AN UPPER BOUND FOR THE DISTRIBUTION OVERLAP INDEX AND ITS APPLICATIONS

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ABSTRACT

The overlap index between two probability distributions has various applications in statistics, machine learning, and other scientific research. However, approximating the overlap index is challenging when the probability distributions are unknown (i.e., distribution-free settings). This paper proposes an easy-to-compute upper bound for the overlap index without requiring any knowledge of the distribution models. We first utilize the bound to find the upper limit for the accuracy of a trained machine learning model when a domain shift exists. We additionally employ this bound to study the distribution membership classification of given samples. Specifically, we build a novel, distribution-free, computation-efficient, memory-efficient one-class classifier by converting the bound into a confidence score function. The proposed classifier does not need to train any parameters and requires only a small number of in-class samples to be accurate. The classifier shows its efficacy on various datasets and outperforms many state-of-the-art methods in different one-class classification scenarios, including novelty detection, out-of-distribution detection, and backdoor detection. The obtained results show significant promise toward broadening the applications of overlap-based metrics.

1 INTRODUCTION

The distribution overlap index refers to the area intersected by two probability density functions (i.e., Fig. 1(a)) and measures the similarity between the two distributions. A high overlap index value implies a high similarity. Although the overlap index has various applications in many areas, such as biology (Langøy et al., 2012; Utne et al., 2012), economics (Milanovic & Yitzhaki, 2002), and statistics (Inman & Bradley Jr, 1989), the literature on approximating it under distribution-free settings is thin. This work proposes an upper bound for the overlap index with distribution-free settings to broaden the potential applications of overlap-based metrics. The bound is easy to compute and contains three terms: a constant number, the norm of the difference between the two distributions' means, and a variation distance between the two distributions over a subset. Even though finding such an upper bound for the distribution overlap index is already valuable, we further explore two additional applications of our bound as discussed below.

One application of our bound is for domain shift analysis. Specifically, a domain shift is a change in the dataset distribution between a model's training dataset and the testing dataset encountered during implementation (i.e., the overlap index value between the distributions of the two datasets is less than 1). We calculated the model's testing accuracy in terms of the overlap index between the distributions of the training and testing datasets and further found the upper limit of the accuracy using our bound for the overlap index. Knowing the upper bound for a model's testing accuracy helps measure the model's potential and compare it with other models.

Another application of our bound is for one-class classification. Specifically, one-class classification refers to a model that outputs positive for in-class samples and negative for out-class samples that are absent, poorly sampled, or not well defined (i.e., Fig. 1(b)). We propose a novel one-class classifier by converting our bound into a confidence score function to evaluate if a sample is in-class or out-class. The proposed classifier has many advantages. For example, implementing deep neural network-based classifiers requires training thousands of parameters and large memory, whereas implementing our classifier does not. It only needs sample norms to calculate the confidence score. Besides, deep neural network-based classifiers need relatively large amounts of data

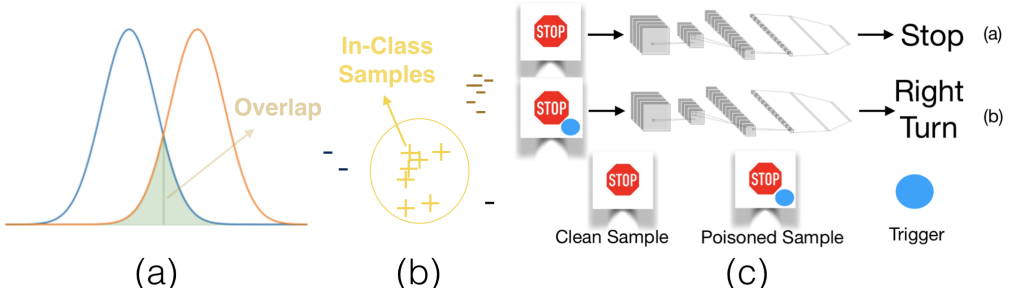

Figure 1: (a): Overlap of two distributions. (b): One-class classification. (c): Backdoor attack.

to avoid under-fitting or over-fitting, whereas our method requires only a small number of in-class samples to be accurate. Therefore, our classifier is computation-efficient, memory-efficient, and data-efficient. Additionally, compared with other traditional one-class classifiers, such as Gaussian distribution-based classifier, Mahalanobis distance-based classifier (Lee et al., 2018) and one-class support vector machine (Schölkopf et al., 2001), our classifier is distribution-free, explainable, and easy to understand.

