COMAL: A Convergent Meta-Algorithm for Aligning LLMs with General Preferences

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Abstract

Many alignment methods, including reinforcement learning from human feedback (RLHF), rely on the Bradley-Terry reward assumption, which is insufficient to capture the full range of general human preferences. To achieve robust alignment with general preferences, we model the alignment problem as a two-player zero-sum game, where the Nash equilibrium policy guarantees a 50% win rate against any competing policy. However, previous algorithms for finding the Nash policy either diverge or converge to a Nash policy in a modified game, even in a simple synthetic setting, thereby failing to maintain the 50% win rate guarantee against all other policies. We propose a meta-algorithm, **Co**nvergent **M**eta **Al**ignment Algorithm (COMAL), for language model alignment with general preferences, inspired by convergent algorithms in game theory. Theoretically, we prove that our meta-algorithm converges to an exact Nash policy in the last iterate. Additionally, our meta-algorithm is simple and can be integrated with many existing methods designed for RLHF and preference optimization with minimal changes. Experimental results demonstrate the effectiveness of the proposed framework when combined with existing preference policy optimization methods.

1 Introduction

Large Language Models (LLMs) [Brown et al., 2020, OpenAI, 2023, Dubey et al., 2024] have fundamentally transformed the fields of natural language processing and artificial intelligence. They excel in tasks ranging from text generation and translation to complex question answering and interactive dialogue systems. As these models become more integrated into daily life, a key challenge is ensuring they achieve high levels of alignment with human values and preferences.

One of the most widely adopted approaches to addressing this challenge is Reinforcement Learning from Human Feedback (RLHF) [Christiano et al., 2017, Ouyang et al., 2022]. This framework consists of two steps: first, learning a reward model from a dataset containing human preferences, and second, optimizing the LLM using the proximal policy optimization (PPO) algorithm [Schulman et al., 2017]. Recently, Rafailov et al. [2024] observed that the first step can be bypassed, proposing the direct preference optimization (DPO) algorithm, directly optimizing the LLM from the dataset.

However, the aforementioned approaches crucially rely on the assumption that human preferences can be expressed using the Bradley-Terry (BT) model [Bradley and Terry, 1952]. Unfortunately, the BT model is too restrictive to capture the richness and complexity of human preferences. Specifically, the BT model can only induce *transitive* preferences – i.e., if more people favor A over B, and B over C, then more people must favor A over C. Such transitivity may not hold in the presence of

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diverse populations and is also incompatible with evidence from human decision-making [May, 1954, Tversky, 1969].

To overcome this limitation, recent research has begun to explore alignment under general preferences. Munos et al. [2024] formulate this alignment problem as a symmetric two-player zero-sum game, where both players' strategies are LLMs, and their payoffs are determined by the win rate against the opponent's LLM according to the preference model. The objective is to identify a Nash equilibrium policy that guarantees at least a 50% win rate against any other policy [Munos et al., 2024, Swamy et al., 2024, Azar et al., 2024, Calandriello et al., 2024], a property we refer to as *robust alignment*. However, all the proposed algorithms either diverge or converge to the Nash policy of a modified game, thereby failing to maintain the 50% win rate guarantee against all other policies.

Our Contribution. We introduce a novel meta-algorithm, **Co**nvergent **M**eta **Al**ignment **Al**gorithm (COMAL), inspired by the conceptual prox-method, a convergent algorithm for solving two-player zero-sum games [Nemirovski, 2004]. Our first observation is that many existing algorithms, including PPO [Schulman et al., 2017], DPO [Rafailov et al., 2024], IPO [Azar et al., 2024], SPPO [Wu et al., 2024], REBEL [Gao et al., 2024], DRO [Richemond et al., 2024], INPO [Zhang et al., 2024], etc., can be interpreted as implementations of the Prox operator [Nemirovski, 2004]. COMAL employs the Prox operator as its fundamental building block and provably *converges* to the Nash equilibrium policy in the *last iterate*, assuming the Prox operator can be computed exactly, thus achieving *robust alignment*. This approach allows us to leverage many existing algorithms in a black-box manner. While several algorithms, e.g., IPO, SPPO, etc., in the literature demonstrate average-iterate convergence to the Nash equilibrium policy, they all diverge in the last iterate. Unfortunately, iterate averaging can be cumbersome, particularly when deep-learning components are involved, as it may not be feasible to average the outputs of LLMs.[1] For the more desirable last-iterate convergence [Munos et al., 2024, Zhang et al., 2024], existing algorithms only guarantee convergence to a KL-regularized Nash equilibrium, which does not have the robust alignment property. Compared to these algorithms, COMAL is the first to provably converge to a Nash equilibrium policy in the last iterate, thus guaranteeing robust alignment.

In addition to our theoretical analysis, we validate the effectiveness of COMAL through both synthetic and LLM-based experiments.

Synthetic experiments. We construct a 3×3 two-player zero-sum preference game and compare COMAL with a wide range of algorithms proposed in the literature. The result clearly shows that COMAL is the only algorithm that converges to the Nash equilibrium of the game in the last iterate.

LLM-based experiments. Furthermore, we evaluate the performance of COMAL against existing preference optimization algorithms under a practical setting, where a pre-trained LLM, Qwen2-1.5B [Yang et al., 2024] is fine-tuned using different algorithms on the UltraFeedback [Cui et al., 2023] dataset, which is commonly used for alignment fine-tuning of LLMs. We run iterative algorithms up to 42 iterations and compare both the best and the last checkpoints. Our experimental results demonstrate the advantages of COMAL: it consistently achieves a win rate strictly above 50% compared to baseline algorithms, including DPO [Rafailov et al., 2024] and iterative algorithms such as iterative IPO [Azar et al., 2024] and INPO [Zhang et al., 2024].[2]

2 Background

We use $\Delta(\mathcal{Z})$ to denote a distribution over a set $\mathcal{Z}$. We denote $x \in \mathcal{X}$ as an instruction where $\mathcal{X}$ is the instruction set. We assume a fixed distribution $\rho \in \Delta(\mathcal{X})$ over the instruction set. We denote $\mathcal{Y}$ as the response set and $y \in \mathcal{Y}$ as one response. Given any instruction $x \in \mathcal{X}$, an LLM policy π specifies the output distribution $\pi(\cdot \mid x) \in \Delta(\mathcal{Y})$. For distributions $p, q \in \Delta(\mathcal{Z})$, the Kullback-Leibler (KL) divergence is defined as $\mathrm{KL}(p||q) := \sum_{z \in \mathcal{Z}} p(z) \log \frac{p(z)}{q(z)}$. The sigmoid function is $\sigma(x) := \frac{e^x}{e^x+1}$. We use $\mathrm{supp}(p)$ to denote the support of a distribution p.

[1] Obtaining the average output from multiple LLMs requires serving all LLMs simultaneously, which can be highly compute-inefficient and, to our knowledge, has not been implemented.

[2] Our codebase and trained models are available at https://github.com/yale-nlp/COMAL.

Preference Models In this paper, we focus on general preference models.

Definition 1 (General Preference Model)**.** *A general preference model* $\mathbb{P} : \mathcal{X} \times \mathcal{Y} \times \mathcal{Y} \to [0, 1]$ *satisfies* $\mathbb{P}(y_1 \succ y_2 \mid x) = 1 - \mathbb{P}(y_2 \succ y_1 \mid x)$*. When we query* $\mathbb{P}$ *with* (x, y_1, y_2)*, it outputs* 1 *with probability* $\mathbb{P}(y_1 \succ y_2 \mid x)$ *meaning* y_1 *is preferred over* y_2*, and it outputs* 0 *otherwise.*

We define $\mathbb{P}(\pi_1 \succ \pi_2) := \mathbb{E}_{x\sim\rho}[\mathbb{E}_{y_1\sim\pi_1, y_2\sim\pi_2}[\mathbb{P}(y_1 \succ y_2 \mid x)]]$ as the *win rate* of π_1 over π_2 under preference model $\mathbb{P}$. We denote the preference distribution $\lambda_{\mathbb{P}}(y, y')$ as a binary distribution:

$$\lambda_{\mathbb{P}}(y, y') = \begin{cases} (y, y') \text{ with probability } \mathbb{P}[y \succ y'] \\ (y', y) \text{ with probability } 1 - \mathbb{P}[y \succ y'] \end{cases} \tag{1}$$

A special case of the general preference model is the Bradley-Terry (BT) model, which assumes a reward function parameterizes the preference.

Definition 2 (Bradley-Terry Model)**.** *A preference model* $\mathbb{P}$ *satisfies the* Bradley-Terry (BT) assumption *if there exists a reward function* $r^* : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ *such that*

$$\mathbb{P}(y_1 \succ y_2 \mid x) = \frac{\exp(r^*(x, y_1))}{\exp(r^*(x, y_1)) + \exp(r^*(x, y_2))} = \sigma(r^*(x, y_1) - r^*(x, y_2)).$$

2.1 Alignment under the Bradley-Terry Model Assumption

RLHF The canonical formulation of Reinforcement Learning from Human Feedback (RLHF) is to first learn a reward function r under the BT model and then find the optimal KL regularized policy π^* with respect to the learned reward function r:

$$\pi^* := \arg\max_{\pi} \mathbb{E}_{x\sim\rho, y\sim\pi(\cdot|x)}\left[r(x, y) - \eta^{-1}\,\mathrm{KL}(\pi(\cdot \mid x)||\pi_{\mathrm{ref}}(\cdot \mid x))\right], \tag{2}$$

where $\eta^{-1} > 0$ controls the regularization, and π_{ref} is the initial reference model, usually the policy π_{sft} obtained from pre-training and supervised fine-tuning.

DPO Rafailov et al. [2024] observe that the regularized optimization problem (2) has a closed-form solution: for any x and y,

$$\pi^*(y \mid x) = \frac{\pi_{\mathrm{ref}}(y \mid x)\exp(\eta r(x, y))}{Z_x}, \tag{3}$$

where $Z_x = \mathbb{E}_{y\sim\pi_{\mathrm{ref}}(\cdot|x)}[\exp(\frac{1}{\eta}r(y, x))]$ is the normalization constant known as the partition function. In (3), we see that π^* implicitly parameterizes the reward function r. Rafailov et al. [2024] propose direct preference optimization (DPO) to learn the optimal policy using the maximum likelihood objective directly:

$$\ell_{\mathrm{DPO}}(\pi; \pi_{\mathrm{ref}}) = -\mathbb{E}_{(x,y_w,y_l)\sim\mathcal{D}}\left[\log\sigma\left(\eta^{-1}\log\frac{\pi(y_w \mid x)}{\pi_{\mathrm{ref}}(y_w \mid x)} - \eta^{-1}\log\frac{\pi(y_l \mid x)}{\pi_{\mathrm{ref}}(y_l \mid x)}\right)\right],$$

where $\mathcal{D}$ is a data set containing win-loss pair of responses $\{y_w, y_l\}$ given prompt x.

2.2 Robust Alignment with General Preference Models

The BT model assumption is insufficient to capture the full range of general human preferences [Munos et al., 2024, Swamy et al., 2024]. To achieve robust alignment with general preferences, we model the policy optimization problem as a two-player zero-sum game with the objective function as follows:[3]

$$J(\pi_1, \pi_2) := \mathbb{P}(\pi_1 \succ \pi_2) - \frac{1}{2} = \mathbb{E}_{x\sim\rho}[\mathbb{E}_{y_1\sim\pi_1, y_2\sim\pi_2}[\mathbb{P}(y_1 \succ y_2 \mid x)]] - \frac{1}{2}. \tag{4}$$

In this game, the max-player controls π_1 and tries to maximize $J(\pi_1, \pi_2)$ while the min-player controls π_2 and tries to minimize $J(\pi_1, \pi_2)$. We focus only on policies with $\Pi := \{\pi : \mathrm{supp}(\pi) \subseteq \mathrm{supp}(\pi_{\mathrm{sft}})\}$ in the support of the initial SFT policy. A Nash equilibrium policy $(\pi_1^\star, \pi_2^\star)$ satisfies

$$\pi_1^\star, \pi_2^\star \in \operatorname*{argmax}_{\pi_1\in\Pi}\operatorname*{argmin}_{\pi_2\in\Pi} J(\pi_1, \pi_2), \quad J(\pi_1, \pi_2^\star) \le J(\pi_1^\star, \pi_2^\star) \le J(\pi_1^\star, \pi_2), \forall \pi_1, \pi_2 \in \Pi.$$

[3] We introduce the constant $\frac{1}{2}$ only to ensure the game is zero-sum and it has no effect on its Nash equilibria.

