rangle \right| \\
=&\left| \langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), T_{\boldsymbol{w}_t}(\boldsymbol{p}) \rangle - \left\langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\rangle + \left\langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\rangle - \left\langle B_{\boldsymbol{w}_{b,\frac{\theta}{2}}}(\boldsymbol{s}), T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\rangle \right| \\
\leq&\left| \left\langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), T_{\boldsymbol{w}_t}(\boldsymbol{p}) - T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\rangle \right| + \left| \left\langle T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}), B_{\boldsymbol{w}_b}(\boldsymbol{s}) - B_{\boldsymbol{w}_{b,\frac{\theta}{2}}}(\boldsymbol{s}) \right\rangle \right|
\end{aligned}$$

To upperbound the above we recall (a) the definition of $\mathcal{C}$ from Definitions 2 and (b) that for any two $q-$dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we have, $|\langle \boldsymbol{a},\boldsymbol{b} \rangle| \leq \sum_{i=1}^q |a_i||b_i| \leq (\max_{i=1,\ldots,q} |b_i|) \sum_{i=1}^q |a_i|$. Thus we have,

$$\forall (\boldsymbol{s},\boldsymbol{p}),\ \left| \langle B_{\boldsymbol{w}_b}(\boldsymbol{s}), T_{\boldsymbol{w}_t}(\boldsymbol{p}) \rangle - \left\langle B_{\boldsymbol{w}_{b,\frac{\theta}{2}}}(\boldsymbol{s}), T_{\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{p}) \right\rangle \right| \leq 2 \cdot \left(\frac{J\theta}{2} \cdot q \cdot \mathcal{C} \right) \tag{6}$$

$$\implies \forall (\boldsymbol{s},\boldsymbol{p}),\ \left| h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s},\boldsymbol{p}) - h_{\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}}}(\boldsymbol{s},\boldsymbol{p}) \right| \leq q \cdot \mathcal{C} \cdot J\theta \tag{7}$$

Define, $r_{1,i} := \left(y_i - h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) \right)$ and $r_{2,i} := \left(y_i - h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) \right)$

Now,

$$\begin{aligned}
r_{1,i}^2 - r_{2,i}^2 &= (h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)^2 - h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i)^2) + 2y_i \left(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) - h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) \right) \\
&\leq \left(\left| h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) - h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) \right| \right) \left(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) + h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) \right) + 2 \cdot B \cdot q \cdot \mathcal{C} \cdot J\theta \\
&\leq \left(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) + h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) \right) \cdot q \cdot \mathcal{C} \cdot J\theta + B \cdot q \cdot \mathcal{C} \cdot J\theta \\
&\leq q \cdot \mathcal{C} \cdot J\theta \cdot \left((h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) + h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) + B \right)
\end{aligned}$$

Averaging the above over all training data we get,

$$\frac{1}{n}\sum_{i=1}^n r_{1,i}^2 \leq \frac{1}{n}\sum_{i=1}^n r_{2,i}^2 + \frac{1}{n}\sum_{i=1}^n q \cdot \mathcal{C} \cdot J\theta \cdot \left((h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) + h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)) + B \right) \tag{8}$$

Using Cauchy-Schwarz over the inner-product in the definition of h, we get,

$$\left| h_{(w_b,w_t)}(s_i,p_i) \right| \leq \sqrt{q}\mathcal{C} \cdot \sqrt{q}\mathcal{C} \leq q \cdot \mathcal{C}^2 \implies \left((h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) + h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)) \right) \leq 2q \cdot \mathcal{C}^2 \tag{9}$$

Substituting the above into equation 8 and invoking the definition of $\hat{\mathcal{R}}$,

$$\begin{aligned}\hat{\mathcal{R}}(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}) &\leq \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}) + \frac{1}{n}\sum_{i=1}^{n} q\cdot\mathcal{C}\cdot J\theta\cdot\left((h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}(\boldsymbol{s}_i,\boldsymbol{p}_i) + h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)) + B\right)\\ &\leq \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}) + (q\cdot\mathcal{C}\cdot J\theta\cdot B) + (q\cdot\mathcal{C}\cdot J\theta)\cdot\left(2q\cdot\mathcal{C}^2\right)\\ &\leq \hat{\mathcal{R}}(h_{(\boldsymbol{w}_b,\boldsymbol{w}_t)}) + q\mathcal{C}J\theta\cdot\left(B + 2q\mathcal{C}^2\right)\end{aligned}$$

The above is what we set out to prove.

