ASYNCHRONOUS DISTRIBUTED BILEVEL OPTIMIZATION

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ABSTRACT

Bilevel optimization plays an essential role in many machine learning tasks, ranging from hyperparameter optimization to meta-learning. Existing studies on bilevel optimization, however, focus on either centralized or synchronous distributed setting. The centralized bilevel optimization approaches require collecting massive amount of data to a single server, which inevitably incur significant communication expenses and may give rise to data privacy risks. Synchronous distributed bilevel optimization algorithms, on the other hand, often face the straggler problem and will immediately stop working if a few workers fail to respond. As a remedy, we propose **A**synchronous **D**istributed **B**ilevel **O**ptimization (ADBO) algorithm. The proposed ADBO can tackle bilevel optimization problems with both nonconvex upper-level and lower-level objective functions, and its convergence is theoretically guaranteed. Furthermore, it is revealed through theoretic analysis that the iteration complexity of ADBO to obtain the ϵ-stationary point is upper bounded by $\mathcal{O}(\frac{1}{\epsilon^2})$. Thorough empirical studies on public datasets have been conducted to elucidate the effectiveness and efficiency of the proposed ADBO.

1 INTRODUCTION

Recently, bilevel optimization has emerged due to its popularity in various machine learning applications, *i.e.*, hyperparameter optimization (Khanduri et al., 2021; Liu et al., 2021a), meta learning (Likhosherstov et al., 2021; Ji et al., 2020), reinforcement learning (Hong et al., 2020), and neural architecture search (Jiang et al., 2020). In bilevel optimization, one optimization problem is embedded or nested with another. Specifically, the outer optimization problem is called the upper-level optimization problem and the inner optimization problem is called the lower-level optimization problem. A general form of the bilevel optimization problem can be written as,

$$\begin{aligned} \min \quad & F(\boldsymbol{x}, \boldsymbol{y}) \\ \text{s.t.} \quad & \boldsymbol{y} = \arg\min_{\boldsymbol{y}'} f(\boldsymbol{x}, \boldsymbol{y}') \\ \text{var.} \quad & \boldsymbol{x}, \boldsymbol{y}, \end{aligned} \tag{1}$$

where F and f denote the upper-level and lower-level objective functions, respectively. $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ represent the upper-level and lower-level variables, respectively. Bilevel optimization can be treated as a special case of constrained optimization since the lower-level optimization problem can be viewed as a constraint for the upper-level optimization problem (Sinha et al., 2017).

The proliferation of smartphones and Internet of Things (IoT) devices has generated a plethora of data in various real-world applications. Centralized bilevel optimization approaches require collecting a massive amount of data from distributed edge devices and passing them to a centralized server for model training. These methods, however, may give rise to data privacy risks (Subramanya & Riggio, 2021) and encounter communication bottlenecks (Subramanya & Riggio, 2021). To tackle these challenges, recently, distributed algorithms have been developed to solve the decentralized bilevel optimization problems (Yang et al., 2022; Chen et al., 2022b; Lu et al., 2022). Tarzanagh et al. (2022) and Li et al. (2022) study the bilevel optimization problems under a federated setting.

Specifically, the distributed bilevel optimization problem can be given by

$$\begin{aligned} &\min \;\; F(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{N} G_i(\boldsymbol{x},\boldsymbol{y}) \\ &\text{s.t. } \boldsymbol{y} = \arg\min_{\boldsymbol{y}'} f(\boldsymbol{x},\boldsymbol{y}') = \sum_{i=1}^{N} g_i(\boldsymbol{x},\boldsymbol{y}') \\ &\text{var.} \qquad \boldsymbol{x},\boldsymbol{y}, \end{aligned} \tag{2}$$

where N is the number of workers (devices), G_i and g_i denote the local upper-level and lower-level objective functions, respectively. Although existing approaches have shown their success in resolving distributed bilevel optimization problems, they only focus on the synchronous distributed setting. Synchronous distributed methods may encounter the straggler problem (Jiang et al., 2021) and its speed is limited by the worker with maximum delay (Chang et al., 2016). Moreover, synchronous distributed method will immediately stop working if a few workers fail to respond (Zhang & Kwok, 2014) (which is common in large-scale distributed systems). The aforementioned issues give rise to the following question:

Can we design an asynchronous distributed algorithm for bilevel optimization?

To this end, we develop an **A**synchronous **D**istributed **B**ilevel **O**ptimization (ADBO) algorithm which is a single-loop algorithm and computationally efficient. First, we reformulate Eq. (2) as a consensus problem (Matamoros, 2017; Chang et al., 2016),

$$\begin{aligned} &\min \;\; F(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z}) = \sum_{i=1}^{N} G_i(\boldsymbol{x},\boldsymbol{y}) \\ &\text{s.t.} \quad \boldsymbol{x}_i = \boldsymbol{v}, i = 1,\cdots,N \\ &\qquad \{\boldsymbol{y}_i\},\boldsymbol{z} = \arg\min_{\{\boldsymbol{y}_i'\},\boldsymbol{z}'} f(\boldsymbol{v},\{\boldsymbol{y}_i'\},\boldsymbol{z}') = \sum_{i=1}^{N} g_i(\boldsymbol{v},\boldsymbol{y}_i') \\ &\qquad\qquad\qquad \boldsymbol{y}_i' = \boldsymbol{z}', i = 1,\cdots,N \\ &\text{var.} \qquad \{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z}, \end{aligned} \tag{3}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n$ and $\boldsymbol{y}_i \in \mathbb{R}^m$ are local variables in the ith worker, $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{z} \in \mathbb{R}^m$ are the consensus variables in the master node. Most of the existing bilevel optimization works in machine learning only consider the bilevel programs without upper-level and lower-level constraints (Franceschi et al., 2018; Ji et al., 2021; Yang et al., 2021; Chen et al., 2022a) or bilevel programs with only upper-level (or lower-level) constraint (Zhang et al., 2022; Mehra & Hamm, 2021). On the contrary, we focus on the bilevel programs (i.e., Eq. (3)) with *both* lower-level and upper-level constraints, which is more challenging. The proposed ADBO regards the lower-level optimization problem as a constraint to the upper-level optimization problem, and utilizes cutting planes to approximate this constraint. Then, the approximate problem is solved in an ***asynchronous distributed manner*** by the proposed ADBO. We prove that even if both the upper-level and lower-level objectives are *non-convex*, the proposed ADBO is guaranteed to converge. The iteration complexity of ADBO is also theoretically derived. To facilitate the comparison, we not only present a centralized bilevel optimization algorithm in Appendix A, but also compare the convergence results of ADBO to state-of-the-art bilevel optimization algorithms with both centralized and distributed settings in Table 1.

Contributions. Our contributions can be summarized as:

1. We propose a novel algorithm, ADBO, to solve the bilevel optimization problem in an ***asynchronous distributed manner***. ADBO is a single-loop algorithm and is computationally efficient. To the best of our knowledge, it is the first work in tackling asynchronous distributed bilevel optimization problem.

2. We demonstrate that the proposed ADBO can be applied to bilevel optimization with *non-convex* upper-level and lower-level objectives *with constraints*. We also theoretically derive that the iteration complexity for the proposed ADBO to obtain the ϵ-stationary point is upper bounded by $\mathcal{O}(\frac{1}{\epsilon^2})$.