Overall, the contributions of this paper include:

- Finding a distribution-free upper bound for the overlap index.
- Applying this bound to the domain shift analysis problem.
- Proposing a novel one-class classifier with the bound being the confidence score function.
- Evaluating the proposed one-class classifier through comparison with various state-of-the-art methods on several datasets, including UCI datasets, CIFAR-100, sub-ImageNet, etc., and in different one-class classification scenarios, such as novelty detection, out-of-distribution detection, and neural network backdoor detection.

1.1 Background and Related Works

Measuring the similarity between distributions: Gini & Livada (1943) and Weitzman (1970) introduced the concept of the distribution overlap index. Other measurements for the similarity between distributions include the total variation distance, Kullback-Leibler divergence (Kullback & Leibler, 1951), Bhattacharyya's distance (Bhattacharyya, 1943), and Hellinger distance (Hellinger, 1909). In psychology, some effect size measures' definitions involve the concept of the distribution overlap index, such as Cohen's U index (Cohen, 2013), McGraw and Wong's CL measure (McGraw & Wong, 1992), and Huberty's I degree of non-overlap index (Huberty & Lowman, 2000). However, they all have strong distribution assumptions (e.g., symmetry or unimodality) regarding the overlap index. Pastore & Calcagnì (2019) approximates the overlap index via kernel density estimators.

One-class classification: Moya & Hush (1996) coined the term one-class classification. One-class classification intersects with novelty detection, anomaly detection, out-of-distribution detection, and outlier detection. Yang et al. (2021) explains the differences among these detection areas. Khan & Madden (2014) discusses many traditional non neural network-based one-class classifiers, such as one-class support vector machine (Schölkopf et al., 2001), decision-tree (Comité et al., 1999), and one-class nearest neighbor (Tax, 2002). Two neural network-based one-class classifiers are (Ruff et al., 2018) and OCGAN (Perera et al., 2019). Morteza & Li (2022) introduces a Gaussian mixture-based energy measurement and compares it with several other score functions, including maximum softmax score (Hendrycks & Gimpel, 2017), maximum Mahalanobis distance (Lee et al., 2018), and energy score (Liu et al., 2020a) for one-class classification.

Neural network backdoor attack and detection: Gu et al. (2019) and Liu et al. (2018b) mentioned the concept of the neural network backdoor attack. The attack contains two steps: during training, the attacker injects triggers into the training dataset; during testing, the attacker leads the network to misclassify by presenting the triggers (i.e., Fig. 1(c)). The data poisoning attack (Biggio et al., 2012) and adversarial attack (Goodfellow et al., 2014) overlap with the backdoor attack. Some proposed trigger types are Wanet (Nguyen & Tran, 2021), invisible sample-specific (Li et al., 2021), smooth

(Zeng et al., 2021), and reflection (Liu et al., 2020b). Some methods protecting neural networks from backdoor attacks include neural cleanse (Wang et al., 2019), fine-pruning (Liu et al., 2018a), and STRIP (Gao et al., 2019). NNoculation (Veldanda et al., 2021) and RAID (Fu et al., 2022) utilize online samples to improve their detection methods. The backdoor detection problem also intersects with one-class classification. Therefore, some one-class classifiers can detect poisoned samples against the neural network backdoor attack.

1.2 Organization of the Paper

We first provide preliminaries and derive the proposed upper bound for the overlap index in Sec. 2. We next apply our bound to domain shift analysis in Sec. 3. We then propose, analyze, and evaluate our novel one-class classifier in Sec. 4. We finally conclude the paper in Section 5.

2 An Upper Bound for the Overlap Index

2.1 Preliminaries

For simplicity, we consider the $\mathbb{R}^n$ space and continuous random variables.

Definition 1 (Overlap Index)**.** *With P and Q being two probability distributions in $\mathbb{R}^n$ and with f_P and f_Q being their probability density functions, the overlap $\eta : \mathbb{R}^n \times \mathbb{R}^n \to [0, 1]$ of the two distributions is defined as*

$$\eta(P, Q) = \int_{\mathbb{R}^n} \min[f_P(x), f_Q(x)]dx. \tag{1}$$

Definition 2 (Total Variation Distance)**.** *With P and Q being defined in* **Definition** *1, the total variation distance $\delta : \mathbb{R}^n \times \mathbb{R}^n \to [0, 1]$ of the two distributions is defined as*

$$\delta(P, Q) = \frac{1}{2} \int_{\mathbb{R}^n} |f_P(x) - f_Q(x)|\, dx. \tag{2}$$