Table 1: Property comparison of different preference optimization algorithms. The algorithms are compared based on whether they work for ***general preferences***, whether they exhibit ***last-iterate convergence*** in two-player zero-sum games, and whether the output policy achieves ***robust alignment***, i.e., a 50% win rate against other policies. ✓̸ : convergence only in the modified KL-regularized game $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$ (5) but not in $J(\pi_1, \pi_2)$ (4).

Algorithm	General Preference	Last-Iterate Convergence	Robust Alignment
DPO [Rafailov et al., 2024]	✗	✗	✗
IPO [Azar et al., 2024]	✓	✗	✗
SPPO [Wu et al., 2024]	✓	✗	✗
Nash-MD [Munos et al., 2024]	✓	✓̸	✗
INPO [Zhang et al., 2024]	✓	✓̸	✗
COMAL (Algorithm 1)	✓	✓	✓

Since $J(\pi_1, \pi_2)$ is symmetric, the game has a symmetric Nash equilibrium $(\pi^\star, \pi^\star)$. Moreover, the Nash equilibrium policy $\pi^\star$ guarantees that for any other policy π, its win rate is at least $\mathbb{P}(\pi^\star \succ \pi) \geq \mathbb{P}(\pi^\star \succ \pi^\star) = 50\%$. We call this property *robust alignment*. Our goal is to find a policy with robust alignment.

Existing online iterative preference optimization methods designed for or applicable to the original game, including iterative IPO [Azar et al., 2024] and SPPO [Wu et al., 2024], are based on Multiplicative Weights Update (MWU, definition in Section 3.2), and thus *diverge in the last iterate* as we show in Section 4.[4] There is also a line of works including Nash-MD [Munos et al., 2024, Ye et al., 2024], Online IPO [Calandriello et al., 2024], INPO [Zhang et al., 2024] aim to find the Nash equilibrium of a modified KL-regularized game:

$$J_\tau(\pi_1, \pi_2, \pi_{\text{ref}}) := J(\pi_1, \pi_2) - \tau \mathbb{E}_{x\sim\rho}[\text{KL}(\pi_1(\cdot \mid x)||\pi_{\text{ref}}(\cdot \mid x))] + \tau \mathbb{E}_{x\sim\rho}[\text{KL}(\pi_2(\cdot \mid x)||\pi_{\text{ref}}(\cdot \mid x))]. \quad (5)$$

The additional KL regularization terms in the objective are introduced for training stability. However, the Nash equilibrium of the modified game no longer achieves robust alignment, i.e., it has a win rate of at least 50% against any competing policy. We present comparison of these algorithms in Table 1.

Moreover, most existing theoretical convergence guarantees only hold for the average iterate, i.e., the uniform mixture of training iterates, which is not used in practice. We focus on designing algorithms with provable last-iterate convergence to Nash equilibrium, which aligns with practice and is more space-efficient [Munos et al., 2024].

As we show in the next section, our meta-algorithm COMAL can also be implemented with black-box access to algorithms that solve the regularized game $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$.

3 A Convergent Meta-Algorithm for Alignment

We propose a simple meta-algorithm, **Co**nvergent **M**eta **Al**ignment **Al**gorithm (COMAL, Algorithm 1), for robustly aligning LLMs with general preferences by solving the unregularized game $J(\pi_1, \pi_2)$ (4). In Section 3.1 and 3.2, we present the theoretical foundations of COMAL and analyze its convergence properties. Section 3.3 describes its practical implementation that integrates COMAL with existing preference learning methods.

3.1 COMAL

COMAL (Algorithm 1) is an online iterative algorithm inspired by the classic conceptual prox-method [Nemirovski, 2004] first introduced in the optimization theory community. This method has recently been applied to finding a Nash equilibrium in zero-sum games [Perolat et al., 2021, Abe et al., 2024] and has had notable success in training advanced game AI models [Perolat et al., 2022].

[4]The MWU algorithm only has a weaker average-iterate convergence, i.e., $\frac{1}{T}\sum_{t=1}^{T} \pi^t$ converges.

Algorithm 1: **Co**nvergent **M**eta **Al**ignment **Al**gorithm (COMAL) for solving $J(\pi_1, \pi_2)$ (4)

Input: Initial policy π_{sft}, preference oracle $\mathbb{P}$, regularization $\tau > 0$, number of iterations $T \geq 1$
Output: Optimized policy π^T
Initialize $\pi^1, \pi_{\text{ref}} \leftarrow \pi_{\text{sft}}$
for $t = 1, 2, \ldots, T-1$ **do**
  $\pi^{t+1} \leftarrow \text{argmax}_{\pi_1} \min_{\pi_2} J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$ using Algorithm 2 (discussed in Section 3.2)
  $\pi_{\text{ref}} \leftarrow \pi^{t+1}$
return π^T

Update Rule of COMAL In each iteration t, COMAL uses a regularized game solver (Algorithm 2) to update the next-iteration policy π^{t+1} as the Nash equilibrium policy of a regularized game $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$ using the current policy as reference $\pi_{\text{ref}} = \pi^t$. We defer further discussion of Algorithm 2 to Section 3.2 for clarity. The rationale behind COMAL is simple: update the reference policy when no further progress can be made, which occurs when the algorithm reaches the Nash equilibrium of the regularized game. Denote $\pi^\star$ a Nash equilibrium of the original game. We show that KL divergence to $\pi^\star$ is monotonically decreasing: $\text{KL}(\pi^\star || \pi^{t+1}) \leq \text{KL}(\pi^\star || \pi^t)$. Since π^{t+1} is closer to the Nash equilibrium than π^t, COMAL updates the reference policy from π^t to π^{t+1} for further optimization. We also note that in COMAL, $\text{KL}(\pi^\star || \pi^{t+1}) \leq \text{KL}(\pi^\star || \pi^t)$ holds for any $\tau > 0$, allowing us to choose the regularization parameter $\tau_t > 0$ adaptively during the training process, without requiring it to decrease over time.

Implementation of COMAL Each iteration of COMAL requires solving a zero-sum game with additional KL regularization $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$. We will show momentarily that many existing policy optimization methods for alignment can be applied to the KL regularized game and have exponentially fast convergence. We also present one practical implementation of COMAL integrated with INPO [Zhang et al., 2024] as the regularized game solver in Algorithm 4.

Last-Iterate Convergence We prove that the meta-algorithm COMAL achieves last-iterate convergence to a Nash equilibrium, thereby ensuring robust alignment, which, to our knowledge, is the first result of its kind in the context of LLM alignment. The proof is in Appendix B.

Theorem 1. *We assume that there exists a Nash equilibrium $\pi^\star$ of $J(\pi_1, \pi_2)$ (defined in (4)) such that $\text{supp}(\pi^\star) = \text{supp}(\pi_{\text{sft}})$. In every iteration $t \geq 1$, it holds that $\text{KL}(\pi^\star || \pi^{t+1}) \leq \text{KL}(\pi^\star || \pi^t)$. Moreover, COMAL has last-iterate convergence, i.e., $\lim_{t\to\infty} \pi^t$ exists and is a Nash equilibrium.*

3.2 Solving a Regularized Game

We present Mirror Descent (MD) in Algorithm 2 to compute a Nash equilibrium of the regularized game $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$. MD uses the prox operator as building blocks and we later show how to implement the prox operator using existing policy optimization algorithms. For simplicity, we consider policy $\pi \in \Delta(\mathcal{Y})$ and omit the dependence on the instruction x. All discussions can be extended to the contextual setting in a straightforward way.

Algorithm 2: Regularized game solver for $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$ – $\text{argmax}_{\pi_1} \min_{\pi_2} J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$

Input: Reference policy π_{ref}, preference oracle $\mathbb{P}$, regularization $\tau > 0$, step size $\eta > 0$, number of iterations $K \geq 1$
Output: Regularized Nash equilibrium policy μ_K
Initialize $\mu^1 \leftarrow \pi_{\text{ref}}$
for $k = 1, 2, \ldots, K-1$ **do**
  $g_\tau^k \leftarrow \nabla_\mu(\mathbb{P}(\mu \succ \mu_k) - \tau\,\text{KL}(\mu || \pi_{\text{ref}})) = \mathbb{P}(\cdot \succ \mu_k) - \tau(\log \frac{\mu_k(\cdot)}{\pi_{\text{ref}}(\cdot)} + 1)$ `// Gradient`
  $\mu^{k+1} \leftarrow \text{Prox}(\mu_k, \eta g_\tau^k)$
return μ_K

Mirror Descent and Multiplicative Weights Update Mirror Descent (MD) is a classical family of optimization algorithms [Nemirovskij and Yudin, 1983]. An important member of this family is the

Multiplicative Weights Update (MWU) algorithm [Arora et al., 2012], which is MD with negative entropy regularization. For a maximization problem $\max_\pi f(\pi)$, given an existing policy π^t, MWU computes the update π^{t+1} as follows:

$$\pi^{t+1} := \operatorname*{argmax}_{\pi} \left\langle \nabla f(\pi^t), \pi \right\rangle - \eta^{-1} \cdot \mathrm{KL}(\pi || \pi^t). \tag{6}$$

Note that RLHF in (2) is equivalent to MWU if we interpret $f(\pi)$ as the expected reward under π $\mathbb{E}_{y\sim\pi}[r(y)]$, and the gradient ∇f corresponds directly to r.

We note that the update rule of MWU can be succinctly expressed using the *prox operator* as shown in Algorithm 2.[5] Therefore, our analysis will consider the general case of the Prox Operator.

Prox operator. Fix a 1-strongly convex function $\varphi : \mathcal{Z} \to \mathbb{R}$ over a closed convex set $\mathcal{Z} \subset \mathbb{R}^n$. The *Bregman divergence* induced by φ is

$$D_\varphi(\cdot||\cdot) : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}_{\geq 0},$$
$$D_\varphi(z||z') := \varphi(z) - \varphi(z') - \langle \nabla\varphi(z'), z - z' \rangle.$$

Given a reference point $z \in \mathcal{Z}$ and a vector $g \in \mathbb{R}^n$, the prox operator $\mathrm{Prox}(z, g)$ generalizes the notion of a gradient ascent step from z in the direction of g.

Definition 3 (Prox Operator)**.** *For a strongly convex regularizer φ, the prox operator is defined as*

$$\mathrm{Prox}(z, g) := \operatorname*{argmax}_{z'} \langle g, z' \rangle - D_\varphi(z'||z) = \operatorname*{argmax}_{z'} \langle g + \nabla\varphi(z), z' \rangle - \varphi(z').$$

When $\varphi(z) = \frac{1}{2}\|z\|_2^2$ is the ℓ_2 regularizer, the prox operator $\mathrm{Prox}(z, g) = \Pi_{\mathcal{Z}}[z + g]$ is the exactly the projected gradient ascent step. In this paper, without additional notes, we choose $\varphi = \sum_{i=1}^n z[i] \ln z[i]$ as the negative entropy regularizer and the corresponding Bregman divergence D_φ is the KL divergence. The update rule of MWU in (6) is equivalent to $\pi^{t+1} = \mathrm{Prox}(\pi^t, \eta\nabla f(\pi^t))$

Exponentially Fast Convergence Denote $\pi_\tau^\star$ the Nash equilibrium of the KL regularized game $J_\tau(\pi_1, \pi_2, \pi_{\mathrm{ref}})$, which is τ-strongly monotone. We can apply existing results to show that MWU (Algorithm 2) achieves linear last-iterate convergence rate: the KL divergence to the Nash equilibrium $\pi_\tau^\star$ decreases exponentially fast. The proof is in Appendix C.

Theorem 2. *For step size $0 < \eta \leq \frac{\tau}{\tau^2+0.5}$, Algorithm 2 guarantees for every $k \geq 1$,* $\mathrm{KL}(\pi_\tau^\star||\mu^{k+1}) \leq (1 - \frac{\eta\tau}{2})^k \,\mathrm{KL}(\pi_\tau^\star||\pi_{\mathrm{ref}})$.

3.3 Practical methods for computing the prox operator

We show how to implement COMAL in practical large-scale applications like LLM alignment by computing the prox operator. Specifically, we observe that many existing algorithms designed for RLHF and preference optimization with neural network parameters can be adapted to solve the prox operator $\mathrm{Prox}(\pi, \eta g)$ ($\eta > 0$ is the step size). These algorithms include RL algorithms like PPO [Schulman et al., 2017] and loss-minimization algorithms like, DPO [Rafailov et al., 2024], IPO [Azar et al., 2024], SPPO [Wu et al., 2024], REBEL [Gao et al., 2024], DRO [Richemond et al., 2024], INPO [Zhang et al., 2024]. Each of them may be preferred in certain settings. Due to space limit, we only present IPO and INPO here but defer discussion of other methods to Appendix D.

