Risk Quadrangle and Robust Optimization Based on Extended φ-Divergence

Anonymous Authors[1]

Abstract

The Fundamental Risk Quadrangle (FRQ) is a unified framework linking risk management, statistical estimation, and optimization. Distributionally robust optimization (DRO) based on φ-divergence minimizes the maximal expected loss, where the maximum is over a φ-divergence ambiguity set. This paper introduces the extended φ-divergence and the extended φ-divergence quadrangle, which integrates DRO into the FRQ framework. We derive the primal and dual representations of the quadrangle elements (risk, deviation, regret, error, and statistic). The dual representation provides an interpretation for classification, portfolio optimization, and regression as robust optimization based on the extended φ-divergence. The primal representation offers tractable formulations of these robust optimizations as convex optimization. We provide illustrative examples showing that many common problems, such as least-squares regression, quantile regression, support vector machines, and CVaR optimization, fall within this framework. Additionally, we conduct a case study to visualize the optimal solution of the inner maximization in robust optimization.

1. Introduction

Distributionally robust optimization with φ-divergence ambiguity set (Ben-Tal et al., 2013) minimizes the worst-case expected loss over all probability measures within an ambiguity set defined by φ-divergence. Ahmadi-Javid (2012) shows that the inner maximization of DRO is a coherent risk measure (Artzner et al., 1999), establishing a direct connection between DRO and risk-averse optimization. We refer to it as the φ-divergence risk measure in this study.

The Fundamental Risk Quadrangle (Rockafellar & Uryasev, 2013) combines regular risk measures with regular measures of deviation, error and regret in a framework that links optimization with statistics. Defined through axioms, FRQ clarifies the relationships between objective functions used in tasks such as classification, portfolio optimization, and regression. In particular, regression is shown to be equivalent to deviation minimization. Through the dual representation of the quadrangle, these tasks can be interpreted as robust optimization.

It is natural to consider integrating DRO into FRQ by completing the quadrangle for φ-divergence risk measure. The purpose of the integration is two-fold: FRQ provides insights into DRO by connecting objective functions in various tasks and interprets them as DRO, while the φ-divergence risk measure inspires the construction of new risk quadrangles. However, φ-divergence risk measure is coherent, which excludes some risk measures widely used in machine learning and finance, such as the mean-standard deviation risk measure. By extending the divergence function to the entire real line, we introduce a family of regular risk measures and associated quadrangles, which accommodates the excluded measures.

Main Contributions. (i) **Extension of φ-divergence:** We define the extended φ-divergence and its associated risk measure, allowing for negative values in the worst-case weight. The extension recovers risk measures used as objective functions in various tasks. A notable example is the mean-standard deviation risk measure associated with the extended χ^2-divergence. (ii) **Completion and interpretation of Quadrangle:** For the extended φ-divergence risk measure, we complete the risk quadrangle and derive primal and dual representations for risk, deviation, regret and error. The primal representation facilitates convex optimization formulations. The dual representation provides a robust optimization (RO) interpretation for the measures associated with extended φ-divergence, and a DRO interpretation for the measures associated with φ-divergence. Furthermore, the RO objective functions are conservative versions of their DRO counterparts. (iii) **Examples:** We provide a range of examples to illustrate that the extended φ-divergence quadrangle recovers many important quadrangles. The quadrangle elements are used as objective functions in various learning tasks, such as least-squares regression, quantile regression, support vector machines, and CVaR optimiza-

[1]Anonymous Institution, Anonymous City, Anonymous Region, Anonymous Country. Correspondence to: Anonymous Author <anon.email@domain.com>.

Preliminary work. Under review by the International Conference on Machine Learning (ICML). Do not distribute.

tion. Through the dual representation, these tasks has a novel RO/DRO interpretation. Furthermore, viewing RO as conservative DRO, we recover a well-known example that the mean-standard deviation risk measure bounds the χ^2-divergence risk measure.

Literature Review. The connection between DRO and coherent risk measure has been extensively studied in Bayraksan & Love (2015); Dommel & Pichler (2020); Kuhn et al. (2024). The dual representation of regular risk measure is studied in Rockafellar & Uryasev (2013). The dual representation of coherent regret is studied in Sun et al. (2020); Rockafellar (2020); Fröhlich & Williamson (2022a;b); Rockafellar (2023). Gotoh & Uryasev (2017) studies classification as a risk minimization problem. Our study is the first to define the extended φ-divergence, complete the corresponding risk quadrangle, and unify these developments with robust optimization interpretations for various learning tasks.

2. Preliminaries

This section provides the necessary background on the φ-divergence risk measure and the FRQ. We adopt the following standard notations throughout. Let (Ω, Σ, P_0) be a probability space, where P_0 is a reference measure. The set of all probability measures on (Ω, Σ) is denoted by $\mathcal{P}(\Sigma)$. The extended real number set is denoted by $\overline{\mathbb{R}} = \mathbb{R} \cup +\infty$. Let $X \in \mathcal{L}^2$ be a real-valued random variable. The expectation and standard deviation of X with respect to P_0 are denoted by $\mathbb{E}[X]$ and $\sigma(X)$, respectively. The $\mathcal{L}^p$-norm of X is denoted by $||X||_p$. $const$ denotes a constant.

2.1. φ-Divergence and φ-Divergence Risk Measure

Definition 2.1 (divergence function)**.** A convex lower semi-continuous function $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$ is a divergence function if $(i)\varphi(1) = 0, (ii)\,\mathrm{dom}(\varphi) = \mathbb{R}, (iii)\varphi(x) = +\infty$ for $x < 0, (iv)1 \in \mathrm{int}(\{x : \varphi(x) < +\infty\}), (v)0 \in \partial\varphi(1)$, where int denotes the interior, ∂ denotes the subgradient.

Definition 2.2 (φ-divergence (Csiszár, 1963; Morimoto, 1963))**.** Consider probability measures P and P_0, where P is dominated by P_0. For a divergence function $\varphi(x)$, the φ-divergence of P from P_0 is defined by $D_\varphi(P||P_0) := \int_\Omega \varphi\left(dP/dP_0\right) dP_0$. Let Q be the Radon–Nikodym derivative dP/dP_0. We have $D_\varphi(P||P_0) = \mathbb{E}[\varphi(Q)]$.

Definition 2.3 (φ-divergence risk measure (Ahmadi-Javid, 2012; Dommel & Pichler, 2020))**.** Consider a divergence function $\varphi(x)$. The φ-divergence risk measure is defined by $\mathcal{R}_{\varphi,\beta}(X) = \sup_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[X]$, where $\mathcal{P}_{\varphi,\beta} = \{P \in \mathcal{P}(\Sigma) : D_\varphi(P||P_0) \le \beta\}$.

The condition (v) in Definition 2.1 ensures that the other elements in φ-divergence quadrangle developed in subsequent sections satisfy the defining axioms. The condition (iv) ensures that $\mathcal{R}_{\varphi,\beta}(X) > \mathbb{E}[X]$. The family of divergence functions of the form $\varphi(x)+k(x-1)$, where $k \in \mathbb{R}$, defines the same φ-divergence and φ-divergence risk measure.

2.2. The Fundamental Risk Quadrangle Framework

FRQ framework studies closed and convex functionals of random variables. A functional $\rho : \mathcal{L}^2 \to \overline{\mathbb{R}}$ is called convex if $\rho\left(\mu X + (1-\mu)Y\right) \le \mu\rho(X) + (1-\mu)\rho(Y)$, $\forall\ X, Y \in \mathcal{L}^2$, $\mu \in [0,1]$, and closed if $\left\{X \in \mathcal{L}^2 | \rho(X) \le c\right\}$ is a closed set $\forall\, c < \infty$.

A risk measure aggregates the overall uncertain cost in X into a number, so that the inequality $\mathcal{R}(X) < C$ models that X is adequately smaller than C. For a bet that loses a constant amount, the risk equals the constant. Since a risk measures the undesired outcome, it is more conservative than the expectation. An example is the Markowitz risk $\mathbb{E}[X] + \lambda\sigma(X)$, $\lambda > 0$.

Definition 2.4 (regular risk measure)**.** A closed convex functional $\mathcal{R} : \mathcal{L}^2 \to \overline{\mathbb{R}}$ is a regular measure of risk if it satisfies: (1) $\mathcal{R}(C) = C$, $\forall\ C = const$, (2) $\mathcal{R}(X) > \mathbb{E}[X]$, $\forall\ X \neq const$.

A deviation measure quantifies nonconstancy as the uncertainty in X by measuring deviation from the expectation. Intuitively, a bet that loses a constant amount has constant risk but zero uncertainty. An example is the (scaled) standard deviation $\lambda\sigma(X)$.

Definition 2.5 (regular deviation measure)**.** A closed convex functional $\mathcal{D} : \mathcal{L}^2 \to \overline{\mathbb{R}}^+$ is a regular measure of deviation if it satisfies: (1) $\mathcal{D}(C) = 0$, $\forall\ C = const$, (2) $\mathcal{D}(X) > 0$, $\forall\ X \neq const$.

Regret quantifies the displeasure associated with the mixture of potential positive, zero and negative outcomes of a random variable. An example is $\mathbb{E}[X] + \lambda||X||_2$.

Definition 2.6 (regular regret measure)**.** A closed convex functional $\mathcal{V} : \mathcal{L}^2 \to \overline{\mathbb{R}}$ is a regular measure of regret if it satisfies: (1) $\mathcal{V}(0) = 0$, (2) $\mathcal{V}(X) > \mathbb{E}[X]$, $\forall\ X \neq const$.

An error quantifies nonzeroness. In regression, it measures how wrong an estimate is compared to the true value. It is therefore nonnegative. When the estimate is precise, the error is zero. An example is the scaled $\mathcal{L}^2$ norm $\lambda||X||_2$ used in least squares regression.

Definition 2.7 (regular error measure)**.** A closed convex functional $\mathcal{E} : \mathcal{L}^2 \to \overline{\mathbb{R}}^+$ is a regular measure of error if it satisfies: (1) $\mathcal{E}(0) = 0$,, (2) $\mathcal{E}(X) > 0$, $\forall\ X \neq const$.

The measures defined above are intrinsically connected. There is a one-to-one correspondence between risk and deviation, and between regret and error. Risk and deviation can be derived from one-dimensional minimization prob-

lems involving regret and error, respectively. When reviewing the axioms, it is helpful to use the provided examples as a guide: risk $\mathbb{E}[X] + \lambda\sigma(X)$, deviation $\lambda\sigma(X)$, regret $\mathbb{E}[X] + \lambda||X||_2$, and error $\lambda||X||_2$.

Definition 2.8 (regular Risk Quadrangle)**.** A quartet $(\mathcal{R}, \mathcal{D}, \mathcal{V}, \mathcal{E})$ of regular measures of risk, deviation, regret, and error satisfying the following relationships is called a regular risk quadrangle:

(Q1) error projection: $\mathcal{D}(X) = \inf_C \{ \mathcal{E}(X - C) \}$;

(Q2) certainty equivalence: $\mathcal{R}(X) = \inf_C \{ C + \mathcal{V}(X - C) \}$;

(Q3) centerness: $\mathcal{R}(X) = \mathcal{D}(X) + \mathbb{E}[X]$, $\mathcal{V}(X) = \mathcal{E}(X) + \mathbb{E}[X]$.

Moreover, the quartet $(\mathcal{R}, \mathcal{D}, \mathcal{V}, \mathcal{E})$ is bound by the statistic $\mathcal{S}(X)$ satisfying $\mathcal{S}(X) = \operatorname{argmin}_{C\in\mathbb{R}} \{ \mathcal{E}(X - C) \} = \operatorname{argmin}_{C\in\mathbb{R}} \{ C + \mathcal{V}(X - C) \}$.

A regression problem is defined by minimizing the error of the residual. Error minimization is equivalent to deviation minimization, connecting regression problem to deviation and risk measures.

Definition 2.9 (regression)**.** Let $Z_f = Y - f(X) - C$, $\bar{Z}_f = Y - f(X)$, where $C \in \mathbb{R}$, f belongs to a class of functions $\mathcal{F}$. A regression problem is defined as $\min_{f\in\mathcal{F},C} \mathcal{E}(Z_f)$.

