One-Step Distributional Reinforcement Learning

Anonymous authors
Paper under double-blind review

Abstract

Reinforcement learning (RL) allows an agent interacting sequentially with an environment to maximize its long-term expected return. In the distributional RL (DistrRL) paradigm, the agent goes beyond the limit of the expected value, to capture the underlying probability distribution of the return *across all time steps*. The set of DistrRL algorithms has led to improved empirical performance. Nevertheless, the theory of DistrRL is still not fully understood, especially in the control case. In this paper, we present the simpler *one-step distributional reinforcement learning* (OS-DistrRL) framework encompassing only the randomness induced by the *one-step dynamics* of the environment. Contrary to DistrRL, we show that our approach comes with a unified theory for both policy evaluation and control. Indeed, we propose two OS-DistrRL algorithms for which we provide an almost sure convergence analysis. The proposed approach compares favorably with categorical DistrRL on various environments.

1 Introduction

In reinforcement learning (RL), a decision-maker or *agent* sequentially interacts with an unknown and uncertain environment in order to optimize some performance criterion (Sutton & Barto, 2018). At each time step, the agent observes the current state of the environment, then takes an action that influences both its immediate reward and the next state. In other words, RL refers to learning through trial-and-error a strategy (or *policy*) mapping states to actions that maximizes the *long-term cumulative reward* (or *return*): this is the so-called control task. More accurately, this return is a random variable – due to the random transitions across states – and classical RL only focuses on its expected value. On the other hand, the policy evaluation task aims at assessing the quality of any given policy (not necessarily optimal as in control) by computing its expected return in each initial state, also called *value function*. For both evaluation and control, when the *model* (i.e. reward function and transition probabilities between states) is known, these value functions can be seen as fixed points of some operators and computed by dynamic programming (DP) under the Markov decision process (MDP) formalism (Puterman, 2014). Nevertheless, the model in RL is typically unknown and the agent can only approximate the DP approach based on empirical trajectories. The TD(0) algorithm for policy evaluation and Q-learning for control, respectively introduced in (Sutton, 1988) and (Watkins & Dayan, 1992), are flagship examples of the RL paradigm. Formal convergence guarantees were provided for both of these methods, see (Dayan, 1992; Tsitsiklis, 1994; Jaakkola et al., 1993). In many RL applications, the number of states is very large and thus prevents the use of the aforementioned tabular RL algorithms. In such a situation, one should rather use function approximation to approximate the value functions, as achieved by the DQN algorithm (Mnih et al., 2013; 2015) combining ideas from Q-learning and deep learning. More recently, the distributional reinforcement learning (DistrRL) framework was proposed by Bellemare et al. (2017); see also (Morimura et al., 2010a;b). In DistrRL, the agent is optimized to model the whole probability distribution of the return, not just its expectation. In this new paradigm, several distributional procedures where proposed as extensions of the classic (non-distributional) RL methods, leading to improved empirical performance. In most cases, a DistrRL algorithm is composed of two main ingredients: i) a parametric family of distributions serving as proxies for the distributional returns, and ii) a metric measuring approximation errors between original distributions and parametric proxies.

Related work. We recall a few existing DistrRL approaches, all of which considering mixtures of Dirac measures as their parametric family of proxy distributions. Indeed, the Categorical DistrRL (CDRL) parametrization used in the C51 algorithm proposed in (Bellemare et al., 2017) was shown in (Rowland et al., 2018) to correspond to orthogonal projections derived from the Cramér distance[1]. The CDRL approach learns the categorical probabilities $p_1(x,a),\dots,p_K(x,a)$ of a distribution $\sum_{k=1}^K p_k(x,a)\delta_{z_k}$ with fixed predefined support values $z_1,\dots,z_K$. On the other hand, the quantile regression approach was proposed in (Dabney et al., 2018), where the support $\{Q_1(x,a),\dots,Q_N(x,a)\}$ of the proxy distribution $\frac{1}{N}\sum_{i=1}^N \delta_{Q_i(x,a)}$ is learned for fixed uniform probabilities $\frac{1}{N}$ over the N atoms. In (Dabney et al., 2018), these atoms result from Wasserstein-1 projections, and correspond to quantiles of the unprojected distribution. For policy evaluation, the convergence analysis of CDRL and of the quantile approach were respectively derived in (Rowland et al., 2018) and (Rowland et al., 2023). Nevertheless, the theory of DistrRL is more challenging in the control case because the corresponding operator is not a contraction[2] and does not necessarily admits a fixed point. For that reason, Rowland et al. (2018) proved the convergence of CDRL for control only under the restrictive assumption that the optimal policy is unique. Hence, it seems natural to ask the following question: "Is there another formulation of DistrRL in which both the evaluation and the control tasks lead to contractive operators?". The answer provided by this paper is "Yes, via a one-step approach!". Contrary to classic DistrRL dealing with the randomness across all time steps, our method only cares about the one-step dynamics of the environment.

Contributions. Our main contributions are as follows:

- We introduce a one-step variant of the DistrRL framework: we call that approach *one-step distributional reinforcement learning* (OS-DistrRL).
- We show that our new method solves the well-known instability issue of DistrRL in the control case.
- We provide a unified almost-sure convergence analysis for both evaluation and control.
- We experimentally show the competitive performance of our new (deep learning-enhanced) algorithms in various environments.

The paper is organized as follows. In Section 2, we recall a few standard RL tools and notations as well as their DistrRL generalization. Then, our one-step approach is defined in Section 3. Section 4 introduces new OS-DistrRL algorithms along with theoretical convergence guarantees. Finally, numerical experiments are provided for illustration purpose in Section 5. The main proofs are deferred to the Supplementary Material.

Notations. The indicator function of any event E is denoted by $\mathbb{I}\{E\}$. We let $\mathcal{P}_b(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$ having bounded support, and $\mathcal{P}(\mathcal{E})$ the set of probability mass functions on any finite set $\mathcal{E}$, whose cardinality is denoted by $|\mathcal{E}|$. The support of any discrete distribution $q \in \mathcal{P}(\mathcal{E})$ is $\mathrm{support}(q) = \{y \in \mathcal{E} \colon q(y) > 0\}$; the supremum norm of any function $h \colon \mathcal{E} \to \mathbb{R}$ is $\|h\|_\infty = \max_{y\in\mathcal{E}} |h(y)|$. The cumulative distribution function (CDF) of a probability measure $\nu \in \mathcal{P}_b(\mathbb{R})$ is the mapping $F(z) = \mathbb{P}_{Z\sim\nu}(Z \le z)$ $(\forall z \in \mathbb{R})$, and we denote its generalized inverse distribution function (a.k.a. quantile function) by $F^{-1} \colon \tau \in (0,1] \mapsto \inf\{z \in \mathbb{R}, F(z) \ge \tau\}$. Given $\nu_1, \nu_2 \in \mathcal{P}_b(\mathbb{R})$ with respective CDFs F_1, F_2, we denote $\nu_2 \le \nu_1$ and say that ν_1 *stochastically dominates* ν_2 if $F_1(z) \le F_2(z)$ for all $z \in \mathbb{R}$. For any probability measure $\nu \in \mathcal{P}_b(\mathbb{R})$ and measurable function $f \colon \mathbb{R} \to \mathbb{R}$, the *pushforward measure* $f_\#\nu$ is defined for any Borel set $A \subseteq \mathbb{R}$ by $f_\#\nu(A) = \nu(f^{-1}(A)) = \nu(\{z \in \mathbb{R} \colon f(z) \in A\})$. In this article, we only need the affine case $f_{r_0,\gamma}(z) = r_0 + \gamma z$ (with $r_0 \in \mathbb{R}, \gamma \in [0,1)$) for which $f_{r_0,\gamma\#}\nu \in \mathcal{P}_b(\mathbb{R})$ and $f_{r_0,\gamma\#}\nu(A) = \nu(\{\frac{z-r_0}{\gamma} \colon z \in A\})$ if $\gamma \neq 0$, or $f_{r_0,\gamma\#}\nu = \delta_{r_0}$ is the Dirac measure at r_0 if $\gamma = 0$.

[1] The Cramér distance ℓ_2 between two probability distributions ν_1, ν_2 with CDFs F_1, F_2 is equal to $\ell_2(\nu_1,\nu_2) = \sqrt{\int_{\mathbb{R}} (F_1(x) - F_2(x))^2 dx}$.

[2] A function mapping a metric space to itself is called a γ-contraction if it is Lipschitz continuous with Lipschitz constant $\gamma < 1$.

2 Background on distributional reinforcement learning

In this section, we recall some standard notations, tools and algorithms used in RL and DistrRL.