3. Our thorough empirical studies justify the superiority of the proposed ADBO over the existing state of the art methods.

2 RELATED WORK

Bilevel optimization: The bilevel optimization problem was firstly introduced by Bracken & McGill (1973). In recent years, many approaches have been developed to solve this problem and they can be divided into three categories (Gould et al., 2016). The first type of approaches assume there is an analytic solution to the lower-level optimization problem (*i.e.*, $\phi(\boldsymbol{x}) = \arg\min_{\boldsymbol{y}'} f(\boldsymbol{x}, \boldsymbol{y}')$) (Zhang et al., 2021). In this case, the bilevel optimization problem can be simplified to a single-level optimization problem (*i.e.*, $\min_{\boldsymbol{x}} F(\boldsymbol{x}, \phi(\boldsymbol{x}))$. Nevertheless, finding the analytical solution for the lower-level optimization problem is often very difficult, if not impossible. The second type of approaches replace the lower-level optimization problem with the sufficient conditions for optimality (*e.g.*, KKT conditions) (Biswas & Hoyle, 2019; Sinha et al., 2017). Then, the bilevel program can be reformulated as a single-level constrained optimization problem. However, the resulting problem could be hard to solve since it often involves a large number of constraints (Ji et al., 2021; Gould et al., 2016). The third type of approaches are gradient-based methods (Ghadimi & Wang, 2018; Hong et al., 2020; Liao et al., 2018) that compute the hypergradient (or the estimation of hypergradient), *i.e.*, $\frac{\partial F(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}} + \frac{\partial F(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{y}} \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}$, and use gradient descent to solve the bilevel optimization problems. Most of the existing bilevel optimziation methods focus on centralized settings and require collecting a massive amount of data from distributed edge devices (workers). This may give rise to data privacy risks (Subramanya & Riggio, 2021) and encounter communication bottlenecks (Subramanya & Riggio, 2021).

Asynchronous distributed optimization: To alleviate the aforementioned issues in the centralized setting, various distributed optimization methods can be employed. Distributed optimization methods can be generally divided into synchronous distributed methods and asynchronous distributed methods (Assran et al., 2020). For synchronous distributed methods (Boyd et al., 2011), the master needs to wait for the updates from all workers before it proceeds to the next iteration. Therefore, it may suffer from the straggler problem and the speed is limited by the worker with maximum delay (Chang et al., 2016). For asynchronous distributed methods (Chen et al., 2020; Matamoros, 2017), the master can update its variables once it receives updates from S workers, *i.e.*, active workers ($1 \leq S \leq N$, where N is the number of all workers). The asynchronous distributed algorithm is strongly preferred for large scale distributed systems in practice since it does not suffer from the straggler problem (Jiang et al., 2021). Since the action orders of each worker are different in the asynchronous distributed setting, which will result in complex interaction dynamics (Jiang et al., 2021), the theoretical analysis for asynchronous distributed algorithms is usually more challenging than that of the synchronous distributed algorithms. So far, existing works for distributed bilevel optimization only focus on the synchronous setting (Tarzanagh et al., 2022; Li et al., 2022; Chen et al., 2022b), how to design an asynchronous algorithm for distributed bilevel optimization remains under-explored. To the best of our knowledge, this is the first work that designs an asynchronous algorithm for distributed bilevel optimization.

3 ASYNCHRONOUS DISTRIBUTED BILEVEL OPTIMIZATION

In this section, we propose **A**synchronous **D**istributed **B**ilevel **O**ptimization (ADBO) to solve the distributed bilevel optimization problem (i.e., Eq. (3)) in an asynchronous distributed manner. By defining $\phi(\boldsymbol{v}) = \underset{\{\boldsymbol{y}_i'\},\boldsymbol{z}'}{\arg\min}\{\sum_{i=1}^{N} g_i(\boldsymbol{v}, \boldsymbol{y}_i') : \boldsymbol{y}_i' = \boldsymbol{z}', i = 1, \cdots, N\}$ and $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) = || \begin{bmatrix} \{\boldsymbol{y}_i\} \\ \boldsymbol{z} \end{bmatrix} - \phi(\boldsymbol{v})||_1$, we can reformulate problem in Eq. (3) as:

$$
\begin{aligned}
\min \quad & F(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z}) = \sum_{i=1}^{N} G_i(\boldsymbol{x}_i, \boldsymbol{y}_i) \\
\text{s.t.} \quad & \boldsymbol{x}_i = \boldsymbol{v}, i = 1, \cdots, N \\
& h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) = 0 \\
\text{var.} \quad & \{\boldsymbol{x}_i\}, \{\boldsymbol{y}_i\}, \boldsymbol{v}, \boldsymbol{z},
\end{aligned} \tag{4}
$$

A consensus problem (*i.e.*, the lower-level optimization problem in Eq. (3)) needs to be solved in a distributed manner if an exact $\phi(\boldsymbol{v})$ is desired. Following existing works (Li et al., 2022; Gould et al., 2016; Yang et al., 2021) for bilevel optimization, instead of pursuing the exact $\phi(\boldsymbol{v})$, an approximated $\phi(\boldsymbol{v})$ could be utilized. For this purpose, we first obtain the first-order Taylor

Table 1: Convergence results of bilevel optimization algorithms (with centralized and distributed setting).

Method	Centralized	Synchronous (Distributed)	Asynchronous (Distributed)
AID-BiO (Ghadimi & Wang, 2018)	$\mathcal{O}(\frac{1}{\epsilon^{1.25}})$	NA	NA
AID-BiO (Ji et al., 2021)	$\mathcal{O}(\frac{1}{\epsilon^{1}})$	NA	NA
ITD-BiO (Ji et al., 2021)	$\mathcal{O}(\frac{1}{\epsilon^{1}})$	NA	NA
STABLE (Chen et al., 2022a)	$\mathcal{O}(\frac{1}{\epsilon^{2}})$[1]	NA	NA
stocBio (Ji et al., 2021)	$\mathcal{O}(\frac{1}{\epsilon^{2}})$[1]	NA	NA
VRBO (Yang et al., 2021)	$\mathcal{O}(\frac{1}{\epsilon^{1.5}})$[1]	NA	NA
FEDNEST (Tarzanagh et al., 2022)	NA	$\mathcal{O}(\frac{1}{\epsilon^{2}})$[1]	NA
SPDB (Lu et al., 2022)	NA	$\mathcal{O}(\frac{1}{\epsilon^{2}})$[1]	NA
DSBO (Yang et al., 2022)	NA	$\mathcal{O}(\frac{1}{\epsilon^{2}})$[1]	NA
Proposed Method	$\mathcal{O}(\frac{1}{\epsilon^{1}})$	NA	$\mathcal{O}(\frac{1}{\epsilon^{2}})$

[1] Stochastic optimization algorithm.