Our contribution here is not proposing new algorithms but unifying existing diverse preference methods through the perspective of computing the prox operator. This perspective opens the possibility of applying other algorithms from online learning and optimization to robust LLM alignment. We include implementations for two other last-iterate convergent algorithms, the Mirror-Prox algorithm [Nemirovski, 2004] and the Optimistic Multiplicative Weights Update algorithm [Rakhlin and Sridharan, 2013, Syrgkanis et al., 2015], in Appendix E.

IPO for computing Prox **for unregularized preferences** Before we provide the a practical implementation of Algorithm 2, we first show that the IPO loss could be used to solve $\pi_\theta = \mathrm{Prox}(\pi, \eta g_\mu)$ where g is the unregularized win-rate against a reference policy μ such that $g_\mu(y) =$

[5] The prox operator is also called the prox-mapping [Nemirovski, 2004].

$\mathbb{P}[y \succ \mu] := \mathbb{E}_{y' \sim \mu}[\mathbb{P}[y \succ y']]$. Given a dataset of win-lose pairs sampled from μ: $\{y_w, y_l \sim \mu\}$, the (population) IPO loss [Azar et al., 2024] is

$$\ell_{\text{IPO}}(\theta) := \mathbb{E}_{\substack{(y_w, y_l) \sim \mu \\ (y^+, y^-) \sim \lambda_{\mathbb{P}}(y_w, y_l)(1)}} \left[\left(\log \frac{\pi_\theta(y^+)}{\pi_\theta(y^-)} - \log \frac{\pi(y^+)}{\pi(y^-)} - \frac{\eta}{2} \right)^2 \right].$$

Azar et al. [2024] have shown that the minimizer of the $\ell_{\text{IPO}}(\theta, \mu)$ satisfies

$$\pi_\theta(y) \propto \pi(y) \exp\left(-\eta \mathbb{P}[y \succ \mu]\right) \Leftrightarrow \pi_\theta = \text{Prox}(\pi, \eta g_\mu).$$

Thus we can compute the prox operator $\text{Prox}(\pi, \eta g_\mu)$ where $g_\mu = \mathbb{P}(\cdot \succ \mu)$ by minimizing the IPO loss against policy μ.

INPO for computing Prox **for regularized preferences** The Iterative Nash Policy Optimization (INPO) loss [Zhang et al., 2024] is a generalization of the IPO loss to the regularized preference setting. We show that INPO could be used to compute $\text{Prox}(\mu, \eta g_\mu^\tau)$, where $g_\mu^\tau := \nabla_\pi J_\tau(\pi, \mu, \pi_{\text{ref}}) = \mathbb{P}(\cdot \succ \mu) - \tau \log \frac{\mu(\cdot)}{\pi_{\text{ref}}(\cdot)}$ is the gradient of the regularized objective (5). Given a win-loss pair data set $\{y_w, y_l \sim \mu\}$, the INPO loss is

$$\ell_{\text{INPO}}(\pi) := \mathbb{E}_{\substack{(y_w, y_l) \sim \mu \\ (y^+, y^-) \sim \lambda_{\mathbb{P}}(y_w, y_l)(1)}} \left[\left(\log \frac{\pi(y^+)}{\pi(y^-)} - \eta\tau \log \frac{\pi_{\text{ref}}(y^+)}{\pi_{\text{ref}}(y^-)} - (1 - \eta\tau) \log \frac{\mu(y^+)}{\mu(y^-)} - \frac{\eta}{2} \right)^2 \right].$$

It has been proved that the minimizer of the INPO loss is $\text{Prox}(\mu, \eta g_\mu^\tau)$ [Zhang et al., 2024]. Thus we can use INPO in Algorithm 2 as a regularized game solver, as we show in Algorithm 3.

Algorithm 3: INPO [Zhang et al., 2024] for solving $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$

Input: Reference policy π_{ref}, regularization $\tau > 0$, step size $\eta > 0$, number of rounds $K \geq 1$, preference oracle $\mathbb{P}$.
Output: Approximate regularized Nash equilibrium policy μ^K
Initialize $\mu^1 \leftarrow \pi_{\text{ref}}$
for $k = 1, 2, \ldots, K-1$ **do**
  Generate response pairs $\{y_1^{(i)}, y_2^{(i)}\}_{i=1}^n$ where $y_1^{(i)}, y_2^{(i)} \sim \mu^k$
  Query preference oracle $\mathbb{P}$ to get preference data $\mathcal{D}_k = \{y_w^{(i)}, y_l^{(i)}\}_{i=1}^n$
  Compute $\mu^{k+1} = \text{argmin}_{\pi \in \Pi} \mathbb{E}_{(y_w, y_l) \sim \mathcal{D}_k} \ell_{\text{INPO}}(\pi)$ where
  $\ell_{\text{INPO}}(\pi) := \mathbb{E}_{\substack{(y^+, y^-) \sim \lambda_{\mathbb{P}}(y_w, y_l) \\ (1)}} \left[\left(\log \frac{\pi(y^+)}{\pi(y^-)} - \eta\tau \log \frac{\pi_{\text{ref}}(y^+)}{\pi_{\text{ref}}(y^-)} - (1 - \eta\tau) \log \frac{\mu^t(y^+)}{\mu^t(y^-)} - \frac{\eta}{2} \right)^2 \right]$
return μ^K

Practical Implementation of COMAL We present an implementation of COMAL in Algorithm 4 using the INPO [Zhang et al., 2024] as a subgame solver. We remark that COMAL can also be implemented using PPO or many other preference learning algorithms, as we show in Section 3.3 and Appendix D. Given the implementation of these existing methods, our meta-algorithm requires minimal change but achieves last-iterate convergence to a Nash equilibrium.

Algorithm 4: Practical Implementation of COMAL integrated with INPO (Algorithm 3)

Input: Initial policy π_{sft}, regularization $\{\tau_t > 0\}$, step size $\{\eta_t > 0\}$, number of iterations $T \geq 1$, number of inner optimization steps $\{K_t \geq 1\}$, preference oracle $\mathbb{P}$.
Output: Optimized policy π^T
Initialize $\pi^1, \pi_{\text{ref}} \leftarrow \pi_{\text{sft}}$
for $t = 1, 2, \ldots, T-1$ **do**
  $\pi^{t+1} \leftarrow \text{INPO}(\pi_{\text{ref}}, \tau_t, \eta_t, K_t, \mathbb{P})$ defined in Algorithm 3
  $\pi_{\text{ref}} \leftarrow \pi^{t+1}$
return π^T

4 Synthetic Experiments

We conduct experiments on a simple bandit problem with $\mathcal{Y} = \{y_a, y_b, y_c\}$ and non-BT preference model over $\mathcal{Y}$. Specifically, we set $\mathbb{P}[y_b \succ y_a] = \mathbb{P}[y_c \succ y_b] = 0.9$ and $\mathbb{P}[y_a \succ y_c] = 0.8$. Observe that the preference is intransitive and exhibits a preference cycle $y_c \succ y_b \succ y_a \succ y_c$. The setup for the synthetic experiment is included in Appendix F.

Experiments using noiseless gradient We present numerical results of mirror-descent (MD) algorithms (equivalent to MWU) and COMAL (Algorithm 1) in Figure 1. We can see that the MD algorithm diverges from the unique Nash equilibrium and suffers a large equilibrium gap, while COMAL achieves fast last-iterate convergence to the Nash equilibrium, aligned with our theoretical results (Theorem 1).

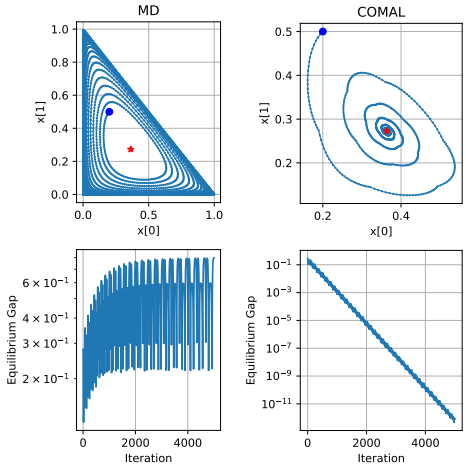

Figure 1: Dyanmics on a simple 3-dimensional preference game. The unique Nash equilibrium is $[4/11, 3/11, 4/11]$ represented as red star. We initialize all algorithms at the blue dot point $[0.2, 0.5, 0.3]$.

Experiements using preference samples Since the popular iterative DPO algorithm does not contain a gradient step, we also conduct experiments with only Oracle query access to the preference model. We compare the performance of various algorithms, including iterative DPO, iterative IPO, SPPO, and INPO and present results in Figure 2. The sample-only setting is also more aligned with what happens in practice. We use a sufficient number of samples in each iteration for every algorithm. As a result, the COMAL performs the same as in the noiseless gradient setting, while the iterative IPO algorithm becomes equivalent to the MD algorithm. We note the following:

Iterative DPO: We observe that iterative DPO diverges and cycles between extreme policies (e.g., outputting y_a with probability close to 1). This is aligned with [Azar et al., 2024], where they found DPO will converge to the deterministic policy regardless of the regularization parameter in extreme preference settings. The cycling behavior of iterative DPO may be explained as follows: in each iteration, DPO converges to a nearly deterministic policy output y; then the new preference data shows that $y' \neq y$ is more preferred; finally, iterative DPO cycles over $\mathcal{Y}$ since the preference itself exhibits a cycle and there is no clear winner.

Iterative IPO [Azar et al., 2024, Calandriello et al., 2024]: The IPO loss is a variant of the DPO loss, but it does not rely on the BT model assumption and works for a general preference model. However, as we have discussed before, (exactly) minimizing the IPO loss is equivalent to performing one MD step, and thus, iterative IPO is equivalent to MD up to sampling error. As a result, we observe that iterative IPO also exhibits cycling behavior.

SPPO [Wu et al., 2024]: The SPPO algorithm (see Appendix D) is not exactly the same as MWU since SPPO assumes the partition function is always $Z = \log \frac{\eta}{2}$ which may not be the case. We observe that SPPO exhibits very similar cycling behavior as MD. We conclude that SPPO approximates MD very well in this instance and exhibits similar behavior.

INPO [Zhang et al., 2024]: The INPO algorithm is designed for finding the Nash equilibrium of the KL regularized game $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$. As we proved in Theorem 2, INPO does not diverge and exhibits last-iterate convergence. However, it converges to a point that differs from the Nash equilibrium of the game $J(\pi_1, \pi_2)$ and, as a result, lacks the robust alignment property.

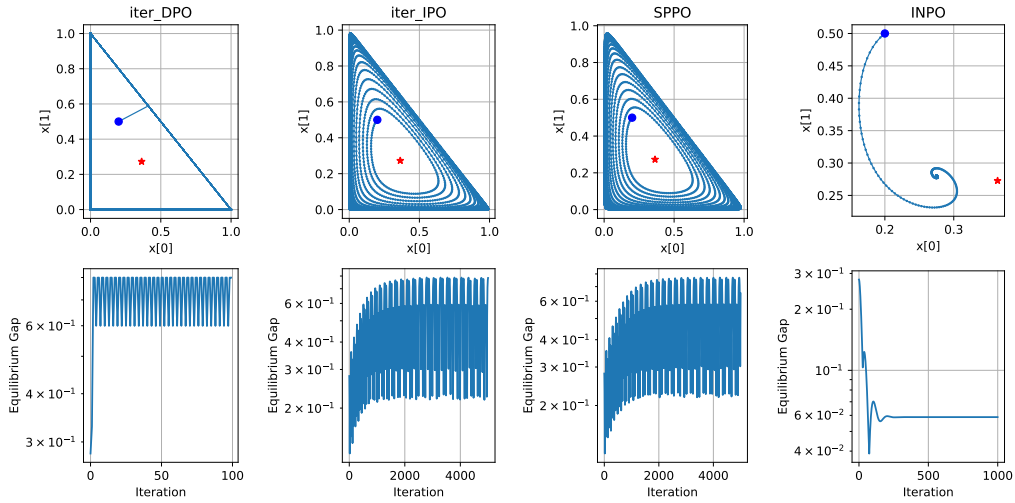

Figure 2: Dyanmics on a simple 3-dimensional preference game. The unique Nash equilibrium is $[4/11, 3/11, 4/11]$ represented as red star. We initialize all algorithms at the blue dot point $[0.2, 0.5, 0.3]$.

5 LLM-Based Experiments

Apart from the controlled synthetic experiments, we conduct experiments with a pre-trained LLM, Qwen2-1.5B [Yang et al., 2024], on a commonly used dataset UltraFeedback [Cui et al., 2023] to show the effectiveness of COMAL under the practical preference optimization setting.