Theorem 2.10 (error shaping decomposition of regression (Theorem 3.2, Rockafellar et al. (2008))**.** *The solution to regression in Definition 2.9 is characterized by*

$$f, C \in \operatorname*{argmin}_{f,C} \mathcal{E}(Z_f) \text{ iff } f \in \operatorname*{argmin}_{f} \mathcal{D}(\bar{Z}_f), C \in \mathcal{S}(\bar{Z}_f).$$

The dual representation of Risk Quadrangle involves the definition of risk envelope. Section 4.2 discusses its connection to ambiguity set in robust optimization.

Definition 2.11 (conjugate functional, risk envelope, risk identifier (Rockafellar & Uryasev, 2013))**.** Let $\rho : \mathcal{L}^2 \to \overline{\mathbb{R}}$ be a closed convex functional. Then a functional $\rho^* : \mathcal{L}^2 \to \overline{\mathbb{R}}$ is said to be conjugate to ρ if $\rho(X) = \sup_{Q\in\mathcal{Q}}\{\mathbb{E}[XQ] - \rho^*(Q)\}, \forall\, X \in \mathcal{L}^2$, where $\mathcal{Q} = \operatorname{dom}(\rho^*)$ is called the risk envelope associated with ρ, and Q furnishing the maximum in the conjugate ρ^* is called a risk identifier for X.

3. Functional Space Setting

This section discusses the logic behind choosing $\mathcal{L}^2$ as a working space. The choice of $\mathcal{L}^p := \mathcal{L}^p(\Omega, \Sigma, P_0)$, $p \in [1, \infty)$ seems to be reasonable, however, one still has to be careful since if $\mathcal{R} : \mathcal{L}^p \to \overline{\mathbb{R}}$ is a proper convex risk measure, then either $\mathcal{R}(\cdot)$ is finite valued and continuous on $\mathcal{L}^p$ or $\mathcal{R}(X) = +\infty$ on a dense set of points $X \in \mathcal{L}^p$ (cf. Shapiro et al. (2014), Proposition 6.8). Therefore, for some risk measures, it may be even impossible to find an appropriate space.

For φ-divergence risk measures, the natural choice of a functional space can be an Orlicz space paired with a divergence function satisfying $\varphi(0) < +\infty$, $\lim_{x\to+\infty} \varphi(x)/x = +\infty$, suggested by Dommel & Pichler (2020) and adopted by Fröhlich & Williamson (2022b). However, this particular space excludes important divergence functions such as the total variation distance (TVD). The TVD fits in the framework of Shapiro (2017), which uses $\mathcal{L}^p$ in general and switches to $\mathcal{L}^\infty$ for certain divergence functions. Of course, the simplest way would be to work with finite Ω. Then every function $X : \Omega \to \mathbb{R}$ is measurable, and the space of all such functions can be identified with the Euclidean space. Such an approach was taken by Bayraksan & Love (2015).

In light of everything mentioned above, we follow (Rockafellar & Uryasev, 2013) and take $\mathcal{L}^2$ as our working space assuming finiteness where needed. This choice allows us to apply the extensive FRQ theory developed in this setting.

4. Extended φ-Divergence Quadrangle

4.1. Dual Representation of Extended φ-Divergence Quadrangle

This section defines the extended divergence function and its associated risk measure, and completes the risk quadrangle in dual representation for the risk measure. Throughout this paper, if extension is not explicitly mentioned for divergence function, divergence and risk quadrangle, then the referred object is associated with the non-extended version.

We define the extended divergence function by removing the condition $\varphi(x) = +\infty$ for $x < 0$ in Definition 2.1. We frequently work with natural extensions of the divergence function in this study. For example, the divergence function for the χ^2-divergence is $\varphi(x) = x^2$ for $x > 0$; we extend this by using $\varphi(x) = x^2$ for $x \in \mathbb{R}$ to define the extended χ^2-divergence function.

Definition 4.1 (extended divergence function)**.** A convex lower semi-continuous function $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$ is an extended divergence function if $(i)\varphi(1) = 0, (ii)\operatorname{dom}(\varphi) = \mathbb{R}, (iii)1 \in \operatorname{int}(\{x : \varphi(x) < +\infty\}), (iv)0 \in \partial\varphi(1)$.

Next, we define the extended φ-divergence risk measure. We will see that the φ-divergence risk measure (Definition 2.3) is a special case of the extended version.

Definition 4.2 (extended φ-divergence risk measure)**.** The extended φ-divergence risk measure is defined by $\mathcal{R}_{\varphi,\beta}(X) = \sup_{\mathcal{Q}\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[XQ]$, where $\mathcal{Q}^1_{\varphi,\beta} = \{Q \in \mathcal{L}^2 : \mathbb{E}[Q] = 1, \mathbb{E}[\varphi(Q)] \le \beta\}$.

We complete the risk quadrangle in dual representation for the extended φ-divergence risk measure.

Definition 4.3 (dual representation of extended φ-divergence quadrangle)**.** For an extended φ-divergence

function and $X \in \mathcal{L}^2$, the dual extended φ-divergence quadrangle is defined by

$$\mathcal{R}_{\varphi,\beta}(X) = \sup_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[XQ], \quad (1) \qquad \mathcal{D}_{\varphi,\beta}(X) = \sup_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[X(Q-1)], \quad (3)$$

$$\mathcal{V}_{\varphi,\beta}(X) = \sup_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[XQ], \quad (2) \qquad \mathcal{E}_{\varphi,\beta}(X) = \sup_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[X(Q-1)], \quad (4)$$

$$\mathcal{S}_{\varphi,\beta}(X) = \operatorname*{argmin}_{C\in\mathbb{R}} \sup_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[(X-C)(Q-1)], \quad (5)$$

where

$$\mathcal{Q}^1_{\varphi,\beta} = \{Q \in \mathcal{L}^2 : \mathbb{E}[Q] = 1, \mathbb{E}[\varphi(Q)] \le \beta\}, \quad (6)$$

$$\mathcal{Q}_{\varphi,\beta} = \{Q \in \mathcal{L}^2 : \mathbb{E}[\varphi(Q)] \le \beta\} \quad (7)$$

are the envelopes associated with $\mathcal{R}_{\varphi,\beta}(X)$ and $\mathcal{V}_{\varphi,\beta}(X)$ respectively.

The next theorem proves that the dual representation above satisfies the axioms in Definition 2.8.

Theorem 4.4 (extended φ-divergence quadrangle)**.** *Let $\varphi(x)$ be an extended φ-divergence function, $X \in \mathcal{L}^2$. The quartet $(\mathcal{R}_{\varphi,\beta}, \mathcal{D}_{\varphi,\beta}, \mathcal{V}_{\varphi,\beta}, \mathcal{E}_{\varphi,\beta})$ defined by (1)–(4) is a regular risk quadrangle with the statistic (5).*

The proof verifying the axioms, based on Ang et al. (2018); Sun et al. (2020), is in Appendix B. It will be clear that Theorem 4.4 integrates DRO into the FRQ framework after the discussion of the φ-divergence ambiguity set and the risk envelope $\mathcal{Q}$ in Section 4.2. The coherent risk measure in DRO is a special case of the extended φ-divergence risk measure. New quadrangles can be constructed by plugging extended φ-divergences into Definition 4.3. The dual representation provides a robust optimization interpretation for many well-known optimization problems (Section 6).

4.2. Discussion on Risk Identifier Q

This section interprets the risk identifier Q, showing that minimizing elements in the extended φ-divergence quadrangle corresponds to DRO, whereas the non-extended quadrangle corresponds to robust optimization (RO).

Consider the (non-extended) φ-divergence quadrangle. Although there is no requirement in the envelope (6) that $Q \ge 0$, the conditions $\varphi(x) = +\infty$ for $x < 0$ and $\mathbb{E}[\varphi(Q)] \le \beta$ imply that $Q \ge 0$ almost surely. Define indicator function $\mathcal{I}_A(x) = 1$ if $x \in A$ and 0 otherwise. For every $Q \in \mathcal{Q}^1_{\varphi,\beta}$, we can define a probability measure on (Ω, Σ) by $P_Q(A) = \mathbb{E}[\mathcal{I}_A(\omega)Q(\omega)], A \in \Sigma$. Let $Q_0(\omega) = 1$ be the constant random variable. We have $Q_0 \in \mathcal{Q}^1_{\varphi,\beta}$. Define $P_0 = P_{Q_0}$. By definition, Q is the Radon–Nikodym derivative dP_Q/dP_0. The condition $\mathbb{E}[\varphi(Q)] \le \beta$ can be equivalently expressed by $D_\varphi(P||P_0) \le \beta$. The envelope $\mathcal{Q}^1_{\varphi,\beta}$ has a one-to-one correspondence to a set of probability measures $\mathcal{P}_{\varphi,\beta} = \{P \in \mathcal{P}(\Sigma) : D_\varphi(P||P_0) \le \beta\}$. The dual representations (1) (3) can be equivalently written as

$$\mathcal{R}_{\varphi,\beta}(X) = \sup_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[X],$$
$$\mathcal{D}_{\varphi,\beta}(X) = \sup_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[X] - \mathbb{E}[X].$$

Next, consider the extended φ-divergence quadrangle. By extending the divergence function, Q can take negative values. Note that the envelope with $Q \ge 0$ is the necessary and sufficient condition for monotonicity of the convex homogeneous functional associated with such envelope (Rockafellar et al., 2006; Rockafellar & Uryasev, 2013). We therefore forgo the interpretation of Q as a Radon-Nikodym derivative, and the monotonicity of the associated risk measure. Instead, Q can be viewed as (potentially negative) weight on samples. In this case, the minimization of (1) can still be interpreted as a robust optimization, where the maximum is over a set of weights. Also, the covariance between random variables X and Q is $\mathrm{cov}(X,Q) = \mathbb{E}[(X-\mathbb{E}X)(Q-\mathbb{E}Q)]$. Since $\mathbb{E}Q = 1$ by (6), $\mathrm{cov}(X,Q) = \mathbb{E}[X(Q-1)]$. The deviation (3) can be written as $\sup_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathrm{cov}(X,Q)$. Thus the worst-case Q^* tracks X as closely as possible.

4.3. Primal Representation of Extended φ-Divergence Quadrangle

This section derives the primal representations of extended φ-divergence quadrangle in Definition 4.5 from the dual representations in Definition 4.3. The proof is in Appendix C.

Definition 4.5 (primal representation of extended φ-divergence quadrangle)**.** For an extended divergence function $\varphi(x)$ and $X \in \mathcal{L}^2$, the primal extended φ-divergence quadrangle is defined by

$$\mathcal{R}_{\varphi,\beta}(X) = \inf_{\substack{C\in\mathbb{R},\\ t>0}} t\left\{C + \beta + \mathbb{E}\left[\varphi^*\left(\frac{X}{t} - C\right)\right]\right\}, \quad (8)$$

$$\mathcal{D}_{\varphi,\beta}(X) = \inf_{\substack{C\in\mathbb{R},\\ t>0}} t\left\{C + \beta + \mathbb{E}\left[\varphi^*\left(\frac{X}{t} - C\right) - \frac{X}{t}\right]\right\}, \quad (9)$$

$$\mathcal{V}_{\varphi,\beta}(X) = \inf_{t>0} t\left\{\beta + \mathbb{E}\left[\varphi^*\left(\frac{X}{t}\right)\right]\right\}, \quad (10)$$

$$\mathcal{E}_{\varphi,\beta}(X) = \inf_{t>0} t\left\{\beta + \mathbb{E}\left[\varphi^*\left(\frac{X}{t}\right) - \frac{X}{t}\right]\right\}, \quad (11)$$

$$\mathcal{S}_{\varphi,\beta}(X) = \operatorname*{argmin}_{C\in\mathbb{R}} \inf_{t>0} t\left\{\frac{C}{t} + \beta + \mathbb{E}\Big[\varphi^*\Big(\frac{X-C}{t}\Big)\Big]\right\}. \tag{12}$$

Theorem 4.6 (primal extended φ-divergence quadrangle)**.** *Let $\varphi(x)$ be an extended divergence function, $X \in \mathcal{L}^2$. Elements of the dual extended φ-divergence quadrangle in Theorem (4.4) can be presented as (8)–(12) in Definition 4.5. The optimal t and C in (8)–(12) are attainable.*

The quadrangle elements in primal representation facilitates optimization, since the minimax problem of minimizing the worst-case expectation becomes a minimization with an additional variable. Furthermore, substituting important extended φ-divergence functions into the definitions, we recover many risk quadrangles with interpretable expressions (Section 5).