□

7.3 Proof of Lemma 5.3

Proof. Recall that for each data i, we had defined the random variable, $z_i := y_i - g(\boldsymbol{s}_i,\boldsymbol{p}_i)$. Since $g(\boldsymbol{s},\boldsymbol{p}) = \mathbb{E}[y \mid (\boldsymbol{s},\boldsymbol{p})]$, we can note that $\mathbb{E}[z_i] = 0$. Further,

$$z_i^2 = (y_i - g(\boldsymbol{s}_i,\boldsymbol{p}_i))^2 \leq y_i^2 - 2\cdot y_i\cdot g(\boldsymbol{s}_i,\boldsymbol{p}_i) + g(\boldsymbol{s}_i,\boldsymbol{p}_i)^2 \leq 4B^2 \tag{10}$$

Recall from equation 9. that $\left|h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right| \leq q\cdot\mathcal{C}^2$

For each data i, we further define the random variable, $Y_{\theta,i} := \left((h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) - \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}])z_i\right)$

Now note that,

$$\begin{aligned}\mathbb{E}[Y_{\theta,i}] &= \mathbb{E}\left[(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) - \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}])z_i\right]\\ &= \mathbb{E}\left[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\cdot y_i\right] - \mathbb{E}\left[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\cdot g(\boldsymbol{s}_i,\boldsymbol{p}_i)\right]\end{aligned}$$

Next, we use the tower property of conditional expectation to expand the first term,

$$\begin{aligned}\mathbb{E}\left[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\cdot y_i\right] &= \mathbb{E}\left[\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\, y_i \mid (\boldsymbol{s}_i,\boldsymbol{p}_i)]\right] = \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\,\mathbb{E}[y \mid (\boldsymbol{s}_i,\boldsymbol{p}_i)]]\\ &= \mathbb{E}\left[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\cdot g(\boldsymbol{s}_i,\boldsymbol{p}_i)\right]\end{aligned}$$

Substituting this back into the previous equation, we get,

$$\mathbb{E}[Y_{\theta,i}] = 0$$

Further,

$$\begin{aligned}|Y_{\theta,i}| &= \left|h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i) - \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]\right|\cdot|z_i| \leq \left(\left|h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)\right| + \left|\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]\right|\right)\cdot 2B\\ &\leq 4\cdot\mathcal{C}^2\cdot B\cdot q\end{aligned}$$

Applying Hoeffding's inequality [1] on $Y_{\theta,i}$, we will get,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}\left((h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)-\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]\right)z_i\right)\geq t\right)\leq \exp\left(-\frac{2nt^2}{(8\cdot B\cdot q\mathcal{C}^2)^2}\right) \tag{11}$$

We choose $t=\frac{\theta}{8}$ to get,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}\left((h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)-\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]\right)z_i\right)\right|\geq\frac{\theta}{8}\right)\leq 2\cdot\exp\left(-\frac{2n\theta^2}{8^4\cdot(B\cdot q\mathcal{C}^2)^2}\right)$$

We define two events,

$$\mathbf{E}_5\coloneqq\left\{\left|\frac{1}{n}\sum_{i=1}^{n}z_i\right|\geq\frac{\theta}{8\cdot q\mathcal{C}^2}\right\}\ \&\ \mathbf{E}_6\coloneqq\left\{\exists\ h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}\in\mathcal{H}_\theta\mid\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]z_i\geq\frac{\theta}{8}\right\}$$

Recalling the bound on the h function we have, $\frac{1}{q\cdot\mathcal{C}^2}\cdot|\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]|\in[0,1]$, we have that if $\mathbf{E}_5^c$ happens then for such a sample of $\{z_i,i=1,\ldots,n\}$,

$$\begin{aligned}&\forall h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}\in\mathcal{H}_\theta,\\&\frac{\theta}{8\cdot q\mathcal{C}^2}>\left|\frac{1}{n}\sum_{i=1}^{n}z_i\right|\geq\frac{1}{q\cdot\mathcal{C}^2}\cdot|\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]|\left|\frac{1}{n}\sum_{i=1}^{n}z_i\right|\geq\frac{1}{n\cdot q\mathcal{C}^2}\sum_{i=1}^{n}\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]z_i\end{aligned}$$