5.1 Experimental Settings

Datasets We use the UltraFeedback dataset, specifically its binarized version for preference fine-tuning.[6] It contains 64K data examples consisting of a user instruction and a positive-negative output pair annotated by GPT-4. The instructions in this dataset span a wide range of types, making it well-suited for studying preference optimization in practical settings. Since we focus on online and iterative preference optimization, only the instructions are used because the output pairs will be generated and annotated online. In addition, to reduce the computational cost, the instructions are randomly split into 6 equal-size subsets. Each subset therefore contains around 10K instructions and is used in one training iteration.

Preference Oracle The preference oracle we used is Llama-3-OffsetBias-8B [Park et al., 2024], which is a pairwise preference model that predicts which output is better given an instruction and a pair of outputs. Fine-tuned from Meta-Llama-3-8B-Instruct [Dubey et al., 2024], it achieves strong performance on various human preference alignment benchmarks in RewardBench [Lambert et al., 2024]. We selected it as the preference oracle for its balance of computational efficiency and alignment with human preferences, making it suitable for iterative preference optimization.

Preference Data Generation To construct the preference data, i.e., output pairs with a preference annotation specifying which one is better, we adopt the setting of Zhang et al. [2024] by sampling 5 candidate outputs for each instruction with a temperature of 0.8 and applying the preference oracle to compare all the output pairs constructed. The best and the worst candidate outputs, derived from the pairwise comparison results, are then selected to form a data point.

Baselines We include the following baselines for comparisons with COMAL: (1) SFT, which fine-tunes the pre-trained Qwen2-1.5B on the UltraChat dataset, with the resulting checkpoint serving as the starting point and/or reference policy for the other training algorithms; (2) vanilla DPO [Rafailov et al., 2024] and (3) vanilla IPO [Azar et al., 2024], where one training iteration is performed over the entire instruction set of UltraFeedback with output pairs sampled from the SFT policy; (4) INPO [Zhang et al., 2024], where each iteration of training is performed on a single data split; (5)

[6] https://huggingface.co/datasets/HuggingFaceH4/ultrafeedback_binarized.

iterative IPO, which follows a training setting similar to INPO but without the KL regularization with respect to the reference policy.

Evaluations We use the instructions in a widely used benchmark, AlpacaEval [Li et al., 2023], to construct the test set, since these instructions are diverse and cover various task scenarios. However, instead of using GPT-4, the default evaluator for the AlpacaEval benchmark, we chose to use the same preference oracle used during data generation, Llama-3-OffsetBias-8B, as the evaluator. This decision was made to maintain a controlled experimental setting, ensuring that the preference oracle the model learns to fit is also the one used to evaluate its performance.

Training Details We follow the training recipe proposed in Tunstall et al. [2023] for the experiments. Specifically, at each training iteration, the models are fine-tuned for 3 epochs with a batch size of 32 and maximum learning rate of 5×10^{-7}, using a linear learning rate scheduler where 10% of the steps are for warmup and the rest for linearly decreasing the rate. The checkpoints are selected based on their validation loss on the UltraFeedback dataset. The training is performed on 8 NVIDIA A6000 Ada GPUs with 48GB memory, and one training iteration over the 10K instructions takes around 5 hours. Due to the relatively high computational requirements and the large number of training iterations we tested (up to 42), we opted to use a moderately sized LLM and did not conduct an exhaustive hyper-parameter search, instead referencing settings from previous work when appropriate. To the best of our knowledge, multi-iteration training like ours has rarely been explored in previous work. For example, INPO [Zhang et al., 2024] only performed optimization for up to 3 iterations, which is equivalent to just one full round over UltraFeedback's instructions.

Hyper-Parameters We conduct a grid search for the strength of the KL regularization, η^{-1}, in both vanilla DPO and IPO. We found that DPO achieves the best performance when η^{-1} is set to 0.01, while IPO achieves the best performance when η^{-1} is set within the range of 0.002 - 0.01. We then choose the value of η^{-1} to be 0.002 to encourage larger learning steps.[7] This value of η is also used for iterative IPO. For INPO, we compare two settings where η^{-1} is set to 0.002 and 0.01, corresponding to a small and a large regularization respectively. INPO has another hyper-parameter τ which controls the strength of the KL regularization from the reference policy. We determine its value following the setting of Zhang et al. [2024], where $\eta\tau$ is set to a fixed ratio, $1/3$. Regarding COMAL, which is implemented based on INPO as outlined in Algorithm 4, η^{-1} is also set to 0.002 at the beginning of the training. The reference policy used in COMAL is updated when the first optimization step begins to converge or overfit, and η^{-1} is increased to 0.01 to improve training stability.

5.2 Result Analysis

Figure 3 presents the training dynamics of three iterative preference optimization algorithms we compared: iterative IPO (Iter-IPO), INPO with a small and a large regularization (INPO-Small and INPO-Large), and COMAL, which are demonstrated by their checkpoints' win rates against the *best checkpoints* produced by 7 different algorithms: SFT, IPO, DPO, Iter-IPO, INPO-Small, INPO-Large, COMAL, and the average lengths of their outputs. For iterative algorithms, the model is trained for up to 42 iterations, equivalent to 7 training rounds over the entire instruction set since it has been split into 6 subsets. We note that:

(1) Iter-IPO shows a quicker improvement rate at the beginning of the training, but its performance begins to lag behind other algorithms after the first training round with a rapid increase in output length, which indicates the inherent instability of this training algorithm.

(2) INPO achieves stronger performance and larger improvement rates compared to Iter-IPO. However, the win rates of both INPO-Small and INPO-Large start to decrease after 5 training rounds. We suspect this suggests that INPO has started to converge and/or overfit. Moreover, for INPO-Small, its performance shows only a minor improvement and even a slight decline during training rounds 2 to 4 (iterations 12 - 24). Therefore, for COMAL, which shares the same training trajectory as INPO-Small for the first two training rounds, we update the reference policy at the beginning of the third training round, following the optimization process described in Algorithm 4.

[7] More details are in Appendix G.

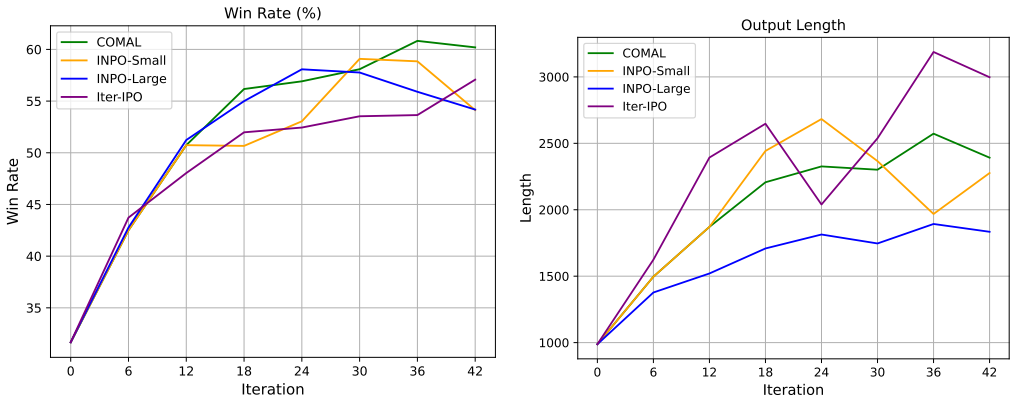

Figure 3: Comparisons of Iterative IPO (Iter-IPO), INPO, and COMAL. The average win rates of the trained checkpoints against the best checkpoints of each training algorithm, and the average lengths of the outputs are compared. For INPO, two variations with a small regularization ($\eta^{-1} = 0.002$, INPO-Small) and a large regularization ($\eta^{-1} = 0.01$, INPO-Large) are compared.

Table 2: Performance comparison of different training algorithms. The row v.s. column win rate (%) is reported. The *best* checkpoints produced by each training algorithm are compared. For INPO, we report two variations with a small regularization ($\eta^{-1} = 0.002$, INPO-Small) and a large regularization ($\eta^{-1} = 0.01$, INPO-Large).

Row/Column	SFT	DPO	IPO	Iter-IPO	INPO-Large	INPO-Small	COMAL	Avg
Iter-IPO	70.81	64.35	61.99	50.00	52.17	47.20	48.94	56.50
INPO-Large	**77.02**	69.81	67.83	47.83	50.00	46.21	44.84	57.65
INPO-Small	73.66	66.21	66.46	**52.80**	53.79	50.00	48.70	58.80
COMAL	74.53	**70.56**	**68.82**	51.06	**55.16**	**51.30**	**50.00**	**60.20**

(3) COMAL is able to further improve the model performance with the updated reference policy. Notably, its performance continues to improve up until the 6th training round, when the other algorithms begin to degrade, demonstrating the benefit of updating the reference policy.

Table 2 provides pairwise comparisons between the *best* checkpoints of the iterative preference optimization algorithms and a few baselines. It demonstrates the clear advantage of COMAL, which is able to achieve a win rate that is strictly above 50% against all the other checkpoints. The comparison of the *final* checkpoints of different algorithms after the last iteration is presented in Appendix H, where COMAL is able to achieve significantly better performance thanks to its stability.

6 Related Work

Alignment under Preference models Most existing approaches adopt the Bradley-Terry (BT) preference model [Bradley and Terry, 1952, Christiano et al., 2017], which involves first learning a preference model and then optimizing the objective function with a KL divergence penalty relative to the original language model. For example, RLHF [Ouyang et al., 2022] aims to ensure that LLMs follow instructions by initially learning a BT model and subsequently fine-tuning the model based on the learned reward while regularizing it with the original LLM.

Building on this framework, Rafailov et al. [2024] introduces Direct Preference Optimization (DPO) that maintains the assumption of the BT model for preferences but eliminates the preference learning step by reformulating the objective and optimizing it directly. Additionally, Ethayarajh et al. [2024] diverges from the traditional BT-based methods by deriving algorithms that bypass the preference modeling step altogether. Instead, they model user preferences based on Kahneman and Tversky's utility theory.

Alignment Solution Concepts under General Preferences Azar et al. [2024] is the first to consider general preferences. They propose the IPO algorithm, an offline algorithm that directly optimizes the

win rate of the model penalized by the KL divergence with respect to the original model. Munos et al. [2024] also consider general preferences and aim to find the *von Neumann winner*, which corresponds to the Nash equilibrium of a game played between the two LLMs over the win rate. They propose a variant of the Mirror Descent (MD) algorithm called Nash-MD and show last-iterate convergence in the KL-regularized game. Concurrently, Swamy et al. [2024] study the same solution concept focusing more on sequential games. Calandriello et al. [2024] proved that the objective of the the IPO algorithm coincides with the Nash policy under a proper choice of the parameter that controls the regularization.

Iterative Self-Play Algorithms Apart from the aforementioned works, recent research has also proposed practical implementations of Mirror Descent (MD) algorithms, which can be used to learn Nash equilibria through self-play. Rosset et al. [2024] propose Direct Nash Optimization (DNO), where, at each iteration, the model regresses predicted preferences against actual preferences using cross-entropy loss. Similarly, Wu et al. [2024] introduces the Self-Play Preference Optimization (SPPO) method, Gao et al. [2024] introduces Reinforcement Learning via Regressing Relative Rewards (REBEL), and Richemond et al. [2024] introduces the Direct Reward Optimization (DRO) which regresses the loss using the ℓ_2 distance at each iteration. Since these algorithms simulate the MD update, when applied in a two-player zero-sum game, they only have average-iterate convergence but all *diverge in the last iterate*. Moreover, all these methods require the estimation of the win rate, which can be computationally expensive.

Most closely related to our work is Iterative Nash Policy Optimization (INPO) by Zhang et al. [2024], which continues to use ℓ_2 distance regression. However, by further reformulating and simplifying the objective in a manner similar to IPO, INPO eliminates the need to estimate the expected win rate. The primary distinction between our approach and INPO is that INPO is designed for the KL-regularized game and is equivalent to MD; while our algorithm COMAL is inspired by the Conceptual Prox algorithm and guarantees last-iterate convergence in the original game. This fundamental difference allows COMAL to achieve more favorable convergence properties and outperform INPO, achieving a win rate strictly greater than 50% against it.