2.1 Markov decision process

Throughout the paper, we consider a Markov decision process (MDP) characterized by the tuple $(\mathcal{X}, \mathcal{A}, P, r, \gamma)$ with finite state space $\mathcal{X}$, finite action space $\mathcal{A}$, transition kernel $P\colon \mathcal{X} \times \mathcal{A} \to \mathcal{P}(\mathcal{X})$, reward function $r\colon \mathcal{X} \times \mathcal{A} \times \mathcal{X} \to \mathbb{R}$, and discount factor $0 \leq \gamma < 1$. If the agent takes some action $a \in \mathcal{A}$ while the environment is in state $x \in \mathcal{X}$, then the next state X_1 is sampled from the distribution $P(\cdot|x, a)$ and the immediate reward is equal to $r(x, a, X_1)$. In the discounted MDP setting, the agent seeks a policy $\pi\colon \mathcal{X} \to \mathcal{P}(\mathcal{A})$ maximizing its expected long-term return for each pair (x, a) of initial state and action:

$$Q^\pi(x, a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(X_t, A_t, X_{t+1}) \;\middle|\; X_0 = x, A_0 = a\right],$$

where $X_{t+1} \sim P(\cdot|X_t, A_t)$ and $A_{t+1} \sim \pi(\cdot|X_{t+1})$. $Q^\pi(x, a)$ and $V^\pi(x) = \sum_{a\in\mathcal{A}} \pi(a|x)Q^\pi(x, a)$ are respectively called the state-action value function and the value function of the policy π. Further, each of these functions can be seen as the unique fixed point of a so-called Bellman operator (Bellman, 1966), that we denote by T^π for Q-functions. For any $Q\colon \mathcal{X} \times \mathcal{A} \to \mathbb{R}$, the image of Q by T^π is another Q-function given by:

$$(T^\pi Q)(x, a) = \sum_{x'} P(x'|x, a)\big[r(x, a, x') + \gamma \sum_{a'} \pi(a'|x')Q(x', a')\big].$$

This Bellman operator T^π has several nice properties: in particular, it is a γ-contraction in $\|\cdot\|_\infty$ and thus admits a unique fixed point (by Banach's fixed point theorem), namely $Q^\pi = T^\pi Q^\pi$. It is also well-known from (Bellman, 1966) that there always exists at least one policy π^* that is optimal uniformly for all initial conditions (x, a):

$$Q^*(x, a) \;:=\; Q^{\pi^*}(x, a) \;=\; \sup_\pi Q^\pi(x, a) \quad \text{and} \quad V^*(x) \;:=\; \max_a Q^*(x, a) \;=\; V^{\pi^*}(x) \;=\; \sup_\pi V^\pi(x).$$

Similarly, this optimal Q-function Q^* is the unique fixed point of some operator T called the Bellman optimality operator and defined by:

$$(TQ)(x, a) = \sum_{x'} P(x'|x, a)\big[r(x, a, x') + \gamma \max_{a'} Q(x', a')\big],$$

which is also a γ-contraction in $\|\cdot\|_\infty$. Noteworthy, knowing Q^* is sufficient to behave optimally: indeed, a policy π^* is optimal if and only if in every state x, $\mathrm{support}\big(\pi^*(\cdot|x)\big) \subseteq \arg\max_a Q^*(x, a)$.

2.2 The distributional Bellman operator

In distributional RL, we replace scalar-valued functions Q by functions μ taking values that are entire probability distributions: $\mu^{(x,a)} \in \mathcal{P}_b(\mathbb{R})$ for each pair (x, a). In other words, μ is a collection of distributions indexed by states and actions. We recall below the definition of the distributional Bellman operator, which generalizes the Bellman operator to distributions.

Definition 2.1 (Distributional Bellman operator, Bellemare et al. (2017))**.** Let π be a policy. The distributional Bellman operator $\mathcal{T}^\pi\colon \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}} \to \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}}$ is defined for any distribution function $\mu = (\mu^{(x,a)})_{x,a}$ by

$$(\mathcal{T}^\pi \mu)^{(x,a)} = \sum_{x',a'} P(x'|x, a)\pi(a'|x') f_{r(x,a,x'),\gamma\#}\mu^{(x',a')}.$$

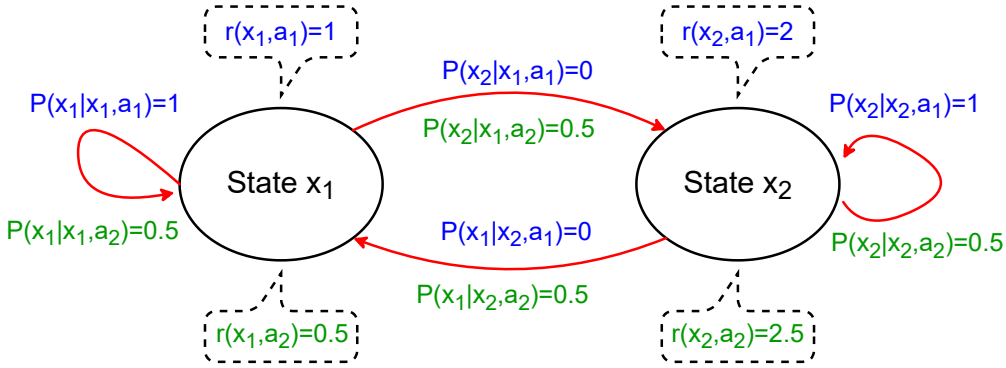


Figure 1: Markov decision process with two states $\mathcal{X} = \{x_1, x_2\}$, two actions $\mathcal{A} = \{a_1, a_2\}$ and reward function independent of the next state: $r(x, a, \cdot) \equiv r(x, a)$.

We know from Bellemare et al. (2017) that $\mathcal{T}^\pi$ is a γ-contraction in the maximal p-Wasserstein metric[3]

$$\overline{W}_p(\mu_1, \mu_2) = \max_{(x,a)\in\mathcal{X}\times\mathcal{A}} W_p(\mu_1^{(x,a)}, \mu_2^{(x,a)})$$

at any order $p \in [1, +\infty]$. Consequently, $\mathcal{T}^\pi$ has a unique fixed point $\mu_\pi = (\mu_\pi^{(x,a)})_{x,a}$ equal to the collection of the probability distributions of the returns:

$$\mu_\pi^{(x,a)} = \mathrm{Distr}\left(\sum_{t=0}^{\infty} \gamma^t r(X_t, A_t, X_{t+1}) \,\middle|\, X_0 = x, A_0 = a\right) . \tag{1}$$

Applying $\mathcal{T}^\pi$ may be prohibitive in terms of space complexity: if μ is discrete, then $\mathcal{T}^\pi\mu$ is still discrete but with up to $|\mathcal{X}| \cdot |\mathcal{A}|$ times more atoms, as shown in the following example.

Example 2.2. Let $\mu = (\mu^{(x,a)})_{x,a}$ be the collection of the atomic distributions

$$\mu^{(x,a)} = \sum_{k=1}^{K} p_k(x, a)\delta_{Q_k(x,a)} ,$$

where $K \geq 1$, $p_k(x, a) \geq 0$ and $p_1(x, a) + \cdots + p_K(x, a) = 1$. Then, $\mathcal{T}^\pi\mu$ is also a collection of atomic distributions with (at most) $|\mathcal{X}| \cdot |\mathcal{A}|$ times more atoms:

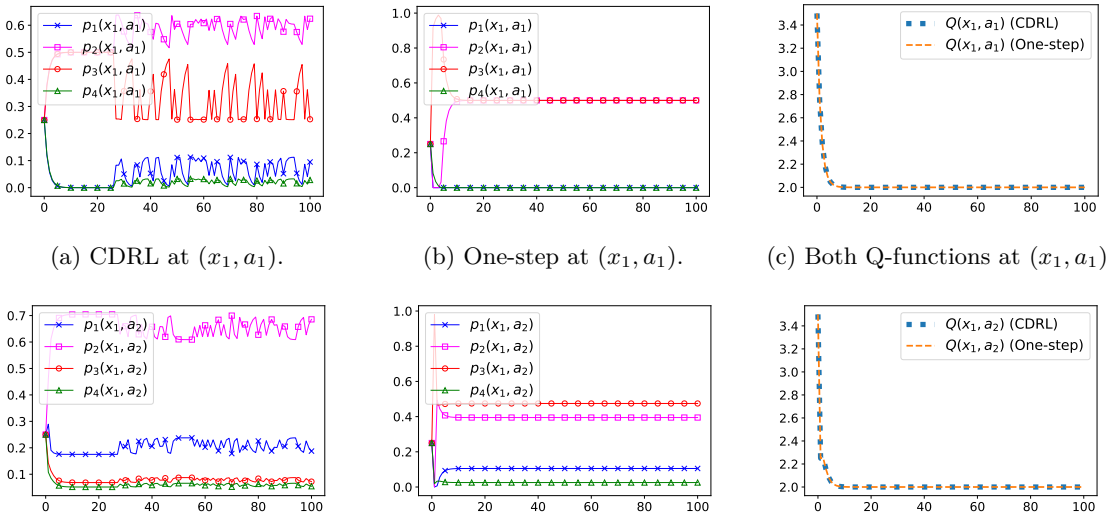
$$(\mathcal{T}^\pi\mu)^{(x,a)} = \sum_{x',a'} P(x'|x, a)\pi(a'|x') \sum_{k=1}^{K} p_k(x', a')\delta_{r(x,a,x')+\gamma Q_k(x',a')} .$$

Projected operators. Motivated by this space complexity issue, projected DistrRL operators were proposed to ensure a predefined and fixed space complexity budget. A projected DistrRL operator is the composition of a DistrRL operator with a projection over some parametric family of distributions. The Cramér distance projection $\Pi_\mathcal{C}$ has been considered in (Bellemare et al., 2017; Rowland et al., 2018; Bellemare et al., 2019): we recall its definition below.

Definition 2.3 (Cramér projection)**.** Let $K \geq 2$ and $z_1 < \cdots < z_K$ be real numbers defining the support of the categorical distributions. The Cramér projection is then defined by: for all $z' \in \mathbb{R}$,

$$\Pi_\mathcal{C}(\delta_{z'}) = \begin{cases} \delta_{z_1} & \text{if } \; z' \leq z_1 \\ \frac{z_{j+1}-z'}{z_{j+1}-z_j}\delta_{z_j} + \frac{z'-z_j}{z_{j+1}-z_j}\delta_{z_{j+1}} & \text{if } \; z_j < z' \leq z_{j+1} \\ \delta_{z_K} & \text{if } \; z' > z_K \end{cases}$$

[3]For $p \geq 1$, we recall that the p-Wasserstein distance between two probability distributions ν_1, ν_2 on $\mathbb{R}$ with CDFs F_1, F_2 is defined as $W_p(\nu_1, \nu_2) = \left(\int_{\tau=0}^{1} \left|F_1^{-1}(\tau) - F_2^{-1}(\tau)\right|^p d\tau\right)^{\frac{1}{p}}$. If $p = \infty$, $W_\infty(\nu_1, \nu_2) = \sup_{\tau\in(0,1)} |F_1^{-1}(\tau) - F_2^{-1}(\tau)|$.