approximation of $g_i(\boldsymbol{v}, \{\boldsymbol{y}_i'\})$ with respect to $\boldsymbol{v}$, *i.e.*, for a given point $\overline{\boldsymbol{v}}$, $\widetilde{g}_i(\boldsymbol{v}, \{\boldsymbol{y}_i'\}) = g_i(\overline{\boldsymbol{v}}, \{\boldsymbol{y}_i'\}) + \nabla_{\boldsymbol{v}} g_i(\overline{\boldsymbol{v}}, \{\boldsymbol{y}_i'\})^\top (\boldsymbol{v} - \overline{\boldsymbol{v}})$. Then, similar to many works that use K steps of gradient descent (GD) to approximate the optimal solution of lower-level optimization problem (Ji et al., 2021; Yang et al., 2021; Liu et al., 2021b), we utilize the results after K communication rounds between workers and master to approximate $\phi(\boldsymbol{v})$. In this paper, we consider solving the lower-level optimization problem approximately in an asynchronous distributed manner. Specifically, given $\widetilde{g}_i(\boldsymbol{v}, \{\boldsymbol{y}_i'\})$, the Lagrangian function of the lower-level optimization problem in Eq. (3)) can be expressed as,

$$g_p(\boldsymbol{v}, \{\boldsymbol{y}_i'\}, \boldsymbol{z}', \{\boldsymbol{\varphi}_i\}) = \sum_{i=1}^{N} \left(\widetilde{g}_i(\boldsymbol{v}, \boldsymbol{y}_i') + \boldsymbol{\varphi}_i^\top (\boldsymbol{y}_i' - \boldsymbol{z}') \right), \tag{5}$$

where $\boldsymbol{\varphi}_i \in \mathbb{R}^m$ is the vector containing dual variables. We assume that the master will update its variables once it receives updates from S workers (*i.e.*, active workers, and $1 \leq S \leq N$) every iteration. We also assume that every worker has to communicate with the master at least once every τ iterations. In $(k+1)^{\text{th}}$ iteration, let $\boldsymbol{\mathcal{Q}}_{k+1}$ be the index set of active workers, we have,

(1) Active workers update their local variables as follows,

$$\boldsymbol{y}_{i,k+1}' = \begin{cases} \boldsymbol{y}_{i,k}' - \eta_{\boldsymbol{y}} \nabla_{\boldsymbol{y}_i} g_p(\boldsymbol{v}, \{\boldsymbol{y}_{i,\hat{k}_i}'\}, \boldsymbol{z}_{\hat{k}_i}', \{\boldsymbol{\varphi}_{i,\hat{k}_i}\}), i \in \boldsymbol{\mathcal{Q}}_{k+1} \\ \boldsymbol{y}_{i,k}', i \notin \boldsymbol{\mathcal{Q}}_{k+1} \end{cases}, \tag{6}$$

where $\hat{k}_i$ denotes the last iteration during which worker i was active. Then, active workers transmit the local variable $\boldsymbol{y}_{i,k+1}'$ to the master.

(2) After receiving updates from active workers, the master updates variables as follows,

$$\boldsymbol{z}_{k+1}' = \boldsymbol{z}_k' - \eta_{\boldsymbol{z}} \nabla_{\boldsymbol{z}} g_p(\boldsymbol{v}, \{\boldsymbol{y}_{i,k+1}'\}, \boldsymbol{z}_k', \{\boldsymbol{\varphi}_{i,k}\}), \tag{7}$$

$$\boldsymbol{\varphi}_{i,k+1} = \begin{cases} \boldsymbol{\varphi}_{i,k} + \eta_{\boldsymbol{\varphi}} \nabla_{\boldsymbol{\varphi}} g_p(\boldsymbol{v}, \{\boldsymbol{y}_{i,k+1}'\}, \boldsymbol{z}_{k+1}', \{\boldsymbol{\varphi}_{i,k}\}), i \in \boldsymbol{\mathcal{Q}}_{k+1} \\ \boldsymbol{\varphi}_{i,k}, i \notin \boldsymbol{\mathcal{Q}}_{k+1} \end{cases}. \tag{8}$$

Next, the master broadcasts $\boldsymbol{z}_{k+1}'$ and $\boldsymbol{\varphi}_{i,k+1}$ to worker $i, i \in \boldsymbol{\mathcal{Q}}_{k+1}$ (*i.e.*, active workers).

As mentioned above, we utilize the results after K communication rounds to approximate $\phi(\boldsymbol{v})$, *i.e.*,

$$\phi(\boldsymbol{v}) = \begin{bmatrix} \{\boldsymbol{y}_{i,0}' - \sum_{k \in \mathcal{V}_i^K} \eta_{\boldsymbol{y}} \nabla_{\boldsymbol{y}_i} g_p(\boldsymbol{v}, \{\boldsymbol{y}_{i,\hat{k}_i}'\}, \boldsymbol{z}_{\hat{k}_i}', \{\boldsymbol{\varphi}_{i,\hat{k}_i}\})\} \\ \boldsymbol{z}_0' - \sum_{k=0}^{K-1} \eta_{\boldsymbol{z}} \nabla_{\boldsymbol{z}} g_p(\boldsymbol{v}, \{\boldsymbol{y}_{i,k}'\}, \boldsymbol{z}_{i,k}', \{\boldsymbol{\varphi}_{i,k}\}) \end{bmatrix}, \tag{9}$$

where $\mathcal{V}_i^K$ denotes the index set that worker i is active during K iterations. Considering $\phi(\boldsymbol{v})$ in Eq. (9), the relaxed problem with respect to the problem in Eq. (4) can be expressed as,

$$\begin{aligned} \min \quad & F(\{\boldsymbol{x}_i\}, \{\boldsymbol{y}_i\}, \boldsymbol{v}, \boldsymbol{z}) = \sum_{i=1}^{N} G_i(\boldsymbol{x}_i, \boldsymbol{y}_i) \\ \text{s.t.} \quad & \boldsymbol{x}_i = \boldsymbol{v}, i = 1, \cdots, N \\ & h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \leq \varepsilon \\ \text{var.} \quad & \{\boldsymbol{x}_i\}, \{\boldsymbol{y}_i\}, \boldsymbol{v}, \boldsymbol{z}, \end{aligned} \tag{10}$$

where $\varepsilon > 0$ is a constant. It is easy to check that the Hessian matrix of $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z})$ is a positive semidefinite matrix. Thus we can conclude that $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z})$ is a convex function with respect to $(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z})$. Since the sublevel set of a convex function is convex (Boyd et al., 2004), the feasible set with respect to constraint $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \leq \varepsilon$ is a convex set. In this paper, inspired by the cutting plane method (Boyd & Vandenberghe, 2007; Michalka, 2013; Franc et al., 2011), a set of cutting planes is utilized to approximate the feasible region with respect to constraint $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \leq \varepsilon$ in Eq. (10). The set of cutting planes forms a polytope, let $\boldsymbol{\mathcal{P}}^t$ denotes the polytope in $(t+1)^{\text{th}}$ iteration, which can be expressed as,

$$\boldsymbol{\mathcal{P}}^t = \{\boldsymbol{a}_l{}^\top \boldsymbol{v} + \sum_{i=1}^{N} \boldsymbol{b}_{i,l}{}^\top \boldsymbol{y}_i + \boldsymbol{c}_l{}^\top \boldsymbol{z} + \kappa_l \leq 0,\ l = 1, \cdots, |\boldsymbol{\mathcal{P}}^t|\}, \tag{11}$$