Hence $\mathbf{E}_5^c\implies\mathbf{E}_6^c$ and hence $\mathbb{P}(\mathbf{E}_6)\leq\mathbb{P}(\mathbf{E}_5)$ i.e

$$\mathbb{P}\left(\exists\ h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}\in\mathcal{H}\mid\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]z_i\geq\frac{\theta}{8}\right)\leq\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}z_i\right|\geq\frac{\theta}{8\cdot q\cdot\mathcal{C}^2}\right) \tag{12}$$

Recalling that $(|z_i|\leq 2B)$, by Hoeffding's inequality we have,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}z_i\right|\geq\frac{\theta}{8\cdot q\cdot\mathcal{C}^2}\right)\leq 2\exp\left(\frac{-n\theta^2}{8^3\cdot B^2\cdot q^2\cdot\mathcal{C}^4}\right) \tag{13}$$

Now we define three events $\mathbf{E}_7$,$\mathbf{E}_8$ and $\mathbf{E}_9$ as follows,

$$\mathbf{E}_7\coloneqq\left\{\forall\ h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}\in\mathcal{H}_\theta,\frac{1}{n}\sum_{i=1}^{n}\left(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i,\boldsymbol{p}_i)-\mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]\right)z_i\leq\frac{\theta}{8}\right\}$$

[1]

Theorem 7.1. *(Hoeffding's inequality). Let $Z_1,\ldots,Z_n$ be independent bounded random variables with $Z_i\in[a,b]$ for all i, where $-\infty<a\leq b<\infty$. Then*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(Z_i-\mathbb{E}[Z_i])\geq t\right)\leq\exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$

and

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(Z_i-\mathbb{E}[Z_i])\leq -t\right)\leq\exp\left(-\frac{2nt^2}{(b-a)^2}\right)$$

for all $t\geq 0$.

$$\mathbf{E}_8 \coloneqq \left\{ \forall\ h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta, \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}] z_i \leq \frac{\theta}{8} \right\}$$

$$\mathbf{E}_9 \coloneqq \left\{ \forall\ h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta, \frac{1}{n}\sum_{i=1}^{n} h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i)\, z_i \leq \frac{\theta}{4} \right\}$$

Observe that, if $\mathbf{E}_7$ and $\mathbf{E}_8$ hold then $\mathbf{E}_9$ will also hold.

Hence,

$$\mathbb{P}(\mathbf{E}_7 \cap \mathbf{E}_8) \leq \mathbb{P}(\mathbf{E}_9) \implies \mathbb{P}(\mathbf{E}_9^c) \leq \mathbb{P}(\mathbf{E}_7^c) + \mathbb{P}(\mathbf{E}_8^c)$$

Thus, we can invoke equations 12 and 13 to get,

$$\mathbb{P}\left(\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \frac{1}{n}\sum_{i=1}^{n} h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i)\, z_i \geq \frac{\theta}{4} \right)$$

$$\leq \mathbb{P}\left(\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \frac{1}{n}\left| \sum_{i=1}^{n} \left(h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i) - \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}] \right) z_i \right| \geq \frac{\theta}{8} \right) + \mathbb{P}\left(\left| \frac{1}{n}\sum_{i=1}^{n} z_i \right| \geq \frac{\theta}{8 \cdot q \cdot \mathcal{C}^2} \right)$$

$$\leq \mathbb{P}\left(\bigcup_{h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta} \left\{ \frac{1}{n}\left| \sum_{i=1}^{n} (h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i) - \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]) z_i \right| \geq \frac{\theta}{8} \right\} \right) + 2\exp\left(\frac{-n\theta^2}{8^3 \cdot B^2 \cdot q^2 \cdot \mathcal{C}^4} \right)$$

$$\leq \sum_{h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta} \mathbb{P}\left(\frac{1}{n}\left| \sum_{i=1}^{n} (h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i) - \mathbb{E}[h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}]) z_i \right| \geq \frac{\theta}{8} \right) + 2\exp\left(\frac{-n\theta^2}{8^3 \cdot B^2 \cdot q^2 \cdot \mathcal{C}^4} \right)$$

