FACTOR GRAPH OPTIMIZATION OF ERROR-CORRECTING CODES FOR BELIEF PROPAGATION DECODING

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ABSTRACT

The design of optimal linear block codes capable of being efficiently decoded is of major concern, especially for short block lengths. As near capacity-approaching codes, Low-Density Parity-Check (LDPC) codes possess several advantages over other families of codes, the most notable being its efficient decoding via Belief Propagation. While many LDPC code design methods exist, the development of efficient sparse codes that meet the constraints of modern short code lengths and accommodate new channel models remains a challenge. In this work, we propose for the first time a gradient-based data-driven approach for the design of sparse codes. We develop locally optimal codes with respect to Belief Propagation decoding via the learning of the Factor graph under channel noise simulations. This is performed via a novel complete graph tensor representation of the Belief Propagation algorithm, optimized over finite fields via backpropagation and coupled with an efficient line-search method. The proposed approach is shown to outperform the decoding performance of existing popular codes by orders of magnitude and demonstrates the power of data-driven approaches for code design.

1 INTRODUCTION

Reliable digital communication is of major importance in the modern information age and involves the design of codes that can be robustly and efficiently decoded despite noisy transmission channels. Over the last half-century, significant research has been dedicated to the study of capacity-approaching Error Correcting Codes (ECC) (Shannon, 1948). Despite the initial focus on short and medium-length linear block codes (Berlekamp, 1974), the development of long channel codes (Forney, 1966; Costello & Forney, 2007) has emerged as a viable approach to approaching channel capacity (Berrou et al., 1993; MacKay, 1999; Richardson et al., 2001; Richardson & Urbanke, 2001; Arikan, 2008; Luby et al., 2001; Kudekar et al., 2011).

While the NP-hard maximum likelihood rule defines the target decoding of a given code, developing more practical solutions generally relies on theories grounded upon asymptotic analysis over conventional communication channels. However, modern communication systems rely on the design of short and medium-block-length codes (Liva et al., 2016) and the latest communication settings provide new types of channels. This is mainly due to emergent applications in the modern wireless realm requiring the transmission of short data units, such as remote command links, Internet of Things, and messaging services (De Cola et al., 2011; Boccardi et al., 2014; Paolini et al., 2015; Durisi et al., 2016; ESTI, 2021). These challenges call for the formulation of data-driven solutions, capable of adapting to various settings of interest and constraints, generally uncharted by existing theories.

The vast majority of existing machine-learning solutions to the ECC problem concentrate on the design of *neural decoders*. The first neural models focused on the implementation of parameterized versions of the legacy Belief Propagation (BP) decoder (Nachmani et al., 2016; 2018; Lugosch & Gross, 2017; Nachmani & Wolf, 2019; Buchberger et al., 2020). Recently, state-of-the-art learning-based de novo decoders have been introduced, borrowing from well-proven architectures from other domains. A Transformer-based decoder that incorporates the code into the architecture has been recently proposed by (Choukroun & Wolf, 2022a), outperforming existing methods by sizable margins and at a fraction of their time complexity. This architecture has been subsequently integrated

into a denoising diffusion models paradigm, further improving results (Choukroun & Wolf, 2022b). Subsequently, a universal neural decoder has been proposed in (Choukroun & Wolf, 2024c), capable of unified decoding of codes from different families, lengths, and rates. Most recently and related to our work, (Choukroun & Wolf, 2024a) developed an end-to-end learning framework capable of co-learning binary linear block codes along with the neural decoder.

However, neural decoding methods require increased computational and memory complexity compared to their well-established classical counterparts. Due to these challenges, and the non-trivial acceleration and implementation required, neural decoders were never deployed in real-world systems, as far as we know.

In this work, given the ubiquity and advantages of the Belief Propagation (BP) algorithm (Pearl, 1988; Richardson et al., 2001) for sparse codes, we consider the optimization of codes with respect to BP via the learning of the underlying factor/Tanner graph. From a graphical probabilistic model perspective (Koller & Friedman, 2009), BP being a marginalization algorithm, a gradient-based of a score metric method is given for the *structure learning* of BP's underlying Bayesian network in an end-to-end fashion. As far as we can ascertain, this is the first time a gradient-based data-driven solution is given for the design of the codes themselves based on a classical decoder. Such a solution induces a very low overhead (if any) for integration into the existing decoding solutions.

Beyond the conceptual novelty, we make three technical contributions: (i) we formulate the data-driven optimization objective adapted to the setting of interest (e.g., channel noise, code structure), (ii) we reformulate BP in a tensor fashion to learn the connectivity of the factor graph through backpropagation, and (iii) we propose a differentiable and fast optimization approach via a line-search method adapted to the relaxed binary programming setting. Applied to a wide variety of codes, our method produces codes that outperform existing codes on various channel noise settings, demonstrating the power and flexibility of the method in adapting to realistic settings of interest.

2 Related Works

Neural decoder or data-driven contributions generally focus on short and moderate-length codes for two main reasons. First, classical decoders reach the capacity of the channel for large codes, and second, the emergence of short data units applications driven by the Internet of Things (e.g., smart metering networks, messaging services, etc.) requires effective decoders for short to moderate-length codes. For example, 5G Polar codes have code lengths of 32 to 1024 (Liva et al., 2016; ESTI, 2021).

Previous work on neural decoders is generally divided into two main classes: model-free and model-based. Model-free decoders employ general types of neural network architectures (Cammerer et al., 2017; Gruber et al., 2017; Kim et al., 2018b; Bennatan et al., 2018; Jiang et al., 2019a; Choukroun & Wolf, 2022a;b; 2024c;b). Model-based decoders implement parameterized versions of classical Belief Propagation (BP) decoders, where the Tanner graph is unfolded into an NN in which scalar weights are assigned to each variable edge. This results in an improvement in comparison to the baseline BP method for short codes (Nachmani et al., 2016; Nachmani & Wolf, 2019; Raviv et al., 2020; 2023; Kwak et al., 2023). While model-based decoders benefit from a strong theoretical background, the architecture is overly restrictive, which generally enforces its coupling with high-complexity NN (Nachmani & Wolf, 2021). Also, the improvement gain generally vanishes for more iterations and longer codewords (Hoydis et al., 2022) and the integration cost remains very high due to both computational and memory requirements.

While neural decoders show improved performance in various communication settings, there has been very limited success in the design of novel neural coding methods, which remain impracticable for deployment (O'Shea & Hoydis, 2017; Kim et al., 2018a; Jiang et al., 2019b). Recently, (Choukroun & Wolf, 2024a) provided a new differentiable way of designing binary linear block codes (i.e., parity-check matrices) for a given neural decoder also showing improved performance with classical decoders.

Belief-propagation decoding has multiple advantages for LDPC codes (Gallager, 1962; Richardson & Urbanke, 2001; Richardson et al., 2001). A large number of LDPC code (parity check matrix) design techniques exist in the literature, depending on the design criterion. Among them, Gallager (Gallager, 1962) developed the first regular LDPC codes as the concatenations of permuted sub-matrices. MacKay (MacKay & Neal, 1995) demonstrated the ability of sparse codes to reach near-

capacity limits via semi-randomly generated matrices. Irregular LDPC codes have been developed by (Richardson et al., 2001; Luby et al., 2001; Chung et al., 2001) where the decoding threshold can be optimized via density-evolution. Progressive Edge Growth (Hu et al., 2001; 2005) has been proposed to design large girth codes. Certain classes of LDPC array codes have been presented in (Eleftheriou & Olcer, 2002) and LDPC codes with combinatorial design constraints have been developed in (Vasic & Milenkovic, 2004). Finite geometry codes have been developed in (Lucas et al., 2000; Kou et al., 2001) and repeat-accumulate codes have been proposed by (Yang et al., 2004; Jin et al., 2000; Narayanaswami, 2001). However, the classical methods are not data-driven and are difficult to adapt to the design of codes under constrained settings of interest (e.g., short codes, modern channels, structure constraints, etc.). Most related to our work are methods for structure learning (Koller & Friedman, 2009) for Bayesian networks such as the Chow-Liu Algorithm (Chow & Liu, 1968) or search-based methods (Tian, 2013). Related to greedy search-based methods, Elkelesh et al. (2019) recently suggested the application of classical genetic algorithms for the discovery of better IRA codes.

3 BACKGROUND

We assume a standard transmission protocol using a linear block code C. The code is defined by a generator matrix $G \in \{0,1\}^{n\times k}$ and the parity check matrix $H \in \{0,1\}^{(n-k)\times n}$ is defined such that $HG^T = 0$ over the order 2 Galois field $GF(2)$. The parity check matrix H entails what is known as a Tanner graph (Tanner, 1981), which consists of n variable nodes and $(n-k)$ check nodes. The edges of this bipartite graph correspond to the on-bits of the matrix H.

The input message $m \in \{0,1\}^k$ (column vector) is encoded by G to a codeword $c \in C \subset \{0,1\}^n$ satisfying $Hc = H(Gm) = 0$ and transmitted via a Binary-Input Symmetric-Output channel, e.g., an AWGN channel. Let y denote the channel output represented as $y = c_s + \varepsilon$, where c_s denotes the transmission modulation of c (e.g., Binary Phase Shift Keying (BPSK)), and ε is random noise independent of the transmitted c. The main goal of the decoder $f_H : \mathbb{R}^n \to \mathbb{R}^n$ conditioned on the code (i.e., H) is to provide a soft approximation $\hat{x} = f_H(y)$ of the codeword.

The Belief Propagation algorithm allows the iterative transmission (propagation) of a current codeword estimate (belief) via a Trellis graph determined according to a *factor graph* defined by the code (i.e., the Tanner graph). The factor graph is unrolled into a Trellis graph, initiated with n variable nodes, and composed of two types of interleaved layers defined by the check/factor nodes and variable nodes. An illustration of the Tanner graph unrolled to the Trellis graph is given in Figure 1.

