A 2-PARAMETER PERSISTENCE LAYER FOR LEARNING

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ABSTRACT

1-parameter persistent homology, a cornerstone in Topological Data Analysis (TDA), studies the evolution of topological features such as cycle basis hidden in data. It has found its application in strengthening the representation power of deep learning models like Graph Neural Networks (GNN). To enrich the representations of topological features, here we propose to study 2-parameter persistence modules induced by bi-filtration functions. In order to incorporate these representations into machine learning models, we introduce a novel vectorization on 2-parameter persistence modules called Generalized Rank Invariant Landscape (GRIL). We show that this vector representation is stable and differentiable with respect to underlying filtration functions and can be easily integrated into machine learning models to augment encoding topological features. We present an algorithm to compute the vectorization and its gradients. We also test our methods on synthetic graph datasets and compare the results with some popular graph neural networks.

1 INTRODUCTION

Machine Learning models such as Convolutional Neural Networks (CNNs) (Lecun & Bengio, 1995) and Graph Neural Networks (GNNs) (Gori et al., 2005; Scarselli et al., 2009; Kipf & Welling, 2017; Xu et al., 2019) are well-known successful tools in the deep learning community. The representation power of such models can be augmented by infusing topological information of the underlying space hidden in data. Many recent works have successfully integrated topological information with machine learning models by differentiably vectorizing Persistent Homology (Carrière et al., 2020; Kim et al., 2020; Gabrielsson et al., 2020; Hofer et al., 2020; Horn et al., 2021; Swenson et al., 2020; Bouritsas et al., 2020). In (Corbet et al., 2018; Carrière & Blumberg, 2020), the authors vectorize a 2-parameter persistence module. In all of these works, the authors either consider 1-parameter persistence module or slice a 2-parameter persistence module into a collection of 1-parameter modules. Multiparameter Persistence Landscapes were defined in (Vipond, 2020) which generalize 1-parameter persistence landscapes (Bubenik, 2015). Multiparameter Persistence Landscapes do not need slicing of a 2-parameter persistence module. However, it can not be used directly in a differentiable manner in machine learning models. In this paper we propose a novel vector representation for 2-paramter persistence modules based on the recently introduced concept of *generalized rank invariant* (Kim & Mémoli, 2021) and its computation (Dey et al., 2022). We call it *Generalized Rank Invariant Landscape* (GRIL). The construction of GRIL is also inspired by persistence landscape (Bubenik, 2015; Vipond, 2020). It can be viewed as a generalization of persistence landscape for both 1- and 2-parameter persistence modules and encodes richer information. We will also show that the vector representation GRIL is 1-Lipschitz and differentiable with respect to the filtration function f, which allows us to build a differentiable topological layer, PERSGRIL, in a machine learning pipeline. We demonstrate its use on synthetic datasets[1]. From the perspective of direct use of 2-parameter persistence modules into machine learning models, to the best of our knowledge, this is the first work of its kind.

Persistent homology is a useful tool for characterizing the shape of data. Rooted in the theory of algebraic topology and algorithms, it has spawned the flourishing area of Topological Data Analysis(TDA). The classical persistent homology, also known as, 1-parameter persistence module, has attracted plenty of attention from both theory (Edelsbrunner & Harer, 2010; Oudot, 2015; Carlsson & Vejdemo-Johansson, 2021; Dey & Wang, 2022; Hofer et al., 2017; Ye et al., 2020; Dey & Wang,

[1]the code for full implementation will be available after review process is completed.

2022; Mémoli et al., 2022) and applications (Yan et al., 2021; Zhao et al., 2020; Yang et al., 2021b;a; Banerjee et al., 2020; Wu et al., 2020; Wang et al., 2020; Chen et al., 2021; Hu et al., 2021; Yan et al., 2022). The standard pipeline of 1-parameter persistence module is as follows: Given a domain of interest $\mathcal{X}$ (e.g. a topological space, point cloud data, a graph, or a simplicial complex) with a scalar function $f : \mathcal{X} \to \mathbb{R}$, one filters the domain $\mathcal{X}$ by the sublevel sets $\mathcal{X}_\alpha \triangleq \{x \in \mathcal{X} \mid f(x) \leq \alpha\}$ along with a continuously increasing threshold $\alpha \in \mathbb{R}$. The collection $\{\mathcal{X}_\alpha\}$, which is called a *filtration*, forms an increasing sequence of subspaces $\emptyset = \mathcal{X}_{-\infty} \subseteq \mathcal{X}_{\alpha_1} \subseteq \cdots \subseteq \mathcal{X}_{+\infty} = \mathcal{X}$. Along with the filtration, topological features appear, persist, and disappear over some intervals. We consider p-homology groups $H_p(-)$ (over a field, see (Hatcher, 2000)) of the subspaces in this filtration, which results into a sequence of vector spaces. These vector spaces are connected by inclusion-induced linear maps forming an algebraic structure $0 = H_p(\mathcal{X}_{-\infty}) \to H_p(\mathcal{X}_{\alpha_1}) \to \cdots \to H_p(\mathcal{X}_{+\infty})$. (Hatcher, 2000)). This algebraic structure, known as 1-parameter persistence module induced by f and denoted as M^f, can be uniquely decomposed into a collection of atomic modules called interval modules, which completely characterizes the topological features in regard to the three behaviors–appearance, persistence, and disappearance of all p-dimensional cycles. This unique decomposition of 1-parameter persistence module is commonly summarized as *persistence diagram* (Edelsbrunner et al., 2000) or *barcode* (Zomorodian & Carlsson, 2005). Figure 1 (left) shows a filtration of a simplicial complex which induces a 1-parameter persistence module and its decomposition into bars.

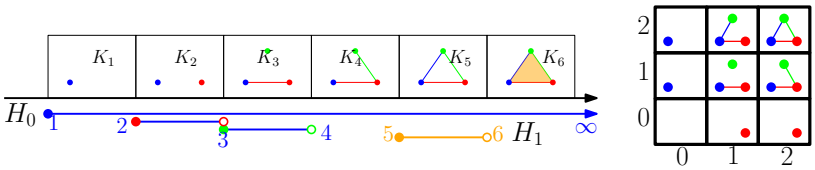

Figure 1: (left) 1-parameter filtration and bars; (right) a 2-parameter filtration inducing a 2-parameter persistence module whose decomposition is not shown.

Some problems in practice may demand tracking the topological information in a filtration that is not necessarily linear. For example, in (Adcock et al., 2014), 2-parameter persistence module is shown to be better for classifying hepatic lesions compared to 1-parameter persistence. In (Keller et al., 2018), a virtual screening system based on 2-parameter persistence modules are shown to be effective for searching new candidate drugs. In such applications, instead of studying a sequential filtration filtered by a scalar function, one may study a grid-filtration induced by a $\mathbb{R}^2$-valued bi-filtration function $f : \mathcal{X} \to \mathbb{R}^2$ with $\mathbb{R}^2$ equipped with partial order $\boldsymbol{u} \leq \boldsymbol{v} : u_1 \leq v_1, u_2 \leq v_2$; see Figure 1(right) for an example of 2-parameter filtration. Following a similar pipeline as the 1-parameter persistence module, one will get a collection of vector spaces $\{M^f_{\boldsymbol{u}}\}_{\boldsymbol{u}\in\mathbb{R}^2}$ indexed by vectors $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$ and linear maps $\{M^f(\boldsymbol{u} \leq \boldsymbol{v}) : M^f_{\boldsymbol{u}} \to M^f_{\boldsymbol{v}} \mid \boldsymbol{u} \leq \boldsymbol{v} \in \mathbb{R}^2\}$ for all comparable $\boldsymbol{u} \leq \boldsymbol{v}$. The entire structure M^f, in analogy to the 1-parameter case, is called a 2-parameter persistence module induced from f. Unlike 1-parameter case, the algebraic structure of 2-parameter persistence modules is much more complicated. There is no *complete* discrete invariant like persistence diagrams or barcodes for 2-parameter persistence modules (Carlsson & Zomorodian, 2009). A good non-complete invariant for 2-parameter persistence modules should characterize as many non-isomorphic topological features as possible. At the same time it should be stable with respect to small perturbations of filtration functions, which guarantees its important properties of continuity and differentiability for machine learning models. Therefore, how to build a good summary for 2-parameter persistence modules which is also applicable to machine learning models is an important problem.