4.4. Relation between φ-Divergence Risk Quadrangle and Extended φ-Divergence Risk Quadrangle

This section discusses the interpretation of RO as a conservative version of DRO.

Proposition 4.7 (RO as conservative DRO)**.** *The risk, regret, deviation and error in the φ-divergence quadrangle are bounded from above by their counterparts in the extended φ-divergence quadrangle. For sufficiently small β, the φ-divergence quadrangle is equivalent to the extended φ-divergence quadrangle.*

This follows from the observation that for fixed parameter β, the risk envelopes (6), (7) with divergence function φ are subsets of the risk envelopes with the extended version. The φ-divergence measures are thus smaller than their extended counterparts. Therefore, when the quadrangle elements are used as objective function, the RO is a more conservative version of the corresponding DRO. Furthermore, the conditions $\mathbb{E}[\varphi(Q)] \le \beta$ and $\mathbb{E}[Q] = 1$ imply that for sufficiently small β, the risk identifier Q cannot be negative. Since the risk envelopes (6), (7) are equivalent to their extended versions with the additional constraint $Q \ge 0$, the quadrangles are equivalent.

We can recover a well-known bound in Theorem 8.2 in Kuhn et al. (2024) as a special case of Proposition 4.7. Consider χ^2-divergence and its extended version.

Example 4.8 (χ^2-divergence risk measure bounded by mean-standard deviation risk measure)**.** Consider the risk measures generated by $\varphi(x) = (x-1)^2$ for $x > 0$ and $+\infty$ if $x \le 0$, and its extended version. The risk measure generated by the extended χ^2-divergence is the mean-standard deviation risk. The risk measure generated by χ^2-divergence is the second-order superquantile. See Example 5.2 and 5.6 for details. We have

$$\sup_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[X] \le \mathbb{E}[X] + \sqrt{\beta}\sigma(X).$$

4.5. Expression for Risk Identifier

We derive an expression for the risk identifier, which allows us to directly calculate the risk identifier (worst-case weight) given the solution to the problem in primal representation.

Proposition 4.9. *Denote by C^* and t^* the optimal C and t in the primal representation 8 of extended φ-divergence risk measure. The risk identifier of risk measure $\mathcal{R}_{\varphi,\beta}(X)$ can be expressed as $Q^*(\omega) \in \partial\varphi^*\left(X(\omega)/t^* - C^*\right)$. Denote by C^* the optimal C in the primal representation 11 of extended φ-divergence error measure. The risk identifier of extended φ-divergence error measure $\mathcal{E}_{\varphi,\beta}(X)$ can be expressed as $Q^*(\omega) \in \partial\varphi^*\left(X(\omega)/t^*\right)$.*

The proof is in Appendix G. Proposition 4.9 will be used to calculate the worst-case weight after solving the primal problem in Section 8.

5. Examples of Extended φ-Divergence Quadrangle

This section presents important examples of extended φ-divergences and their corresponding φ-divergence quadrangles. These quadrangles are derived by substituting the convex conjugates of various extended φ-divergence functions into the primal representation in Definition 4.5. More examples are listed in Appendix D. The derivations are in Appendix E. For all examples, the quadrangle establishes a novel connection between regression and DRO (Section 6).

5.1. Extended φ-Divergence Quadrangles

We show that two important risk quadrangles are generated by the extended total variation distance (TVD) and extended χ^2-divergence.

Example 5.1 (range-based quadrangle generated by extended TVD)**.** Consider the following extended divergence function and its convex conjugate

$$\varphi(x) = |x-1|, \quad x\in\mathbb{R}, \quad \varphi^*(z) = \begin{cases} z, & z\in[-1,1] \\ +\infty, & z\notin[-1,1]\,. \end{cases}$$

The complete quadrangle is as follows

$$\mathcal{R}_{\varphi,\beta}(X) = \frac{\beta}{2}(\operatorname{ess\,sup} X - \operatorname{ess\,inf} X) + \mathbb{E}[X],$$
$$\mathcal{V}_{\varphi,\beta}(X) = \beta \operatorname{ess\,sup}|X| + \mathbb{E}[X],$$
$$\mathcal{D}_{\varphi,\beta}(X) = \frac{\beta}{2}(\operatorname{ess\,sup} X - \operatorname{ess\,inf} X),$$

$$\mathcal{E}_{\varphi,\beta}(X) = \beta \operatorname{ess\,sup} |X|,$$
$$\mathcal{S}_{\varphi,\beta}(X) = \frac{1}{2}(\operatorname{ess\,sup} X + \operatorname{ess\,inf} X).$$

We recover the range-based quadrangle in Example 4 in Rockafellar & Uryasev (2013). The divergence function is the extended version of TVD in Example 5.5.

Example 5.2 (mean quadrangle generated by extended Pearson χ^2-divergence)**.** Consider the following extended divergence function and its convex conjugate

$$\varphi(x) = (x-1)^2, \quad \varphi^*(z) = \frac{z^2}{4} + z\,.$$

The complete quadrangle is as follows

$$\mathcal{R}_{\varphi,\beta}(X) = \mathbb{E}[X] + \sqrt{\beta}\sigma(X), \quad \mathcal{D}_{\varphi,\beta}(X) = \sqrt{\beta}\sigma(X),$$
$$\mathcal{V}_{\varphi,\beta}(X) = \mathbb{E}[X] + \sqrt{\beta}\,\|X\|_2\,, \quad \mathcal{E}_{\varphi,\beta}(X) = \sqrt{\beta}\,\|X\|_2\,,$$
$$\mathcal{S}_{\varphi,\beta}(X) = \mathbb{E}[X].$$

We recover the mean quadrangle in Example 1 in Rockafellar & Uryasev (2013). The solution to the regression (error minimization) is not dependent on β, since scaling the error function does not change the solution.

The divergence function is the extended version of the χ^2-divergence function in Example 5.6. The connection between variance penalty and DRO is studied in Lam (2016); Duchi & Namkoong (2019).

5.2. φ-Divergence Quadrangles

This section completes the risk quadrangle in primal representation for some well-known φ-divergence risk measures, which, apart from Example 5.3, had not been established.

Example 5.3 (quantile quadrangle generated by indicator divergence)**.** Consider the divergence function and its convex conjugate $\varphi(x) = \mathbf{1}_{[0,(1-\alpha)^{-1}]}(x)$ and $\varphi^*(z) = \max\{0, (1-\alpha)^{-1}z\}$. Denote α-Value-at-Risk by VaR_α.

We obtain the quantile quadrangle:

$$\mathcal{R}_{\varphi,\beta}(X) = \mathrm{CVaR}_\alpha(X),$$
$$\mathcal{D}_{\varphi,\beta}(X) = \mathrm{CVaR}_\alpha(X) - \mathbb{E}[X],$$
$$\mathcal{V}_{\varphi,\beta}(X) = \frac{1}{1-\alpha}\mathbb{E}[X_+],$$
$$\mathcal{E}_{\varphi,\beta}(X) = \mathbb{E}\Big[\frac{\alpha}{1-\alpha}X_+ + X_-\Big],$$
$$\mathcal{S}_{\varphi,\beta}(X) = \mathrm{VaR}_\alpha(X).$$

The derivation of the risk measure is from Ahmadi-Javid (2012); Shapiro (2017). We recover the quantile quadrangle in Example 2 in Rockafellar & Uryasev (2013). Note that α is the parameter of divergence function. The parameter β of the ambiguity set does not appear in the formula in the primal representation. When $\alpha \to 1$, the quadrangle becomes the worst-case-based quadrangle. When $\alpha \to 0$, the risk measure becomes $\mathbb{E}[X]$, which is not risk averse. $\varphi(x)$ in this case violates Definition 2.1.

Example 5.4 (EVaR quadrangle generated by Kullback--Leibler divergence)**.** The divergence function and its convex conjugate are $\varphi(x) = x\ln(x) - x + 1,\ \varphi^*(z) = \exp(z) - 1$.

The complete quadrangle is as follows:

$$\mathcal{R}_{\varphi,\beta}(X) = \inf_{t>0} t\left\{\beta + \ln\mathbb{E}\Big[e^{\frac{X}{t}}\Big]\right\} = \mathrm{EVaR}_{1-e^{-\beta}}(X),$$
$$\mathcal{D}_{\varphi,\beta}(X) = \inf_{t>0} t\left\{\beta + \ln\mathbb{E}\Big[e^{\frac{X-\mathbb{E}[X]}{t}}\Big]\right\},$$
$$\mathcal{V}_{\varphi,\beta}(X) = \inf_{t>0} t\left\{\beta + \mathbb{E}\big[e^{\frac{X}{t}} - 1\big]\right\},$$
$$\mathcal{E}_{\varphi,\beta}(X) = \inf_{t>0} t\left\{\beta + \mathbb{E}\Big[e^{\frac{X}{t}} - \frac{X}{t} - 1\Big]\right\},$$
$$\mathcal{S}_{\varphi,\beta}(X) = t^*\ln\mathbb{E}\big[e^{\frac{X}{t^*}}\big],$$

where t^* is a solution of the following equation $t^*\beta + t^*\ln\mathbb{E}\big[e^{\frac{X}{t^*}}\big] - \mathbb{E}\big[Xe^{\frac{X}{t^*}}\big]/\mathbb{E}\big[e^{\frac{X}{t^*}}\big] = 0$. The risk measure EVaR is studied in Ahmadi-Javid (2012).

Example 5.5 (robustified supremum-based quadrangle generated by total variation distance)**.** Consider the following divergence function and its convex conjugate $\varphi(x) = |x-1|$ for $x \geq 0$ and $+\infty$ if otherwise. $\varphi^*(z) = -1 + [z+1]_+$ if $z \leq 1$ and $+\infty$ if otherwise.

The complete quadrangle is as follows:

$$\mathcal{R}_{\varphi,\beta}(X) = \frac{\beta}{2}\operatorname{ess\,sup}(X) + (1-\frac{\beta}{2})\mathrm{CVaR}_{\frac{\beta}{2}}(X),$$
$$\mathcal{V}_{\varphi,\beta}(X) = \inf_{\substack{t>0\\ t\geq \operatorname{ess\,sup} X}}\left\{t(\beta-1) + \mathbb{E}\Big[X+t\Big]_+\right\},$$
$$\mathcal{D}_{\varphi,\beta}(X) = \mathcal{R}_{\varphi,\beta}(X) - \mathbb{E}[X],$$
$$\mathcal{E}_{\varphi,\beta}(X) = \inf_{t>0}\left\{t(\beta-1) + \mathbb{E}\Big[\Big[X+t\Big]_+ - X\Big]\right\},$$
$$\mathcal{S}_{\varphi,\beta}(X) = \operatorname{ess\,sup}(X) - 2\mathrm{VaR}_{1-\frac{\beta}{2}}(X).$$

The risk measure is studied in Shapiro (2017), Example 3.10.

Example 5.6 (second-order quantile-based quadrangle generated by Pearson χ^2-divergence)**.** The divergence function and its convex conjugate are $\varphi(x) = (x-1)^2$ if $x \geq 0$ and $+\infty$ if otherwise. $\varphi^*(z) = \big(\frac{z}{2}+1\big)^2 I_{z\geq -2} - 1$, where $I_{\{\cdot\}} = 1$ if the argument in the bracket is true and 0 otherwise.

The complete quadrangle is

$$\mathcal{R}_{\varphi,\beta}(X) = \sqrt{(\beta+1)\mathbb{E}\left[(X - q_\beta^{(2)}(X))^2 I_{\{X \geq q_\beta^{(2)}(X)\}}\right]} + q_\beta^{(2)}(X) = \text{second-order superquantile},$$

$$\mathcal{D}_{\varphi,\beta}(X) = \mathcal{R}_{\varphi,\beta}(X) - \mathbb{E}[X] = \text{ second-order superquantile deviation},$$

$$\mathcal{V}_{\varphi,\beta}(X) = \inf_{t>0} t\beta + \frac{1}{4t}\mathbb{E}\left[\left((X+2t)^2 - 4t^2\right) I_{\{X+2t>0\}}\right],$$

$$\mathcal{E}_{\varphi,\beta}(X) = \inf_{t>0} t\beta + \frac{1}{4t}\mathbb{E}\left[\left((X+2t)^2 - 4t^2\right) I_{\{X+2t>0\}}\right] - \mathbb{E}[X],$$

$$\mathcal{S}_{\varphi,\beta}(X) = q_\alpha^{(2)}(X) = \text{ second-order quantile},$$

where $\sqrt{1+\beta} = (1-\alpha)^{-1}$, and the statistic $q_\alpha^{(2)}(X)$ is characterized by the equation $1 - \alpha = ||(X - q_\alpha^{(2)}(X))_+||_1(||(X - q_\alpha^{(2)}(X))_+||_2)^{-1}$.