As a message-passing algorithm, Belief Propagation operates on the Trellis graph by propagating the messages from variable nodes to check nodes and from check nodes to variable nodes, in an alternative and iterative fashion. The input layer generally corresponds to the vector of log-likelihood ratios (LLR) $L \in \mathbb{R}^n$ of the channel output y defined as

$$L_v = \log\left(\frac{\Pr\left(c_v = 1|y_v\right)}{\Pr\left(c_v = 0|y_v\right)}\right).$$

Here, we describe ECC's classical notation of BP with, $v \in \{1,\ldots,n\}$ denotes the index corresponding to the vth element of the channel output y, for the corresponding bit c_v we wish to recover.

Let x^i be the vector of messages that a column/layer in the Trellis graph propagates to the next one. At the first round of message passing, a variable node type of computation is performed such that

$$x_e^{2k+1} = x_{(c,v)}^{2k+1} = L_v + \sum_{e' \in N(v)\setminus\{(c,v)\}} x_{e'}^{2k}. \tag{1}$$

Here, each message indexed by the edge $e = (c,v)$ on the Tanner graph and $N(v) = \{(c,v)|H(c,v) = 1\}$, i.e, the set of all edges in which v participates. By definition $x^0 = 0$ such that the messages are directly determined by the vector L for $k = 0$.

For even layers, the check layer performs the following

$$x_e^{2k} = x_{(c,v)}^{2k} = 2\text{arctanh}\left(\prod_{e' \in N(c)\setminus\{(c,v)\}} \tanh\left(\frac{x_{e'}^{2k-1}}{2}\right)\right) \tag{2}$$

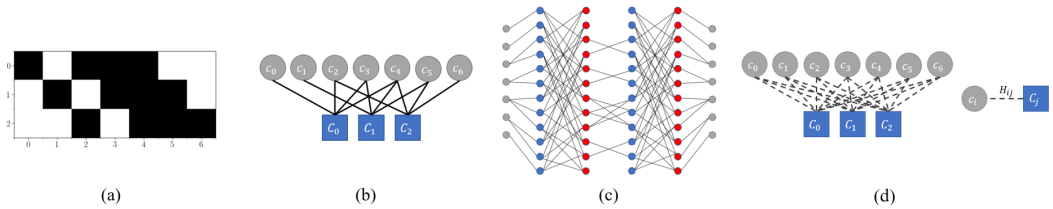


Figure 1: For the Hamming(7,4) Code: (a) parity check matrix, the induced (b) Tanner graph, and (c) the corresponding unrolled Trellis graph with two iterations, with odd layers in blue and even layers in red. In (d) we present our approach for structure learning via the learned binary weighting of the edges of the *complete* bipartite factor graph unlike the conventional sparse representation.

where $N(c) = \{(c, v) | H(c, v) = 1\}$ is the set of edges in the Tanner graph in which row c of the parity check matrix H participates.

The final v^{th} output layer of the BP algorithm, which corresponds to the soft-decision output of the codeword, is given by

$$o_v = L_v + \sum_{e' \in N(v)} x_{e'}^{2L} \tag{3}$$

4 METHOD

The performance of BP is strongly tied to the underlying Tanner graph induced by the code. BP and its variants are generally implemented over a *fixed* sparse graph, such that the only degree of freedom resides in the number of decoding iterations. While several recent contributions (Nachmani et al., 2016; Nachmani & Wolf, 2019) aim to enhance the BP algorithm by augmenting the Trellis graph with neural networks, these approaches assume and maintain fixed codes. Here, we propose optimizing the code for the BP algorithm on a decoding setting of interest. Given a trainable binary parity check matrix H, we wish to obtain BP-optimized codes by solving the following parameterized optimization problem

$$H^* = \underset{H \in \{0,1\}^{(n-k) \times n}}{\arg\min} \; \mathbb{E}_{m \sim \text{Bern}^k(1/2), \varepsilon \sim \mathcal{Z}, T \in \mathbb{N}_+} \mathcal{D}\Big(f_{H,T}\big(\phi(G(H)m) + \varepsilon\big), m \Big) + \mathcal{R}(H) \tag{4}$$

Here, $G(H)$ denotes a generator matrix defined by H (i.e., $HG^T = 0$), ϕ denotes the modulation function such that $c_s = \phi(c)$, and $\mathcal{Z}$ is the channel noise distribution. $f_{H,T}$ denotes the BP decoder built upon H with T discrete iterations (sampled uniformly from a given set), $\mathcal{D}$ denotes the distance metric of interest, and $\mathcal{R}$ denotes the potential hard/soft regularization of interest, e.g., sparsity or constraints on the code structure.

Several challenges arise from this optimization problem: (i) the optimization is highly non-differentiable and results in an NP-hard binary non-linear integer programming problem, (ii) the computation of the codewords $c = Gm$ is both highly non-differentiable (matrix-vector multiplication over $GF(2)$ in case symmetry is not maintained during the optimization(Richardson & Urbanke, 2001)) and computationally expensive (inverse via Gaussian elimination of H), (iii) the modulation $\phi(\cdot)$ can be non-differentiable, and last but most important, (iv) BP assumes a *fixed* code (i.e., the factor graph edges) upon which the decoder is implemented.

Learning the Factor graph via *Tensor* Belief *Back*propagation To obtain BP codes, we propose a *structure/Tanner graph learning* approach, where the bipartite graph is assumed as **complete** with **learnable** binary-weighted edges. This way, the tensor reformulation of BP weighted by H allows a differentiable optimization of the Tanner graph itself. The two alternating stages of BP can now be represented in a differentiable matrix form rather than its static graph formulation, where the variable layers can be rewritten as

$$Q_{ij} = L_i + \sum_{j' \in C_i \setminus j} R_{j'i} \equiv L_i + \sum_{j'} R_{j'i} H_{j'i} - R_{ji} \,, \tag{5}$$

Algorithm 1: Tensor Belief Propagation

```
function BP(llr, H, iters, eps=1e-7)
    // llr is the batched LLR matrix, (B, n)
    // H is the binary parity-check matrix, (n-k,n)
    // iters is the number of BP iterations
    H = H.unsqueeze(dim=0).T
    C = llr.unsqueeze(dim=-1)
    for t in range(iters) do
        Q = C if t == 0 else C + sum(R*H,dim=-1).unsqueeze(dim=-1) - R
        tmp = tanh(0.5*Q)
        R = 2*atanh( prod(tmp*H+(1-H),dim=1)/tmp )
    return C.squeeze()+sum(R*H,dim=-1)
```

where R_{ij} are the check layers, which are now represented as

$$\begin{aligned} R_{ji} &= 2\text{arctanh}\left(\prod_{i' \in V_j \setminus i} \tanh\left(\frac{Q_{i'j}}{2}\right)\right) \\ &= 2\text{arctanh}\left(\prod_{i'}\left(\tanh\left(\frac{Q_{i'j}H_{ji'}}{2}\right) + (1 - H_{ji'})\right) / \tanh\left(\frac{Q_{ij}}{2}\right)\right), \end{aligned} \tag{6}$$

where C_i and V_j correspond to the non-zero elements in column i and row j of H, respectively, while the ones elements in $(1 - H) \in \{0, 1\}^{(n-k)\times n}$ satisfy the identity element of multiplication. Potential zero denominators have not been observed but can be handled via regularization or omission. As we can observe, BP remains differentiable with respect to H as a composition of differentiable functions.

We provide in Algorithm 1 the pseudo-code for the tensor formulation of the BP algorithm, implementing Eq. 6 and 5.

Belief Propagation Codes Optimization The tensor reformulation solves the major challenge of graph learning (challenge (iv)). Challenges (ii) and (iii) are also eliminated in our formulation. First, since for any given H the conditional independence of error probability under symmetry (Richardson & Urbanke, 2001) is satisfied for message passing algorithms, it is enough to optimize the zero codeword only, i.e., $c = Gm = 0$, removing then any dependency on G in the objective (challenge (ii)). As a byproduct, we obtain that the optimization problem is invariant to the choice of modulation, whether differentiable or not (challenge (iii)).

To optimize H (challenge (i)) we relax the NP-hard binary programming problems to an unconstrained objective where, given a parameter matrix $\Omega \in \mathbb{R}^{(n-k)\times n}$, we have $H := H(\Omega) = \text{bin}(\Omega)$. Here $\text{bin}(\cdot)$ refers to the element-wise binarization operator implemented via the shifted straight-through-estimator (STE) (Bengio et al., 2013) defined such that

$$\text{bin}(u) = (1 - \text{sign}(u))/2\,, \quad \partial\,\text{bin}(u)/\partial u := -0.5\mathbb{1}_{|u|\leq 1} \tag{7}$$

Finally, opting for the binary cross-entropy loss (BCE) as the Bit Error rate (BER) discrepancy measure $\mathcal{D} = \text{BCE}$, we obtain the following empirical risk objective

$$\mathcal{L}(\Omega) = \sum_{t=1}^{T}\sum_{i=1}^{n} \text{BCE}\Big(f_{\text{bin}(\Omega),t}\big(c_s + \varepsilon_i\big), c\Big) + \mathcal{R}(\text{bin}(\Omega)) \tag{8}$$

where $c_s = \phi(c)$ denotes the modulated *zero codeword* and ε_i denotes the ith noise sample drawn from the channel noise distribution. This objective aims to provide optimal decoding on different numbers of (variable) decoding iterations t (Nachmani et al., 2016).