2 2-PARAMETER PERSISTENCE LANDSCAPE

From the perspective of representation learning, a persistence module can be viewed as a special representation of a discrete topological space, like point cloud data or graph embedding, which captures geometric and topological information. 1-parameter persistence module captures information about topological features that persist across different scales. Here, we consider a bi-filtration which leads to a 2-parameter persistence module. To better utilize the richer information captured by 2-

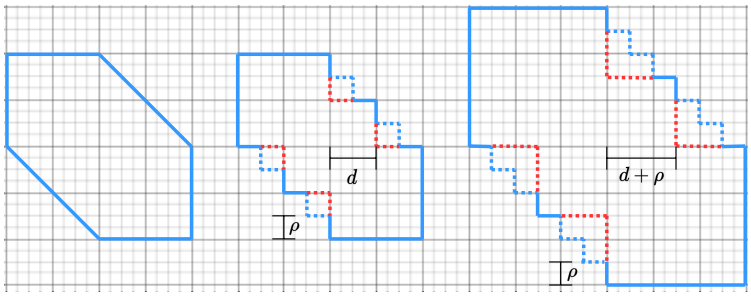

Figure 2: A 2-worm, discretized 2-worm and expanded discretized 2-worm. ρ denotes grid resolution. The blue dotted lines show the intermediate staircase with step-size ρ. The red dotted lines form parts of the squares with size d which are replaced by the blue dotted lines in the worm. The last figure shows the expanded 2-worm with red and blue dotted lines. The expanded 2-worm has width $d + \rho$ which is the one step expansion of the worm with width d.

parameter persistence modules, here we propose GRIL (Generalized Rank Invariant Landscape), a vectorized representation of a 2-parameter persistence module.

Let $M = M^f$ be a 2-parameter persistence module induced by some filtration function f. Let $I \subseteq \mathbb{R}^2$ be any connected partially ordered subset. Then, the restriction of M to I, denoted as $M|_I$, is the collection of vector spaces $\{M_{\boldsymbol{u}} \mid \boldsymbol{u} \in I\}$ along with linear maps $\{M(\boldsymbol{u} \leq \boldsymbol{v}) \mid \boldsymbol{u}, \boldsymbol{v} \in I)\}$. We can define generalized rank $\mathsf{rk}^M(I)$ (see Definition A.6) for any such $M|_I$ which generalizes the idea of the rank of a persistence module between two points to the rank of a persistence module over a connected partially ordered set (Kim & Mémoli, 2021). When $I = [\boldsymbol{u}, \boldsymbol{v}] \triangleq \{\boldsymbol{w} \in \mathbb{R}^2 \mid \boldsymbol{u} \leq \boldsymbol{w} \leq \boldsymbol{v}\}$ is a rectangle subset in $\mathbb{R}^2$, $\mathsf{rk}^M(I)$ equals the rank of the linear map $M(\boldsymbol{u} \leq \boldsymbol{v})$. The basic idea of GRIL is to compute a collection of generalized ranks $\{\mathsf{rk}^M(I)\}_{I \in \mathcal{W}}$ over some covering set $\mathcal{W}$ on $\mathbb{R}^2$, which is called a *generalized rank invariant* of M over $\mathcal{W}$. See Definition A.6 in Appendix for exact definition of generalized rank and generalized rank invariant.

This gives us a vector representation for the persistence module M. We choose $\mathcal{W}$ to be a set of *Worms* defined as follows:

$$\mathcal{W} \triangleq \left\{ \boxed{\boldsymbol{p}}_\delta^\ell \mid \delta > 0, \ell \geq 1, \boldsymbol{p} \in \mathbb{R}^2 \right\}$$

where

$$\boxed{\boldsymbol{p}}_\delta^\ell \triangleq \{\boldsymbol{q} \mid \exists \alpha \in \mathbb{R}, |\alpha| \leq (\ell - 1)\delta : \|\boldsymbol{q} - \boldsymbol{p} - (\alpha, -\alpha)\|_\infty \leq \delta\}.$$

Specially, for $\ell = 1$, $\boxed{\boldsymbol{p}}_\delta^1 = \boxed{\boldsymbol{p}}_\delta \triangleq \{\boldsymbol{q} : \|\boldsymbol{p} - \boldsymbol{q}\|_\infty \leq \delta\}$ which is a δ-square with side 2δ centered at $\boldsymbol{p}$. In general, for any $\ell \geq 1$, $\delta > 0$, ℓ-worm $\boxed{\boldsymbol{p}}_\delta^\ell$ is the union of all δ-squares $\boxed{\boldsymbol{q}}_\delta$ centered at some point $\boldsymbol{q}$ on the off-diagonal line segment $\boldsymbol{p} + \alpha(1, -1)$ with $|\alpha| \leq (\ell - 1)\delta$. Therefore, we can also equivalently write $\boxed{\boldsymbol{p}}_\delta^\ell = \bigcup_{\substack{\boldsymbol{q} = \boldsymbol{p} + (\alpha, -\alpha) \\ |\alpha| \leq (l-1)\delta}} \boxed{\boldsymbol{q}}_\delta$. See Figure 2 for an illustration of a 2-worm example and its discretization that we use in the algorithm. Specifics of the discretization are described in Section 3. We now define GRIL.

Definition 2.1 (Generalized Rank Invariant Landscape). For a persistence module M, the *Generalized Rank Invariant Landscape (*GRIL*)* of M is a function $\lambda^M : \mathbb{R}^2 \times \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R}$ defined as

$$\lambda^M(\boldsymbol{p}, k, \ell) \triangleq \sup_{\delta \geq 0} \{\mathsf{rk}^M(\boxed{\boldsymbol{p}}_\delta^\ell) \geq k\}. \tag{1}$$

In practice, we choose $\boldsymbol{p}$ from a finite subset of $\mathcal{P} \subset \mathbb{R}^2$, e.g. a finite uniform grid in $\mathbb{R}^2$, and consider $k \leq K, \ell \leq L$ for some $K, L \in \mathbb{N}_+$. Then λ^M can be viewed as a vector of dimension $|\mathcal{P}| \times K \times L$. See Figure 3 as an illustration. Figure 4 shows the discriminating power of GRIL where we see that GRIL can differentiate between shapes that are topologically non-isomorphic.

Stability and Differentiability of GRIL. An important property of GRIL is its stability property which makes it immune to small perturbations of the input bi-filtration function while still retaining

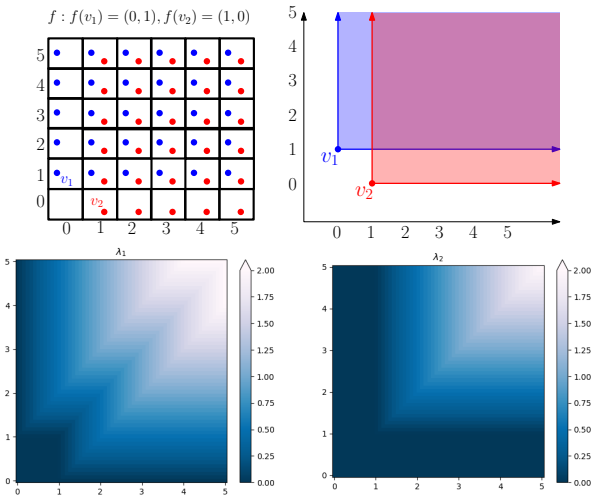

Figure 3: The top left figure is the simplicial filtration of a simplicial complex consisting of only two vertices $\{v_1, v_2\}$ with filtration values $f(v_1) = (0,1), f(v_2) = (1,0)$. The top right figure is the induced persistence module for 0-dimensional homology of this simplicial filtration. It consists of two interval modules on rectangular intervals starting from $(0,1)$ and $(1,0)$. The bottom two figures show the GRIL values for generalized ranks $k = 1, 2$ with $\ell = 2$ fixed.