The risk measure is a special case of the higher-moment coherent risk measure studied in Krokhmal (2007). The risk, deviation and statistic are the same as those of second-order quantile-based quadrangle in Example 12 in Rockafellar & Uryasev (2013).

6. Robust Optimization Interpretation for Various Applications

The primal representation of the extended φ-divergence quadrangle recovers many important quadrangles, whose elements are used in various tasks such as classification, portfolio optimization and regression. As is discussed in Section 4.1 and 4.2, the dual representation of (extended) φ-divergence quadrangle provides the interpretation as RO/DRO. For the extended φ-divergence quadrangle, the dual representation provides the interpretation as robust optimization which reweights the samples. We formalize the statement and illustrate it with two important examples.

We consider classification (13), portfolio optimization (14), and regression (15). In portfolio optimization, the portfolio loss is $\boldsymbol{w}^\top \boldsymbol{L}$, where $\boldsymbol{w}$ is the portfolio weight, $\boldsymbol{L}$ is the random loss vector. In classification, given attribute $\boldsymbol{X}$, label Y and decision vector $\boldsymbol{w}$, the margin is defined by $L(\boldsymbol{w}, b) = Y(\boldsymbol{w}^\top \boldsymbol{X} - b)$. $\gamma(\boldsymbol{w})$ is the regularization term. In regression, consider a dependent variable (regressant) Y, a vector of independent variables (regressors) $\boldsymbol{X} = (X_1, \ldots, X_d)$, a class of function $\mathcal{F}$ and intercept $C \in \mathbb{R}$. The regression residual is defined by $Z_f = Y - f(\boldsymbol{X}) - C$, and the residual without intercept C is defined by $\bar{Z}_f = Y - f(\boldsymbol{X})$.

Each problem in (13), (14), and (15) has a corresponding RO interpretation given by (16), (18), and (20), respectively, through the dual representations. The equivalence between (15) and (20) follows from Theorem 2.10. Furthermore, when $\varphi(x)$ is a divergence function as defined in Definition 2.1, these problems also have a DRO interpretation, given by (17), (19), and (21), respectively.

Classification

$$\min_{\boldsymbol{w}} \mathcal{R}_{\varphi,\beta}(-L(\boldsymbol{w}, b)) + \gamma(\boldsymbol{w}), \tag{13}$$

Portfolio Optimization

$$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \mathcal{R}_{\varphi,\beta}(\boldsymbol{w}^\top \boldsymbol{L}), \tag{14}$$

Regression

$$\min_{f\in\mathcal{F},C} \mathcal{E}_{\varphi,\beta}(Z_f)), \tag{15}$$

Robust expected margin maximization

$$\min_{\boldsymbol{w}} \max_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[-QL(\boldsymbol{w}, b)] + \gamma(\boldsymbol{w}). \tag{16}$$

DR expected margin minimization

$$\min_{\boldsymbol{w}} \max_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[-L(\boldsymbol{w}, b)] + \gamma(\boldsymbol{w}). \tag{17}$$

Robust loss minimization

$$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \max_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[Q\boldsymbol{w}^\top \boldsymbol{L}]. \tag{18}$$

DRO

$$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \max_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[\boldsymbol{w}^\top \boldsymbol{L}] \tag{19}$$

Deviation minimization

$$\min_f \left\{ \max_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[Q\bar{Z}_f] - \mathbb{E}[\bar{Z}_f] \right\} \quad \text{calculate } C = \mathcal{S}(\bar{Z}_f) \tag{20}$$

Deviation minimization

$$\min_f \left\{ \max_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[\bar{Z}_f] - \mathbb{E}[\bar{Z}_f] \right\} \quad \text{calculate } C = \mathcal{S}(\bar{Z}_f) \tag{21}$$

6.1. Examples of Robust Optimization Interpretation

Example 6.1 (mean quadrangle)**.** The risk and error measure in mean quadrangle are objective functions for Large Margin Distribution Machine (Zhang & Zhou, 2014), Markowitz portfolio optimization (Markowitz, 1952), and least squares regression. We have the following robust optimization interpretation.

Large Margin Distribution Machine

$$\min_{\boldsymbol{w},b} \mathbb{E}[-L(\boldsymbol{w}, b)] + \sqrt{\beta}\sigma(-L(\boldsymbol{w}, b)) + \gamma(\boldsymbol{w}),$$

Robust expected margin maximization

$$\min_{\boldsymbol{w},b} \max_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[-QL(\boldsymbol{w}, b)] + \gamma(\boldsymbol{w}).$$

Markowitz portfolio optimization

$$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \mathbb{E}[\boldsymbol{w}^\top \boldsymbol{L}] + \sqrt{\beta}\sigma(\boldsymbol{w}^\top \boldsymbol{L}),$$

Robust expected loss minimization

$$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \max_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[Q(\boldsymbol{w}^\top \boldsymbol{L})].$$

Least squares regression	**Deviation minimization**
$\min_{f\in\mathcal{F},C\in\mathbb{R}} \sqrt{\beta}\|\|Z_f\|\|_2$,	$\min_{f\in\mathcal{F}} \max_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[Q\bar{Z}_f]-\mathbb{E}[\bar{Z}_f]$ calculate $C=\mathbb{E}[\bar{Z}_f]$.

Example 6.2 (quantile quadrangle)**.** The risk and error measure in quantile quadrangle are objective functions of ν-support vector machine (Schölkopf et al., 2000), CVaR optimization (Rockafellar & Uryasev, 2000), and quantile regression (Koenker & Bassett Jr, 1978). Let $\nu = 1-\alpha$. The equivalence of ν-SVM and CVaR optimization is studied by Gotoh & Takeda (2004); Takeda & Sugiyama (2008). We have the following DRO interpretation.

CVaR portfolio optimization	**DR loss minimization**
$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \mathrm{CVaR}_\alpha(X(\boldsymbol{w}))$,	$\min_{\mathbf{1}^\top \boldsymbol{w}=1} \max_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[X(\boldsymbol{w})]$,
ν-SVM	**DR expected margin maximization**
$\min_{\boldsymbol{w},b} \mathrm{CVaR}_\alpha(-L(\boldsymbol{w},b)) + \gamma(\boldsymbol{w})$,	$\min_{\boldsymbol{w}} \max_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[-L(\boldsymbol{w},b)] + \gamma(\boldsymbol{w})$.
Quantile regression	**Deviation minimization**
$\min_{f\in\mathcal{F},C\in\mathbb{R}} \mathcal{E}_\alpha(Z_f)$,	$\min_{f\in\mathcal{F}} \max_{P\in\mathcal{P}_{\varphi,\beta}} \mathbb{E}_P[\bar{Z}_f]-\mathbb{E}[\bar{Z}_f]$ calculate $C\in\mathrm{VaR}_\alpha[\bar{Z}_f]$,

where $X_+ = \max\{0, X\}$, $X_- = \max\{0, -X\}$, $\mathcal{E}(X) = \left[\frac{\alpha}{1-\alpha}X_+ + X_-\right]$ is the normalized Koenker-Bassett error, VaR is Value-at-Risk.

7. Recovering φ-Divergence from Quadrangle Elements

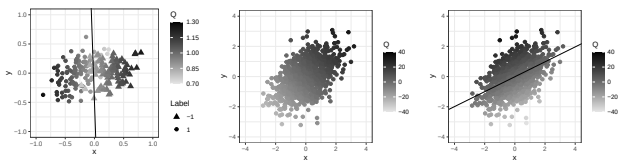
This work has focused on developing new risk quadrangles induced by a specified φ-divergence. It is of independent interest to establish the duality between divergence and risk quadrangle, which allows for recovering the φ-divergence from the elements of the corresponding φ-divergence quadrangle. The proof, based on Föllmer & Knispel (2011), is provided in the Appendix H.

Proposition 7.1. *Let $\varphi(x)$ be a divergence function. φ-divergence can be recovered from the elements in φ-divergence quadrangle by*

$$
\begin{aligned}
D_\varphi(P\|P_0) &= \sup_{\substack{X\in\mathcal{L}^2\\ \beta>0}} \{\mathbb{E}[XQ]-\mathcal{R}_{\varphi,\beta}(X)-\beta\}\\
&= \sup_{\substack{X\in\mathcal{L}^2\\ \beta>0}} \{\mathbb{E}[X(Q-1)]-\mathcal{D}_{\varphi,\beta}(X)-\beta\}\\
&= \sup_{\substack{X\in\mathcal{L}^2\\ \beta>0,C}} \{\mathbb{E}[XQ]-\mathcal{V}_{\varphi,\beta}(X-C)+C-\beta\}\\
&= \sup_{\substack{X\in\mathcal{L}^2\\ \beta>0,C}} \{\mathbb{E}[X(Q-1)]-\mathcal{E}_{\varphi,\beta}(X-C)-\beta\}.
\end{aligned}
$$

8. Risk Identifier Visualization

This section contains three case studies visualizing the risk envelope (Definition 2.11) in classification (16), portfolio optimization (18), and regression (20). We focus on the mean quadrangle (Example 5.2) in this case study. The details of the experiments are specified in Appendix J.



(a) Risk envelope in Large Margin Distribution Machine. (b) Risk envelope in Markowitz portfolio optimization. (c) Risk envelope in least squares regression.

Figure 1: Darker points correspond to higher values of $Q^*(w)$ in all figures. In **(a)**, the circles represent samples with label 1, while the diamonds represent samples with label -1. The optimal decision line is $y = -28.826x - 0.486$. In **(b)**, optimal portfolio weights = $(0.4999038, 0.5000962)$. In **(c)**, the straight line is the least squares regression line $y = 0.495x - 0.0127$.

From the minimax formulation in the dual representation (4.3), we observe that a larger incurred loss corresponds to a larger weight being assigned to the data point. This observation is confirmed by the figures. In Figure 1a, misclassified points with a large margin are assigned larger weights. In Figure 1b, points in the upper-right corner, corresponding to large portfolio losses, are assigned larger weights. In Figure 1c, points further above the regression line are assigned larger weights.

9. Conclusion

We introduce the extended φ-divergence risk measure and complete its associated risk quadrangle. The inner maximization problem of DRO is integrated as a special case of the risk measure. The extended φ-divergence quadrangle encompasses many important quadrangles, whose elements are used as objective functions in well-known learning tasks in classification, portfolio optimization, and regression. The FRQ framework connects the elements axiomatically and provides a RO/DRO interpretation to the tasks. A case study is conducted to visualize the worst-case weight.

Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none which we feel must be specifically highlighted here.

References

Ahmadi-Javid, A. Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization Theory and Applications*, 155:1105–1123, 2012.

Ang, M., Sun, J., and Yao, Q. On the dual representation of coherent risk measures. *Annals of Operations Research*, 262:29–46, 2018.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. Coherent measures of risk. *Mathematical finance*, 9(3):203–228, 1999.

Bauschke, H. H. and Combettes, P. L. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, New York, NY, 2011. ISBN 978-1-4419-9466-8.

Bayraksan, G. and Love, D. K. Data-driven stochastic programming using phi-divergences. In *The operations research revolution*, pp. 1–19. INFORMS, 2015.

Bellini, F., Klar, B., Müller, A., and Gianin, E. R. Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, 54:41–48, 2014.

Ben-Tal, A. and Teboulle, M. An old-new concept of convex risk measures: The optimized certainty equivalent. *Mathematical Finance*, 17(3):449–476, 2007.

Ben-Tal, A., Den Hertog, D., De Waegenaere, A., Melenberg, B., and Rennen, G. Robust solutions of optimization problems affected by uncertain probabilities. *Management Science*, 59(2):341–357, 2013.

Csiszár, I. Eine informations theoretische ungleichung und ihre anwendung auf den beweis der ergodizitat von markoffschen ketten. *Mayar Tud. Acad. Mat. Kutato Int. Kozl.*, 8:85–108, 1963.