While highly non-convex, the objective is (sub)differentiable when considering the STE definition of the gradient (Bengio et al., 2013; Yin et al., 2019) and thus optimizable via classical first-order

Algorithm 2: Belief Propagation Codes Optimization

```
function Loss(H, x, y, BPiters=5)
    // H is the initial binary parity-check matrix, (n-k,n)
    // BPiters is the number of BP iterations
    return BCE(BP(computeLLR(y), H, BPiters),x)
function BPCodesOptimization(H, iters)
    // H is the initial binary parity-check matrix, (n-k,n)
    // iters is the number of optimization iterations
    Ω = 1-2*H
    for t in range(iters) do
        x,y = getData()
        Loss(bin(Ω),x,y).backward()
        lambdas = Ω/Ω.grad
        lambdas = sorted(lambdas[lambdas>0].view(-1))[:50]
        Ω = Ω - Ω.grad*lambdas[argmin([Loss(bin(Ω -lambda*Ω.grad),x,y)
         for lambda in lambdas]]
        if converged:  break
    return bin(Ω)
```

methods. Since H is binary, only changes in the sign of Ω are relevant for the optimization, so most gradient descent iterations remain ineffective in reducing the objective using conventional small learning-rate regimes. Thus, given the gradient $\nabla_\Omega \mathcal{L}$ computed on sufficiently representative batch, we propose a line-search procedure capable of finding the optimal step size.

Binary Line-Search Conventional first-order optimization methods with small learning rate regimes have two major drawbacks with binarization (Rastegari et al., 2016; Courbariaux et al., 2016). First, they are generally slow since only gradient steps modifying the sign of the binarized tensor induce a modification of the loss. Second, they have difficulties in converging to local minima because of oscillating behavior around zero.

In general, efficient line search methods (Nocedal & Wright, 2006) assume local convexity or a smooth objective (Wolf, 1978) or, alternatively, apply exhaustive search on a given interval. Since this is not our case, we propose a novel efficient *grid*-search approach optimized to our binary programming setting. While classical grid search methods look for the optimal step size on handcrafted predefined sample points, in our binary setting we can search only for the step sizes inducing a flip of the sign in Ω, provably limiting the maximum number of *relevant* grid samples to $n(n-k)$. Thus, the line-search problem is now given by

$$\lambda^* = \arg\min_{\lambda \in \mathcal{I}_\Omega} \mathcal{L}(\Omega - \lambda \nabla_\Omega \mathcal{L}), \qquad \mathcal{I}_\Omega = \{s_{ij} = \frac{(\Omega)_{ij}}{(\nabla_\Omega \mathcal{L})_{ij}} | s_i > 0\}, \tag{9}$$

which corresponds to the (parallelizable) objective evaluation on the obtained grid. The same formulation can support other more practical line-search objectives instead of the cross-entropy loss $\mathcal{L}$, such as the non-differentiable BER or Frame Error Rate (FER) instead of the Bayesian BCE loss.

Training The optimization parameters are the following: the initial H (i.e., initial Ω), the maximum number of optimization steps (if convergence is not reached), the number and quality of the data samples, the grid search length, and the number of BP iterations.

We assume that an initial H is given by the user as the code to be improved. The number of optimization steps is set to 20 iterations. The training noise is sampled randomly per batch in the $\{3, \ldots, 7\}$ normalized SNR (i.e. E_b/N_0) range but can be modified according to the noise setting of interest. The number of data samples per optimization iteration is set to 4.9M for every code as sufficient gradient estimation, and the data samples are required to have non-zero syndrome. Because of computational constraints, the number of BP iterations during training is fixed and set to 5, while other ranges or values of interest can be used instead. For faster optimization, the grid search is heuristically restricted to the first 50 smallest step sizes as the optimal step size is generally in the vicinity of the working point (Appendix C) . Training and experiments are performed on 8×12GB GeForce RTX 2080 Ti GPUs and require 2.96 minutes on average per optimization step.

Table 1: A comparison of the negative natural logarithm of Bit Error Rate (BER) for several normalized SNR values of our method with classical codes. Higher is better. BP results are provided for 5 iterations in the first row and 15 in the second row. The best results are in bold. PEGX means the degree of each node is of $X\%$ under the Progressive Edge Growth construction.

Channel	AWGN						Fading						Bursting					
Method	BP			Our			BP			Our			BP			Our		
E_b/N_0	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6
BCH(63,45)	4.06	4.91	6.04	**5.44**	**6.93**	**8.60**	3.09	3.46	3.90	**3.96**	**4.58**	**5.27**	3.60	4.32	5.19	**4.05**	**5.07**	**6.27**
	4.21	5.24	6.59	**5.70**	**7.35**	**9.16**	3.13	3.55	4.04	**4.10**	**4.80**	**5.56**	3.67	4.52	5.59	**4.21**	**5.40**	**6.85**
CCSDS(128,64)	6.46	9.61	13.99	**7.34**	**10.48**	**14.37**	5.72	7.42	9.47	**6.73**	**8.45**	**10.45**	5.29	7.81	11.25	**6.23**	**8.80**	**11.90**
	7.32	10.83	15.43	**8.61**	**12.26**	**16.00**	6.43	8.29	10.28	**8.05**	**10.07**	**12.37**	5.98	8.85	12.53	**7.39**	**10.43**	**13.28**
LDPC(121,60)	4.81	7.17	10.75	**7.70**	**10.87**	**14.25**	4.10	5.23	6.68	**6.68**	**8.47**	**10.50**	3.97	5.75	8.40	**6.23**	**8.89**	**11.98**
	5.31	7.96	11.85	**8.86**	**11.91**	**14.41**	4.42	5.61	7.04	**7.71**	**9.67**	**11.76**	4.31	6.37	9.25	**7.26**	**10.03**	**12.88**
LDPC(121,80)	6.59	9.68	13.43	**7.77**	**11.21**	**15.06**	4.60	5.80	7.22	**5.55**	**6.90**	**8.36**	5.30	7.60	10.66	**6.23**	**8.87**	**12.19**
	7.35	10.94	15.46	**8.75**	**12.45**	**15.67**	4.97	6.29	7.82	**6.25**	**7.80**	**9.47**	5.81	8.50	12.15	**6.99**	**10.09**	**13.74**
LDPC(128,64)	3.66	4.65	5.80	**5.54**	**7.37**	**9.44**	3.22	3.80	4.44	**4.86**	**5.94**	**7.15**	3.23	4.08	5.09	**3.72**	**5.00**	**6.54**
	4.00	5.16	6.42	**6.56**	**8.70**	**10.81**	3.51	4.18	4.84	**5.64**	**6.85**	**8.14**	3.48	4.51	5.66	**4.13**	**5.72**	**7.66**
LDPC(32,16)	4.36	5.59	7.18	**5.48**	**7.02**	**8.92**	4.03	4.70	5.47	**5.26**	**6.02**	**6.82**	3.88	4.89	6.18	**4.77**	**6.02**	**7.52**
	4.64	6.07	7.94	**5.76**	**7.44**	**9.41**	4.29	5.06	5.90	**5.43**	**6.23**	**6.97**	4.09	5.26	6.76	**5.01**	**6.35**	**7.96**
LDPC(96,48)	6.73	9.48	12.98	**7.22**	**9.96**	**13.37**	3.83	4.57	5.35	**5.37**	**6.51**	**7.71**	5.68	7.94	10.90	**5.90**	**8.19**	**10.91**
	7.50	10.61	**14.26**	**8.29**	**11.12**	14.06	4.17	4.94	5.73	**6.14**	**7.38**	**8.65**	6.33	8.91	**11.99**	**6.71**	**9.28**	11.75
LTE(132,40)	2.94	3.32	3.57	**3.25**	**3.71**	**4.04**	3.17	3.45	3.67	**4.49**	**4.99**	**5.47**	2.75	3.17	3.47	**2.99**	**3.44**	**3.78**
	3.37	3.79	4.09	**3.93**	**4.49**	**4.89**	3.60	3.82	4.01	**5.32**	**5.81**	**6.31**	3.17	3.62	3.96	**3.53**	**4.03**	**4.41**
MACKAY(96,48)	6.75	9.45	**12.85**	**7.03**	**9.63**	12.78	6.28	7.86	9.55	**6.53**	**8.06**	**9.77**	5.72	7.97	10.81	**5.95**	**8.23**	**10.91**
	7.59	10.52	**14.09**	**7.99**	**10.97**	14.05	7.04	8.76	10.64	**7.47**	**9.32**	**11.19**	6.39	8.90	11.91	**6.82**	**9.41**	**12.71**
POLAR(128,86)	3.76	4.17	4.58	**4.83**	**5.87**	**6.58**	3.15	3.53	3.91	**3.64**	**4.28**	**4.94**	3.48	3.96	4.37	**3.69**	**4.51**	**5.18**
	4.02	4.67	5.38	**5.37**	**6.88**	**8.10**	3.28	3.73	4.18	**3.92**	**4.70**	**5.52**	3.65	4.31	4.97	**3.87**	**4.91**	**5.91**
RS(60,52)	4.41	5.32	6.41	**5.02**	**6.38**	**7.99**	3.11	3.41	3.77	**3.37**	**3.73**	**4.12**	3.85	4.58	5.44	**4.17**	**5.18**	**6.40**
	4.54	5.52	6.64	**5.07**	**6.47**	**8.12**	3.13	3.43	3.81	**3.38**	**3.75**	**4.15**	3.91	4.72	5.67	**4.21**	**5.27**	**6.56**
LDPC PEG2(64,32)	4.38	5.12	6.04	**4.45**	**5.19**	**6.10**	4.08	4.44	4.81	**4.10**	**4.46**	**4.85**	4.07	4.69	5.43	**4.07**	**4.70**	**5.43**
	4.38	5.13	6.04	**4.44**	**5.19**	**6.10**	4.08	4.44	4.81	**4.10**	**4.47**	**4.85**	4.06	4.69	5.43	**4.06**	**4.69**	**5.44**
LDPC PEG5(64,32)	6.02	8.20	10.95	**6.53**	**8.73**	**11.56**	5.63	6.86	8.31	**6.22**	**7.48**	**8.82**	5.18	6.97	9.34	**5.59**	**7.41**	**9.51**
	6.63	9.06	**12.30**	**7.13**	**9.48**	12.20	6.19	7.52	9.02	**6.96**	**8.34**	**9.85**	5.68	7.75	**10.19**	**6.12**	**8.06**	10.13
LDPC PEG10(64,32)	3.98	5.17	6.70	**5.56**	**7.22**	**9.13**	3.52	4.18	4.95	**5.02**	**6.00**	**7.11**	3.48	4.47	5.75	**4.26**	**5.50**	**7.01**
	4.27	5.77	7.67	**6.25**	**8.28**	**10.59**	3.71	4.47	5.30	**5.60**	**6.72**	**7.90**	3.67	4.90	6.46	**4.73**	**6.22**	**8.01**