Definition 3 (Variation Distance on Subsets)**.** *Given the P and Q defined in* **Definition** *1, and a subset A from $\mathbb{R}^n$, we define $\delta_A : \mathbb{R}^n \times \mathbb{R}^n \to [0, 1]$ to be the variation distance of the two distributions on A, which is*

$$\delta_A(P, Q) = \frac{1}{2} \int_A |f_P(x) - f_Q(x)|dx. \tag{3}$$

Remark 1. *One can prove that η and δ satisfy the following equation:*

$$\eta(P, Q) = 1 - \delta(P, Q) = 1 - \delta_A(P, Q) - \delta_{\mathbb{R}^n \setminus A}(P, Q). \tag{4}$$

The quantity δ_A defined in (3) will play an important role in deriving our upper bound for η.

2.2 The Upper Bound for the Overlap Index

We now proceed with deriving our proposed upper bound.

Theorem 1. *Without loss of generality, assume D^+ and D^- are two probability distributions on a bounded domain $B \subset \mathbb{R}^n$ with defined norm $||\cdot||$ [1] (i.e., $\sup_{x \in B} ||x|| < +\infty$), then for any subset $A \subset B$ with its complementary set $A^c = B \setminus A$, we have*

$$\eta(D^+, D^-) \leq 1 - \frac{1}{2r_{A^c}} ||\mu_{D^+} - \mu_{D^-}|| - \frac{r_{A^c} - r_A}{r_{A^c}} \delta_A \tag{5}$$

where $r_A = \sup_{x \in A} ||x||$ and $r_{A^c} = \sup_{x \in A^c} ||x||$, μ_{D^+} and μ_{D^-} are the means of D^+ and D^-, and δ_A is the variation distance on set A defined in **Definition** *3. Moreover, let $r_B = \sup_{x \in B} ||x||$, then we have*

$$\eta(D^+, D^-) \leq 1 - \frac{1}{2r_B} ||\mu_{D^+} - \mu_{D^-}|| - \frac{r_B - r_A}{r_B} \delta_A. \tag{6}$$

Since (6) holds for any A, a tighter bound can be written as

$$\eta(D^+, D^-) \leq 1 - \frac{1}{2r_B} ||\mu_{D^+} - \mu_{D^-}|| - \max_A \frac{r_B - r_A}{r_B} \delta_A. \tag{7}$$

[1] In this paper, we use the L_2 norm. However, the choice of the norm is not unique and the analysis can be carried out using other norms as well.

Proof. Let f_{D^+} and f_{D^-} be the probability density functions for D^+ and D^-. From (4), we have

$$\eta(D^+, D^-) = 1 - \delta_A(D^+, D^-) - \delta_{A^c}(D^+, D^-). \tag{8}$$

Using (8), triangular inequality, and boundedness, we obtain

$$||\mu_{D^+} - \mu_{D^-}|| = ||\int_B x\left(f_{D^+}(x) - f_{D^-}(x)\right)dx|| \leq \int_B ||x(f_{D^+}(x) - f_{D^-}(x))||dx \tag{9}$$

$$= \int_A ||x|| \cdot |f_{D^+}(x) - f_{D^-}(x)|dx + \int_{A^c} ||x|| \cdot |f_{D^+}(x) - f_{D^-}(x)|dx \tag{10}$$

$$\leq 2r_A\delta_A + 2r_{A^c}\delta_{A^c} = 2r_A\delta_A + 2r_{A^c}(1 - \delta_A - \eta(D^+, D^-)) \tag{11}$$

which implies (5). Replacing r_{A^c} with r_B in (11) implies (6). □

Remark 2. *The only assumption in this theorem is that the probability distribution domain is bounded. However, almost all real-world applications satisfy the boundedness assumption since the data is bounded. Therefore, r_B can always be found (or at least a reasonable approximation can be found). Additionally, we can constrain A to be a bounded ball so that r_A is also known. Although the proof of this theorem involves probability density functions, the computation does not require knowing the probability density functions but only finite samples because we can use the law of large numbers to estimate $||\mu_{D^+} - \mu_{D^-}||$ and δ_A, which will be shown next.*