(a) CDRL at (x_1, a_1). (b) One-step at (x_1, a_1). (c) Both Q-functions at (x_1, a_1).

(d) CDRL at (x_1, a_2). (e) One-step at (x_1, a_2). (f) Both Q-functions at (x_1, a_2).

Figure 2: Instability of categorical distributional control compared to the convergence of our one-step approach. The two leftmost plots are the iterations of the Cramér-projected "full" optimality operator $\Pi_{\mathcal{C}}\mathcal{T}$, while the center plots are obtained with our one-step operator $\Pi_{\mathcal{C}}\mathcal{T}$. The rightmost plots correspond to the Q-functions $Q(x, a) = \sum_{k=1}^{4} p_k(x, a) z_k$.

and extended linearly to mixtures of Dirac measures.

We will also apply $\Pi_{\mathcal{C}}$ entrywise to collections of distributions: if $\mu = (\mu^{(x,a)})_{x,a}$, then $\Pi_{\mathcal{C}}\mu = (\Pi_{\mathcal{C}}\mu^{(x,a)})_{x,a}$. The Cramér projection satisfies an important *mean-preserving property*: for any discrete distribution q with support included in the interval $[z_1, z_K]$, then:

$$\mathbb{E}_{Z\sim\Pi_{\mathcal{C}}(q)}[Z] = \mathbb{E}_{Z\sim q}[Z] \ . \tag{2}$$

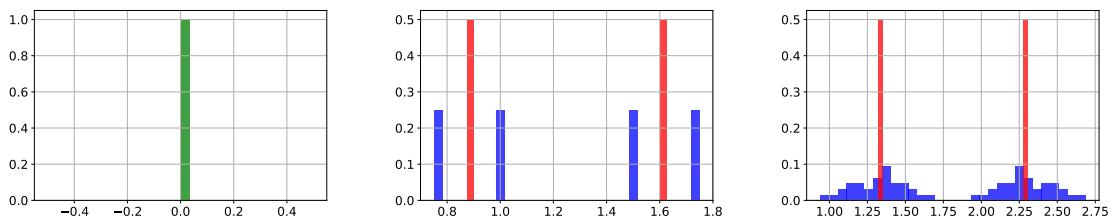
In this line of work, the proxy distributions $\sum_{k=1}^{K} p_k(x, a)\delta_{z_k}$ are parametrized by the categorical probabilities $p_1(x, a), \ldots, p_K(x, a)$. In (Dabney et al., 2018), the Wasserstein-1 projection Π_{W_1} over atomic distributions $\frac{1}{N}\sum_{i=1}^{N} \delta_{Q_i(x,a)}$ is used. For this specific projection, the atoms that best approximate some CDF $F_{x,a}$ are obtained through quantile regression with $Q_i(x, a) = F_{x,a}^{-1}(\frac{2i-1}{2N})$.

2.3 Instability of distributional control

Bellemare et al. (2017) have also proposed DistrRL optimality operators $\mathcal{T}$, defined such that $\mathcal{T}\mu = \mathcal{T}^{\pi_G}\mu$ where π_G is a greedy policy with respect to the expectation of μ. The authors showed in their Propositions 1-2 that, unfortunately, $\mathcal{T}$ is not a contraction and does not necessarily admits a fixed point. For that reason, the convergence analysis of the control case in DistRL is more challenging than for the evaluation task. Rowland et al. (2018) circumvent this issue by reducing the control task to evaluation under the assumption that the optimal policy is unique. We start our investigation by verifying that this uniqueness hypothesis is critical to ensure convergence in DistrRL control. For that purpose, we choose a toy example – fully described in Section 5 – with (infinitely) many optimal policies and we run a CDRL procedure over it. As expected, we observe in Figure 2 the unstable behavior of the DistrRL paradigm for the control task. In the next section, we introduce our new "one-step" distributional approach converging even in this situation.

3 One-step distributional operators

This section introduces the building blocks of our new framework. The main goal here is to show that our one-step approach is theoretically sound, though being less ambitious than "full" DistrRL. The main benefit



(a) $\mu_0^{(x_1,a_2)}$ (b) $\mu_2^{(x_1,a_2)}$ (c) $\mu_4^{(x_1,a_2)}$

Figure 3: Illustration of Examples 2.2-3.3 in the two-states two-actions MDP depicted in Figure 1, with discount factor $\gamma = \frac{1}{2}$, for the stochastic policy $\pi(a|x) = \frac{1}{2}$ for all x, a. The probability distribution $\mu_j^{(x_1,a_2)}$ with histogram represented in red (resp. in blue) is obtained by applying j successive times the distributional operator $\mathcal{T}^\pi$ (resp. $\mathcal{T}^\pi$) to the initial distributions $\mu_0^{(x,a)} = \delta_0$.

of this simplification is that it will allow us to derive in the next section a convergence result holding for both evaluation and control. In particular in the control case, we will *not* need any additional assumption such as the uniqueness of the optimal policy as in CDRL (Rowland et al., 2018).

3.1 Formal definitions

Let us now define the one-step DistrRL operators. Intuitively, they are similar to the "full" DistrRL operator $\mathcal{T}^\pi$ except that they average the randomness after the first random transition and are thus oblivious to the randomness induced by the remaining time steps.

Definition 3.1 (One-step distributional Bellman operator)**.** Let π be a policy. The one-step distributional Bellman operator $\mathfrak{T}^\pi \colon \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}} \to \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}}$ is defined for any distribution function μ by

$$(\mathfrak{T}^\pi \mu)^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \sum_{a'} \pi(a'|x')\mathbb{E}_{Z\sim\mu(x',a')}[Z]} \ .$$

Definition 3.2 (One-step distributional Bellman optimality operator)**.** The one-step distributional Bellman optimality operator $\mathfrak{T} \colon \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}} \to \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}}$ is defined for any μ by

$$(\mathfrak{T} \mu)^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \max_{a'} \mathbb{E}_{Z\sim\mu(x',a')}[Z]} \ .$$

Example 3.3. Let μ be the same discrete distribution function as in Example 2.2. Then,

$$(\mathfrak{T}^\pi \mu)^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \sum_{a'} \pi(a'|x')Q(x',a')}$$

$$\text{and} \qquad (\mathfrak{T} \mu)^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \max_{a'} Q(x',a')} \ ,$$

where $Q(x',a') = \sum_{k=1}^K p_k(x',a')Q_k(x',a')$.

Similarly to $\mathcal{T}$, the one-step operators $\mathfrak{T}$ and $\mathfrak{T}^\pi$ lead to a space complexity issue by producing distributions with a number $|\mathcal{X}|$ of atoms, which can be too demanding for a large state space.

3.2 Main properties

Let us now discuss some key properties satisfied by our new one-step distributional operators. We first show that, similarly to the non-distributional setting, our approach comes with contractive operators in both evaluation and control.

Proposition 3.1 (Contractivity)**.** *Let π be a policy.*

	DistrRL	One-step DistrRL
Evaluation	$\text{Distr}\left(\sum_{t=0}^{\infty}\gamma^t r(X_t, A_t, X_{t+1})\right)$	$\sum_{x'} P(x'\|x,a)\delta_{r(x,a,x')+\gamma V^\pi(x')}$
Control	does not necessarily exist	$\sum_{x'} P(x'\|x,a)\delta_{r(x,a,x')+\gamma V^*(x')}$

Table 1: Comparison of the fixed points of the distributional Bellman operators in DistrRL versus one-step DistrRL.

(i) For any $1 \leq p \leq \infty$, the one-step operators $\mathcal{T}^\pi$ and $\mathcal{T}$ are γ-contractions in $\overline{W}_p$.

(ii) The Cramér-projected one-step operators $\Pi_{\mathcal{C}} \circ \mathcal{T}^\pi$ and $\Pi_{\mathcal{C}} \circ \mathcal{T}$ are γ-contractions in $\overline{W}_1$.

We stress that Proposition 3.1 highly contrasts with classic DistrRL where the control operator $\mathcal{T}$ is not a contraction in any metric. A major consequence of contractivity is the existence and uniqueness of fixed points, whose explicit formulas are gathered in the next proposition and in Table 1.

Proposition 3.2 (Fixed points)**.** *Let π be a policy and consider $\Pi_{\mathcal{C}}$ from Definition 2.3.*

(i) The unique fixed point of $\mathcal{T}^\pi$ is ν_π given by

$$\nu_\pi^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma V^\pi(x')}\,.$$

(ii) The unique fixed point of $\mathcal{T}$ is ν_* given by

$$\nu_*^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma V^*(x')}\,.$$

(iii) If $z_1 \leq r(x,a,x') + \gamma V^\pi(x') \leq z_K$ for all triplets (x,a,x'), then the unique fixed point of $\Pi_{\mathcal{C}} \circ \mathcal{T}^\pi$ is $\eta_\pi = \Pi_{\mathcal{C}}(\nu_\pi)$.

