CHARACTERIZING THE SPECTRUM OF THE NTK VIA A POWER SERIES EXPANSION

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ABSTRACT

Under mild conditions on the network initialization we derive a power series expansion for the Neural Tangent Kernel (NTK) of arbitrarily deep feedforward networks in the infinite width limit. We provide expressions for the coefficients of this power series which depend on both the Hermite coefficients of the activation function as well as the depth of the network. We observe faster decay of the Hermite coefficients leads to faster decay in the NTK coefficients. Using this series, first we relate the effective rank of the NTK to the effective rank of the input-data Gram. Second, for data drawn uniformly on the sphere we derive an explicit formula for the eigenvalues of the NTK, which shows faster decay in the NTK coefficients implies a faster decay in its spectrum. From this we recover existing results on eigenvalue asymptotics for ReLU networks and comment on how the activation function influences the RKHS. Finally, for generic data and activation functions with sufficiently fast Hermite coefficient decay, we derive an asymptotic upper bound on the spectrum of the NTK.

1 INTRODUCTION

Neural networks currently dominate modern artificial intelligence, yet it is an active challenge to establish a principled basis for their understanding. The key difficulties are that neural networks induce nonconvex optimization objectives (Sontag & Sussmann, 1989) and typically operate in an overparameterized regime which precludes classical statistical learning theory (Anthony & Bartlett, 2002). The persistent success of overparameterized models tuned via nonconvex optimization suggests that the relationship between the parameterization, optimization, and generalization is more sophisticated than what classical theories can address.

A recent breakthrough on the success of overparameterized networks was established through the Neural Tangent Kernel (NTK) (Jacot et al., 2018). In the infinite width limit the optimization dynamics are described entirely by the NTK and the parameterization behaves like a linear model (Lee et al., 2019). In this regime explicit guarantees for the optimization and generalization can be obtained (Du et al., 2019a;b; Arora et al., 2019a; Allen-Zhu et al., 2019). While one must be judicious when extrapolating insights from the NTK to finite width networks (Lee et al., 2020), the NTK remains one of the most promising avenues to understand deep learning on a principled basis.

The spectrum of the NTK is fundamental to both the optimization and generalization of wide networks. Bounding the smallest eigenvalue of the NTK Gram matrix is a staple technique for establishing convergence guarantees for the optimization (Du et al., 2019a;b; Oymak & Soltanolkotabi, 2020). Furthermore, the full spectrum of the NTK Gram matrix governs the dynamics of the empirical risk (Arora et al., 2019b), and the eigenvalues of the associated integral operator characterize the dynamics of the generalization error outside the training set (Bowman & Montúfar, 2022a;b). Moreover, the decay rate of the generalization error for Gaussian process regression using the NTK can be characterized by the decay rate of the spectrum (Caponnetto & De Vito, 2007; Cui et al., 2021; Jin et al., 2022).

The importance of the spectrum of the NTK has led to a variety of efforts to characterize its structure via random matrix theory and other tools (Yang & Salman, 2019; Fan & Wang, 2020). There is a broader body of work studying the closely related Conjugate Kernel, Fisher Information Matrix, and Hessian (Poole et al., 2016a; Pennington & Worah, 2017; 2018; Louart et al., 2018; Karakida et al., 2020). These results often require complex random matrix theory or operate in a regime where the

input dimension is sent to infinity. By contrast, using a just a power series expansion we are able to characterize a variety of attributes of the spectrum for fixed input dimension and recover results from prior work.

1.1 CONTRIBUTIONS

In Theorem 3.1 we derive coefficients for the power series expansion of the NTK under unit variance initialization, Assumption 2. Consequently we are able to derive insights into the NTK spectrum, explaining both the outlier eigenvalues and the asymptotic decay of the spectrum.

- In Theorem 4.1 and Observation 4.2 we demonstrate that the largest eigenvalue $\lambda_1(\mathbf{K})$ of the NTK takes up an $\Omega(1)$ proportion of the trace and that there are $O(1)$ outlier eigenvalues of the same order as $\lambda_1(\mathbf{K})$.
- In Theorem 4.3 and Theorem 4.5 we show that the effective rank $Tr(\mathbf{K})/\lambda_1(\mathbf{K})$ of the NTK is upper bounded by a constant multiple of the effective rank $Tr(\mathbf{X}\mathbf{X}^T)/\lambda_1(\mathbf{X}\mathbf{X}^T)$ of the input data Gram matrix for both infinite and finite width networks.
- In Theorem 4.6 and Theorem 4.8 we characterize the asymptotic behavior of the NTK spectrum for both uniform and nonuniform data distributions.

1.2 RELATED WORK

Neural Tangent Kernel The NTK was introduced by Jacot et al. (2018), who demonstrated that in the infinite width limit neural network optimization is described via a kernel gradient descent. As a consequence, when the network is polynomially wide in the number of samples, global convergence guarantees for gradient descent can be obtained (Du et al., 2019a;b; Allen-Zhu et al., 2019; Zou & Gu, 2019; Lee et al., 2019; Zou et al., 2020; Oymak & Soltanolkotabi, 2020; Nguyen & Mondelli, 2020; Nguyen, 2021). Furthermore the connection between infinite width networks and Gaussian processes, which traces back to Neal (1996), has been reinvigorated in light of the NTK. Recent investigations include those of Lee et al. (2018); de G. Matthews et al. (2018); Novak et al. (2019).

Analysis of NTK Spectrum Theoretical analysis of the NTK spectrum via random matrix theory was investigated by Yang & Salman (2019); Fan & Wang (2020) in the high dimensional limit. Velikanov & Yarotsky (2021) demonstrate that for ReLU networks the spectrum of the NTK integral operator asymptotically follows a power law, which is consistent with our results for the uniform distribution. Basri et al. (2019) calculate the NTK spectrum for shallow ReLU networks under the uniform distribution, which was then expanded to the nonuniform case by Basri et al. (2020). Geifman et al. (2020) and Bietti & Bach (2021) analyze the reproducing kernel Hilbert spaces of the NTK for ReLU networks and the Laplace kernel by the decay rate of the spectrum of the kernel. In contrast to previous works, we are able to address the spectrum in the finite dimensional setting and characterize the impact of different activation functions on it.

Hermite Expansion To our knowledge Oymak & Soltanolkotabi (2020) were the first to use the Hermite expansion to study the NTK, through which they establish a quantitative bound on the smallest eigenvalue for shallow networks. This approach was incorporated by Nguyen & Mondelli (2020) to handle convergence for deep networks, with sharp bounds on the smallest NTK eigenvalue for deep ReLU networks provided by Nguyen et al. (2021). The Hermite approach was utilized by Panigrahi et al. (2020) to analyze the smallest NTK eigenvalue of shallow networks under various activations. Rather than focus on the smallest eigenvalue, we employ the Hermite expansion to characterize both the outlier and asymptotic portions of the spectrum for both shallow and deep networks under general activations.

2 PRELIMINARIES

For our notation, lower case letters, e.g., x, y, denote scalars, lower case bold characters, e.g., $\mathbf{x}, \mathbf{y}$ are for vectors, and upper case bold characters, e.g., $\mathbf{X}, \mathbf{Y}$, are for matrices. For natural numbers $k_1, k_2 \in \mathbb{N}$ we let $[k_1] := \{1, \ldots, k_1\}$ and $[k_2, k_1] = \{k_2, \ldots, k_1\}$. If $k_2 > k_1$ then $[k_2, k_1]$ is the empty set. We use $\|\cdot\|_p$ to denote the p-norm of the matrix or vector in question and as default use $\|\cdot\|$

as the operator or 2-norm respectively. We use $\mathbf{1}_{m\times n} \in \mathbb{R}^{m\times n}$ to denote the matrix whose entries are all equal to one. We define $\delta_{p=c}$ to take value 1 if $p = c$ and zero otherwise. We will frequently overload scalar functions $\phi : \mathbb{R} \to \mathbb{R}$ by applying them elementwise to vectors and matrices. The entry in the ith row and jth column we access using the notation $[\mathbf{X}]_{ij}$. The Hadamard or entrywise product of two matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m\times n}$ we denote $\mathbf{X} \odot \mathbf{Y}$ as is standard. The pth Hadamard power we denote $\mathbf{X}^{\odot p}$ and define it as the Hadamard product of $\mathbf{X}$ with itself p times,

$$\mathbf{X}^{\odot p} := \mathbf{X} \odot \mathbf{X} \odot \cdots \odot \mathbf{X}.$$

Given a Hermitian or symmetric matrix $\mathbf{X} \in \mathbb{R}^{n\times n}$, we adopt the convention that $\lambda_i(\mathbf{X})$ denotes the ith largest eigenvalue,

$$\lambda_1(\mathbf{X}) \geq \lambda_2(\mathbf{X}) \geq \cdots \geq \lambda_n(\mathbf{X}).$$

Finally, for a square matrix $\mathbf{X} \in \mathbb{R}^{n\times n}$ we let $Tr(\mathbf{X}) = \sum_{i=1}^{n} [\mathbf{X}]_{ii}$ denote the trace.

2.1 Hermite Expansion

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is square integrable with respect to the standard Gaussian measure $\gamma(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ if $\mathbb{E}_{X\sim\mathcal{N}(0,1)}[f(X)^2] < \infty$. We denote by $L^2(\mathbb{R}, \gamma)$ the space of all such functions. The normalized probabilist's Hermite polynomials are defined as

$$H_k(x) = \frac{(-1)^k e^{x^2/2}}{\sqrt{k!}} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k = 0, 1, \ldots$$

and form a complete orthonormal basis in $L^2(\mathbb{R}, \gamma)$ (O'Donnell, 2014, §11). The Hermite expansion of a function $\phi \in L^2(\mathbb{R}, \gamma)$ is given by $\phi(x) = \sum_{k=0}^{\infty} \mu_k(\phi) h_k(x)$, where $\mu_k(\phi) = \mathbb{E}_{X\sim\mathcal{N}(0,1)}[\phi(X)h_k(X)]$ is the kth normalized probabilist's Hermite coefficient of ϕ.

2.2 NTK Parametrization

In what follows, for $n, d \in \mathbb{N}$ let $\mathbf{X} \in \mathbb{R}^{n\times d}$ denote a matrix which stores n points in $\mathbb{R}^d$ row-wise. Unless otherwise stated, we assume $d \leq n$ and denote the ith row of $\mathbf{X}_n$ as $\mathbf{x}_i$. In this work we consider fully-connected neural networks of the form $f^{(L+1)} : \mathbb{R}^d \to \mathbb{R}$ with $L \in \mathbb{N}$ hidden layers and a linear output layer. For a given input vector $\mathbf{x} \in \mathbb{R}^d$, the activation $f^{(l)}$ and preactivation $g^{(l)}$ at each layer $l \in [L + 1]$ are defined via the recurrence relation

$$\begin{aligned}
&g^{(1)}(\mathbf{x}) = \gamma_w \mathbf{W}^{(1)}\mathbf{x} + \gamma_b \mathbf{b}^{(1)},\ f^{(1)}(\mathbf{x}) = \phi\left(g^{(1)}(\mathbf{x})\right),\\
&g^{(l)}(\mathbf{x}) = \frac{\sigma_w}{\sqrt{m_{l-1}}} \mathbf{W}^{(l)} f^{(l-1)}(\mathbf{x}) + \sigma_b \mathbf{b}^{(l)},\ f^{(l)}(\mathbf{x}) = \phi\left(g^{(l)}(\mathbf{x})\right),\ \forall l \in [2, L],\\
&g^{(L+1)}(\mathbf{x}) = \frac{\sigma_w}{\sqrt{m_L}} \mathbf{W}^{(L+1)} f^{(L)}(\mathbf{x}),\ f^{(L+1)}(\mathbf{x}) = g^{(L+1)}(\mathbf{x}).
\end{aligned} \tag{1}$$

The parameters $\mathbf{W}^{(l)} \in \mathbb{R}^{m_l\times m_{l-1}}$ and $\mathbf{b}^{(l)} \in \mathbb{R}^{m_l}$ are the weight matrix and bias vector at the lth layer respectively, $m_0 = d$, $m_{L+1} = 1$, and $\phi : \mathbb{R} \to \mathbb{R}$ is the activation function applied elementwise. The variables $\gamma_w, \sigma_w \in \mathbb{R}_{>0}$ and $\gamma_b, \sigma_b \in \mathbb{R}_{\geq 0}$ correspond to weight and bias hyperparameters respectively. Let $\theta_l \in \mathbb{R}^p$ denote a vector storing the network parameters $(\mathbf{W}^{(h)}, \mathbf{b}^{(h)})_{h=1}^{l}$ up to and including the lth layer. The Neural Tangent Kernel (Jacot et al., 2018) $\tilde{\Theta}^{(l)} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ associated with $f^{(l)}$ at layer $l \in [L + 1]$ is defined as

$$\tilde{\Theta}^{(l)}(\mathbf{x}, \mathbf{y}) := \langle \nabla_{\theta_l} f^{(l)}(\mathbf{x}), \nabla_{\theta_l} f^{(l)}(\mathbf{y}) \rangle. \tag{2}$$

We will mostly study the NTK under the following standard assumptions.

Assumption 1. *NTK initialization.*

1. *At initialization all network parameters are distributed as $\mathcal{N}(0, 1)$ and are mutually independent.*
2. *The activation function satisfies $\phi \in L^2(\mathbb{R}, \gamma)$, is differentiable almost everywhere and its derivative, which we denote ϕ', also satisfies $\phi' \in L^2(\mathbb{R}, \gamma)$.*
3. *The widths are sent to infinity in sequence, $m_1 \to \infty, m_2 \to \infty, \ldots, m_L \to \infty$. We refer to this regime as the sequential infinite width limit.*

Under Assumption 1, for any $l \in [L+1]$, $\tilde{\Theta}^{(l)}(\mathbf{x}, \mathbf{y})$ converges in probability to a deterministic limit $\Theta^{(l)} \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (Jacot et al., 2018). Under Assumption 1 we emphasize that the network behaves like a kernelized linear predictor during training; see, e.g., Arora et al. (2019b); Lee et al. (2019); Woodworth et al. (2020). Given access to the rows $(\mathbf{x}_i)_{i=1}^n$ of $\mathbf{X}$ the NTK matrix at layer $l \in [L+1]$, which we denote $\mathbf{K}_l$, is the $n \times n$ matrix with entries defined as

$$[\mathbf{K}_l]_{ij} = \frac{1}{n}\Theta^{(l)}(\mathbf{x}_i, \mathbf{x}_j), \ \forall (i,j) \in [n] \times [n]. \tag{3}$$

3 Expressing the NTK as a power series

We derive a power series for the NTK under the following assumptions on the network initialization hyperparameters.

Assumption 2. *The hyperparameters of the network satisfy $\gamma_w^2 + \gamma_b^2 = 1$, $\sigma_w^2 \mathbb{E}_{Z\sim\mathcal{N}(0,1)}[\phi(Z)^2] \leq 1$ and $\sigma_b^2 = 1 - \sigma_w^2 \mathbb{E}_{Z\sim\mathcal{N}(0,1)}[\phi(Z)^2]$. The data is normalized so that $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$.*

In order to gain insight into this assumption, recall under Assumption 1 that the preactivations of the network are centered Gaussian processes (Neal, 1996; Lee et al., 2018). Assuming the data lies on the unit sphere, Assumption 2 therefore ensures the preactivation of each neuron has unit variance with respect to the network parameters. We refer the reader to Appendix A.1 for a more thorough discussion. Notably, under Assumption 2 we will show it is possible to write the NTK not only as a dot-product kernel, but also as an analytic power series on $[-1, 1]$. In order to state this result we require a few pieces of notation. Recall, given a function $f \in L^2(\mathbb{R}, \gamma)$, that we denote the pth normalized probabilist's Hermite coefficient of f as $\mu_p(f)$. We refer the reader to Appendix A.4 for an overview of the Hermite polynomials and their properties. Furthermore, letting $\bar{a} = (a_j)_{j=0}^{\infty}$ denote a sequence of real numbers, then for any $p, k \in \mathbb{Z}_{\geq 0}$ we define

$$F(p, k, \bar{a}) = \begin{cases} 1, & k = 0 \text{ and } p = 0, \\ 0, & k = 0 \text{ and } p \geq 1, \\ \sum_{(j_i) \in \mathcal{J}(p,k)} \prod_{i=1}^k a_{j_i}, & k \geq 1 \text{ and } p \geq 0, \end{cases} \tag{4}$$

where

$$\mathcal{J}(p, k) := \left\{ (j_i)_{i \in [k]} \ : \ j_i \geq 0 \ \forall i \in [k], \ \sum_{i=1}^k j_i = p \right\} \quad \text{for all } p \in \mathbb{Z}_{\geq 0}, k \in \mathbb{N}.$$

Here $\mathcal{J}(p, k)$ is the set of all k-tuples of nonnegative integers which sum to p and $F(p, k, \bar{a})$ is therefore the sum of all ordered products of k elements of $\bar{a}$ whose indices sum to p. We are now ready to state the key result of this section, Theorem 3.1, whose proof is provided in Appendix B.1.

Theorem 3.1. *Under Assumptions 1 and 2, for all $l \in [L+1]$*

$$n\mathbf{K}_l = \sum_{p=0}^{\infty} \kappa_{p,l} \left(\mathbf{X}\mathbf{X}^T \right)^{\odot p}. \tag{5}$$

The series for each entry $n[\mathbf{K}_l]_{ij}$ converges absolutely and the coefficients $\kappa_{p,l}$ are nonnegative and can be evaluated using the recurrence relationships

$$\kappa_{p,l} = \begin{cases} \delta_{p=0}\gamma_b^2 + \delta_{p=1}\gamma_w^2, & l = 1, \\ \alpha_{p,l} + \sum_{q=0}^p \kappa_{q,l-1} \upsilon_{p-q,l}, & l \in [2, L+1], \end{cases} \tag{6}$$

where

$$\alpha_{p,l} = \begin{cases} \sigma_w^2 \mu_p^2(\phi) + \delta_{p=0}\sigma_b^2, & l = 2, \\ \sum_{k=0}^{\infty} \alpha_{k,2} F(p, k, \bar{\alpha}_{l-1}), & l \geq 3, \end{cases} \tag{7}$$

and

$$\upsilon_{p,l} = \begin{cases} \sigma_w^2 \mu_p^2(\phi'), & l = 2, \\ \sum_{k=0}^{\infty} \upsilon_{k,2} F(p, k, \bar{\alpha}_{l-1}), & l \geq 3, \end{cases} \tag{8}$$

are likewise nonnegative for all $p \in \mathbb{Z}_{\geq 0}$ and $l \in [2, L+1]$.

To compute the coefficients of the NTK as per Theorem 3.1, the Hermite coefficients of both ϕ and ϕ' are required. Under Assumption 3 below, which has minimal impact on the generality of our results, this calculation can be simplified. In short, under Assumption 3 $\upsilon_{p,2} = (p+1)\alpha_{p+1,2}$ and therefore only the Hermite coefficients of ϕ are required. We refer the reader to Lemma B.3 in Appendix B.1 for further details.

Assumption 3. *The activation function $\phi\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous on $[-a, a]$ for all $a > 0$, differentiable almost everywhere, and is polynomially bounded, i.e., $|\phi(x)| = \mathcal{O}(|x|^\beta)$ for some $\beta > 0$. Further, the derivative $\phi'\colon \mathbb{R} \to \mathbb{R}$ satisfies $\phi' \in L^2(\mathbb{R}, \gamma)$.*

We remark that ReLU, Tanh, Sigmoid, Softplus and many other commonly used activation functions satisfy Assumption 3. In order to understand the relationship between the Hermite coefficients of the activation function and the coefficients of the NTK, we first consider the simple two-layer case with $L = 1$ hidden layers. From Theorem 3.1

$$\kappa_{p,2} = \sigma_w^2(1 + \gamma_w^2 p)\mu_p^2(\phi) + \sigma_w^2\gamma_b^2(1+p)\mu_{p+1}^2(\phi) + \delta_{p=0}\sigma_b^2. \tag{9}$$

As per Table 1, a general trend we observe across all activation functions is that the first few coefficients account for the large majority of the total NTK coefficient series.

Table 1: Percentage of $\sum_{p=0}^{\infty} \kappa_{p,2}$ accounted for by the first $T+1$ NTK coefficients assuming $\gamma_w^2 = 1$, $\gamma_b^2 = 0$, $\sigma_w^2 = 1$ and $\sigma_b^2 = 1 - \mathbb{E}[\phi(Z)^2]$.

$T =$	0	1	2	3	4	5
ReLU	43.944	77.277	93.192	93.192	95.403	95.403
Tanh	41.362	91.468	91.468	97.487	97.487	99.090
Sigmoid	91.557	99.729	99.729	99.977	99.977	99.997
Gaussian	95.834	95.834	98.729	98.729	99.634	99.634

However, the asymptotic rate of decay of the NTK coefficients varies significantly by activation function due to the varying behavior of their tails. In Lemma 3.2 we choose ReLU, Tanh and Gaussian as prototypical examples of activations functions with growing, constant, and decaying tails respectively, and analyze the corresponding NTK coefficients in the two layer setting. For typographical ease we denote the zero mean Gaussian density function with variance σ^2 as $\omega_\sigma(z) := (1/\sqrt{2\pi\sigma^2}) \exp\left(-z^2/(2\sigma^2)\right)$.

Lemma 3.2. *Under Assumptions 1 and 2,*

1. *if $\phi(z) = ReLU(z)$, then $\kappa_{p,2} = \delta_{(\gamma_b>0)\cup(p \text{ even})}\Theta(p^{-3/2})$,*
2. *if $\phi(z) = Tanh(z)$, then $\kappa_{p,2} = \mathcal{O}\left(\exp\left(-\frac{\pi\sqrt{p-1}}{2}\right)\right)$,*
3. *if $\phi(z) = \omega_\sigma(z)$, then $\kappa_{p,2} = \delta_{(\gamma_b>0)\cup(p \text{ even})}\Theta(p^{1/2}(\sigma^2+1)^{-p})$.*

The trend we observe from Lemma 3.2 is that activation functions whose Hermite coefficients decay quickly, such as ω_σ, result in a faster decay of the NTK coefficients. We remark that analyzing the rates of decay in the deep setting is challenging due to the calculation of $F(p, k, \bar{\alpha}_{l-1})$ (4) and therefore leave this study to future work.

Finally, we briefly highlight here the potential for using a truncation of (5) in order to perform efficient numerical approximation of the infinite width NTK. As per our observations thus far that the coefficients of the NTK power series (5) typically decay quite rapidly, one might consider instead approximating $\Theta^{(l)}$ by computing the first few terms in each series of (5). Noting under Assumption 1 and 2 that the NTK can be treated as a univariate function of the correlation $\rho \in [-1, 1]$ between the two inputs, Figure 1 displays the absolute error between the truncated ReLU NTK and the analytical expression for the ReLU NTK defined in Appendix B.3. The key takeaway from Figure 1 is that while for $|\rho|$ close to one the approximation is poor, for $|\rho| < 0.5$ (which is arguably more realistic for real-world data) with just 50 coefficients machine level precision can be achieved. Due to space requirements we delay a detailed discussion of this figure to Appendix B.1.

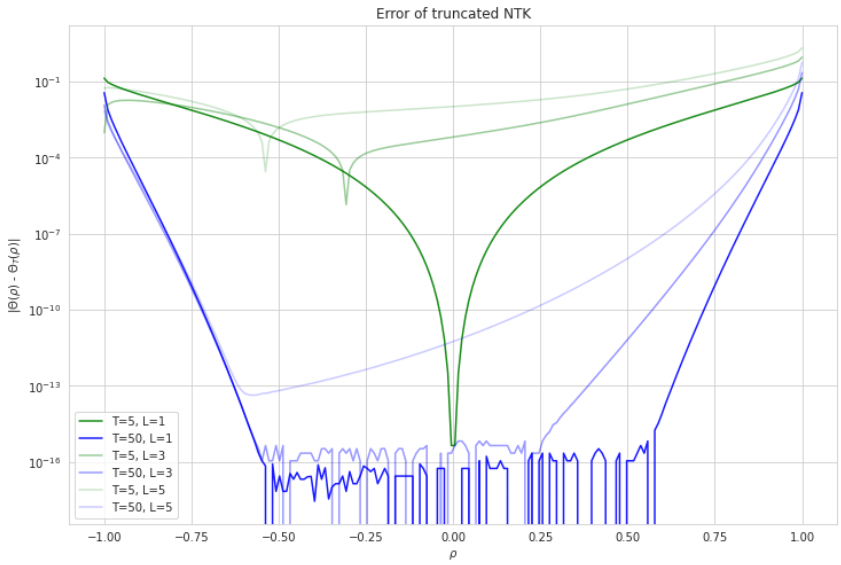

Figure 1: Absolute error between the analytical ReLU NTK and the truncated ReLU NTK power series for two values of the truncation point T of the power series and the depth L of the network, depending on the input correlation ρ. Although the truncated NTK achieves a uniform approximation error of only 10^{-1} on $[-1, 1]$, for $|\rho| \leq 0.5$ which is more typical for real data $T = 50$ suffices for the truncated NTK to achieve machine level precision.

4 Analyzing the Spectrum of the NTK via its Power Series

In this section, we consider a general kernel matrix power series $n\mathbf{K} = \sum_{p=0}^{\infty} c_p(\mathbf{X}\mathbf{X}^T)^{\odot p}$ where $\{c_p\}_{p=0}^{\infty}$ are coefficients and $\mathbf{X}$ is the data matrix. According to Theorem 3.1, the coefficients of the NTK power series (5) are always nonnegative, thus we only consider the case when c_p are nonnegative. We will also consider the kernel function power series $K(x_1, x_2) = \sum_{p=0}^{\infty} c_p \langle x_1, x_2 \rangle^p$. Later on we will analyze the spectrum of kernel matrix $\mathbf{K}$ and kernel function K.

4.1 Analysis of the Upper Spectrum and Effective Rank

In this section we analyze the upper part of the spectrum of the NTK, corresponding to the large eigenvalues, using the power series given in Theorem 3.1. Our first result concerns the *effective rank* (Huang et al., 2022) of the NTK. Given a positive semidefinite matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ we define the effective rank of $\mathbf{A}$ to be

$$\text{eff}(\mathbf{A}) = \frac{Tr(\mathbf{A})}{\lambda_1(\mathbf{A})}.$$

The effective rank quantifies how many eigenvalues are on the order of the largest eigenvalue. This follows from the Markov-like inequality

$$|\{p : \lambda_p(\mathbf{A}) \geq c\lambda_1(\mathbf{A})\}| \leq c^{-1}\text{eff}(\mathbf{A}) \tag{10}$$

and the eigenvalue bound

$$\frac{\lambda_p(\mathbf{A})}{\lambda_1(\mathbf{A})} \leq \frac{\text{eff}(\mathbf{A})}{p}.$$

Our first result is that the effective rank of the NTK can be bounded in terms of a ratio involving the power series coefficients. Based on the results in Section 3 we can assume that the kernel Gram matrix $\mathbf{K}$ has the form $n\mathbf{K} = \sum_{p=0}^{\infty} c_p(\mathbf{X}\mathbf{X}^T)^{\odot p}$ for some set of nonnegative coefficients $\{c_p\}_{p=0}^{\infty}$. Furthermore we assume that the data is normalized so that $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$. Then we observe by the linearity of the trace

$$Tr(n\mathbf{K}) = \sum_{p=0}^{\infty} c_p Tr((\mathbf{X}\mathbf{X}^T)^{\odot p}) = n\sum_{p=0}^{\infty} c_p,$$

where we have used the fact that $Tr((\mathbf{X}\mathbf{X}^T)^{\odot p}) = n$ for all $p \in \mathbb{N}$. On the other hand,

$$\lambda_1(n\mathbf{K}) \geq \lambda_1(c_0(\mathbf{X}\mathbf{X}^T)^0) = \lambda_1(c_0\mathbf{1}_{n\times n}) = nc_0.$$

Combining these two results we get the following theorem.