Dacorogna, B. and Maréchal, P. The role of perspective functions in convexity, polyconvexity, rank-one convexity and separate convexity. *Journal of Convex Analysis*, 15 (2):271–284, 2008.

Dommel, P. and Pichler, A. Convex Risk Measures based on Divergence. Papers 2003.07648, arXiv.org, March 2020. URL https://ideas.repec.org/p/arx/papers/2003.07648.html.

Duchi, J. and Namkoong, H. Variance-based regularization with convex objectives. *Journal of Machine Learning Research*, 20(68):1–55, 2019. URL http://jmlr.org/papers/v20/17-750.html.

Föllmer, H. and Knispel, T. Entropic risk measures: Coherence vs. convexity, model ambiguity and robust large deviations. *Stochastics and Dynamics*, 11(02n03):333–351, 2011.

Fröhlich, C. and Williamson, R. C. Risk measures and upper probabilities: Coherence and stratification. *arXiv preprint arXiv:2206.03183*, 2022a.

Fröhlich, C. and Williamson, R. C. Tailoring to the tails: Risk measures for fine-grained tail sensitivity. *arXiv preprint arXiv:2208.03066*, 2022b.

Gotoh, J.-y. and Takeda, A. A linear classification model based on conditional geometric score. 2004.

Gotoh, J.-y. and Uryasev, S. Support vector machines based on convex risk functions and general norms. *Annals of Operations Research*, 249:301–328, 2017.

Koenker, R. and Bassett Jr, G. Regression quantiles. *Econometrica: journal of the Econometric Society*, pp. 33–50, 1978.

Krokhmal, P. A. Higher moment coherent risk measures. 2007.

Kuhn, D., Shafiee, S., and Wiesemann, W. Distributionally robust optimization, 2024. URL https://arxiv.org/abs/2411.02549.

Lam, H. Robust sensitivity analysis for stochastic systems. *Mathematics of Operations Research*, 41(4):1248–1275, 2016. doi: 10.1287/moor.2015.0776. URL https://doi.org/10.1287/moor.2015.0776.

Love, D. and Bayraksan, G. Phi-divergence constrained ambiguous stochastic programs for data-driven optimization. *Technical report, Department of Integrated Systems Engineering, The Ohio State University, Columbus, Ohio*, 2015.

Malandii, A., Kuzmenko, V., and Uryasev, S. Expectile risk quadrangles and applications. *Working Paper*, 2024.

Markowitz, H. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952. ISSN 00221082, 15406261. URL http://www.jstor.org/stable/2975974.

Morimoto, T. Markov processes and the h-theorem. *Journal of the Physical Society of Japan*, 18(3):328–331, 1963.

Pflug, G. C. and Ruszczynski, A. A risk measure for income processes. 2004.

Rockafellar, R. and Wets, R. J.-B. *Variational Analysis*. Springer Verlag, Heidelberg, Berlin, New York, 1998.

Rockafellar, R. T. *Convex Analysis*. Princeton University Press, Princeton, NJ, 1970.

Rockafellar, R. T. Integral functionals, normal integrands and measurable selections. In *Nonlinear Operators and the Calculus of Variations*, pp. 157–207. Springer Berlin Heidelberg, Berlin, Heidelberg, 1976. ISBN 978-3-540-38075-7.

Rockafellar, R. T. Risk and utility in the duality framework of convex analysis. *From Analysis to Visualization: A Celebration of the Life and Legacy of Jonathan M. Borwein, Callaghan, Australia, September 2017*, pp. 21–42, 2020.

Rockafellar, R. T. Distributional robustness, stochastic divergences, and the quadrangle of risk. 2023.

Rockafellar, R. T. and Uryasev, S. Optimization of conditional value-at-risk. *Journal of risk*, 2:21–42, 2000.

Rockafellar, R. T. and Uryasev, S. The fundamental risk quadrangle in risk management, optimization and statistical estimation. *Surveys in Operations Research and Management Science*, 18(1-2):33–53, oct 2013. doi: 10.1016/j.sorms.2013.03.001. URL https://doi.org/10.1016/j.sorms.2013.03.001.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. Generalized deviations in risk analysis. *Finance and Stochastics*, 10:51–74, 2006.

Rockafellar, R. T., Uryasev, S., and Zabarankin, M. Risk tuning with generalized linear regression. *Mathematics of Operations Research*, 33(3):712–729, 2008.

Royset, J. O. and Wets, R. J.-B. *An optimization primer*. Springer, 2022.

Schölkopf, B., Smola, A. J., Williamson, R. C., and Bartlett, P. L. New support vector algorithms. *Neural computation*, 12(5):1207–1245, 2000.

Shapiro, A. Distributionally robust stochastic programming. *SIAM Journal on Optimization*, 27(4):2258–2275, 2017.

Shapiro, A., Dentcheva, D., and Ruszczyński, A. *Lectures on Stochastic Programming: Modeling and Theory, Second Edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2014. doi: 10.1137/1.9781611973433. URL https://epubs.siam.org/doi/abs/10.1137/1.9781611973433.

Sun, J., Yang, X., Yao, Q., and Zhang, M. Risk minimization, regret minimization and progressive hedging algorithms. *Mathematical Programming*, 181:509–530, 2020. doi: 10.1007/s10107-020-01471-8.

Takeda, A. and Sugiyama, M. ν-support vector machine as conditional value-at-risk minimization. In *Proceedings of the 25th international conference on Machine learning*, pp. 1056–1063, 2008.

Zhang, T. and Zhou, Z.-H. Large margin distribution machine. In *Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining*, pp. 313–322, 2014.

A. Appendix

Rockafellar & Uryasev (2013) introduced measures of uncertainty that are built upon the concept of regularity, which is closely linked to convexity and closedness.

Uncertainty can be modeled via random variables and by studying and estimating the statistical properties of these random variables, we can estimate the risk in one form or the other. When the aim is to estimate the risk, it is convenient to think of the the random variable as 'loss' or 'cost'. There are various ways in which risk can be quantified and expressed. One such framework developed by Rockafellar & Uryasev (2013) is called the *Risk Quadrangle*, which is shown in Figure 2.

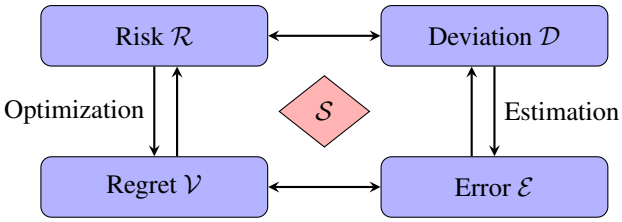


Figure 2: Risk Quadrangle Flowchart

The quadrangle begins from the upper left corner which depicts the *measure of risk* denoted by $\mathcal{R}$. It aggregates the uncertainty in losses into a numerical value $\mathcal{R}(X)$ by the inequality $\mathcal{R}(X) \leq C$ where C is the tolerance level for the risk. The next term is in the upper-right corner called the *measure of deviation* denoted by $\mathcal{D}$ and it quantifies the nonconstancy of the random variable. The lower-left corner depicts *measure of regret* denoted by $\mathcal{V}$. It stands for the net displeasure perceived in the potential mix of outcomes of a random variable "loss" which can be bad (> 0) or acceptable/good (≥ 0). The last measure is the *measure or error* which sits as the right-bottom of the quadrangle denoted by $\mathcal{E}$. Error quantifies the non-zeroness in the random variable.

Theorem A.1 (Quadrangle Theorem, Rockafellar & Uryasev (2013))**.** *The theorem states the following:*

(a) The centerness relations $\mathcal{D}(X) = \mathcal{R}(X) - \mathbb{E}[X]$ and $\mathcal{R}(X) = \mathbb{E}[X] + \mathcal{D}(X)$ give a one-to-one correspondence between regular measures of risk $\mathcal{R}$ and regular measures of deviation $\mathcal{D}$. In this correspondence, $\mathcal{R}$ is positively homogeneous if and only if $\mathcal{D}$ is positively homogeneous. On the other hand, $\mathcal{R}$ is monotonic if and only if $\mathcal{D}(X) \leq \sup X - \mathbb{E}[X]$ for all X.

(b) The relations $\mathcal{E}(X) = \mathcal{V}(X) - \mathbb{E}[X]$ and $\mathcal{V}(X) = \mathbb{E}[X] + \mathcal{E}(X)$ give a one-to-one correspondence between regular measures of regret V and regular measures of error $\mathcal{E}$. In this correspondence, $\mathcal{V}$ is positively homogeneous if and only if $\mathcal{E}$ is positively homogeneous. On the other hand, $\mathcal{V}$ is monotonic if and only if $\mathcal{E}(X) \leq |\mathbb{E}[X]|$ for $X \leq 0$.

(c) For any regular measure of regret $\mathcal{V}$, a regular measure of risk $\mathcal{E}$ is obtained by:

$$\mathcal{R}(X) = \min_{C \in \mathbb{R}} \{C + \mathcal{V}(X - C)\} \ .$$

If $\mathcal{V}$ is positively homogeneous, $\mathcal{R}$ is positively homogeneous. If $\mathcal{V}$ is monotonic, $\mathcal{R}$ is monotonic.

(d) For any regular measure of error $\mathcal{E}$, a regular measure of deviation $\mathcal{D}$ is obtained by

$$\mathcal{D}(X) = \min_{C \in \mathbb{R}} \{\mathcal{E}(X - C)\} \ .$$

If $\mathcal{E}$ is positively homogeneous, $\mathcal{D}$ is positively homogeneous. If $\mathcal{E}$ satisfies the condition $\mathcal{E}(X) \leq |\mathbb{E}[X]|$ for $X \leq 0$, then $\mathcal{D}$ satisfies the condition $\mathcal{D}(X) \leq \sup X - \mathbb{E}[X]$ for all X.

(e) In both (c) and (d), as long as the expression being minimized is finite for some C , the set of C values for which the minimum is attained is a nonempty, closed, bounded interval. Moreover when $\mathcal{V}$ and $\mathcal{E}$ are paired as in (b), the interval comes out the same and gives the associated statistic:

$$\operatorname*{argmin}_{C \in \mathbb{R}} \{C + \mathcal{V}(X - C)\} = \mathcal{S}(X) = \operatorname*{argmin}_{C \in \mathbb{R}} \{\mathcal{E}(X - C)\} \ .$$

Theorem A.2 (Error Projection and Certainty Equivalence)**.** *In order to establish the validity of a given quartet $(\mathcal{R}, \mathcal{D}, \mathcal{V}, \mathcal{E})$ as a quadrangle, it is sufficient to demonstrate the satisfaction of either conditions (Q1) and (Q3), or conditions (Q2) and (Q3), as conditions (Q1) and (Q2) are intrinsically linked through the condition (Q3). Indeed, $\mathcal{R}(X) = \inf_C \{ C + \mathcal{V}(X - C) \} = \inf_C \{ \mathcal{E}(X - C) \} + \mathbb{E}[X] = \mathcal{D}(X) + \mathbb{E}[X]$.*

B. Proof of Theorem 4.4

Proof. First, we verify the conditions for regular risk measure in Definition 2.4.

Closedness and Convexity: Since the envelope $\mathcal{Q}$ is closed and convex ((Rockafellar et al., 2006; Rockafellar & Uryasev, 2013)), then $\mathcal{R}_{\varphi,\beta}(X)$ is closed (lower semicontinuous) and convex as a maximum of continuous affine functions.

Constancy: Constancy is implied by the condition $\mathbb{E}Q = 1$,

$$\sup_{Q\in\mathcal{Q}^1_{\varphi,\beta}} \mathbb{E}[CQ] = \sup_{Q\in\mathcal{Q}^1_{\varphi,\beta}} C\,\mathbb{E}[Q] = C\ .$$

Risk aversity: We can construct a Q_0 such that the strict inequality holds for $\mathcal{R}_{\varphi,\beta}(X) > \mathbb{E}[X]$. As a function of r, $P(X \le r)$ is a nondecreasing, right-continuous function with a range in $[0,1]$. Thus for a nonconstant X, there exists $r \in \mathbb{R}$, $p \in (0,1)$ such that $P(X \le r) = p$, $P(X > r) = 1-p$. By convexity of $\varphi(x)$ and $1 \subset \mathrm{int}(\{x : \varphi(x) < +\infty\})$, there exists $\delta > 0$ such that $\varphi(x) \le \beta$ for $x \in (1-\delta, 1+\delta)$. Then, there exists $\delta_1 \in (0,\delta)$, $\delta_2 \in (0,\delta)$ such that $\delta_1 = \frac{1-p}{p}\delta_2$. Define Q_0 by

$$Q_0(\omega) = \begin{cases} 1-\delta_1, & \omega : X(\omega) \le r \\ 1+\delta_2, & \omega : X(\omega) > r \end{cases}\ . \tag{22}$$

The feasibility can be checked by $\mathbb{E}[\varphi(Q_0)] \le \beta$, $\mathbb{E}Q_0 = 1$.