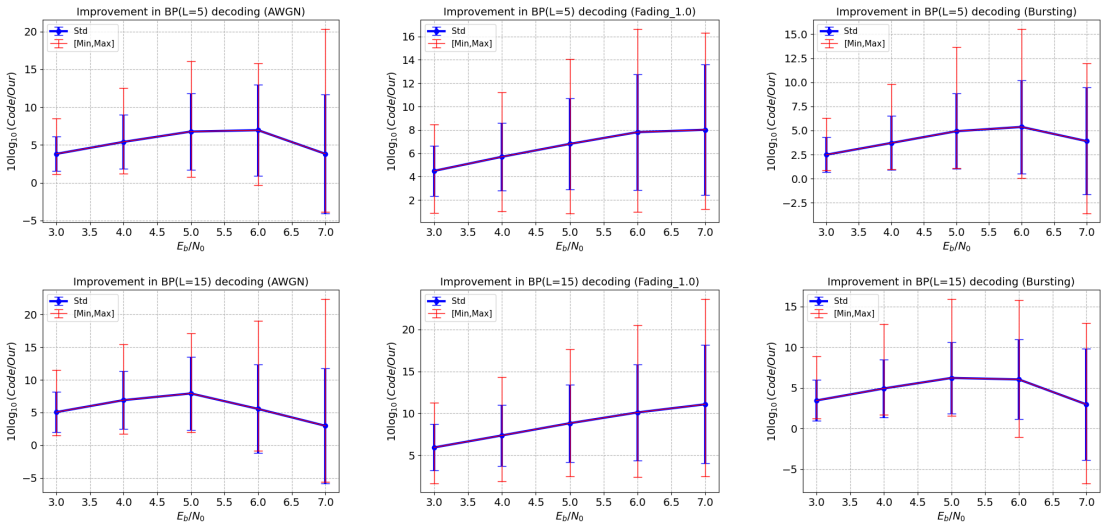
The full training algorithm (pseudocode) is given in Algorithm 2. Given an initial parity check matrix, the algorithm optimizes H iteratively upon convergence. At each iteration, after computing the gradient on sufficiently large statistics (line 7), the line search procedure (line 10) searches for the optimal step size among those that flip the values of H (line 9).

5 EXPERIMENTS

We evaluate our framework on five classes of linear codes: various Low-Density Parity Check (LDPC) codes (Gallager, 1962; Abu-Surra et al., 2010), Polar codes (Arikan, 2008), Reed Solomon codes (Reed & Solomon, 1960), Bose–Chaudhuri–Hocquenghem (BCH) codes (Bose & Ray-Chaudhuri, 1960) and random codes. All the parity check matrices are taken from (Helmling et al., 2019) except the LDPC codes created using the popular Progressive Edge Growth framework (Hu et al., 2005; MacKay).

We consider three types of channel noise under BPSK modulation. We first test our framework with the canonical AWGN channel given as $y = c_s + \varepsilon$ with $\varepsilon \sim \mathcal{N}(0, \sigma I_n)$. We also consider the Rayleigh fading channel, where $y = h \odot c_s + \varepsilon$, with h the iid Rayleigh distributed fading vector with coefficient 1 and ε the regular AWGN noise, where we assume ideal channel state information. Finally, we consider the AWGN channel with Gaussian mixture channel (also referred to as bursty noise (Kurmukova & Gunduz, 2024)) simulating wireless channel interference as $y = c_s + \varepsilon + \zeta$ with ε the AWGN and $\zeta_i \sim \mathcal{N}(0, \sqrt{2}\sigma)$ with probability $\rho = 0.1$ and $\zeta_i = 0$ with probability $1 - \rho$.

The results are reported as negative natural logarithm bit error rates (BER) for three different normalized SNR values (E_b/N_0), following the conventional testing benchmark, e.g., (Nachmani & Wolf, 2019; Choukroun & Wolf, 2022a). BP-based results are obtained after $\ell = 5$ BP iterations in the first row and $\ell = 15$ in the second row of the results tables. During testing, at least 10^5 random



(a) (b) (c)

Figure 2: Statistics of improvement in dB for the (a) AWGN, (b) fading, and (c) bursting channel on the *sparse codes* only. We provide the mean and standard deviation as well as the minimum and maximum improvements.

codewords are decoded, to obtain at least 50 frames with errors at each SNR value. For this section, we performed a small hyperparameter search as reported in Appendix A, where the final code is selected to have the lowest average BER on the SNR test range.

The results are provided in Table 1, where *our* method means BP applied on the learned code initialized by the given classical code. We also provide in Appendix B the same table with a broader SNR range ($E_b/N_0 \in \{3, \ldots, 7\}$). We provide in Appendix J the standard visualization of the BER and BLER vs E_b/N_0 curves on several codes. We further show the performance of the proposed method on the state-of-the-art 5G NR LDPC short blocklenght codes (Richardson & Kudekar, 2018; 3GPP, 2018) We provide the overall improvement statistics (i.e., mean, std, min, max) in dB on all the sparse codes in Figure 2 and we extend the analysis to all the codes in Appendix F. For completeness, we provide in Appendix H a comparison with the genetic algorithm of Elkelesh et al. (2019) where our method demonstrates much better performance while being faster by orders of magnitude. We also show in Appendix I that our method can outperform the performance of the SCL algorithm (Tal & Vardy, 2015) even on very short codes where the performances are close to ML decoding.

Evidently, our method improves by large margins all code families on the three different channel noise scenarios and with both numbers of decoding iterations, demonstrating the capacity of the framework to provide improved codes on multiple settings of interest.

6 Analysis

Initialization and Random Codes We provide in Figure 3 the performance of the proposed method on random codes initialized with different sparsity rates. The parity check matrix is initialized in a systematic form $H = [I_{n-k}, P]$ for full rank initialization, where $P \sim \text{Bern}^{(n-k)\times k}(p)$. We can observe that the framework can greatly improve the performance of the original random code. Most importantly, we can observe that different initializations provide convergence to different local optima and that better initialization generally induces convergence to a better minimum. Performance on other code lengths is provided in Appendix G. Finally, good initialization (i.e., (large) well-performing sparse codes under BP decoding) requires perturbation at initialization or during training (c.f., Appendix A) in order to get extracted from local minima.

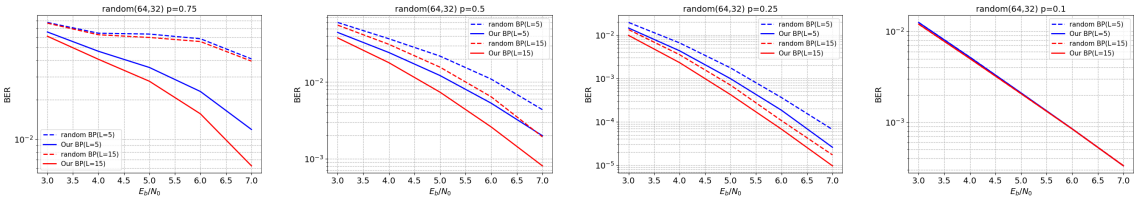


Figure 3: Performance of the method on random codes under different sparsity rate initialization p.

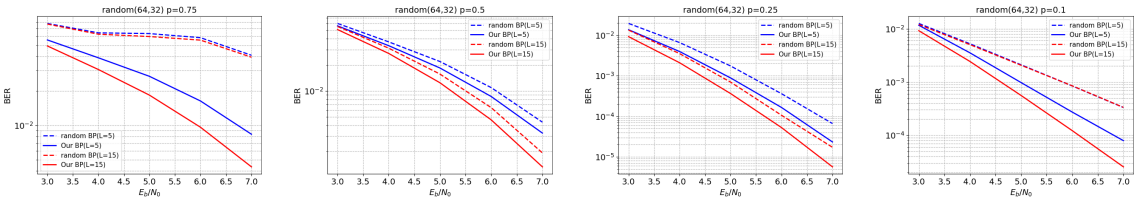


Figure 4: Performance of the method on constrained systematic random codes under different sparsity rate initialization p on the AWGN channel.

Constrained Codes In Figure 4 we provide the performance of the method on constrained systematic random codes as described in the previous paragraph, while here, we constrain them to maintain their systematic form during the optimization, i.e., only the parity matrix elements of P are optimized (via Ω). The optimization is performed by backpropagating over the P tensor only, similarly to having a hard structure constraint on the identity part of H. While maintaining a structure of interest, we can observe this regularization can further improve the convergence quality (e.g., $p = 0.1$) compared to the unconstrained setting of Figure 3. Performance on other code lengths is provided in Appendix G. Any constraint (e.g., dual diagonal) can be similarly added to the code.

In Figure 5 we present the sparsification of the codes created by the framework. Here, $\Delta = 100(S_b - S_o)/S_b$ represents the sparsity ratio, with S_b and S_o being the sparsity of the baseline code and our code, respectively. We can observe that optimization always provides sparser codes. Nevertheless, we also observed that the optimization does not modify the girth of the code. We provide in Appendix F the column weights distribution of the initial and learned parity check matrices.

In Figure 6 we present the performance of the method on random codes with sparsity constraint, i.e., $\mathcal{R}(H) = \lambda\|H\|_1$, with $\lambda \in \mathbb{R}_+$. We can observe that adding a sparsity constraint is generally not profitable, since the optimization over BP already induces sparse codes.