Last-Iterate Convergence in Games Mirror Descent fails to converge in simple zero-sum games, often resulting in cycling behavior [Mertikopoulos et al., 2018]. In contrast, several algorithms have been shown to achieve last-iterate convergence including the Proximal Point (PP) method [Rockafellar, 1976], Extra-Gradient (EG) [Korpelevich, 1976], Optimistic Gradient Descent (OGD) [Popov, 1980, Rakhlin and Sridharan, 2013], and the Conceptual Prox/Mirror Prox methods [Nemirovski, 2004]. The asymptotic convergence properties of these algorithms have been extensively studied [Popov, 1980, Facchinei and Pang, 2003, Iusem et al., 2003, Nemirovski, 2004, Daskalakis and Panageas, 2018]. Recently, there has been a growing focus on establishing finite-time convergence guarantees for these methods, addressing the practical necessity of understanding their performance within a limited number of iterations (see e.g., [Mokhtari et al., 2020b,a, Golowich et al., 2020b,a, Bauschke et al., 2021, Wei et al., 2021, Cai et al., 2022, Gorbunov et al., 2022, Cai and Zheng, 2023a,b, Cai et al., 2023, 2024b,a] and references therein).

7 Conclusion

We have proposed COMAL, a meta-algorithm for preference optimization that provably converges to the Nash equilibrium policy in the last iterate. We have provided a theoretical analysis of the properties of COMAL and have empirically demonstrated its effectiveness under both synthetic and real-world experimental settings. We believe COMAL has significant potential to enhance the performance of LLMs in the alignment fine-tuning setting, due to its theoretical guarantees and flexibility, as it can be integrated with existing learning algorithms while overcoming their limitations.

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A Properties of the Prox Operator

Recall that $\mathrm{Prox}(z, g) = \mathrm{argmax}_{z' \in \mathcal{Z}} \langle g, z' \rangle - D_\varphi(z' || z) = \mathrm{argmax}_{z' \in \mathcal{Z}} \langle g + \nabla\varphi(z), z' \rangle - \varphi(z')$. The following properties of the prox operator are well-known in the literature(e.g., [Nemirovski, 2004])

Lemma 1. $\mathrm{Prox}(z, g) = z'$ *if and only if* $\langle g + \nabla\varphi(z) - \nabla\varphi(z'), z' - z^* \rangle \geq 0$ *for all* $z^* \in \mathcal{Z}$.

Corollary 1. *Let* $\mathrm{Prox}(z, g) = z'$*, then*

$$\langle g, z^* - z' \rangle \leq D_\varphi(z^* || z) - D_\varphi(z^* || z') - D_\varphi(z' || z), \quad \forall z^* \in \mathcal{Z}$$

B Proof of Theorem 1

The proof of Theorem 1 is largely inspired by existing results for the conceptual prox algorithm in the literature [Facchinei and Pang, 2003, Nemirovski, 2004]. We first consider the case where each step of COMAL, $\pi^{t+1} \leftarrow \mathrm{argmax}_{\pi_1} \min_{\pi_2} J_\tau(\pi_1, \pi_2, \pi_{\mathrm{ref}})$, can be solved *exactly* in Appendix B.1. We then extend the proof to the case where we only solve the regularized game *approximately* in Appendix B.2. In both cases, we prove last-iterate convergence to Nash equilibrium, i.e., $\lim_{t\to\infty} \pi^t$ exists and is a Nash equilibrium. The proof for the latter case seems to be the first in the literature.

In Theorem 1, we make the following assumption.

Assumption 1. *We assume there exists a Nash equilibrium* $\pi^\star$ *such that* $\mathrm{supp}(\pi^\star) = \mathrm{supp}(\pi_{\mathrm{sft}})$.

This assumption is mild and **much weaker** than the "Bounded Log Density" assumptions used in previous works [Rosset et al., 2024, Zhang et al., 2024], which directly assumes $|\log \frac{\pi^t}{\pi_{\mathrm{sft}}}|$ is bounded.

B.1 Last-Iterate Convergence under Exact Solutions

Recall that $\Pi := \{\pi : \mathrm{supp}(\pi) \subseteq \mathrm{supp}(\pi_{\mathrm{sft}})\}$. Then $\mathrm{KL}(\pi || \pi_{\mathrm{sft}}) \leq D := \max_{y:\pi_{\mathrm{sft}}(y)>0} \log \pi_{\mathrm{sft}}(y)$ is bounded for any $\pi \in \Pi$. We first prove $\mathrm{KL}(\pi^\star || \pi^{t+1}) \leq \mathrm{KL}(\pi^\star || \pi^t)$ for any $t \geq 1$.

Lemma 2. *Let* $\pi^\star$ *be an Nash equilibrium of* $J(\pi_1, \pi_2)$*. Then for any* $\tau > 0$*, if*

$$(\pi, \pi) = \operatorname*{argmax}_{\pi_1 \in \Pi} \operatorname*{argmin}_{\pi_2 \in \Pi} J_\tau(\pi_1, \pi_2, \pi_{\mathrm{ref}}),$$

then

$$\mathrm{KL}(\pi^\star || \pi) \leq \mathrm{KL}(\pi^\star || \pi_{\mathrm{ref}}) - \mathrm{KL}(\pi || \pi_{\mathrm{ref}})$$

Proof. By definition of the prox operator, we have

$$\begin{aligned} \pi &= \operatorname*{argmax}_{\pi_1 \in \Pi} J_\tau(\pi_1, \pi, \pi_{\mathrm{ref}}) \\ &= \operatorname*{argmax}_{\pi_1 \in \Pi} \mathbb{P}(\pi_1 \succ \pi) - \tau\, \mathrm{KL}(\pi_1, \pi_{\mathrm{ref}}) \\ &= \mathrm{Prox}(\pi_{\mathrm{ref}}, \frac{1}{\tau}\mathbb{P}(\cdot \succ \pi)). \end{aligned} \tag{7}$$

Using Corollary 1, we have for any $\pi' \in \Pi$,

$$\frac{1}{\tau}(\mathbb{P}(\pi' \succ \pi) - \mathbb{P}(\pi \succ \pi)) \leq \mathrm{KL}(\pi' || \pi_{\mathrm{ref}}) - \mathrm{KL}(\pi' || \pi) - \mathrm{KL}(\pi || \pi_{\mathrm{ref}}). \tag{8}$$

Plugging $\pi' = \pi^\star$ into the above inequality and noting that $\mathbb{P}(\pi \succ \pi) = \frac{1}{2}$, we get

$$\frac{1}{\tau}\left(\mathbb{P}(\pi^\star \succ \pi) - \frac{1}{2}\right) \leq \mathrm{KL}(\pi^\star || \pi_{\mathrm{ref}}) - \mathrm{KL}(\pi^\star || \pi) - \mathrm{KL}(\pi || \pi_{\mathrm{ref}}).$$

Since $\pi^\star$ is a Nash equilibrium and thus $\mathbb{P}(\pi^\star \succ \pi) \geq \frac{1}{2}$, the lefthand side of the above inequality is ≥ 0. Then we have

$$\mathrm{KL}(\pi^\star || \pi) \leq \mathrm{KL}(\pi^\star || \pi_{\mathrm{ref}}) - \mathrm{KL}(\pi || \pi_{\mathrm{ref}}).$$

□

Lemma 2 implies the following properties on the trajectory $\{\pi^t\}$.

Corollary 2. *Denote* $\pi^\star$ *an Nash equilibrium such that* $\mathrm{supp}(\pi^\star) = \mathrm{supp}(\pi_{\mathrm{sft}})$ *as guaranteed by Assumption 1. Then the following holds for the trajectory* $\{\pi^t\}$ *produced by COMAL:*

1. $\mathrm{KL}(\pi^\star||\pi^{t+1}) \leq \mathrm{KL}(\pi^\star||\pi^t)$ *for all* $t \geq 1$.
2. $\sum_{t=1}^{\infty} \mathrm{KL}(\pi^{t+1}||\pi^t) \leq \mathrm{KL}(\pi^\star||\pi_{\mathrm{sft}}) < +\infty$.
3. *For all* $t \geq 1$*, it holds that for* $y \in \mathrm{supp}(\pi_{\mathrm{sft}})$*,* $\pi^t(y) \geq c > 0$ *where* c *is some constant* c *depends only on* $\pi^\star$ *and* π_{sft}*. This also holds even for any limit point of* $\{\pi^t\}$*.*

Proof. The first item is direct from Lemma 2. The second item is also direct by applying Lemma 2 for $t \geq 1$:

$$\sum_{t=1}^{\infty} \mathrm{KL}(\pi^{t+1}||\pi^t) \leq \sum_{t=1}^{\infty} \mathrm{KL}(\pi^\star||\pi^t) - \mathrm{KL}(\pi^\star||\pi^{t+1}) \leq \mathrm{KL}(\pi^\star||\pi_{\mathrm{sft}}) \leq D < \infty.$$

Now we consider the third item. Define $D := \mathrm{KL}(\pi^\star||\pi_{\mathrm{sft}})$ and $p_{\min} := \min_{y \in \mathrm{supp}(\pi^\star)} \pi^\star(y)$. By Assumption 1, $p_{\min} > 0$. Then

$$\begin{aligned}\mathrm{KL}(\pi^\star||\pi^t) \leq D &\Rightarrow p_{\min} \log \frac{p_{\min}}{\pi^t(y)} \leq D, \forall y \in \mathrm{supp}(\pi^\star) \\ &\Rightarrow \pi^t(y) \geq \frac{p_{\min}}{\exp(D/p_{\min})}, \forall y \in \mathrm{supp}(\pi^\star).\end{aligned}$$

Since the above holds for all π^t, it also holds for any limit point of $\{\pi^t\}$. □

Since the sequence $\{\pi^t\}$ is bounded (all lies in the simplex), it has at least one limit point $\hat{\pi}$. The next lemma shows that a limit point must be a Nash equilibrium.

Lemma 3. *If* $\hat{\pi}$ *is a limit point of* $\{\pi^t\}$*, then* $\hat{\pi}$ *is a Nash equilibrium of* $J(\pi_1, \pi_2)$*.*

Proof. By item 2 in Corollary 2, we have $\lim_{t\to\infty} \mathrm{KL}(\pi^{t+1}||\pi^t) = 0$. This implies $\lim_{t\to\infty} \|\pi^{t+1} - \pi^t\| = 0$. As $\hat{\pi}$ is a limit point of $\{\pi^t\}$, we let $\{\pi^k : k \in \kappa\}$ be the subsequence that converges to $\hat{\pi}$. Then by Equation (7), we have

$$\begin{aligned}&\lim_{k\in\kappa, k\to\infty} \pi^{k+1} = \lim_{k\in\kappa, k\to\infty} \mathrm{Prox}(\pi^k, \frac{1}{\tau}\mathbb{P}(\cdot \succ \pi^{k+1})) \\ \Rightarrow &\hat{\pi} = \mathrm{Prox}(\hat{\pi}, \frac{1}{\tau}\mathbb{P}(\cdot \succ \hat{\pi})).\end{aligned}$$

Thus $\hat{\pi}$ is a fixed point of $\mathrm{Prox}(\pi, \frac{1}{\tau}\mathbb{P}(\cdot \succ \pi)$. Moreover, by item 3 in Corollary 2, we have $\mathrm{supp}(\hat{\pi}) = \mathrm{supp}(\pi_{\mathrm{sft}})$. Now consider both the max and min player running MWU initialized with $\pi^1 = \hat{\pi}$. Then we have $\pi^t = \hat{\pi}$ for all $t \geq 1$. By Equation (8), we have for any $\pi' \in \Pi$,

$$\frac{1}{\tau}\sum_{t=1}^{\infty}\left(\mathbb{P}(\pi' \succ \hat{\pi}) - \frac{1}{2}\right) \leq \mathrm{KL}(\pi'||\hat{\pi}) < \infty,$$

where the inequality holds since $\mathrm{supp}(\pi') \subseteq \mathrm{supp}(\hat{\pi})$. As a result, we get

$$\mathbb{P}(\pi' \succ \hat{\pi}) \leq \frac{1}{2}, \forall \pi' \in \Pi \Leftrightarrow \mathbb{P}(\hat{\pi} \succ \pi') \geq \frac{1}{2}, \forall \pi' \in \Pi$$

Thus $\hat{\pi}$ is a Nash equilibrium of $J(\pi_1, \pi_2)$. □

Proof of Theorem 1 By Lemma 3, we know a limit point $\hat{\pi}$ is a Nash equilibrium. Then by Corollary 2, $\{\mathrm{KL}(\hat{\pi}||\pi^t) \geq 0\}$ is a decreasing sequence. Thus $\{\mathrm{KL}(\hat{\pi}||\pi^t)\}$ converges. Let $\{\pi^k : k \in \kappa\}$ be a subsequence that converges to $\hat{\pi}$. Then we have

$$\lim_{t\to\infty} \mathrm{KL}(\hat{\pi}||\pi^t) = \lim_{k\in\kappa, k\to\infty} \mathrm{KL}(\hat{\pi}||\pi^k) = \mathrm{KL}(\hat{\pi}||\hat{\pi}) = 0.$$

Thus we have $\lim_{t\to\infty} \pi^t = \hat{\pi}$ is a Nash equilibrium. This completed the proof of Theorem 1.