Theorem 4.1. *Assume that we have a kernel Gram matrix $\mathbf{K}$ of the form $n\mathbf{K} = \sum_{p=0}^{\infty} c_p(\mathbf{X}\mathbf{X}^T)^{\odot p}$ where $c_0 \neq 0$. Furthermore, assume the input data $\mathbf{x}_i$ are normalized so that $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$. Then*

$$\text{eff}(\mathbf{K}) \leq \frac{\sum_{p=0}^{\infty} c_p}{c_0}.$$

By Theorem 3.1 $c_0 \neq 0$ provided the network has biases or the activation function has nonzero Gaussian expectation (i.e., $\mu_0(\phi) \neq 0$). Thus we have that the effective rank of $\mathbf{K}$ is bounded by an $O(1)$ quantity. In the case of ReLU for example, as evidenced by Table 1, the effective rank will be roughly 2.5 for a shallow network. By contrast, a well-conditioned matrix would have an effective rank that is $\Omega(n)$. Combining Theorem 4.1 and the Markov-type bound (10) we make following important observation.

Observation 4.2. *The largest eigenvalue $\lambda_1(\mathbf{K})$ of the NTK takes up an $\Omega(1)$ fraction of the entire trace and there are $O(1)$ eigenvalues on the same order of magnitude as $\lambda_1(\mathbf{K})$.*

While the constant term $c_0\mathbf{1}_{n\times n}$ in the kernel leads to a significant outlier in the spectrum of $\mathbf{K}$, it is rather uninformative beyond this. What interests us is how the structure of the data $\mathbf{X}$ manifests in the spectrum of the kernel matrix $\mathbf{K}$. For this reason we will examine the centered kernel matrix $\widetilde{\mathbf{K}} := \mathbf{K} - \frac{c_0}{n}\mathbf{1}_{n\times n}$. By a very similar argument as before we get the following result.

Theorem 4.3. *Assume that we have a kernel Gram matrix $\mathbf{K}$ of the form $n\mathbf{K} = \sum_{p=0}^{\infty} c_p(\mathbf{X}\mathbf{X}^T)^{\odot p}$ where $c_1 \neq 0$. Furthermore, assume the input data $\mathbf{x}_i$ are normalized so that $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$. Then the centered kernel $\widetilde{\mathbf{K}} := \mathbf{K} - \frac{c_0}{n}\mathbf{1}_{n\times n}$ satisfies*

$$\text{eff}(\widetilde{\mathbf{K}}) \leq \text{eff}(\mathbf{X}\mathbf{X}^T)\frac{\sum_{p=1}^{\infty} c_p}{c_1}.$$

Thus we have that the effective rank of the centered kernel $\widetilde{\mathbf{K}}$ is upper bounded by a constant multiple of the effective rank of the input data Gram $\mathbf{X}\mathbf{X}^T$. Furthermore, we can take the ratio $\frac{c_1}{\sum_{p=1}^{\infty} c_p}$ as a measure of how much the NTK inherits the behavior of the linear kernel $\mathbf{X}\mathbf{X}^T$. Again from Table 1, in the shallow setting we see that this ratio tends to be significant for common activations. For example for ReLU it is roughly 0.6. From Theorem 4.5 we make the important observation.

Observation 4.4. *Whenever the input data are approximately low rank, the centered kernel matrix $\widetilde{\mathbf{K}} = \mathbf{K} - \frac{c_0}{n}\mathbf{1}_{n\times n}$ is also approximately low rank.*

It turns out that this phenomenon also holds for finite-width networks. Consider the shallow model

$$\sum_{\ell=1}^{m} a_\ell \phi(\langle \mathbf{w}_\ell, \mathbf{x}\rangle),$$

where $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{w}_\ell \in \mathbb{R}^d$, $a_\ell \in \mathbb{R}$ for all $\ell \in [m]$. The following theorem demonstrates that when the width m is linear in the number of samples n then $\text{eff}(\mathbf{K})$ is upper bounded by a constant multiple of $\text{eff}(\mathbf{X}\mathbf{X}^T)$.

Theorem 4.5. *Assume $\phi(x) = ReLU(x)$ and $n \geq d$. Fix $\epsilon > 0$ small. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_m \sim N(0, \nu_1^2 I_d)$ i.i.d. and $a_1, \ldots, a_m \sim N(0, \nu_2^2)$. Set $M = \max_{i\in[n]} \|\mathbf{x}_i\|_2$, and let*

$$\Sigma := \mathbb{E}_{\mathbf{w}\sim N(0,\nu_1^2 I)}[\phi(\mathbf{X}\mathbf{w})\phi(\mathbf{w}^T\mathbf{X}^T)].$$

Then

$$m = \Omega\left(\max(\lambda_1(\Sigma)^{-2}, 1)\max(n, \log(1/\epsilon))\right), \quad \nu_1 = O(1/M\sqrt{m})$$

suffices to ensure that, with probability at least $1-\epsilon$ over the sampling of the parameter initialization,

$$\text{eff}(\mathbf{K}) \leq C \cdot \text{eff}(\mathbf{X}\mathbf{X}^T),$$

where $C > 0$ is an absolute constant.

Many works consider the model where the outer layer weights are fixed and have constant magnitude and only the inner layer weights are trained. This is the setting considered by Xie et al. (2017), Arora et al. (2019a), Du et al. (2019b), Oymak et al. (2019), Li et al. (2020), and Oymak & Soltanolkotabi (2020). In this setting we can reduce the dependence on the width m to only be logarithmic in the number of samples n, and we have an accompanying lower bound. See Theorem C.5 in the Appendix C.2.3 for details.

In Figure 2 we empirically validate our theory by computing the NTK for MNIST and CIFAR10. We plot the spectrum of the NTK Gram matrix $\mathbf{K}$ and the data Gram matrix $\mathbf{X}\mathbf{X}^T$. We observe that the NTK has a small number of outlier eigenvalues and long tail of small eigenvalues. Furthermore, the low rank structure of the data Gram matrix $\mathbf{X}\mathbf{X}^T$ is inherited by the kernel $\mathbf{K}$.

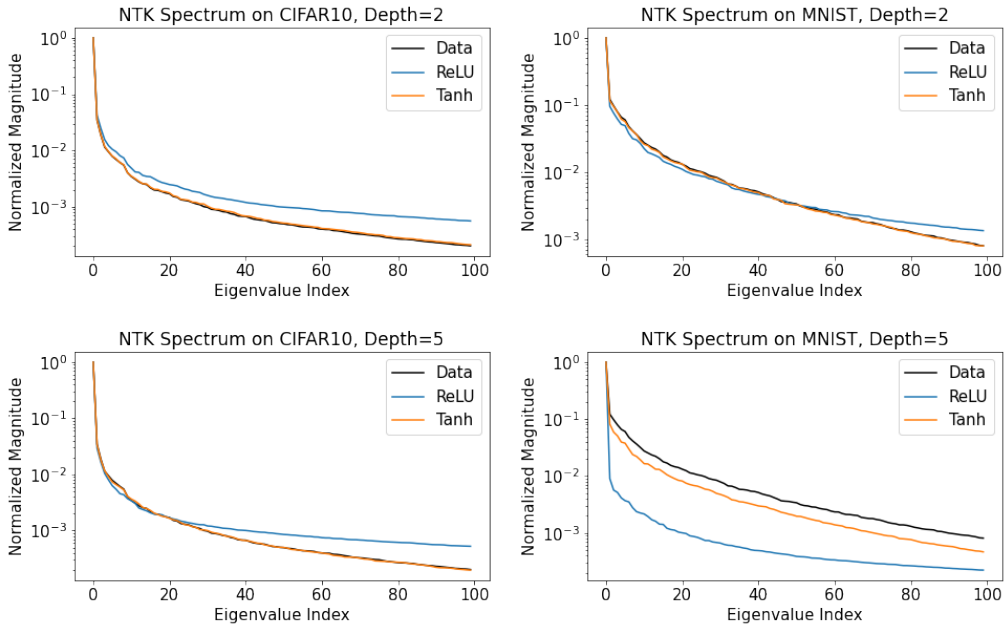

Figure 2: We plot the normalized eigenvalues λ_p/λ_1 of the NTK Gram matrix $\mathbf{K}$ and the data Gram matrix $\mathbf{X}\mathbf{X}^T$ for the datasets CIFAR10 and MNIST. To compute the NTK we randomly initialize a network of width 1000. We consider the depths 2 and 5. We set $n = 1000$ and plot the first 100 eigenvalues. We observe that the NTK inherits the low rank structure of the data Gram $\mathbf{X}\mathbf{X}^T$ as predicted by our theory.

4.2 Analysis of the lower spectrum

In this section, we analyze the lower part of the spectrum using the power series. We first analyze the kernel function K which has the form $K(x_1, x_2) = \sum_{p=0}^{\infty} c_p \langle x_1, x_2 \rangle^p$. As the kernel K is a dot-product kernel, when the training data is uniformly distributed on a hypersphere it was shown by Basri et al. (2019); Bietti & Mairal (2019) that the eigenfunctions of K are the spherical harmonics. The following theorem gives the eigenvalues of the kernel K in this setting.

Theorem 4.6. *Suppose that the training data are uniformly sampled from the unit hypersphere $\mathbb{S}^d$, $d \geq 2$. If the dot-product kernel function has the expansion $K(x_1, x_2) = \sum_{p=0}^{\infty} c_p \langle x_1, x_2 \rangle^p$ where $c_p \geq 0$, then the eigenvalue of every spherical harmonic of frequency k is given by*

$$\overline{\lambda_k} = \frac{\pi^{d/2}}{2^{k-1}} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} c_p \frac{\Gamma(p+1)\Gamma(\frac{p-k+1}{2})}{\Gamma(p-k+1)\Gamma(\frac{p-k+1}{2} + k + d/2)},$$

where Γ is the gamma function.

A proof of Theorem 4.6 is provided in Appendix C.3. This theorem gives the connection between the coefficients c_p of the kernel power series and the eigenvalues $\overline{\lambda_k}$ of the kernel. Given a specific decay rate of c_p, we have the decay rate of $\overline{\lambda_k}$ as follows:

Corollary 4.7. *Under the same setting as in Theorem 4.6,*

1. *if* $c_p = \Theta(p^{-a})$ *where* $a \geq 1$, *then* $\overline{\lambda_k} = \Theta(k^{-d-2a+2})$,
2. *if* $c_p = \delta_{(p\ even)}\Theta(p^{-a})$, *then* $\overline{\lambda_k} = \delta_{(k\ even)}\Theta(k^{-d-2a+2})$,
3. *if* $c_p = \mathcal{O}\left(\exp\left(-a\sqrt{p}\right)\right)$, *then* $\overline{\lambda_k} = \mathcal{O}\left(k^{-d+1/2}\exp\left(-a\sqrt{k}\right)\right)$,
4. *if* $c_p = \Theta(p^{1/2}a^{-p})$, *then* $\overline{\lambda_k} = \mathcal{O}\left(k^{-d+1}a^{-k}\right)$ *and* $\overline{\lambda_k} = \Omega\left(k^{-d/2+1}2^{-k}a^{-k}\right)$.

For the NTK of a two-layer ReLU network with $\gamma_b > 0$, then according to Lemma 3.2 we have $c_p = \kappa_{p,2} = \Theta(p^{-3/2})$. Therefore using Corollary 4.7 $\overline{\lambda_k} = \Theta(k^{-d-1})$. Notice here that k refers to the frequency, and the number of spherical harmonics of frequency $\leq k$ is $\Theta(k^d)$. Therefore for the lth largest eigenvalue λ_l we have $\lambda_l = \Theta(l^{-(d+1)/d})$. This rate agrees with Basri et al. (2019) and Velikanov & Yarotsky (2021). For the NTK of a two-layer ReLU network with $\gamma_b = 0$, the eigenvalues corresponding to the even frequencies are 0, which also agrees with Basri et al. (2019). Corollary 4.7 also shows the decay rates of eigenvalues for the NTK of two-layer networks with Tanh activation and Gaussian activation. We observe that when the coefficients of the kernel power series decay quickly then the eigenvalues of the kernel also decay quickly. As faster decay of the eigenvalues of the kernel implies a smaller RKHS (Geifman et al., 2020), Corollary 4.7 demonstrates that using ReLU results in a larger RKHS relative to using either Tanh or Gaussian activations.

We now briefly touch on how one can relate the coefficients of a power series kernel to its spectrum without making any assumptions on the data distribution.

Theorem 4.8 (Informal)**.** *Let the rows of* $\mathbf{X} \in \mathbb{R}^{n\times d}$ *be arbitrary points on the unit sphere. Consider the kernel matrix* $n\mathbf{K} = \sum_{p=0}^{\infty} c_p \left(\mathbf{X}\mathbf{X}^T\right)^{\odot p}$, *then*

1. *if* $c_p = \mathcal{O}(p^{-\alpha})$ *with* $\alpha > r(n) + 1$ *for all* $n \in \mathbb{Z}_{\geq 0}$ *then* $\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(n^{-\frac{\alpha-1}{r(n)}}\right)$,
2. *if* $c_p = \mathcal{O}(e^{-\alpha\sqrt{p}})$ *then* $\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(n^{\frac{1}{2r(n)}}\exp\left(-\alpha' n^{\frac{1}{2r(n)}}\right)\right)$ *for any* $\alpha' < \alpha$,
3. *if* $c_p = \mathcal{O}(e^{-\alpha p})$ *then* $\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(\exp\left(-\alpha n^{\frac{1}{r(n)}}\right)\right)$.

Although the presence of the factor $1/r(n)$ in the exponents of n in these bounds is a weakness, Theorem 4.8 still illustrates how, in a highly general setting, the asymptotic decay of the coefficients of the power series ensures a certain asymptotic decay in the eigenvalues of the kernel matrix. A formal version of this result is provided in Appendix C.4 along with further discussion.

5 Conclusion

Using a power series expansion we derived a number of insights into both the outliers as well as the asymptotic decay of the spectrum of the NTK. We are able to perform our analysis without recourse to a high dimensional limit or complex random matrix theory. Interesting avenues for future work include better characterizing the role of depth and performing the same analysis for networks with convolutional or residual layers.

Reproducibility Statement To ensure reproducibility, we make the code public at `https://anonymous.4open.science/r/data_ntk-6944/README.md`.

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APPENDIX

The appendix is organized as follows.

- Appendix A gives background material on Gaussan kernels, NTK, unit variance intitialization, and Hermite polynomial expansions.
- Appendix B provides details for Section 3.
- Appendix C provides details for Section 4.

A BACKGROUND MATERIAL

A.1 GAUSSIAN KERNEL

Observe by construction that the flattened collection of preactivations at the first layer $(g^{(1)}(\mathbf{x}_i))_{i=1}^n$ form a centered Gaussian process, with the covariance between the αth and βth neuron being described by

$$\Sigma^{(1)}_{\alpha\beta}(\mathbf{x}_i, \mathbf{x}_j) := \mathbb{E}[g^{(1)}_\alpha(\mathbf{x}_i) g^{(1)}_\beta(\mathbf{x}_j)] = \delta_{\alpha=\beta}\left(\gamma_w^2 \mathbf{x}_i^T \mathbf{x}_j + \gamma_b^2\right).$$

Under the Assumption 1, the preactivations at each layer $l \in [L+1]$ converge also in distribution to centered Gaussian processes Neal (1996); Lee et al. (2018). We remark that the sequential width limit condition of Assumption 1 is not necessary for this behavior, for example the same result can be derived in the setting where the widths of the network are sent to infinity simultaneously under certain conditions on the activation function de G. Matthews et al. (2018). However, as our interests lie in analyzing the limit rather than the conditions for convergence to said limit, for simplicity we consider only the sequential width limit. As per (Lee et al., 2018, Eq. 4), the covariance between the preactivations of the αth and βth neurons at layer $l \geq 2$ for any input pair $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ are described by the following kernel,

$$\begin{aligned}\Sigma^{(l)}_{\alpha\beta}(\mathbf{x}, \mathbf{y}) &:= \mathbb{E}[g^{(l)}_\alpha(\mathbf{x}) g^{(l)}_\beta(\mathbf{y})] \\ &= \delta_{\alpha=\beta}\left(\sigma_w^2 \mathbb{E}_{g^{(l-1)}\sim\mathcal{GP}(0,\Sigma^{l-1})}[\phi(g^{(l-1)}_\alpha(\mathbf{x}))\phi(g^{(l-1)}_\beta(\mathbf{y}))] + \sigma_b^2\right).\end{aligned}$$

We refer to this kernel as the Gaussian kernel. As each neuron is identically distributed and the covariance between pairs of neurons is 0 unless $\alpha = \beta$, moving forward we drop the subscript and discuss only the covariance between the preactivations of an arbitrary neuron given two inputs. As per the discussion in (Lee et al., 2018, Section 2.3), the expectations involved in the computation of these Gaussian kernels can be computed with respect to a bivariate Gaussian distribution, whose covariance matrix has three distinct entries: the variance of a preactivation of $\mathbf{x}$ at the previous layer, $\Sigma^{(l-1)}(\mathbf{x}, \mathbf{x})$, the variance of a preactivation of $\mathbf{y}$ at the previous layer, $\Sigma^{(l)}(\mathbf{y}, \mathbf{y})$, and the covariance between preactivations of $\mathbf{x}$ and $\mathbf{y}$, $\Sigma^{(l-1)}(\mathbf{x}, \mathbf{y})$. Therefore the Gaussian kernel, or covariance function, and its derivative, which we will require later for our analysis of the NTK, can be computed via the the following recurrence relations, see for instance (Lee et al., 2018; Jacot et al., 2018; Arora et al., 2019b; Nguyen et al., 2021),

$$\begin{aligned}&\Sigma^{(1)}(\mathbf{x}, \mathbf{y}) = \gamma_w^2 \mathbf{x}^T \mathbf{x} + \gamma_b^2, \\ &\mathbf{A}^{(l)}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix}\Sigma^{(l-1)}(\mathbf{x}, \mathbf{x}) & \Sigma^{(l-1)}(\mathbf{x}, \mathbf{y}) \\ \Sigma^{(l-1)}(\mathbf{y}, \mathbf{x}) & \Sigma^{(l-1)}(\mathbf{x}, \mathbf{x})\end{bmatrix} \\ &\Sigma^{(l)}(\mathbf{x}, \mathbf{y}) = \sigma_w^2 \mathbb{E}_{(B_1,B_2)\sim\mathcal{N}(0,\mathbf{A}^{(l)}(\mathbf{x},\mathbf{y}))}[\phi(B_1)\phi(B_2)] + \sigma_b^2, \\ &\dot{\Sigma}^{(l)}(\mathbf{x}, \mathbf{y}) = \sigma_w^2 \mathbb{E}_{(B_1,B_2)\sim\mathcal{N}(0,\mathbf{A}^{(l)}(\mathbf{x},\mathbf{y}))}\left[\phi'(B_1)\phi'(B_2)\right].\end{aligned} \tag{11}$$

A.2 NEURAL TANGENT KERNEL (NTK)

As discussed in the Section 1, under Assumption 1 $\tilde{\Theta}^{(l)}$ converges in probability to a deterministic limit, which we denote $\Theta^{(l)}$. This deterministic limit kernel can be expressed in terms of the Gaussian kernels and their derivatives from Section A.1 via the following recurrence relationships (Jacot

et al., 2018, Theorem 1),

$$\begin{aligned}\Theta^{(1)}(\mathbf{x},\mathbf{y}) &= \Sigma^{(1)}(\mathbf{x},\mathbf{y}),\\ \Theta^{(l)}(\mathbf{x},\mathbf{y}) &= \Theta^{(l-1)}(\mathbf{x},\mathbf{y})\dot{\Sigma}^{(l)}(\mathbf{x},\mathbf{y}) + \Sigma^{(l)}(\mathbf{x},\mathbf{y})\\ &= \Sigma^{(l)}(\mathbf{x},\mathbf{y}) + \sum_{h=1}^{l-1}\Sigma^{(h)}(\mathbf{x},\mathbf{y})\left(\prod_{h'=h+1}^{l}\dot{\Sigma}^{(h')}(\mathbf{x},\mathbf{y})\right)\ \forall l\in[2,L+1].\end{aligned} \tag{12}$$

A useful expression for the NTK matrix, which is a straightforward extension and generalization of Nguyen et al. (2021, Lemma 3.1), is provided in Lemma A.1 below.

Lemma A.1. *(Based on Nguyen et al. 2021, Lemma 3.1) Under Assumption 1, a sequence of positive semidefinite matrices $(\mathbf{G}_l)_{l=1}^{L}$ in $\mathbb{R}^{n\times n}$, and the related sequence $(\dot{\mathbf{G}}_l)_{l=2}^{L}$ also in $\mathbb{R}^{n\times n}$, can be constructed via the following recurrence relationships,*

$$\begin{aligned}\mathbf{G}_1 &= \gamma_w^2\mathbf{X}_n\mathbf{X}_n^T + \gamma_b^2\boldsymbol{I}_{n\times n},\\ \mathbf{G}_2 &= \sigma_w^2\mathbb{E}_{\mathbf{w}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I}_d)}[\phi(\mathbf{X}\mathbf{w})\phi(\mathbf{X}\mathbf{w})^T] + \sigma_b^2\boldsymbol{I}_{n\times n},\\ \dot{\mathbf{G}}_2 &= \sigma_w^2\mathbb{E}_{\mathbf{w}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I}_n)}[\phi'(\mathbf{X}\mathbf{w})\phi'(\mathbf{X}\mathbf{w})^T],\\ \mathbf{G}_l &= \sigma_w^2\mathbb{E}_{\mathbf{w}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I}_n)}[\phi(\sqrt{\mathbf{G}_{l-1}}\mathbf{w})\phi(\sqrt{\mathbf{G}_{l-1}}\mathbf{w})^T] + \sigma_b^2\boldsymbol{I}_{n\times n},\ l\in[3,L+1],\\ \dot{\mathbf{G}}_l &= \sigma_w^2\mathbb{E}_{\mathbf{w}\sim\mathcal{N}(\boldsymbol{0},\boldsymbol{I}_n)}[\phi'(\sqrt{\mathbf{G}_{l-1}}\mathbf{w})\phi'(\sqrt{\mathbf{G}_{l-1}}\mathbf{w})^T],\ l\in[3,L+1].\end{aligned} \tag{13}$$

The sequence of NTK matrices $(\mathbf{K}_l)_{l=1}^{L+1}$ can in turn be written using the following recurrence relationship,

$$\begin{aligned}n\mathbf{K}_1 &= \mathbf{G}_1,\\ n\mathbf{K}_l &= \mathbf{G}_l + n\mathbf{K}_{l-1}\odot\dot{\mathbf{G}}_l\\ &= \mathbf{G}_l + \sum_{i=1}^{l-1}\left(\mathbf{G}_i\odot\left(\odot_{j=i+1}^{l}\dot{\mathbf{G}}_j\right)\right).\end{aligned} \tag{14}$$

Proof. For the sequence $(\mathbf{G}_l)_{l=1}^{L+1}$ it suffices to prove for any $i,j\in[n]$ and $l\in[L+1]$ that

$$[\mathbf{G}_l]_{i,j} = \Sigma^{(l)}(\mathbf{x}_i,\mathbf{x}_j)$$

and $\mathbf{G}_l$ is positive semi-definite. We proceed by induction, considering the base case $l=1$ and comparing (13) with (11) then it is evident that

$$[\mathbf{G}_1]_{i,j} = \Sigma^{(1)}(\mathbf{x}_i,\mathbf{x}_j).$$

In addition, $\mathbf{G}_1$ is also clearly positive semi-definite as for any $\mathbf{u}\in\mathbb{R}^n$

$$\mathbf{u}^T\mathbf{G}_1\mathbf{u} = \gamma_w^2\left\|\mathbf{X}^T\mathbf{u}\right\|^2 + \gamma_b^2\left\|\mathbf{1}_n^T\mathbf{u}\right\|^2 \geq 0.$$

We now assume the induction hypothesis is true for $\mathbf{G}_{l-1}$. We will need to distinguish slightly between two cases, $l=2$ and $l\in[3,L+1]$. The proof of the induction step in either case is identical. To this end, and for notational ease, let $\mathbf{V}=\mathbf{X}$, $\mathbf{w}\sim\mathcal{N}(0,\mathbf{I}_d)$ when $l=2$, and $\mathbf{V}=\sqrt{\mathbf{G}_{l-1}}$, $\mathbf{w}\sim\mathcal{N}(0,\mathbf{I}_n)$ for $l\in[3,L+1]$. In either case we let $\mathbf{v}_i$ denote the ith row of $\mathbf{V}$. For any $i,j\in[n]$

$$[\mathbf{G}_l]_{ij} = \sigma_w^2\mathbb{E}_{\mathbf{w}}[\phi(\mathbf{v}_i^T\mathbf{w})\phi(\mathbf{v}_j^T\mathbf{w})] + \sigma_b^2.$$

Now let $B_1=\mathbf{v}_i^T\mathbf{w}$, $B_2=\mathbf{v}_j^T\mathbf{w}$ and observe for any $\alpha_1,\alpha_2\in\mathbb{R}$ that $\alpha_1B_1+\alpha_2B_2=\sum_k^n(\alpha_1v_{ik}+\alpha_2v_{jk})w_k\sim\mathcal{N}(0,\|\alpha_1\mathbf{v}_i+\alpha_2\mathbf{v}_j\|^2)$. Therefore the joint distribution of (B_1,B_2) is a mean 0 bivariate normal distribution. Denoting the covariance matrix of this distribution as $\tilde{\mathbf{A}}\in\mathbb{R}^{2\times2}$, then $[\mathbf{G}_l]_{ij}$ can be expressed as

$$[\mathbf{G}_l]_{ij} = \sigma_w^2\mathbb{E}_{(B_1,B_2)\sim\tilde{\mathbf{A}}}[\phi(B_1)\phi(B_2)] + \sigma_b^2.$$