2.3 Approximating the Bound with Finite Samples

Let $g : B \to \{0, 1\}$ be a condition function[2] and define $A = \{x \mid g(x) = 1, x \in B\}$. According to the definition of δ_A and triangular inequality, we have

$$\delta_A(D^+, D^-) = \frac{1}{2}\int_A |f_{D^+}(x) - f_{D^-}(x)|dx \geq \frac{1}{2}|\int_A f_{D^+}(x) - f_{D^-}(x)dx| \tag{12}$$

$$= \frac{1}{2}\left|\int_B f_{D^+}(x)g(x)dx - \int_B f_{D^-}(x)g(x)dx\right| = \frac{1}{2}\left|\mathbb{E}_{D^+}[g] - \mathbb{E}_{D^-}[g]\right|. \tag{13}$$

Calculating $\mathbb{E}_{D^+}[g]$ and $\mathbb{E}_{D^-}[g]$ is easy: one just needs to draw samples from D^+ and D^-, and then average their g values. Applying (13) into **Theorem** 1 gives the following corollary:

Corollary 1. *Given D^+, D^-, B, and $||\cdot||$ used in* **Theorem** *1, let $A(g) = \{x \mid g(x) = 1, x \in B\}$ with any condition function $g : B \to \{0, 1\}$. Then, an upper bound for $\eta(D^+, D^-)$ that can be obtained by our approximation is*

$$\eta(D^+, D^-) \leq 1 - \frac{1}{2r_B}||\mu_{D^+} - \mu_{D^-}|| - \max_g \frac{r_B - r_{A(g)}}{2r_B}\left|\mathbb{E}_{D^+}[g] - \mathbb{E}_{D^-}[g]\right|. \tag{14}$$

Given several condition functions $\{g_j\}_{j=1}^k$ and finite sample sets (i.e., $\{x_i^+\}_{i=1}^n \sim D^+$ and $\{x_i^-\}_{i=1}^m \sim D^-$), Alg. 1 shows how to compute the RHS of (14).

Algorithm 1 ComputeBound($\{x_i^+\}_{i=1}^n$, $\{x_i^-\}_{i=1}^m$, $\{g_j\}_{j=1}^k$)

$B \leftarrow \{x_1^+, x_2^+, ..., x_n^+, x_1^-, x_2^-, ..., x_m^-\}$ and $r_B \leftarrow \max_{x \in B} ||x||$
$\Delta_\mu \leftarrow \left|\left|\frac{1}{n}\sum_{i=1}^n x_i^+ - \frac{1}{m}\sum_{i=1}^m x_i^-\right|\right|$
for $j = 1 \to k$ **do**
  $A = \{x \mid g_j(x) = 1, x \in B\}$ and $r_A \leftarrow \max_{x \in A} ||x||$
  $s_j \leftarrow \left(1 - \frac{r_A}{r_B}\right)\left|\frac{1}{n}\sum_{i=1}^n g_j(x_i^+) - \frac{1}{m}\sum_{i=1}^m g_j(x_i^-)\right|$
end for
Return: $1 - \frac{1}{2r_B}\Delta_\mu - \frac{1}{2}\max_j s_j$

Remark 3. *The choice of condition functions is not unique. In this work, we use the indicator function $g(x) = \mathbb{1}\{||x|| \leq r\}$, which outputs 1 if $||x|| \leq r$ and 0 otherwise. By setting different values for r, we generate a family of condition functions.*

[2]The condition function is an indicator function $\mathbb{1}\{condition\}$ that outputs 1 when the input satisfies the given condition and 0 otherwise.

3 APPLICATION OF OUR BOUND TO DOMAIN SHIFT ANALYSIS

We now apply our bound to domain shift analysis.

Theorem 2. *Assume that D and D^* are two different data distributions (i.e., $\eta(D, D^*) < 1$). If a model is trained on D with p accuracy on D (including the overlap area between D and D^*) and q accuracy on D^* (excluding the overlap area between D and D^*), then the overall accuracy of the model on D^* is $p\eta(D, D^*) + q(1 - \eta(D, D^*))$ and is upper bounded due to (14).*

Remark 4. *To prove the theorem, let f_D and f_{D^*} be their probability density functions, then*

$$\textit{Accuracy} = \int_{x \sim D^*} \left(p \frac{\min\{f_D(x), f_{D^*}(x)\}}{f_{D^*}(x)} + q \left(1 - \frac{\min\{f_D(x), f_{D^*}(x)}{f_{D^*}(x)} \right) \right) f_{D^*}(x) dx \quad (15)$$

$$= p\eta(D, D^*) + q(1 - \eta(D, D^*)) = (p - q)\eta(D, D^*) + q. \quad (16)$$

Without loss of generality, assume $p > q$. Then (16) shows that a large domain shift (i.e., a small $\eta(D, D^*)$) leads to a low overall accuracy of the model on the testing data distribution D^*. If $p = 1$ and $q = 0$, then the overall accuracy is equal to $\eta(D, D^*)$. With a finite-sample dataset and (14), one can upper bound the overall accuracy of the model when the domain shift exists.