(iv) If $z_1 \leq r(x,a,x') + \gamma V^*(x') \leq z_K$ for all triplets (x,a,x'), then the unique fixed point of $\Pi_{\mathcal{C}} \circ \mathcal{T}$ is $\eta_* = \Pi_{\mathcal{C}}(\nu_*)$.

Interestingly, Proposition 3.2-(iii)-(iv) shows that, in our one-step framework, the fixed point of a projected operator is simply the projection of the fixed point of the unprojected operator. Although this fact seems natural, it is not necessarily true in DistrRL. Indeed, the proof of Proposition 3 in (Rowland et al., 2018) suggests that the fixed point of $\Pi_{\mathcal{C}} \circ \mathcal{T}^\pi$ is a worse approximation of μ_π than is $\Pi_{\mathcal{C}}\mu_\pi$, by a multiplicative factor $\sqrt{1/(1-\gamma)}$ in terms of Cramér distance. Equipped with our projected one-step operators, we derive in the next section variants of the CDRL algorithms.

4 One-step DistrRL algorithms

This section introduces new categorical algorithms based on the Cramér projection together with formal convergence guarantees.

4.1 One-Step CDRL

We propose two algorithms, for policy evaluation and control respectively, that are described in Algorithm 1. These new categorical methods are derived from the stochastic approximation of the projected operators $\Pi_{\mathcal{C}} \circ \mathcal{T}^\pi$ and $\Pi_{\mathcal{C}} \circ \mathcal{T}$. For each state-action pair (x,a), both methods learn a discrete probability distribution $p_1(x,a), \ldots, p_K(x,a)$ over some fixed support $z_1 < \cdots < z_K$. As in classic RL algorithms, we consider a sequence of stepsizes $\alpha_t(x,a) \geq 0$ indexed by states, actions and time steps $t \geq 0$. At each time t, we perform a mixture update between the current distribution and a distributional target which is the Cramér projection of a single Dirac mass located at the target of TD(0) or Q-learning, computed from a single

	CDRL	One-step CDRL
Categorical target	$\Pi_{\mathcal{C}}\left(\sum_{k=1}^{K} p_{t,k}(x_{t+1}, a^*)\delta_{r_t+\gamma z_k}\right)$	$\Pi_{\mathcal{C}}\left(\delta_{r_t+\gamma Q_t(x_{t+1},a^*)}\right)$
Time complexity	$\mathcal{O}(K \log K)$	$\mathcal{O}(\log K)$

Table 2: Comparison of the categorical targets in CDRL versus one-step CDRL. The notation a^* refers to a greedy action, namely $a^* \in \arg\max_{a'} Q_t(x_{t+1}, a')$.

transition (x_t, a_t, r_t, x_{t+1}). The only difference with tabular CDRL lies in the target as shown in Table 2. Contrary to the original CDRL target, ours remains the same whatever the greedy action a^* we choose inside the set $\arg\max_{a'} Q_t(x_{t+1}, a')$: this explains why our approach is stable even when the optimal policy is not unique as illustrated in Figure 2. Moreover, our categorical target is faster to compute than in CDRL, where the time complexity $\mathcal{O}(K \log(K))$ pays the price for inserting every atom $r_t + \gamma z_k$ into the sorted array $(z_1, \ldots, z_K)$. Before moving forward to the convergence analysis of Algorithm 1, we propose its deep

Algorithm 1 Tabular one-step categorical DistrRL

Input: $\eta_t^{(x,a)} = \sum_{k=1}^{K} p_{t,k}(x,a)\delta_{z_k}$ for all (x,a)
Sample transition: (x_t, a_t, r_t, x_{t+1})
Estimate Q-values: $Q_t(x_{t+1}, a) \leftarrow \sum_{k=1}^{K} p_{t,k}(x_{t+1}, a) \cdot z_k$
if policy evaluation **then**
$\widehat{\eta}_t^{(x_t,a_t)} \leftarrow \Pi_{\mathcal{C}}(\delta_{r_t+\gamma \sum_{a'} \pi(a'|x_{t+1}) Q_t(x_{t+1},a')})$
else if control **then**
$\widehat{\eta}_t^{(x_t,a_t)} \leftarrow \Pi_{\mathcal{C}}(\delta_{r_t+\gamma \max_{a'} Q_t(x_{t+1},a')})$
end if
Mixture update: $\eta_{t+1}^{(x_t,a_t)} \leftarrow (1-\alpha_t(x_t,a_t))\eta_t^{(x_t,a_t)} + \alpha_t(x_t,a_t)\widehat{\eta}_t^{(x_t,a_t)}$
$\eta_{t+1}^{(x,a)} \leftarrow \eta_t^{(x,a)}$, $\forall (x,a) \neq (x_t, a_t)$
Output: η_{t+1}

RL counterpart in Algorithm 2 based on the minimization of a Kullback-Leibler (KL) loss.

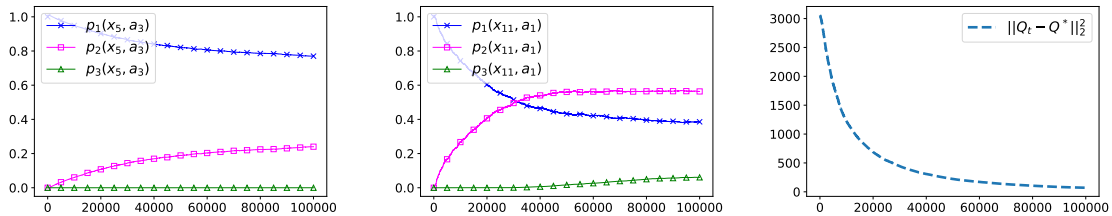
Deep one-step CDRL. The main challenge in a non-tabular context is to learn the distributions in a compact and efficient way. For that purpose, we use the same deep categorical approach as in C51 (Bellemare et al., 2017). As shown in Table 1, the one-step method aims at learning a much simpler distribution than full DistrRL. This suggests choosing a small number of categories, e.g. $K = 4$ used in Section 5, compared to what is commonly used in CDRL. Due to its similarity with C51, we call this new algorithm "OS-C51".

Algorithm 2 OS-C51 (single update)

Input: categorical distributions $\eta_\theta^{(x,a)} = \sum_{k=1}^{K} p_{\theta,k}(x,a)\delta_{z_k}$ and a transition (x_t, a_t, r_t, x_{t+1})
Compute Q-function in next state: $Q(x_{t+1}, a') \leftarrow \sum_{k=1}^{K} p_{\theta,k}(x_{t+1}, a')z_k$
Compute categorical target: $\widehat{\eta}^{(x_t,a_t)} \leftarrow \Pi_{\mathcal{C}}(\delta_{r_t+\gamma \max_{a'} Q(x_{t+1},a')})$
Output: $\mathrm{KL}(\widehat{\eta}^{(x_t,a_t)} \| \eta_\theta^{(x_t,a_t)})$

4.2 Convergence analysis

We now provide convergence guarantees for our tabular one-step DistrRL algorithms. The major difference with the analysis of CDRL (Rowland et al., 2018) is that we do not require the uniqueness of the optimal policy in the case of control. We also rely on the existing analysis of non-distributional RL (Dayan, 1992; Tsitsiklis, 1994; Jaakkola et al., 1993). In particular, we require the following standard assumption.



(a) (x_5, a_3) (b) (x_{11}, a_1) (c) $\|Q_t - Q^*\|_2^2$

Figure 4: Learning dynamics of the atomic probability distribution $(p_{t,k}(x,a))_{1\leq k\leq K=3}$ produced by Algorithm 1 (control) on the Frozen Lake environment.

Assumption 4.1. For any pair $(x,a) \in \mathcal{X} \times \mathcal{A}$, the stepsizes $(\alpha_t(x,a))_{t\geq 0}$ satisfy the Robbins-Monro conditions:

$$\begin{cases} \sum_{t=0}^{\infty} \alpha_t(x,a) = \infty \\ \sum_{t=0}^{\infty} \alpha_t(x,a)^2 < \infty \end{cases} \quad \text{almost surely.}$$

Equipped with stepsizes $(\alpha_t(x,a))$ satisfying the Robbins-Monro conditions, we are now ready to state our main theoretical contribution: namely, the convergence of Algorithm 1.

Theorem 4.1. *Consider Algorithm 1 and let us assume that Assumption 4.1 holds for the stepsizes* $(\alpha_t(x,a))_{t,x,a}$.

(i) In the case of evaluation of a policy π*,*

$$\overline{W}_1(\eta_t, \eta_\pi) \xrightarrow{t\to\infty} 0 \quad \textit{almost surely},$$

where η_π *is defined in Proposition 3.2-(iii).*

(ii) In the case of control,

$$\overline{W}_1(\eta_t, \eta_*) \xrightarrow{t\to\infty} 0 \quad \textit{almost surely},$$

where η_* *is defined in Proposition 3.2-(iv).*

The proof of Theorem 4.1 is deferred to the Supplementary Material: it follows the same steps as the proofs of Theorem 2 in (Tsitsiklis, 1994) and Theorem 1 in (Rowland et al., 2018). Notably, our analysis remains the same for evaluation and control contrary to (Rowland et al., 2018), where the control case requires additional assumptions (namely, uniqueness of the optimal policy and for all $t \geq 0$, for all $1 \leq k \leq K$ such that $p_{t,k}(x_{t+1}, a^*) \neq 0$, $r_t + \gamma z_k \in [z_1, z_K]$ almost surely).