Learned Codes Visualization We depict in Figure 7 the learned codes via the visualization of the parity-check matrices. We can observe that for low-density codes the modifications remain small, since the code is already near local optimum, while for denser codes the change can be substantial.

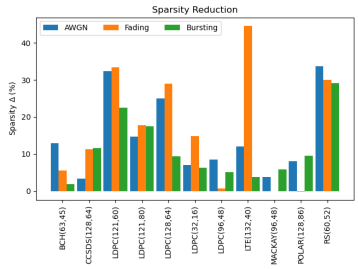


Figure 5: Sparsity reduction of the proposed codes.

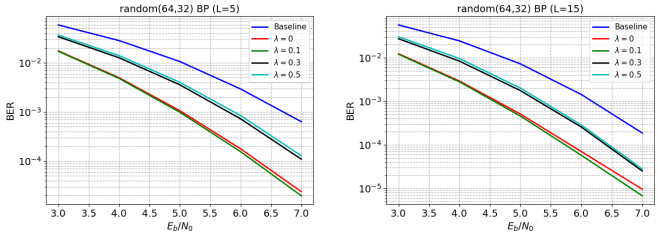


Figure 6: Performance of the method with L_1 regularization for different values of the regularization factor λ for random codes with $p = 0.25$ on the AWGN channel.

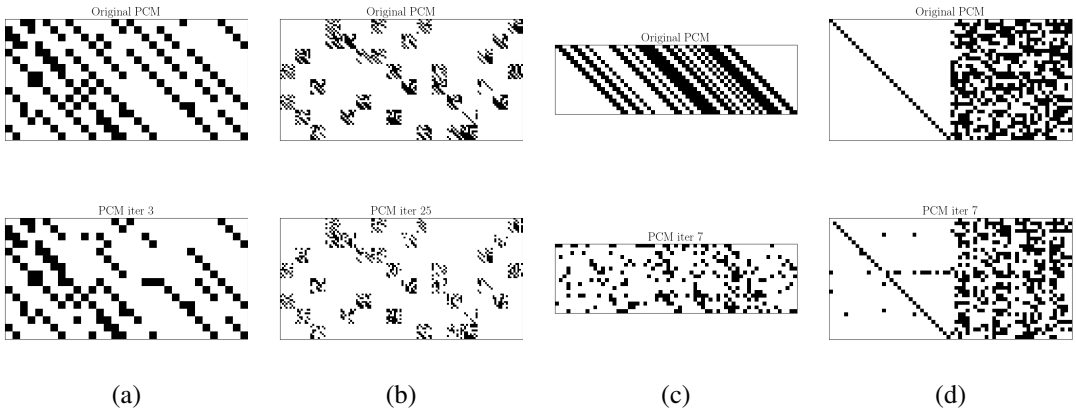


Figure 7: Visualization of the original (first row) and the learned parity check matrices (second row) for (a) LDPC(32,16), (b) LDPC(128,64), (c) BCH(63,45) and (d) Random(64,32,$p = 0.5$). "PCM iter X" denotes the final iteration Parity Check Matrix of the optimization.

It further shows SGD's inductive bias seems not to impact BP inductive bias assuming sparse (i.e., no-loop or tree) connections. Also, the optimized codes tend to be more sparse than the original.

We provide in Appendix C visualizations of the **line search optimization**, demonstrating the high non-convexity and the proximity of the optimum to the current estimate. We provide in Appendix D statistics on **convergence rates** and typical convergence curves demonstrating the fast and monotonic convergence. Finally, we provide in Appendix E the performance of the learned code on the efficient **Min-Sum approximation** of the BP algorithm and show that the learned code outperforms the baseline codes over the Min-Sum framework as well.

7 Conclusions

We present a novel gradient-based optimization method of binary linear block codes for the Belief Propagation algorithm. The proposed framework enables the differentiable optimization of the factor graph via weighted tensor representation. The optimization is efficiently carried out via a tailor-made grid search procedure that is aware of the binary constraint of the optimization problem.

A common criticism of ML-based ECC is that the neural decoder cannot be deployed directly without the application of massive deep-learning acceleration methods. Here, we show that the code can be designed efficiently in a data-driven fashion on differentiable formulations of classical decoders. The optimization of codes may open the door to the establishment of new industry standards and the creation of new families of codes. Future work includes the development of more efficient optimization methods, able to define better initializations and to escape bad local minima.

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A HYPER-PARAMETER TUNING

Under the problem's stochastic optimization, we provide here the different modifications used to obtain better performance. The first set of training/optimization hyperparameters is the E_b/N_0 range defined as $(u, 7)$ with $u \in \{3, 4, 5\}$. The second set of hyperparameters is the data sampling, where we experimented with random data (i.e., classical setting) and data with non-zero syndromes only. Finally, for better backpropagation, we also experimented with a soft approximation $\tilde{H}$ of the binary H during the optimization, defined as

$$\tilde{H}_{ij} = \begin{cases} (-1)^z \epsilon, & if \quad H_{ij} = 0 \\ 1, & else \end{cases}$$

where $z \sim Bern(0.5)$ and ϵ is a small scalar (10^{-7} in our experiments). We note we only used a size 15 random subset of all the possible permutations of the hyperparameters mentioned above.

B MORE SNR RESULTS

We provide results on a larger range of SNRs in Table 2.

Table 2: A comparison of the negative natural logarithm of Bit Error Rate (BER) for several normalized SNR values of our method with classical codes. Higher is better. BP results are provided for 5 iterations in the first row and 15 in the second row. The best results are in bold.