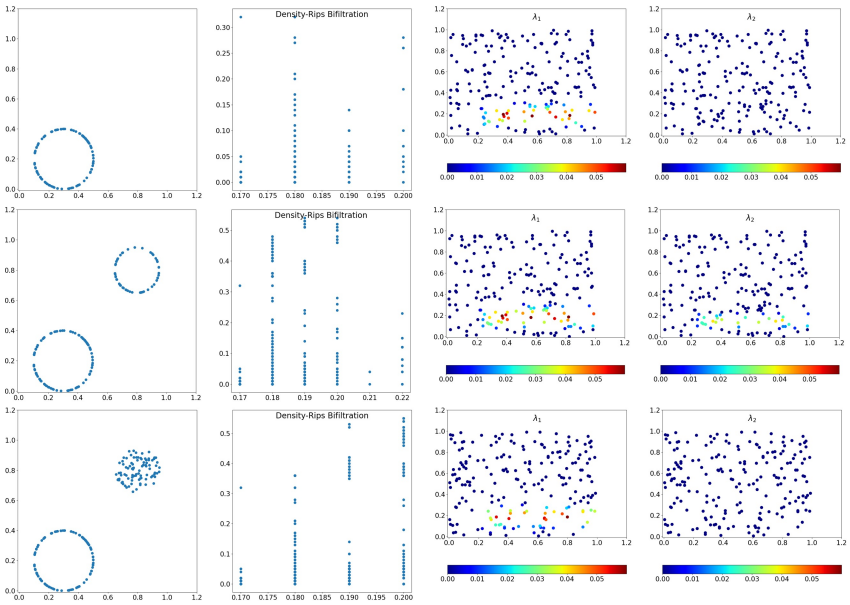

Figure 4: GRIL as a topological discriminator: each row shows a point cloud, its density-Rips bifiltration, GRIL value heatmap for 1-dimensional homology and generalized ranks $k = 1$ and $k = 2$ named as λ_1 and λ_2 respectively. First Betti number (β_1) of a circle is 1 which is reflected in λ_1 being non-zero. β_1 for two circles is 2 which is reflected in both λ_1 and λ_2 being non-zero. Similarly, β_1 of a circle and disk together is 1 which is reflected in λ_1 being non-zero but λ_2 being zero for this point cloud.

the ability to distinguish different topologies. We prove this stability result in Appendix B. It says GRIL is 1-Lipschitz (stable) with respect to filtrations in the following sense:

Proposition 2.1. *For two filtration functions* $f, f' : \mathcal{X} \to \mathbb{R}^2$, $\|\lambda^{M_f} - \lambda^{M_{f'}}\|_\infty \leq ||f - f'||_\infty$.

Proof of Proposition 2.1 is given in Appendix B. Proposition 2.1 shows that GRIL is stable under small perturbations of the input.

Remark 2.1. We note that when $\mathcal{X}$ is a finite space (e.g. finite simplicial complex (see Definition A.1), point cloud) with $|\mathcal{X}| = n$ then, any $f\colon \mathcal{X} \to \mathbb{R}^2$ can be considered as an $n \times 2$ matrix which can be linearized into a vector in $\mathbb{R}^{2n}$.

We now define PERSGRIL.

Definition 2.2 (PERSGRIL)**.** Given $k, \ell, \boldsymbol{p}$, PERSGRIL is a function $\Lambda_{\boldsymbol{p}}^{k,\ell}$

$$\Lambda_{\boldsymbol{p}}^{k,\ell}\colon \{\mathcal{X} \to \mathbb{R}^2\} \to \mathbb{R}$$

that takes in a bi-filtration function f as input and outputs the GRIL value at $\boldsymbol{p}$ given by $\lambda^{M^f}(k, \ell, \boldsymbol{p})$.

Remark 2.2. PERSGRIL can, thus, be considered as a map from $\mathbb{R}^{2n}$ to $\mathbb{R}$ when $|\mathcal{X}| = n$.

Proposition 2.2. PERSGRIL *is Lipschitz continuous with respect to the bi-filtration functions on finite spaces.*

Proof of Proposition 2.2 is given in Appendix B.

Corollary 2.3. PERSGRIL is differentiable almost everywhere.

Proof. Since PERSGRIL is Lipschitz continuous, by Rademacher's theorem (Evans & Gariepy, 2015), we get that PERSGRIL is differentiable almost everywhere. □

The differentiability of PERSGRIL in Corollary 2.3 refers to the existence of all directional derivatives. But the existence of a steepest direction as the "gradient" of PERSGRIL might not be unique. We propose an algorithm to efficiently compute one specific steepest direction based on the following theorem.

Theorem 2.4. *Consider the space of all filtration functions $\{f : \mathcal{X} \to \mathbb{R}^2\}$ on a finite space $\mathcal{X}$ with $|\mathcal{X}| = n$, which is equivalent to $\mathbb{R}^{2n}$. For fixed $k, \ell, \boldsymbol{p}$, there exists a measure-zero subset $Z \subseteq \mathbb{R}^{2n}$ such that for any $f \in \mathbb{R}^{2n} \setminus Z$ satisfying the following generic condition: $\forall x \neq y \in \mathcal{X}, f(x)_1 \neq f(y)_1, f(x)_2 \neq f(y)_2$, there exists an assignment $s : \mathcal{X} \to \{\pm 1, 0, \pm \ell\}^2$ such that $\nabla_s \Lambda_{\boldsymbol{p}}^{k,\ell}(f) \triangleq \lim_{\alpha \to 0} \frac{\Lambda_{\boldsymbol{p}}^{k,\ell}(f+\alpha s) - \Lambda_{\boldsymbol{p}}^{k,\ell}(f)}{\alpha \|s\|_\infty} = \max_{g \in \mathcal{X}} \nabla_g \Lambda_{\boldsymbol{p}}^{k,\ell}(f)$.*

The proof of Theorem 2.4 which also shows how to find the assignment s with the corresponding set of *supporting* simplices are in Appendix B. This result leads us to update the simplicial filtration with such an assignment s. See the description of enhancing topological features in section 4.

3 ALGORITHM

In practice, we are usually presented with a piecewise linear (PL) approximation $\hat{f}$ of a $\mathbb{R}^2$-valued function f on a discretized domain such as a finite simplicial complex. The PL-approximation $\hat{f}$ itself is $\mathbb{R}^2$-valued. Discretizing the parameter space $\mathbb{R}^2$ by a grid, we consider a *lower star* bi-filtration of the simplicial complex. Analogous to the 1-parameter case, a lower star bi-filtration is obtained by assigning every simplex the maximum of the values over all of its vertices in each of the two co-ordinates. With appropriate scaling, these (finite) values can be mapped to a subset of points in a uniform finite grid over $[0, 1] \times [0, 1]$. Observe that because of the maximization of values over all vertices, we have the property that two simplices $\sigma \subseteq \tau$ have values $\hat{f}(\sigma) \in \mathbb{R}^2$ and $\hat{f}(\tau) \in \mathbb{R}^2$ where $\hat{f}(\sigma) \leq \hat{f}(\tau)$. A partial order of the simplices according to these values provide a bi-filtration over the grid $[0, 1] \times [0, 1]$.

Constructing worms. Let us denote the chosen grid as $\text{Grid} = \{(\frac{m}{M}, \frac{n}{M}) \mid m, n \in \{0, 1, \ldots, M\}\}$ for some $M \in \mathbb{Z}_+$. We take a subset $\mathcal{P} \subseteq \text{Grid}$ as the collection of center points of worms used to build persistence landscape. Then our persistence landscape function $\lambda : \mathcal{P} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ only checks the values on $\mathcal{P}$.

Discretized ℓ-worms. We saw the definition of ℓ-worm in the previous section. However, in practice, since we work with a discrete grid rather than $\mathbb{R}^2$, we use *discretized* ℓ-worms. A discretized ℓ-worm centered at $\boldsymbol{p}$ with width d is the union of $2\ell - 1$ squares with centers

at $\boldsymbol{p} + (kd, -kd)$ and $\boldsymbol{p} - (kd, -kd)$ where $k \in \{0, 1, \ldots, \ell - 1\}$ along with the intermediate staircases between two squares of step-size equal to *grid resolution* (ρ). Figure 2 shows a discretized 2-worm. This construction is sensitive to the grid resolution. In what follows, we drop the word *discretized*, and write simply ℓ-worm. We fix ℓ to be 2, which allows us to initiate the worm with its center at $\boldsymbol{p}$ as a union of 3 ($= 2\ell - 1$) squares of side lengths equal to ρ with their centers at $\boldsymbol{p}, \boldsymbol{p} + (\rho, -\rho), \boldsymbol{p} - (\rho, -\rho)$ respectively. These initial worms are expanded further according to the values of generalized ranks that we compute on them.