Hence,

$$\mathbb{P}\left(\exists h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})} \in \mathcal{H}_\theta \mid \frac{1}{n}\sum_{i=1}^{n} h_{(\boldsymbol{w}_{b,\frac{\theta}{2}},\boldsymbol{w}_{t,\frac{\theta}{2}})}(\boldsymbol{s}_i, \boldsymbol{p}_i)\, z_i \geq \frac{\theta}{4} \right)$$
$$\leq \frac{2^{2(d_B+d_T)}}{\theta^{(d_B+d_T)}} \cdot \left(W_B\sqrt{d_B} \right)^{d_B} \cdot \left(W_T\sqrt{d_T} \right)^{d_T} \cdot 2 \cdot \exp\left(-\frac{2n\theta^2}{8^4 \cdot (B \cdot q\mathcal{C}^2)^2} \right)$$
$$+ 2\exp\left(\frac{-n\theta^2}{8^3 \cdot B^2 \cdot q^2 \cdot \mathcal{C}^4} \right)$$

And the above is what we set out to prove. □

7.4 Proof of Lemma 5.4

Proof. Recall the definition of z_i from the previous proof and and from the assumptions in Theorem 4.2 we have, $\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[z_i^2\right] = \sigma^2$. Recalling that $z_i \in [-2B, 2B]$ and they are i.i.d. we can invoke Hoeffding's Lemma 7.1 (with $t = \frac{\theta}{6}, b = 2B, a = -2B$) to get,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} z_i^2 \leq \sigma^2 - \frac{\theta}{6} \right) \leq \exp\left(-\frac{n\theta^2}{288B^2} \right) \tag{14}$$

Further note that, $z_i \cdot g(\boldsymbol{s}_i, \boldsymbol{p}_i)$ is i.i.d with mean 0 since $\mathbb{E}[z_i \mid (\boldsymbol{s}_i, \boldsymbol{p}_i)] = 0$ and $|z_i \cdot g(\boldsymbol{s}_i, \boldsymbol{p}_i)| \leq 2B$
Applying Hoeffding's inequality again,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} z_i g\left(\boldsymbol{s}_i, \boldsymbol{p}_i\right) \leq -\frac{\theta}{6}\right) \leq \exp\left(-\frac{n\theta^2}{288B^2}\right) \tag{15}$$

Given a $h_{\boldsymbol{w}_b,\boldsymbol{w}_t} \in \mathcal{H}$, we define the following vector random variables,

$$Z := \frac{1}{\sqrt{n}}\left(z_1, z_2, \cdots, z_n\right) \tag{16}$$

$$G = \frac{1}{\sqrt{n}}\left(g\left(\boldsymbol{s}_1, \boldsymbol{p}_1\right), g\left(\boldsymbol{s}_2, \boldsymbol{p}_2\right), \cdots, g\left(\boldsymbol{s}_n, \boldsymbol{p}_n\right)\right) \tag{17}$$

$$F = \frac{1}{\sqrt{n}}\left(h_{\boldsymbol{w}_b,\boldsymbol{w}_t}\left(\boldsymbol{s}_1, \boldsymbol{p}_1\right), h_{\boldsymbol{w}_b,\boldsymbol{w}_t}\left(\boldsymbol{s}_2, \boldsymbol{p}_2\right), \cdots, h_{\boldsymbol{w}_b,\boldsymbol{w}_t}\left(\boldsymbol{s}_n, \boldsymbol{p}_n\right)\right) \tag{18}$$

Note that,

$$\|G + Z - F\|^2 = \|\frac{1}{\sqrt{n}}\left(g\left(\boldsymbol{s}_1, \boldsymbol{p}_1\right), \cdots, g\left(\boldsymbol{s}_n, \boldsymbol{p}_n\right)\right) + \frac{1}{\sqrt{n}}\left(z_1, \ldots, z_n\right) - \frac{1}{\sqrt{n}}(h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s}_1, \boldsymbol{p}_1), \cdots h_{\boldsymbol{w}_b,\boldsymbol{w}_t}\left(\boldsymbol{s}_n, \boldsymbol{p}_n\right))\|^2 \tag{19}$$