where $\boldsymbol{a}_l \in \mathbb{R}^n$, $\boldsymbol{b}_{i,l} \in \mathbb{R}^m$, $\boldsymbol{c}_l \in \mathbb{R}^m$ and $\kappa_l \in \mathbb{R}^1$ are the parameters in l-th cutting plane, and $|\boldsymbol{\mathcal{P}}^t|$ denotes the number of cutting planes in $\boldsymbol{\mathcal{P}}^t$. Thus, the approximated problem in $(t+1)^{\text{th}}$ iteration can be expressed as follows,

$$\begin{aligned} \min \quad & F(\{\boldsymbol{x}_i\}, \{\boldsymbol{y}_i\}, \boldsymbol{v}, \boldsymbol{z}) = \sum_{i=1}^{N} G_i(\boldsymbol{x}_i, \boldsymbol{y}_i) \\ \text{s.t.} \quad & \boldsymbol{x}_i = \boldsymbol{v}, i = 1, \cdots, N \\ & \boldsymbol{a}_l{}^\top \boldsymbol{v} + \sum_{i=1}^{N} \boldsymbol{b}_{i,l}{}^\top \boldsymbol{y}_i + \boldsymbol{c}_l{}^\top \boldsymbol{z} + \kappa_l \leq 0, l = 1, \cdots, |\boldsymbol{\mathcal{P}}^t| \\ \text{var.} \quad & \{\boldsymbol{x}_i\}, \{\boldsymbol{y}_i\}, \boldsymbol{v}, \boldsymbol{z}, \end{aligned} \tag{12}$$

The Lagrangian function of Eq. (12) can be written as:

$$L_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\}) = \sum_{i=1}^{N} G_i(\boldsymbol{x}_i, \boldsymbol{y}_i) + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \lambda_l \left(\boldsymbol{a}_l{}^\top \boldsymbol{v} + \sum_{i=1}^{N} \boldsymbol{b}_{i,l}{}^\top \boldsymbol{y}_i + \boldsymbol{c}_l{}^\top \boldsymbol{z} + \kappa_l \right) + \sum_{i=1}^{N} \boldsymbol{\theta}_i{}^\top (\boldsymbol{x}_i - \boldsymbol{v}), \tag{13}$$

where $\lambda_l \in \mathbb{R}^1$ and $\boldsymbol{\theta}_i \in \mathbb{R}^n$ are dual variables. Following (Xu et al., 2020), the regularized version of Eq. (13) is employed to update all variables as follows,

$$\widetilde{L}_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\}) = L_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\}) - \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \frac{c_1^t}{2} ||\lambda_l||^2 - \sum_{i=1}^{N} \frac{c_2^t}{2} ||\boldsymbol{\theta}_i||^2, \tag{14}$$

where c_1^t and c_2^t denote the regularization terms in $(t+1)^{\text{th}}$ iteration. In each iteration, we set that $|\boldsymbol{\mathcal{P}}^t| \leq M, \forall t$. $c_1^t = \frac{1}{\eta_\lambda (t+1)^{\frac{1}{4}}} \geq \underline{c}_1$, $c_2^t = \frac{1}{\eta_{\boldsymbol{\theta}} (t+1)^{\frac{1}{4}}} \geq \underline{c}_2$ are two nonnegative non-increasing sequences, where η_λ and $\eta_{\boldsymbol{\theta}}$ are positive constants, and $\underline{c}_1 > 0$, $\underline{c}_2 > 0$ meet $M\underline{c}_1{}^2 + N\underline{c}_2{}^2 \leq \frac{\epsilon}{4}$ ($\epsilon > 0$ is the error-tolerance which will be introduced in Definition 2).

In the proposed ADBO, we solve problem in Eq. (12) in an asynchronous distributed manner. With the same asynchronous setting as described above, we set that the master updates its variables once it receives updates from S active workers every iteration and assume every worker has to communicate with the master at least once every τ iterations. Let $\boldsymbol{\mathcal{Q}}^{t+1}$ denote the index set of active workers in $(t+1)^{\text{th}}$ iteration. In $(t+1)^{\text{th}}$ iteration, the proposed optimization algorithm proceeds as follows,

(1) Each worker updates the local variables as follows,

$$\boldsymbol{x}_i^{t+1} = \begin{cases} \boldsymbol{x}_i^t - \eta_{\boldsymbol{x}} \nabla_{\boldsymbol{x}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^{\hat{t}_i}\},\{\boldsymbol{y}_i^{\hat{t}_i}\},\boldsymbol{v}^{\hat{t}_i},\boldsymbol{z}^{\hat{t}_i},\{\lambda_l^{\hat{t}_i}\},\{\boldsymbol{\theta}_i^{\hat{t}_i}\}), i \in \boldsymbol{\mathcal{Q}}^{t+1} \\ \boldsymbol{x}_i^t, i \notin \boldsymbol{\mathcal{Q}}^{t+1} \end{cases}, \tag{15}$$

$$\boldsymbol{y}_i^{t+1} = \begin{cases} \boldsymbol{y}_i^t - \eta_{\boldsymbol{y}} \nabla_{\boldsymbol{y}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^{\hat{t}_i}\},\{\boldsymbol{y}_i^{\hat{t}_i}\},\boldsymbol{v}^{\hat{t}_i},\boldsymbol{z}^{\hat{t}_i},\{\lambda_l^{\hat{t}_i}\},\{\boldsymbol{\theta}_i^{\hat{t}_i}\}), i \in \boldsymbol{\mathcal{Q}}^{t+1} \\ \boldsymbol{y}_i^t, i \notin \boldsymbol{\mathcal{Q}}^{t+1} \end{cases}, \tag{16}$$

where $\hat{t}_i$ denotes the last iteration during which worker i was active. Then, the active workers transmit the variables $\boldsymbol{x}_i^{t+1}$ and $\boldsymbol{y}_i^{t+1}$ to the master.

(2) After receiving the updates from active workers, the master updates its variables as follows,

$$\boldsymbol{v}^{t+1} = \boldsymbol{v}^t - \eta_{\boldsymbol{v}} \nabla_{\boldsymbol{v}} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}), \tag{17}$$

$$\boldsymbol{z}^{t+1} = \boldsymbol{z}^t - \eta_{\boldsymbol{z}} \nabla_{\boldsymbol{z}} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^t,\{\lambda_l{}^t\},\{\boldsymbol{\theta}_i^t\}), \tag{18}$$

$$\lambda_l^{t+1} = \lambda_l^t + \eta_\lambda \nabla_\lambda \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}), \tag{19}$$

$$\boldsymbol{\theta}_i^{t+1} = \begin{cases} \boldsymbol{\theta}_i^t + \eta_{\boldsymbol{\theta}} \nabla_{\boldsymbol{\theta}} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l{}^{t+1}\},\{\boldsymbol{\theta}_i^t\}), i \in \boldsymbol{\mathcal{Q}}^{t+1} \\ \boldsymbol{\theta}_i^t, i \notin \boldsymbol{\mathcal{Q}}^{t+1} \end{cases}, \tag{20}$$

Next, the master broadcasts $\boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \boldsymbol{\theta}_i^{t+1}$ and $\{\lambda_l^{t+1}\}$ to worker $i, i \in \boldsymbol{\mathcal{Q}}^{t+1}$ (active workers).