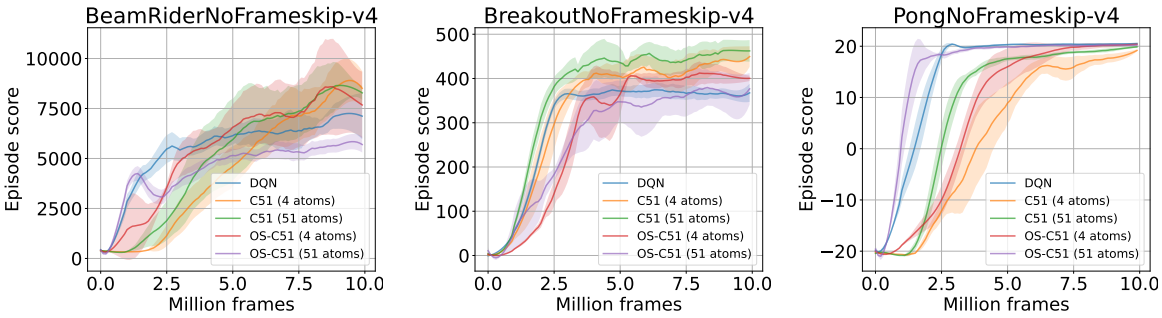
5 Numerical experiments

In this section, we present numerical experiments on both tabular and Atari games environments.

5.1 Tabular setting

We describe our tabular experiments: first in a dynamic programming context i.e. knowing the transition kernel P and the reward function r, then in the "Frozen Lake" environment by only observing empirical transitions.

Distributional dynamic programming. Figures 2-3 are obtained by exact dynamic programming in the MDP in Figure 1 with $\gamma = \frac{1}{2}$: in this specific case, all policies are optimal. As discussed in subsection 2.3, this handcrafted example reveals the instability of classic CDRL in Figure 2 contrary to our one-step



(a) Beamrider (b) Breakout (c) Pong

Figure 5: Average evaluation episodic return over 10 million frames for three Atari games.

approach: for both methods we consider $K = 4$ categories with $(z_1, z_2, z_3, z_4) = (0, 1.9, 2.1, 10)$. Figure 3 illustrates the space complexity issue of DistrRL without projection: while the number of atoms is multiplied by at most $|\mathcal{X}||\mathcal{A}|$ after each application of $\mathcal{T}^\pi$, our one-step operators produce distributions with (at most) as many atoms as there are states, i.e. two in this example.

Frozen Lake. We also consider the Frozen Lake environment from OpenAI Gym (Brockman et al., 2016) with discount factor $\gamma = 0.95$. It is characterized by 16 states $x_1, \ldots, x_{16}$ and four actions $a_1, \ldots, a_4$. Plus, this is a stochastic environment i.e. the transition probabilities $P(x'|x, a)$ are not all equal to either 0 or 1, which justifies a distributional approach. In Figure 4, we plot the iterates $p_{t,k}(x, a)$ over 100000 steps generated by our tabular Algorithm 1 with $K = 3$ atoms and $(z_1, z_2, z_3) = (0, 10, 20)$, constant stepsize $\alpha = 0.6$ and ϵ-greedy exploration with ϵ exponentially decaying from 1 to 0.25. We average the results over 100 seeds. Given a pair (x, a), we observe the joint convergence of the three probabilities. As in CDRL, and because of the mean-preserving property (Eq. 2), the average $Q_t(x, a) = \sum_{k=1}^{3} p_{t,k}(x, a) z_k$ coincides with the Q-learning iterates and converges to Q^*.

5.2 Atari games

For the experiments on Atari games (Bellemare et al., 2013), we implement the OS-C51 agent on top of the "cleanrl" codebase (Huang et al., 2022). We compare the OS-C51 algorithm against C51 for two different number of atoms: $K = 4$ or $K = 51$. In each of the two cases, we choose the support to be evenly spread over the interval $[-10, 10]$: $z_1 = -10 < \cdots < z_K = 10$. We use the same architecture as C51: a deep neural network, parameterized by θ, takes an observation as input and outputs a vector of K logits. For the optimization of OS-C51, we use the Adam optimizer (Kingma & Ba, 2014) with learning rate set to 0.00025 and batch size equal to 32. As shown in Figure 5, the performance of OS-C51 is comparable with C51 on the Beamrider, Breakout and Pong games. The results are averaged over 3 seeds.

6 Conclusion

We proposed new distributional RL algorithms that naturally extend TD(0) and Q-learning in the tabular setting. The main novelty in our approach is the use of new one-step distributional operators circumventing the instability issues of DistrRL control. We provided both theoretical convergence analysis and empirical proof-of-concept. Future research could investigate the generalization of our method based on the dynamics of several successive steps instead of a single one.

References

Marc G Bellemare, Yavar Naddaf, Joel Veness, and Michael Bowling. The arcade learning environment: An evaluation platform for general agents. *Journal of Artificial Intelligence Research*, 47:253–279, 2013.

Marc G Bellemare, Will Dabney, and Rémi Munos. A distributional perspective on reinforcement learning. In *International Conference on Machine Learning*, pp. 449–458. PMLR, 2017.

Marc G Bellemare, Nicolas Le Roux, Pablo Samuel Castro, and Subhodeep Moitra. Distributional reinforcement learning with linear function approximation. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pp. 2203–2211. PMLR, 2019.

Richard Bellman. Dynamic programming. *Science*, 153(3731):34–37, 1966.

Greg Brockman, Vicki Cheung, Ludwig Pettersson, Jonas Schneider, John Schulman, Jie Tang, and Wojciech Zaremba. Openai gym. *arXiv preprint arXiv:1606.01540*, 2016.

Will Dabney, Mark Rowland, Marc Bellemare, and Rémi Munos. Distributional reinforcement learning with quantile regression. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32, 2018.

Peter Dayan. The convergence of td (λ) for general λ. *Machine learning*, 8(3):341–362, 1992.

Shengyi Huang, Rousslan Fernand Julien Dossa, Chang Ye, Jeff Braga, Dipam Chakraborty, Kinal Mehta, and João G.M. Araújo. Cleanrl: High-quality single-file implementations of deep reinforcement learning algorithms. *Journal of Machine Learning Research*, 23(274):1–18, 2022.

Tommi Jaakkola, Michael Jordan, and Satinder Singh. Convergence of stochastic iterative dynamic programming algorithms. *Advances in neural information processing systems*, 6, 1993.

Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Alex Graves, Ioannis Antonoglou, Daan Wierstra, and Martin Riedmiller. Playing atari with deep reinforcement learning. *arXiv preprint arXiv:1312.5602*, 2013.

Volodymyr Mnih, Koray Kavukcuoglu, David Silver, Andrei A Rusu, Joel Veness, Marc G Bellemare, Alex Graves, Martin Riedmiller, Andreas K Fidjeland, Georg Ostrovski, et al. Human-level control through deep reinforcement learning. *Nature*, 518(7540):529–533, 2015.

Tetsuro Morimura, Masashi Sugiyama, Hisashi Kashima, Hirotaka Hachiya, and Toshiyuki Tanaka. Nonparametric return distribution approximation for reinforcement learning. In *ICML*, 2010a.

Tetsuro Morimura, Masashi Sugiyama, Hisashi Kashima, Hirotaka Hachiya, and Toshiyuki Tanaka. Parametric return density estimation for reinforcement learning. In *Proceedings of the Twenty-Sixth Conference on Uncertainty in Artificial Intelligence*, pp. 368–375, 2010b.

Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

Mark Rowland, Marc Bellemare, Will Dabney, Rémi Munos, and Yee Whye Teh. An analysis of categorical distributional reinforcement learning. In *International Conference on Artificial Intelligence and Statistics*, pp. 29–37. PMLR, 2018.

Mark Rowland, Rémi Munos, Mohammad Gheshlaghi Azar, Yunhao Tang, Georg Ostrovski, Anna Harutyunyan, Karl Tuyls, Marc G Bellemare, and Will Dabney. An analysis of quantile temporal-difference learning. *arXiv preprint arXiv:2301.04462*, 2023.

Richard S Sutton. Learning to predict by the methods of temporal differences. *Machine learning*, 3(1):9–44, 1988.

Richard S Sutton and Andrew G Barto. *Reinforcement learning: An introduction*. MIT press, 2018.

John N Tsitsiklis. Asynchronous stochastic approximation and q-learning. *Machine learning*, 16(3):185–202, 1994.

Christopher JCH Watkins and Peter Dayan. Q-learning. *Machine learning*, 8(3):279–292, 1992.