4 APPLICATION OF OUR BOUND TO ONE-CLASS CLASSIFICATION

As a second application of our bound, we now consider one-class classification.

4.1 PROBLEM FORMULATION FOR ONE-CLASS CLASSIFICATION

Given $\mathbb{R}^d$ space and n samples $\{x_i\}_{i=1}^n$ that lie in an unknown probability distribution, we would like to build a test $\Psi : \mathbb{R}^d \to \{\pm 1\}$ so that for any new input x, $\Psi(x)$ outputs 1 when x is from the same unknown probability distribution, and outputs -1, otherwise. Some applications of Ψ are novelty detection, out-of-distribution detection, and backdoor detection (e.g., Fig. 1(b, c)).

4.2 A NOVEL CONFIDENCE SCORE FUNCTION

Given some in-class samples $\{x_i\}_{i=1}^n$, one can pick several condition functions $\{g_j\}_{j=1}^k$, where $g_j(x) = \mathbb{1}\{||x|| \leq r_j\}$ for different r_j, so that $f(x) = ComputeBound(\{x\}, \{x_i\}_{i=1}^n, \{g_j\}_{j=1}^k)$ defined in Alg. 1 is a score function that measures the likelihood of any input, x, being an in-class sample. Alg. 2 shows the overall one-class classification algorithm.

Algorithm 2 The Novel One-Class Classifier for the Input x

Given in-class samples $\{x_i\}_{i=1}^n$, select several condition functions $\{g_j\}_{j=1}^k$, set a threshold T_0
if $ComputeBound(\{x\}, \{x_i\}_{i=1}^n, \{g_j\}_{j=1}^k) \geq T_0$ **then**
  x is an in-class sample
else
  x is an out-class sample
end if

Remark 5. *The score function f measures the maximum similarity between the new input x and the available in-class samples $\{x_i\}_{i=1}^n$. Different T_0 lead to different detection accuracy. However, we will show that the proposed one-class classifier has an overall high accuracy under different T_0.*

4.3 COMPUTATION AND SPACE COMPLEXITIES

Our algorithm can pre-compute and store $\frac{1}{n}\sum_{i=1}^n x_i$ and $\frac{1}{n}\sum_{i=1}^n g_j(x_i)$ with $j = 1, 2, ..., k$. Therefore, the total space complexity is $\mathcal{O}(k+1)$. Assume that the total number of new online inputs is l; then, for every new input x, our algorithm needs to calculate $||x||$ once and s_j for k times. Therefore, the total computation complexity is $\mathcal{O}(l(k+1))$. Empirically, we restricted k to be a small number (e.g., 10) so that even devices without strong computation power can run our algorithm efficiently. Therefore, our classifier is computation-efficient and memory-efficient.

Table 1: Means and standard deviations of AUROC (%) for different methods on 100 UCI datasets.

Ours	L1-Ball	K-Center	Parzen	Gaussian	K-Mean
79 ± 15.6	65.4 ± 25.4	72.2 ± 17.6	77.6 ± 18.6	81.1 ± 16.6	76.3 ± 17.9
1-Nearest Neighbor	K-Nearest Neighbor	Auto-Encoder Network	Linear Programming	Principal Component	Lof Range
78.8 ± 18.3	79.2 ± 18.3	73.7 ± 17.9	77.5 ± 18.9	75.1 ± 18.3	56.5 ± 30.8
Nearest Neighbor Distance	Minimum Spanning Tree	Minimum Covariance Determinant	Self Organizing Map	Support Vector Machine	Minimax Probability Machine
77.5 ± 16.6	78.8 ± 17.4	80.4 ± 17.1	77.6 ± 17.4	69.4 ± 24.6	66.4 ± 33.3
Mixture Gaussians	Local Outlier Factor	Naive Parzen	Local Correlation Integral		
78.7 ± 18.1	76.1 ± 19.7	78.4 ± 16.2	60.7 ± 23.9		