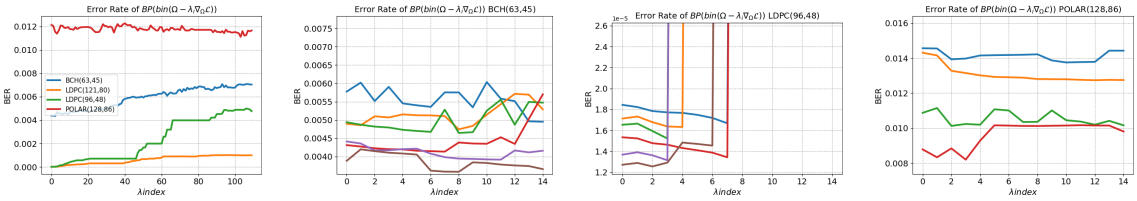
Channel	AWGN										Fading										Bursting									
Method	BP					Our					BP					Our					BP					Our				
E_b/N_0	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7
BCH(63,45)	3.35	4.06	4.91	6.04	7.47	**4.23**	**5.44**	**6.93**	**8.60**	**10.27**	2.77	3.09	3.46	3.90	4.37	**3.42**	**3.96**	**4.58**	**5.27**	**5.99**	3.00	3.60	4.32	5.19	6.25	**3.24**	**4.05**	**5.07**	**6.27**	**7.67**
	3.40	4.21	5.24	6.59	8.35	**4.36**	**5.70**	**7.35**	**9.16**	**11.10**	2.79	3.13	3.55	4.04	4.61	**3.50**	**4.10**	**4.80**	**5.56**	**6.36**	3.02	3.67	4.52	5.59	6.93	**3.31**	**4.21**	**5.40**	**6.85**	**8.55**
CCSDS(128,64)	4.32	6.46	9.61	13.99	**18.27**	**4.99**	**7.34**	**10.48**	**14.37**	17.38	4.37	5.72	7.42	9.47	11.84	**5.22**	**6.73**	**8.45**	**10.45**	**12.36**	3.62	5.29	7.81	11.25	**15.59**	**4.32**	**6.23**	**8.80**	**11.90**	14.76
	4.82	7.32	10.83	15.43	**18.51**	**5.80**	**8.61**	**12.26**	**16.00**	18.15	4.89	6.43	8.29	10.28	12.88	**6.22**	**8.05**	**10.07**	**12.37**	**14.87**	3.97	5.98	8.85	12.53	**17.10**	**5.05**	**7.39**	**10.43**	**13.28**	15.56
LDPC(121,60)	3.33	4.81	7.17	10.75	15.69	**5.29**	**7.70**	**10.87**	**14.25**	**16.82**	3.24	4.10	5.23	6.68	8.56	**5.18**	**6.68**	**8.47**	**10.50**	**12.31**	2.87	3.97	5.75	8.40	12.16	**4.32**	**6.23**	**8.89**	**11.98**	**14.91**
	3.53	5.31	7.96	11.85	17.01	**6.18**	**8.86**	**11.91**	**14.41**	**17.04**	3.44	4.42	5.61	7.04	8.77	**6.04**	**7.71**	**9.67**	**11.76**	**14.21**	2.98	4.31	6.37	9.25	13.10	**5.02**	**7.26**	**10.03**	**12.88**	**15.14**
LDPC(121,80)	4.50	6.59	9.68	13.43	**18.51**	**5.28**	**7.77**	**11.21**	**15.06**	18.40	3.67	4.60	5.80	7.22	8.95	**4.42**	**5.55**	**6.90**	**8.36**	**10.00**	3.74	5.30	7.60	10.66	14.88	**4.32**	**6.23**	**8.87**	**12.19**	**16.31**
	4.85	7.35	10.94	15.46	**19.61**	**5.85**	**8.75**	**12.45**	**15.67**	18.30	3.89	4.97	6.29	7.82	9.58	**4.89**	**6.25**	**7.80**	**9.47**	**11.21**	3.94	5.81	8.50	12.15	16.45	**4.73**	**6.99**	**10.09**	**13.74**	**17.55**
LDPC(128,64)	2.88	3.66	4.65	5.80	7.03	**4.07**	**5.54**	**7.37**	**9.44**	**11.71**	2.73	3.22	3.80	4.44	5.14	**3.95**	**4.86**	**5.94**	**7.15**	**8.38**	2.58	3.23	4.08	5.09	6.21	**2.80**	**3.72**	**5.00**	**6.54**	**8.30**
	3.04	4.00	5.16	6.42	7.77	**4.71**	**6.56**	**8.70**	**10.81**	**12.92**	2.90	3.51	4.18	4.84	5.54	**4.55**	**5.64**	**6.85**	**8.14**	**9.59**	2.68	3.48	4.51	5.66	6.88	**2.97**	**4.13**	**5.72**	**7.66**	**9.86**
LDPC(32,16)	3.45	4.36	5.59	7.18	9.19	**4.29**	**5.48**	**7.02**	**8.92**	**11.23**	3.44	4.03	4.70	5.47	6.31	**4.53**	**5.26**	**6.02**	**6.82**	**7.61**	3.10	3.88	4.89	6.18	7.82	**3.78**	**4.77**	**6.02**	**7.52**	**9.22**
	3.59	4.64	6.07	7.94	10.23	**4.47**	**5.76**	**7.44**	**9.41**	**12.03**	3.62	4.29	5.06	5.90	6.83	**4.67**	**5.43**	**6.23**	**6.97**	**7.81**	3.21	4.09	5.26	6.76	8.58	**3.93**	**5.01**	**6.35**	**7.96**	**9.72**
LDPC(96,48)	4.72	6.73	9.48	12.98	**16.87**	**5.17**	**7.22**	**9.96**	**13.37**	16.45	3.19	3.83	4.57	5.35	6.17	**4.38**	**5.37**	**6.51**	**7.71**	**8.94**	4.03	5.68	7.94	10.90	**14.27**	**4.23**	**5.90**	**8.19**	**10.91**	13.61
	5.20	7.50	10.61	**14.26**	**17.80**	**5.85**	**8.29**	**11.12**	14.06	17.19	3.44	4.17	4.94	5.73	6.58	**4.99**	**6.14**	**7.38**	**8.65**	**9.77**	4.40	6.33	8.91	**11.99**	**15.55**	**4.71**	**6.71**	**9.28**	11.75	14.06
LTE(132,40)	2.49	2.94	3.32	3.57	3.81	**2.72**	**3.25**	**3.71**	**4.04**	**4.36**	2.82	3.17	3.45	3.67	3.89	**3.97**	**4.49**	**4.99**	**5.47**	**5.96**	2.30	2.75	3.17	3.47	3.70	**2.48**	**2.99**	**3.44**	**3.78**	**4.05**
	2.85	3.37	3.79	4.09	4.32	**3.26**	**3.93**	**4.49**	**4.89**	**5.22**	3.29	3.60	3.82	4.01	4.21	**4.78**	**5.32**	**5.81**	**6.31**	**6.84**	2.63	3.17	3.62	3.96	4.19	**2.92**	**3.53**	**4.03**	**4.41**	**4.70**
MACKAY(96,48)	4.77	6.75	9.45	**12.85**	**16.37**	**5.03**	**7.03**	**9.63**	12.78	16.11	4.98	6.28	7.86	9.55	11.30	**5.18**	**6.53**	**8.06**	**9.77**	**11.58**	4.08	5.72	7.97	10.81	13.92	**4.28**	**5.95**	**8.23**	**10.91**	**14.02**
	5.28	7.59	10.52	**14.09**	17.43	**5.63**	**7.99**	**10.97**	14.05	**17.49**	5.55	7.04	8.76	10.64	12.58	**5.93**	**7.47**	**9.32**	**11.19**	**13.14**	4.47	6.39	8.90	11.91	15.23	**4.79**	**6.82**	**9.41**	**12.71**	**15.81**
POLAR(128,86)	3.25	3.76	4.17	4.58	5.12	**3.72**	**4.83**	**5.87**	**6.58**	**7.21**	2.80	3.15	3.53	3.91	4.26	**3.10**	**3.64**	**4.28**	**4.94**	**5.58**	2.95	3.48	3.96	4.37	4.78	**2.96**	**3.69**	**4.51**	**5.18**	**5.73**
	3.36	4.02	4.67	5.38	6.19	**3.96**	**5.37**	**6.88**	**8.10**	**9.00**	2.87	3.28	3.73	4.18	4.60	**3.26**	**3.92**	**4.70**	**5.52**	**6.30**	3.02	3.65	4.31	4.97	5.66	**3.03**	**3.87**	**4.91**	**5.91**	**6.88**
RS(60,52)	3.65	4.41	5.32	6.41	7.80	**3.98**	**5.02**	**6.38**	**7.99**	**9.73**	2.86	3.11	3.41	3.77	4.16	**3.05**	**3.37**	**3.73**	**4.12**	**4.53**	3.26	3.85	4.58	5.44	6.43	**3.42**	**4.17**	**5.18**	**6.40**	**7.75**
	3.70	4.54	5.52	6.64	8.04	**4.00**	**5.07**	**6.47**	**8.12**	**9.80**	2.87	3.13	3.43	3.81	4.24	**3.06**	**3.38**	**3.75**	**4.15**	**4.56**	3.27	3.91	4.72	5.67	6.78	**3.42**	**4.21**	**5.27**	**6.56**	**8.01**
PGE2(64,32)	3.78	4.38	5.12	6.04	7.17	**3.84**	**4.45**	**5.19**	**6.10**	**7.22**	3.74	4.08	4.44	4.81	5.20	**3.75**	**4.10**	**4.46**	**4.85**	**5.24**	**3.54**	4.07	4.69	5.43	6.29	**3.54**	**4.07**	**4.70**	**5.43**	**6.30**
	3.78	4.38	5.13	6.04	7.16	**3.84**	**4.44**	**5.19**	**6.10**	**7.23**	3.74	4.08	4.44	4.81	5.20	**3.75**	**4.10**	**4.47**	**4.85**	**5.24**	**3.54**	4.06	4.69	5.43	6.29	**3.54**	**4.06**	**4.69**	**5.44**	**6.30**
PGE5(64,32)	4.41	6.02	8.20	10.95	14.40	**4.82**	**6.53**	**8.73**	**11.56**	**14.41**	4.56	5.63	6.86	8.31	9.78	**5.09**	**6.22**	**7.48**	**8.82**	**10.16**	3.84	5.18	6.97	9.34	**12.25**	**4.19**	**5.59**	**7.41**	**9.51**	11.84
	4.78	6.63	9.06	**12.30**	**15.75**	**5.26**	**7.13**	**9.48**	12.20	14.90	4.98	6.19	7.52	9.02	10.76	**5.67**	**6.96**	**8.34**	**9.85**	**11.24**	4.13	5.68	7.75	**10.19**	**13.40**	**4.55**	**6.12**	**8.06**	10.13	12.27
PGE10(64,32)	3.08	3.98	5.17	6.70	8.49	**4.21**	**5.56**	**7.22**	**9.13**	**11.11**	2.96	3.52	4.18	4.95	5.81	**4.16**	**5.02**	**6.00**	**7.11**	**8.33**	2.75	3.48	4.47	5.75	7.28	**3.30**	**4.26**	**5.50**	**7.01**	**8.80**
	3.20	4.27	5.77	7.67	9.72	**4.62**	**6.25**	**8.28**	**10.59**	**12.84**	3.08	3.71	4.47	5.30	6.24	**4.60**	**5.60**	**6.72**	**7.90**	**9.24**	2.82	3.67	4.90	6.46	8.26	**3.57**	**4.73**	**6.22**	**8.01**	**10.06**

C LINE SEARCH OPTIMIZATION

In Figure 8 we provide visualizations of the line search procedure. We provide BER with respect to the step size λ_i indexed by i ($\lambda_0 \equiv 0$). We can observe the high non-convexity of the objective, with the presence of several local minima. We can also notice the proximity of the optimum to the current parity-check estimate (i.e., λ_0).

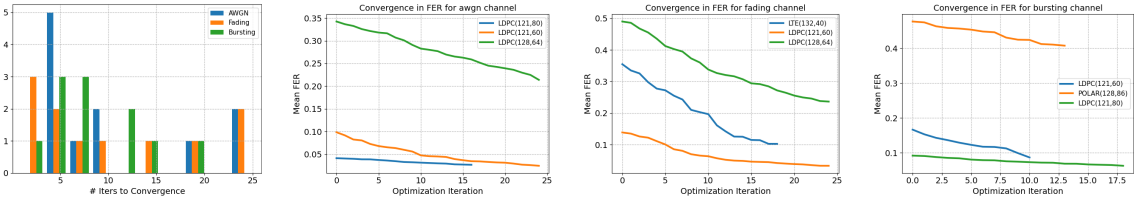
D CONVERGENCE RATE

In Figure 9 we provide statistics on the number of optimization iterations for convergence (a). We also provide (b,c,d) typical convergence. We can observe that the framework typically converges within a few iterations and that the loss decreases monotonically.



(a) (b) (c) (d)

Figure 8: BER in function of the step size index i on AWGN channel. (a) Averaged BER over the optimization iterations for 4 codes. (b,c,d) BER per optimization iteration for the first 5 optimization iterations and the first 10 indices for three different codes. Here $\lambda_0 = 0$ denotes the original BER.



(a) (b) (c) (d)

Figure 9: (a) Histogram of the number of required iterations until convergence. (b) Convergence rate of the Frame Error Rate for three codes on (b) AWGN, (c) fading, and (d) bursting channel. We selected the three codes with the largest number of iterations. The FER is averaged over all the tested $E_b/N_0 = \{3, \dots 7\}$ range.

E Impact on other BP Variants

In Table 3 we provide the performance of the learned code on the more efficient Min-Sum approximation of the Sum-Product algorithm. We can observe that the codes learned with BP consistently outperform the performance of the Min-Sum approximation as well. For some codes, the training range may need to be adjusted. We note our method can be applied to neural BP decoders as well. The direct optimization over BP approximations and augmentations is left for future work.