To prove $[\mathbf{G}_l]_{i,j} = \Sigma^{(l)}$ it therefore suffices to show that $\tilde{\mathbf{A}} = \mathbf{A}^{(l)}$ as per (11). This follows by the induction hypothesis as

$$\mathbb{E}[B_1^2] = \mathbf{v}_i^T \mathbf{v}_i = [\mathbf{G}_{l-1}]_{ii} = \Sigma^{(l-1)}(\mathbf{x}_i, \mathbf{x}_i),$$
$$\mathbb{E}[B_2^2] = \mathbf{v}_j^T \mathbf{v}_j = [\mathbf{G}_{l-1}]_{jj} = \Sigma^{(l-1)}(\mathbf{x}_j, \mathbf{x}_j),$$
$$\mathbb{E}[B_1 B_2] = \mathbf{v}_i^T \mathbf{v}_j = [\mathbf{G}_{l-1}]_{ij} = \Sigma^{(l-1)}(\mathbf{x}_i, \mathbf{x}_j).$$

Finally, $\mathbf{G}_l$ is positive semi-definite as long as $\mathbb{E}_{\mathbf{w}}[\phi(\mathbf{V}\mathbf{w})\phi(\mathbf{V}\mathbf{w})^T]$ is positive semi-definite. Let $M(\mathbf{w}) = \phi(\mathbf{V}\mathbf{w}) \in \mathbb{R}^{n \times n}$ and observe for any $\mathbf{w}$ that $M(\mathbf{w})M(\mathbf{w})^T$ is positive semi-definite. Therefore $\mathbb{E}_{\mathbf{w}}[M(\mathbf{w})M(\mathbf{w})^T]$ must also be positive semi-definite. Thus the inductive step is complete and we may conclude for $l \in [L+1]$ that

$$[\mathbf{G}_l]_{i,j} = \Sigma^{(l)}(\mathbf{x}_i, \mathbf{x}_j). \tag{15}$$

For the proof of the expression for the sequence $(\dot{\mathbf{G}}_l)_{l=2}^{L+1}$ it suffices to prove for any $i, j \in [n]$ and $l \in [L+1]$ that

$$[\dot{\mathbf{G}}_l]_{i,j} = \dot{\Sigma}^{(l)}(\mathbf{x}_i, \mathbf{x}_j).$$

By comparing (13) with (11) this follows immediately from (15). Therefore with (13) proven (14) follows from (12). □

A.3 Unit variance initialization

The initialization scheme for a neural network, particularly a deep neural network, needs to be designed with some care in order to avoid either vanishing or exploding gradients during training Glorot & Bengio (2010); He et al. (2015); Mishkin & Matas (2016); LeCun et al. (2012). Some of the most popular initialization strategies used in practice today, in particular LeCun et al. (2012) and Glorot & Bengio (2010) initialization, first model the preactivations of the network as Gaussian random variables and then select the network hyperparameters in order that the variance of these idealized preactivations is fixed at one. Under Assumption 1 this idealized model on the preactivations is actually realized and if we additionally assume the conditions of Assumption 2 hold then likewise the variance of the preactivations at every layer will be fixed at one. To this end, and as in Poole et al. (2016b); Murray et al. (2022), consider the function $V \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined as

$$V(q) = \sigma_w^2 \mathbb{E}_{Z \sim \mathcal{N}(0,1)} \left[\phi\left(\sqrt{q} Z\right)^2 \right] + \sigma_b^2. \tag{16}$$

Noting that V is another expression for $\Sigma^{(l)}(\mathbf{x}, \mathbf{x})$, derived via a change of variables as per Poole et al. (2016b), the sequence of variances $(\Sigma^{(l)}(\mathbf{x}, \mathbf{x}))_{l=2}^{L}$ can therefore be generated as follows,

$$\Sigma^{(l)}(\mathbf{x}, \mathbf{x}) = V(\Sigma^{(l-1)}(\mathbf{x}, \mathbf{x})). \tag{17}$$

The linear correlation $\rho^{(l)} : \mathbb{R}^d \times \mathbb{R}^d \to [-1, 1]$ between the preactivations of two inputs $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we define as

$$\rho^{(l)}(\mathbf{x}, \mathbf{y}) = \frac{\Sigma^{(l)}(\mathbf{x}, \mathbf{y})}{\sqrt{\Sigma^{(l)}(\mathbf{x}, \mathbf{x})\Sigma^{(l)}(\mathbf{y}, \mathbf{y})}}. \tag{18}$$

Assuming $\Sigma^{(l)}(\mathbf{x}, \mathbf{x}) = \Sigma^{(l)}(\mathbf{y}, \mathbf{y}) = 1$ for all $l \in [L+1]$, then $\rho^{(l)}(\mathbf{x}, \mathbf{y}) = \Sigma^{(l)}(\mathbf{x}, \mathbf{y})$. Again as in Murray et al. (2022) and analogous to (16), with $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ independent, $U_1 := Z_1$, $U_2(\rho) := (\rho Z_1 + \sqrt{1 - \rho^2} Z_2)$ [1] we define the correlation function $R : [-1, 1] \to [-1, 1]$ as

$$R(\rho) = \sigma_w^2 \mathbb{E}[\phi(U_1)\phi(U_2(\rho))] + \sigma_b^2. \tag{19}$$

Noting under these assumptions that R is equivalent to $\Sigma^{(l)}(\mathbf{x}, \mathbf{y})$, the sequence of correlations $(\rho^{(l)}(\mathbf{x}, \mathbf{y}))_{l=2}^{L}$ can thus be generated as

$$\rho^{(l)}(\mathbf{x}, \mathbf{y}) = R(\rho^{(l-1)}(\mathbf{x}, \mathbf{y})).$$

As observed in Poole et al. (2016b); Schoenholz et al. (2017), $R(1) = V(1) = 1$, hence $\rho = 1$ is a fixed point of R. We remark that as all preactivations are distributed as $\mathcal{N}(0, 1)$, then a correlation

[1] We remark that U_1, U_2 are dependent and identically distributed as $U_1, U_2 \sim \mathcal{N}(0, 1)$.

of one between preactivations implies they are equal. The stability of the fixed point $\rho = 1$ is of particular significance in the context of initializing deep neural networks successfully. Under mild conditions on the activation function one can compute the derivative of R, see e.g., Poole et al. (2016b); Schoenholz et al. (2017); Murray et al. (2022), as follows,

$$R'(\rho) = \sigma_w^2 \mathbb{E}[\phi'(U_1)\phi'(U_2(\rho))]. \tag{20}$$

Observe that the expression for $\dot{\Sigma}^{(l)}$ and R' are equivalent via a change of variables (Poole et al., 2016b), and therefore the sequence of correlation derivatives may be computed as

$$\dot{\Sigma}^{(l)}(\mathbf{x}, \mathbf{y}) = R'(\rho^{(l)}(\mathbf{x}, \mathbf{y})).$$

With the relevant background material now in place we are in a position to prove Lemma A.2.

Lemma A.2. *Under Assumptions 1 and 2 and defining $\chi = \sigma_w^2 \mathbb{E}_{Z\sim\mathcal{N}(0,1)}[\phi'(Z)^2] \in \mathbb{R}_{>0}$, then for all $i, j \in [n]$, $l \in [L+1]$*

- $[\mathbf{G}_{n,l}]_{ij} \in [-1, 1]$ *and* $[\mathbf{G}_{n,l}]_{ii} = 1$,
- $[\dot{\mathbf{G}}_{n,l}]_{ij} \in [-\chi, \chi]$ *and* $[\dot{\mathbf{G}}_{n,l}]_{ii} = \chi$.

Furthermore, the NTK is a dot product kernel, meaning $\Theta(\mathbf{x}_i, \mathbf{x}_j)$ can be written as a function of the inner product between the two inputs, $\Theta(\mathbf{x}_i^T \mathbf{x}_j)$.

Proof. Recall from Lemma A.1 and its proof that for any $l \in [L+1]$, $i, j \in [n]$ $[\mathbf{G}_{n,l}]_{ij} = \Sigma^{(l)}(\mathbf{x}_i, \mathbf{x}_j)$ and $[\dot{\mathbf{G}}_{n,l}]_{ij} = \dot{\Sigma}^{(l)}(\mathbf{x}_i, \mathbf{x}_j)$. We first prove by induction $\Sigma^{(l)}(\mathbf{x}_i, \mathbf{x}_i) = 1$ for all $l \in [L+1]$. The base case $l = 1$ follows as

$$\Sigma^{(1)}(\mathbf{x}, \mathbf{x}) = \gamma_w^2 \mathbf{x}^T \mathbf{x} + \gamma_b^2 = \gamma_w^2 + \gamma_b^2 = 1.$$

Assume the induction hypothesis is true for layer $l - 1$. With $Z \sim \mathcal{N}(0, 1)$, then from (16) and (17)

$$\begin{aligned}
\Sigma^{(l)}(\mathbf{x}, \mathbf{x}) &= V(\Sigma^{(l-1)}(\mathbf{x}, \mathbf{x})) \\
&= \sigma_w^2 \mathbb{E}\left[\phi^2\left(\sqrt{\Sigma^{(l-1)}(\mathbf{x}, \mathbf{x})} Z\right)\right] + \sigma_b^2 \\
&= \sigma_w^2 \mathbb{E}\left[\phi^2(Z)\right] + \sigma_b^2 \\
&= 1,
\end{aligned}$$

thus the inductive step is complete. As an immediate consequence it follows that $[\mathbf{G}_l]_{ii} = 1$. Also, for any $i, j \in [n]$ and $l \in [L+1]$,

$$\Sigma^{(l)}(\mathbf{x}_i, \mathbf{x}_j) = \rho^{(l)}(\mathbf{x}_i, \mathbf{x}_j) = R(\rho^{(l-1)}(\mathbf{x}_i, \mathbf{x}_j)) = R(...R(R(\mathbf{x}_i^T \mathbf{x}_j))).$$

Thus we can consider $\Sigma^{(l)}$ as a univariate function of the input correlation $\Sigma : [-1, 1] \to [-1, 1]$ and also conclude that $[\mathbf{G}_l]_{ij} \in [-1, 1]$. Furthermore,

$$\dot{\Sigma}^{(l)}(\mathbf{x}_i, \mathbf{x}_j) = R'(\rho^{(l)}(\mathbf{x}_i, \mathbf{x}_j)) = R'(R(...R(R(\mathbf{x}_i^T \mathbf{x}_j)))),$$

which likewise implies $\dot{\Sigma}$ is a dot product kernel. Recall now the random variables introduced to define R: $Z_1, Z_2 \sim \mathcal{N}(0, 1)$ are independent and $U_1 = Z_1$, $U_2 = (\rho Z_1 + \sqrt{1 - \rho^2} Z_2)$. Observe U_1, U_2 are dependent but identically distributed as $U_1, U_2 \sim \mathcal{N}(0, 1)$. For any $\rho \in [-1, 1]$ then applying the Cauchy-Schwarz inequality gives

$$|R'(\rho)|^2 = \sigma_w^4 \left|\mathbb{E}[\phi'(U_1)\phi'(U_2)]\right|^2 \leq \sigma_w^4 \mathbb{E}[\phi'(U_1)^2]\mathbb{E}[\phi'(U_2)^2] = \sigma_w^4 \mathbb{E}[\phi'(U_1)^2]^2 = |R'(1)|^2.$$

As a result, under the assumptions of the lemma $\dot{\Sigma}^{(l)} : [-1, 1] \to [-\chi, \chi]$ and $\dot{\Sigma}^{(l)}(\mathbf{x}_i, \mathbf{x}_i) = \chi$. From this it immediately follows that $[\dot{\mathbf{G}}_l]_{ij} \in [-\chi, \chi]$ and $[\dot{\mathbf{G}}_l]_{ii} = \chi$ as claimed. Finally, as $\Sigma : [-1, 1] \to [-1, 1]$ and $\dot{\Sigma} : [-1, 1] \to [-\chi, \chi]$ are dot product kernels, then from (12) the NTK must also be a dot product kernel and furthermore a univariate function of the pairwise correlation of its input arguments. □

The following corollary, which follows immediately from Lemma A.2 and (14), characterizes the trace of the NTK matrix in terms of the trace of the input gram.

Corollary A.3. *Under the same conditions as Lemma A.2, suppose ϕ and σ_w^2 are chosen such that $\chi = 1$. Then*

$$Tr(\mathbf{K}_{n,l}) = l. \tag{21}$$

A.4 HERMITE EXPANSIONS

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is square integrable w.r.t. the standard Gaussian measure $\gamma = e^{-x^2/2}/\sqrt{2\pi}$ if $\mathbb{E}_{x\sim\mathcal{N}(0,1)}[f(x)^2] < \infty$. We denote by $L^2(\mathbb{R}, \gamma)$ the space of all such functions. The probabilist's Hermite polynomials are given by

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}, \quad k = 0, 1, \ldots.$$

The first three Hermite polynomials are $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = (x^2 - 1)$. Let $h_k(x) = \frac{H_k(x)}{\sqrt{k!}}$ denote the normalized probabilist's Hermite polynomials. The normalized Hermite polynomials form a complete orthonormal basis in $L^2(\mathbb{R}, \gamma)$ (O'Donnell, 2014, §11): in all that follows, whenever we reference the Hermite polynomials, we will be referring to the normalized Hermite polynomials. The Hermite expansion of a function $\phi \in L^2(\mathbb{R}, \gamma)$ is given by

$$\phi(x) = \sum_{k=0}^{\infty} \mu_k(\phi) h_k(x), \tag{22}$$

where

$$\mu_k(\phi) = \mathbb{E}_{X\sim\mathcal{N}(0,1)}[\phi(X)h_k(X)] \tag{23}$$

is the kth normalized probabilist's Hermite coefficient of ϕ. In what follows we shall make use of the following identities.

$$\forall k \geq 1,\ h_k'(x) = \sqrt{k} h_{k-1}(x), \tag{24}$$

$$\forall k \geq 1,\ x h_k(x) = \sqrt{k+1} h_{k+1}(x) + \sqrt{k} h_{k-1}(x). \tag{25}$$

$$h_k(0) = \begin{cases} 0, & \text{if } k \text{ is odd} \\ \frac{1}{\sqrt{k!}}(-1)^{\frac{k}{2}}(k-1)!! & \text{if } k \text{ is even} \end{cases},$$
$$\text{where } k!! = \begin{cases} 1, & k \leq 0 \\ k \cdot (k-2) \cdots 5 \cdot 3 \cdot 1, & k > 0 \text{ odd} \\ k \cdot (k-2) \cdots 6 \cdot 4 \cdot 2, & k > 0 \text{ even}. \end{cases} \tag{26}$$

We also remark that the more commonly encountered physicist's Hermite polynomials, which we denote $\tilde{H}_k$, are related to the normalized probablist's polynomials as follows,

$$h_k(z) = \frac{2^{-k/2}\tilde{H}_k(z/\sqrt{2})}{\sqrt{k!}}.$$

The Hermite expansion of the activation function deployed will play a key role in determining the coefficients of the NTK power series. In particular, the Hermite coefficients of ReLU are as follows.

Lemma A.4. *Daniely et al. (2016) For $\phi(z) = \max\{0, z\}$ the Hermite coefficients are given by*

$$\mu_k(\phi) = \begin{cases} 1/\sqrt{2\pi}, & k = 0, \\ 1/2, & k = 1, \\ (k-3)!!/\sqrt{2\pi k!}, & k \text{ even and } k \geq 2, \\ 0, & k \text{ odd and } k > 3. \end{cases} \tag{27}$$

B EXPRESSING THE NTK AS A POWER SERIES

B.1 DERIVING A POWER SERIES FOR THE NTK

We will require the following minor adaptation of Nguyen & Mondelli (2020, Lemma D.2). We remark this result was first stated for ReLU and Softplus activations in the work of Oymak & Soltanolkotabi (2020, Lemma H.2).

Lemma B.1. *For arbitrary $n, d \in \mathbb{N}$, let $\mathbf{A} \in \mathbb{R}^{n\times d}$. For $i \in [n]$, we denote the ith row of $\mathbf{A}$ as $\mathbf{a}_i$, and further assume that $\|\mathbf{a}_i\| = 1$. Let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy $\phi \in L^2(\mathbb{R}, \gamma)$ and define*

$$\mathbf{M} = \mathbb{E}_{\mathbf{w}\sim\mathcal{N}(0,\boldsymbol{I}_n)}[\phi(\mathbf{A}\mathbf{w})\phi(\mathbf{A}\mathbf{w})^T] \in \mathbb{R}^{n\times n}.$$

Then the matrix series

$$\mathbf{S}_K = \sum_{k=0}^{K} \mu_k^2(\phi) \left(\mathbf{A}\mathbf{A}^T\right)^{\odot k}$$

converges uniformly to $\mathbf{M}$ *as* $K \to \infty$.

The proof of Lemma B.1 follows exactly as in (Nguyen & Mondelli, 2020, Lemma D.2), and is in fact slightly simpler due to the fact we assume the rows of $\mathbf{A}$ are unit length and $\mathbf{w} \sim \mathcal{N}(0, \mathbf{I}_d)$ instead of $\sqrt{d}$ and $\mathbf{w} \sim \mathcal{N}(0, \frac{1}{d}\mathbf{I}_d)$ respectively. For the ease of the reader, we now recall the following definitions, which are also stated in Section 3. Letting $\bar{\alpha}_l := (\alpha_{p,l})_{p=0}^{\infty}$ denote a sequence of real coefficients, then

$$F(p, k, \bar{\alpha}_l) := \begin{cases} 1 & k = 0 \text{ and } p = 0, \\ 0 & k = 0 \text{ and } p \geq 1, \\ \sum_{(j_i) \in \mathcal{J}(p,k)} \prod_{i=1}^{k} \alpha_{j_i,l} & k \geq 1 \text{ and } p \geq 0, \end{cases} \tag{28}$$

where

$$\mathcal{J}(p, k) := \{(j_i)_{i \in [k]} \ : \ j_i \geq 0 \ \forall i \in [k], \ \sum_{i=1}^{k} j_i = p\}$$

for all $p \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$.

We are now ready to derive power series for elements of $(\mathbf{G}_l))_{l=1}^{L+1}$ and $(\dot{\mathbf{G}}_l))_{l=2}^{L+1}$.

Lemma B.2. *Under Assumptions 1 and 2, for all* $l \in [2, L+1]$

$$\mathbf{G}_l = \sum_{k=0}^{\infty} \alpha_{k,l} (\mathbf{X}\mathbf{X}^T)^{\odot k}, \tag{29}$$

where the series for each element $[\mathbf{G}_l]_{ij}$ *converges absolutely and the coefficients* $\alpha_{p,l}$ *are nonnegative. The coefficients of the series* (29) *for all* $p \in \mathbb{Z}_{\geq 0}$ *can be expressed via the following recurrence relationship,*

$$\alpha_{p,l} = \begin{cases} \sigma_w^2 \mu_p^2(\phi) + \delta_{p=0}\sigma_b^2, & l = 2, \\ \sum_{k=0}^{\infty} \alpha_{k,2} F(p, k, \bar{\alpha}_{l-1}), & l \geq 3. \end{cases} \tag{30}$$

Furthermore,

$$\dot{\mathbf{G}}_l = \sum_{k=0}^{\infty} \upsilon_{k,l} (\mathbf{X}\mathbf{X}^T)^{\odot k}, \tag{31}$$

where likewise the series for each entry $[\dot{\mathbf{G}}_l]_{ij}$ *converges absolutely and the coefficients* $\upsilon_{p,l}$ *for all* $p \in \mathbb{Z}_{\geq 0}$ *are nonnegative and can be expressed via the following recurrence relationship,*

$$\upsilon_{p,l} = \begin{cases} \sigma_w^2 \mu_p^2(\phi'), & l = 2, \\ \sum_{k=0}^{\infty} \upsilon_{k,2} F(p, k, \bar{\alpha}_{l-1}), & l \geq 3. \end{cases} \tag{32}$$

Proof. We start by proving (29) and (30). Proceeding by induction, consider the base case $l = 2$. From Lemma A.1

$$\mathbf{G}_2 = \sigma_w^2 \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d)} [\phi(\mathbf{X}\mathbf{w})\phi(\mathbf{X}\mathbf{w})^T] + \sigma_b^2 \mathbf{1}_{n \times n}.$$

By the assumptions of the lemma, the conditions of Lemma B.1 are satisfied and therefore

$$\begin{aligned} \mathbf{G}_2 &= \sigma_w^2 \sum_{k=0}^{\infty} \mu_k^2(\phi) \left(\mathbf{X}\mathbf{X}^T\right)^{\odot k} + \sigma_b^2 \mathbf{1}_{n \times n} \\ &= \alpha_{0,2} \mathbf{1}_{n \times n} + \sum_{k=1}^{\infty} \alpha_{k,2} \left(\mathbf{X}\mathbf{X}^T\right)^{\odot k}. \end{aligned}$$

Observe the coefficients $(\alpha_{k,2})_{k \in \mathbb{Z}_{\geq 0}}$ are nonnegative. Therefore, for any $i, j \in [n]$ using Lemma A.2 the series for $[\mathbf{G}_l]_{ij}$ satisfies

$$\sum_{k=0}^{\infty} |\alpha_{k,2}| \left|\langle \mathbf{x}_i, \mathbf{x}_j \rangle^k\right| \leq \sum_{k=0}^{\infty} \alpha_{k,2} \langle \mathbf{x}_i, \mathbf{x}_i \rangle^k \quad = [\mathbf{G}_l]_{ii} = 1 \tag{33}$$

and so must be absolutely convergent. With the base case proved we proceed to assume the inductive hypothesis holds for arbitrary $\mathbf{G}_l$ with $l \in [2, L]$. Observe

$$\mathbf{G}_{l+1} = \sigma_w^2 \mathbb{E}_{\mathbf{w}\sim\mathcal{N}(\mathbf{0},\mathbf{I}_n)}[\phi(\mathbf{A}\mathbf{w})\phi(\mathbf{A}\mathbf{w})^T] + \sigma_b^2 \mathbf{1}_{n\times n},$$

where $\mathbf{A}$ is a matrix square root of $\mathbf{G}_l$, meaning $\mathbf{G}_l = \mathbf{A}\mathbf{A}$. Recall from Lemma A.1 that $\mathbf{G}_l$ is also symmetric and positive semi-definite, therefore we may additionally assume, without loss of generality, that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, which conveniently implies $\mathbf{G}_{n,l} = \mathbf{A}\mathbf{A}^T$. Under the assumptions of the lemma the conditions for Lemma A.2 are satisfied and as a result $[\mathbf{G}_{n,l}]_{ii} = \|\mathbf{a}_i\| = 1$ for all $i \in [n]$, where we recall $\mathbf{a}_i$ denotes the ith row of $\mathbf{A}$. Therefore we may again apply Lemma A.1,

$$\begin{aligned}
\mathbf{G}_{l+1} &= \sigma_w^2 \sum_{k=0}^{\infty} \mu_k^2(\phi) \left(\mathbf{A}\mathbf{A}^T\right)^{\odot k} + \sigma_b^2 \mathbf{1}_{n\times n} \\
&= (\sigma_w^2 \mu_0^2(\phi) + \sigma_b^2)\mathbf{1}_{n\times n} + \sigma_w^2 \sum_{k=1}^{\infty} \mu_k^2(\phi) \left(\mathbf{G}_{n,l}\right)^{\odot k} \\
&= (\sigma_w^2 \mu_0^2(\phi) + \sigma_b^2)\mathbf{1}_{n\times n} + \sigma_w^2 \sum_{k=1}^{\infty} \mu_k^2(\phi) \left(\sum_{m=0}^{\infty} \alpha_{m,l} (\mathbf{X}\mathbf{X}^T)^{\odot m}\right)^{\odot k},
\end{aligned}$$

where the final equality follows from the inductive hypothesis. For any pair of indices $i, j \in [n]$

$$[\mathbf{G}_{l+1}]_{ij} = (\sigma_w^2 \mu_0^2(\phi) + \sigma_b^2) + \sigma_w^2 \sum_{k=1}^{\infty} \mu_k^2(\phi) \left(\sum_{m=0}^{\infty} \alpha_{m,l} \langle \mathbf{x}_i, \mathbf{x}_j\rangle^m\right)^k.$$

By the induction hypothesis, for any $i, j \in [n]$ the series $\sum_{m=0}^{\infty} \alpha_{m,l} \langle \mathbf{x}_i, \mathbf{x}_j\rangle^m$ is absolutely convergent. Therefore, from the Cauchy product of power series and for any $k \in \mathbb{Z}_{\geq 0}$ we have

$$\left(\sum_{m=0}^{\infty} \alpha_{m,l} \langle \mathbf{x}_i, \mathbf{x}_j\rangle^m\right)^k = \sum_{p=0}^{\infty} F(p, k, \bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j\rangle^p, \tag{34}$$

where $F(p, k, \bar{\alpha}_l)$ is defined in (4). By definition, $F(p, k, \bar{\alpha}_l)$ is a sum of products of positive coefficients, and therefore $|F(p, k, \bar{\alpha}_l)| = F(p, k, \bar{\alpha}_l)$. In addition, recall again by Assumption 2 and Lemma A.2 that $[\mathbf{G}_l]_{ii} = 1$. As a result, for any $k \in \mathbb{Z}_{\geq 0}$, as $|\langle \mathbf{x}_i, \mathbf{x}_j\rangle| \leq 1$

$$\sum_{p=0}^{\infty} |F(p, k, \bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j\rangle^p| \leq \left(\sum_{m=0}^{\infty} \alpha_{m,l}\right)^k = [\mathbf{G}_{n,l}]_{ii} = 1 \tag{35}$$

and therefore the series $\sum_{p=0}^{\infty} F(p, k, \bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j\rangle^p$ converges absolutely. Recalling from the proof of the base case that the series $\sum_{p=1}^{\infty} \alpha_{p,2}$ is absolutely convergent and has only nonnegative elements, we may therefore interchange the order of summation in the following,

$$\begin{aligned}
[\mathbf{G}_{l+1}]_{ij} &= (\sigma_w^2 \mu_0^2(\phi) + \sigma_b^2) + \sigma_w^2 \sum_{k=1}^{\infty} \mu_k^2(\phi) \left(\sum_{p=0}^{\infty} F(p, k, \bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j\rangle^p\right) \\
&= \alpha_{0,2} + \sum_{k=1}^{\infty} \alpha_{k,2} \left(\sum_{p=0}^{\infty} F(p, k, \bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j\rangle^p\right) \\
&= \alpha_{0,2} + \sum_{p=0}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_{k,2} F(p, k, \bar{\alpha}_l)\right) \langle \mathbf{x}_i, \mathbf{x}_j\rangle^p.
\end{aligned}$$

Recalling the definition of $F(p,k,l)$ in (4), in particular $F(0,0,\bar{\alpha}_l) = 1$ and $F(p,0,\bar{\alpha}_l) = 0$ for $p \in \mathbb{Z}_{\geq 1}$, then

$$\begin{aligned}
[\mathbf{G}_{l+1}]_{ij} &= \left(\alpha_{0,2} + \sum_{k=1}^{\infty} \alpha_{k,2} F(0,k,\bar{\alpha}_l)\right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^0 + \sum_{p=1}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_{k,2} F(p,k,\bar{\alpha}_l)\right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \\
&= \left(\sum_{k=0}^{\infty} \alpha_{k,2} F(0,k,\bar{\alpha}_l)\right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^0 + \sum_{p=1}^{\infty} \left(\sum_{k=0}^{\infty} \alpha_{k,2} F(p,k,\bar{\alpha}_l)\right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \\
&= \sum_{p=0}^{\infty} \left(\sum_{k=0}^{\infty} \alpha_{k,2} F(p,k,\bar{\alpha}_l)\right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \\
&= \sum_{p=0}^{\infty} \alpha_{p,l+1} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p.
\end{aligned}$$

As the indices $i, j \in [n]$ were arbitrary we conclude that

$$\mathbf{G}_{l+1} = \sum_{p=0}^{\infty} \alpha_{p,l+1} \left(\mathbf{X}\mathbf{X}^T\right)^{\odot p}$$

as claimed. In addition, by inspection and using the induction hypothesis it is clear that the coefficients $(\alpha_{p,l+1})_{p=0}^{\infty}$ are nonnegative. Therefore, by an argument identical to (33), the series for each entry of $[\mathbf{G}_{l+1}]_{ij}$ is absolutely convergent. This concludes the proof of (29) and (30).