Expanding/Contracting the worms. As part of computing the value of generalized ranks, we need to expand the 2-worms because we need to find the maximum width at which the rank of the zigzag filtration on the boundary of the 2-worm equals k, where k is one of the arguments of GRIL function $\lambda(\boldsymbol{p}, k, 2)$ as defined in the previous section. Construction of the expanded worm is very similar to that of standard 2-worm with the difference that the width has increased. As a consequence, the expanded worm will have more steps in the intermediary staircase between two consecutive squares. Figure 2 shows an expanded worm. Similarly, one can contract the worm by reducing the width of the worm, consequently reducing the number of steps in the intermediary staircase between two consecutive squares.

Computing generalized ranks. We need to compute the generalized rank $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d)$ for every worm $\boxed{\boldsymbol{p}}^2_d$ to decide whether to expand it or not. We use a result of (Dey et al., 2022) to compute $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d)$. It says that $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d)$ can be computed by considering a zigzag module and computing the number of full bars (bars that begin at the start of the zigzag filtration and persist until the end of the filtration) in its decomposition. This zigzag module decomposition can be obtained by restricting the bi-filtration on the boundary of $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d)$ and using any of the zigzag persistence algorithms on the resulting zigzag filtration. We use the recently published efficient algorithm and its associated software (Dey & Hou, 2022) for computing zigzag persistence.

Computing the value of GRIL using binary search. For a worm $\boxed{\boldsymbol{p}}^2_d$ and a given $k \geq 1$, we apply binary search to compute the value of GRIL. Let us denote the grid resolution by ρ. We do the binary search for d in the range $[d_{\min}, d_{\max}]$ where $d_{\min} = \rho$ and $d_{\max} = length\ of\ the\ grid$. In each iteration, we compute $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d)$ for $d = (d_{\min} + d_{\max})/2$ and check if $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d) \geq k$. We expand the worm by updating $d_{\min}$ to be d if $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d) \geq k$. Otherwise, we contract the worm by updating $d_{\max}$ to be d. The binary search stops and returns d when $(d_{\max} - d_{\min}) \leq \rho$. This ensures that we have searched through all possible values of d for which $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^2_d) \geq k$ and returned the maximum of these values.

We make use of subroutines COMPUTE-ℓ-WORM and BD-FIL during this binary search to compute the boundary of the worm and to compute the zigzag filtration along the boundary respectively.

COMPUTE-ℓ-WORM: This subroutine takes in $\boldsymbol{p} = (p_1, p_2)$ and d as inputs and outputs the list of points that make up the boundary of the 2-worm centered at $\boldsymbol{p}$ with width d. This subroutine is based on computing the ℓ-worm as defined in Definition 2.1. It follows that the first point on the boundary of the worm would be $(p_1, p_2 - 2d)$, the second $(p_1, p_2 - 2d + \rho$, the third $(p_1 - \rho, p_2 - 2d + \rho)$ and so on. Here, ρ denotes grid resolution. Refer Figure 5 (left).

BD-FIL: This subroutine takes the list of points $\mathcal{L}$, output by COMPUTE-ℓ-WORM along with the bi-filtration function f as the input and outputs the zigzag filtration along the boundary cap of the worm. At each point q of $\mathcal{L}$, this subroutine checks for any new simplices that have been added or any existing simplices that have been deleted while moving from the previous point in $\mathcal{L}$ to the current point q. These simplices are added/deleted accordingly at q. Refer Figure 5 (right). This way, we form the zigzag filtration which is used by FAST-ZIGZAG.

Algorithm 1 Persistence landscape computation

Input: f : Bi-filtration function, $\ell \geq 0, k \geq 1, p \in \mathcal{P} \subseteq$ Grid
Output: $\lambda(p, k, l)$: Persistence landscape at a point p for fixed k and ℓ

```
step ← m/M                                          ▷ step is grid resolution
d_min ← step, d_max ← 1
while d_min ≤ d_max do
    d ← (d_min + d_max)/2
    L ← COMPUTE-ℓ-WORM(p, d)
    F ← BD-FIL(L, f)
    B ← FASTZIGZAG(F)
    r ← # full bars in B
    if r = k then
        rk ← d
        d_min ← d + step
    else if r > k then
        d_min ← d + step
    else if r < k then
        d_max ← d − step
return rk
```

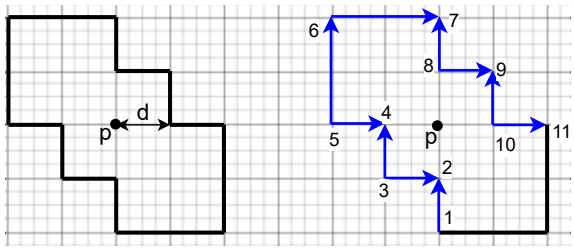

Figure 5: (Left) The figure shows the 2-worm centered at p with width d. In the second figure, the highlighted part denotes the boundary cap of the worm. COMPUTE-l-WORM enlists the coordinates of the points along this boundary cap, given p and d. (Right) The arrows in the figure denote the direction of arrows in the zigzag filtration that BD-FIL procedure outputs.

Time complexity. Assuming a grid with t nodes and a bi-filtration of a complex with n simplices on it, one can observe that each probe in the binary search takes $O((t+n)^{\omega})$ time. This is because each probe generates a zigzag filtration of length $O(t)$ with $O(n)$ simplices. Therefore, the binary search takes $O((t+n)^{\omega} \log t)$ time giving a total time complexity of $O(t((t+n)^{\omega} \log t))$ that accounts for $O(t)$ worms.

4 EXPERIMENTS

We create a differentiable topological layer based on GRIL named PERSGRIL which is in line with Definition 2.2. In essence, PERSGRIL takes in a bi-filtration function as input and gives the value of GRIL on the persistence module generated by the filtration function as output.

We compare PERSGRIL with several commonly used GNN models from the literature including: Graph Convolutional Networks (GCN)(Kipf & Welling, 2017), Graph Isomorphism Networks (GIN) (Xu et al., 2019).

Experiment with HourGlass dataset. We test our model on a synthetic dataset (HourGlass) that entails a binary graph classification problem over a collection of attributed undirected graphs. Essentially the structures of graphs from these two classes are the same. Each graph G is composed with two circulant subgraphs G_1, G_2 connected by some cross edges. Node attributes are order indices generated by two different traversals T_1, T_2. See Figure 6 as an illustration. More details can be found in Appendix C.0.1. We denote HourGlass[a,b] as the dataset of graphs generated with node size of each circulant subgraphs in range $[a, b]$. We generate three datasets with different sizes:

HourGlass[10,20], HourGlass[21,30], HourGlass[31,40]. Each dataset contains roughly 400 graphs. We evenly split HourGlass[21,30] into balanced training set and testing set on which we compare our method with 3-layer GCN, GIN, and persistence image (PersImg) (Adams et al., 2017), a 1-parameter persistence homology vector representation, on 1-parameter persistence modules. After that we also test these trained models on HourGlass[10,20] and HourGlass[31,40] to check if they can generalize well on smaller and larger graphs. The experiment results are shown in Table 1. We can see that PERSGRIL higher testing accuracy than other models. Also, it generalizes well.

Testing accuracy of models on HourGlass				
Model	**GCN**	**GIN**	**PersImg**	**PERSGRIL**
HourGlass[21,30]	87.25	84.00	74.00	100.0
HourGlass[10,20]	67.31	62.98	50.33	99.79
HourGlass[31,40]	87.75	79.10	86.95	100.0

Table 1: Table of testing results from different models. Last two rows show the testing results on HourGlass[10,20] and HourGlass[31,40] of models trained on HourGlass[21,30].