Recalling that $z_i := y_i - g\left(\boldsymbol{s}_i, \boldsymbol{p}_i\right)$ and the definition of the empirical risk of the predictor, $\hat{R}(h_{\boldsymbol{w}_b,\boldsymbol{w}_t}) := \frac{1}{n}\sum_{i=1}^{n}(y_i - h_{\boldsymbol{w}_b,\boldsymbol{w}_t}(\boldsymbol{s}_i, \boldsymbol{p}_i))^2$, we realize that,

$$\|Z + G - F\|^2 = \hat{R}(h_{\boldsymbol{w}_b,\boldsymbol{w}_t})$$

Suppose, $\|Z\|^2 \geqslant \sigma^2 - \frac{\theta}{6}$ and $\langle Z, G\rangle \geqslant -\frac{\theta}{6}$. Then we have,

$$\begin{aligned}\|Z + G - F\|^2 &= \|Z\|^2 + 2\langle Z, G - F\rangle + \|G - F\|^2 = \|Z\|^2 + 2\langle Z, G\rangle - 2\langle Z, F\rangle + \|G - F\|^2 \\ &\geq \sigma^2 - \frac{\theta}{6} - 2\frac{\theta}{6} - 2\langle Z, F\rangle \geq \sigma^2 - \frac{\theta}{2} - 2\langle Z, F\rangle.\end{aligned}$$

If further we have, $\|Z + G - F\|^2 \leq \sigma^2 - \theta$ then we have from above, $\langle F, Z\rangle \geq \frac{\theta}{4}$

Motivated by the above, we define the following 4 events, namely $\mathbf{E}_i, i = 1, \ldots, 4$

$$\mathbf{E}_1 := \left\{\|Z\|^2 \geq \sigma^2 - \frac{\theta}{6}\right\},\ \mathbf{E}_2 := \left\{\langle Z, G\rangle \geq -\frac{\theta}{6}\right\}, \mathbf{E}_3 := \left\{\exists\ h_{\boldsymbol{w}_b,\boldsymbol{w}_t} \in \mathcal{H} \mid \hat{\mathcal{R}} \leq \sigma^2 - \theta\right\}\ \&\ \mathbf{E}_4 := \left\{\exists\ h_{\boldsymbol{w}_b,\boldsymbol{w}_t} \in \mathcal{H} \mid \langle F, Z\rangle \geq \frac{\theta}{4}\right\}$$

Thus our above argument can be summarized to say that if the events $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ hold, then $\mathbf{E}_4$ will also hold. This we can write as, $\mathbb{P}(\mathbf{E}_1 \cap \mathbf{E}_2 \cap \mathbf{E}_3) \leq \mathbb{P}(\mathbf{E}_4)$. This implies, $\mathbb{P}(\mathbf{E}_4) \geq 1 - \mathbb{P}((\mathbf{E}_1 \cap \mathbf{E}_2 \cap \mathbf{E}_3)^c)$. But, by union bounding, $\mathbb{P}\left(\mathbf{E}_1^c \cup \mathbf{E}_2^c \cup \mathbf{E}_3^c\right) \leq \mathbb{P}(\mathbf{E}_1^c) + \mathbb{P}(\mathbf{E}_2^c) + \mathbb{P}(\mathbf{E}_3^c) \leq 3 - (\mathbb{P}(\mathbf{E}_1) + \mathbb{P}(\mathbf{E}_2) + \mathbb{P}(\mathbf{E}_3))$. Hence combining we have, $\mathbb{P}(\mathbf{E}_4) \geq -2 + (\mathbb{P}(\mathbf{E}_1) + \mathbb{P}(\mathbf{E}_2) + \mathbb{P}(\mathbf{E}_3))$

From equations 14 and 15 we obtain, that, $(1 - \mathbb{P}(\mathbf{E}_1)) \leq \exp\left(-\frac{n\theta^2}{288B^2}\right)$ and similarly for $(1 - \mathbb{P}(\mathbf{E}_2))$.