B.2 Last-Iterate Convergence under Approximate Solutions

This section considers the case where we can not solve the regularized game $J_\tau(\pi_1, \pi_2, \pi_{\text{ref}})$ exactly but only compute an approximate solution. Specifically, we consider the following inexact COMAL update: denote $\hat{\pi}^{t+1} = \text{argmax}_{\pi_1 \in \Pi} \min_{\pi_2 \in \Pi} J_\tau(\pi_1, \pi_2, \pi^t)$ the exactly solution; the algorithm updates the next iterate π^{t+1} as an ε_t-approximate solution such that

$$\text{KL}(\hat{\pi}^{t+1}, \pi^{t+1}) \leq \varepsilon_t = O\left(\frac{1}{t^4}\right). \tag{9}$$

We note that we can compute π^{t+1} within ε_t error using $O(\log \frac{1}{\varepsilon_t}) = O(\log t)$ iterations of Algorithm 2 (Theorem 2).

We denote $\Pi^\star$ the set of Nash equilibria such that each $\pi^\star \in \Pi^\star$ has support $\text{supp}(\pi^\star) = \text{supp}(\pi_{\text{sft}})$ as guaranteed by Assumption 1. We introduce a few quantities that depend on the Nash equilibria and the initial policy.

Definition 4. *We define the following constants.*

1. $p_{\text{sft}} := \max\{p > 0 : \forall y \in \text{supp}(\pi_{\text{sft}}), \pi_{\text{sft}}(y) \geq p\}$; $D := |\mathcal{Y}| \log \frac{1}{p_{\text{sft}}}$ *so that* $\text{KL}(\pi || \pi_{\text{sft}}) \leq D$ *for all* $\pi \in \Pi$
2. $p_{\min} := \max\{p > 0 : \exists \pi^\star \in \Pi^\star, \forall y \in \text{supp}(\pi_{\text{sft}}), \pi^\star(y) \geq p\}$; *Let* $\pi^\star \in \Pi^\star$ *be a Nash equilibrium so that* $\pi^\star(y) \geq p_{\min}$ *holds for all* y *in its support.*
3. $c_1 := \frac{p_{\min}}{\exp(D+2)/p_{\min}}$ *and* $c_2 := \frac{c_1}{\exp(1/c_1)}$.

Our main result is that if each optimization problem at iteration t can be solved within approximation error $\varepsilon_t \leq \frac{c_1}{3t^2}$, then COMAL converges in last-iterate to a Nash equilibrium.

Theorem 3 (COMAL with approximate regularized game solver)**.** *Assume Assumption 1 holds. If in each iteration* $t \geq 1$*, the returned iterate* π^{t+1} *is an* ε_t*-approximate solution to* $J_\tau(\pi_1, \pi_2, \pi^t)$ *as defined in* (9) *with* $\varepsilon_t \leq \frac{c_1^2}{9t^4}$ *(*c_1 *defined in Definition 4), then* $\{x^t\}$ *converges to a Nash equilibrium of* $J(\pi_1, \pi_2)$.

We need the following technical lemma in the proof of Theorem 3.

Lemma 4. *Let* $\varepsilon_t \leq \frac{c_1^2}{9t^4}$. *Then for all* $t \geq 1$,

1. $\text{KL}(\pi^\star || \pi^{t+1}) \leq \text{KL}(\pi^\star || \pi^t) - \text{KL}(\pi^{t+1} || \pi^t) + \frac{1}{t^2}$.
2. $\min_{y \in \text{supp}(\pi_{\text{sft}})} \pi^t(y) \geq c_2$.
3. $\lim_{t \to \infty} \|\pi^{t+1} - \pi^t\| = 0$.
4. *For any Nash equilibrium* $\hat{\pi} \in \Pi$ *and* $t \geq 1$*, we have* $\text{KL}(\hat{\pi} || \pi^{t+1}) \leq \text{KL}(\hat{\pi} || \pi^t) + \frac{1}{t^2}$

Proof. By Lemma 2, we have $\hat{\pi}^{t+1} = \text{Prox}(\pi^t, \mathbb{P}(\cdot \succ \hat{\pi}^{t+1}))$ and

$$\text{KL}(\pi^\star || \hat{\pi}^{t+1}) \leq \text{KL}(\pi^\star || \pi^t) - \text{KL}(\hat{\pi}^{t+1} || \pi^t). \tag{10}$$

The above implies

$$\begin{aligned}\text{KL}(\pi^\star || \pi^{t+1}) \leq \text{KL}(\pi^\star || \pi^t) - \text{KL}(\pi^{t+1} || \pi^t) + \underbrace{\text{KL}(\pi^\star || \pi^{t+1}) - \text{KL}(\pi^\star || \hat{\pi}^{t+1})}_{E_1} \\ + \underbrace{\text{KL}(\pi^{t+1} || \pi^t) - \text{KL}(\hat{\pi}^{t+1} || \pi^t)}_{E_2}.\end{aligned} \tag{11}$$

Now, we use induction to prove the claim. For the base case, we define $\pi^0 := \pi^1$ and $\varepsilon_t = 0$, then

Base Case: $t = 0$ Since $\pi^0 = \pi^1$, we have $\text{KL}(\pi^1 || \pi^0) = 0$. Then it is clear that

$$\text{KL}(\pi^\star || \pi^1) \leq \text{KL}(\pi^\star || \pi^0) - \text{KL}(\pi^1 || \pi^0).$$

Moreover, by Proposition 1 and $D \geq \text{KL}(\pi^\star || \pi_{\text{sft}})$, we have $\min_{y \in \text{supp}(\pi^1)} \pi^1(y) \geq c_1 \geq c_2$.

Induction: $t \geq 1$ We have

$$\begin{aligned}
\mathrm{KL}(\pi^\star || \hat{\pi}^{t+1}) &\leq \mathrm{KL}(\pi^\star || \pi^t) && ((10)) \\
&\leq \mathrm{KL}(\pi^\star || \pi_{\mathrm{sft}}) + \sum_{t=1}^{t-1} \frac{1}{t^2} && \text{(inductive hypothesis)} \\
&\leq D + 2. && (D \geq \mathrm{KL}(\pi^\star || \pi_{\mathrm{sft}}))
\end{aligned}$$

Using Proposition 1, we have $\min_{y \in \mathrm{supp}(\pi_{\mathrm{sft}})} \hat{\pi}^{t+1}(y) \geq c_1$. By $\mathrm{KL}(\hat{\pi}^{t+1} || \pi^{t+1}) \leq \varepsilon_t \leq 1$ and Proposition 1 again, we get $\min_{y \in \mathrm{supp}(\pi_{\mathrm{sft}})} \pi^{t+1}(y) \geq c_2 := \frac{c_1}{\exp(1/c_1)}$. Thus, both $\hat{\pi}^{t+1}$ and π^{t+1} are bounded away from the boundary in their support. Further by $\mathrm{KL}(\hat{\pi}^{t+1} || \pi^{t+1}) \leq \varepsilon_t$, we have

$$\sum_y \hat{\pi}^{t+1}(y) \log \frac{\hat{\pi}^{t+1}(y)}{\pi^{t+1}(y)} \leq \varepsilon_t \Rightarrow \max_y \log \frac{\hat{\pi}^{t+1}(y)}{\pi^{t+1}(y)} \leq \frac{\varepsilon_t}{c_1}.$$

As a result, we can bound

$$\begin{aligned}
E_1 &= \mathrm{KL}(\pi^\star || \pi^{t+1}) - \mathrm{KL}(\pi^\star || \hat{\pi}^{t+1}) \\
&= \sum_y \pi^\star(y) \log \frac{\hat{\pi}^{t+1}(y)}{\pi^{t+1}(y)} \\
&\leq \max_y \log \frac{\hat{\pi}^{t+1}(y)}{\pi^{t+1}(y)} \\
&\leq \frac{\varepsilon_t}{c_1}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
E_2 &= \mathrm{KL}(\pi^{t+1} || \pi^t) - \mathrm{KL}(\hat{\pi}^{t+1} || \pi^t) \\
&= \sum_y (\pi^{t+1}(y) - \hat{\pi}^{t+1}(y)) \log \frac{\pi^{t+1}(y)}{\pi^t(y)} - \mathrm{KL}(\hat{\pi}^{t+1} || \pi^{t+1}) \\
&\leq \left\| \pi^{t+1} - \hat{\pi}^{t+1} \right\|_1 \cdot \max_y |\log \frac{\pi^{t+1}(y)}{\pi^t(y)}| \\
&\leq \sqrt{\mathrm{KL}(\hat{\pi}^{t+1} || \pi^{t+1})} \cdot \log \frac{1}{c_2} && \text{(Pinsker's Inequality)} \\
&\leq \frac{2\sqrt{\varepsilon_t}}{c_1}
\end{aligned}$$

Combining the above two inequalities with (11) and noting the fact that $\varepsilon_t \leq \sqrt{\varepsilon_t}$ gives

$$\mathrm{KL}(\pi^\star || \pi^{t+1}) \leq \mathrm{KL}(\pi^\star || \pi^t) - \mathrm{KL}(\pi^{t+1} || \pi^t) + \frac{3\sqrt{\varepsilon_t}}{c_1}.$$

We conclude the claim since $\varepsilon_t \leq \frac{c_1^2}{9t^4}$. This completes the proof for item 1 and item 2.

For item 3, we have $\sum_{t=1}^\infty \|\pi^{t+1} - \pi^t\| \leq \sum_{t=1}^\infty \mathrm{KL}(\pi^{t+1} || \pi^t) \leq D + 2$. Thus $\lim_{t \to \infty} \|\pi^{t+1} - \pi^t\| = 0$.

For item 4, we can use Lemma 2 and $\hat{\pi}^{t+1} = \mathrm{Prox}(\pi^t, \mathbb{P}(\cdot \succ \hat{\pi}^{t+1}))$ to get

$$\begin{aligned}
\mathrm{KL}(\hat{\pi} || \pi^{t+1}) \leq \mathrm{KL}(\hat{\pi} || \pi^t) - \mathrm{KL}(\pi^{t+1} || \pi^t) + \underbrace{\mathrm{KL}(\hat{\pi} || \pi^{t+1}) - \mathrm{KL}(\hat{\pi} || \hat{\pi}^{t+1})}_{E_1} \\
+ \underbrace{\mathrm{KL}(\pi^{t+1} || \pi^t) - \mathrm{KL}(\hat{\pi}^{t+1} || \pi^t)}_{E_2}. \qquad (12)
\end{aligned}$$

We note that $E_2 \leq \frac{2\sqrt{\varepsilon_t}}{c_1}$ has been proved in the above. For E_1, we have

$$\begin{aligned} E_1 &= \mathrm{KL}(\hat{\pi}||\pi^{t+1}) - \mathrm{KL}(\hat{\pi}||\hat{\pi}^{t+1}) \\ &= \sum_y \hat{\pi}(y) \log \frac{\hat{\pi}^{t+1}(y)}{\pi^{t+1}(y)} \\ &\leq \max_y \log \frac{\hat{\pi}^{t+1}(y)}{\pi^{t+1}(y)} \\ &\leq \frac{\varepsilon_t}{c_1}. \end{aligned}$$

Thus we have $\mathrm{KL}(\hat{\pi}||\pi^{t+1}) \leq \mathrm{KL}(\hat{\pi}||\pi^t) + \frac{1}{t^2}$ as $\varepsilon_t \leq \frac{c_1^2}{9t^4}$. □

Proof of Theorem 3

Proof. Since the sequence $\{\pi^t\}$ is bounded, it has at least one limit point $\hat{\pi}$. By item 2 in Lemma 4, we know $\hat{\pi}(y) \geq c_2$ for all $y \in \mathrm{supp}(\pi_{\mathrm{sft}})$. By item 3 in Lemma 4, we have $\lim_{t\to\infty} \|\pi^{t+1} - \pi^t\| = 0$. Denote $\{\pi^k : k \in \kappa\}$ a subsequence that converges to $\hat{\pi}$. Then we have

$$\begin{aligned} \hat{\pi} &= \lim_{k\in\kappa,\kappa\to\infty} \pi^{k+1} \\ &= \lim_{k\in\kappa,\kappa\to\infty} \hat{\pi}^{k+1} && (\mathrm{KL}(\hat{\pi}^{k+1}, \pi^{k+1}) \leq \varepsilon_k \text{ and } \lim_{t\to\infty} \varepsilon_t = 0) \\ &= \lim_{k\in\kappa,\kappa\to\infty} \mathrm{Prox}(\pi^k, \frac{1}{\tau}\mathbb{P}(\cdot \succ \hat{\pi}^{k+1})) \\ &= \lim_{k\in\kappa,\kappa\to\infty} \mathrm{Prox}(\pi^{k+1}, \frac{1}{\tau}\mathbb{P}(\cdot \succ \hat{\pi}^{k+1})) && (\lim_{t\to\infty} \|\pi^{t+1} - \pi^t\| = 0) \\ &= \lim_{k\in\kappa,\kappa\to\infty} \mathrm{Prox}(\pi^{k+1}, \frac{1}{\tau}\mathbb{P}(\cdot \succ \pi^{k+1})) && (\mathrm{KL}(\hat{\pi}^{k+1}, \pi^{k+1}) \leq \varepsilon_k \text{ and } \lim_{t\to\infty} \varepsilon_t = 0) \\ &= \mathrm{Prox}(\hat{\pi}, \frac{1}{\tau}\mathbb{P}(\cdot \succ \hat{\pi})). \end{aligned}$$

Since $\hat{\pi}$ is a fixed point of $\mathrm{Prox}(\pi, \frac{1}{\tau}\mathbb{P}(\cdot \succ \pi))$ and $\mathrm{supp}(\hat{\pi}) = \mathrm{supp}(\pi_{\mathrm{sft}})$, we can use the same proof in Lemma 3 to show that $\hat{\pi}$ is a Nash equilibrium of $J(\pi_1, \pi_2)$.