Every k_{pre} iterations ($k_{\text{pre}} > 0$ is a pre-set constant, which can be controlled flexibly), the cutting planes are updated based on the following two steps (a) and (b):

(a) Removing the inactive cutting planes and update the dual variable set,

$$\boldsymbol{\mathcal{P}}^{t+1} = \begin{cases} \text{Drop}(\boldsymbol{\mathcal{P}}^t, cp_l), \text{if } \lambda_l^{t+1} \text{ and } \lambda_l^t = 0, l = 1,\cdots,|\boldsymbol{\mathcal{P}}^t| \\ \boldsymbol{\mathcal{P}}^t, \text{if } \lambda_l^{t+1} \neq 0 \text{ or } \lambda_l^t \neq 0, \forall l = 1,\cdots,|\boldsymbol{\mathcal{P}}^t| \end{cases}, \tag{21}$$

where cp_l represents the l-th cutting plane in $\boldsymbol{\mathcal{P}}^t$ and $\text{Drop}(\boldsymbol{\mathcal{P}}^t, cp_l)$ represents removing the l-th cutting plane cp_l from $\boldsymbol{\mathcal{P}}^t$. The dual variable set $\{\lambda^{t+1}\}$ will be updated as follows,

$$\{\lambda^{t+1}\} = \begin{cases} \text{Drop}(\{\lambda^t\}, \lambda_l), \text{if } \lambda_l^{t+1} \text{ and } \lambda_l^t = 0, l = 1,\cdots,|\boldsymbol{\mathcal{P}}^t| \\ \{\lambda^t\}, \text{if } \lambda_l^{t+1} \neq 0 \text{ or } \lambda_l^t \neq 0, \forall l = 1,\cdots,|\boldsymbol{\mathcal{P}}^t| \end{cases}, \tag{22}$$

where $\{\lambda^{t+1}\}$ and $\{\lambda^t\}$ represent the dual variable set in $(t+1)^{\text{th}}$ and t^{th} iterations, respectively . $\text{Drop}(\{\lambda^t\}, \lambda_l)$ represents remove λ_l from the dual variable set $\{\lambda^t\}$.

(b) Adding new cutting planes. Firstly, we investigate whether $(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})$ is feasible for the constraint $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \leq \varepsilon$. We can obtain $h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})$ according to $\phi(\boldsymbol{v}^{t+1})$ in Eq. (9). If $(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})$ is not a feasible solution to the original problem (Eq. (10)), *i.e.*, $h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) > \varepsilon$, new cutting plane cp_{new}^{t+1} will be generated to separate the point $(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})$ from the feasible region of constraint $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \leq \varepsilon$. Thus, the *valid* cutting plane (Boyd & Vandenberghe, 2007) $\boldsymbol{a}_l{}^\top \boldsymbol{v} + \sum_{i=1}^N \boldsymbol{b}_{i,l}{}^\top \boldsymbol{y}_i + \boldsymbol{c}_l{}^\top \boldsymbol{z} + \kappa_l \leq 0$ must satisfy that,

$$\begin{cases} \boldsymbol{a}_l{}^\top \boldsymbol{v} + \sum\limits_{i=1}^N \boldsymbol{b}_{i,l}{}^\top \boldsymbol{y}_i + \boldsymbol{c}_l{}^\top \boldsymbol{z} + \kappa_l \leq 0, \forall (\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \text{ satisfies } h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \leq \varepsilon \\ \boldsymbol{a}_l{}^\top \boldsymbol{v}^{t+1} + \sum\limits_{i=1}^N \boldsymbol{b}_{i,l}{}^\top \boldsymbol{y}_i^{t+1} + \boldsymbol{c}_l{}^\top \boldsymbol{z}^{t+1} + \kappa_l > 0 \end{cases}. \tag{23}$$

Since $h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z})$ is a convex function, we have that,

$$h(\boldsymbol{v}, \{\boldsymbol{y}_i\}, \boldsymbol{z}) \geq h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) + \begin{bmatrix} \frac{\partial h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})}{\partial \boldsymbol{v}} \\ \{\frac{\partial h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})}{\partial \boldsymbol{y}_i}\} \\ \frac{\partial h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})}{\partial \boldsymbol{z}} \end{bmatrix}^\top \left(\begin{bmatrix} \boldsymbol{v} \\ \{\boldsymbol{y}_i\} \\ \boldsymbol{z} \end{bmatrix} - \begin{bmatrix} \boldsymbol{v}^{t+1} \\ \{\boldsymbol{y}_i^{t+1}\} \\ \boldsymbol{z}^{t+1} \end{bmatrix} \right). \tag{24}$$

Combing Eq. (24) with Eq. (23), we have that a valid cutting plane (at point $(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})$) can be expressed as,

$$h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) + \begin{bmatrix} \frac{\partial h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})}{\partial \boldsymbol{v}} \\ \{\frac{\partial h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})}{\partial \boldsymbol{y}_i}\} \\ \frac{\partial h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1})}{\partial \boldsymbol{z}} \end{bmatrix}^\top \left(\begin{bmatrix} \boldsymbol{v} \\ \{\boldsymbol{y}_i\} \\ \boldsymbol{z} \end{bmatrix} - \begin{bmatrix} \boldsymbol{v}^{t+1} \\ \{\boldsymbol{y}_i^{t+1}\} \\ \boldsymbol{z}^{t+1} \end{bmatrix} \right) \leq \varepsilon, \tag{25}$$

For brevity, we utilize cp_{new}^{t+1} to denote the new added cutting plane. Thus the polytope $\boldsymbol{\mathcal{P}}^{t+1}$ will be updated as follows,

$$\boldsymbol{\mathcal{P}}^{t+1} = \begin{cases} \text{Add}(\boldsymbol{\mathcal{P}}^{t+1}, cp_{new}^{t+1}), \text{if } h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) > \varepsilon \\ \boldsymbol{\mathcal{P}}^{t+1}, \text{if } h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) \leq \varepsilon \end{cases}, \tag{26}$$

where $\text{Add}(\boldsymbol{\mathcal{P}}^{t+1}, cp_{new}^{t+1})$ represents that new cutting plane cp_{new}^{t+1} is added to polytope $\boldsymbol{\mathcal{P}}^{t+1}$. The dual variable set $\{\lambda^{t+1}\}$ is updated as follows,

$$\{\lambda^{t+1}\} = \begin{cases} \text{Add}(\{\lambda^{t+1}\}, \lambda_{|\boldsymbol{\mathcal{P}}^{t+1}|}^{t+1}), \text{if } h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) > \varepsilon \\ \{\lambda^{t+1}\}, \text{if } h(\boldsymbol{v}^{t+1}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{z}^{t+1}) \leq \varepsilon \end{cases}, \tag{27}$$

where $\text{Add}(\{\lambda^{t+1}\}, \lambda_{|\boldsymbol{\mathcal{P}}^{t+1}|}^{t+1})$ represents that $\lambda_{|\boldsymbol{\mathcal{P}}^{t+1}|}^{t+1}$ will be added into the dual variable set $\{\lambda^{t+1}\}$. Finally, master broadcasts the updated $\boldsymbol{\mathcal{P}}^{t+1}$ and $\{\lambda_l^{t+1}\}$ to all workers. The details of the proposed algorithm are summarized in Algorithm 1.