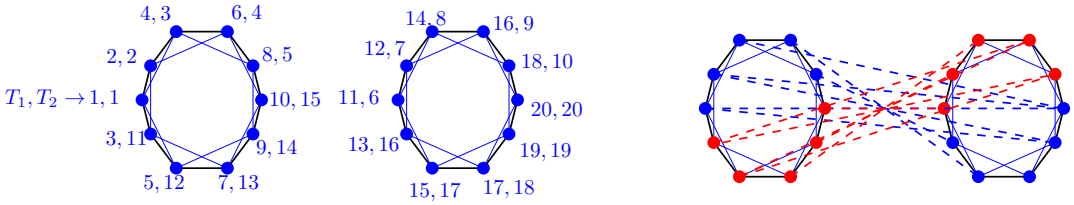

Figure 6: (Left) An example of a graph consisting of two circulant subgraphs. The pair of indices on each node represents the its order on the traversals T_1 and T_2 respectively. Both traversals start from the left node as the root node. (Right) Cross edges placed across two subgraphs.

Enhancing topological features. In the previous experiments we showed how PERSGRIL can be used to obtain topological signatures from graphs to facilitate a specific downstream task, which in our case is a graph classification problem. In that application, we build PERSGRIL on a static filtration function. By static, we mean that we computed the topological features and used them as an input to a classifier. In this experiment, we demonstrate, as a *proof of concept*, how PERSGRIL can be easily integrated in a differentiable framework (with the theoretical foundation laid in Sec.2, specifically Theorem 2.4) like the standard neural network architectures. We show this by rearranging the positions of input points, i.e. encouraging formation of clusters, holes by choosing suitable loss functions.

As shown in Figure 7, input to PERSGRIL is points sampled non-uniformly from two circles. Recall that GRIL is defined over a 2-parameter persistence module induced by some filtration function $f = (f_x, f_y)$. For every vertex v, we assign $f_x(v) = 1 - \exp(\frac{1}{\alpha}\sum_{i=1}^{\alpha} d(v, v_i))$, where v_i denotes i-th nearest neighbor of the vertex v and $d(v, v_i)$ denotes the distance between v and v_i. For our experiments we fix $\alpha = 5$. We set $f_y(v) = 0$. We compute ALPHACOMPLEX filtration (Edelsbrunner & Harer, 2010) of the points and for each edge $e := (u, v)$ we assign $f_x(e) = \max(f_x(u), f_x(v))$ and $f_y(e) = 1 - \exp(d(u, v))$. To obtain a valid bi-filtration function on the simplicial complex we extend the bi-filtration function from 1-simplices to 2-simplices, i.e. triangles. We pass f as an input to PERSGRIL, coded with the framework PYTORCH (Paszke et al., 2019), that computes persistence landscapes. PERSGRIL uniformly samples n center points from the grid $[0, 1]^2$. Since GRIL value computation can be done independently for each k and a center point, we take advantage of parallel computation and implement the code in a parallel manner. In the forward pass we get GRIL values $\lambda(\boldsymbol{p}, k, \ell)$ for generalized rank $k = 1, 2$, worm size $\ell = 2$ and homology of dimension 1 while varying $\boldsymbol{p}$ over all the sampled center points. After we get the GRIL values, we compute the assignment s according to Theorem 2.4. During the backward pass, we utilize this assignment to compute the derivative of PERSGRIL with respect to the filtration function and consequently update it. We get n values of $\lambda(\cdot, 1, 2)$ for n center points. We treat these n values as a vector and denote is as λ_1. Similarly, we use λ_2 to denote the vector formed by values $\lambda(\cdot, 2, 2)$. We minimize the loss $L = -(\|\lambda_1\|_2^2 + \|\lambda_2\|_2^2)$. Figure 7 shows the result after running PERSGRIL for 200 epochs. The

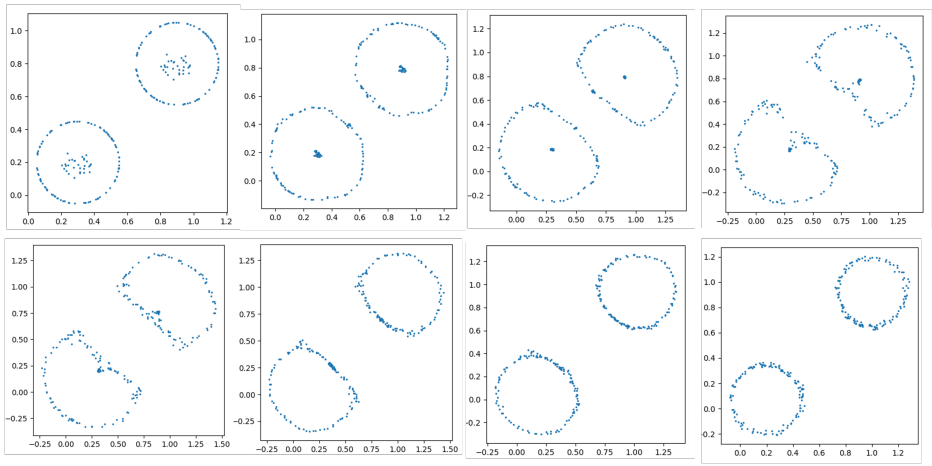

Figure 7: The figures show the rearrangement of points according to the loss function, which in our case is increasing the norm of λ_1 and λ_2 vectors. We start with two circles containing some noisy points inside. We see that the points rearrange to form two circles because that increases the norm of λ_1 and λ_2 vectors.

optimizer we use to optimize the loss function is Adam (Kingma & Ba, 2015) with a learning rate of 0.01.

5 Conclusion

This work presents GRIL, a 2-parameter persistence vectorization based on generalized rank invariant which we show is stable and differentiable with respect to the bi-filtration functions. Further, we present an algorithm for computing GRIL which is a synergistic confluence of the recent developments in computing generalized rank invariant of a 2-parameter module and a fast algorithm for computing zigzag persistence. We propose PERSGRIL, a differentiable topological layer, which can be used as a topological feature extractor in a differentiable manner. As a topological feature extractor, PERSGRIL can perform better than Graph Convolutional Networks (GCNs) and Graph Isomorphism Networks (GINs) on some synthetic datasets. Further, we give a proof of concept for the differentiability of PERSGRIL by rearranging the point cloud to enhance its topological features. We believe that the additional topological information that a 2-parameter persistence module encodes, as compared to a 1-parameter persistence module, can be leveraged to learn better representations. Further directions of research include using PERSGRIL with GNNs for filtration learning to learn more powerful representations. We hope that this work motivates further research into exploring this direction.

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A BACKGROUND AND DEFINITIONS

Here, we give the detailed definitions of all the concepts explained in the paper. We begin by defining a simplicial complex.

Definition A.1 (Simplicial Complex)**.** An abstract simplicial complex is a pair (V, Σ) where V is a finite set and Σ is a collection of non-empty subsets of V such that if $\sigma \in \Sigma$ and if $\tau \subseteq \sigma$ then $\tau \in \Sigma$. A topological space $|(V, \Sigma)|$ can be associated with the simplicial complex which can be defined using a bijection $t\colon V \to \{1, 2, \ldots, |V|\}$ as the subspace of $\mathbb{R}^{|V|}$ formed by the union $\bigcup_{\sigma \in \Sigma} h(\sigma)$, where $h(\sigma)$ denotes the convex hull of the set $\{e_{t(s)}\}_{s \in \sigma}$, where e_i denotes the standard basis vector in $\mathbb{R}^{|V|}$.

We shall now define a zigzag filtration and the zigzag persistence module induced by it.

Definition A.2. A *zigzag filtration* is a sequence of simplicial complexes where both insertions and deletions of simplices are allowed, the possibility of which we indicate with double arrows:

$$X_0 \leftrightarrow X_1 \leftrightarrow \cdots \leftrightarrow X_n = \mathcal{X}.$$

Applying homology functor on such a filtration we get a zigzag persistence module that is a sequence of vector spaces connected either by forward or backward linear maps:

$$H_*(X_0) \leftrightarrow H_*(X_1) \leftrightarrow \cdots \leftrightarrow H_*(X_n).$$

Now, we give the definition of 2-parameter filtration over $\mathbb{R}^2$ and the 2-parameter persistence module induced by it.