Thus substituting in above we get,

$$\mathbb{P}(\mathbf{E}_3) \leq 2\exp\left(-\frac{n\theta^2}{288B^2}\right) + \mathbb{P}(\mathbf{E}_4)$$

Thus we have proven what we had set out to prove, □

8 The Experiment Set-up

In this section we shall demonstrate that at a fixed number of total parameters, increasing the output dimension(q) arbitrarily high keeps errors down. We shall also show an ablation study on $\frac{q}{\sqrt{n}}$

The advection-diffusion-reaction partial differential equation (PDE) (Rahaman et al., 2022) plays a crucial role in modeling various physical, chemical, and biological processes. This PDE is important as it allows us to understand and predict the behavior of substances or quantities that are transported, diffused, and react within a system.

A advection-diffusion-reaction system with a source term $f(x)$ is described by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ku^2 + f(x), \quad x \in [0,1], t \in [0,1]$$

with zero initial/boundary conditions, where $D = 0.01$ is the diffusion coefficient, and $k = 0.01$ is the reaction rate. We use DeepONets to learn the operator mapping from $f(x)$ to the PDE solution $u(x,t)$. In this case the operator $\mathcal{G}_\theta$ will map the source terms $f(x)$ to the PDE solution $u(x,t)$. As above. given a choice of m sensor points in the domain of the solutions, we shall denote a discretize a f onto the sensor points as the vector $\mathbf{f} \in \mathbb{R}^m$. Recalling the DeepONet operator loss, we realize that minimizing that is trying to induce, $\mathcal{G}_\theta(\mathbf{f}^{(i)})(x,t) \approx \mathcal{G}(f^{(i)})(x,t) = f^{(i)}(x,t), \ \forall i.$

Hence each training data can be seen as a 3-tuple, given by $(\mathbf{f}, \mathbf{p}, \mathbf{y})$, where $\mathbf{f} = (f(x_1), f(x_2), ..., f(x_m))$, For sampling f we have considered Gaussian random field(GRF) distribution. Here we have used the mean-zero GRF, $f \sim \mathcal{G}\left(0, k_l\left(x_1, x_2\right)\right)$ where the covariance kernel $k_l\left(x_1, x_2\right) = \exp\left(-\left\|x_1 - x_2\right\|^2 / 2l^2\right)$ is the radial-basis function (RBF) kernel with a length-scale parameter $l > 0$. For our experiments we have taken $l = 10^{-3}$. After sampling f from the chosen function spaces, we solve the PDE by a second-order finite difference method to obtain the reference solutions.

For n training data samples, the ℓ_2 empirical loss being minimized is, $\hat{\mathcal{L}}_{\text{DeepONet}} \coloneqq \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \mathcal{G}_\theta\left(f_i\right)\left(p_i\right)\right)^2$, where p_i is a randomly sampled point in the (x,t) space and y_i is the approximate PDE solution at p_i corresponding to $\boldsymbol{f}_i$ – which we recall was obtained from a conventional solver.

8.1 Implementations & Results

We created 10 DeepOnet models in each experimental setting such that each model has a depth of 5 and width has been varied from 24 to 50 for each layer while keeping the total number of training parameters approximately equal for each of those 10 models. For each case the branch input dimension is 40(i.e number of sensor points), and trunk input dimension is 2. We have taken the starting value of the training data size (n) as 10^4 and for the rest of the models we have varied the output dimension q. And twice we make this choice of 10 different (q, n) parameterized learning setups, once keeping the ratio $\frac{q}{\sqrt{n}}$ approximately constant and then holding the ratio $\frac{q}{n^{\frac{2}{3}}}$) almost fixed.

The code for this experiment can be found in our GitHub repository (link). The DeepONet model is trained by stochastic Adam optimizer

Experiment in $\frac{q}{\sqrt{n}}$ fixed setting. In this setting, the q value was varied from 5 to 50, in increments of 5. We have taken the starting value of n as 10^4. In Figure 2 we have plotted the training loss dynamics for these 10 models being trained over 200 epochs.