We now turn our attention to proving the (31) and (32). Under the assumptions of the lemma the conditions for Lemmas A.1 and B.1 are satisfied and therefore for the base case $l = 2$

$$\begin{aligned}
\dot{\mathbf{G}}_2 &= \sigma_w^2 \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)} [\phi'(\mathbf{X}\mathbf{w}) \phi'(\mathbf{X}\mathbf{w})^T] \\
&= \sigma_w^2 \sum_{k=0}^{\infty} \mu_k^2(\phi') \left(\mathbf{X}\mathbf{X}^T\right)^{\odot k} \\
&= \sum_{k=0}^{\infty} \upsilon_{k,2} \left(\mathbf{X}\mathbf{X}^T\right)^{\odot k}.
\end{aligned}$$

By inspection the coefficients $(\upsilon_{p,2})_{p=0}^{\infty}$ are nonnegative and as a result by an argument again identical to (33) the series for each entry of $[\dot{\mathbf{G}}_2]_{ij}$ is absolutely convergent. For $l \in [2, L]$, from (29) and its proof there is a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ such that $\mathbf{G}_l = \mathbf{A}\mathbf{A}^T$. Again applying Lemma B.1

$$\begin{aligned}
\dot{\mathbf{G}}_{n,l+1} &= \sigma_w^2 \mathbb{E}_{\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)} [\phi'(\mathbf{A}\mathbf{w}) \phi'(\mathbf{A}\mathbf{w})^T] \\
&= \sigma_w^2 \sum_{k=0}^{\infty} \mu_k^2(\phi') \left(\mathbf{A}\mathbf{A}^T\right)^{\odot k} \\
&= \sum_{k=0}^{\infty} \upsilon_{k,2} \left(\mathbf{G}_{n,l}\right)^{\odot k} \\
&= \sum_{k=0}^{\infty} \upsilon_{k,2} \left(\sum_{p=0}^{\infty} \alpha_{p,l} \left(\mathbf{X}\mathbf{X}^T\right)^{\odot p}\right)^{\odot k}
\end{aligned}$$

Analyzing now an arbitrary entry $[\dot{\mathbf{G}}_{l+1}]_{ij}$, by substituting in the power series expression for $\mathbf{G}_l$ from (29) and using (34) we have

$$\begin{aligned}
[\dot{\mathbf{G}}_{l+1}]_{ij} &= \sum_{k=0}^{\infty} \upsilon_{k,2} \left(\sum_{p=0}^{\infty} \alpha_{p,l} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \right)^k \\
&= \sum_{k=0}^{\infty} \upsilon_{k,2} \left(\sum_{p=0}^{\infty} F(p,k,\bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \right) \\
&= \sum_{p=0}^{\infty} \left(\sum_{k=0}^{\infty} \upsilon_{k,2} F(p,k,\bar{\alpha}_l) \right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \\
&= \sum_{p=0}^{\infty} \upsilon_{p,l+1} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p .
\end{aligned}$$

Note that exchanging the order of summation in the third equality above is justified as for any $k \in \mathbb{Z}_{\geq 0}$ by (35) we have $\sum_{p=0}^{\infty} F(p,k,\bar{\alpha}_l)|\langle \mathbf{x}_i, \mathbf{x}_j \rangle|^p \leq 1$ and therefore $\sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \upsilon_{k,2} F(p,k,\bar{\alpha}_l) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p$ converges absolutely. As the indices $i, j \in [n]$ were arbitrary we conclude that

$$\dot{\mathbf{G}}_{l+1} = \sum_{p=0}^{\infty} \upsilon_{p,l+1} \left(\mathbf{X}\mathbf{X}^T \right)^{\odot p}$$

as claimed. Finally, by inspection the coefficients $(\upsilon_{p,l+1})_{p=0}^{\infty}$ are nonnegative, therefore, and again by an argument identical to (33), the series for each entry of $[\dot{\mathbf{G}}_{n,l+1}]_{ij}$ is absolutely convergent. This concludes the proof. □

We are now prove the key result of Section 3.

Theorem 3.1. *Under Assumptions 1 and 2, for all $l \in [L+1]$*

$$n\mathbf{K}_l = \sum_{p=0}^{\infty} \kappa_{p,l} \left(\mathbf{X}\mathbf{X}^T \right)^{\odot p} . \tag{5}$$

The series for each entry $n[\mathbf{K}_l]_{ij}$ converges absolutely and the coefficients $\kappa_{p,l}$ are nonnegative and can be evaluated using the recurrence relationships

$$\kappa_{p,l} = \begin{cases} \delta_{p=0}\gamma_b^2 + \delta_{p=1}\gamma_w^2, & l = 1, \\ \alpha_{p,l} + \sum_{q=0}^{p} \kappa_{q,l-1} \upsilon_{p-q,l}, & l \in [2, L+1], \end{cases} \tag{6}$$

where

$$\alpha_{p,l} = \begin{cases} \sigma_w^2 \mu_p^2(\phi) + \delta_{p=0}\sigma_b^2, & l = 2, \\ \sum_{k=0}^{\infty} \alpha_{k,2} F(p,k,\bar{\alpha}_{l-1}), & l \geq 3, \end{cases} \tag{7}$$

and

$$\upsilon_{p,l} = \begin{cases} \sigma_w^2 \mu_p^2(\phi'), & l = 2, \\ \sum_{k=0}^{\infty} \upsilon_{k,2} F(p,k,\bar{\alpha}_{l-1}), & l \geq 3, \end{cases} \tag{8}$$

are likewise nonnegative for all $p \in \mathbb{Z}_{\geq 0}$ and $l \in [2, L+1]$.

Proof. We proceed by induction. The base case $l = 1$ follows trivially from Lemma A.1. We therefore assume the induction hypothesis holds for an arbitrary $l - 1 \in [1, L]$. From (14) and Lemma B.2

$$\begin{aligned}
n\mathbf{K}_l &= \mathbf{G}_l + n\mathbf{K}_{l-1} \odot \dot{\mathbf{G}}_l \\
&= \left(\sum_{p=0}^{\infty} \alpha_{p,l} \left(\mathbf{X}\mathbf{X}^T \right)^{\odot p} \right) + \left(n \sum_{q=0}^{\infty} \kappa_{q,l-1} \left(\mathbf{X}\mathbf{X}^T \right)^{\odot q} \right) \odot \left(\sum_{w=0}^{\infty} \upsilon_{w,l} \left(\mathbf{X}\mathbf{X}^T \right)^{\odot w} \right) .
\end{aligned}$$

Therefore, for arbitrary $i, j \in [n]$

$$[n\mathbf{K}_l]_{ij} = \sum_{p=0}^{\infty} \alpha_{p,l} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p + \left(n \sum_{q=0}^{\infty} \kappa_{q,l-1} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^q \right) \left(\sum_{w=0}^{\infty} \upsilon_{w,l} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^w \right).$$

Observe $n \sum_{q=0}^{\infty} \kappa_{q,l-1} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^q = \Theta^{(l-1)}(\mathbf{x}_i, \mathbf{x}_j)$ and therefore the series must converge due to the convergence of the NTK. Furthermore, $\sum_{w=0}^{\infty} \upsilon_{w,l} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^w = [\dot{\mathbf{G}}_{n,l}]_{ij}$ and therefore is absolutely convergent by Lemma B.2. As a result, by Merten's Theorem the product of these two series is equal to their Cauchy product. Therefore

$$\begin{aligned}
[n\mathbf{K}_l]_{ij} &= \sum_{p=0}^{\infty} \alpha_{p,l} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p + \sum_{p=0}^{\infty} \left(\sum_{q=0}^{p} \kappa_{q,l-1} \upsilon_{p-q,l} \right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \\
&= \sum_{p=0}^{\infty} \left(\alpha_{p,l} + \sum_{q=0}^{p} \kappa_{q,l-1} \upsilon_{p-q,l} \right) \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p \\
&= \sum_{p=0}^{\infty} \kappa_{p,l} \langle \mathbf{x}_i, \mathbf{x}_j \rangle^p,
\end{aligned}$$

from which the (5) immediately follows. □

B.2 Analyzing the coefficients of the NTK power series

In this section we study the coefficients of the NTK power series stated in Theorem 3.1. Our first observation is that, under additional assumptions on the activation function ϕ, the recurrence relationship (6) can be simplified in order to depend only on the Hermite expansion of ϕ.

Lemma B.3. *Under Assumption 3 the Hermite coefficients of ϕ' satisfy*

$$\mu_k(\phi') = \sqrt{k+1}\mu_{k+1}(\phi)$$

for all $k \in \mathbb{Z}_{\geq 0}$.

Proof. Note for each $n \in \mathbb{N}$ as ϕ is absolutely continuous on $[-n, n]$ it is differentiable a.e. on $[-n, n]$. It follows by the countable additivity of the Lebesgue measure that ϕ is differentiable a.e. on $\mathbb{R}$. Furthermore, as ϕ is polynomially bounded we have $\phi \in L^2(\mathbb{R}, e^{-x^2/2}/\sqrt{2\pi})$. Fix $a > 0$. Since ϕ is absolutely continuous on $[-a, a]$ it is of bounded variation on $[-a, a]$. Also note that $h_k(x)e^{-x^2/2}$ is of bounded variation on $[-a, a]$ due to having a bounded derivative. Thus we have by Lebesgue-Stieltjes integration-by-parts (see e.g. Folland 1999, Chapter 3)

$$\begin{aligned}
&\int_{-a}^{a} \phi'(x) h_k(x) e^{-x^2/2} dx \\
&= \phi(a) h_k(a) e^{-a^2/2} - \phi(-a) h_k(-a) e^{-a^2/2} + \int_{-a}^{a} \phi(x) [x h_k(x) - h_k'(x)] e^{-x^2/2} dx \\
&= \phi(a) h_k(a) e^{-a^2/2} - \phi(-a) h_k(-a) e^{-a^2/2} + \int_{-a}^{a} \phi(x) \sqrt{k+1} h_{k+1}(x) e^{-x^2/2} dx,
\end{aligned}$$

where in the last line above we have used the fact that (24) and (25) imply that $x h_k(x) - h_k'(x) = \sqrt{k+1} h_{k+1}(x)$. Thus we have shown

$$\begin{aligned}
&\int_{-a}^{a} \phi'(x) h_k(x) e^{-x^2/2} dx \\
&= \phi(a) h_k(a) e^{-a^2/2} - \phi(-a) h_k(-a) e^{-a^2/2} + \int_{-a}^{a} \phi(x) \sqrt{k+1} h_{k+1}(x) e^{-x^2/2} dx.
\end{aligned}$$

We note that since $|\phi(x) h_k(x)| = \mathcal{O}(|x|^{\beta+k})$ we have that as $a \to \infty$ the first two terms above vanish. Thus by sending $a \to \infty$ we have

$$\int_{-\infty}^{\infty} \phi'(x) h_k(x) e^{-x^2/2} dx = \int_{-\infty}^{\infty} \sqrt{k+1} \phi(x) h_{k+1}(x) e^{-x^2/2} dx.$$

After dividing by $\sqrt{2\pi}$ we get the desired result. □

In particular, under Assumption 3, and as highlighted by Corollary B.4, which follows directly from Lemmas B.2 and B.3, the NTK coefficients can be computed only using the Hermite coefficients of ϕ.

Corollary B.4. *Under Assumptions 1, 2 and 3, for all $p \in \mathbb{Z}_{\geq 0}$*

$$\upsilon_{p,l} = \begin{cases} (p+1)\alpha_{p+1,2}, & l = 2, \\ \sum_{k=0}^{\infty} \upsilon_{k,2} F(p,k,\bar{\alpha}_{l-1}), & l \geq 3. \end{cases} \tag{36}$$

With these results in place we proceed to analyze the decay of the coefficients of the NTK for depth two networks. As stated in the main text, the decay of the NTK coefficients depends on the decay of the Hermite coefficients of the activation function deployed. This in turn is strongly influenced by the behavior of the tails of the activation function. To this end we roughly group activation functions into three categories: growing tails, flat or constant tails and finally decaying tails. Analyzing each of these groups in full generality is beyond the scope of this paper, we therefore instead study the behavior of ReLU, Tanh and Gaussian activation functions, being prototypical and practically used examples of each of these three groups respectively. We remark that these three activation functions satisfy Assumption 3. For typographical ease we let $\omega_\sigma(z) := (1/\sqrt{2\pi\sigma^2}) \exp\left(-z^2/(2\sigma^2)\right)$ denote the Gaussian activation function with variance σ^2.

Lemma B.5. *Under Assumptions 1 and 2,*

1. *if $\phi(z) = ReLU(z)$, then $\kappa_{p,2} = \delta_{(\gamma_b>0)\cup(p \text{ even})}\Theta(p^{-3/2})$,*
2. *if $\phi(z) = Tanh(z)$, then $\kappa_{p,2} = \mathcal{O}\left(\exp\left(-\frac{\pi\sqrt{p-1}}{2}\right)\right)$,*
3. *if $\phi(z) = \omega_\sigma(z)$, then $\kappa_{p,2} = \delta_{(\gamma_b>0)\cup(p \text{ even})}\Theta(p^{1/2}(\sigma^2+1)^{-p})$.*

Proof. Recall (9),

$$\kappa_{p,2} = \sigma_w^2(1+\gamma_w^2 p)\mu_p^2(\phi) + \sigma_w^2\gamma_b^2(1+p)\mu_{p+1}^2(\phi) + \delta_{p=0}\sigma_b^2.$$

In order to bound $\kappa_{p,2}$ we proceed by using Lemma A.4 to bound the square of the Hermite coefficients. We start with ReLU. Note Lemma A.4 actually provides precise expressions for the Hermite coefficients of ReLU, however, these are not immediately easy to interpret. Observe from Lemma A.4 that above index $p = 2$ all odd indexed Hermite coefficients are 0. It therefore suffices to bound the even indexed terms, given by

$$\mu_p(ReLU) = \frac{1}{\sqrt{2\pi}} \frac{(p-3)!!}{\sqrt{p!}}.$$

Observe from (26) that for p even

$$h_p(0) = (-1)^{p/2} \frac{(p-1)!!}{\sqrt{p!}},$$

therefore

$$\mu_p(ReLU) = \frac{1}{\sqrt{2\pi}} \frac{(p-3)!!}{\sqrt{p!}} = \frac{1}{\sqrt{2\pi}} \frac{|h_p(0)|}{p-1}.$$

Analyzing now $|h_p(0)|$,

$$\frac{(p-1)!!}{\sqrt{p!}} = \frac{\prod_{i=1}^{p/2}(2i-1)}{\sqrt{\prod_{i=1}^{p/2}(2i-1)2i}} = \sqrt{\frac{\prod_{i=1}^{p/2}(2i-1)}{\prod_{i=1}^{p/2}2i}} = \sqrt{\frac{(p-1)!!}{p!!}}.$$

Here, the expression inside the square root is referred to in the literature as the Wallis ratio, for which the following lower and upper bounds are available Kazarinoff (1956),

$$\sqrt{\frac{1}{\pi(p+0.5)}} < \frac{(p-1)!!}{p!!} < \sqrt{\frac{1}{\pi(p+0.25)}}. \tag{37}$$

As a result

$$|h_p(0)| = \Theta(p^{-1/4})$$

and therefore

$$\mu_p(ReLU) = \begin{cases} \Theta(p^{-5/4}), & p \text{ even}, \\ 0, & p \text{ odd}. \end{cases}$$

As $(p+1)^{-3/2} = \Theta(p^{-3/2})$, then from (9)

$$\begin{aligned} \kappa_{p,2} &= \Theta((p\mu_p^2(ReLU) + \delta_{\gamma_b>0}(p+1)\mu_{p+1}^2(ReLU))) \\ &= \Theta((\delta_{p \text{ even}} p^{-3/2} + \delta_{(p \text{ odd})\cap(\gamma_b>0)}(p+1)^{-3/2})) \\ &= \Theta\left(\delta_{(p \text{ even})\cup((p \text{ odd})\cap(\gamma_b>0))} p^{-3/2}\right) \\ &= \delta_{(p \text{ even})\cup(\gamma_b>0)} \Theta\left(p^{-3/2}\right) \end{aligned}$$

as claimed in item *1*.

We now proceed to analyze $\phi(z) = Tanh(z)$. From Panigrahi et al. (2020, Corollary F.7.1)

$$\mu_p(Tanh') = \mathcal{O}\left(\exp\left(-\frac{\pi\sqrt{p}}{4}\right)\right).$$

As Tanh satisfies the conditions of Lemma B.3

$$\mu_p(Tanh) = p^{-1/2}\mu_{p-1}(Tanh') = \mathcal{O}\left(p^{-1/2}\exp\left(-\frac{\pi\sqrt{p-1}}{4}\right)\right).$$

Therefore the result claimed in item *2.* follows as

$$\begin{aligned} \kappa_{p,2} &= \mathcal{O}((p\mu_p^2(Tanh) + (p+1)\mu_{p+1}^2(Tanh))) \\ &= \mathcal{O}\left(\exp\left(-\frac{\pi\sqrt{p-1}}{2}\right) + \exp\left(-\frac{\pi\sqrt{p}}{2}\right)\right) \\ &= \mathcal{O}\left(\exp\left(-\frac{\pi\sqrt{p-1}}{2}\right)\right). \end{aligned}$$

Finally, we now consider $\phi(z) = \omega_\sigma(z)$ where $\omega_\sigma(z)$ is the density function of $\mathcal{N}(0, \sigma^2)$. Similar to ReLU, analytic expressions for the Hermite coefficients of $\omega_\sigma(z)$ are known (see e.g., Davis, 2021, Theorem 2.9),

$$\mu_p^2(\omega_\sigma) = \begin{cases} \frac{p!}{((p/2)!)^2 2^p 2\pi(\sigma^2+1)^{p+1}}, & \text{p even}, \\ 0, & \text{p odd}. \end{cases}$$

For p even

$$(p/2)! = p!!2^{-p/2}.$$

Therefore

$$\frac{p!}{(p/2)!(p/2)!} = 2^p \frac{p!}{p!!p!!} = 2^p \frac{(p-1)!!}{p!!}.$$

As a result, for p even and using (37), it follows that

$$\mu_p^2(\omega_\sigma) = \frac{(\sigma^2+1)^{-(p+1)}}{2\pi}\frac{(p-1)!!}{p!!} \quad = \Theta(p^{-1/2}(\sigma^2+1)^{-p}).$$

Finally, since $(p+1)^{1/2}(\sigma^2+1)^{-p-1} = \Theta(p^{1/2}(\sigma^2+1)^{-p})$, then from (9)

$$\begin{aligned} \kappa_{p,2} &= \Theta((p\mu_p^2(\omega_\sigma) + \delta_{\gamma_b>0}(p+1)\mu_{p+1}^2(\omega_\sigma))) \\ &= \Theta\left(\delta_{(p \text{ even})\cup((p \text{ odd})\cap(\gamma_b>0))} p^{1/2}(\sigma^2+1)^{-p}\right) \\ &= \delta_{(p \text{ even})\cup(\gamma_b>0)} \Theta\left(p^{1/2}(\sigma^2+1)^{-p}\right) \end{aligned}$$

as claimed in item *3*. □

B.3 NUMERICAL APPROXIMATION VIA A TRUNCATED NTK POWER SERIES AND INTERPRETATION OF 1

Currently, computing the infinite width NTK requires either a) explicit evaluation of the Gaussian integrals highlighted in (13), b) numerical approximation of these same integrals such as in Lee et al. (2018), or c) approximation via a sufficiently wide yet still finite width network, see for instance Engel et al. (2022); Novak et al. (2022). These Gaussian integrals (13) can be solved solved analytically only for a minority of activation functions, notably ReLU as discussed for example by Arora et al. (2019b), while the numerical integration and finite width approximation approaches are relatively computationally expensive. The truncated NTK power series we define as analogous to (5) but with the series involved being computed only up to the Tth element. Once the top T coefficients are computed, then for any input correlation the NTK can be approximated by evaluating the corresponding finite degree T polynomial.

Definition B.6. *For an arbitrary pair $\mathbf{x}, \mathbf{y} \in \mathbb{S}^{d-1}$ let $\rho = \mathbf{x}^T\mathbf{y}$ denote their linear correlation. Under Assumptions 1, 2 and 3, for all $l \in [2, L+1]$ the T-truncated NTK power series $\hat{\Theta}_T^{(l)} : [-1, 1] \to \mathbb{R}$ is defined as*

$$\Theta_T^{(l)}(\rho) = \sum_{p=0}^{T} \hat{\kappa}_{p,l}\rho^p. \tag{38}$$

and whose coefficients are defined via the following recurrence relation,

$$\hat{\kappa}_{p,l} = \begin{cases} \delta_{p=0}\gamma_b^2 + \delta_{p=1}\gamma_w^2, & l = 1, \\ \hat{\alpha}_{p,l} + \sum_{q=0}^{p} \hat{\kappa}_{q,l-1}\hat{v}_{p-q,l}, & l \in [2, L+1]. \end{cases} \tag{39}$$

Here, with $\bar{\hat{\alpha}}_{l-1} = (\hat{\alpha}_{p,l-1})_{p=0}^T$,

$$\hat{\alpha}_{p,l} := \begin{cases} \sigma_w^2\mu_p^2(\phi) + \delta_{p=0}\sigma_b^2, & l = 2, \\ \sum_{k=0}^{T} \hat{\alpha}_{k,2}F(p, k, \bar{\hat{\alpha}}_{l-1}), & l \geq 3 \end{cases} \tag{40}$$

and

$$\hat{v}_{p,l} := \begin{cases} \sqrt{p+1}\hat{\alpha}_{p+1,2}, & l = 2, \\ \sum_{k=0}^{T} \sqrt{k+1}\hat{\alpha}_{p+1,2}F(p, k, \bar{\hat{\alpha}}_{l}), & l \geq 3. \end{cases} \tag{41}$$

In order to analyze the performance and potential of the truncated NTK for numerical approximation, we compute it for ReLU and compare it with its analytical expression Arora et al. (2019b). To recall this result, let

$$R(\rho) := \frac{\sqrt{1-\rho^2} + \rho \cdot \arcsin(\rho)}{\pi} + \frac{\rho}{2},$$
$$R'(\rho) := \frac{\arcsin(\rho)}{\pi} + \frac{1}{2}.$$

Under Assumptions 1 and 2, with $\phi(z) = ReLU(z)$, $\gamma_w^2 = 1$, $\sigma_w^2 = 2$, $\sigma_b^2 = \gamma_b^2 = 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$ and $\rho_1 := \mathbf{x}^T\mathbf{y}$, then $\Theta_1(\mathbf{x}, \mathbf{y}) = \rho$ and for all $l \in [2, L+1]$

$$\begin{aligned} \rho_l &= R(\rho_{l-1}), \\ \Theta_l(\mathbf{x}, \mathbf{y}) &= \rho_l + \rho_{l-1}R'(\rho_{l-1}). \end{aligned} \tag{42}$$

Turning our attention to Figure 1, we observe particularly for input correlations $|\rho| \approx 0.5$ and below then the truncated ReLU NTK power series achieves machine level precision. For $|\rho| \approx 1$ higher order coefficients play a more significant role. As the truncated ReLU NTK power series approximates these coefficients less well the overall approximation of the ReLU NTK is worse. We remark also that negative correlations have a smaller absolute error as odd indexed terms cancel with even index terms: we emphasize again that in Figure 1 we plot the absolute not relative error. In addition, for $L = 1$ there is symmetry in the absolute error for positive and negative correlations as $\alpha_{p,2} = 0$ for all odd p. One also observes that approximation accuracy goes down with depth, which is due to the error in the coefficients at the previous layer contributing to the error in the coefficients at the next, thereby resulting in an accumulation of error with depth. Also, and certainly as one might expect, a larger truncation point T results in overall better approximation. Finally, as the decay in the Hermite coefficients for ReLU is relatively slow, see e.g., Table 1 and Lemma 3.2, we expect the truncated ReLU NTK power series to perform worse relative to the truncated NTK's for other activation functions.

C ANALYZING THE SPECTRUM OF THE NTK VIA ITS POWER SERIES

C.1 EFFECTIVE RANK OF POWER SERIES KERNELS

Recall that for a positive semidefinite matrix $\mathbf{A}$ we define the *effective rank* Huang et al. (2022) via the following ratio

$$\text{eff}(\mathbf{A}) := \frac{Tr(\mathbf{A})}{\lambda_1(\mathbf{A})}.$$

We consider a kernel Gram matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$ that has the following power series representation in terms of an input gram matrix $\mathbf{X}\mathbf{X}^T$

$$n\mathbf{K} = \sum_{i=0}^{\infty} c_i(\mathbf{X}\mathbf{X}^T)^{\odot i}.$$

Whenever $c_0 \neq 0$ the effective rank of $\mathbf{K}$ is $O(1)$, as displayed in the following theorem.