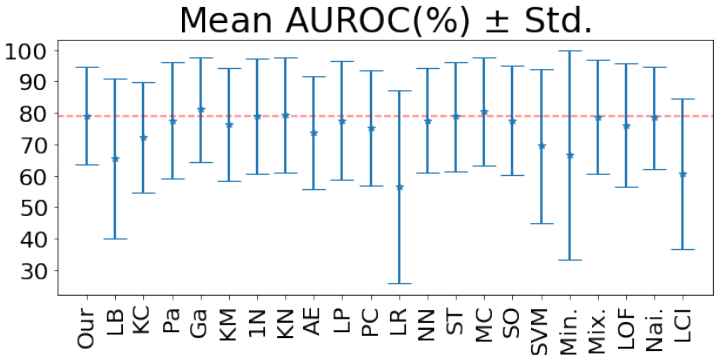

Figure 2: The mean and standard deviation of each method listed in Table 1 in the same order.

4.4 Evaluation

4.4.1 One-Class Classification for Novelty Detection

We evaluated our classifier on 100 small UCI datasets (UCI; Dua & Graff, 2017) and recorded the area under the receiver operating characteristic curve (AUROC). Both Table 1 and Fig. 2 show the mean and standard deviation of AUROC for ours and other classifiers. The detailed results for each individual dataset and the implementation code are in the supplementary material.

From the table, our classifier is the most consistent with the smallest standard deviation. Except for *Gaussian*, *K-Nearest Neighbor*, and *Minimum Covariance Determinant*, our classifier outperforms the other methods by showing the highest AUROC mean and the lowest AUROC standard deviation. Among *Gaussian*, *K-Nearest Neighbor*, and *Minimum Covariance Determinant*, *Gaussian* is the best classifier by having the highest mean and lowest standard deviation of AUROC. However, our method is comparable to *Gaussian* by showing a close mean and a smaller standard deviation.

Besides the results, our classifier is distribution-free, computation-efficient, and memory-efficient, whereas some other classifiers do not. Additionally, our method is also easy to explain and understand: the score measures the maximum similarity between the new input and the available in-class samples. Therefore, we conclude that our classifier is valid for novelty detection.

4.4.2 One-Class Classification for Out-of-Distribution Detection

We used CIFAR-10 and CIFAR-100 (Krizhevsky et al., 2009) as the in-distribution datasets. The compared methods contain MSP (Hendrycks & Gimpel, 2017), Mahalanobis (Lee et al., 2018),

Energy score (Liu et al., 2020a), and GEM (Morteza & Li, 2022). We used WideResNet (Zagoruyko & Komodakis, 2016) to extract features from the raw data. To fit the model parameters for all the methods, we formed a small dataset by randomly selecting 10 samples from each class. The out-of-distribution datasets include Textures (Cimpoi et al., 2014), SVHN (Netzer et al., 2011), LSUN-Crop (Yu et al., 2015), LSUN-Resize (Yu et al., 2015), and iSUN (Xu et al., 2015). We used three metrics: the detection accuracy for out-of-distribution samples when the detection accuracy for in-distribution samples is 95% (TPR95), AUROC, and area under precision and recall (AUPR).

Table 2: Results for CIFAR-10 in-distribution case (higher number implies higher accuracy). **Boldface** shows the best performing algorithm, whereas underline shows the second best algorithm.

Out-of-Distribution Datasets	Method	TPR95 (%)	AUROC (%)	AUPR (%)
Texture	Ours	58.41	90.97	85.64
	MSP	40.75	88.31	97.08
	Mahalanobis	62.38	94.46	98.75
	Energy Score	47.47	85.47	95.58
	GEM	**72.61**	**94.59**	**98.79**
SVHN	Ours	**88.31**	**96.23**	98.49
	MSP	52.41	92.11	98.32
	Mahalanobis	79.34	95.72	**99.04**
	Energy Score	64.20	91.05	97.66
	GEM	79	95.65	99.01
LSUN-Crop	Ours	79.45	94.33	94.60
	MSP	69.07	95.64	99.13
	Mahalanobis	30.06	86.15	97.05
	Energy Score	**91.89**	**98.40**	**99.67**
	GEM	30.20	86.09	97.03
LSUN-Resize	Ours	69.92	92.03	92.54
	MSP	47.45	91.30	98.11
	Mahalanobis	35.64	88.12	97.45
	Energy Score	**71.75**	**94.12**	**98.64**
	GEM	35.45	88.09	97.43
iSUN	Ours	65.60	90.25	90.33
	MSP	43.40	89.72	97.72
	Mahalanobis	26.77	87.87	97.33
	Energy Score	**66.27**	**92.56**	**98.25**
	GEM	36.80	87.85	93.33
Average Performance	Ours	**72.33**	**92.76**	92.32
	MSP	50.63	91.46	**98.07**
	Mahalanobis	46.83	90.46	97.92
	Energy Score	68.31	92.32	97.96
	GEM	50.81	90.45	97.91