Given that $\hat{\pi}$ is a Nash equilibrium of the original game, we can apply item 4 in Lemma 4 and get

$$\mathrm{KL}(\hat{\pi}||\pi^{t+1}) \leq \mathrm{KL}(\hat{\pi}||\pi^t) + \frac{1}{t^2}.$$

Now we show the sequence $\{x^t\}$ converges to $\hat{\pi}$. Fix any $\epsilon > 0$. Let $T_1 \geq 1$ such that $\sum_{t=T_1}^{\infty} \frac{1}{t^2} < \frac{\epsilon}{2}$, Since $\hat{\pi}$ is a limit point of $\{x^t\}$, there exists $T_2 \geq T_1$ such that $\mathrm{KL}(\hat{\pi}||\pi^{T_2}) \leq \frac{\epsilon}{2}$. Then for any $t \geq T_2$, we have

$$\mathrm{KL}(\hat{\pi}||\pi^{t+1}) \leq \mathrm{KL}(\hat{\pi}||\pi^{T_2}) + \sum_{t=T_2}^{\infty} \frac{1}{t^2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since the above holds for any $\varepsilon > 0$, we know $\lim_{t\to\infty} \mathrm{KL}(\hat{\pi}||\pi^t) = 0$ and thus $\{x^t\}$ converges to $\hat{\pi}$. This completes the proof. □

B.3 Auxiliary propostion

Proposition 1. *Let π_1 and π_2 be two distributions with the same support. If there exists $p, D > 0$ such that $\min_{y\in\mathrm{supp}(\pi_1)} \pi_1(y) \geq p$ and $\mathrm{KL}(\pi_1||\pi_2) \leq D$, then $\mathrm{supp}(\pi_2) = \mathrm{supp}(\pi_1)$ and*

$$\min_{y\in\mathrm{supp}(\pi_1)} \pi_2(y) \geq \frac{p}{\exp(D/p)}.$$

Proof. We have

$$\mathrm{KL}(\pi_1||\pi_2) \le D \Rightarrow p \log \frac{p}{\pi_2(y)} \le D, \forall y \in \mathrm{supp}(\pi_1)$$
$$\Rightarrow \pi_2(y) \ge \frac{p}{\exp(D/p)}, \forall y \in \mathrm{supp}(\pi_1).$$

□

C Proof of Theorem 2

We show that MWU (Algorithm 2) has linear convergence to the unique Nash equilibrium of a KL-regularized zero-sum game $J(\pi_1, \pi_2, \pi_{\mathrm{ref}})$. We denote $\mu^\star = \pi_\tau^\star$ its unique Nash equilibrium. We note that there are existing results showing linear convergence of MWU (e.g., [Abe et al., 2024, Lemma F.1]). We include a slightly simpler proof for our setting for completeness.

We prove the following descent lemma, which immediately implies Theorem 2.

Lemma 5. *If we choose $\eta \in (0, \frac{\tau}{\tau^2+\frac{1}{2}}]$ in Algorithm 2, then we have for every $k \ge 1$*

$$\mathrm{KL}(\mu^\star, \mu^{k+1}) \le \left(1 - \frac{\eta\tau}{2}\right) \mathrm{KL}(\mu^\star, \mu^k).$$

Proof. We define the gradient operator $G : \Pi \to \mathbb{R}^{|\mathcal{Y}|}$ of $J(\pi_1, \pi_2)$ and the gradient operator $A : \Pi \to \mathbb{R}^{|\mathcal{Y}|}$ of the KL regularization $\mathrm{KL}(\pi, \pi_{\mathrm{ref}})$ as follows.

$$G(\pi) := \mathbb{P}(\cdot \succ \pi)$$
$$A(\pi) := \nabla_\pi \mathrm{KL}(\pi, \pi_{\mathrm{ref}}) = \log \frac{\pi(\cdot)}{\pi_{\mathrm{ref}}(\cdot)}.$$

We define the composite operator $F = G - \tau A$. Then MWU update in Algorithm 2 is equivalent to

$$\mu^{k+1} = \mathrm{Prox}(\mu^k, \eta F(\mu^k)).$$

Using Corollary 1, we have

$$\left\langle \eta F(\mu^k), \mu^\star - \mu^{k+1} \right\rangle \le \mathrm{KL}(\mu^\star||\mu^k) - \mathrm{KL}(\mu^\star||\mu^{k+1}) - \mathrm{KL}(\mu^{k+1}||\mu^k)$$

We focus on the lefthand side of the above inequality. Since $\mu^\star$ is a Nash equilibrium of the regularized game with gradient F, we have $\langle \eta F(\mu^\star), \mu^\star - \mu^{k+1} \rangle \ge 0$ and thus

$$\begin{aligned}
&\left\langle \eta F(\mu^k), \mu^\star - \mu^{k+1} \right\rangle \\
&\ge \left\langle \eta F(\mu^k), \mu^\star - \mu^{k+1} \right\rangle - \left\langle \eta F(\mu^\star), \mu^\star - \mu^{k+1} \right\rangle \\
&= \underbrace{\eta \left\langle G(\mu^k) - G(\mu^{k+1}), \mu^\star - \mu^{k+1} \right\rangle}_{\mathrm{term}_1} + \underbrace{\eta\tau \left\langle A(\mu^k) - A(\mu^\star), \mu^{k+1} - \mu^\star \right\rangle}_{\mathrm{term}_2} \\
&\quad + \underbrace{\eta \left\langle G(\mu^{k+1}) - G(\mu^\star), \mu^\star - \mu^{k+1} \right\rangle}_{\mathrm{term}_3 = 0}.
\end{aligned}$$

We note that $\mathrm{term}_3 = 0$ since G is the gradient of a zero-sum game:

$$\begin{aligned}
&\left\langle G(\mu^{k+1}) - G(\mu^\star), \mu^\star - \mu^{k+1} \right\rangle \\
&= \mathbb{P}(\mu^\star \succ \mu^{k+1}) + \mathbb{P}(\mu^{k+1} \succ \mu^\star) - \frac{1}{2} - \frac{1}{2} = 0.
\end{aligned}$$

For term_2, we can apply the three-point identity for the Bregman divergence as follows:

$$\begin{aligned}
\mathrm{term}_2 &= \eta\tau \left\langle A(\mu^k) - A(\mu^\star), \mu^{k+1} - \mu^\star \right\rangle \\
&= \eta\tau \left\langle \log \frac{\mu^k}{\mu^\star}, \mu^{k+1} - \mu^\star \right\rangle \\
&= \eta\tau \big(\mathrm{KL}(\mu^\star||\mu^k) - \mathrm{KL}(\mu^{k+1}||\mu^k) + \mathrm{KL}(\mu^{k+1}||\mu^\star) \big) \\
&\ge \eta\tau \big(\mathrm{KL}(\mu^\star||\mu^k) - \mathrm{KL}(\mu^{k+1}||\mu^k) \big).
\end{aligned}$$

For term_1, we will use the 1-Lipschitzness of G and Cauchy-Swarz inequality:

$$\begin{aligned}
\mathrm{term}_1 &= \eta\big\langle G(\mu^k) - G(\mu^{k+1}), \mu^\star - \mu^{k+1}\big\rangle \\
&\geq -\eta\left(\frac{1}{2\tau}\big\|G(\mu^k) - G(\mu^{k+1})\big\|_\infty^2 + \frac{\tau}{2}\big\|\mu^\star - \mu^{k+1}\big\|_1^2\right) \\
&\geq -\eta\left(\frac{1}{2\tau}\big\|\mu^k - \mu^{k+1}\big\|_1^2 + \frac{\tau}{2}\big\|\mu^\star - \mu^{k+1}\big\|_1^2\right) && (G \text{ is 1-Lipschitz}) \\
&\geq -\frac{\eta}{2\tau}\,\mathrm{KL}(\mu^{k+1}||\mu^k) - \frac{\eta\tau}{2}\,\mathrm{KL}(\mu^\star||\mu^{k+1})
\end{aligned}$$

Combining the above gives

$$(1 - \frac{\eta\tau}{2})\,\mathrm{KL}(\mu^\star||\mu^{k+1}) \leq (1 - \eta\tau)\,\mathrm{KL}(\mu^\star||\mu^k) - (1 - \eta\tau - \frac{\eta}{2\tau})\,\mathrm{KL}(\mu^{k+1}||\mu^k)$$

Let $\eta \leq \frac{1}{\tau + \frac{1}{2\tau}} = \frac{\tau}{\tau^2 + \frac{1}{2}}$, then we have $1 - \eta\tau - \frac{\eta}{2\tau} \geq 0$ and thus

$$\mathrm{KL}(\mu^\star||\mu^{k+1}) \leq \frac{1 - \eta\tau}{1 - \frac{\eta\tau}{2}}\,\mathrm{KL}(\mu^\star||\mu^k) \leq \left(1 - \frac{\eta\tau}{2}\right)\mathrm{KL}(\mu^\star||\mu^k).$$

This completes the proof. □

D Computing the Prox Operator using Preference Learning Methods

We include additional examples showing how existing algorithms designed for RLHF and preference optimization with neural network parameters can be adapted to solve the prox operator $\mathrm{Prox}(\pi, \eta g)$ ($\eta > 0$ is the step size). These algorithms include RL algorithms like PPO and loss-minimization algorithms like DPO, IPO, SPPO, DRO, INPO, each of which may be preferred in certain settings.

Reinforcement Learning algorithms We can use the Proximal Policy Optimization (PPO) algorithm [Schulman et al., 2017] to solve $\mathrm{Prox}(\pi, \eta g)$. Observe that

$$\begin{aligned}
\mathrm{Prox}(\pi, \eta g) &= \operatorname*{argmax}_{\pi'}\{\langle \eta g, \pi'\rangle - \mathrm{KL}(\pi'||\pi)\} \\
&= \operatorname*{argmax}_{\pi'} \mathbb{E}_{y\sim\pi'}\big[g[y] - \eta^{-1}\cdot \mathrm{KL}(\pi'||\pi)\big]
\end{aligned}$$

shares the same form as the objective in (2). Typically, we parameterize $\pi' = \pi_\theta$ with neural network parameters θ and optimize over θ.

Loss minimization algorithms Let us denote $\hat{\pi}$ the prox operator $\mathrm{Prox}(\pi, \eta g)$, then we have

$$\hat{\pi}[y] = \frac{\pi(y)\exp(\eta g(y))}{Z} \Leftrightarrow \log\frac{\hat{\pi}(y)}{\pi(y)} - \eta g(y) + \log Z = 0,$$

where $Z = \mathbb{E}_{y\sim\pi}[\exp(\eta g(y))]$ is the partition function. We can directly compute the partition function Z and thus $\hat{\pi}$ in small tabular cases. However, the partition function is hard to compute in general large-scale applications. Several works have recently proposed to solve the above equality by optimizing the corresponding L_2 loss.