Algorithm 1 Asynchronous Distributed Bilevel Optimization

Initialization: master iteration $t=0$, variables $\{\boldsymbol{x}_i^0\}$, $\{\boldsymbol{y}_i^0\}$, $\boldsymbol{v}^0$, $\boldsymbol{z}^0$, $\{\lambda_l^0\}$, $\{\boldsymbol{\theta}_i^0\}$ and polytope $\boldsymbol{\mathcal{P}}^0$.
repeat
  for each worker **do**
    updates variable $\boldsymbol{x}_i^{t+1}$, $\boldsymbol{y}_i^{t+1}$ according to Eq. (15) and (16).
  end for
  Active workers transmit their local variables to master;
  for master **do**
    updates variable $\boldsymbol{v}^{t+1}$, $\boldsymbol{z}^{t+1}$, $\{\lambda_l^{t+1}\}$, $\{\boldsymbol{\theta}_i^{t+1}\}$ according to Eq. (17), (18), (19) and (20);
  end for
  master broadcasts variables to active workers;
  if t mod $k_{\text{pre}} == 0$ **then**
    master computes $\phi(\boldsymbol{v}^{t+1})$ according to Eq. (9);
    master updates $\boldsymbol{\mathcal{P}}^{t+1}$ and $\{\lambda_l^{t+1}\}$ according to Eq. (21), (22), (26) and (27);
    master broadcasts $\boldsymbol{\mathcal{P}}^{t+1}$ and $\{\lambda_l^{t+1}\}$ to all workers;
  end if
  $t = t + 1$;
until termination.

4 Discussion

Theorem 1 ***(Convergence)*** *As the cutting plane continues to be added to the polytope, the optimal objective value of approximate problem in Eq. (12) converges monotonically.*
The proof of Theorem 1 is presented in Appendix C.

Definition 1 ***(Stationarity gap)*** *Following (Xu et al., 2020; Lu et al., 2020), the stationarity gap of our problem at t^{th} iteration is defined as:*

$$\nabla G^t = \begin{bmatrix} \{\nabla_{\boldsymbol{x}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \\ \{\nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \\ \nabla_{\boldsymbol{v}} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}) \\ \nabla_{\boldsymbol{z}} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}) \\ \{\nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \\ \{\nabla_{\boldsymbol{\theta}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \end{bmatrix}. \tag{28}$$

Definition 2 ***(ϵ-stationary point)*** *$(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})$ is an ϵ-stationary point ($\epsilon \geq 0$) of a differentiable function L_p, if $||\nabla G^t|| \leq \epsilon$. $T(\epsilon)$ is the first iteration index such that $||\nabla G^t|| \leq \epsilon$, i.e., $T(\epsilon) = \min\{t \mid ||\nabla G^t|| \leq \epsilon\}$.*

Assumption 1 ***(Smoothness/Gradient Lipschitz)*** *Following (Ji et al., 2021), we assume that L_p has Lipschitz continuous gradients, i.e., for any $\boldsymbol{\omega}, \boldsymbol{\omega}'$, we assume that there exists $L > 0$ satisfying that,*

$$||\nabla L_p(\boldsymbol{\omega}) - \nabla L_p(\boldsymbol{\omega}')|| \leq L||\boldsymbol{\omega} - \boldsymbol{\omega}'||, \tag{29}$$

Assumption 2 ***(Boundedness)*** *Following (Qian et al., 2019), we assume that variables are bounded, i.e., $||\boldsymbol{x}_i||^2 \leq \alpha_1, ||\boldsymbol{v}||^2 \leq \alpha_1, ||\boldsymbol{y}_i||^2 \leq \alpha_2, ||\boldsymbol{z}||^2 \leq \alpha_2, ||\lambda_l||^2 \leq \alpha_3, ||\boldsymbol{\theta}_i||^2 \leq \alpha_4$. And we assume that before obtaining the ε-stationary point (i.e., $t \leq T(\varepsilon) - 1$), the variables in master satisfy that $||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 + \sum_l ||\lambda_l^{t+1} - \lambda_l^t||^2 \geq \vartheta$, where $\vartheta > 0$ is a relative small constant. The change of the variables in master is upper bounded within τ iterations:*

$$||\boldsymbol{v}^t - \boldsymbol{v}^{t-k}||^2 \leq \tau k_1 \vartheta, \quad ||\boldsymbol{z}^t - \boldsymbol{z}^{t-k}||^2 \leq \tau k_1 \vartheta, \quad \textstyle\sum_l ||\lambda_l^t - \lambda_l^{t-k}||^2 \leq \tau k_1 \vartheta, \forall 1 \leq k \leq \tau, \tag{30}$$

where $k_1 > 0$ is a constant.

Theorem 2 ***(Iteration complexity)*** *Suppose Assumption 1 and 2 hold, let $\eta_{\boldsymbol{x}} = \eta_{\boldsymbol{y}} = \eta_{\boldsymbol{v}} = \eta_{\boldsymbol{z}} = \frac{2}{L + \eta_\lambda |\boldsymbol{\mathcal{P}}^t| L^2 + \eta_{\boldsymbol{\theta}} N L^2 + 8(\frac{|\boldsymbol{\mathcal{P}}^t| \gamma L^2}{\eta_\lambda (c_1^t)^2} + \frac{N \gamma L^2}{\eta_{\boldsymbol{\theta}} (c_2^t)^2})}$. And $\eta_\lambda \leq \min\{\frac{2}{L + 2c_1^0}, \frac{1}{15\tau k_1 N L^2}\}$ and $\eta_{\boldsymbol{\theta}} \leq \frac{2}{L + 2c_2^0}$ are positive constants. For a given ϵ, we have:*

$$T(\epsilon) \sim \mathcal{O}(\max\{(\frac{4M\alpha_3}{\eta_\lambda{}^2} + \frac{4N\alpha_4}{\eta_{\boldsymbol{\theta}}{}^2})^2 \frac{1}{\epsilon^2}, (\frac{4(d_7 + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2})^2 (\bar{d} + k_d \tau(\tau - 1)) d_6}{\epsilon} + 2)^2\}), \tag{31}$$

where α_3, α_4, γ, k_d, $\bar{d}$, d_6 and d_7 are constants. The detailed proof is given in Appendix B.

5 EXPERIMENT

In this section, experiments are conducted on two hyperparameter optimization tasks (*i.e.*, data hyper-cleaning task and regularization coefficient optimization task) in the distributed setting to evaluate the performance of the proposed ADBO. The proposed ADBO is compared with the state-of-the-art distributed bilevel optimization method FEDNEST (Tarzanagh et al., 2022). In data hyper-cleaning task, experiments are carried out on MNIST (LeCun et al., 1998) and Fashion MNIST (Xiao et al., 2017) datasets. In coefficient optimization task, following (Chen et al., 2022a), experiments are conducted on Covertype (Blackard & Dean, 1999) and IJCNN1 (Prokhorov, 2001) datasets.