Theorem 4.1. *Assume that we have a kernel Gram matrix $\mathbf{K}$ of the form $n\mathbf{K} = \sum_{p=0}^{\infty} c_p(\mathbf{X}\mathbf{X}^T)^{\odot p}$ where $c_0 \neq 0$. Furthermore, assume the input data $\mathbf{x}_i$ are normalized so that $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$. Then*

$$\text{eff}(\mathbf{K}) \leq \frac{\sum_{p=0}^{\infty} c_p}{c_0}.$$

Proof. By linearity of trace we have that

$$Tr(n\mathbf{K}) = \sum_{i=0}^{\infty} c_i Tr((\mathbf{X}\mathbf{X}^T)^{\odot i}) = n\sum_{i=0}^{\infty} c_i$$

where we have used the fact that $Tr((\mathbf{X}\mathbf{X}^T)^{\odot i}) = n$ for all $i \in \mathbb{N}$. On the other hand

$$\lambda_1(n\mathbf{K}) \geq \lambda_1(c_0(\mathbf{X}\mathbf{X}^T)^0) = \lambda_1(c_0\mathbf{1}_{n\times n}) = nc_0.$$

Thus we have that

$$\text{eff}(\mathbf{K}) = \frac{Tr(\mathbf{K})}{\lambda_1(\mathbf{K})} = \frac{Tr(n\mathbf{K})}{\lambda_1(n\mathbf{K})} \leq \frac{\sum_{i=0}^{\infty} c_i}{c_0}.$$

□

The above theorem demonstrates that the constant term $c_0\mathbf{1}_{n\times n}$ in the kernel leads to a significant outlier in the spectrum of $\mathbf{K}$. However this fails to capture how the structure of the input data $\mathbf{X}$ manifests in the spectrum of $\mathbf{K}$. For this we will examine the centered kernel matrix $\widetilde{\mathbf{K}} := \mathbf{K} - \frac{c_0}{n}\mathbf{1}\mathbf{1}^T$. Using a very similar argument as before we can demonstrate that the effective rank of $\widetilde{\mathbf{K}}$ is controlled by the effective rank of the input data gram $\mathbf{X}\mathbf{X}^T$. This is formalized in the following theorem.

Theorem 4.3. *Assume that we have a kernel Gram matrix $\mathbf{K}$ of the form $n\mathbf{K} = \sum_{p=0}^{\infty} c_p(\mathbf{X}\mathbf{X}^T)^{\odot p}$ where $c_1 \neq 0$. Furthermore, assume the input data $\mathbf{x}_i$ are normalized so that $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$. Then the centered kernel $\widetilde{\mathbf{K}} := \mathbf{K} - \frac{c_0}{n}\mathbf{1}_{n\times n}$ satisfies*

$$\text{eff}(\widetilde{\mathbf{K}}) \leq \text{eff}(\mathbf{X}\mathbf{X}^T)\frac{\sum_{p=1}^{\infty} c_p}{c_1}.$$

Proof. By the linearity of the trace we have that

$$Tr(n\widetilde{\mathbf{K}}) = \sum_{i=1}^{\infty} c_i Tr((\mathbf{X}\mathbf{X}^T)^{\odot i}) = Tr(\mathbf{X}\mathbf{X}^T)\sum_{i=1}^{\infty} c_i$$

where we have used the fact that $Tr((\mathbf{X}\mathbf{X}^T)^{\odot i}) = Tr(\mathbf{X}\mathbf{X}^T) = n$ for all $i \in [n]$. On the other hand we have that

$$\lambda_1(n\widetilde{\mathbf{K}}) \geq \lambda_1(c_1\mathbf{X}\mathbf{X}^T) = c_1\lambda_1(\mathbf{X}\mathbf{X}^T).$$

Thus we conclude

$$\text{eff}(\widetilde{\mathbf{K}}) = \frac{Tr(\widetilde{\mathbf{K}})}{\lambda_1(\widetilde{\mathbf{K}})} = \frac{Tr(n\widetilde{\mathbf{K}})}{\lambda_1(n\widetilde{\mathbf{K}})} \leq \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}\frac{\sum_{i=1}^{\infty} c_i}{c_1}.$$

□

C.2 EFFECTIVE RANK OF THE NTK FOR FINITE WIDTH NETWORKS

C.2.1 NOTATION AND DEFINITIONS

We will let $[k] := \{1, 2, \ldots, k\}$. We consider a neural network

$$\sum_{\ell=1}^{m} a_\ell \phi(\langle \mathbf{w}_\ell, \mathbf{x} \rangle)$$

where $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{w}_\ell \in \mathbb{R}^d$, $a_\ell \in \mathbb{R}$ for all $\ell \in [m]$ and ϕ is a scalar valued activation function. The network we present here does not have any bias values in the inner-layer, however the results we will prove later apply to the nonzero bias case by replacing $\mathbf{x}$ with $[\mathbf{x}^T, 1]^T$. We let $\mathbf{W} \in \mathbb{R}^{m \times d}$ be the matrix whose ℓ-th row is equal to $\mathbf{w}_\ell$ and $\mathbf{a} \in \mathbb{R}^m$ be the vector whose ℓ-th entry is equal to a_ℓ. We can then write the neural network in vector form

$$f(\mathbf{x}; \mathbf{W}, \mathbf{a}) = \mathbf{a}^T \phi(\mathbf{W}\mathbf{x})$$

where ϕ is understood to be applied entry-wise.

Suppose we have n training data inputs $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^d$. We will let $\mathbf{X} \in \mathbb{R}^{n \times d}$ be the matrix whose i-th row is equal to $\mathbf{x}_i$. Let $\theta_{inner} = vec(\mathbf{W})$ denote the row-wise vectorization of the inner-layer weights. We consider the Jacobian of the neural networks predictions on the training data with respect to the inner layer weights:

$$\mathbf{J}_{inner}^T = \left[\frac{\partial f(\mathbf{x}_1)}{\partial \theta_{inner}}, \frac{\partial f(\mathbf{x}_2)}{\partial \theta_{inner}}, \ldots, \frac{\partial f(\mathbf{x}_n)}{\partial \theta_{inner}} \right]$$

Similarly we can look at the analagous quantity for the outer layer weights

$$\mathbf{J}_{outer}^T = \left[\frac{\partial f(\mathbf{x}_1)}{\partial \mathbf{a}}, \frac{\partial f(\mathbf{x}_2)}{\partial \mathbf{a}}, \ldots, \frac{\partial f(\mathbf{x}_n)}{\partial \mathbf{a}} \right] = \phi\left(\mathbf{W}\mathbf{X}^T\right).$$

Our first observation is that the per-example gradients for the inner layer weights have a nice Kronecker product representation

$$\frac{\partial f(\mathbf{x})}{\partial \theta_{inner}} = \begin{bmatrix} a_1 \phi'(\langle \mathbf{w}_1, \mathbf{x} \rangle) \\ a_2 \phi'(\langle \mathbf{w}_2, \mathbf{x} \rangle) \\ \cdots \\ a_m \phi'(\langle \mathbf{w}_m, \mathbf{x} \rangle) \end{bmatrix} \otimes \mathbf{x}.$$

For convenience we will let

$$\mathbf{Y}_i := \begin{bmatrix} a_1 \phi'(\langle \mathbf{w}_1, \mathbf{x}_i \rangle) \\ a_2 \phi'(\langle \mathbf{w}_2, \mathbf{x}_i \rangle) \\ \cdots \\ a_m \phi'(\langle \mathbf{w}_m, \mathbf{x}_i \rangle) \end{bmatrix}.$$

where the dependence of $\mathbf{Y}_i$ on the parameters $\mathbf{W}$ and $\mathbf{a}$ is suppressed (formally $\mathbf{Y}_i = \mathbf{Y}_i(\mathbf{W}, \mathbf{a})$). This way we may write

$$\frac{\partial f(\mathbf{x}_i)}{\partial \theta_{inner}} = \mathbf{Y}_i \otimes \mathbf{x}_i.$$

We will study the NTK with respect to the inner-layer weights

$$\mathbf{K}_{inner} = \mathbf{J}_{inner} \mathbf{J}_{inner}^T$$

and the same quantity for the outer-layer weights

$$\mathbf{K}_{outer} = \mathbf{J}_{outer} \mathbf{J}_{outer}^T.$$

For a hermitian matrix $\mathbf{A}$ we will let $\lambda_i(\mathbf{A})$ denote the ith largest eigenvalue of $\mathbf{A}$ so that $\lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \geq \cdots \geq \lambda_n(\mathbf{A})$. Similarly for an arbitrary matrix $\mathbf{A}$ we will let $\sigma_i(\mathbf{A})$ to the ith largest singular value of $\mathbf{A}$. For a matrix $\mathbf{A} \in \mathbb{R}^{r \times k}$ we will let $\sigma_{min}(\mathbf{A}) = \sigma_{\min(r,k)}$.

C.2.2 EFFECTIVE RANK

For a positive semidefinite matrix $\mathbf{A}$ we define the *effective rank* (Huang et al., 2022) of $\mathbf{A}$ to be the quantity

$$\text{eff}(\mathbf{A}) := \frac{Tr(\mathbf{A})}{\lambda_1(\mathbf{A})}.$$

The effective rank quantifies how many eigenvalues are on the order of the largest eigenvalue. We have the Markov-like inequality

$$|\{i : \lambda_i(\mathbf{A}) \geq c\lambda_1(\mathbf{A})\}| \leq c^{-1}\frac{Tr(\mathbf{A})}{\lambda_1(\mathbf{A})}$$

and the eigenvalue bound

$$\frac{\lambda_i(\mathbf{A})}{\lambda_1(\mathbf{A})} \leq \frac{1}{i}\frac{Tr(\mathbf{A})}{\lambda_1(\mathbf{A})}.$$

Let $\mathbf{A}$ and $\mathbf{B}$ be positive semidefinite matrices. Then we have

$$\frac{Tr(\mathbf{A}+\mathbf{B})}{\lambda_1(\mathbf{A}+\mathbf{B})} \leq \frac{Tr(\mathbf{A})+Tr(\mathbf{B})}{\max\left(\lambda_1(\mathbf{A}),\lambda_1(\mathbf{B})\right)} \leq \frac{Tr(\mathbf{A})}{\lambda_1(\mathbf{A})} + \frac{Tr(\mathbf{B})}{\lambda_1(\mathbf{B})}.$$

Thus the effective rank is subadditive for positive semidefinite matrices.

We will be interested in bounding the effective rank of the NTK. Let $\mathbf{K} = \mathbf{J}\mathbf{J}^T = \mathbf{J}_{outer}\mathbf{J}_{outer}^T + \mathbf{J}_{inner}\mathbf{J}_{inner}^T = \mathbf{K}_{outer} + \mathbf{K}_{inner}$ be the NTK matrix with respect to all the network parameters. Note that by subadditivity

$$\frac{Tr(\mathbf{K})}{\lambda_1(\mathbf{K})} \leq \frac{Tr(\mathbf{K}_{outer})}{\lambda_1(\mathbf{K}_{outer})} + \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})}.$$

In this vein we will control the effective rank of $\mathbf{K}_{inner}$ and $\mathbf{K}_{outer}$ separately.

C.2.3 EFFECTIVE RANK OF INNER-LAYER NTK

We will show that the effective rank of inner-layer NTK is bounded by a multiple of the effective rank of the data input gram $\mathbf{X}\mathbf{X}^T$. We introduce the following meta-theorem that we will use to prove various corollaries later

Theorem C.1. *Set* $\alpha := \sup_{\|\mathbf{b}\|=1}\left[\min_{j\in[n]}|\langle \mathbf{Y}_j, \mathbf{b}\rangle|\right]$. *Assume* $\alpha > 0$. *Then*

$$\frac{\min_{i\in[n]}\|\mathbf{Y}_i\|_2^2\, Tr(\mathbf{X}\mathbf{X}^T)}{\max_{i\in[n]}\|\mathbf{Y}_i\|_2^2\,\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1\left(\mathbf{K}_{inner}\right)} \leq \frac{\max_{i\in[n]}\|\mathbf{Y}_i\|_2^2}{\alpha^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

Proof. We will first prove the upper bound. We first observe that

$$\begin{aligned} Tr(\mathbf{K}_{inner}) &= \sum_{i=1}^n \left\|\frac{\partial f(\mathbf{x}_i)}{\partial \theta_{inner}}\right\|_2^2 = \sum_{i=1}^n \|\mathbf{Y}_i\otimes\mathbf{x}_i\|_2^2 = \sum_{i=1}^n \|\mathbf{Y}_i\|_2^2\,\|\mathbf{x}_i\|_2^2 \\ &\leq \max_{j\in[n]}\|\mathbf{Y}_j\|_2^2\sum_{i=1}^n\|\mathbf{x}_i\|_2^2 = \max_{j\in[n]}\|\mathbf{Y}_j\|_2^2\, Tr(\mathbf{X}\mathbf{X}^T) \end{aligned}$$

Recall that

$$\lambda_1\left(\mathbf{K}_{inner}\right) = \lambda_1\left(\mathbf{J}_{inner}\mathbf{J}_{inner}^T\right) = \lambda_1\left(\mathbf{J}_{inner}^T\mathbf{J}_{inner}\right).$$

Well

$$\begin{aligned} \mathbf{J}_{inner}^T\mathbf{J}_{inner} &= \sum_{i=1}^n \frac{\partial f(\mathbf{x}_i)}{\partial\theta_{inner}}\frac{\partial f(\mathbf{x}_i)}{\partial\theta_{inner}}^T = \sum_{i=1}^n\left[\mathbf{Y}_i\otimes\mathbf{x}_i\right]\left[\mathbf{Y}_i\otimes\mathbf{x}_i\right]^T \\ &= \sum_{i=1}^n\left[\mathbf{Y}_i\mathbf{Y}_i^T\right]\otimes\left[\mathbf{x}_i\mathbf{x}_i^T\right] \end{aligned}$$

Well then we may use the fact that

$$\lambda_1(\mathbf{J}_{inner}^T\mathbf{J}_{inner}) = \max_{\|\mathbf{b}\|_2=1} \mathbf{b}^T\mathbf{J}_{inner}^T\mathbf{J}_{inner}\mathbf{b}$$

Let $\mathbf{b}_1 \in \mathbb{R}^m$ and $\mathbf{b}_2 \in \mathbb{R}^d$ be vectors that we will optimize later satisfying $\|\mathbf{b}_1\|_2 \|\mathbf{b}_2\|_2 = 1$. Then we have that $\|\mathbf{b}_1 \otimes \mathbf{b}_2\| = 1$ and

$$\begin{aligned}(\mathbf{b}_1 \otimes \mathbf{b}_2)^T\mathbf{J}_{inner}^T\mathbf{J}_{inner}(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \sum_{i=1}^n (\mathbf{b}_1 \otimes \mathbf{b}_2)^T \left(\left[\mathbf{Y}_i\mathbf{Y}_i^T\right] \otimes \left[\mathbf{x}_i\mathbf{x}_i^T\right]\right)(\mathbf{b}_1 \otimes \mathbf{b}_2) \\ &= \sum_{i=1}^n \left[\mathbf{b}_1^T\mathbf{Y}_i\mathbf{Y}_i^T\mathbf{b}_1\right]\left[\mathbf{b}_2^T\mathbf{x}_i\mathbf{x}_i^T\mathbf{b}_2\right] \geq \left[\min_{j\in[n]} \mathbf{b}_1^T\mathbf{Y}_j\mathbf{Y}_j^T\mathbf{b}_1\right] \sum_{i=1}^n \mathbf{b}_2^T\mathbf{x}_i\mathbf{x}_i^T\mathbf{b}_2 \\ &= \left[\min_{j\in[n]} \mathbf{b}_1^T\mathbf{Y}_j\mathbf{Y}_j^T\mathbf{b}_1\right] \mathbf{b}_2^T \left[\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^T\right] \mathbf{b}_2 = \left[\min_{j\in[n]} \mathbf{b}_1^T\mathbf{Y}_j\mathbf{Y}_j^T\mathbf{b}_1\right] \mathbf{b}_2\mathbf{X}^T\mathbf{X}\mathbf{b}_2\end{aligned}$$

Pick $\mathbf{b}_2$ so that $\|\mathbf{b}_2\| = 1$ and

$$\mathbf{b}_2\mathbf{X}^T\mathbf{X}\mathbf{b}_2 = \lambda_1(\mathbf{X}^T\mathbf{X}) = \lambda_1(\mathbf{X}\mathbf{X}^T).$$

Thus for this choice of $\mathbf{b}_2$ we have

$$\begin{gathered}\lambda_1(\mathbf{J}_{inner}^T\mathbf{J}_{inner}) \geq (\mathbf{b}_1 \otimes \mathbf{b}_2)^T\mathbf{J}_{inner}^T\mathbf{J}_{inner}(\mathbf{b}_1 \otimes \mathbf{b}_2) \geq \\ \left[\min_{j\in[n]} \mathbf{b}_1^T\mathbf{Y}_j\mathbf{Y}_j^T\mathbf{b}_1\right] \mathbf{b}_2\mathbf{X}^T\mathbf{X}\mathbf{b}_2 = \left[\min_{j\in[n]} \mathbf{b}_1^T\mathbf{Y}_j\mathbf{Y}_j^T\mathbf{b}_1\right] \lambda_1(\mathbf{X}\mathbf{X}^T)\end{gathered}$$

Now note that $\alpha^2 = \sup_{\|\mathbf{b}_1\|=1} \left[\min_{j\in[n]} \mathbf{b}_1^T\mathbf{Y}_j\mathbf{Y}_j^T\mathbf{b}_1\right]$. Thus by taking the sup over $\mathbf{b}_1$ in our previous bound we have

$$\lambda_1(\mathbf{K}_{inner}) = \lambda_1(\mathbf{J}_{inner}^T\mathbf{J}_{inner}) \geq \alpha^2\lambda_1(\mathbf{X}\mathbf{X}^T).$$

Thus combined with our previous result we have

$$\frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \leq \frac{\max_{i\in[n]} \|\mathbf{Y}_i\|_2^2}{\alpha^2} \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

We now prove the lower bound.

$$\begin{aligned}Tr(\mathbf{K}_{inner}) &= \sum_{i=1}^n \left\|\frac{\partial f(\mathbf{x}_i)}{\partial \theta_{inner}}\right\|_2^2 = \sum_{i=1}^n \|\mathbf{Y}_i \otimes \mathbf{x}_i\|_2^2 = \sum_{i=1}^n \|\mathbf{Y}_i\|_2^2 \|\mathbf{x}_i\|_2^2 \\ &\geq \min_{j\in[n]} \|\mathbf{Y}_j\|_2^2 \sum_{i=1}^n \|\mathbf{x}_i\|_2^2 = \min_{j\in[n]} \|\mathbf{Y}_j\|_2^2 Tr(\mathbf{X}\mathbf{X}^T)\end{aligned}$$

Let $\mathbf{Y} \in \mathbb{R}^{n\times m}$ be the matrix whose ith row is equal to $\mathbf{Y}_i$. Then observe that

$$\mathbf{K}_{inner} = [\mathbf{Y}\mathbf{Y}^T] \odot [\mathbf{X}\mathbf{X}^T]$$

where $\odot$ denotes the entry-wise Hadamard product of two matrices. We now recall that if $\mathbf{A}$ and $\mathbf{B}$ are two positive semidefinite matrices we have (Oymak & Soltanolkotabi, 2020, Lemma 2)

$$\lambda_1(\mathbf{A} \odot \mathbf{B}) \leq \max_{i\in[n]} \mathbf{A}_{i,i}\lambda_1(\mathbf{B}).$$

Applying this to $\mathbf{K}_{inner}$ we get that

$$\lambda_1(\mathbf{K}_{inner}) \leq \max_{i\in[n]} \|\mathbf{Y}_i\|_2^2 \lambda_1(\mathbf{X}\mathbf{X}^T)$$

Combining this with our previous result we get

$$\frac{\min_{i\in[n]} \|\mathbf{Y}_i\|_2^2 Tr(\mathbf{X}\mathbf{X}^T)}{\max_{i\in[n]} \|\mathbf{Y}_i\|_2^2 \lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})}$$

□

We can immediately get a useful corollary that applies to the ReLU activation function

Corollary C.2. *Set* $\alpha := \sup_{\|\mathbf{b}\|=1} \left[\min_{j\in[n]} |\langle \mathbf{Y}_j, \mathbf{b}\rangle|\right]$ *and* $\gamma_{max} := \sup_{x\in\mathbb{R}} |\phi'(x)|$. *Assume* $\alpha > 0$ *and* $\gamma_{max} < \infty$. *Then*

$$\frac{\alpha^2}{\gamma_{max}^2 \|\mathbf{a}\|_2^2} \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \leq \frac{\gamma_{max}^2 \|\mathbf{a}\|_2^2}{\alpha^2} \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

Proof. Note that the hypothesis on $|\phi'|$ gives $\|\mathbf{Y}_i\|_2^2 \leq \gamma_{max}^2 \|\mathbf{a}\|_2^2$ for all $i \in [n]$. Moreover by Cauchy-Schwarz we have that $\min_{i\in[n]} \|\mathbf{Y}_i\|_2 \geq \alpha$. Thus by theorem C.1 we get the desired result. □

If ϕ is a leaky ReLU type activation (say like those used in Nguyen & Mondelli (2020)) Theorem C.1 translates into an even simpler bound

Corollary C.3. *Suppose* $\phi'(x) \in [\gamma_{min}, \gamma_{max}]$ *for all* $x \in \mathbb{R}$ *where* $\gamma_{min} > 0$. *Then*

$$\frac{\gamma_{min}^2 Tr(\mathbf{X}\mathbf{X}^T)}{\gamma_{max}^2 \lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \leq \frac{\gamma_{max}^2}{\gamma_{min}^2} \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

Proof. We will lower bound

$$\alpha := \sup_{\|\mathbf{b}\|=1} \left[\min_{j\in[n]} |\langle \mathbf{Y}_j, \mathbf{b}\rangle|\right]$$

so that we can apply Corollary C.2. Set $\mathbf{b} = \mathbf{a}/\|\mathbf{a}\|_2$. Then we have that

$$\langle \mathbf{Y}_j, \mathbf{b}\rangle = \sum_{\ell=1}^m a_\ell \phi'(\langle \mathbf{w}_\ell, \mathbf{x}_j\rangle) a_\ell / \|\mathbf{a}\|_2 \geq \frac{\gamma_{min}}{\|\mathbf{a}\|_2} \sum_{\ell=1}^m a_\ell^2 = \gamma_{min} \|\mathbf{a}\|_2$$

Thus $\alpha \geq \gamma_{min} \|\mathbf{a}\|_2$. The result then follows from Corollary C.2 □

To control α in Theorem C.1 when ϕ is the ReLU activation function requires a bit more work. To this end we introduce the following lemma.

Lemma C.4. *Assume* $\phi(x) = ReLU(x)$. *Let* $R_{min}, R_{max} > 0$ *and define* $\tau = \{\ell \in [m] : |a_\ell| \in [R_{min}, R_{max}]\}$. *Set* $T = \min_{i\in[n]} \sum_{\ell\in\tau} \mathbb{I}[\langle \mathbf{x}_i, \mathbf{w}_\ell\rangle \geq 0]$. *Then*

$$\alpha := \sup_{\|\mathbf{b}\|=1} \left[\min_{i\in[n]} |\langle \mathbf{Y}_i, \mathbf{b}\rangle|\right] \geq \frac{R_{min}^2}{R_{max}} \frac{T}{|\tau|^{1/2}}$$

Proof. Let $\mathbf{a}_\tau$ be the vector such that $(\mathbf{a}_\tau)_\ell = a_\ell \mathbb{I}[\ell \in \tau]$. Then note that

$$\langle \mathbf{Y}_j, \mathbf{a}_\tau / \|\mathbf{a}_\tau\|_2\rangle = \frac{1}{\|\mathbf{a}_\tau\|} \sum_{\ell\in\tau} a_\ell^2 \mathbb{I}[\langle \mathbf{w}_\ell, \mathbf{x}_j\rangle \geq 0] \geq$$
$$\frac{R_{min}^2}{\|\mathbf{a}_\tau\|} \sum_{\ell\in\tau} \mathbb{I}[\langle \mathbf{w}_\ell, \mathbf{x}_j\rangle \geq 0] \geq \frac{R_{min}^2}{\|\mathbf{a}_\tau\|_2} T \geq \frac{R_{min}^2}{R_{max}|\tau|^{1/2}} T.$$

□

Roughly what Lemma C.4 says is that α is controlled when there is a set of inner-layer neurons that are active for each data point whose outer layer weights are similar in magnitude. Note that in Du et al. (2019b), Arora et al. (2019a), Oymak et al. (2019), Li et al. (2020), Xie et al. (2017) and Oymak & Soltanolkotabi (2020) the outer layer weights all have fixed constant magnitude. Thus in that case we can set $R_{min} = R_{max}$ in Lemma C.4 so that $\tau = [m]$. In this setting we have the following result.

Theorem C.5. *Assume $\phi(x) = ReLU(x)$. Suppose $|a_\ell| = R > 0$ for all $\ell \in [m]$. Furthermore suppose $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are independent random vectors such that $\mathbf{w}_\ell / \|\mathbf{w}_\ell\|$ has the uniform distribution on the sphere for each $\ell \in [m]$. Also assume $m \geq \frac{4\log(n/\epsilon)}{\delta^2}$ for some $\delta, \epsilon \in (0,1)$. Then with probability at least $1-\epsilon$ we have that*

$$\frac{(1-\delta)^2}{4}\text{eff}(\mathbf{X}\mathbf{X}^T) \leq \text{eff}(\mathbf{K}_{inner}) \leq \frac{4}{(1-\delta)^2}\text{eff}(\mathbf{X}\mathbf{X}^T).$$

Proof. Fix $j \in [n]$. Note by the assumption on the $\mathbf{w}_\ell$'s we have that $\mathbb{I}[\langle \mathbf{w}_1, \mathbf{x}_j\rangle \geq 0], \ldots, \mathbb{I}[\langle \mathbf{w}_m, \mathbf{x}_j\rangle \geq 0]$ are i.i.d. Bernouilli random variables taking the values 0 and 1 with probability $1/2$. Thus by the Chernoff bound for Binomial random variables we have that

$$\mathbb{P}\left(\sum_{\ell=1}^m \mathbb{I}[\langle \mathbf{w}_\ell, \mathbf{x}_j\rangle \geq 0] \leq \frac{m}{2}(1-\delta)\right) \leq \exp\left(-\delta^2\frac{m}{4}\right).$$

Thus taking the union bound over every $j \in [n]$ we get that if $m \geq \frac{4\log(n/\epsilon)}{\delta^2}$ then

$$\min_{j\in[n]} \sum_{\ell=1}^m \mathbb{I}[\langle \mathbf{w}_\ell, \mathbf{x}_j\rangle \geq 0] \geq \frac{m}{2}(1-\delta)$$

holds with probability at least $1-\epsilon$. Now note that if we set $R_{min} = R_{max} = R$ we have that $\tau = [m]$ where τ is defined as it is in Lemma C.4. In this case by our previous bound we have that T as defined in Lemma C.4 satisfies $T \geq \frac{m}{2}(1-\delta)$ with probability at least $1-\epsilon$. In this case the conclusion of Lemma C.4 gives us

$$\alpha \geq Rm^{1/2}\frac{(1-\delta)}{2} = \|\mathbf{a}\|_2 \frac{(1-\delta)}{2}.$$

Thus by Corollary C.2 and the above bound for α we get the desired result. □

We will now use Lemma C.4 to prove a bound in the case of Gaussian initialization.