Table 2 shows the results for CIFAR-10, and Table 3 shows the results for CIFAR-100. Our method outperforms the other methods by showing the highest AUROC on average. For the CIFAR-10 case, our method also shows the highest average TPR95, and the average AUPR of our method is over 92%. For the CIFAR-100 case, the average TPR95 of our method is 0.3% close to the highest average TPR95, and the average AUPR of our method is over 85%. For each individual out-of-distribution dataset, our method always outperforms no less than half of the methods in TPR95 and AUROC, and the total average AUPR of our method over all cases is 89.02%. Note that the out-of-distribution datasets are much smaller in size than the in-distribution datasets. Therefore, we see a potential improvement in our method to increase the AUPR for heavily imbalanced problems in our prospective work. In summary, our classifier is a valid out-of-distribution detector.

4.4.3 One-Class Classification for Backdoor Detection

The utilized datasets are MNIST (LeCun et al., 2010), CIFAR-10 (Krizhevsky et al., 2009), GTSRB (Stallkamp et al., 2011), YouTube Face (Wolf et al., 2011), and sub-ImageNet (Deng et al., 2009). The adopted backdoor attacks include naive triggers, all-label attacks (Gu et al., 2019), moving

Table 3: Results for CIFAR-100 in-distribution case (higher number implies higher accuracy). **Boldface** shows the best performing algorithm, whereas underline shows the second best algorithm.

Out-of-Distribution Datasets	Method	TPR95 (%)	AUROC (%)	AUPR (%)
Texture	Ours	46.13	87.52	80.57
	MSP	16.71	73.58	93.02
	Mahalanobis	**57.62**	90.14	97.62
	Energy Score	20.38	76.46	93.68
	GEM	57.40	**90.17**	**97.63**
SVHN	Ours	**90.69**	**96.41**	**98.58**
	MSP	15.66	71.37	92.89
	Mahalanobis	51.35	89.25	97.52
	Energy Score	14.59	74.10	93.65
	GEM	51.51	89.40	97.57
LSUN-Crop	Ours	27.93	80.09	79.32
	MSP	33.44	83.71	96.32
	Mahalanobis	1.53	58.48	89.73
	Energy Score	**64.01**	**93.41**	**98.59**
	GEM	1.70	58.42	89.70
LSUN-Resize	Ours	39.80	88.15	86.79
	MSP	16.54	75.32	94.03
	Mahalanobis	**67.20**	93.97	**98.70**
	Energy Score	21.38	79.29	94.97
	GEM	67.09	**94.01**	**98.70**
iSUN	Ours	35.87	86.01	83.37
	MSP	17.02	75.87	94.20
	Mahalanobis	64.07	92.69	**98.32**
	Energy Score	19.20	78.98	94.90
	GEM	**64.10**	**92.73**	**98.32**
Average Performance	Ours	48.08	**87.63**	85.72
	MSP	19.87	75.97	94.09
	Mahalanobis	48.35	84.90	96.37
	Energy Score	27.91	80.44	95.15
	GEM	**48.36**	84.94	**96.38**

triggers (Fu et al., 2020), Wanet (Nguyen & Tran, 2021), combination attacks, large-sized triggers, filter triggers, and invisible sample-specific triggers (Li et al., 2021), as listed in Fig. 3. The neural network architecture includes Network in Network (Lin et al., 2014), Resnet (He et al., 2016), and other networks from (Wang et al., 2019; Gu et al., 2019). For each backdoor attack, we assume that a small clean validation dataset is available (i.e., 10 samples from each class) at the beginning. Therefore, the poisoned samples (i.e., samples attached with triggers) can be considered out-class samples, whereas the clean samples can be considered in-class samples. We used the backdoored network to extract data features. Then, we evaluated our one-class classifier and compared it with STRIP (Gao et al., 2019) in the feature space. The metrics used are the same: TPR95 (i.e., the detection accuracy for poisoned samples when the detection accuracy for clean samples is 95%), AUROC, and AUPR. Table 4 shows the results.