We have

$$\begin{aligned}
\mathbb{E}[XQ_0] =& \mathbb{E}[XQ_0|X \le r]P(X \le r) + \mathbb{E}[XQ_0|X > r]P(X > r) && (23)\\
=& p(1-\delta_1)\mathbb{E}[X|X \le r] + (1-p)(1+\delta_2)\mathbb{E}[X|X > r] && (24)\\
=& p\mathbb{E}[X|X \le r] + (1-p)\mathbb{E}[X|X > r] - p\delta_1\mathbb{E}[X|X \le r] + (1-p)\delta_2\mathbb{E}[X|X > r] && (25)\\
=& \mathbb{E}[X] + p\delta_1(\mathbb{E}[X|X > r] - \mathbb{E}[X|X \le r]) && (26)\\
>& \mathbb{E}[X]\ . && (27)
\end{aligned}$$

Thus $\mathcal{R}_{\varphi,\beta}(X)$ is a regular risk measure.

Next, we verify the conditions for regular regret measure.

Closedness and Convexity: Same with the proof above for regular risk measure.

Risk aversity: For $X \ne const$,

$$\mathcal{V}_{\beta,\varphi}(X) \ge \mathcal{R}_{\beta,\varphi}(X) > \mathbb{E}X. \tag{28}$$

The first inequality is due to $\mathcal{Q}^1_{\varphi,\beta} \subset \mathcal{Q}_{\varphi,\beta}$.

Zeroness:

$$\sup_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[0\cdot Q] = 0. \tag{29}$$

The proof of Theorem 1 in Sun et al. (2020) (which works on coherent risk measure) can be applied here to show that a regular regret measure can be obtained by removing condition $\mathbb{E}Q = 1$ in 6. Thus the risk 1 and regret 2 satisfies (Q2) in Definition 2.8.

Deviation 3 and error 4 measure are obtained by centerness formulae (Q3) (see Definition 2.8). With Theorem A.1, we can show the regularity of deviation and error, and that the minimum in C for a regular regret measure is attainable. The optimal C is $\mathcal{S}_{\varphi,\beta}(X)$. □

The proof of aversity of the risk measure constructs a feasible random variable inspired by Ang et al. (2018). The proof of the relation between risk and regret follows Sun et al. (2020). Ang et al. (2018); Sun et al. (2020) work with coherent risk measures. The proving techniques are of broader interest. Ang et al. (2018) proves that 1 being a relative interior point of

the envelope $\mathcal{Q}$ is sufficient for a coherent risk measure to be risk averse. Sun et al. (2020) proves that removing $\mathbb{E}Q = 1$ in the envelope of coherent risk measure generates a coherent regret measure.

Proposition 4.1 in Artzner et al. (1999) proves the coherency of risk measures that have representation $\sup_{P\in\mathcal{P}} \mathbb{E}_P[X]$ for any set $\mathcal{P}$. The setting in Artzner et al. (1999) is finite $\mathcal{R}(X)$ and finite Ω.

An alternative proof of risk 1 and regret 2 satisfying (Q2) in Definition 2.8 can be obtained from the primal representations in Section 4.3. The relation (Q2) can be directly observed from the primal risk and regret.

C. Proof of Theorem 4.6

Proof. Consider the regret 2

$$\mathcal{V}_{\varphi,\beta} = \sup_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[XQ] = -\inf_{Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[-QX]. \tag{30}$$

Consider the Lagrangian dual problem of $\inf_{Q:Q\in\mathcal{Q}_{\varphi,\beta}} \mathbb{E}[-QX]$

$$\sup_{t\geq 0}\inf_{Q} \left\{\mathbb{E}[-XQ] + t\left(\mathbb{E}[\varphi(Q)] - \beta\right)\right\}. \tag{31}$$

Denote the optimal t by t^*. If $t^* = 0$, then

$$\sup_{t\geq 0}\inf_{Q} \left\{\mathbb{E}[-XQ] + t\left(\mathbb{E}[\varphi(Q)] - \beta\right)\right\} = \inf_{Q} \mathbb{E}[-XQ] = -\infty. \tag{32}$$

Thus for all $t \geq 0$,

$$\inf_{Q} \left\{\mathbb{E}[-XQ] + t\left(\mathbb{E}[\varphi(Q)] - \beta\right)\right\} = -\infty\ . \tag{33}$$

Thus if $t^* = 0$, the optimum is also attained at $t > 0$. If $t^* > 0$, $t > 0$ and $t \geq 0$ are the same for the problem. Thus, we can substitute $t \geq 0$ with $t > 0$ in the Lagrange dual problem.

Then,

$$\sup_{t>0}\inf_{Q} \left\{\mathbb{E}[-XQ] + t\left(\mathbb{E}[\varphi(Q)] - \beta\right)\right\} \tag{34}$$

$$= \sup_{t>0}\inf_{Q} (-t)\left\{\mathbb{E}\left[\frac{X}{t}Q - \varphi(Q)\right] + \beta\right\} \tag{35}$$

$$= -\inf_{t>0}\sup_{Q} t\left\{\mathbb{E}\left[\frac{X}{t}Q - \varphi(Q)\right] + \beta\right\}\ . \tag{36}$$

Next, we prove that

$$-\inf_{t>0}\sup_{Q} t\left\{\mathbb{E}\left[\frac{X}{t}Q - \varphi(Q)\right] + \beta\right\} = -\inf_{t>0} t\left\{\beta + \mathbb{E}\varphi^*\left(\frac{X}{t}\right)\right\}. \tag{37}$$

We consider two cases where the following condition is satisfied and not satisfied

$$\sup_{Q}\left\{\mathbb{E}\left[\frac{X}{t}Q - \varphi(Q)\right]\right\} < +\infty \quad \text{for some } t\ . \tag{38}$$

When 38 is satisfied, since $XQ/t - \varphi(Q)$ is a normal convex integrand (Shapiro, 2017), sup and expectation in 36 are exchangeable by Theorem 3A in Rockafellar (1976). Thus, 37 holds.

When 38 is not satisfied, $\sup_Q\{\mathbb{E}[XQ/t - \varphi(Q)]\} = +\infty$ for all t. We have

$$-\inf_{t>0}\sup_{Q} t\left\{\mathbb{E}\left[\left(\frac{X}{t}\right)Q - \varphi(Q)\right] + \beta\right\} = -\infty.$$

We also have that

$$t\left(\mathbb{E}\varphi^*\left(\frac{X}{t}\right)+\beta\right) = t\left(\mathbb{E}\left[\sup_Q\left\{\left(\frac{X}{t}\right)Q-\varphi(Q)\right\}\right]+\beta\right) \tag{39}$$

$$\geq \sup_Q t\left\{\mathbb{E}\left[\left(\frac{X}{t}\right)Q-\varphi(Q)\right]+\beta\right\} \tag{40}$$

$$= +\infty. \tag{41}$$

Thus

$$-\inf_{t>0} t\left(\mathbb{E}\varphi^*\left(\frac{X}{t}\right)+\beta\right) = -\infty. \tag{42}$$

We see that 37 holds with or without the condition 38. With 34–36, 37, we obtain

$$\sup_{t>0}\inf_Q\left\{\mathbb{E}[-XQ]+t\left(\mathbb{E}[\varphi(Q)]-\beta\right)\right\} = -\inf_{t>0} t\left\{\beta+\mathbb{E}\varphi^*\left(\frac{X}{t}\right)\right\}. \tag{43}$$

Strong duality for the convex problem holds since the following Slater's condition is valid for $Q=1$

$$\exists Q\ :\ Q\in\mathcal{Q}_{\varphi,\beta},\ \mathbb{E}[\varphi(Q)]<\beta\,. \tag{44}$$

Thus

$$\mathcal{V}_{\varphi,\beta} = -\inf_{Q\in\mathcal{Q}_{\varphi,\beta}}\mathbb{E}[-QX] = \inf_{t>0} t\left\{\beta+\mathbb{E}\varphi^*\left(\frac{X}{t}\right)\right\}. \tag{45}$$

By regularity, the statistic $S_{\varphi,\beta}(X)$ is attainable. Denote the optimal C and t by C^* and t^*. If $t^*>0$, $\frac{S_{\varphi,\beta}(X)}{t^*}$ is attainable. We showed that if $t^*=0$, any $t>0$ is also optimal. $\frac{S_{\varphi,\beta}(X)}{t^*}$ is attainable. By change of variable, C^* in 8,9 equals $\frac{S_{\varphi,\beta}(X)}{t^*}$. Thus t^* and C^* in 8–12 are attainable.

The primal representation of the other elements can be obtained similarly by Lagrange dual problem, or by direct calculation using the quadrangle relations in Definition 2.8. □

The primal representation of the risk measure 8 has been studied in the literature under different technical conditions. Fröhlich & Williamson (2022b) starts with the primal representation of coherent regret and obtains the coherent risk with (Q3) centerness relation in Definition 2.8.

D. More Examples

Example D.1 (Expectile Quadrangle Generated by Generalized Pearson χ^2-divergence)**.** Let $0<p<1$. Consider the following extended divergence function and its convex conjugate

$$\varphi(x)=\begin{cases}\frac{1}{q}(x-1)^2, & x>1\\ \frac{1}{1-q}(x-1)^2, & x\leq 1\end{cases}\quad\text{and}\quad \varphi^*(z)=\begin{cases}\frac{qz^2}{4}+z, & z>0\\ \frac{(1-q)z^2}{4}+z, & z\leq 0\end{cases}. \tag{46}$$

The complete quadrangle is as follows

$$\mathcal{R}_{\varphi,\beta}(X) = q\mathbb{E}[(((X-e_q(X))_+)^2]+(1-q)\mathbb{E}[(((X-e_q(X))_-)^2]+\mathbb{E}[X]$$

$$\mathcal{V}_{\varphi,\beta}(X) = \mathbb{E}[X]+\sqrt{\beta E\left[qX_-^2+(1-q)X_+^2\right]}$$

$$\mathcal{D}_{\varphi,\beta}(X) = q\mathbb{E}[(((X-e_q(X))_+)^2]+(1-q)\mathbb{E}[(((X-e_q(X))_-)^2]$$

$$\mathcal{E}_{\varphi,\beta}(X) = \sqrt{\beta E\left[qX_-^2+(1-q)X_+^2\right]} = \text{ asymmetric squared loss, scaled}$$

$$\mathcal{S}_{\varphi,\beta}(X) = e_q(X) = \text{ expectile}$$

We recover one version of expectile quadrangle in Malandii et al. (2024). The divergence function $\varphi(x)$ gives rise to a generalized Pearson χ^2-divergence. Example 5.2 is a special case of this quadrangle with $q=0.5$.

Example D.2 (Example Generated by finite-interval-indicator Divergence)**.** Let $0 < a < 1 < b$. The divergence function and its convex conjugate are

$$\varphi(x) = \begin{cases} +\infty, & x \in [0, a) \\ 0, & x \in [a, b] \\ +\infty, & x \in (b, +\infty) \end{cases}, \quad \varphi^*(z) = \begin{cases} az, & z < 0 \\ bz, & z \geq 0 \end{cases}. \tag{47}$$

The error measure is

$$\mathcal{E}_{\varphi,\beta}(X) = \mathbb{E}[(1-a)X_- + (b-1)X_+] \,. \tag{48}$$

The complete quadrangle is

$$\begin{aligned} \mathcal{R}_{\varphi,\beta}(X) &= (1-a)\mathrm{CVaR}_{\frac{b-1}{b-a}}(X) + a\mathbb{E}[X], & \mathcal{V}_{\varphi,\beta}(X) &= \mathbb{E}[(2-a)X_- + bX_+], \\ \mathcal{D}_{\varphi,\beta}(X) &= (1-a)\mathrm{CVaR}_{\frac{b-1}{b-a}}(X) + (a-1)\mathbb{E}[X], & \mathcal{E}_{\varphi,\beta}(X) &= \mathbb{E}[(1-a)X_- + (b-1)X_+], \end{aligned}$$
$$\mathcal{S}_{\varphi,\beta}(X) = \mathrm{argmin}_{C\in\mathbb{R}}\, \mathbb{E}[(1-a)(X-C)_- + (b-1)(X-C)_+] \,,$$

The risk measure in this quadrangle is studied in Pflug & Ruszczynski (2004), Ben-Tal & Teboulle (2007) (see Example 2.3), Love & Bayraksan (2015) (see Example 3). CVaR is a special case of this risk measure for $a = 0$. When $\alpha/(1-\alpha) = (b-1)/(1-a)$, the quadrangle is a scaled version of Example 5.3.