Lemma C.6. *Assume $\phi(x) = ReLU(x)$. Suppose that $a_\ell \sim N(0, \nu^2)$ for each $\ell \in [m]$ i.i.d. Furthermore suppose $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are random vectors independent of each other and $\mathbf{a}$ such that $\mathbf{w}_\ell / \|\mathbf{w}_\ell\|$ has the uniform distribution on the sphere for each $\ell \in [m]$. Set $p = \mathbb{P}_{z\sim N(0,1)}(|z| \in [1/2, 1]) \approx 0.3$. Assume*

$$m \geq \frac{4\log(n/\epsilon)}{\delta^2(1-\delta)p}$$

for some $\epsilon, \delta \in (0,1)$. Then with probability at least $(1-\epsilon)^2$ we have that

$$\alpha := \sup_{\|\mathbf{b}\|=1}\left[\min_{i\in[n]} |\langle \mathbf{Y}_i, \mathbf{b}\rangle|\right] \geq \frac{\nu}{8}(1-\delta)^{3/2}p^{1/2}m^{1/2}$$

Proof. Set $R_{min} = \nu/2$ and $R_{max} = \nu$. Now set

$$\begin{aligned} p = \mathbb{P}_{a\sim N(0,\nu^2)}(|a| \in [R_{min}, R_{max}]) &= 2\mathbb{P}_{z\sim N(0,1)}\left(z \in \left[\frac{R_{min}}{\nu}, \frac{R_{max}}{\nu}\right]\right) \\ &= 2\mathbb{P}_{z\sim N(0,1)}(z \in [1/2, 1]) \approx 0.3. \end{aligned}$$

Now define $\tau = \{\ell \in [m] : |a_\ell| \in [R_{min}, R_{max}]\}$. We have by the Chernoff bound for binomial random variables

$$\mathbb{P}(|\tau| \leq (1-\delta)mp) \leq \exp\left(-\delta^2\frac{mp}{2}\right).$$

Thus if $m \geq \log\left(\frac{1}{\epsilon}\right)\frac{2}{p\delta^2}$ (a weaker condition than the hypothesis on m) then we have that $|\tau| \geq (1-\delta)mp$ with probability at least $1-\epsilon$. From now on assume such a τ has been observed and view it as fixed so that the only remaining randomness is over the $\mathbf{w}_\ell$'s. Now set $T = \min_{i\in[n]}\sum_{\ell\in\tau}\mathbb{I}[\langle \mathbf{x}_i, \mathbf{w}_\ell\rangle \geq 0]$. By the Chernoff bound again we get that for fixed $i \in [n]$

$$\mathbb{P}\left(\sum_{\ell\in\tau}\mathbb{I}[\langle \mathbf{x}_i, \mathbf{w}_\ell\rangle \geq 0] \leq \frac{(1-\delta)}{2}|\tau|\right) \leq \exp\left(-\delta^2\frac{|\tau|}{4}\right).$$

Thus by taking the union bound over $i \in [n]$ we get

$$\mathbb{P}\left(T \leq \frac{(1-\delta)}{2}|\tau|\right) \leq n \exp\left(-\delta^2 \frac{|\tau|}{4}\right)$$
$$\leq n \exp\left(-\delta^2 \frac{(1-\delta)mp}{4}\right)$$

Thus if we consider τ as fixed and $m \geq \frac{4\log(n/\epsilon)}{\delta^2(1-\delta)p}$ then with probability at least $1-\epsilon$ over the sampling of the $\mathbf{w}_\ell$'s we have that

$$T \geq \frac{(1-\delta)}{2}|\tau|$$

In this case by lemma C.4 we have that

$$\alpha := \sup_{\|\mathbf{b}\|=1}\left[\min_{i\in[n]} |\langle \mathbf{Y}_i, \mathbf{b}\rangle|\right] \geq \frac{R_{min}^2}{R_{max}} \frac{T}{|\tau|^{1/2}}$$
$$\geq \frac{\nu}{8}(1-\delta)^{3/2} m^{1/2} p^{1/2}.$$

Thus the above holds with probability at least $(1-\epsilon)^2$. □

This lemma now allows us to bound the effective rank of $\mathbf{K}_{inner}$ in the case of Gaussian initialization.

Theorem C.7. *Assume $\phi(x) = ReLU(x)$. Suppose that $a_\ell \sim N(0, \nu^2)$ for each $\ell \in [m]$ i.i.d. Furthermore suppose $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are random vectors independent of each other and $\mathbf{a}$ such that $\mathbf{w}_\ell / \|\mathbf{w}_\ell\|$ has the uniform distribution on the sphere for each $\ell \in [m]$. Set $p = \mathbb{P}_{z\sim N(0,1)}(|z| \in [1/2, 1]) \approx 0.3$. Let $\epsilon, \delta \in (0,1)$. Then there exists absolute constants $c, K > 0$ such that if*

$$m \geq \frac{4\log(n/\epsilon)}{\delta^2(1-\delta)p}$$

then with probability at least $1 - 3\epsilon$ we have that

$$\frac{1}{C}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \leq C\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

where

$$C = \frac{64}{(1-\delta)^3 p}\left[1 + \frac{\max\{c^{-1}K\log(1/\epsilon), mK\}}{m}\right].$$

Proof. By Bernstein's inequality

$$\mathbb{P}\left(\|\mathbf{a}/\nu\|_2^2 - m \geq t\right) \leq \exp\left[-c \cdot \min\left(\frac{t^2}{mK^2}, \frac{t}{K}\right)\right]$$

where c is an absolute constant. Set $t = \max\{c^{-1}K\log(1/\epsilon), mK\}$ so that the right hand side of the above inequality is bounded by ϵ. Thus by Lemma C.6 and the union bound we can ensure that with probability at least

$$1 - \epsilon - [1 - (1-\epsilon)^2] = 1 - 3\epsilon + \epsilon^2 \geq 1 - 3\epsilon$$

that $\|\mathbf{a}/\nu\|_2^2 \leq m + t$ and the conclusion of Lemma C.6 hold simultaneously. In that case

$$\frac{\|\mathbf{a}\|_2^2}{\alpha^2} \leq \frac{\nu^2[m+t]}{\frac{\nu^2}{64}(1-\delta)^3 mp} = \frac{64}{(1-\delta)^3 p}\left[1 + \frac{t}{m}\right] = C.$$

Thus by Corollary C.2 we get the desired result. □

By fixing $\delta > 0$ in the previous theorem we get the immediate corollary

Corollary C.8. *Assume $\phi(x) = ReLU(x)$. Suppose that $a_\ell \sim N(0, \nu^2)$ for each $\ell \in [m]$ i.i.d. Furthermore suppose $\mathbf{w}_1, \ldots, \mathbf{w}_m$ are random vectors independent of each other and $\mathbf{a}$ such that $\mathbf{w}_\ell / \|\mathbf{w}_\ell\|$ has the uniform distribution on the sphere for each $\ell \in [m]$. Then there exists an absolute constant $C > 0$ such that $m = \Omega(\log(n/\epsilon))$ ensures that with probability at least $1 - \epsilon$*

$$\frac{1}{C}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \leq C\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

C.2.4 EFFECTIVE RANK OF OUTER-LAYER NTK

Throughout this section $\phi(x) = ReLU(x)$. Our goal of this section, similar to before, is to bound the effective rank of $\mathbf{K}_{outer}$ by the effective rank of the input data gram $\mathbf{X}\mathbf{X}^T$. In this section we will use often make use of the basic identities

$$\|\mathbf{AB}\|_F \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_F$$

$$\|\mathbf{AB}\|_F \leq \|\mathbf{A}\|_F \|\mathbf{B}\|_2$$

$$Tr(\mathbf{A}\mathbf{A}^T) = Tr(\mathbf{A}^T\mathbf{A}) = \|\mathbf{A}\|_F^2$$

$$\|\mathbf{A}\|_2 = \left\|\mathbf{A}^T\right\|_2$$

$$\lambda_1(\mathbf{A}^T\mathbf{A}) = \lambda_1(\mathbf{A}\mathbf{A}^T) = \|\mathbf{A}\|_2^2 .$$

To begin bounding the effective rank of $\mathbf{K}_{outer}$, we prove the following lemma.

Lemma C.9. *Assume $\phi(x) = ReLU(x)$ and $\mathbf{W}$ is full rank with $m \geq d$. Then*

$$\frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\left[\|\phi(\mathbf{W}\mathbf{X}^T)\|_2 + \|\phi(-\mathbf{W}\mathbf{X}^T)\|_2\right]^2} \leq \frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2} \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

Proof. First note that

$$\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2 \leq \left\|\mathbf{W}\mathbf{X}^T\right\|_F^2 \leq \|\mathbf{W}\|_2^2 \left\|\mathbf{X}^T\right\|_F^2 = \|\mathbf{W}\|_2^2 Tr(\mathbf{X}\mathbf{X}^T).$$

Pick $\mathbf{b} \in \mathbb{R}^d$ such that $\|\mathbf{b}\|_2 = 1$ and $\|\mathbf{X}\mathbf{b}\|_2 = \|\mathbf{X}\|_2$. Since $\mathbf{W}^T$ is full rank we may set $\mathbf{u} = (\mathbf{W}^T)^\dagger \mathbf{b}$ so that $\mathbf{W}^T\mathbf{u} = \mathbf{b}$ where $\|\mathbf{u}\|_2 \leq \sigma_{min}(\mathbf{W}^T)^{-1}$ where $\sigma_{min}(\mathbf{W}^T)$ is the smallest *nonzero* singular value of $\mathbf{W}^T$. Well then

$$\begin{aligned}\|\mathbf{X}\|_2 = \|\mathbf{X}\mathbf{b}\|_2 = \left\|\mathbf{X}\mathbf{W}^T\mathbf{u}\right\|_2 \leq \left\|\mathbf{X}\mathbf{W}^T\right\|_2 \|\mathbf{u}\|_2 \leq \left\|\mathbf{X}\mathbf{W}^T\right\|_2 \sigma_{min}(\mathbf{W}^T)^{-1} \\ = \left\|\mathbf{W}\mathbf{X}^T\right\|_2 \sigma_{min}(\mathbf{W})^{-1}\end{aligned}$$

Now using the fact that $x = \phi(x) - \phi(-x)$ we have that

$$\left\|\mathbf{W}\mathbf{X}^T\right\|_2 = \left\|\phi(\mathbf{W}\mathbf{X}^T) - \phi(-\mathbf{W}\mathbf{X}^T)\right\|_2 \leq \left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2 + \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2$$

Thus combined with our previous results gives

$$\|\mathbf{X}\|_2 \leq \sigma_{min}(\mathbf{W})^{-1} \left[\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2 + \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2\right]$$

Therefore

$$\begin{aligned}\frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\sigma_{min}(\mathbf{W})^{-2}\left[\|\phi(\mathbf{W}\mathbf{X}^T)\|_2 + \|\phi(-\mathbf{W}\mathbf{X}^T)\|_2\right]^2} &\leq \frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\|\mathbf{X}\|_2^2} \\ &\leq \frac{\|\mathbf{W}\|_2^2 Tr(\mathbf{X}\mathbf{X}^T)}{\|\mathbf{X}\|_2^2} = \|\mathbf{W}\|_2^2 \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}\end{aligned}$$

which gives us the desired result. □

Corollary C.10. *Assume $\phi(x) = ReLU(x)$ and $\mathbf{W}$ is full rank with $m \geq d$. Then*

$$\frac{\max\left(\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2, \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_F^2\right)}{\max\left(\|\phi(\mathbf{W}\mathbf{X}^T)\|_2^2, \|\phi(-\mathbf{W}\mathbf{X}^T)\|_2^2\right)} \leq 4\frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2} \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

Proof. Using the fact that

$$\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2 + \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2 \leq 2\max\left(\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2, \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2\right)$$

and lemma C.9 we have that

$$\frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{4\max\left(\|\phi(\mathbf{W}\mathbf{X}^T)\|_2^2, \|\phi(-\mathbf{W}\mathbf{X}^T)\|_2^2\right)} \leq \frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

Note that the right hand side and the denominator of the left hand side do not change when you replace $\mathbf{W}$ with $-\mathbf{W}$. Therefore by using the above bound for both $\mathbf{W}$ and $-\mathbf{W}$ as the weight matrix separately we can conclude

$$\frac{\max\left(\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2, \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_F^2\right)}{4\max\left(\|\phi(\mathbf{W}\mathbf{X}^T)\|_2^2, \|\phi(-\mathbf{W}\mathbf{X}^T)\|_2^2\right)} \leq \frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

□

Corollary C.11. *Assume $\phi(x) = ReLU(x)$ and $m \geq d$. Suppose $\mathbf{W}$ and $-\mathbf{W}$ have the same distribution. Then conditioned on $\mathbf{W}$ being full rank we have that with probability at least $1/2$*

$$\frac{Tr(\mathbf{K}_{outer})}{\lambda_1(\mathbf{K}_{outer})} \leq 4\frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

Proof. Fix $\mathbf{W}$ where $\mathbf{W}$ is full rank. We have by corollary C.10 that either

$$\frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\|\phi(\mathbf{W}\mathbf{X}^T)\|_2^2} \leq 4\frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

holds or

$$\frac{\left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\|\phi(-\mathbf{W}\mathbf{X}^T)\|_2^2} \leq 4\frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

(the first holds in the case where $\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2^2 \geq \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2^2$ and the second in the case $\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2^2 < \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2^2$). Since $\mathbf{W}$ and $-\mathbf{W}$ have the same distribution, it follows that the first inequality must hold at least $1/2$ of the time. From

$$\frac{Tr(\mathbf{K}_{outer})}{\lambda_1(\mathbf{K}_{outer})} = \frac{\left\|\mathbf{J}_{outer}^T\right\|_F^2}{\left\|\mathbf{J}_{outer}^T\right\|_2^2} = \frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\|\phi(\mathbf{W}\mathbf{X}^T)\|_2^2}$$

we get the desired result. □

We now note that when $\mathbf{W}$ is rectangular shaped and the entries of $\mathbf{W}$ are i.i.d. Gaussians that $\mathbf{W}$ is full rank with high probability and $\sigma_{min}(\mathbf{W})^{-2}\|\mathbf{W}\|_2^2$ is well behaved. We recall the result from Vershynin (2012)

Theorem C.12. *Let $\mathbf{A}$ be a $N \times n$ matrix whose entries are independent standard normal random variables. Then for every $t \geq 0$, with probability at least $1 - 2\exp(-t^2/2)$ one has*

$$\sqrt{N} - \sqrt{n} - t \leq \sigma_{min}(\mathbf{A}) \leq \sigma_1(\mathbf{A}) \leq \sqrt{N} + \sqrt{n} + t$$

Corollary C.11 gives us a bound that works at least half the time. However, we would like to derive a bound that holds with high probability. We will have that when $m \gtrsim n$ we have sufficient concentration of the largest singular value of $\phi(\mathbf{W}\mathbf{X}^T)$ to prove such a bound. We recall the result from Vershynin (2012) (Remark 5.40)

Theorem C.13. *Assume that $\mathbf{A}$ is an $N \times n$ matrix whose rows $\mathbf{A}_i$ are independent sub-gaussian random vectors in $\mathbb{R}^n$ with second moment matrix Σ. Then for every $t \geq 0$, the following inequality holds with probability at least $1 - 2\exp(-ct^2)$*

$$\left\|\frac{1}{N}\mathbf{A}^*\mathbf{A} - \Sigma\right\|_2 \leq \max(\delta, \delta^2) \quad \textit{where} \quad \delta = C\sqrt{\frac{n}{N}} + \frac{t}{\sqrt{N}}$$

where $C = C_K, c = c_K > 0$ depend only on $K := \max_i \|\mathbf{A}_i\|_{\psi_2}$.

We will use theorem C.13 in the following lemma.

Lemma C.14. *Assume $\phi(x) = ReLU(x)$. Let $\mathbf{A} = \phi(\mathbf{W}\mathbf{X}^T)$ and $M = \max_{i\in[n]} \|\mathbf{x}_i\|_2$. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_m \sim N(0, \nu^2 I_d)$ i.i.d. Set $K = M\nu\sqrt{n}$ and define*

$$\Sigma := \mathbb{E}_{\mathbf{w}\sim N(0,\nu^2 I)}[\phi(\mathbf{X}\mathbf{w})\phi(\mathbf{w}^T\mathbf{X}^T)]$$

Then for every $t \geq 0$ the following inequality holds with probability at least $1 - 2\exp(-c_K t^2)$

$$\left\|\frac{1}{m}\mathbf{A}^T\mathbf{A} - \Sigma\right\|_2 \leq \max(\delta, \delta^2) \quad \textit{where} \quad \delta = C_K\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}},$$

where $c_K, C_K > 0$ are absolute constants that depend only on K.

Proof. We will let $\mathbf{A}_{\ell:}$ denote the ℓth row of $\mathbf{A}$ (considered as a column vector). Note that

$$\mathbf{A}_{\ell:} = \phi(\mathbf{X}\mathbf{w}_\ell).$$

We immediately get that the rows of $\mathbf{A}$ are i.i.d. We will now bound $\|\mathbf{A}_{\ell:}\|_{\psi_2}$. Let $\mathbf{b} \in \mathbb{R}^n$ such that $\|\mathbf{b}\|_2 = 1$. Then

$$\begin{aligned}
\|\langle\phi(\mathbf{X}\mathbf{w}_\ell), \mathbf{b}\rangle\|_{\psi_2} &= \left\|\sum_{i=1}^n \phi(\langle\mathbf{x}_i, \mathbf{w}_\ell\rangle)b_i\right\|_{\psi_2} \\
&\leq \sum_{i=1}^n |b_i| \, \|\phi(\langle\mathbf{x}_i, \mathbf{w}_\ell\rangle)\|_{\psi_2} \leq \sum_{i=1}^n |b_i| \, \|\langle\mathbf{x}_i, \mathbf{w}_\ell\rangle\|_{\psi_2} \\
&\leq \sum_{i=1}^n |b_i| C \|\mathbf{x}_i\|_2 \nu \leq CM\nu \|\mathbf{b}\|_1 \leq CM\nu\sqrt{n}
\end{aligned}$$

where $C > 0$ is an absolute constant. Set $K := M\nu\sqrt{n}$. Well then by theorem C.13 we have the following. For every $t \geq 0$ the following inequality holds with probability at least $1 - 2\exp(-c_K t^2)$

$$\left\|\frac{1}{m}\mathbf{A}^T\mathbf{A} - \Sigma\right\|_2 \leq \max(\delta, \delta^2) \quad \text{where} \quad \delta = C_K\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}$$

□

We are now ready to prove a high probability bound for the effective rank of $\mathbf{K}_{outer}$.

Theorem C.15. *Assume $\phi(x) = ReLU(x)$ and $m \geq d$. Let $M = \max_{i\in[n]} \|\mathbf{x}_i\|_2$. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_m \sim N(0, \nu^2 I_d)$ i.i.d. Set $K = M\nu\sqrt{n}$*

$$\Sigma := \mathbb{E}_{\mathbf{w}\sim N(0,\nu^2 I)}[\phi(\mathbf{X}\mathbf{w})\phi(\mathbf{w}^T\mathbf{X}^T)]$$

$$\delta = C_K\left[\sqrt{\frac{n}{m}} + \sqrt{\frac{\log(2/\epsilon)}{m}}\right]$$

where $\epsilon > 0$ is small. Now assume

$$\sqrt{m} > \sqrt{d} + \sqrt{2\log(2/\epsilon)}$$

and

$$\max(\delta, \delta^2) \leq \frac{1}{2}\lambda_1(\Sigma)$$

Then with probability at least $1 - 3\epsilon$

$$\frac{Tr(\mathbf{K}_{outer})}{\lambda_1(\mathbf{K}_{outer})} \leq 12\left(\frac{\sqrt{m} + \sqrt{d} + t_1}{\sqrt{m} - \sqrt{d} - t_1}\right)^2 \frac{Tr(\mathbf{X}^T\mathbf{X})}{\lambda_1(\mathbf{X}^T\mathbf{X})}$$

Proof. By theorem C.12 with $t_1 = \sqrt{2\log(2/\epsilon)}$ we have that with probability at least $1-\epsilon$ that

$$\sqrt{m} - \sqrt{d} - t_1 \leq \sigma_{min}(\mathbf{W}/\nu) \leq \sigma_1(\mathbf{W}/\nu) \leq \sqrt{m} + \sqrt{d} + t_1 \tag{43}$$

The above inequalities and the hypothesis on m imply that $\mathbf{W}$ is full rank.

Let $\mathbf{A} = \phi(\mathbf{W}\mathbf{X}^T)$ and $\tilde{\mathbf{A}} = \phi(-\mathbf{W}\mathbf{X}^T)$. Set $t_2 = \sqrt{\frac{\log(2/\epsilon)}{c_K}}$ where c_K is defined as in theorem C.14. Note that $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are identical in distribution. Thus by theorem C.14 and the union bound we get that with probability at least $1-2\epsilon$

$$\left\|\frac{1}{m}\mathbf{A}^T\mathbf{A} - \Sigma\right\|_2, \left\|\frac{1}{m}\tilde{\mathbf{A}}^T\tilde{\mathbf{A}} - \Sigma\right\|_2 \leq \max(\delta, \delta^2) =: \rho \tag{44}$$

where

$$\delta = C_K\sqrt{\frac{n}{m}} + \frac{t_2}{\sqrt{m}}.$$

By our previous results and the union bound we can ensure with probability at least $1-3\epsilon$ that the bounds (43) and (44) all hold simultaneously. In this case we have

$$\left\|\frac{1}{m}\tilde{\mathbf{A}}^T\tilde{\mathbf{A}}\right\|_2 \leq \left\|\frac{1}{m}\mathbf{A}^T\mathbf{A}\right\|_2 + 2\rho$$
$$= \left\|\frac{1}{m}\mathbf{A}^T\mathbf{A}\right\|_2\left[1 + \frac{2\rho}{\left\|\frac{1}{m}\mathbf{A}^T\mathbf{A}\right\|_2}\right] \leq \left\|\frac{1}{m}\mathbf{A}^T\mathbf{A}\right\|_2\left[1 + \frac{2\rho}{\lambda_1(\Sigma) - \rho}\right]$$

Assuming $\rho \leq \lambda_1(\Sigma)/2$ we have by the above bound

$$\left\|\frac{1}{m}\tilde{\mathbf{A}}^T\tilde{\mathbf{A}}\right\|_2 \leq 3\left\|\frac{1}{m}\mathbf{A}^T\mathbf{A}\right\|_2.$$

Now note that

$$\left\|\mathbf{A}^T\mathbf{A}\right\|_2 = \left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2^2 \quad \left\|\tilde{\mathbf{A}}^T\tilde{\mathbf{A}}\right\|_2 = \left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2^2$$

so that our previous bound implies

$$\left\|\phi(-\mathbf{W}\mathbf{X}^T)\right\|_2^2 \leq 3\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2^2$$

then we have by corollary C.10 that

$$\frac{Tr(\mathbf{K}_{outer})}{\lambda_1(\mathbf{K}_{outer})} = \frac{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_F^2}{\left\|\phi(\mathbf{W}\mathbf{X}^T)\right\|_2^2} \leq 12\frac{\|\mathbf{W}\|_2^2}{\sigma_{min}(\mathbf{W})^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$
$$\leq 12\left(\frac{\sqrt{m}+\sqrt{d}+t_1}{\sqrt{m}-\sqrt{d}-t_1}\right)^2\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

□

From the above theorem we get the following corollary.

Corollary C.16. *Assume $\phi(x) = ReLU(x)$ and $n \geq d$. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_m \sim N(0, \nu^2 I_d)$ i.i.d. Fix $\epsilon > 0$ small. Set $M = \max_{i\in[n]}\|\mathbf{x}_i\|_2$. Then*

$$m = \Omega\left(\max(\lambda_1(\Sigma)^{-2}, 1)\max(n, \log(1/\epsilon))\right)$$

and

$$\nu = O(1/M\sqrt{m})$$

suffices to ensure that with probability at least $1-\epsilon$

$$\frac{Tr(\mathbf{K}_{outer})}{\lambda_1(\mathbf{K}_{outer})} \leq C\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

where $C > 0$ is an absolute constant.

C.2.5 BOUND FOR THE COMBINED NTK

Based on the results in the previous two sections, we can now bound the effective rank of the combined NTK gram matrix $\mathbf{K} = \mathbf{K}_{inner} + \mathbf{K}_{outer}$.

Theorem 4.5. *Assume $\phi(x) = ReLU(x)$ and $n \geq d$. Fix $\epsilon > 0$ small. Suppose that $\mathbf{w}_1, \ldots, \mathbf{w}_m \sim N(0, \nu_1^2 I_d)$ i.i.d. and $a_1, \ldots, a_m \sim N(0, \nu_2^2)$. Set $M = \max_{i \in [n]} \|\mathbf{x}_i\|_2$, and let*

$$\Sigma := \mathbb{E}_{\mathbf{w} \sim N(0, \nu_1^2 I)}[\phi(\mathbf{X}\mathbf{w})\phi(\mathbf{w}^T\mathbf{X}^T)].$$

Then

$$m = \Omega\left(\max(\lambda_1(\Sigma)^{-2}, 1)\max(n, \log(1/\epsilon))\right), \quad \nu_1 = O(1/M\sqrt{m})$$

suffices to ensure that, with probability at least $1-\epsilon$ over the sampling of the parameter initialization,

$$\text{eff}(\mathbf{K}) \leq C \cdot \text{eff}(\mathbf{X}\mathbf{X}^T),$$

where $C > 0$ is an absolute constant.

Proof. This follows from the union bound and Corollaries C.8 and C.16. □

C.2.6 MAGNITUDE OF THE SPECTRUM

By our results in sections C.2.3 and C.2.4 we have that $m \gtrsim n$ suffices to ensure that

$$\frac{Tr(\mathbf{K})}{\lambda_1(\mathbf{K})} \lesssim \frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq d$$

Well note that

$$i\frac{\lambda_i(\mathbf{K})}{\lambda_1(\mathbf{K})} \leq \frac{Tr(\mathbf{K})}{\lambda_1(\mathbf{K})} \lesssim d$$

If $i \gg d$ then $\lambda_i(\mathbf{K})/\lambda_1(\mathbf{K})$ is small. Thus the NTK only has $O(d)$ large eigenvalues. The smallest eigenvalue $\lambda_n(\mathbf{K})$ of the NTK has been of interest in proving convergence guarantees (Du et al., 2019a;b; Oymak & Soltanolkotabi, 2020). By our previous inequality

$$\frac{\lambda_n(\mathbf{K})}{\lambda_1(\mathbf{K})} \lesssim \frac{d}{n}$$

Thus in the setting where $m \gtrsim n \gg d$ we have that the smallest eigenvalue will be driven to zero relative to the largest eigenvalue. Alternatively we can view the above inequality as a lower bound on the condition number

$$\frac{\lambda_1(\mathbf{K})}{\lambda_n(\mathbf{K})} \gtrsim \frac{n}{d}$$

We will first bound the analytical NTK in the setting when the outer layer weights have fixed constant magnitude. This is the setting considered by Xie et al. (2017), Arora et al. (2019a), Du et al. (2019b), Oymak et al. (2019), Li et al. (2020), and Oymak & Soltanolkotabi (2020).