F Improvement Statistics on all the Codes

We provide in Figure 10 the statistics of improvement on all the codes presented in Table 2.

G More Random Codes

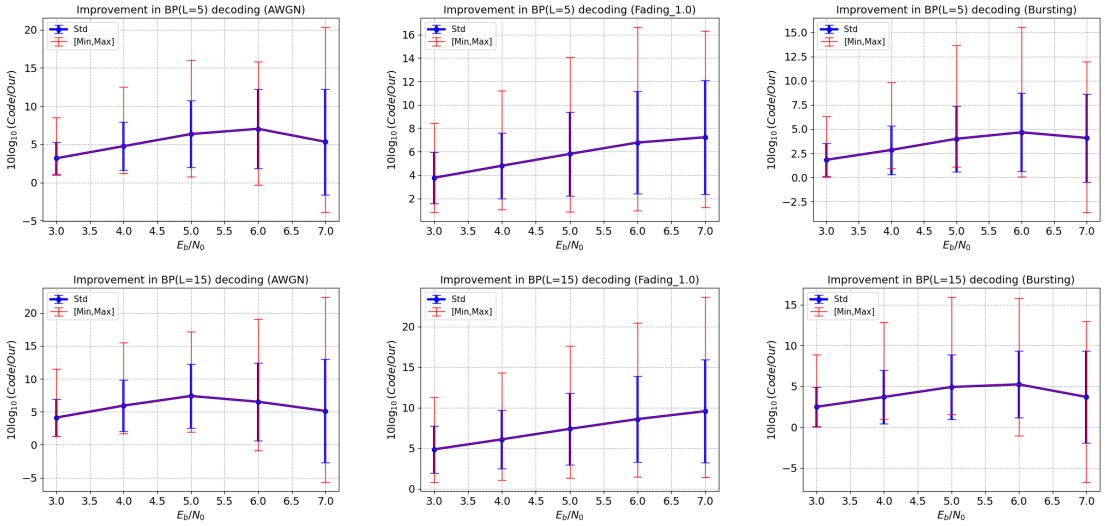
We provide in Figure 11 the performance of the proposed method on random codes initialized with different sparsity rates on different lengths. We also provide in Figure 12 the performance of the proposed method on constrained systematic random codes initialized with different sparsity rates on different lengths.

H Comparison with Genetic Algorithm

We provide in table 4 a comparison with the genetic algorithm of Elkelesh et al. (2019). We note that the method requires 230 offspring/code evaluations per iteration, with 300 iterations (Fig. 7 in (Elkelesh et al., 2019)) or even an infinite loop (cf. the provided MATLAB code). Our algorithm is tested on 50 line-search steps as described in in the paper on 2 to 25 iterations (cf. App. D), which means that Elkelesh et al. (2019) requires approximately 25 to 313 times more computations than our proposed method. The performance presented are for 75 and 150 iterations of the algorithm, representing around 70 and 140 times slower performance than our approach, respectively, while they

Table 3: A comparison of the negative natural logarithm of Bit Error Rate (BER) for five normalized SNR values of our method applied on the Min-Sum BP algorithm. NE = no errors spotted under the testing limits.

BP Method	Sum-Product										Min-Sum									
Method	Baseline					Our					Baseline					Our				
E_b/N_0	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7	3	4	5	6	7
BCH(63,45)	3.35	4.06	4.92	5.98	7.39	3.48	4.30	5.29	6.51	8.12	3.04	3.79	4.89	6.33	8.13	3.21	4.09	5.32	6.84	8.65
	3.40	4.22	5.24	6.60	8.33	3.57	4.49	5.69	7.17	9.17	3.22	4.09	5.41	7.06	9.14	3.40	4.44	5.89	7.60	10.00
CCSDS(128,64)	4.32	6.47	9.62	13.80	18.40	4.44	6.66	9.73	13.60	18.30	4.21	6.62	10.40	15.10	19.40	4.35	6.82	10.50	15.00	21.00
	4.82	7.30	10.70	15.50	17.90	4.99	7.57	11.00	15.60	NE	4.76	7.66	12.20	17.70	NE	4.97	8.03	12.30	17.40	NE
LDPC(32,16)	3.46	4.39	5.60	7.20	9.23	3.62	4.59	5.83	7.45	9.52	3.36	4.38	5.75	7.65	10.10	3.53	4.61	6.01	7.92	10.10
	3.61	4.66	6.07	7.87	10.30	3.80	4.91	6.36	8.16	10.70	3.55	4.69	6.21	8.21	10.90	3.74	4.93	6.50	8.44	11.10
LDPC(96,48)	4.70	6.73	9.52	13.20	17.30	5.01	7.11	9.92	13.50	16.90	4.71	6.96	9.95	14.20	18.30	4.98	7.16	10.10	14.00	15.80
	5.20	7.55	10.70	14.40	18.50	5.70	8.13	11.30	14.70	17.40	5.23	7.89	11.50	15.10	19.10	5.68	8.32	12.00	14.80	15.80



(a) (b) (c)

Figure 10: Statistics of improvement in dB for the (a) AWGN, (b) fading, and (c) bursting channel on all the codes from Table 2. We provide the mean and standard deviation as well as the minimum and maximum improvements.

remain below our performance. We note here, as described in the paper, that combining the methods by allowing the perturbation of the parity-check matrix at a local minima may allow the discovery of other better local optimum.

Table 4: A comparison of the negative natural logarithm of Bit Error Rate (BER) for several normalized SNR values of our method with classical codes. Higher is better. BP results are provided for 5 iterations in the first row and 15 in the second row. The best results are in bold. The training range is defined as $E_b/N_0 = \{5\}$. GA denotes the genetic algorithm of Elkelesh et al. (2019) with k training iterations.

Channel	AWGN											
Method	BP			Our			GA $k = 75$			GA $k = 150$		
E_b/N_0	4	5	6	4	5	6	4	5	6	4	5	6
CCSDS(128,64)	6.46	9.61	13.99	**7.34**	**10.48**	**14.37**	6.86	9.95	13.38	7.09	10.40	14.08
	7.32	10.83	15.43	**8.61**	**12.26**	16.00	7.87	11.31	15.56	8.23	11.79	**16.04**

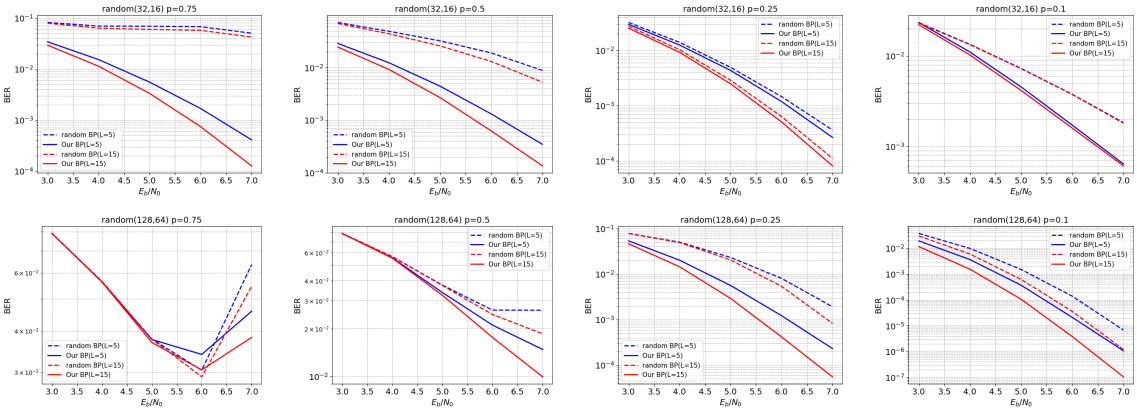


Figure 11: Performance of the method on random codes under different sparsity rate initialization p.

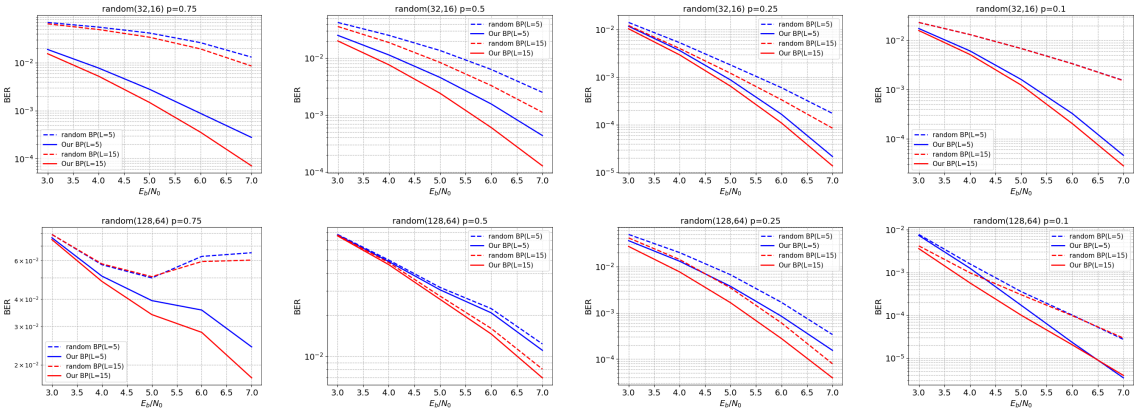


Figure 12: Performance of the method on constrained systematic random codes under different sparsity rate initialization p on the AWGN channel.

I Comparison with Successive Cancelation List

We provide in Table 5 a comparison with the powerful Polar codes (Arikan, 2008) under SCL decoding (Tal & Vardy, 2015) ($\mathcal{O}(LNlog(N))$) for very short-length code (32,16) in which SCL is close to ML decoding. The SCL results are obtained using the implementation of Cassagne et al. (2019).

We provide the performance of BP and of our method with the same Polar code initialization (BP (Polar) and Our(Polar)) and with 5G LDPC code initialization (BP (5G LDPC), Our (5G LDPC)). SCL performance is provided with the corresponding Polar code.