The risk measure provides another way to connect expectile $e_q(X)$ with distributionally robust optimization (see Proposition 9 in Bellini et al. (2014))

$$e_q(X) = \max_{\gamma\in[\frac{1-q}{q},1]} \mathcal{R}_{I_{[\gamma,\gamma\frac{q}{1-q}]},\beta}(X) \,. \tag{49}$$

E. Derivation for Examples

This section contains derivations of the examples in Section 5.

E.1. Example 5.1

The regret measure is given by

$$\begin{aligned} \mathcal{V}_{\varphi,\beta}(X) &= \inf_{\substack{t>0 \\ t\geq -\mathrm{ess\,inf}\, X \\ t\geq \mathrm{ess\,sup}\, X}} \{t\beta + \mathbb{E}[X]\} && (50) \\ &= \beta \max\{0, -\mathrm{ess\,inf}\, X, \mathrm{ess\,sup}\, X\} + \mathbb{E}[X] && (51) \\ &= \beta\, \mathrm{ess\,sup}\, |X| + \mathbb{E}[X] \,. && (52) \end{aligned}$$

The risk measure is given by

$$\begin{aligned} \mathcal{R}_{\varphi,\beta}(X) &= \inf_{\substack{t>0,\, C\in\mathbb{R} \\ t(C-1)\leq \mathrm{ess\,inf}\, X \\ t(C+1)\geq \mathrm{ess\,sup}\, X}} \{t\beta + tC + \mathbb{E}[X - tC]\} && (53) \\ &= \frac{\beta}{2}(\mathrm{ess\,sup}\, X - \mathrm{ess\,inf}\, X) + \mathbb{E}[X] \,. && (54) \end{aligned}$$

From the constraints $t(C-1) \leq \mathrm{ess\,inf}\, X$ and $t(C+1) \geq \mathrm{ess\,sup}\, X$, we have

$$2t \geq \mathrm{ess\,sup}\, X - \mathrm{ess\,inf}\, X,$$

hence the optimal

$$t^* = (\mathrm{ess\,sup}\, X - \mathrm{ess\,inf}\, X)/2.$$

From the constraints, we have

$$2\text{ess sup}\, X/(\text{ess sup}\, X - \text{ess inf}\, X) - 1 \le C \le 2\text{ess inf}\, X/(\text{ess sup}\, X - \text{ess inf}\, X) + 1.$$

Thus,

$$(\text{ess sup}\, X + \text{ess inf}\, X)/(\text{ess sup}\, X - \text{ess inf}\, X) \ge C \ge (\text{ess sup}\, X + \text{ess inf}\, X)/(\text{ess sup}\, X - \text{ess inf}\, X),$$

yelding

$$C^* = (\text{ess sup}\, X + \text{ess inf}\, X)/(\text{ess sup}\, X - \text{ess inf}\, X).$$

Therefore, the statistic

$$\mathcal{S}_{\varphi,\beta} = C^* t^* = (\text{ess sup}\, X + \text{ess inf}\, X)/2.$$

E.2. Example 5.4

The equation for the statistic can be obtained from the second equation in 74 when $\varphi^*(z) = \exp(z) - 1$.

E.3. Example 5.5

The risk measure is given by

$$\begin{aligned}
\mathcal{R}_{\varphi,\beta}(X) &= \inf_{\substack{t>0,\, C\in\mathbb{R} \\ \text{ess sup}(X-C)\le t}} \{t\beta + C - t + \mathbb{E}[X - C + t]_+\} \\
&= \inf_{\substack{t>0,\, C\in\mathbb{R} \\ \text{ess sup}(X-C-t)\le t}} \{t\beta + C + \mathbb{E}[X - C]_+\} \\
&= \inf_{\substack{t>0,\, C\in\mathbb{R} \\ \text{ess sup}(X)-2t\le C}} \{t\beta + C + \mathbb{E}[X - C]_+\}.
\end{aligned}$$

The function being minimized is convex in C. It attains minimum at $C \in (-\infty, \text{ess inf}\, X]$ if there is no constraint on C. Thus the minimum in C is attained at $C^* = \text{ess sup}(X) - 2t$. Suppose that $\beta \in (0, 2)$ (Note that TVD is no larger than 2). Then

$$\begin{aligned}
\mathcal{R}_{\varphi,\beta}(X) &= \text{ess sup}(X) + \inf_{t>0} \{t(\beta - 2) + \mathbb{E}[X - \text{ess sup}(X) + 2t]_+\} \\
&= \text{ess sup}(X) + \inf_{t<0} \left\{t(1 - \frac{\beta}{2}) + \mathbb{E}[X - \text{ess sup}(X) - t]_+\right\} \\
&= \text{ess sup}(X) + (1 - \frac{\beta}{2}) \inf_{t<0} \left\{t + (1 - \frac{\beta}{2})^{-1}\mathbb{E}[X - \text{ess sup}(X) - t]_+\right\}.
\end{aligned}$$

Note that since $X - \text{ess sup}(X) \le 0$, the minimum in the last equation is attained at some $t \le 0$, and this minimum is equal to

$$\text{CVaR}_{\frac{\beta}{2}}(X - \text{ess sup}(X)) = \text{CVaR}_{\frac{\beta}{2}}(X) - \text{ess sup}(X).$$

E.4. Example 5.2

The extended Pearson χ^2-divergence risk measure is given by

$$\begin{aligned}
\mathcal{R}_{\varphi,\beta}(X) &= \inf_{t>0, C\in\mathbb{R}} t\left\{C + \beta + \frac{1}{4t^2}\mathbb{E}[(X - C)^2] + \mathbb{E}[\frac{X - C}{t}]\right\} \\
&= \inf_{t>0, C\in\mathbb{R}} \left\{t\beta + \frac{1}{4t}\mathbb{E}[(X - C)^2] + \mathbb{E}[X]\right\} \\
&= \mathbb{E}[X] + \sqrt{\beta\mathbb{V}[X]},
\end{aligned}$$

where $\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$ is the variance of X and (t^*, C^*), which furnish the minimum are

$$t^* = \sqrt{\frac{\mathbb{V}[X]}{4\beta}}, \qquad C^* = \mathbb{E}[X].$$

Evidently, the corresponding regret is given by

$$\begin{aligned}\mathcal{V}_{\varphi,\beta}(X) &= \mathbb{E}[X] + \sqrt{\beta \mathbb{E}[X^2]} \\ &= \mathbb{E}[X] + \sqrt{\beta}\, \|X\|_2 \,.\end{aligned}$$

E.5. Example D.1

The error measure is given by

$$\mathcal{E}_{\varphi,\beta}(X) = \inf_{t>0}\ t\beta + \mathbb{E}\left[t\varphi^*\left(\frac{X}{t}\right) - X\right] \tag{55}$$

$$= \inf_{t>0}\ t\beta + \frac{1}{4t} E\left[qX_-^2 + (1-q)X_+^2\right] \tag{56}$$

$$= t\beta + \frac{1}{4t} E\left[qX_-^2 + (1-q)X_+^2\right]\Big|_{t=\sqrt{\frac{E\left[qX_-^2+(1-q)X_+^2\right]}{4\beta}}} \tag{57}$$

$$= \sqrt{\beta \mathbb{E}\left[qX_-^2 + (1-q)X_+^2\right]} \tag{58}$$

E.6. Example 5.6

Plugging in the convex conjugate, the regret measure is given by

$$\mathcal{V}_{\varphi,\beta}(X) = \inf_{t>0} t\beta + t\mathbb{E}\left[\left(\frac{1}{4}\left(\frac{X}{t}\right)^2 + \frac{X}{t}\right) I_{\{\frac{X}{t}\geq -2\}}\right] \tag{59}$$

$$= \inf_{t>0} t\beta + \frac{1}{4t}\mathbb{E}\left[\left((X+2t)^2 - 4t^2\right) I_{\{X+2t>0\}}\right]. \tag{60}$$

For the risk measure, we use an equivalent divergence function and its convex conjugate

$$\varphi(x) = \begin{cases} x^2 - 1, & x \geq 0 \\ +\infty, & x < 0 \end{cases}, \quad \varphi^*(z) = \begin{cases} \frac{z^2}{4} + 1, & z \geq 0 \\ 1, & z < 0 \end{cases} = 1 + \frac{z^2}{4} I_{z\geq 0}\,. \tag{61}$$

Plugging in the convex conjugate, the risk measure is given by

$$\mathcal{R}_{\varphi,\beta}(X) = \inf_{C\in\mathbb{R}, t>0} C + t\beta + t\mathbb{E}\left[\left(\frac{1}{4}\left(\frac{X-C}{t}\right)^2 I_{\{\frac{X-C}{t}\geq 0\}} + 1\right)\right] \tag{62}$$

$$= \min_{C\in\mathbb{R}} \inf_{t>0} C + t(\beta+1) + \frac{1}{4t}\mathbb{E}\left[(X-C)^2 I_{\{X-C>0\}}\right] \tag{63}$$

$$= \min_{C\in\mathbb{R}} C + \sqrt{(\beta+1)\mathbb{E}\left[(X-C)^2 I_{\{X-C>0\}}\right]}. \tag{64}$$

The optimal t^* is $\sqrt{\frac{\mathbb{E}\left[(X-C)^2 I_{\{X-C>0\}}\right]}{4(\beta+1)}}$. By Krokhmal (2007), the optimal C^* is the second-order quantile.

F. Proof of Equivalence

The equivalence between 15 and 20 is proved as follows. By Theorem 2.10, Problem 15 is equivalent to

$$\min_f\ \mathcal{D}_{\varphi,\beta}(\bar{Z}_f), \qquad \text{calculate } C = \mathcal{S}(\bar{Z}_f)\,. \tag{65}$$

The equivalence to Problem 20 follows from the dual representation of $\mathcal{D}_{\varphi,\beta}(X)$ in 3.

G. Statistic and Risk Identifier

G.1. Proof of Proposition I.1

Lemma G.1 (Convexity)**.** *Let $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be such that*

$$f(C,t) = C + t\beta + \mathbb{E}\Big[t\varphi^*\Big(\frac{X-C}{t}\Big)\Big]. \tag{66}$$

Then $f(C,t)$ is convex in (C,t) and

$$\partial_{(C,t)}(f(C,t)) = (1,\beta)^\top + \mathbb{E}\left[\partial_{(C,t)}\left(t\varphi^*\left(\frac{X-C}{t}\right)\right)\right], \tag{67}$$

where $\partial_{(C,t)}(f(C,t))$ denotes a subdifferential of a convex function $f(C,t)$ with respect to the vector $(C,t)^\top \in \mathbb{R} \times (0,\infty)$, cf. Definition 23.1 in Rockafellar (1970). The "+" sign in 67 is understood in the sense of the Minkowski sum.

Proof. To prove the first part of the lemma it suffices to establish that the function

$$\psi(z,t) = t\varphi^*(z/t), \quad z \in \mathbb{R},\ t \in (0,\infty)$$

is convex. This follows from the fact that the function $h(z,t) = tg(z/t),\ z \in \mathbb{R}^n,\ t > 0$ is convex if and only if g is convex. Such function h is called a perspective function, cf. Lemma 2.1 in Dacorogna & Maréchal (2008). Hence, since φ^* is convex then ψ is also convex as a perspective function. Therefore, $f(C,t)$ is convex since convexity is preserved under linear transformations.