From the table, our classifier outperforms STRIP in all the datasets with all the triggers by showing higher TPR95, AUROC, and AUPR except for the YouTube Face dataset with sunglasses trigger. Compared with STRIP on the overall average performance, our classifier is 49.8% higher in TPR95, 26.38% higher in AUROC, and 26.66% higher in AUPR. For each individual trigger, the TPR95 of our method is over 96% for most cases, the AUROC of our method is over 97% for most cases, and the AUPR of our method is over 95% for most cases. It is also seen that our classifier is robust against the latest or advanced backdoor attacks, such as Wanet, invisible trigger, all label attack, and filter attack, whereas STRIP shows low accuracy against those attacks. Therefore, we conclude that our classifier is valid for backdoor detection.

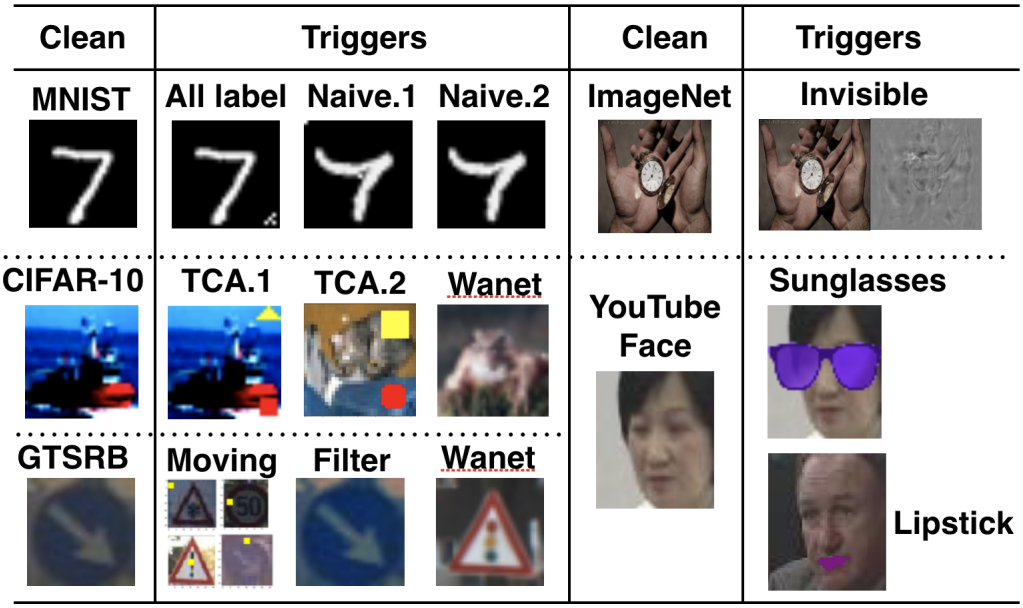

Figure 3: Clean samples and triggers.

Table 4: Comparison results for backdoor detection (higher number implies higher accuracy).

Datasets	Triggers	TPR95 (%)		AUROC (%)		AUPR (%)	
		Ours	STRIP	Ours	STRIP	Ours	STRIP
MNIST	All label	83.05	2.58	96.13	44.69	94.20	35.47
	Naive.1	100	98.85	97.50	97.32	96.17	95.95
	Naive.2	96.53	67.46	97.28	93.67	95.75	89.85
CIFAR-10	TCA.1	100	35.68	97.50	83.00	95.47	73.22
	TCA.2	100	27.86	97.50	68.79	97.63	72.41
	Wanet	37.87	0.07	92.74	34.97	89.95	37.42
GTSRB	Moving	99.99	54.	97.50	91.69	96.96	89.07
	Filter	85.39	7.29	96.54	38.92	95.42	38.81
	Wanet	100	1.24	97.50	36.31	97.62	39.53
YouTube Face	Sunglasses	73.37	83.03	95.21	94.80	93.00	95.54
	Lipstick	96.64	90.14	97.21	93.15	96.30	94.98
sub-ImageNet	Invisible	100	7.01	97.49	66.26	96.53	62.83
Average		89.40	39.60	96.68	70.30	95.42	68.76

5 Conclusion

This paper proposes an easy-to-compute distribution-free upper bound for the distribution overlap index. Two applications of the bound are explored. The first application is for domain shift analysis with a proposed theorem and discussion. The second application is for one-class classification. Specifically, this paper introduces a novel distribution-free one-class classifier with the bound being its confidence score function. The classifier is sample-efficient, computation-efficient, and memory-efficient. The proposed classifier is evaluated on novelty detection, out-of-distribution detection, and backdoor detection on various datasets and compared with many state-of-the-art methods. The obtained results show significant promise toward broadening the application of overlap-based metrics.

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