Theorem C.17. *Let $\phi(x) = ReLU(x)$ and assume $\mathbf{X} \neq 0$. Let $\mathbf{K}_{inner}^{\infty} \in \mathbb{R}^{n \times n}$ be the analytical NTK, i.e.*

$$(\mathbf{K}_{inner}^{\infty})_{i,j} := \langle \mathbf{x}_i, \mathbf{x}_j \rangle \mathbb{E}_{\mathbf{w} \sim N(0, I_d)}\left[\phi'(\langle \mathbf{x}_i, \mathbf{w} \rangle)\phi'(\langle \mathbf{x}_j, \mathbf{w} \rangle)\right].$$

Then

$$\frac{1}{4}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner}^{\infty})}{\lambda_1\left(\mathbf{K}_{inner}^{\infty}\right)} \leq 4\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}.$$

Proof. We consider the setting where $|a_\ell| = 1/\sqrt{m}$ for all $\ell \in [m]$ and $\mathbf{w}_\ell \sim N(0, I_d)$ i.i.d.. As was shown by Jacot et al. (2018), Du et al. (2019b) in this setting we have that if we fix the training data $\mathbf{X}$ and send $m \to \infty$ we have that

$$\|\mathbf{K}_{inner} - \mathbf{K}_{inner}^{\infty}\|_2 \to 0$$

in probability. Therefore by continuity of the effective rank we have that

$$\frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \to \frac{Tr(\mathbf{K}^{\infty}_{inner})}{\lambda_1(\mathbf{K}^{\infty}_{inner})}$$

in probability. Let $\eta > 0$. Then there exists an $M \in \mathbb{N}$ such that $m \geq M$ implies that

$$\left| \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} - \frac{Tr(\mathbf{K}^{\infty}_{inner})}{\lambda_1(\mathbf{K}^{\infty}_{inner})} \right| \leq \eta \tag{45}$$

with probability greater than $1/2$. Now fix $\delta \in (0,1)$. On the other hand by Theorem C.5 with $\epsilon = 1/4$ we have that if $m \geq \frac{4}{\delta^2}\log(4n)$ then with probability at least $3/4$ that

$$\frac{(1-\delta)^2}{4}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}_{inner})}{\lambda_1(\mathbf{K}_{inner})} \leq \frac{4}{(1-\delta)^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}. \tag{46}$$

Thus if we set $m = \max(\frac{4}{\delta^2}\log(4n), M)$ we have with probability at least $3/4 - 1/2 = 1/4$ that (45) and (46) hold simultaneously. In this case we have that

$$\frac{(1-\delta)^2}{4}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} - \eta \leq \frac{Tr(\mathbf{K}^{\infty}_{inner})}{\lambda_1(\mathbf{K}^{\infty}_{inner})} \leq \frac{4}{(1-\delta)^2}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} + \eta$$

Note that the above argument runs through for any $\eta > 0$ and $\delta \in (0,1)$. Thus we may send $\eta \to 0^+$ and $\delta \to 0^+$ in the above inequality to get

$$\frac{1}{4}\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)} \leq \frac{Tr(\mathbf{K}^{\infty}_{inner})}{\lambda_1(\mathbf{K}^{\infty}_{inner})} \leq 4\frac{Tr(\mathbf{X}\mathbf{X}^T)}{\lambda_1(\mathbf{X}\mathbf{X}^T)}$$

□

We thus have the following corollary about the conditioning of the analytical NTK.

Corollary C.18. *Let $\phi(x) = ReLU(x)$ and assume $\mathbf{X} \neq 0$. Let $\mathbf{K}^{\infty}_{inner} \in \mathbb{R}^{n\times n}$ be the analytical NTK, i.e.*

$$(\mathbf{K}^{\infty}_{inner})_{i,j} := \langle \mathbf{x}_i, \mathbf{x}_j \rangle \mathbb{E}_{\mathbf{w}\sim N(0,I_d)}\left[\phi'(\langle \mathbf{x}_i, \mathbf{w}\rangle)\phi'(\langle \mathbf{x}_j, \mathbf{w}\rangle)\right].$$

Then

$$\frac{\lambda_n(\mathbf{K}^{\infty}_{inner})}{\lambda_1(\mathbf{K}^{\infty}_{inner})} \leq 4\frac{d}{n}.$$

C.3 Analysis of the lower spectrum: uniform data

Theorem 4.6. *Suppose that the training data are uniformly sampled from the unit hypersphere $\mathbb{S}^d$, $d \geq 2$. If the dot-product kernel function has the expansion $K(x_1, x_2) = \sum_{p=0}^{\infty} c_p \langle x_1, x_2\rangle^p$ where $c_p \geq 0$, then the eigenvalue of every spherical harmonic of frequency k is given by*

$$\overline{\lambda_k} = \frac{\pi^{d/2}}{2^{k-1}} \sum_{\substack{p\geq k \\ p-k \text{ is even}}} c_p \frac{\Gamma(p+1)\Gamma(\frac{p-k+1}{2})}{\Gamma(p-k+1)\Gamma(\frac{p-k+1}{2}+k+d/2)},$$

where Γ is the gamma function.

Proof. Let $\theta(t) = \sum_{p=0}^{\infty} c_p t^p$, then $K(x_1, x_2) = \theta(\langle x_1, x_2\rangle)$ According to Funk Hecke theorem (Basri et al., 2019, Section 4.2), we have

$$\overline{\lambda_k} = \mathrm{Vol}(\mathbb{S}^{d-1})\int_{-1}^{1} \theta(t) P_{k,d}(t)(1-t^2)^{\frac{d-2}{2}}\mathrm{d}t, \tag{47}$$

where $\mathrm{Vol}(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the volume of the hypersphere $\mathbb{S}^{d-1}$, and $P_{k,d}(t)$ is the Gegenbauer polynomial, given by

$$P_{k,d}(t) = \frac{(-1)^k}{2^k}\frac{\Gamma(d/2)}{\Gamma(k+d/2)}\frac{1}{(1-t^2)^{(d-2)/2}}\frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2},$$

and Γ is the gamma function.

From (47) we have

$$\begin{aligned}\overline{\lambda_k} &= \mathrm{Vol}(\mathbb{S}^{d-1}) \int_{-1}^{1} \theta(t) P_{k,d}(t)(1-t^2)^{\frac{d-2}{2}} \mathrm{d}t \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{-1}^{1} \theta(t) \frac{(-1)^k}{2^k} \frac{\Gamma(d/2)}{\Gamma(k+d/2)} \frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{(-1)^k}{2^k} \frac{\Gamma(d/2)}{\Gamma(k+d/2)} \sum_{p=0}^{\infty} c_p \int_{-1}^{1} t^p \frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t. \end{aligned} \tag{48}$$

Using integration by parts, we have

$$\begin{aligned}&\int_{-1}^{1} t^p \frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t \\ &= t^p \frac{d^{k-1}}{dt^{k-1}}(1-t^2)^{k+(d-2)/2}\Big|_{-1}^{1} - p\int_{-1}^{1} t^{p-1} \frac{d^{k-1}}{dt^{k-1}}(1-t^2)^{k+(d-2)/2} \mathrm{d}t \\ &= -p\int_{-1}^{1} t^{p-1} \frac{d^{k-1}}{dt^{k-1}}(1-t^2)^{k+(d-2)/2} \mathrm{d}t, \end{aligned} \tag{49}$$

where the last line in (49) holds because $\frac{d^{k-1}}{dt^{k-1}}(1-t^2)^{k+(d-2)/2} = 0$ when $t = 1$ or $t = -1$.

When $p < k$, repeat the above procedure (49) p times, we get

$$\begin{aligned}\int_{-1}^{1} t^p \frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t &= (-1)^p p! \int_{-1}^{1} \frac{d^{k-p}}{dt^{k-p}}(1-t^2)^{k+(d-2)/2} \mathrm{d}t \\ &= (-1)^p p! \ \frac{d^{k-p-1}}{dt^{k-p-1}}(1-t^2)^{k+(d-2)/2}\Big|_{-1}^{1} \\ &= 0. \end{aligned} \tag{50}$$

When $p \geq k$, repeat the above procedure (49) k times, we get

$$\int_{-1}^{1} t^p \frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t = (-1)^k p(p-1)\cdots(p-k+1) \int_{-1}^{1} t^{p-k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t. \tag{51}$$

When $p-k$ is odd, $t^{p-k}(1-t^2)^{k+(d-2)/2}$ is an odd function, then

$$\int_{-1}^{1} t^{p-k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t = 0. \tag{52}$$

When $p-k$ is even,

$$\begin{aligned}\int_{-1}^{1} t^{p-k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t &= 2\int_{0}^{1} t^{p-k}(1-t^2)^{k+(d-2)/2} \mathrm{d}t \\ &= \int_{0}^{1} (t^2)^{(p-k-1)/2}(1-t^2)^{k+(d-2)/2} \mathrm{d}t^2 \\ &= B\left(\frac{p-k+1}{2}, k+d/2\right) \\ &= \frac{\Gamma(\frac{p-k+1}{2})\Gamma(k+d/2)}{\Gamma(\frac{p-k+1}{2}+k+d/2)}, \end{aligned} \tag{53}$$

where B is the beta function.

Plugging (53) , (50) and (52) into (51), we get

$$\begin{aligned}&\int_{-1}^{1} t^p \frac{d^k}{dt^k}(1-t^2)^{k+(d-2)/2}\mathrm{d}t \\ &= \begin{cases}(-1)^k p(p-1)\dots(p-k+1)\frac{\Gamma(\frac{p-k+1}{2})\Gamma(k+d/2)}{\Gamma(\frac{p-k+1}{2}+k+d/2)}, & p-k \text{ is even and } p \geq k, \\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{54}$$

Plugging (54) into (48), we get

$$\begin{aligned}\overline{\lambda_k} &= \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{(-1)^k}{2^k}\frac{\Gamma(d/2)}{\Gamma(k+d/2)}\sum_{\substack{p\geq k\\ p-k \text{ is even}}} c_p(-1)^k p(p-1)\dots(p-k+1)\frac{\Gamma(\frac{p-k+1}{2})\Gamma(k+d/2)}{\Gamma(\frac{p-k+1}{2}+k+d/2)} \\ &= \frac{\pi^{d/2}}{2^{k-1}}\sum_{\substack{p\geq k\\ p-k \text{ is even}}} c_p\frac{p(p-1)\dots(p-k+1)\Gamma(\frac{p-k+1}{2})}{\Gamma(\frac{p-k+1}{2}+k+d/2)} \\ &= \frac{\pi^{d/2}}{2^{k-1}}\sum_{\substack{p\geq k\\ p-k \text{ is even}}} c_p\frac{\Gamma(p+1)\Gamma(\frac{p-k+1}{2})}{\Gamma(p-k+1)\Gamma(\frac{p-k+1}{2}+k+d/2)}.\end{aligned}$$

□

Corollary C.19. *Under the same setting as in Theorem 4.6,*

1. *if* $c_p = \Theta(p^{-a})$ *where* $a \geq 1$*, then* $\overline{\lambda_k} = \Theta(k^{-d-2a+2})$*,*
2. *if* $c_p = \delta_{(p\ even)}\Theta(p^{-a})$*, then* $\overline{\lambda_k} = \delta_{(k\ even)}\Theta(k^{-d-2a+2})$*,*
3. *if* $c_p = \mathcal{O}\left(\exp\left(-a\sqrt{p}\right)\right)$*, then* $\overline{\lambda_k} = \mathcal{O}\left(k^{-d+1/2}\exp\left(-a\sqrt{k}\right)\right)$*,*
4. *if* $c_p = \Theta(p^{1/2}a^{-p})$*, then* $\overline{\lambda_k} = \mathcal{O}\left(k^{-d+1}a^{-k}\right)$ *and* $\overline{\lambda_k} = \Omega\left(k^{-d/2+1}2^{-k}a^{-k}\right)$*.*

Proof of Corollary C.3, part 1. We first prove $\overline{\lambda_k} = O(k^{-d-2a+2})$. Suppose that $c_p \leq Cp^{-a}$ for some constant C, then according to Theorem 4.6 we have

$$\overline{\lambda_k} \leq \frac{\pi^{d/2}}{2^{k-1}}\sum_{\substack{p\geq k\\ p-k \text{ is even}}} Cp^{-a}\frac{\Gamma(p+1)\Gamma(\frac{p-k+1}{2})}{\Gamma(p-k+1)\Gamma(\frac{p-k+1}{2}+k+d/2)}.$$

According to Stirling's formula, we have

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1+O\left(\frac{1}{z}\right)\right). \tag{55}$$

Then for any $z \geq \frac{1}{2}$, we can find constants C_1 and C_2 such that

$$C_1\sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z \leq \Gamma(z) \leq C_2\sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z. \tag{56}$$

Then

$$\begin{aligned}
\overline{\lambda_k} &\leq \frac{\pi^{d/2}}{2^{k-1}} \frac{C_2^2}{C_1^2} \sum_{\substack{p\geq k \\ p-k \text{ is even}}} Cp^{-a} \frac{\sqrt{\frac{2\pi}{p+1}} \left(\frac{p+1}{e}\right)^{p+1} \sqrt{\frac{2\pi}{\frac{p-k+1}{2}}} \left(\frac{\frac{p-k+1}{2}}{e}\right)^{\frac{p-k+1}{2}}}{\sqrt{\frac{2\pi}{p-k+1}} \left(\frac{p-k+1}{e}\right)^{p-k+1} \sqrt{\frac{2\pi}{\frac{p-k+1}{2}+k+d/2}} \left(\frac{\frac{p-k+1}{2}+k+d/2}{e}\right)^{\frac{p-k+1}{2}+k+d/2}} \\
&= \frac{\pi^{d/2}}{2^{k-1}} \frac{C_2^2 C}{C_1^2} \sum_{\substack{p\geq k \\ p-k \text{ is even}}} p^{-a} \frac{e^{\frac{d}{2}} \sqrt{\frac{2}{p+1}} (p+1)^{p+1} \left(\frac{p-k+1}{2}\right)^{\frac{p-k+1}{2}}}{(p-k+1)^{p-k+1} \sqrt{\frac{1}{\frac{p-k+1}{2}+k+d/2}} \left(\frac{p-k+1}{2}+k+d/2\right)^{\frac{p-k+1}{2}+k+d/2}} \\
&= \frac{\pi^{d/2}}{2^{k-1}} \frac{C_2^2 C}{C_1^2} \sum_{\substack{p\geq k \\ p-k \text{ is even}}} p^{-a} \frac{e^{\frac{d}{2}} 2^{\frac{-p+k}{2}} (p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} \left(\frac{p-k+1}{2}+k+d/2\right)^{\frac{p-k}{2}+k+d/2}} \\
&= 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_2^2 C}{C_1^2} \sum_{\substack{p\geq k \\ p-k \text{ is even}}} \frac{p^{-a} (p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{p+k+d}{2}}}. \qquad (57)
\end{aligned}$$

We define

$$f_a(p) = \frac{p^{-a} (p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{p+k+d}{2}}}. \qquad (58)$$

By applying the chain rule to $e^{\log f_a(p)}$, we have that the derivative of f_a is

$$\begin{aligned}
f_a'(p) = &\frac{(p+1)^{p+\frac{1}{2}} p^{-a}}{2(p-k+1)^{\frac{p-k+1}{2}} (p+k+d+1)^{\frac{p+k+d}{2}}} \\
&\cdot \left(-\frac{2a}{p} - \frac{k+d}{(p+1)(p+k+d+1)} + \log(1 + \frac{k^2 - d(p-k+1)}{(p-k+1)(p+k+d+1)}) \right). \qquad (59)
\end{aligned}$$

Let $g_a(p) = -\frac{2a}{p} - \frac{k+d}{(p+1)(p+k+d+1)} + \log(1 + \frac{k^2-d(p-k+1)}{(p-k+1)(p+k+d+1)})$. Then $g_a(p)$ and $f_a'(p)$ have the same sign. Next we will show that $g_a(p) \geq 0$ for $k \leq p \leq \frac{k^2}{d+24a}$ when k is large enough.

First when $p \geq k$ and $\frac{k^2-d(p-k+1)}{(p-k+1)(p+k+d+1)} \geq 1$, we have

$$g_a(p) \geq -\frac{2a}{k} - \frac{k+d}{(k+1)(k+k+d+1)} + \log(2) \geq 0, \qquad (60)$$

when k is sufficiently large.

Second when $p \geq k$ and $0 \leq \frac{k^2-d(p-k+1)}{(p-k+1)(p+k+d+1)} \leq 1$, since $\log(1+x) \geq \frac{x}{2}$ for $0 \leq x \leq 1$, we have

$$\begin{aligned}
g_a(p) &\geq -\frac{2a}{p} - \frac{k+d}{(p+1)(p+k+d+1)} + \frac{k^2 - d(p-k+1)}{2(p-k+1)(p+k+d+1)} \\
&\geq -\frac{2a}{p} - \frac{k+d}{(p+1)(p+k+d+1)} + \frac{k^2 - dp}{2p(p+k+d+1)}.
\end{aligned}$$

When $p \leq \frac{k^2}{d+24a}$, we have $k^2 - dp \geq 24ap$. Then

$$\frac{k^2 - dp}{4p(p+k+d+1)} \geq \frac{24ap}{4p(p+k+d+1)} \geq \frac{6ap}{(p+1)(p+k+d+1)} \geq \frac{k+d}{(p+1)(p+k+d+1)}$$

when k is sufficiently large. Also we have

$$\frac{k^2 - dp}{4r(p+k+d+1)} \geq \frac{24ap}{4r(p+k+d+1)} \geq \frac{6a}{p+k+d+1} \geq \frac{2a}{p}$$

when k is sufficiently large.

Combining all the arguments above, we conclude that $g_a(p) \geq 0$ and $f_a'(p) \geq 0$ when $k \leq p \leq \frac{k^2}{d+24a}$. Then when $k \leq p \leq \frac{k^2}{d+24a}$, we have

$$f_a(p) \leq f_a\left(\frac{k^2}{d+24a}\right). \tag{61}$$

When $p \geq \frac{k^2}{d+24a}$, we have

$$\begin{aligned}
f_a(p) &= \frac{p^{-a}\,(p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}}\,(p+k+1+d)^{\frac{p+k+d}{2}}} \\
&= \frac{p^{-a}\,(p+1)^{p+\frac{1}{2}}}{((p+1)^2-k^2+d(p-k+1))^{\frac{p-k+1}{2}}\,(p+k+1+d)^{\frac{2k+d-1}{2}}} \\
&= \frac{p^{-a}\,(p+1)^{-\frac{d}{2}}}{\left(1-\frac{k^2-d(p-k+1)}{(p+1)^2}\right)^{\frac{p-k+1}{2}}\left(1+\frac{k+d}{p+1}\right)^{\frac{2k+d-1}{2}}} \\
&\leq \frac{p^{-a-\frac{d}{2}}}{\left(1-\frac{k^2-d(p-k+1)}{(p+1)^2}\right)^{\frac{p-k+1}{2}}}.
\end{aligned}$$

If $k^2-d(p-k+1)<0$, $\left(1-\frac{k^2-d(p-k+1)}{(p+1)^2}\right)^{\frac{p-k+1}{2}} \geq 1$. If $k^2-d(p-k+1)\geq 0$, i.e., $p \leq \frac{k^2+dk-d}{d}$, for sufficiently large k, we have

$$\begin{aligned}
\left(1-\frac{k^2-d(p-k+1)}{(p+1)^2}\right)^{\frac{p-k+1}{2}} &\geq \left(1-\frac{k^2-d(\frac{k^2}{d+24a}-k+1)}{(\frac{k^2}{d+24a}+1)^2}\right)^{\frac{\frac{k^2+dk-d}{d}-k+1}{2}} \\
&\geq \left(1-\frac{48a(d+24a)}{k^2}\right)^{\frac{k^2}{2d}} \\
&\geq e^{-\frac{k^2}{2d}\frac{48a(d+24a)}{k^2}} = e^{-\frac{48a(d+24a)}{2d}},
\end{aligned}$$

which is a constant independent of k. Then for $p \geq \frac{k^2}{d+24a}$, we have

$$f_a(p) \leq e^{\frac{48a(d+24a)}{2d}} p^{-a-\frac{d}{2}}. \tag{62}$$

Finally we have

$$\begin{aligned}
\overline{\lambda_k} &= 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_2^2C}{C_1^2}\sum_{\substack{p\geq k\\ p-k \text{ is even}}} f_a(p) \\
&\leq O\left(\sum_{\substack{k\leq p\leq \frac{k^2}{d+24a}\\ p-k \text{ is even}}} f_a(p) + \sum_{\substack{p\geq \frac{k^2}{d+24a}\\ p-k \text{ is even}}} f_a(p)\right) \\
&\leq O\left(\left(\frac{k^2}{d+24a}-k+1\right) f_a\left(\frac{k^2}{d+24a}\right) + \sum_{\substack{p\geq \frac{k^2}{d+24a}\\ p-k \text{ is even}}} e^{\frac{48a(d+24a)}{2d}} p^{-a-\frac{d}{2}}\right) \\
&\leq O\left(\left(\frac{k^2}{d+24a}-k+1\right) e^{\frac{48a(d+24a)}{2d}}\left(\frac{k^2}{d+24a}\right)^{-a-\frac{d}{2}} + e^{\frac{48a(d+24a)}{2d}}\frac{1}{a+\frac{d}{2}-1}\left(\frac{k^2}{d+24a}-1\right)^{1-a-\frac{d}{2}}\right) \\
&= O(k^{-d-2a+2}).
\end{aligned}$$

Next we prove $\overline{\lambda_k} = \Omega(k^{-d-2a+2})$. Since c_p are nonnegative and $c_p = \Theta(p^{-a})$, we have that $c_p \geq C' p^{-a}$ for some constant C'. Then we have

$$\overline{\lambda_k} \geq \frac{\pi^{d/2}}{2^{k-1}} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} C' p^{-a} \frac{\Gamma(p+1)\Gamma(\frac{p-k+1}{2})}{\Gamma(p-k+1)\Gamma(\frac{p-k+1}{2}+k+d/2)}. \tag{63}$$

According to Stirling's formula (55) and (56), using the similar argument as (57) we have

$$\overline{\lambda_k} \geq \frac{\pi^{d/2}}{2^{k-1}} \frac{C_1^2}{C_2^2} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} C' p^{-a} \frac{\sqrt{\frac{2\pi}{p+1}} \left(\frac{p+1}{e}\right)^{p+1} \sqrt{\frac{2\pi}{\frac{p-k+1}{2}}} \left(\frac{\frac{p-k+1}{2}}{e}\right)^{\frac{p-k+1}{2}}}{\sqrt{\frac{2\pi}{p-k+1}} \left(\frac{p-k+1}{e}\right)^{p-k+1} \sqrt{\frac{2\pi}{\frac{p-k+1}{2}+k+d/2}} \left(\frac{\frac{p-k+1}{2}+k+d/2}{e}\right)^{\frac{p-k+1}{2}+k+d/2}} \tag{64}$$

$$= 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_1^2 C'}{C_2^2} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} \frac{p^{-a} (p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{p+k+d}{2}}} \tag{65}$$

$$\geq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_1^2 C'}{C_2^2} \sum_{\substack{p \geq k^2 \\ p-k \text{ is even}}} f_a(p), \tag{66}$$

where $f_a(p)$ is defined in (58). When $p \geq k^2$, we have

$$\begin{aligned} f_a(p) &= \frac{p^{-a} (p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{p+k+d}{2}}} \\ &= \frac{p^{-a} (p+1)^{p+\frac{1}{2}}}{((p+1)^2 - k^2 + d(p-k+1))^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{2k+d-1}{2}}} \\ &\geq \frac{(p+1)^{-a-\frac{d}{2}}}{\left(1 - \frac{k^2 - d(p-k+1)}{(p+1)^2}\right)^{\frac{p-k+1}{2}} \left(1 + \frac{k+d}{p+1}\right)^{\frac{2k+d-1}{2}}} \end{aligned}$$

For sufficiently large k, $k^2 - d(p-k+1) < 0$. Then we have

$$\begin{aligned} \left(1 - \frac{k^2 - d(p-k+1)}{(p+1)^2}\right)^{\frac{p-k+1}{2}} &= \left(1 - \frac{k^2 - d(p-k+1)}{(p+1)^2}\right)^{\frac{-(p+1)^2}{k^2 - d(p-k+1)} \cdot \frac{-k^2 + d(p-k+1)}{(p+1)^2} \cdot \frac{p-k+1}{2}} \\ &\leq e^{\frac{-k^2 + d(p-k+1)}{(p+1)^2} \cdot \frac{p-k+1}{2}} \\ &\leq e^{\frac{dp^2}{2p^2}} = e^{\frac{d}{2}} \end{aligned}$$

which is a constant independent of k. Also, for sufficiently large k, we have

$$\begin{aligned} \left(1 + \frac{k+d}{p+1}\right)^{\frac{2k+d-1}{2}} &= \left(1 + \frac{k+d}{p+1}\right)^{\frac{p+1}{k+d} \frac{k+d}{p+1} \frac{2k+d-1}{2}} \\ &\leq e^{\frac{k+d}{p+1} \frac{2k+d-1}{2}} \\ &\leq e^{\frac{3k^2}{2r}} = e^{\frac{3}{2}} \end{aligned}$$

Then for $p \geq k^2$, we have $f_a(p) \geq e^{-\frac{d}{2} - \frac{3}{2}} (p+1)^{-a-\frac{d}{2}}$.