We can observe that even in the extremely short length setting where sparsity is hard to obtain our method is able to remarkably improve the performance over existing short-length low-density codes and get close to the ML bound even within very few number of iterations, even with bad initialization. With good initialization (good sparse code), our method provides state-of-the-art performance.

J BER vs SNR Curves

We provide standard visualizations of the BER and BLER performance with respect to the performance on multiple codes and channels. Figures 15, 14 provide performance on the AWGN and fading channel, respectively.

Table 5: A comparison of the negative natural logarithm of Bit Error Rate (BER) for several normalized SNR values of our method with classical codes. Higher is better. BP results are provided for 5 iterations in the first row and 15 in the second row. The best results are in bold. The training range is defined as $E_b/N_0 = \{5\}$. The first and the second row of the SCL algorithm denote performance with a list length of 1 and 32 respectively.

Method	SCL			BP (Polar)			Our (Polar)			BP (5G LDPC)			Our (5G LDPC)		
E_b/N_0	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6
(32,16)	6.22	8.06	10.28	4.36	5.59	7.18	5.48	7.02	8.92	6.06	7.53	9.23	**6.63**	**8.53**	**10.36**
	6.45	8.37	10.60	4.64	6.07	7.94	5.76	7.44	9.41	6.57	8.17	10.16	**7.11**	**8.92**	**11.31**

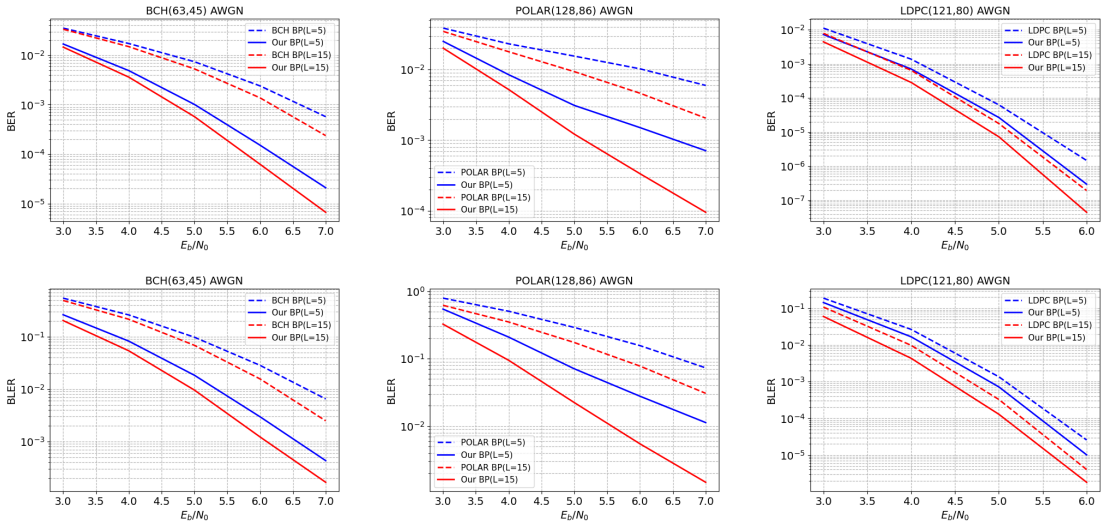


Figure 13: BER and BLER performance of the method on different codes on the AWGN channel.

K Column Weight Distribution

We provide in Figure 15 an analysis of the column weight distribution of the original and learned parity check matrices on several codes and channel settings. While the method modifies substantially the distribution for non-sparse codes, BP's inductive bias seems to push the LDPC codes towards a non-uniform distribution of the variable nodes' degree.

L Performance on 5G LDPC Codes

We provide in Figure 16a performance analysis on short state-of-the-art 5G NR LDPC (protograph based) codes (Richardson & Kudekar, 2018; Ten Brink; 3GPP, 2018).

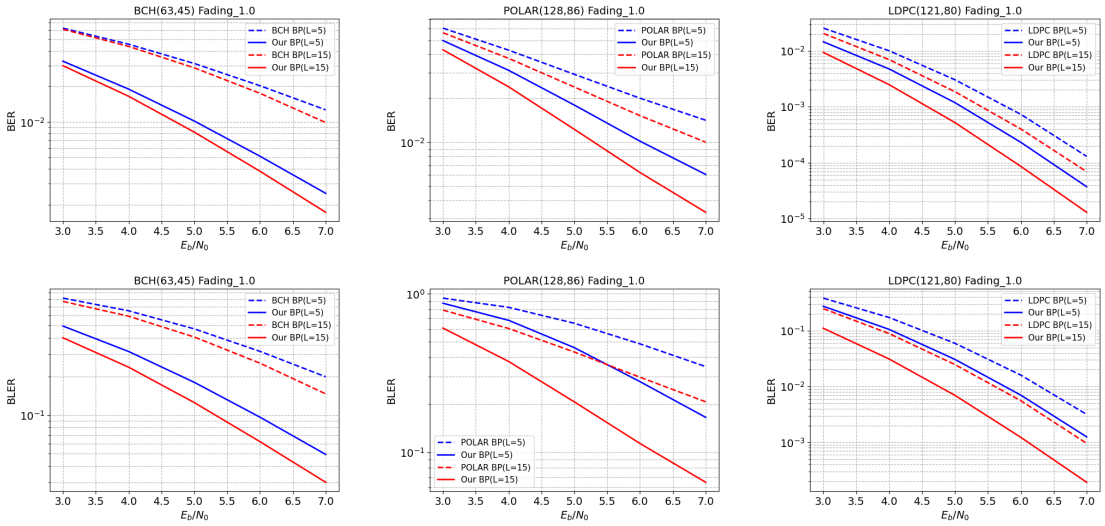
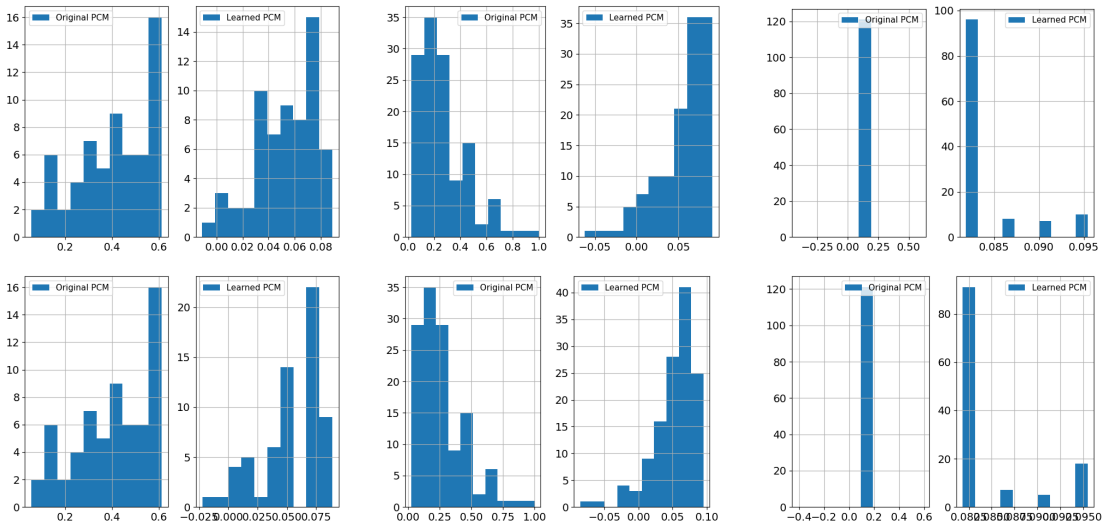


Figure 14: BER and BLER performance of the method on different codes on the Fading channel.



(a) (b) (c)

Figure 15: Column weight distribution of the parity check matrices on the (a) BCH(63,45) (b) POLAR(128,86) and (c) LDPC(121,80) codes. Top row and second row are for the AWGN and Fading channel, respectively.

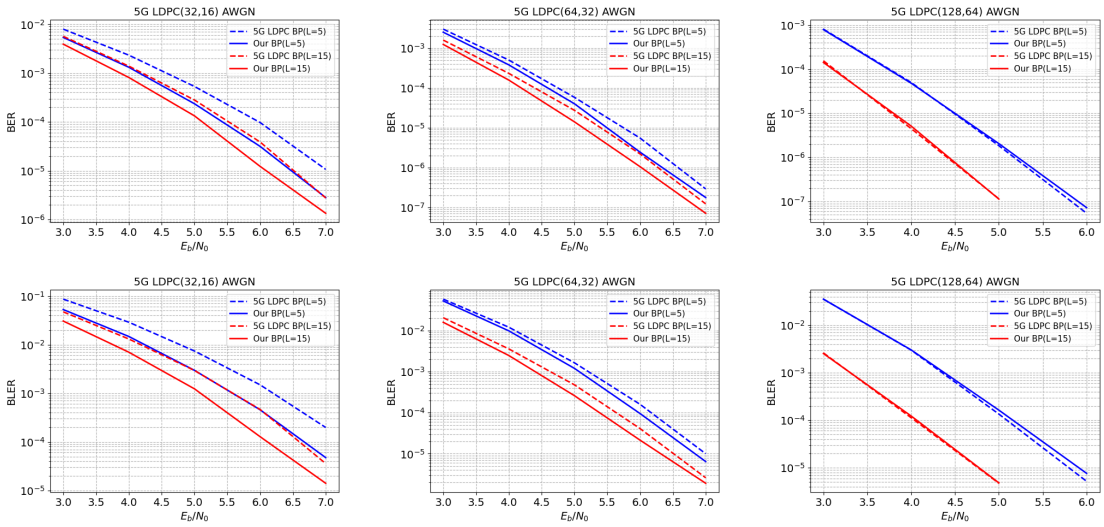


Figure 16: BER and BLER performance of the method on different SOTA 5G LDPC codes.