The Self-Play Preference Optimization (SPPO) loss [Wu et al., 2024] assumes $\log Z = \frac{\eta}{2}$ and optimizes

$$\ell_{\mathrm{SPPO}}(\theta) = \left(\log\frac{\pi_\theta(y)}{\pi(y)} - \eta g(y) - \frac{\eta}{2}\right)^2.$$

The Direct Reward Optimization (DRO) loss [Richemond et al., 2024] parameterizes both $\hat{\pi}$ and $\log Z$ with θ and V_ϕ respectively and optimize[8]

$$\ell_{\mathrm{DRO}}(\theta, \phi) = \left(\log\frac{\pi_\theta(y)}{\pi(y)} - \eta g(y) - \eta V_\phi\right)^2.$$

[8]We modified some constants in the original DRO loss to make it consistent with our presentation. The modification has no other effects.

The REBEL loss [Gao et al., 2024] uses *differences in rewards* to eliminate the partition function Z and optimize the regression loss

$$\ell_{\mathrm{REBEL}}(\theta) = \left(\eta^{-1}\left(\log\frac{\pi_\theta(y)}{\pi(y)} - \log\frac{\pi_\theta(y')}{\pi(y')}\right) - (g(y) - g(y'))\right)^2.$$

All the above approaches can be used to solve $\mathrm{Prox}(\pi, \eta g)$. However, directly applying them iteratively on $J(\pi_1, \pi_2)$ is equivalent to running MWU, which provably diverges. In contrast, we can apply them in Algorithm 2 and then apply our meta-algorithm COMAL to guarantee convergence to a Nash equilibrium with robust alignment.

Remark 1. *The above approaches are versatile and work well for any g that can be evaluated efficiently. In particular, we should consider using them when (1) $g = r$ is a reward function and we can efficiently query r; (2) $g = \mathbb{P}(\cdot \mid \mu)$ is the win rate against a reference policy μ, and we can efficiently sample from μ and have oracle access to $\mathbb{P}$. These two setting are popular and practical in the LLM alignment setting.*

Now we turn attention to the more specific setting where g corresponds to a preference model $\mathbb{P}$ (could be a BT model or a general preference) and that we can collect a win-loss preference data set $\mathcal{D} = \{(y_w, y_l)\}$, which is standard for LLM alignment. Although the abovementioned algorithms apply, they all require estimating g (the win rate) and may be inefficient in practice. In the following, we present algorithms directly working on the sampled dataset $\mathcal{D}$ without further estimation.

Sampled loss based on the BT preference model Assume $g = r$ is the reward of the Bradley-Terry model, and the dataset $\{(y_w, y_l)\}$ consists of win-lose pairs of responses. Then we can solve $\mathrm{Prox}(\pi, \eta g)$ by optimize the DPO loss [Rafailov et al., 2024] defined as

$$\ell_{\mathrm{DPO}}((y_w, y_l); \theta) = -\log\sigma\left(\eta^{-1}\log\frac{\pi_\theta(y_w)}{\pi(y_w)} - \eta^{-1}\log\frac{\pi_\theta(y_l)}{\pi(y_l)}\right).$$

Sampled loss for general preference The DPO loss inspires many other loss functions that work under even weaker assumptions on the preference model. Now, we assume a general preference model $\mathbb{P}$ over $\mathcal{Y}$ (not necessarily the BT model). We assume g is the win-rate against some policy μ such that $g_\mu(y) = \mathbb{P}[y \succ \mu] := \mathbb{E}_{y'\sim\mu}[\mathbb{P}[y \succ y']]$ (think of μ as the reference policy π_{ref} or other online policy π_t). We assume the dataset contains win-lose pairs sampled from μ: $\{y_w, y_l \sim \mu\}$. Recall the preference distribution $\lambda_{\mathbb{P}}(y, y')$ is a binary distribution:

$$\lambda_{\mathbb{P}}(y, y') = \begin{cases} (y, y') \text{ with probability } \mathbb{P}[y \succ y'] \\ (y', y) \text{ with probability } 1 - \mathbb{P}[y \succ y'] \end{cases}$$

The (population) IPO loss [Tang et al., 2024, Calandriello et al., 2024] is defined as

$$\ell_{\mathrm{IPO}}(\theta, \mu) := \mathbb{E}_{(y_w, y_l)\sim\mu, (y^+, y^-)\sim\lambda_{\mathbb{P}}(y_w, y_l)}\left[\left(\log\frac{\pi_\theta(y^+)}{\pi(y^+)} - \log\frac{\pi_\theta(y^-)}{\pi(y^-)} - \frac{\eta}{2}\right)^2\right].$$

It has been proved that the minimizer of the $\ell_{\mathrm{IPO}}(\theta, \mu)$ satisfies

$$\pi_\theta(y) \propto \pi(y)\exp\left(-\eta\mathbb{P}[y \succ \mu]\right) \Leftrightarrow \pi_\theta = \mathrm{Prox}(\pi, \eta g_\mu).$$

Thus we can compute the prox operator $\mathrm{Prox}(\pi, \eta g_\mu)$ where $g_\mu = \mathbb{P}(\cdot \succ \mu)$ by minimizing the IPO loss against policy μ.

A variant of the IPO loss applied to the regularized preference setting is the Iterative Nash Policy Optimization (INPO) loss [Zhang et al., 2024]. Here, we define g_μ^τ the gradient $\nabla_\pi J_\tau(\pi, \mu, \pi_{\mathrm{ref}}) = \mathbb{P}(\cdot \succ \mu) - \tau\log\frac{\mu(\cdot)}{\pi_{\mathrm{ref}}(\cdot)}$ of the regularized objective. The corresponding INPO loss is

$$\ell_{\mathrm{INPO}} := \mathbb{E}_{(y_w, y_l)\sim\mu, (y^+, y^-)\sim\lambda_{\mathbb{P}}(y_w, y_l)}\left[\left(\log\frac{\pi_\theta(y^+)}{\pi_\theta(y^-)} - \eta\tau\log\frac{\pi_{\mathrm{ref}}(y^+)}{\pi_{\mathrm{ref}}(y^-)} - (1 - \eta\tau)\log\frac{\mu(y^+)}{\mu(y^-)} - \frac{\eta}{2}\right)^2\right].$$

Similarly, it has been shown that the INPO loss minimizer corresponds to the prox operator's solution $\mathrm{Prox}(\pi, \eta g_\mu^\tau)$. Thus we can use the INPO in Algorithm 2 directly.

E Implementation of Mirror-Prox and Optimistic Multiplicative Weights Update

We note that there are other algorithms that has provable last-iterate convergence to Nash equilibrium in (unregularized) zero-sum games, including the Mirror-Prox algorithm [Nemirovski, 2004] and Optimistic Multiplicative Weights Update (OMWU) algorithm [Rakhlin and Sridharan, 2013, Syrgkanis et al., 2015, Hsieh et al., 2021]. We present practical implementations of these two algorithms in the context of LLM alignment for solving $J(\pi_1, \pi_2)$ (4), where we use preference optimization algorithms to solve the prox operator as shown in Section 3.3 and Appendix D.

We denote the gradient $g(\pi) := \mathbb{P}(\cdot \succ \pi)$.

Mirror-Prox The Mirror-Prox algorithm [Nemirovski, 2004] initialized $\pi^1 = \pi_{\text{sft}}$ and updates in each iteration $t \geq 1$:

$$\pi^{t+\frac{1}{2}} = \text{Prox}(\pi^t, \eta g(\pi^t))$$
$$\pi^{t+1} = \text{Prox}(\pi^t, \eta g(\pi^{t+\frac{1}{2}}))$$

We can implement Mirror-Prox using PPO/DPO/IPO/SPPO/DRO/REBEL to compute the prox operator. Specifically, we could sample from π^t and construct a preference dataset D_t and optimize certain regression loss (IPO/DRO/REBEL) to compute $\pi^{t+\frac{1}{2}} = \text{Prox}(\pi^t, \eta g(\pi^t))$. The procedure applies to the second step in each iteration. Thus in such an implementation, we require two sampling and two optimization procedures in each iteration.

Optimistic Multiplicative Weights Update (OMWU) The OMWU algorithm [Rakhlin and Sridharan, 2013] is an optimistic variant of the MWU algorithm. Although MWU diverges in zero-sum games, it has been shown that OMWU has last-iterate convergence to Nash equilibrium [Wei et al., 2021, Hsieh et al., 2021]. Initialized with $\pi^1 = \pi^{\frac{1}{2}} = \pi_{\text{sft}}$, OMWU updates in each iteration $t \geq 1$:

$$\pi^{t+\frac{1}{2}} = \text{Prox}(\pi^t, \eta g(\pi^{t-\frac{1}{2}}))$$
$$\pi^{t+1} = \text{Prox}(\pi^t, \eta g(\pi^{t+\frac{1}{2}}))$$

Similarly, we can implement OMWU to solve $J(\pi_1, \pi_2)$ using preference methods to compute the prox operator as shown in Section 3.3. Moreover, OMWU has an equivalent update rule: initialize $\pi^1 = \pi^0 = \pi_{\text{sft}}$

$$\pi^{t+1} = \text{Prox}(\pi^t, 2\eta g(\pi^t) - \eta g(\pi^{t-1})),$$

which requires computing only one prox operator in each iteration.

We leave testing the practical performance of Mirror-Prox and OMWU for large-scale applications, including LLM alignment, as future works.

F Setup for Synthetic Experiments

Recall that we set $\mathbb{P}[y_b \succ y_a] = \mathbb{P}[y_c \succ y_b] = 0.9$ and $\mathbb{P}[y_a \succ y_c] = 0.8$. This results in the following zero-sum game: we have policies $\Pi = \Delta(\{y_a, y_b, y_c\})$ and objective

$$J(\pi_1, \pi_2) = \pi_1^\top A \pi_2 - 0.5, \text{ where } A = \begin{bmatrix} 0.5 & 0.1 & 0.8 \\ 0.9 & 0.5 & 0.1 \\ 0.2 & 0.9 & 0.5 \end{bmatrix}.$$

The game has a unique Nash equilibrium $[4/11, 3/11, 4/11]$. We set the initial policy to be $\pi^1 = [0.2, 0.5, 0.3]$ for all algorithms. We choose $\eta = 0.3$ for iterative DPO, iterative IPO, and SPPO. We choose $\eta = 0.3$ and $\tau = 0.1$ for INPO and COMAL. For COMAL (Algorithm 4), we set $T = 200$ and $K_t = 25$ so the total number of iterations is $T \cdot K_t = 5000$.

G Hyperparameter Search for LLM-Based Experiments

Here we outline the results of the hyperparameter search we conducted in Section 5.1 for identifying the optimal value of η for DPO and IPO. Table 3 reports the win rates of different checkpoints trained

with different values of η against the SFT policy. It shows that DPO achieves the best performance when η^{-1} is set to 0.01. On the other hand, IPO achieves a relatively stable and strong performance when η^{-1} is set within the range of 0.002-0.01. However, when compared against the best DPO checkpoint, we found that IPO trained with $\eta^{-1} = 0.002$ achieves the highest win rate (51.43%), therefore we chose it as the default value for the rest of the experiments.

Table 3: Results of the hyperparameter search for DPO and IPO regarding the strength of the KL constraint η^{-1}. The checkpoints' win rates against the SFT policy are reported.

η^{-1}	IPO	DPO
0.02	64.34	67.32
0.01	69.06	69.44
0.005	68.44	65.71
0.002	68.94	61.49
0.001	58.01	53.29

H Additional Results for LLM-Based Experiments

Table 4: Performance comparison of different training algorithms. The row v.s. column win rate (%) is reported. The *last* checkpoints produced by each training algorithm are compared. For INPO, we report two variations with a small regularization ($\eta^{-1} = 0.002$, INPO-Small) and a large regularization ($\eta^{-1} = 0.01$, INPO-Large).

Row/Column	SFT	DPO	IPO	Iter-IPO	INPO-Large	INPO-Small	COMAL	Avg
Iter-IPO	70.81	64.35	61.99	50.00	53.79	50.43	46.83	56.89
INPO-Large	70.43	62.98	61.61	46.21	50.00	48.07	41.61	54.42
INPO-Small	68.57	61.12	59.88	49.57	51.93	50.00	43.23	54.90
COMAL	**74.53**	**67.83**	**65.09**	**53.17**	**58.39**	**56.77**	**50.00**	**60.83**

In Section 5.2, the effectiveness of different iterative algorithms are compared using the performance of their *best* checkpoints (Table 2). Here, we provide an additional comparison among the *last* checkpoints produced by different algorithms. Table 4 shows that COMAL is able to achieve significantly better performance at the last iteration, demonstrating its superior stability.