Finally we have

$$\overline{\lambda_k} \geq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_1^2 C'}{C_2^2} \sum_{\substack{p \geq k^2 \\ p-k \text{ is even}}} f_a(p) \tag{67}$$

$$\geq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_1^2 C'}{C_2^2} \sum_{\substack{p \geq k^2 \\ p-k \text{ is even}}} e^{-\frac{d}{2}-\frac{3}{2}} (p+1)^{-a-\frac{d}{2}} \tag{68}$$

$$\geq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_1^2 C'}{C_2^2} e^{-\frac{d}{2}-\frac{3}{2}} \frac{1}{2(a+\frac{d}{2}-1)} (k^2+2)^{1-a-\frac{d}{2}} \tag{69}$$

$$= \Omega(k^{-d-2a+2}). \tag{70}$$

Overall, we have $\overline{\lambda_k} = \Theta(k^{-d-2a+2})$. □

Proof of Corollary C.3, part 2. It is easy to verify that $\overline{\lambda_k} = 0$ when k is even because $c_p = 0$ when $p \geq k$ and $p - k$ is even. When k is odd, the proof of Theorem 4.6 still applies. □

Proof of Corollary C.3, part 3. Since $c_p = \mathcal{O}\left(\exp\left(-a\sqrt{p}\right)\right)$, we have that $c_p \leq Ce^{-a\sqrt{p}}$ for some constant C. Similar to (57), we have

$$\overline{\lambda_k} \leq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_2^2 C}{C_1^2} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} \frac{e^{-a\sqrt{p}} (p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{p+k+d}{2}}}. \tag{71}$$

Use the definition in (58) and let $a = 0$, we have

$$f_0(p) = \frac{(p+1)^{p+\frac{1}{2}}}{(p-k+1)^{\frac{p-k+1}{2}} (p+k+1+d)^{\frac{p+k+d}{2}}}. \tag{72}$$

Then according to (61) and (62), for sufficiently large k, we have $f_0(p) \leq f_0\left(\frac{k^2}{d}\right)$ when $k \leq p \leq \frac{k^2}{d}$ and $f_0(p) \leq C_3 p^{-\frac{d}{2}}$ for some constant C_3 when $p \geq \frac{k^2}{d}$. Then when $k \leq p \leq \frac{k^2}{d}$, we have $f_0(p) \leq f_0\left(\frac{k^2}{d}\right) \leq C_3 \left(\frac{k^2}{d}\right)^{-\frac{d}{2}}$. When $p \geq \frac{k^2}{d}$, we have $f_0(p) \leq C_3 p^{-\frac{d}{2}} \leq C_3 \left(\frac{k^2}{d}\right)^{-\frac{d}{2}}$. Overall, for all $p \geq k$, we have

$$f_0(p) \leq C_3 \left(\frac{k^2}{d}\right)^{-\frac{d}{2}}. \tag{73}$$

Then we have

$$\overline{\lambda_k} \leq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_2^2 C}{C_1^2} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} e^{-a\sqrt{p}} f_0(p) \tag{74}$$

$$\leq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_2^2 C_3 C}{C_1^2} \left(\frac{k^2}{d}\right)^{-\frac{d}{2}} \sum_{\substack{p \geq k \\ p-k \text{ is even}}} e^{-a\sqrt{p}} \tag{75}$$

$$\leq 2\pi^{d/2} \frac{2^{\frac{d}{2}} e^{\frac{d}{2}} C_2^2 C_3 C}{C_1^2} \left(\frac{k^2}{d}\right)^{-\frac{d}{2}} \frac{2e^{-a\sqrt{k-1}}(a\sqrt{k-1}+1)}{a^2} \tag{76}$$

$$= \mathcal{O}\left(k^{-d+1/2} \exp\left(-a\sqrt{k}\right)\right) \tag{77}$$

□

Proof of Corollary C.3, part 4. Since $c_p = \Theta(p^{1/2}a^{-p})$, we have that $c_p \leq Cp^{1/2}a^{-p}$ for some constant C. Similar to (57), we have

$$\overline{\lambda_k} \leq 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_2^2C}{C_1^2}\sum_{\substack{p\geq k\\ p-k \text{ is even}}}\frac{p^{1/2}a^{-p}\left(p+1\right)^{p+\frac{1}{2}}}{\left(p-k+1\right)^{\frac{p-k+1}{2}}\left(p+k+1+d\right)^{\frac{p+k+d}{2}}}. \tag{78}$$

Use the definition in (58) and let $a = 0$, we have

$$f_0(p) = \frac{\left(p+1\right)^{p+\frac{1}{2}}}{\left(p-k+1\right)^{\frac{p-k+1}{2}}\left(p+k+1+d\right)^{\frac{p+k+d}{2}}}. \tag{79}$$

Then according to (61) and (62), for sufficiently large k, we have $f_0(p) \leq f_0\left(\frac{k^2}{d}\right)$ when $k \leq p \leq \frac{k^2}{d}$ and $f_0(p) \leq C_3p^{-\frac{d}{2}}$ for some constant C_3 when $p \geq \frac{k^2}{d}$. Then when $k \leq p \leq \frac{k^2}{d}$, we have $p^{1/2}f_0(p) \leq p^{1/2}f_0\left(\frac{k^2}{d}\right) \leq C_3\left(\frac{k^2}{d}\right)^{1/2}\left(\frac{k^2}{d}\right)^{-\frac{d}{2}}$. When $p \geq \frac{k^2}{d}$, we have $p^{1/2}f_0(p) \leq C_3p^{1/2}p^{-\frac{d}{2}} \leq C_3\left(\frac{k^2}{d}\right)^{-\frac{d}{2}+\frac{1}{2}}$. Overall, for all $p \geq k$, we have

$$p^{1/2}f_0(p) \leq C_3\left(\frac{k^2}{d}\right)^{-\frac{d}{2}+\frac{1}{2}}. \tag{80}$$

Then we have

$$\begin{aligned}
\overline{\lambda_k} &\leq 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_2^2C}{C_1^2}\sum_{\substack{p\geq k\\ p-k \text{ is even}}}p^{1/2}a^{-p}f_0(p) && (81)\\
&\leq 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_2^2C_3C}{C_1^2}\left(\frac{k^2}{d}\right)^{-\frac{d}{2}+\frac{1}{2}}\sum_{\substack{p\geq k\\ p-k \text{ is even}}}a^{-p} && (82)\\
&\leq 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_2^2C_3C}{C_1^2}\left(\frac{k^2}{d}\right)^{-\frac{d}{2}+\frac{1}{2}}\frac{1}{\log a}a^{-(k-1)} && (83)\\
&= \mathcal{O}\left(k^{-d+1}a^{-k}\right). && (84)
\end{aligned}$$

On the other hand, since $c_p = \Theta(p^{1/2}a^{-p})$, we have that $c_p \geq C'p^{1/2}a^{-p}$ for some constant C'. Similar to (65), we have

$$\begin{aligned}
\overline{\lambda_k} &\geq 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_1^2C'}{C_2^2}\sum_{\substack{p\geq k\\ p-k \text{ is even}}}\frac{p^{1/2}a^{-p}\left(p+1\right)^{p+\frac{1}{2}}}{\left(p-k+1\right)^{\frac{p-k+1}{2}}\left(p+k+1+d\right)^{\frac{p+k+d}{2}}} && (85)\\
&\geq 2\pi^{d/2}\frac{2^{\frac{d}{2}}e^{\frac{d}{2}}C_1^2C'}{C_2^2}\frac{k^{1/2}a^{-k}\left(k+1\right)^{k+\frac{1}{2}}}{\left(k-k+1\right)^{\frac{k-k+1}{2}}\left(k+k+1+d\right)^{\frac{k+k+d}{2}}} && (86)\\
&= \Omega\left(\frac{k^{-d/2+1}a^{-k}\left(k+1\right)^k}{\left(k+k+1+d\right)^k}\right). && (87)
\end{aligned}$$

Since $\left(k+1\right)^k = k^k(1+1/k)^k = \Theta(k^k)$. Similarly, $\left(k+k+1+d\right)^k = \Theta((2k)^k)$. Then we have

$$\begin{aligned}
\overline{\lambda_k} &= \Omega\left(\frac{k^{-d/2+1}a^{-k}\left(k+1\right)^k}{\left(k+k+1+d\right)^k}\right) && (88)\\
&= \Omega\left(\frac{k^{-d/2+1}a^{-k}k^k}{(2k)^k}\right) && (89)\\
&= \Omega\left(k^{-d/2+1}2^{-k}a^{-k}\right). && (90)
\end{aligned}$$

□

C.4 ANALYSIS OF THE LOWER SPECTRUM: NON-UNIFORM DATA

The purpose of this section is to prove a formal version of Theorem 4.8. In order to prove this result we first need the following lemma.

Lemma C.20. *Let the coefficients $(c_j)_{j=0}^{\infty}$ with $c_j \in \mathbb{R}_{\geq 0}$ for all $j \in \mathbb{Z}_{\geq 0}$ be such that the series $\sum_{j=0}^{\infty} c_j \rho^j$ converges for all $\rho \in [-1, 1]$. Given a data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ with $\|\mathbf{x}_i\| = 1$ for all $i \in [n]$, define $r := \operatorname{rank}(\mathbf{X}) \geq 2$ and the gram matrix $\mathbf{G} := \mathbf{X}\mathbf{X}^T$. Consider the kernel matrix*

$$n\mathbf{K} = \sum_{j=0}^{\infty} a_j \mathbf{G}^{\odot j}.$$

For arbitrary $m \in \mathbb{Z}_{\geq 1}$, let the eigenvalue index k satisfy $n \geq k > \operatorname{rank}(\mathbf{H}_m)$, where $\mathbf{H}_m := \sum_{j=0}^{m-1} a_j \mathbf{G}^{\odot j}$. Then

$$\lambda_k(\mathbf{K}) \leq \frac{\|\mathbf{G}^{\odot m}\|}{n} \sum_{j=m}^{\infty} c_j. \tag{91}$$

Proof. We start our analysis by considering $\lambda_k(n\mathbf{K})$ for some arbitrary $k \in \mathbb{N}_{\leq n}$. Let

$$\mathbf{H}_m := \sum_{j=0}^{m-1} c_j \mathbf{G}^{\odot j},$$

$$\mathbf{T}_m := \sum_{j=m}^{\infty} c_j \mathbf{G}^{\odot j}$$

be the m-head and m-tail of the Hermite expansion of $n\mathbf{K}$: clearly $n\mathbf{K} = \mathbf{H}_m + \mathbf{T}_m$ for any $m \in \mathbb{N}$. Recall that a constant matrix is symmetric and positive semi-definite. Furthermore by the Schur product theorem, the Hadamard product of two positive semi-definite matrices is positive semi-definite. As a result, $\mathbf{G}^{\odot j}$ is symmetric and positive semi-definite for all $j \in \mathbb{Z}_{\geq 0}$ and therefore $\mathbf{H}_m$ and $\mathbf{T}_m$ are also symmetric positive semi-definite matrices. From Weyl's inequality (Weyl, 1912, Satz 1) it follows that

$$n\lambda_k(\mathbf{K}_n) \leq \lambda_k(\mathbf{H}_m) + \lambda_1(\mathbf{T}_m). \tag{92}$$

In order to upper bound $\lambda_1(\mathbf{T}_m)$, observe, as $\mathbf{T}_m$ is square, symmetric and positive semi-definite, that $\lambda_1(\mathbf{T}_m) = \|\mathbf{T}_m\|$. Using the non-negativity of the coefficients $(c_j)_{j=0}^{\infty}$ and the triangle inequality we have

$$\lambda_1(\mathbf{T}_m) = \left\| \sum_{j=m}^{\infty} a_j \mathbf{G}^{\odot j} \right\| \leq \sum_{j=m}^{\infty} a_j \left\| \mathbf{G}^{\odot j} \right\|$$

By the assumptions of the lemma $[\mathbf{G}]_{ii} = 1$ and therefore $[\mathbf{G}]_{ii}^j = 1$ for all $j \in \mathbb{Z}_{\geq 0}$. Furthermore, for any pair of positive semi-definite matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ and $k \in [n]$

$$\lambda_1(\mathbf{A} \odot \mathbf{B}) \leq \max_{i \in [n]} [\mathbf{A}]_{ii} \lambda_1(\mathbf{B}), \tag{93}$$

Schur (1911). Therefore as $\max_{i \in [n]} [\mathbf{G}]_{ii} = 1$,

$$\left\| \mathbf{G}^{\odot j} \right\| = \lambda_1(\mathbf{G}^{\odot j}) = \lambda_1(\mathbf{G} \odot \mathbf{G}^{\odot (j-1)}) \leq \lambda_1(\mathbf{G}^{\odot (j-1)}) = \left\| \mathbf{G}^{\odot (j-1)} \right\|$$

for all $j \in \mathbb{N}$. As a result

$$\lambda_1(\mathbf{T}_m) \leq \left\| \mathbf{G}^{\odot m} \right\| \sum_{j=m}^{\infty} a_j.$$

Finally, we now turn our attention to the analysis of $\lambda_k(\mathbf{H}_m)$. Upper bounding a small eigenvalue is challenging, however the problem simplifies when and k exceeds the rank of $\mathbf{H}_m$, as is assumed here, as this implies $\lambda_k(\mathbf{H}_m) = 0$. Therefore for $k > \operatorname{rank}(\mathbf{H}_m)$

$$\lambda_k(\mathbf{K}) \leq \frac{\|\mathbf{G}^m\|}{n} \sum_{j=m}^{\infty} a_j$$

as claimed. □

In order to use Lemma C.20 we require an upper bound on the rank of $\mathbf{H}_m$. This is the subject of Lemma C.21 given below.

Lemma C.21. *Let $\mathbf{G} \in \mathbb{R}^{n\times n}$ be a symmetric, positive semi-definite matrix of rank $2 \leq r \leq d$. Define $\mathbf{H}_m \in \mathbb{R}^{n\times n}$ as*

$$\mathbf{H}_m = \sum_{j=0}^{m-1} c_j \mathbf{G}^{\odot j} \tag{94}$$

where $(c_j)_{j=0}^{m-1}$ is a sequence of real coefficients. Then

$$\begin{aligned} \textit{rank}\,(\mathbf{H}_m) \leq & 1 + \min\{r-1, m-1\}(2e)^{r-1} \\ & + \max\{0, m-r\}\left(\frac{2e}{r-1}\right)^{r-1}(m-1)^{r-1}. \end{aligned} \tag{95}$$

Proof. As $\mathbf{G}$ is a symmetric and positive semi-definite matrix its eigenvalues are real and non-negative and its eigenvectors are orthogonal. Let $\{\mathbf{v}_i\}_{i=1}^r$ be a set of orthogonal eigenvectors for $\mathbf{G}$ and γ_i the eigenvalue associated with $\mathbf{v}_i \in \mathbb{R}^n$. Then $\mathbf{G}$ may be written as a sum of rank one matrices as follows,

$$\mathbf{G} = \sum_{i=1}^{r} \gamma_i \mathbf{v}_i \mathbf{v}_i^T.$$

As the Hadamard product is commutative, associative and distributive over addition, for any $j \in \mathbb{Z}_{\geq 0}$ then $\mathbf{G}^{\odot j}$ can be expressed as a sum of rank 1 matrices,

$$\begin{aligned} \mathbf{G}^{\odot j} &= \left(\sum_{i=1}^{r} \gamma_i \mathbf{v}_i \mathbf{v}_i^T\right)^{\odot j} \\ &= \left(\sum_{i_1=1}^{r} \gamma_{i_1} \mathbf{v}_{i_1} \mathbf{v}_{i_1}^T\right) \odot \left(\sum_{i_2=1}^{r} \gamma_{i_2} \mathbf{v}_{i_2} \mathbf{v}_{i_2}^T\right) \odot \cdots \odot \left(\sum_{i_j=1}^{r} \gamma_{i_j} \mathbf{v}_{i_j} \mathbf{v}_{i_j}^T\right) \\ &= \sum_{i_1, i_2 \ldots i_j = 1}^{r} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_r} \left(\mathbf{v}_{i_1}\mathbf{v}_{i_1}^T\right) \odot \left(\mathbf{v}_{i_2}\mathbf{v}_{i_2}^T\right) \odot \cdots \odot \left(\mathbf{v}_{i_j}\mathbf{v}_{i_j}^T\right) \\ &= \sum_{i_1, i_2, \ldots, i_j = 1}^{r} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_j} \left(\mathbf{v}_{i_1} \odot \mathbf{v}_{i_2} \odot \cdots \odot \mathbf{v}_{i_j}\right)\left(\mathbf{v}_{i_1} \odot \mathbf{v}_{i_2} \odot \cdots \odot \mathbf{v}_{i_j}\right)^T. \end{aligned}$$

Note the fourth equality in the above follows from $\mathbf{v}_i\mathbf{v}_i^T = \mathbf{v}_i \otimes \mathbf{v}_i$ and an application of the mixed-product property of the Hadamard product. As matrix rank is sub-additive, the rank of $\mathbf{G}^{\odot j}$ is less than or equal to the number of distinct rank-one matrix summands. This quantity in turn is equal to the number of vectors of the form $\left(\mathbf{v}_{i_1} \odot \mathbf{v}_{i_2} \odot \cdots \odot \mathbf{v}_{i_j}\right)$, where $i_1, i_2, \ldots, i_j \in [r]$, which is equivalent to computing the number of j-combinations with repetition from r objects. Via a stars and bars argument this is equal to $\binom{r+j-1}{j} = \binom{r+j-1}{r(n)-1}$. It follows that

$$\begin{aligned} \text{rank}(\mathbf{G}^{\odot j}) &\leq \binom{r+j-1}{r-1} \\ &\leq \left(\frac{e(r+j-1)}{r-1}\right)^{r-1} \\ &\leq e^{r-1}\left(1 + \frac{j}{r-1}\right)^{r-1} \\ &\leq (2e)^{r-1}\left(\delta_{j\leq r-1} + \delta_{j>r-1}\left(\frac{j}{r-1}\right)^{r-1}\right). \end{aligned}$$

The rank of $\mathbf{H}_m$ can therefore be bounded via subadditivity of the rank as

$$\begin{aligned}\operatorname{rank}(\mathbf{H}_m) =& \operatorname{rank}\left(a_0\mathbf{1}_{n\times n} + \sum_{j=1}^{m-1} c_j \mathbf{G}^{\odot j}\right)\\ \leq& 1 + \sum_{j=1}^{m-1} \operatorname{rank}\left(\mathbf{G}^{\odot j}\right)\\ \leq& 1 + \sum_{j=1}^{m-1} (2e)^{r-1}\left(\delta_{j\leq r-1} + \delta_{j>r-1}\left(\frac{j}{r-1}\right)^{r-1}\right)\\ \leq& 1 + \min\{r-1, m-1\}(2e)^{r-1}\\ &+ \max\{0, m-r\}\left(\frac{2e}{r-1}\right)^{r-1}(m-1)^{r-1}.\end{aligned} \tag{96}$$

□

As we are interested in understanding the small eigenvalues, as n grows we need both k and therefore m to grow as well and therefore will be operating in the regime where $m > r$. To this end we provide the following corollary.

Corollary C.22. *Under the same conditions and setup as Lemma C.21 with $m > r \geq 7$ then*

$$\operatorname{rank}(\mathbf{H}_m) < m^r,$$

where $C_d := d(2e)^d$ is a constant.

Proof. If $r \geq 7 > 2e + 1$ and $m > r$ then $m - 1 > r - 1 > 2e$. As a result from Lemma C.21

$$\begin{aligned}\operatorname{rank}(\mathbf{H}_m) &\leq 1 + (r-1)(2e)^{r-1} + (m-r)\left(\frac{2e}{r-1}\right)^{r-1}(m-1)^{r-1}\\ &< (m-1)\left((2e)^{r-1} + (m-1)^{r-1}\right)\\ &< m^r\end{aligned}$$

as claimed. □

Corollary C.22 implies that for any $k \geq m^r$ then we can apply Lemma C.20 to upper bound the size of $\lambda_k(\mathbf{K})$. However, observe that the right hand side of (91) in Lemma C.20 depends on only m, not k, and is uniform across all $k \geq m^r$. For k much larger than m we would expect this not to be very informative: for example, if we kept m fixed as $k, n \to \infty$ then this bound would be meaningless in terms of showing the asymptotic behavior of the spectrum. Our goal then is to choose m as large as possible while still ensuring $m^r \leq n$. Perhaps the natural choice, and the one we adopt moving forward, is to study the nth eigenvalue by choosing the truncation point $m = \lfloor n^{1/r} \rfloor$. In order to state the next lemma we introduce the following piece of notation: with $\mathcal{L} := \{\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\}$ define $U : \mathcal{L} \times \mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 0}$ as

$$U(\ell, m) = \int_{m-1}^{\infty} \ell(x)dx.$$

Lemma C.23. *Given a sequence of data points $(\mathbf{x}_i)_{i\in\mathbb{Z}_{\geq 1}}$ with $\mathbf{x}_i \in \mathbb{S}^d$ for all $i \in \mathbb{Z}_{\geq 1}$, construct a sequence of row-wise data matrices $(\mathbf{X}_n)_{n\in\mathbb{Z}_{\geq 1}}$, $\mathbf{X}_n \in \mathbb{R}^{n\times d}$, with $\mathbf{x}_i$ corresponding to the ith row of $\mathbf{X}_n$. The corresponding sequence of gram matrices we denote $\mathbf{G}_n := \mathbf{X}_n\mathbf{X}_n^T$. Let $m(n) := \lfloor n^{1/r(n)} \rfloor$ where $r(n) := \operatorname{rank}(\mathbf{X}_n)$ and suppose for all sufficiently large n that $r(n) \geq 7$. Let the coefficients $(c_j)_{j=0}^{\infty}$ with $c_j \in \mathbb{R}_{\geq 0}$ for all $j \in \mathbb{Z}_{\geq 0}$ be such that 1) the series $\sum_{j=0}^{\infty} c_j\rho^j$ converges for all $\rho \in [-1, 1]$ and 2) $(c_j)_{j=0}^{\infty} = \mathcal{O}(\ell(j))$ where $\ell \in \mathcal{L}$ satisfies $U(\ell, m(n)) < \infty$ for all n and is monotonically decreasing. Consider the sequence of kernel matrices*

$$n\mathbf{K}_n = \sum_{j=0}^{\infty} c_j \mathbf{G}_n^{\odot j}.$$

With $\nu : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ *suppose* $\left\| \mathbf{G}_n^{\odot m(n)} \right\| = \mathcal{O}(n^{-\nu(n)+1})$*, then*

$$\lambda_n(\mathbf{K}_n) = \mathcal{O}(n^{-\nu(n)} U(\ell, m(n))). \tag{97}$$

Proof. From the assumptions of the Lemma, applying Lemma C.20 and Corollary C.22 with $k = n$ then for all sufficiently large n

$$\lambda_n(\mathbf{K}_n) \leq \frac{\left\| \mathbf{G}_n^{\odot m(n)} \right\|}{n} \sum_{j=m(n)}^{\infty} c_j = \mathcal{O}(n^{-\nu(n)}) \sum_{j=m(n)}^{\infty} c_j.$$

Additionally, as $(c_j)_{j=0}^{\infty} = \mathcal{O}(\ell(j))$ then

$$\begin{aligned} \lambda_n(\mathbf{K}_n) &= \mathcal{O}\left(n^{-\nu(n)} \sum_{j=m(n)}^{\infty} \ell(j) \right) \\ &= \mathcal{O}\left(n^{-\nu(n)} \int_{m(n)-1}^{\infty} \ell(x) dx \right) \\ &= \mathcal{O}\left(n^{-\nu(n)} U(\ell, m(n)) \right) \end{aligned}$$

as claimed. □

Based on Lemma C.20 we provide Theorem C.24, which considers three scenarios inspired by Lemma 3.2.

Theorem C.24. *Given a sequence of data points* $(\mathbf{x}_i)_{i \in \mathbb{Z}_{\geq 1}}$ *with* $\mathbf{x}_i \in \mathbb{S}^d$ *for all* $i \in \mathbb{Z}_{\geq 1}$*, construct a sequence of row-wise data matrices* $(\mathbf{X}_n)_{n \in \mathbb{Z}_{\geq 1}}$*,* $\mathbf{X}_n \in \mathbb{R}^{n \times d}$*, with* $\mathbf{x}_i$ *corresponding to the* i*th row of* $\mathbf{X}_n$*. The corresponding sequence of gram matrices we denote* $\mathbf{G}_n := \mathbf{X}_n \mathbf{X}_n^T$*. Let* $m(n) := \lfloor n^{1/r(n)} \rfloor$ *and* $r(n) := \operatorname{rank}(\mathbf{X}_n)$ *and suppose for all sufficiently large* n *that* $r(n) \geq 7$*. Let the coefficients* $(c_j)_{j=0}^{\infty}$ *with* $c_j \in \mathbb{R}_{\geq 0}$ *for all* $j \in \mathbb{Z}_{\geq 0}$ *be such that 1) the series* $\sum_{j=0}^{\infty} c_j \rho^j$ *converges for all* $\rho \in [-1, 1]$ *and 2)* $(c_j)_{j=0}^{\infty} = \mathcal{O}(\ell(j))$ *where* $\ell \in \mathcal{L}$ *satisfies* $U(\ell, m(n)) < \infty$ *for all* n*. Consider the sequence of kernel matrices*

$$n\mathbf{K}_n = \sum_{p=0}^{\infty} c_p \mathbf{G}_n^{\odot p}.$$

Then

1. *if* $c_p = \mathcal{O}(p^{-\alpha})$ *with* $\alpha > r(n) + 1$ *for all* $n \in \mathbb{Z}_{\geq 0}$ *then* $\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(n^{-\frac{\alpha-1}{r(n)}} \right)$*,*
2. *if* $c_p = \mathcal{O}(e^{-\alpha\sqrt{p}})$*, then* $\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(n^{\frac{1}{2r(n)}} \exp\left(-\alpha' n^{\frac{1}{2r(n)}} \right) \right)$ *for any* $\alpha' < \alpha$*,*
3. *if* $c_p = \mathcal{O}(e^{-\alpha p})$*, then* $\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(\exp\left(-\alpha n^{\frac{1}{r(n)}} \right) \right)$*.*

Proof. First, as $[\mathbf{G}_n]_{ij} \leq 1$ then

$$\frac{\|\mathbf{G}^{\odot m}\|}{n} \leq \frac{\text{Trace}(\mathbf{G}^{\odot m})}{n} = 1.$$

Therefore to recover the three results listed we now apply Lemma C.23 with $\nu(n) = 0$. First, to prove i), under the assumption $\ell(x) = x^{-\alpha}$ with $\alpha > 0$ then

$$\int_{m(n)-1}^{\infty} x^{-\alpha} dx = \frac{(m(n)-1)^{-\alpha+1}}{\alpha - 1}.$$

As a result

$$\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(n^{-\frac{\alpha-1}{r(n)}} \right).$$

To prove ii), under the assumption $\ell(x) = e^{-\alpha\sqrt{x}}$ with $\alpha > 0$ then

$$\int_{m(n)-1}^{\infty} e^{-\alpha\sqrt{x}} dx = \frac{2\exp(-\alpha(\sqrt{m(n)-1})(\alpha\sqrt{m(n)-1}+1)}{\alpha^2}.$$

As a result

$$\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(n^{\frac{1}{2r(n)}} \exp\left(-\alpha' n^{\frac{1}{2r(n)}}\right)\right).$$

for any $\alpha' < \alpha$. Finally, to prove iii), under the assumption $\ell(x) = e^{-\alpha x}$ with $\alpha > 0$ then

$$\int_{m(n)-1}^{\infty} e^{-\alpha x} dx = \frac{\exp(-\alpha(m(n)-1)}{\alpha}.$$

Therefore

$$\lambda_n(\mathbf{K}_n) = \mathcal{O}\left(\exp\left(-\alpha n^{\frac{1}{r(n)}}\right)\right).$$

□

Unfortunately the curse of dimensionality is present in these results due to the $1/r(n)$ factor in the exponents of n. Therefore, although perhaps somewhat loose, these results are certainly non-trivial. To observe this note that a standard bound for $\lambda_n(\mathbf{K}_n)$ is given by $\lambda_n(\mathbf{K}_n) \leq Tr(\mathbf{K}_n)/n = \mathcal{O}(n^{-1})$. For example, even concerning power law decay as per point 1. our result improves upon this trivial bound as long as $\alpha > r(n) + 1$, albeit this admittedly is a strict assumption. For points 2. and 3. the results stated are significantly stronger.