Definition A.3 (2-parameter simplicial filtration over $\mathbb{R}^2$)**.** A 2-parameter simplicial filtration, also called bi-filtration, over $\mathbb{R}^2$ is a collection of simplicial complexes $\{X_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}^2}$ with inclusion maps $X_{\mathbf{u}} \hookrightarrow X_{\mathbf{v}}$ for $\mathbf{u} \leq \mathbf{v}$, that is, $u_1 \leq u_2$ and $v_1 \leq v_2$ where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

Definition A.4 (2-parameter Persistence Module)**.** Given a bi-filtration $\{X_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}^2}$, we get a collection of vector spaces $\{M_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}^2}$ along with linear maps $M_{\mathbf{u}} \to M_{\mathbf{v}}$ for $\mathbf{u} \leq \mathbf{v}$ by considering the homology of the simplicial complexes in the bi-filtration over a finite field, such as $\mathbb{Z}_2$. Each inclusion map in the bi-filtration induces a linear map between the corresponding homology vector spaces.

Having defined 2-parameter filtration and 2-parameter persistence module, we now define the notion of an Interval in $\mathbb{R}^2$. In the definition, we shall make use of the standard partial order on $\mathbb{R}^2$, i.e., $\mathbf{u} \leq \mathbf{v}$ if $u_1 \leq v_1$ and $u_2 \leq v_2$ for $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

Definition A.5. An interval in $\mathbb{R}^2$ is a subset $\emptyset \neq I \subseteq \mathbb{R}^2$ that satisfies the following:

1. If $\mathbf{p}, \mathbf{q} \in I$ and $\mathbf{p} \leq \mathbf{r} \leq \mathbf{q}$, then $\mathbf{r} \in I$;
2. If $\mathbf{p}, \mathbf{q} \in I$, then there exists a finite sequence $(\mathbf{p} = \mathbf{p}_0, \mathbf{p}_1, , ..., \mathbf{p}_m = \mathbf{q}) \in I$ so that every consecutive points $\mathbf{p}_i, \mathbf{p}_{i+1}$ are comparable in the partial order for $i \in \{0, \ldots, m-1\}$.

We shall now define generalized rank invariant in $\mathbb{R}^2$. However, generalized rank invariant can be defined over any locally finite connected poset.

Definition A.6 (Generalized Rank (Kim & Mémoli, 2021))**.** Given a 2-parameter persistence module M and an intervals $I \subseteq \mathbb{R}^2$, the *generalized rank* of M restricted to I, $\mathsf{rk}^M(I)$, is defined as

$$\mathsf{rk}^M(I) \triangleq \operatorname{rank}[\varprojlim M|_I \to \varinjlim M|_I].$$

Here $\varprojlim M|_I, \varinjlim M|_I$ denote the limit and colimit of the functor M when restricted to I. We refer the reader to (MacLane, 1971) for the definitions of limit and colimit in category theory.

For a collection of intervals $\mathcal{I}$, the collection $\mathsf{rk}^M_{\mathcal{I}} \triangleq \{\mathsf{rk}^M(I) \mid I \in \mathcal{I}\}$ is called *generalized rank invariant* of M over $\mathcal{I}$.

We can define a metric on the space of persistence modules based on their generalized rank invariants over all intervals in $\mathbb{R}^2$.

Definition A.7 (Erosion Distance (Patel, 2018; Kim & Mémoli, 2021))**.** Let $\mathbf{Int}(\mathbb{R}^2)$ be the collection of all intervals in $\mathbb{R}^2$. Let M and N be two persistence modules. The *erosion distance* is defined as

$$\mathrm{d}_{\mathcal{E}}(M,N) \triangleq \inf_{\varepsilon \geq 0} \{\forall I \in \mathbf{Int}(\mathbb{R}^2), \mathsf{rk}^M(I) \geq \mathsf{rk}^N(I^{+\varepsilon}) \text{ and } \mathsf{rk}^N(I) \geq \mathsf{rk}^M(I^{+\varepsilon})\}.$$

Here $I^{+\varepsilon}$ denotes the ε-extension of the interval I.

B Stability and Differentiability: Proofs

In this section, we provide the proof for stability and differentiability of GRIL. We begin by defining a metric on the space of persistence modules based on GRIL.

Definition B.1. For persistence module M, N with GRILs λ^M, λ^N, define

$$d_{\mathcal{L}}(M,N) \triangleq ||\lambda^M - \lambda^N||_\infty.$$

We shall now look at a property of GRIL that will help in proving the stability.

Definition B.2. Given any interval I and $\varepsilon \geq 0$, let $I^{+\varepsilon}$ be the ε-*extension* of I defined as:

$$I^{+\varepsilon} \triangleq \bigcup_{\mathbf{p} \in I} \boxed{\mathbf{p}}_\varepsilon \tag{2}$$

where $\boxed{\boldsymbol{p}}_\varepsilon \triangleq \{\mathbf{q} : ||\boldsymbol{p} - \boldsymbol{q}||_\infty \leq \varepsilon\}$ is the ∞-norm ε-neighbourhood of x.

Proposition B.1. $\left(\boxed{\boldsymbol{p}}^\ell_\delta\right)^{+\varepsilon} \subseteq \boxed{\boldsymbol{p}}^\ell_{\delta+\varepsilon}$.

In order to better analyze the stability property of persistence landscape, we define a distance in a similar flavour as erosion distance for the underlying collection of all worms.

Notation B.3. Denote the collection of all worms as $\mathcal{W} \triangleq \left\{\boxed{\boldsymbol{p}}^\ell_\delta \mid \delta > 0, l \in \mathbb{N}_+, \boldsymbol{p} \in \mathbb{R}^2\right\}$.

Definition B.4. For $\mathcal{W} \triangleq \left\{\boxed{\boldsymbol{p}}^\ell_\delta \mid \delta > 0, l \in \mathbb{N}_+, \boldsymbol{p} \in \mathbb{R}^2\right\}$, define a distance $\mathrm{d}^{\mathcal{W}}_{\mathcal{E}}$ as follows:

$$\mathrm{d}^{\mathcal{W}}_{\mathcal{E}}(M,N) \triangleq \inf\left\{\varepsilon \mid \forall \boxed{\boldsymbol{p}}^\ell_\delta \in \mathcal{W}, [\mathsf{rk}^M\left(\boxed{\boldsymbol{p}}^\ell_\delta\right) \geq \mathsf{rk}^N\left(\boxed{\boldsymbol{p}}^\ell_{\varepsilon+\delta}\right) \text{ and } \mathsf{rk}^N\left(\boxed{\boldsymbol{p}}^l_\delta\right) \geq \mathsf{rk}^M\left(\boxed{\boldsymbol{p}}^\ell_{\varepsilon+\delta}\right)]\right\}. \tag{3}$$

Proposition B.2. $\mathrm{d}_{\mathcal{L}} = \mathrm{d}^{\mathcal{W}}_{\mathcal{E}} \leq \mathrm{d}_{\mathcal{E}}$*, where* $\mathrm{d}_{\mathcal{E}}$ *is the erosion distance.*

Proof. $\mathrm{d}^{\mathcal{W}}_{\mathcal{E}} \leq \mathrm{d}_{\mathcal{E}}$ is obvious by definition.

To show $\mathrm{d}_{\mathcal{L}} \leq \mathrm{d}^{\mathcal{W}}_{\mathcal{E}}$. Given two persistence modules M, N, assume $\mathrm{d}^{\mathcal{T}}_{\mathcal{E}}(M,N) = \epsilon$. For fixed $\boldsymbol{p}, k, \ell$, let $\lambda^M(\boldsymbol{p}, k, \ell) = \delta_1$ and $\lambda^N(\boldsymbol{p}, k, \ell) = \delta_2$. Without loss of generality, assume $\delta_2 \geq \delta_1$. We want to show that $\delta_2 - \delta_1 \leq \epsilon$. By the construction of $\mathrm{d}^{\mathcal{W}}_{\mathcal{E}}$, we know that for any $\alpha > 0$, $k > \mathsf{rk}^N(\boxed{\boldsymbol{p}}^\ell_{\delta_1+\alpha}(x)) \geq \mathsf{rk}^M(\boxed{\boldsymbol{p}}^\ell_{\delta_1+\epsilon+\alpha}(x))$. One can get $\delta_1 + \epsilon + \alpha > \delta_2 \implies \epsilon + \alpha > \delta_2 - \delta_1$. By taking $\alpha \to 0$, we have $\delta_2 - \delta_1 \leq \epsilon$.