5.1 DATA HYPER-CLEANING

We compare the performance of the proposed ADBO and distributed bilevel optimization FEDNEST on the distributed data hyper-cleaning task (Chen et al., 2022b) on MNIST and Fashion MNIST datasets. Data hyper-cleaning involves training a classifier in a contaminated environment where each training data label is changed to a random class number with a probability (*i.e.*, the corruption rate). In the experiment, the distributed data hyper-cleaning problem is considered, whose formulation can be expressed as,

$$\begin{aligned}
&\min\ F(\boldsymbol{\psi},\boldsymbol{w}) = \sum_{i=1}^{N} \tfrac{1}{|\mathcal{D}_i^{\mathrm{val}}|} \sum_{(\mathbf{x}_j,y_j)\in\mathcal{D}_i^{\mathrm{val}}} \mathcal{L}(\mathbf{x}_j{}^\top \boldsymbol{w}, y_j) \\
&\text{s.t. } \boldsymbol{w} = \arg\min_{\boldsymbol{w}'} f(\boldsymbol{\psi},\boldsymbol{w}') = \sum_{i=1}^{N} \tfrac{1}{|\mathcal{D}_i^{\mathrm{tr}}|} \sum_{(\mathbf{x}_j,y_j)\in\mathcal{D}_i^{\mathrm{tr}}} \sigma(\psi_j)\mathcal{L}(\mathbf{x}_j{}^\top \boldsymbol{w}', y_j) + C_r||\boldsymbol{w}'||^2 \\
&\text{var.} \qquad \boldsymbol{\psi}, \boldsymbol{w},
\end{aligned} \tag{32}$$

where $\mathcal{D}_i^{\mathrm{val}}$ and $\mathcal{D}_i^{\mathrm{tr}}$ represent the training and validation datasets on i^{th} worker, respectively. $(\mathbf{x}_j, y_j)$ denote the jth data and label. $\sigma(.)$ is the sigmoid function, $\mathcal{L}$ is the cross-entropy loss, C_r is a regularization parameter and N is the number of workers in the distributed system. In MNIST and Fashion MNIST datasets, we set $N = 18$ and $S = 9$. According to Cohen et al. (2021), we assume that the communication delay of each worker obeys the heavy-tailed distribution. The proposed ADBO is compared with the state-of-the-art distributed bilevel optimization method FEDNEST and SDBO (Synchronous Distributed Bilevel Optimization, *i.e.*, ADBO without asynchronous setting). The test accuracy versus time is shown in Figure 1, and the test loss versus time is shown in Figure 2. In both figures, we notice that SDBO is relatively more efficient than FEDNEST since each step in SDBO only requires gradient descent/ascent, which is computationally efficient. We can also observe that the proposed ADBO is the most efficient algorithm since 1) the asynchronous setting is considered in ADBO, the master can update its variables once it receives updates from S active workers instead of all workers; and 2) ADBO is a single-loop algorithm and only gradient descent/ascent is required in each iteration, thus ADBO is computationally more efficient.

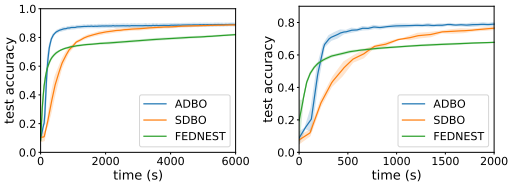
5.2 REGULARIZATION COEFFICIENT OPTIMIZATION

Following (Chen et al., 2022a), we compare the proposed ABDO with baseline algorithms (FEDNEST and SDBO) on the regularization coefficient optimization task with Covertype and IJCNN1 datasets. The distributed regularization coefficient optimization problem is given by,

$$\begin{aligned}
&\min\ F(\boldsymbol{\psi},\boldsymbol{w}) = \sum_{i=1}^{N} \tfrac{1}{|\mathcal{D}_i^{\mathrm{val}}|} \sum_{(\mathbf{x}_j,y_j)\in\mathcal{D}_i^{\mathrm{val}}} \mathcal{L}(\mathbf{x}_j{}^\top \boldsymbol{w}, y_j) \\
&\text{s.t. } \boldsymbol{w} = \arg\min_{\boldsymbol{w}'} f(\boldsymbol{\psi},\boldsymbol{w}') = \sum_{i=1}^{N} \tfrac{1}{|\mathcal{D}_i^{\mathrm{tr}}|} \sum_{(\mathbf{x}_j,y_j)\in\mathcal{D}_i^{\mathrm{tr}}} \mathcal{L}(\mathbf{x}_j{}^\top \boldsymbol{w}', y_j) + \sum_{j=1}^{n} \psi_j (w_j')^2 \\
&\text{var.} \qquad \boldsymbol{\psi}, \boldsymbol{w},
\end{aligned} \tag{33}$$

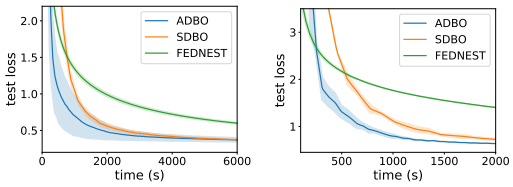
where $\boldsymbol{\psi} \in \mathbb{R}^n$, $\boldsymbol{w} \in \mathbb{R}^n$, $\boldsymbol{w}' = [w_1', \ldots, w_n']$, and $\mathcal{L}$ denote the regularization coefficient, model parameter, and the logistic loss, respectively. In Covertype and IJCNN1 datasets, we set $N = 18$, $S = 9$ and $N = 24$, $S = 12$, respectively. We also assume that the delay of each worker obeys the heavy-tailed distribution. Firstly, we compare the performance of the proposed ADBO, SDBO and FEDNEST in terms of test accuracy and test loss on Covertype and IJCNN1 dataset, which are

shown in Figure 3 and 4. It is seen that the proposed ADBO is most efficient because of the same two reasons we gave in Section 5.1.



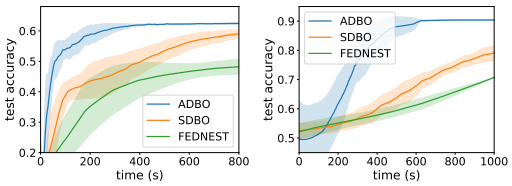
(a) MNIST (b) Fashion MNIST

Figure 1: Test accuracy vs time on (a) MNIST and (b) Fashion MNIST datasets.



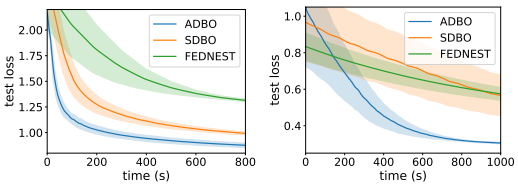
(a) MNIST (b) Fashion MNIST

Figure 2: Test loss vs time on (a) MNIST and (b) Fashion MNIST datasets.



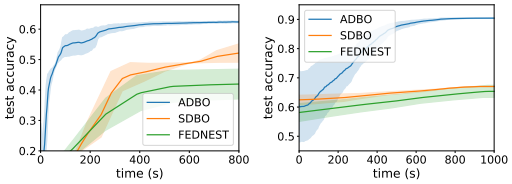
(a) Covertype (b) IJCNN1

Figure 3: Test accuracy vs time on (a) Covertype and (b) IJCNN1 datasets.



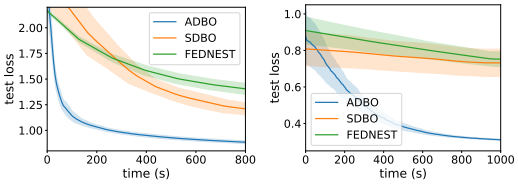
(a) Covertype (b) IJCNN1

Figure 4: Test loss vs time on (a) Covertype and (b) IJCNN1 datasets.