A Proof of Proposition 3.1

i). *Contraction in Wasserstein distance.* Let $1 \leq p < \infty$. Let $\mu_1, \mu_2 \in \mathcal{P}_b(\mathbb{R})^{\mathcal{X}\times\mathcal{A}}$ be two distribution functions. Denoting $Q_1(x,a)$ and $Q_2(x,a)$ the respective expectations of $\mu_1^{(x,a)}$ and $\mu_2^{(x,a)}$, we have for any pair (x,a):

$$
\begin{aligned}
&W_p^p((\mathcal{T}\mu_1)^{(x,a)}, (\mathcal{T}\mu_2)^{(x,a)}) = \\
&\qquad W_p^p(\sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \max_{a'} Q_1(x',a')}, \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \max_{a'} Q_2(x',a')}) \\
&\qquad \leq \sum_{x'} P(x'|x,a) W_p^p(\delta_{r(x,a,x')+\gamma \max_{a'} Q_1(x',a')}, \delta_{r(x,a,x')+\gamma \max_{a'} Q_2(x',a')}) \\
&\qquad = \gamma^p \sum_{x'} P(x'|x,a) |\max_{a'} Q_1(x',a') - \max_{a'} Q_2(x',a')|^p \\
&\leq \gamma^p \sum_{x'} P(x'|x,a) \max_{a'} |Q_1(x',a') - Q_2(x',a')|^p = \gamma^p \sum_{x'} P(x'|x,a) \max_{a'} \Big| \int_{\tau=0}^1 (F^{-1}_{x',a'}(\tau) - \tilde{F}^{-1}_{x',a'}(\tau)) d\tau \Big|^p \\
&\qquad \leq \gamma^p \sum_{x'} P(x'|x,a) \max_{a'} \underbrace{\int_{\tau=0}^1 \big|F^{-1}_{x',a'}(\tau) - \tilde{F}^{-1}_{x',a'}(\tau)\big|^p d\tau}_{W_p^p(\mu_1^{(x',a')}, \mu_2^{(x',a')})} \leq \gamma^p \overline{W}_p^p(\mu_1, \mu_2)\,,
\end{aligned}
$$

where the first inequality follows from the interpretation of the Wasserstein distance as an infimum over couplings, and $F_{x,a}, \tilde{F}_{x,a}$ denote the respective CDFs of $\mu_1^{(x,a)}$ and $\mu_2^{(x,a)}$. Hence, by taking the supremum over all (x,a), we deduce that $\mathcal{T}$ is a γ-contraction in $\overline{W}_p$:

$$
\overline{W}_p(\mathcal{T}\mu_1, \mathcal{T}\mu_2) \leq \gamma \overline{W}_p(\mu_1, \mu_2).
$$

Similarly for $p = \infty$,

$$
\begin{aligned}
&W_\infty((\mathcal{T}\mu_1)^{(x,a)}, (\mathcal{T}\mu_2)^{(x,a)}) = \\
&\qquad W_\infty(\sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \max_{a'} Q_1(x',a')}, \sum_{x'} P(x'|x,a)\delta_{r(x,a,x')+\gamma \max_{a'} Q_2(x',a')}) \\
&\qquad \leq \sum_{x'} P(x'|x,a) W_\infty(\delta_{r(x,a,x')+\gamma \max_{a'} Q_1(x',a')}, \delta_{r(x,a,x')+\gamma \max_{a'} Q_2(x',a')}) \\
&\qquad = \gamma \sum_{x'} P(x'|x,a) |\max_{a'} Q_1(x',a') - \max_{a'} Q_2(x',a')| \\
&\leq \gamma \sum_{x'} P(x'|x,a) \max_{a'} |Q_1(x',a') - Q_2(x',a')| = \gamma \sum_{x'} P(x'|x,a) \max_{a'} \Big| \int_{\tau=0}^1 (F^{-1}_{x',a'}(\tau) - \tilde{F}^{-1}_{x',a'}(\tau)) d\tau \Big| \\
&\qquad \leq \gamma \sum_{x'} P(x'|x,a) \max_{a'} \int_{\tau=0}^1 \big|F^{-1}_{x',a'}(\tau) - \tilde{F}^{-1}_{x',a'}(\tau)\big| d\tau \leq \gamma \overline{W}_\infty(\mu_1, \mu_2)\,.
\end{aligned}
$$

For policy evaluation, one can follow the same proofs by incorporating the following additional step:

$$
\Big|\sum_{a'} \pi(a'|x')(Q_1(x',a') - Q_2(x',a'))\Big| \leq \sum_{a'} \pi(a'|x') \big|Q_1(x',a') - Q_2(x',a')\big| \leq \max_{a'} \big|Q_1(x',a') - Q_2(x',a')\big|\,.
$$

ii). Let us write:

$$
\begin{aligned}
&W_1((\Pi_{\mathcal{C}}\mathcal{T}\mu_1)^{(x,a)}, (\Pi_{\mathcal{C}}\mathcal{T}\mu_2)^{(x,a)}) = \\
&\quad W_1(\sum_{x'} P(x'|x,a)\Pi_{\mathcal{C}}\delta_{r(x,a,x')+\gamma\max_{a'} Q_1(x',a')}, \sum_{x'} P(x'|x,a)\Pi_{\mathcal{C}}\delta_{r(x,a,x')+\gamma\max_{a'} Q_2(x',a')}) \\
&\quad \leq \sum_{x'} P(x'|x,a) W_1(\Pi_{\mathcal{C}}\delta_{r(x,a,x')+\gamma\max_{a'} Q_1(x',a')}, \Pi_{\mathcal{C}}\delta_{r(x,a,x')+\gamma\max_{a'} Q_2(x',a')}) \\
&\quad \leq \gamma \sum_{x'} P(x'|x,a) |\max_{a'} Q_1(x',a') - \max_{a'} Q_2(x',a')| \\
&\leq \gamma \sum_{x'} P(x'|x,a) \max_{a'} |Q_1(x',a') - Q_2(x',a')| = \gamma \sum_{x'} P(x'|x,a) \max_{a'} \Big| \int_{\tau=0}^{1} (F^{-1}_{x',a'}(\tau) - \tilde{F}^{-1}_{x',a'}(\tau)) d\tau \Big| \\
&\quad \leq \gamma \sum_{x'} P(x'|x,a) \max_{a'} \underbrace{\int_{\tau=0}^{1} \big| F^{-1}_{x',a'}(\tau) - \tilde{F}^{-1}_{x',a'}(\tau) \big| d\tau}_{W_1(\mu_1^{(x',a')}, \mu_2^{(x',a')})} \leq \gamma \overline{W}_1(\mu_1, \mu_2)\,,
\end{aligned}
$$

where the second inequality follows from Lemma A.1. Taking the supremum over all (x, a) concludes the proof. The proof for $\Pi_{\mathcal{C}}\mathcal{T}^{\pi}$ is similar.

Lemma A.1. *Let a, b be two real numbers. Then,*

$$W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) \leq |a - b|.$$

Proof. We proceed by exhaustion of all possible (redundant) cases, up to permuting a and b:

(i) a, b are both outside the interval $(z_1, z_K]$: the proof is trivial in this case

(ii) $a \in (z_1, z_K]$, $b \in \{z_1, \ldots, z_K\}$ and $a \leq b$

(iii) $a \in (z_1, z_K]$, $b \in \{z_1, \ldots, z_K\}$ and $a > b$

(iv) $a \in (z_1, z_K]$ and $b \notin (z_1, z_K]$

(v) a, b are both inside the interval $(z_1, z_K]$

Proof in case ii). We assume that b belongs to the support, $b \geq a$ and a lies inside the interval $(z_1, z_K]$:

$$z_j < a \leq z_{j+1} \quad \text{and} \quad b = z_{k+1} \quad \text{with} \quad 1 \leq j \leq k \leq K\ .$$

Then, we have:

$$
\begin{aligned}
W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) &= \frac{z_{j+1} - a}{z_{j+1} - z_j}\big(z_{k+1} - z_j\big) + \frac{a - z_j}{z_{j+1} - z_j}\big(z_{k+1} - z_{j+1}\big) \\
&= (z_{k+1} - z_j)\frac{z_{j+1} - a + a - z_j}{z_{j+1} - z_j} - (z_{j+1} - z_j)\frac{a - z_j}{z_{j+1} - z_j} \\
&= z_{k+1} - z_j - (a - z_j) = z_{k+1} - a = |a - b|.
\end{aligned}
$$

Proof in case iii). Similar to (ii) except that we have $b = z_k$ with $j \geq k$:

$$
\begin{aligned}
W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) &= \frac{z_{j+1} - a}{z_{j+1} - z_j}\big(z_j - z_k\big) + \frac{a - z_j}{z_{j+1} - z_j}\big(z_{j+1} - z_k\big) \\
&= (z_j - z_k)\frac{z_{j+1} - a + a - z_j}{z_{j+1} - z_j} + (z_{j+1} - z_j)\frac{a - z_j}{z_{j+1} - z_j} \\
&= z_j - z_k + a - z_j = a - z_k = |a - b|.
\end{aligned}
$$

Proof in case iv). Now, if b is outside the interval $(z_1, z_k]$, say $b \leq z_1$, we deduce from the previous computation that $W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) = a - z_1 \leq |a - b|$.

Proof in case v). Finally, if both a and b lie in $(z_1, z_K]$,

$$\begin{aligned} W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) &= W_1(\frac{z_{j+1} - a}{z_{j+1} - z_j}\delta_{z_j} + \frac{a - z_j}{z_{j+1} - z_j}\delta_{z_{j+1}}, \Pi_{\mathcal{C}}\delta_b) \\ &\leq \frac{z_{j+1} - a}{z_{j+1} - z_j} W_1(\delta_{z_j}, \Pi_{\mathcal{C}}\delta_b) + \frac{a - z_j}{z_{j+1} - z_j} W_1(\delta_{z_{j+1}}, \Pi_{\mathcal{C}}\delta_b) \\ &= \frac{z_{j+1} - a}{z_{j+1} - z_j}\big|b - z_j\big| + \frac{a - z_j}{z_{j+1} - z_j}\big|b - z_{j+1}\big|, \end{aligned} \tag{3}$$

where we used cases (ii) and (iii) in the last equality. Then, if $b \geq z_{j+1}$, Eq. equation 3 implies

$$\begin{aligned} W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) &\leq \frac{z_{j+1} - a}{z_{j+1} - z_j}(b - z_j) + \frac{a - z_j}{z_{j+1} - z_j}(b - z_{j+1}) \\ &= (b - z_j)\frac{z_{j+1} - a + a - z_j}{z_{j+1} - z_j} - (z_{j+1} - z_j)\frac{a - z_j}{z_{j+1} - z_j} = b - z_j - a + z_j = b - a. \end{aligned}$$

Symmetrically, if $b \leq z_j$, Eq. equation 3 implies

$$W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) \leq \frac{z_{j+1} - a}{z_{j+1} - z_j}(z_j - b) + \frac{a - z_j}{z_{j+1} - z_j}(z_{j+1} - b) = a - b.$$

Lastly, if a, b both belong to the same segment $(z_j, z_{j+1}]$, and say $a \leq b$, we have by direct computation:

$$W_1(\Pi_{\mathcal{C}}\delta_a, \Pi_{\mathcal{C}}\delta_b) = (z_{j+1} - z_j)\frac{z_{j+1} - a - (z_{j+1} - b)}{z_{j+1} - z_j} = b - a\ .$$

□

B Proof of Proposition 3.2

By combining Proposition 3.1 with Banach's fixed point theorem, we deduce the existence and uniqueness of the fixed points.