The second part of the lemma follows from (**?**)Theorem 23]rockafellar1977. Indeed, since the function under the expectation in 66 is convex, hence measurable (cf. Rockafellar & Wets (1998)), the subdifferential can be interchanged with the expectation. □

Proposition G.2 (Proposition I.1)**.** *Let $(\mathcal{R}_{\varphi,\beta}, \mathcal{D}_{\varphi,\beta}, \mathcal{V}_{\varphi,\beta}, \mathcal{E}_{\varphi,\beta})$ be a primal extended φ-divergence quadrangle. Statistic in this quadrangle equals*

$$\mathcal{S}_{\varphi,\beta}(X) = \left\{C \in \mathbb{R} : 0 \in (1,\beta)^\top + \mathbb{E}\left[\partial_{(C,t)}\left(t\varphi^*\left(\frac{X-C}{t}\right)\right)\right]\right\}. \tag{68}$$

Proof. Definition 2.8 implies that the statistic is equal to

$$\begin{aligned}\mathcal{S}_{\varphi,\beta}(X) &= \operatorname*{argmin}_{C\in\mathbb{R}}\{\, C + \mathcal{V}_{\varphi,\beta}(X-C)\,\} \\ &= \operatorname*{argmin}_{C\in\mathbb{R}} \inf_{t>0} f(C,t)\ ,\end{aligned} \tag{69}$$

where $f(C,t) = C + t\beta + \mathbb{E}\Big[t\varphi^*\Big(\frac{X-C}{t}\Big)\Big]$. To find the statistic one has to minimize $f(C,t)$ with respect to (C,t). Since $f(C,t)$ is convex, cf. Lemma G.1, then it reaches the minimum if and only if

$$0 \in \partial_{(C,t)} f(C,t)\ . \tag{70}$$

Therefore, cf. Lemma G.1, condition 70 is equivalent to

$$0 \in (1,\beta)^\top + \mathbb{E}\left[\partial_{(C,t)}\left(t\varphi^*\left(\frac{X-C}{t}\right)\right)\right]. \tag{71}$$

□

If for an extended divergence function $\varphi(x)$, the conjugate $\varphi^*(z)$ is positive homogeneous, then the expression 12 for statistic is reduced to

$$\mathcal{S}_{\varphi,\beta}(X) = \operatorname*{argmin}_{C\in\mathbb{R}}\ C + \mathbb{E}[\varphi^*(X-C)]\ . \tag{72}$$

The Expectation Theorem in (Rockafellar & Uryasev, 2013) in this case implies

$$\mathcal{S}_{\varphi,\beta}(X) = \left\{ C \in \mathbb{R} : \mathbb{E}\left[\frac{\partial^-}{\partial z}\varphi^*(z)\Big|_{x=X-C} \right] \le 1 \le \mathbb{E}\left[\frac{\partial^+}{\partial z}\varphi^*(z)\Big|_{x=X-C} \right] \right\}, \tag{73}$$

where $\frac{\partial^-}{\partial z}, \frac{\partial^+}{\partial z}$ denote left and right derivatives with respect to $z \in \mathbb{R}$. As a finite convex homogeneous function, $\varphi^*(z)$ is the support function of a closed interval (Corollary 13.2.2, Rockafellar (1970)). The convex conjugate of a support function is an indicator function. Since $\varphi(1) = 0$, it must be in the form of the $\varphi(x)$ in Example D.2.

In fact, Dommel & Pichler (2020) provided optimality conditions for (C, t) in 8. For differentiable function φ^*, they developed a set of equations known as the characterizing equations for optimal (C, t). Further, we provide a system of equations similar to the characterizing equations developed by Dommel & Pichler (2020).

Definition G.3 (characterizing equations)**.** Let $\varphi^*(z) \in C^1(\mathbb{R})$. Characterizing system of equations is defined by:

$$\begin{cases} \mathbb{E}\left[\frac{d\varphi^*(z)}{dz}\Big|_{z=\frac{X-C}{t}} \right] = 1 \,, \\ \beta + \mathbb{E}\left[\varphi^*\left(\frac{X-C}{t} \right) \right] - \frac{1}{t}\mathbb{E}\left[(X-C)\frac{d\varphi^*(z)}{dz}\Big|_{z=\frac{X-C}{t}} \right] = 0 \,. \end{cases} \tag{74}$$

The following Corollary G.4 provides an expression for the statistic $\mathcal{S}_{\varphi,\beta}$ with smooth $\varphi^*(z)$.

Corollary G.4 (Characterization of $\mathcal{S}_{\varphi,\beta}$: Smooth Case)**.** *Let $\varphi^*(z) \in C^1(\mathbb{R})$, then the statistic equals*

$$\mathcal{S}_{\varphi,\beta}(X) = \{ C \in \mathbb{R} : (C, t) \text{ is a solution to Characterizing Equations 74 } \}.$$

Proof. Replacing the subdifferential in 71 with the gradient $\nabla_{(C,t)}$ leads to the system of equations 74. □

G.2. Proof of Proposition 4.9

Lemma G.5 (Subgradients of expectation, Bauschke & Combettes (2011))**.** *Let $(\Omega, \mathcal{A}, P_0)$ be a probability space and $\psi : \mathbb{R} \to \overline{\mathbb{R}}$ be a proper, lsc, and convex function. Set*

$$\rho = \mathbb{E}[\psi(X)]. \tag{75}$$

Then ρ is proper, convex lsc functional and, for every $X \in \mathrm{dom}(\rho)$,

$$\partial_X \rho(X) = \{ Q \in \mathcal{L}^2 : Q \in \partial\psi(X)\ \mathbb{P}_0 - a.s. \}. \tag{76}$$

Proposition G.6 (Proposition 4.9)**.** *Denote by C^* and t^* the optimal C and t in the primal representation 8 of extended φ-divergence risk measure. The risk identifier of risk measure $\mathcal{R}_{\varphi,\beta}(X)$ can be expressed as follows*

$$Q^*(\omega) \in \partial\varphi^*\left(\frac{X(\omega)}{t^*} - C^* \right). \tag{77}$$

Denote by C^* the optimal C in the primal representation 11 of extended φ-divergence risk measure. The risk identifier of extended φ-divergence error measure $\mathcal{E}_{\varphi,\beta}(X)$ can be expressed as follows

$$Q^*(\omega) \in \partial\varphi^*\left(\frac{X(\omega)}{t^*} \right). \tag{78}$$

Proof. It is known that the risk identifier is the subgradient of the risk function (see, for example, Proposition 8.36 in Royset & Wets (2022)). Therefore, 77 is obtained by taking the subdifferential of 8 following Lemma G.5. The expression 78 is obtained analogously. □

Note that the envelope $\mathcal{Q}_{\varphi,\beta}$ of error does not have the constraint $\mathbb{E}Q = 1$. However, when we minimize $\mathcal{E}_{\varphi,\beta}(X - C)$ with respect to C to get statistic $\mathcal{S}_{\varphi,\beta}(X)$, the constraint $\mathbb{E}Q = 1$ is satisfied automatically. This can be seen from the necessary condition for saddle point (C^*, Q^*)

$$\frac{\partial}{\partial C}\mathbb{E}[(X - C)(Q^* - 1)]\Big|_{C=C^*} = 0. \tag{79}$$

H. Proof of Proposition 7.1

Proposition H.1 (Proposition 7.1)**.** *Let $\varphi(x)$ be a divergence function. φ-divergence can be recovered from the elements in φ-divergence quadrangle by*

$$\begin{aligned} D_\varphi(P||P_0) &= \sup_{X\in\mathcal{L}^2,\beta>0}\{\mathbb{E}[XQ] - \mathcal{R}_{\varphi,\beta}(X) - \beta\} && (80)\\ &= \sup_{X\in\mathcal{L}^2,\beta>0}\{\mathbb{E}[X(Q-1)] - \mathcal{D}_{\varphi,\beta}(X) - \beta\} && (81)\\ &= \sup_{X\in\mathcal{L}^2,\beta>0,C}\{\mathbb{E}[XQ] - \mathcal{V}_{\varphi,\beta}(X - C) + C - \beta\} && (82)\\ &= \sup_{X\in\mathcal{L}^2,\beta>0,C}\{\mathbb{E}[X(Q-1)] - \mathcal{E}_{\varphi,\beta}(X - C) - \beta\}. && (83) \end{aligned}$$

Proof. From **??**, we have by convex conjugate

$$\mathcal{R}_{t\varphi}(X) = \inf_{\beta>0}\{t\beta + \mathcal{R}_{\varphi,\beta}(X)\}. \tag{84}$$

84 is a generalization of Proposition 3.1 in Föllmer & Knispel (2011).

Next, we have

$$\begin{aligned} \mathbb{E}[\varphi(Q)] &= \sup_{X\in\mathcal{L}^2}\{\mathbb{E}[XQ] - \mathcal{R}_\varphi(X)\} && (85)\\ &= \sup_{X\in\mathcal{L}^2}\{\mathbb{E}[XQ] - \inf_{\beta>0}\{\beta + \mathcal{R}_{\varphi,\beta}(X)\}\} && (86)\\ &= \sup_{X\in\mathcal{L}^2,\beta>0}\{\mathbb{E}[XQ] - \mathcal{R}_{\varphi,\beta}(X) - \beta\}, && (87) \end{aligned}$$

where 85 is by **??**, 86 is by plugging in 84 to **??**.

Since $\varphi(x)$ is a divergence function, $\mathbb{E}[\varphi(Q)] = D_\varphi(P||P_0)$. The rest of the proof is by quadrangle relations. □

I. Characterization of Statistics

Proposition I.1 (Characterization of $\mathcal{S}_{\varphi,\beta}$)**.** *Let $(\mathcal{R}_{\varphi,\beta}, \mathcal{D}_{\varphi,\beta}, \mathcal{V}_{\varphi,\beta}, \mathcal{E}_{\varphi,\beta})$ be a primal extended φ-divergence quadrangle. Statistic in this quadrangle equals*

$$\mathcal{S}_{\varphi,\beta}(X) = \left\{C \in \mathbb{R} : 0 \in (1,\beta)^\top + \mathbb{E}\left[\partial_{(C,t)}\left(t\varphi^*\left(\frac{X-C}{t}\right)\right)\right]\right\}.$$

J. Details of Case Study

Risk Identifier We first solve the problems 14, 13 and 15 in primal representations. With the optimal solutions, we obtain the random variable X in three problems, respectively. By plugging in $\varphi^*(z) = z^2/2 + 1$ to Proposition 4.9, we obtain the risk identifier $Q^* = (X/t^* - C^*)^2/2 + 1$.

Data The data for portfolio optimization and regression are the same: it is generated by drawing 1,000 samples from a bivariate zero-mean Gaussian distribution. The variance of both random variables is 1, while the covariance is 0.5. The data for classification is generated by two normal distributions with different mean and different covariance matrix. The first has mean $(-0.3, 0)$, while the second has mean $(0.3, 0)$. For both distributions, the variance is 0.05 while the covariance is 0.02. Each class has 100 samples. The value of the risk identifier Q^* is represented through the intensity of color. Darker points have larger values.

Portfolio Optimization We illustrate the idea with Markowitz portfolio optimization from the mean quadrangle (Example 5.2). The data points (x, y) represents the loss (negative return) of two assets. We choose $\beta = 100$. The optimal portfolio weight is $(0.4999038, 0.5000962)$. The value of the risk identifier Q^* is represented through the intensity of color in Figure 1b. Darker color corresponds to a larger value. Larger values are assigned to data points incurring larger loss, i.e., points whose both coordinates are larger.

Classification We illustrate the idea with large margin distribution machine from the mean quadrangle (Example 5.2). We choose $\beta = 0.01$ and $\gamma(\boldsymbol{w}) = ||\boldsymbol{w}||_2^2$. The optimal decision line is $y = -28.826x - 0.486$. The circles represent samples with label 1, while the diamonds represent samples with label -1. The value of the risk identifier Q^* is represented through the intensity of color in Figure 1a. A darker spot corresponds to a larger value. Larger values are assigned to data points incurring larger loss (negative margin), i.e., points that are correctly classified and have larger perpendicular distance from the optimal decision line.

Regression We illustrate the idea with least squares regression from the mean quadrangle (Example 5.2). We choose $\beta = 100$. The regression line is $y = 0.495x - 0.0127$. The value of the risk identifier Q^* is represented through the intensity of color in Figure **??**. A darker spot corresponds to a larger value. Larger values are assigned to data points incurring larger loss, i.e., data points further above the regression line.