To show $\mathrm{d}^{\mathcal{W}}_{\mathcal{E}} \leq \mathrm{d}_{\mathcal{L}}$. Let $\mathrm{d}_{\mathcal{L}}(M,N) = \delta$. For any $I = \boxed{\boldsymbol{p}}^\ell_\epsilon \in \mathcal{I}$, we want to show that $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^\ell_\epsilon) \geq \mathsf{rk}^N(\boxed{\boldsymbol{p}}^\ell_{\epsilon+\delta})$ and $\mathsf{rk}^N(\boxed{\boldsymbol{p}}^\ell_\epsilon) \geq \mathsf{rk}^M(\boxed{\boldsymbol{p}}^\ell_{\epsilon+\delta})$. We prove the first inequality. The second one can be proved in a similar way. Let $k = \mathsf{rk}^N(\boxed{\boldsymbol{p}}^\ell_{\epsilon+\delta})$, then $\lambda^N(\boldsymbol{p}, k, \ell) \geq \epsilon + \delta$. By the assumption $\mathrm{d}_{\mathcal{L}}(M,N) = \delta$, we know that $\lambda^N(\boldsymbol{p}, k, \ell) \geq \epsilon$, which implies $\mathsf{rk}^M(\boxed{\boldsymbol{p}}^\ell_\epsilon) \geq k = \mathsf{rk}^N(\boxed{\boldsymbol{p}}^\ell_{\epsilon+\delta})$. □

Proposition B.3. *Persistence landscape is equivalent to the (generalized) rank invariant on* $\mathcal{W}$*. Here equivalence means bijective reconstruction from each other.*

Proof. For any x, ϵ, ℓ, the (generalized) rank invariants $\mathsf{rk}^M_{\mathcal{W}} : \mathcal{W} \to \mathbb{N}$ can be reconstructed by persistence landscapes λ as follows:

$$\mathsf{rk}^M_{\mathcal{W}}(\boxed{\boldsymbol{p}}^{\ell}_{\epsilon}) = \arg\max_k \{\lambda(\boldsymbol{p}, k, \ell) \geq \epsilon\} \tag{4}$$

It is not hard to check that, this construction, combined with the construction of persistence landscape, gives a bijective mapping between (generalized) rank invariants $\mathsf{rk}^M_{\mathcal{W}}$ and persistence landscapes λ. □

By the stability property of erosion distances, we can immediately get the stability of GRIL as follows:

Proposition. 2.1 For two filtration functions $f, f' \colon \mathcal{X} \to \mathbb{R}^2$, $||\lambda^{M_f} - \lambda^{M_{f'}}||_\infty \leq ||f - f'||_\infty$.

Proof. Let M_f and $M_{f'}$ be the persistence modules derived by f and f' respectively. Then, we have the following chain of inequalities:

$$\|\lambda^{M_f} - \lambda^{M_{f'}}\|_\infty = \mathrm{d}_{\mathcal{L}}(M_f, M_{f'}) \leq \mathrm{d}_{\mathcal{E}}(M_f, M_{f'}) \leq d_I(M_f, M_{f'}) \leq \|f - f'\|_\infty$$

The second last inequality has been shown in (Kim & Mémoli, 2021). □

Recall that when $\mathcal{X}$ is a finite space (e.g. finite simplicial complex, point cloud) then, any $f \colon \mathcal{X} \to \mathbb{R}^2$ can be considered as an $n \times 2$ matrix which can be linearized into a vector in $\mathbb{R}^{2n}$. Let us denote that vector by v_f.

Proposition (2.2)**.** PERSGRIL is Lipschitz continuous with respect to bi-filtration functions on finite spaces.

Proof. Given filtration functions f, f' and their corresponding vector representations $v_f, v_{f'} \in \mathbb{R}^{2n}$, it is easy to see that $\|f - f'\|_\infty \leq 2\|v_f - v_{f'}\|_\infty \leq 2\|v_f - v_{f'}\|$. Combining this with the chain of inequalities in the previous proposition, we get that PERSGRIL is Lipschitz continuous with respect to the underlying filtration functions. □

Theorem (2.4)**.** Consider the space of all filtration functions $\{f : \mathcal{X} \to \mathbb{R}^2\}$ on a finite space $\mathcal{X}$ with $|\mathcal{X}| = n$, which is equivalent to $\mathbb{R}^{2n}$. For fixed $k, \ell, \boldsymbol{p}$, there exists a measure-zero subset $Z \subseteq \mathbb{R}^{2n}$ such that for any $f \in \mathbb{R}^{2n} \setminus Z$ satisfying the following generic condition: $\forall x \neq y \in \mathcal{X}, f(x)_1 \neq f(y)_1, f(x)_2 \neq f(y)_2$, there exists an assignment $s : \mathcal{X} \to \{\pm 1, 0, \pm \ell\}^2$ such that

$$\nabla_s \Lambda^{k,\ell}_{\boldsymbol{p}}(f) \triangleq \lim_{\alpha \to 0} \frac{\Lambda^{k,\ell}_{\boldsymbol{p}}(f + \alpha s) - \Lambda^{k,\ell}_{\boldsymbol{p}}(f)}{\alpha \|s\|_\infty} = \max_{g \in \mathcal{X}} \nabla_g \Lambda^{k,\ell}_{\boldsymbol{p}}(f).$$

Proof. By Corollary 2.3 we know there exists some measure-zero set $R \subset \mathbb{R}^{2n}$ such that PERSGRIL is differentiable in $\bar{R} \triangleq \mathbb{R}^{2n} \setminus R$. Let $M = M^f$ be a 2-parameter persistence module induced from some generic filtration function $f \in \bar{R}$ and $I = \boxed{\boldsymbol{p}}^{\ell}_{d}$ be an ℓ-worm in $\mathbb{R}^2$ centered at some point $\boldsymbol{p}$. Let $\partial(I)$ be the boundary of I excluding the right most vertical edge and bottom most vertical edge (See Figure 8 as an illustration). It is shown in (Dey et al., 2022) that, over the boundary $\partial(I)$, a so-called zigzag persistence module can be defined by taking the restricting M to $\partial(I)$ (in practice it is enough to take a zigzag path to approximate the smooth off-diagonal boundary) on which the number of full bars is equal to $\mathsf{rk}^M(I)$. Let $I' = \boxed{\boldsymbol{p}}^{\ell}_{d'}$ be another ℓ-worm centered at $\boldsymbol{p}$ for some $d' \neq d$. One can observe that, if the zigzag filtrations on $\partial(I)$ and $\partial(I')$ have the same order of insertion and deletion of simplices , then the number of full bars on $M|_{\partial(I)}$ and $M|_{\partial(I')}$ are the same, which means $\mathsf{rk}^M(I) = \mathsf{rk}^M(I')$. Now let $d = \lambda^M(k, \ell, \boldsymbol{p}), I = \boxed{\boldsymbol{p}}^{\ell}_{d}, I_- = \boxed{\boldsymbol{p}}^{\ell}_{d-\epsilon}, I_+ = \boxed{\boldsymbol{p}}^{\ell}_{d+\epsilon}$ for some small enough ϵ. Based on the definition of λ^M, we know that $\mathsf{rk}^M(I_-) \geq k$ and $\mathsf{rk}^M(I_+) < k$, which means that zigzag filtrations change on some simplices while moving from $\partial(I_-)$ to $\partial(I_+)$. Either the collection of simplices changes or the order of simplices changes. The former case corresponds to the simplices with x or y-coordinate aligned with some vertical or horizontal edges on the $\partial(I)$. The latter case corresponds to those pairs of simplices (σ, τ) such that

$f(\sigma) \vee f(\tau) \triangleq (\max(f(\sigma)_1, f(\tau)_1), \max(f(\sigma)_2, f(\tau)_2)$ is on some off-diagonal edges on $\partial(I)$. By the generic condition of the filtration function f, we can locate those simplices as the set S, which we call support simplices. The assignment function s is defined on each $\sigma \in S$ by assigning $s(\sigma) = \pm 1$ or $\pm \ell$ which is consistent with the moving direction of the edge from $\partial(I)$ to $\partial(I_+)$. We discuss the assignment values case by case:

We can divide the boundary into four edges: bottom (off-diagonal) edge e_b, top (horizontal) edge e_t, left (vertical) edge e_l, right (off-diagonal) edge e_r.