Experiment in $\frac{q}{n^{\frac{2}{3}}}$ setting. We repeat the above experiment but while appproximately fixing the value of $\frac{q}{n^{\frac{2}{3}}}$. The corresponding plots are shown in Figure 3.

We draw two primary conclusions from the above results. *Firstly*, from Figure 2, we can observe that if q and n increase at a fixed $\frac{q}{\sqrt{n}}$ then performance increases almost monotonically. *Secondly*, from the Figure 3 it is clearly visible that the previous monotonicity is breaking - that is the rate of increase of data size in the later experiment was not sufficient to leverage the increase in the output dimension size of the branch and the trunk as was happening in the first figure.

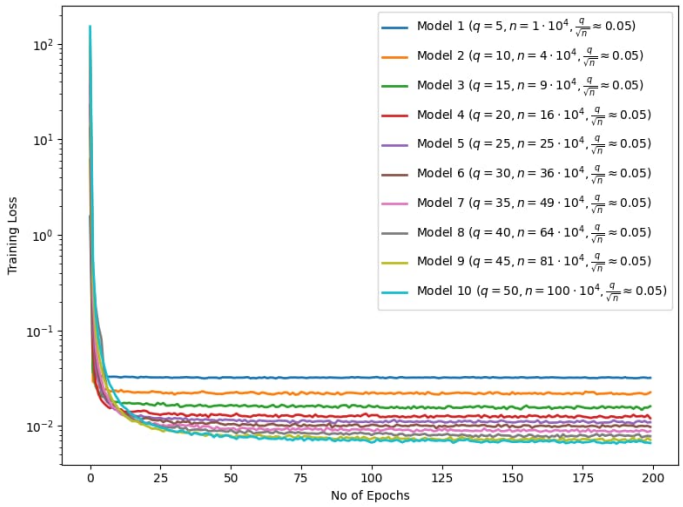

Figure 2: Training Loss vs Epoch in fixed $\frac{q}{\sqrt{n}}$ setting

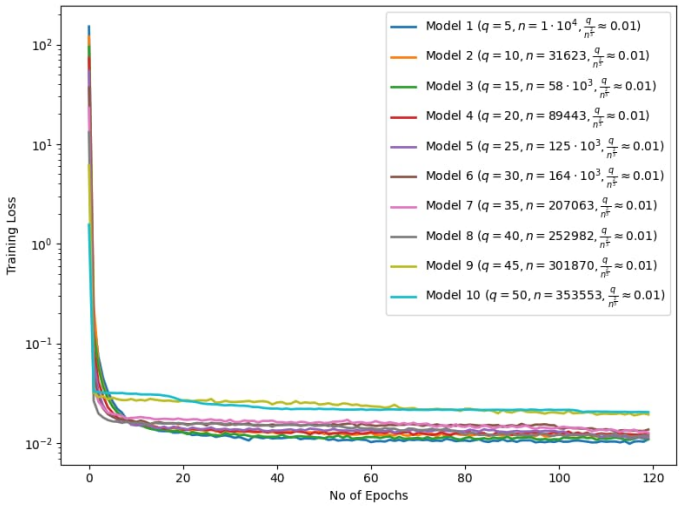

Figure 3: Training Loss vs Epoch in fixed $\frac{q}{n^{\frac{2}{3}}}$ setting

9 Discussion

Our key result Theorem 4.1 shows that a certain data size dependent largeness of q is needed if there has to exist a bounded weight DeepONet at that q which can have their empirical error below the label noise threshold. From our experiments, we have shown that there is some non-trivial range of q (the common output dimension) along which empirical risk improves with q for a fixed model size - if the amount of training data is scaled quadratically with q. We envisage that trying to prove this "scaling law" can be a very interesting direction for future exploration in theory.

Secondly, we note that our result hasn't yet fully exploited the structure of the neural nets used in the branch and the trunk. Also, it might be interesting to understand how to tune the argument specifically for the different variations of this architecture (Kontolati et al., 2023), (Bonev et al., 2023) that are getting deployed, Lastly, we note that our result is currently agnostic to the PDE being attempted to be solved. There is a tantalizing possibility, that methods in this proof could be extended to derive bounds which can distinguish PDEs that are significantly hard for operator learning.

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