For (i), we verify:

$$\begin{aligned} (\mathcal{T}\nu_\pi)^{(x,a)} &= \sum_{x'} P(x'|x,a)\delta_{r(x,a,x') + \gamma \sum_{a'} \pi(a'|x') \sum_{x''} P(x''|x',a')(r(x',a',x'') + \gamma V^\pi(x''))} \\ &= \sum_{x'} P(x'|x,a)\delta_{r(x,a,x') + \gamma \sum_{a'} \pi(a'|x') Q^\pi(x',a')} = \nu_\pi^{(x,a)}\ . \end{aligned}$$

ii). For the control case,

$$\begin{aligned} (\mathcal{T}\nu_*)^{(x,a)} &= \sum_{x'} P(x'|x,a)\delta_{r(x,a,x') + \gamma \max_{a'} \sum_{x''} P(x''|x',a')(r(x',a',x'') + \gamma V^*(x''))} \\ &= \sum_{x'} P(x'|x,a)\delta_{r(x,a,x') + \gamma \max_{a'} Q^*(x',a')} = \nu_*^{(x,a)}\ . \end{aligned}$$

iii). We start by computing:

$$(\mathcal{T}^\pi \circ \Pi_{\mathcal{C}}\nu_\pi)^{(x,a)} = \sum_{x'} P(x'|x,a)\delta_{r(x,a,x') + \gamma \sum_{a'} \pi(a'|x') Q^\pi(x',a')} = \nu_\pi^{(x,a)}\ ,$$

where we used the *mean-preserving property* of the Cramér projection (see Eq. equation 2). Then, we deduce that

$$\Pi_{\mathcal{C}} \circ \mathcal{T}^\pi \circ \Pi_{\mathcal{C}}\nu_\pi = \Pi_{\mathcal{C}}\nu_\pi = \eta_\pi\ .$$

iv). Similarly, one can show that η_* is the fixed point of $\Pi_{\mathcal{C}} \circ \mathcal{T}$.

C Proof of Theorem 4.1

We follow the same steps as in the proofs of Theorem 2 in (Tsitsiklis, 1994) and Theorem 1 in (Rowland et al., 2018). We focus on the control case (ii), though the proof remains similar for policy evaluation. Let us define for each $(x,a) \in \mathcal{X} \times \mathcal{A}$: $L_0^{(x,a)} = \delta_{z_1}$, $U_0^{(x,a)} = \delta_{z_K}$, and for $j \geq 0$,

$$L_{j+1}^{(x,a)} = \frac{1}{2} L_j^{(x,a)} + \frac{1}{2} (\Pi_{\mathcal{C}} \mathcal{T} L_j)^{(x,a)} \quad \text{and} \quad U_{j+1}^{(x,a)} = \frac{1}{2} U_j^{(x,a)} + \frac{1}{2} (\Pi_{\mathcal{C}} \mathcal{T} U_j)^{(x,a)} \ .$$

We now state the two following lemmas and then, before proving them, explain how they allow us to conclude.

Lemma C.1. *(i) In terms of entrywise stochastic dominance, it holds for all $j \geq 0$: $L_{j+1} \geq L_j$ and $U_{j+1} \leq U_j$.*

(ii) (L_j) and (U_j) both converge to η_* in $\overline{W}_1$.

Lemma C.2. *Given $j \geq 0$, there exists a random time $T_j \geq 0$ such that*

$$L_j \leq \eta_t \leq U_j \quad \textit{for all } t \geq T_j \ , \quad \textit{almost surely.}$$

From Lemma C.1-(ii), let $\epsilon > 0$ and take $j \geq 0$ large enough such that

$$\max\{\overline{W}_1(L_j, \eta_*), \overline{W}_1(U_j, \eta_*)\} < \epsilon \ .$$

Then, it follows from Lemma C.2 followed by a triangular inequality that

$$\overline{W}_1(\eta_t, L_j) \leq \overline{W}_1(L_j, U_j) \leq \overline{W}_1(L_j, \eta_*) + \overline{W}_1(U_j, \eta_*) < 2\epsilon \, ,$$

for all $t \geq T_j$, almost surely. Finally,

$$\overline{W}_1(\eta_t, \eta_*) \leq \overline{W}_1(\eta_t, L_j) + \overline{W}_1(L_j, U_j) + \overline{W}_1(U_j, \eta_*) < 5\epsilon \, ,$$

which implies Theorem 4.1. We still need to prove Lemmas C.1-C.2.

C.1 Proof of Lemma C.1

(i). Let us show by induction that $U_{j+1} \leq U_j$. First, the base case $U_1^{(x,a)} \leq U_0^{(x,a)} = \delta_{z_K}$ is true because $U_1^{(x,a)}$ is supported in $\{z_1, \ldots, z_K\}$. Now, let us assume that $U_{j+1} \leq U_j$ for some $j \geq 0$. We recall from Proposition 5 in (Rowland et al., 2018) that $\Pi_{\mathcal{C}}$ is a monotone map for element-wise stochastic dominance. Plus, it is easy to see that both $\mathcal{T}^\pi$ and $\mathcal{T}$ are monotone too. Then, by monotonicity of $\Pi_{\mathcal{C}}\mathcal{T}$, it holds that $\Pi_{\mathcal{C}}\mathcal{T}U_{j+1} \leq \Pi_{\mathcal{C}}\mathcal{T}U_j$ and hence,

$$U_{j+2}^{(x,a)} = \frac{1}{2} U_{j+1}^{(x,a)} + \frac{1}{2} (\Pi_{\mathcal{C}} \mathcal{T} U_{j+1})^{(x,a)} \leq \frac{1}{2} U_j^{(x,a)} + \frac{1}{2} (\Pi_{\mathcal{C}} \mathcal{T} U_j)^{(x,a)} = U_{j+1}^{(x,a)} \, ,$$

which proves the induction step. Symmetrically, one can show that $L_{j+1} \geq L_j$.

(ii). Similarly to Lemma 7 in (Rowland et al., 2018), we formulate the following general result.

Lemma C.3. *Let $(\nu_k)_{k=0}^{\infty}$ be a sequence of probability distributions over $\{z_1, \ldots, z_K\}$ such that $\nu_{k+1} \leq \nu_k$ for all $k \geq 0$. Then, there exists a limit probability distribution $\nu_{\lim}$ over $\{z_1, \ldots, z_K\}$ such that $\nu_k \to \nu_{\lim}$ in W_1.*

Proof. Let us denote F_k the CDF of ν_k and F_k^{-1} its quantile function valued in $\{z_1, \ldots, z_K\}$. The assumption that ν_k stochastically dominates ν_{k+1} reformulates as $F_{k+1}^{-1}(\tau) \leq F_k^{-1}(\tau)$ for all $0 < \tau \leq 1$. Hence for each $\tau \in (0,1]$, the sequence $F_k^{-1}(\tau)$ is non-increasing and lower bounded by z_1. Therefore, this sequence converges: $F_k^{-1}(\tau) \to H_{\lim}(\tau)$. As $F_k^{-1}(\tau)$ can only take discrete values, there exists $N(\tau)$ such that $\forall k \geq N(\tau)$, $F_k^{-1}(\tau) = H_{\lim}(\tau)$. The limit function $H_{\lim}$ is thus non-decreasing, valued in $\{z_1, \ldots, z_K\}$

and right-continuous with left limits. It is therefore the quantile function of a probability distribution $\nu_{\lim}$ supported over $\{z_1, \dots, z_K\}$. Let us now show that $W_1(\nu_k, \nu_{\lim}) \to 0$. Denoting $F_{\lim}$ the CDF of $\nu_{\lim}$ and $p_0 = 0, p_1 = F_{\lim}(z_1), \dots, p_K = F_{\lim}(z_K) = 1$, we have:

$$W_1(\nu_k, \nu_{\lim}) = \int_{\tau=0}^{1} |F_k^{-1}(\tau) - H_{\lim}(\tau)| d\tau = \sum_{1 \leq k \leq K : p_k \neq p_{k-1}} \int_{\tau=p_{k-1}}^{p_k} |F_k^{-1}(\tau) - z_k| d\tau \,. \tag{4}$$

Let $\Delta_p = \min_{1 \leq k \leq K : p_k \neq p_{k-1}} (p_k - p_{k-1})$, $\Delta_z = z_K - z_1$ and $0 < \epsilon < \Delta_p/2$. Then for all $k \geq \max_{1 \leq k \leq K : p_k \neq p_{k-1}} \max\{N(p_{k-1} + \epsilon), N(p_k - \epsilon)\}$, $F_k^{-1}(\tau)$ is constant equal to z_k for any $\tau \in [p_{k-1} + \epsilon, p_k - \epsilon]$ and we have:

$$\begin{aligned} W_1(\nu_k, \nu_{\lim}) &= \sum_{1 \leq k \leq K : p_k \neq p_{k-1}} \int_{\tau=p_{k-1}}^{p_k} |F_k^{-1}(\tau) - z_k| d\tau \\ &= \sum_{1 \leq k \leq K : p_k \neq p_{k-1}} \int_{\tau=p_{k-1}}^{p_{k-1}+\epsilon} \underbrace{|F_k^{-1}(\tau) - z_k|}_{\leq \Delta_z} d\tau + \underbrace{\int_{\tau=p_{k-1}+\epsilon}^{p_k-\epsilon} |F_k^{-1}(\tau) - z_k| d\tau}_{=0} + \int_{\tau=p_k-\epsilon}^{p_k} \underbrace{|F_k^{-1}(\tau) - z_k|}_{\leq \Delta_z} d\tau \\ &\leq 2K\Delta_z \epsilon \,, \end{aligned}$$

which proves the result.