1. $s(\sigma) = (0, +\ell)$ if σ has y-coordinate the same as e_t,
2. $s(\sigma) = (-\ell, 0)$ if σ has x-coordinate the same as e_l,
3. $s(\sigma) = (0, -1), s(\tau) = (-1, 0)$ if $f(\sigma) \vee f(\tau)$ is on e_b and $f(\sigma)_1 \leq f(\tau)_1$,
4. $s(\sigma) = (0, +1), s(\tau) = (+1, 0)$ if $f(\sigma) \vee f(\tau)$ is on e_r and $f(\sigma)_1 \leq f(\tau)_1$,

See Figure 8 as an illustration. We assume f satisfies the condition that the supporting simplices in S either all belong to cases 1 and 2 or all belong to cases 3 and 4, but not a combination of them. It is not hard to see that the collection of f for which this condition does not hold is a measure zero set in $\mathbb{R}^{2n}$. Let us denote the collection of all such f's by F. Then, $Z = F \cup R$ is a measure zero set in $\mathbb{R}^{2n}$ which consists of f's which do not satisfy the condition and those points where PERSGRIL is not differentiable.

Now, check for such a generic $f \notin Z$ so that the directional derivative $\nabla_s \lambda(f)$ is indeed a maximal directional derivative. For the cases 3 and 4, the stability property in Proposition 2.1 implies that, for any $\alpha > 0$ and any direction vector $g \in \mathbb{R}^{2n}$ with $\|g\|_\infty = 1$, we have $\lambda(f + \alpha g) - \lambda(f) \leq \alpha$. Also it is not hard to check that $\lambda(f + \alpha s) - \lambda(f) = \alpha$ for $\alpha > 0$ small enough since the zigzag persistence of $M^{f+\alpha s}|_J$ with $J = \boxed{\boldsymbol{p}}^{\ell}_{d+\alpha}$ has the same collection of simplices and orders as $M^f|_I$ with $I = \boxed{\boldsymbol{p}}^{\ell}_{d}$, which means they have the same rank. Therefore, we have $\forall \|g\|_\infty = 1, \lambda(f + \alpha g) - \lambda(f) \leq \lambda(f + \alpha s) - \lambda(f) \implies \nabla_g \Lambda(f) \leq \nabla_s \Lambda(f)$. For the case 1 (the case 2 is similar), the support simplex is on edge e_t. Now for any direction vector $g \in \mathbb{R}^{2n}$ and $\alpha > 0$ small enough, let $\Delta d = \Lambda(f + \alpha g) - \Lambda(f)$ and let Δy_{e_t} be the difference between y-coordinates of e_t's from $\boxed{\boldsymbol{p}}^{\ell}_{d}$ and $\boxed{\boldsymbol{p}}^{\ell}_{d+\Delta d}$. Note that $\frac{\Delta d}{\Delta y_{e_t}} = \ell$ and $\frac{|\Lambda(f+\alpha g) - \Lambda(f)|}{\alpha \|g\|_\infty} \leq \frac{\Delta d}{\Delta y_{e_t}}$ since in order to change $\Lambda(f)$ by Δd one has to at least move edge e_t by Δy_{e_t}, which correspondingly changes the y-coordinate of $s(\sigma)$ by Δy_{e_t}. From the above argument, we can get the directional derivative $\nabla_g \Lambda(f)$ is bounded from above by the ratio $\frac{\Delta d}{\Delta y_{e_t}} = \frac{1}{\ell} = \nabla_s \Lambda(f)$. The case for $\alpha < 0$ is symmetric.

In summary, $\nabla_s \lambda(f)$ indeed maximizes the directional derivative for f. □

C EXPERIMENTAL SETUP

C.0.1 HOURGLASS PROBLEM

The two traversals T_1 and T_2 are designed as follows: T_1 traverses G_1, then followed by G_2; T_2 traverses upper halves $G_1^\top$ and $G_2^\top$ of G_1 and G_2 sequentially first, then followed by the other halves $G_1^\perp$ and $G_2^\perp$ of G_1 and G_2. For cross edges, we randomly pick $2|V|$ pairs of nodes (with replacement) in $G_1^\top \times G_2^\perp$ on which we place cross edges. We don't place multiple edges on the same pair of nodes. In a similar way we place cross edges on $G_1^\perp \times G_2^\top$. Therefore, G has roughly $6|V|$ cross edges between G_1 and G_2. The (roughly) total number of edges: $|E| \approx 5|V|$. For methods based on persistence modules, we take two filtration functions $f_1, f_2 : V \cup E \to \mathbb{R}$ on G as follows: let $x(v)$ be the node attribute on v given by the order index of the trace. Then

- f_1 is given by $\forall v \in V, f_1(v) = x(v)$ and $\forall e = (v, w) \in E, f_1(e) = \max(x(v), x(w))$.
- f_2 is given by $f_2(v) = 0$ and $f_2 = C(e)$ where $C(e)$ is a curvature value of e. Here we use a version of discrete Ricci called Forman-Ricci curvature (Forman, 2003) computed by the code provided in (Ni et al., 2019).

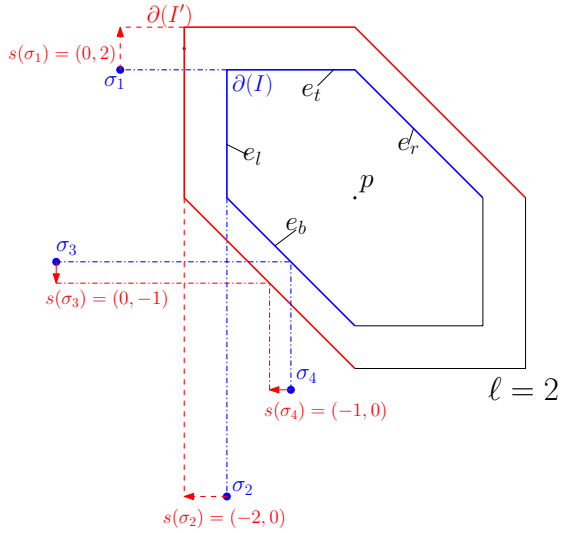

Figure 8: Two examples of 2-worm I, I'. Blue and red lines are boundaries of I and I' respectively on which the zigzag persistence modules are constructed for computing ranks. $\sigma_i, i = 1, 2, 3, 4$ are four support simplices on $\partial(I)$. $s(\sigma_i)$ is the assignment function values on σ_i.

We compute for all points $\boldsymbol{p}$ in a uniform 4×4 grid the PERSGRIL values $\lambda(\boldsymbol{p}, k, \ell)$ for generalized rank $k = 1, 2$, worm size $\ell = 2$, and homology of dimension 0 and 1. Therefore, for each graph our PERSGRIL generates a 64-dimensional vector as representation. For the method based on 1-parameter persistence modules with persistence image vectorization, we compute 1-parameter persistence modules for homology dimension $0, 1$ on f_1 and f_2 independently. Each persistence module will be vectorized on a 4×4 grid. Therefore, it also produces a 64-dimensional vector as representation. We use 3-layer multilayer perceptron (MLP) as classifiers for representations of persistence modules. We fix the hidden dimension dim_h=16. For graph neural networks models, we use 3-layer GCN and 3-layer GNN, followed by sum pooling and one fully-connected layer. We also fix the hidden dimension dim_h=16. We train all the models 100 epochs with cross entropy loss and Adam optimizer (Kingma & Ba, 2015) with learning rate fixed to be lr=0.001. We do 5-fold cross validation and report the mean accuracy.