□

By applying Lemma C.3 to the sequence $(U_k^{(x,a)})_{k \geq 0}$ for each pair (x, a), we deduce the convergence of (U_k) towards some limit distribution function $U_{\lim}$ in $\overline{W}_1$. Finally, by continuity of $\Pi_{\mathcal{C}} \mathcal{T}$ for the metric $\overline{W}_1$, this limit must verify:

$$U_{\lim} = \frac{1}{2} U_{\lim} + \frac{1}{2} \Pi_{\mathcal{C}} \mathcal{T} U_{\lim} \quad \Longrightarrow \quad U_{\lim} = \Pi_{\mathcal{C}} \mathcal{T} U_{\lim} \,,$$

from which we deduce that $U_{\lim} = \eta_*$ by unicity. The proof that (L_j) converges to η_* in $\overline{W}_1$ is analogous.

C.2 Proof of Lemma C.2

Let us prove Lemma C.2 by induction. The base case $j = 0$ is true because for all $t \geq T_0 = 0$, $\eta_t^{(x,a)}$ is supported on $\{z_1, \dots, z_K\}$ and so:

$$L_0^{(x,a)} = \delta_{z_1} \leq \eta_t^{(x,a)} \leq \delta_{z_K} = U_0^{(x,a)} \,.$$

Then, let us assume that the result is true for some $j \geq 0$, i.e. there exists a random time T_j such that $L_j \leq \eta_t \leq U_j$ for all $t \geq T_j$ almost surely. Now, let us show that there exists a random time T_{j+1} such that $\eta_t \leq U_{j+1}$ for all $t \geq T_{j+1}$ almost surely (the proof for $\eta_t \geq L_{j+1}$ is analogous). For each state-action pair (x, a), we define $H_{T_j}^{(x,a)} = U_j^{(x,a)}$ and $W_{T_j}^{(x,a)}$ the zero measure i.e. $W_{T_j}^{(x,a)}(A) = 0$ for all Borel sets $A \subseteq \mathbb{R}$. Then, for $t \geq T_j$,

$$\begin{aligned} H_{t+1}^{(x,a)} &= (1 - \alpha_t(x,a)) H_t^{(x,a)} + \alpha_t(x,a) (\Pi_{\mathcal{C}} \mathcal{T} U_j)^{(x,a)} \\ \text{and} \qquad W_{t+1}^{(x,a)} &= (1 - \alpha_t(x,a)) W_t^{(x,a)} + \alpha_t(x,a) [\Pi_{\mathcal{C}} (\delta_{r(x,a,x') + \gamma \max_{a'} Q_t(x',a')}) - (\Pi_{\mathcal{C}} \mathcal{T} \eta_t)^{(x,a)}] \,, \end{aligned} \tag{5}$$

where $x' \sim P(\cdot|x,a)$, $Q_t(x', a') = \sum_{k=1}^{K} p_{t,k}(x', a') z_k$. Moreover, if $(x, a) = (x_t, a_t)$, then $x' = x_{t+1}$ and $r(x_t, a_t, x_{t+1}) = r_t$. A consequence of Eq. (5) is that $W_t^{(x,a)}$ is a signed measure on $\mathbb{R}$ with total measure equal to zero: $W_t^{(x,a)}(\mathbb{R}) = 0$ for all $t \geq T_j$. Let us now prove by (another) induction that $\eta_t^{(x,a)} \leq H_t^{(x,a)} + W_t^{(x,a)}$ for all $t \geq T_j, (x, a) \in \mathcal{X} \times \mathcal{A}$ almost surely. For $t = T_j$, it is true by assumption that $\eta_{T_j} \leq U_j$ almost surely. Then, suppose that $\eta_t \leq H_t + W_t$ almost surely for some $t \geq T_j$. Recalling that $\alpha_t(x, a) = 0$ for all

$(x,a) \neq (x_t, a_t)$, it holds that:

$$\begin{aligned}
\eta_{t+1}^{(x,a)} &= (1-\alpha_t(x,a))\eta_t^{(x,a)} + \alpha_t(x,a)\Pi_{\mathcal{C}}(\delta_{r(x,a,x')+\gamma \max_{a'} Q_t(x',a')}) \\
&= (1-\alpha_t(x,a))\eta_t^{(x,a)} + \alpha_t(x,a)(\Pi_{\mathcal{C}}\mathcal{T}\eta_t)^{(x,a)} + \alpha_t(x,a)[\Pi_{\mathcal{C}}(\delta_{r(x,a,x')+\gamma \max_{a'} Q_t(x',a')}) - (\Pi_{\mathcal{C}}\mathcal{T}\eta_t)^{(x,a)}] \\
&\leq (1-\alpha_t(x,a))(H_t+W_t)^{(x,a)} + \alpha_t(x,a)(\Pi_{\mathcal{C}}\mathcal{T}U_j)^{(x,a)} + \alpha_t(x,a)[\Pi_{\mathcal{C}}(\delta_{r(x,a,x')+\gamma \max_{a'} Q_t(x',a')}) - (\Pi_{\mathcal{C}}\mathcal{T}\eta_t)^{(x,a)}] \\
&= H_{t+1}^{(x,a)} + W_{t+1}^{(x,a)} \ , \quad (6)
\end{aligned}$$

where the inequality comes from the assumptions $\eta_t \leq H_t + W_t$ and $\eta_t \leq U_j$ combined with the monotonicity of $\Pi_{\mathcal{C}}\mathcal{T}$. Now, observe that H_t can be explicitly expressed as follows:

$$H_t^{(x,a)} = \left(\prod_{t'=T_j}^{t-1}(1-\alpha_{t'}(x,a))\right)U_j + \left(1 - \prod_{t'=T_j}^{t-1}(1-\alpha_{t'}(x,a))\right)(\Pi_{\mathcal{C}}\mathcal{T}U_j)^{(x,a)} \ .$$

Since by Assumption 4.1, it holds that $\sum_{t'=0}^{\infty}\alpha_{t'}(x,a) = \infty$ for all (x,a) almost surely, we deduce that there exists a random time $\widetilde{T}_{j+1} \geq T_j$ such that $\prod_{t'=T_j}^{t-1}(1-\alpha_{t'}(x,a)) \leq \frac{1}{4}$ for all (x,a) and for all $t \geq \widetilde{T}_{j+1}$ almost surely. Then, because Lemma C.1-(i) implies that $\Pi_{\mathcal{C}}\mathcal{T}U_j \leq U_j$, we have for all $t \geq \widetilde{T}_{j+1}$:

$$\begin{aligned}
\eta_t \leq H_t + W_t \leq \frac{1}{4}U_j + \frac{3}{4}\Pi_{\mathcal{C}}\mathcal{T}U_j + W_t &= \frac{1}{2}U_j + \frac{1}{2}\Pi_{\mathcal{C}}\mathcal{T}U_j + W_t - \frac{1}{4}(U_j - \Pi_{\mathcal{C}}\mathcal{T}U_j) \\
&= U_{j+1} + W_t - \frac{1}{4}(U_j - \Pi_{\mathcal{C}}\mathcal{T}U_j) \ . \quad (7)
\end{aligned}$$

We point out that the random noise term appearing in the definition of $W_{t+1}^{(x,a)}$ has zero mean: for all $1 \leq k \leq K$,

$$\mathbb{E}_{x' \sim P(\cdot|x,a)}\left[\Pi_{\mathcal{C}}(\delta_{r(x,a,x')+\gamma \max_{a'} Q_t(x',a')}) - (\Pi_{\mathcal{C}}\mathcal{T}\eta_t)^{(x,a)}\right]((-\infty, z_k]) = 0 \ . \quad (8)$$

Eq. (8) implies via a classic stochastic approximation argument under Assumption 4.1 that $W_t^{(x,a)}((-\infty, z_k]) \to 0$ almost surely, for all (x,a) and $k \in \{1,\dots,K\}$. Finally, we take $T_{j+1} \geq \widetilde{T}_{j+1}$ large enough so that $|W_t^{(x,a)}((-\infty, z_k])| \leq \frac{\Delta}{4}$ for all $t \geq T_{j+1}$ and all (x,a), almost surely, where Δ is given by:

$$\Delta = \inf\left\{\left|\left[U_j^{(x,a)} - (\Pi_{\mathcal{C}}\mathcal{T}U_j)^{(x,a)}\right]((-\infty, z_k])\right| \neq 0 : x \in \mathcal{X}, a \in \mathcal{A}, 1 \leq k \leq K\right\} \ ,$$

which concludes the proof by using Eq. (7). Indeed, if $U_j^{(x,a)}((-\infty, z_k]) = (\Pi_{\mathcal{C}}\mathcal{T}U_j)^{(x,a)}((-\infty, z_k])$, then $U_j^{(x,a)}((-\infty, z_k]) = U_{j+1}^{(x,a)}((-\infty, z_k])$.