(a) Covertype (b) IJCNN1

Figure 5: Test accuracy vs time on (a) Covertype and (b) IJCNN1 datasets when there are stragglers in distributed system.



(a) Covertype (b) IJCNN1

Figure 6: Test loss vs time on (a) Covertype and (b) IJCNN1 datasets when there are stragglers in distributed system.

We also consider the straggler problem, *i.e.*, there exist workers with high delays (stragglers) in the distributed system. In this case, the efficiency of the bilevel optimization method with the synchronous distributed setting will be affected heavily. In the experiment, we assume there are three stragglers in the distributed system, and the mean of (communication + computation) delay of stragglers is four times the delay of normal workers. The results on Covertype and IJCNN1 datasets are reported in Figure 5 and 6. It is seen that the efficiency of the synchronous distributed algorithms (FEDNEST and SDBO) will significantly be affected, while the proposed ADBO does not suffer from the straggler problem since it is an asynchronous method and is able to only consider active workers.

6 Conclusion

Existing bilevel optimization works focus either on the centralized or synchronous distributed setting, which will give rise to data privacy risks and suffer from the straggler problem. As a remedy, we propose ADBO in this paper to solve the bilevel optimization problem in an asynchronous distributed manner. To our best knowledge, this is the first work that devises the asynchronous distributed algorithm for bilevel optimization. We demonstrate that the proposed ADBO can effectively tackle bilevel optimization problems with both nonconvex upper-level and lower-level objective functions. Theoretical analysis has also been conducted to analyze the convergence properties and iteration complexity of ADBO. Extensive empirical studies on real-world datasets demonstrate the efficiency and effectiveness of the proposed ADBO.

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A CUTTING PLANE METHOD FOR BILEVEL OPTIMIZATION

In this section, we introduce the proposed Cutting Plane method for Bileve Optimization (CPBO) in centralized setting. Defining $\phi(\boldsymbol{x}) = \arg\min_{\boldsymbol{y}'} f(\boldsymbol{x}, \boldsymbol{y}')$ and $h(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{y} - \phi(\boldsymbol{x})||_1$, we can reformulate problem in Eq. (1) as:

$$\begin{array}{ll} \min & F(\boldsymbol{x}, \boldsymbol{y}) \\ \text{s.t.} & h(\boldsymbol{x}, \boldsymbol{y}) = 0 \\ \text{var.} & \boldsymbol{x}, \boldsymbol{y}. \end{array} \tag{34}$$

Following the previous works (Li et al., 2022; Gould et al., 2016; Yang et al., 2021) in bilevel optimization, it is not necessary to get the exact $\phi(\boldsymbol{v})$, and the approximate $\phi(\boldsymbol{v})$ is given as follows. Firstly, as many work do (Ji et al., 2021; Yang et al., 2021), we utilize the K steps of gradient descent (GD) to approximate $\phi(\boldsymbol{x})$. And the first-order Taylor approximation of $f(\boldsymbol{x}, \boldsymbol{y}')$ with respect to $\boldsymbol{x}$

is considered, i.e., for a given point $\overline{\boldsymbol{x}}$, $\widetilde{f}(\boldsymbol{x},\boldsymbol{y}')=f(\overline{\boldsymbol{x}},\boldsymbol{y}')+\nabla_{\boldsymbol{x}}f(\overline{\boldsymbol{x}},\boldsymbol{y}')^{\top}(\boldsymbol{x}-\overline{\boldsymbol{x}})$. Thus, we have, $\phi(\boldsymbol{x})=\boldsymbol{y}_0'-\sum_{k=0}^{K-1}\eta\nabla_{\boldsymbol{y}}\widetilde{f}(\boldsymbol{x},\boldsymbol{y}_k')$. The Hessian matrix of $h(\boldsymbol{x},\boldsymbol{y})$ can be expressed as,

$$\boldsymbol{H}=\begin{bmatrix} \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_1\partial \boldsymbol{x}_1} & \cdots & \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_1\partial \boldsymbol{x}_n} & \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_1\partial \boldsymbol{y}_1} & \cdots & \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_1\partial \boldsymbol{y}_m} \\ \vdots & \ddots & & & & \vdots \\ \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_n\partial \boldsymbol{x}_1} & & \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{x}_n\partial \boldsymbol{x}_n} & & & \\ \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{y}_1\partial \boldsymbol{x}_1} & & & \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{y}_1\partial \boldsymbol{y}_1} & & \\ \vdots & & & & \ddots & \vdots \\ \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{y}_m\partial \boldsymbol{x}_1} & \cdots & & & \cdots & \frac{\partial h(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{y}_m\partial \boldsymbol{y}_m} \end{bmatrix} \tag{35}$$

It is easy to see that $\boldsymbol{H}=\boldsymbol{0}\in\mathbb{R}^{(m+n)\times(m+n)}$ is positive semidefinite matrix, and we can conclude that $h(\boldsymbol{x},\boldsymbol{y})$ is a convex function with respect to $(\boldsymbol{x},\boldsymbol{y})$. And the relaxed problem of problem in Eq. (34) is considered as follows,

$$\begin{aligned} \min \quad & F(\boldsymbol{x},\boldsymbol{y}) \\ \text{s.t.} \quad & h(\boldsymbol{x},\boldsymbol{y})\leq\varepsilon \\ \text{var.} \quad & \boldsymbol{x},\boldsymbol{y}. \end{aligned} \tag{36}$$

Since the sublevel set of a convex function is a convex set (Boyd et al., 2004), we have that the feasible set of $(\boldsymbol{x},\boldsymbol{y})$, i.e.,

$$\boldsymbol{Z}^{relax}=\{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^n\times\mathbb{R}^m|h(\boldsymbol{x},\boldsymbol{y})\leq\epsilon\}, \tag{37}$$

is a convex set. This paper utilizes a set of cutting plane constraints (i.e., linear constraints) to approximate the feasible set $\boldsymbol{Z}^{relax}$. The set of cutting plane constraints form a polytope, which can be expressed as follows,

$$\boldsymbol{\mathcal{P}}=\{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^n\times\mathbb{R}^m|\boldsymbol{a}_l{}^{\top}\boldsymbol{x}+\boldsymbol{b}_l{}^{\top}\boldsymbol{y}+\kappa_l\leq 0,\ l=1,\cdots,L\}, \tag{38}$$

where $\boldsymbol{a}_l\in\mathbb{R}^n$, $\boldsymbol{b}_l\in\mathbb{R}^m$ and $\kappa_l\in\mathbb{R}^1$ are parameters in l-th cutting plane, and L represents the number of cutting planes in $\boldsymbol{\mathcal{P}}$. Considering the approximate problem, which can be expressed as follows,

$$\begin{aligned} \min \quad & F(\boldsymbol{x},\boldsymbol{y}) \\ \text{s.t.} \quad & \boldsymbol{a}_l{}^{\top}\boldsymbol{x}+\boldsymbol{b}_l{}^{\top}\boldsymbol{y}+\kappa_l\leq 0,\ l=1,\cdots,|\boldsymbol{\mathcal{P}}^t| \\ \text{var.} \quad & \boldsymbol{x},\boldsymbol{y}, \end{aligned} \tag{39}$$

where $\boldsymbol{\mathcal{P}}^t$ is the polytope in $(t+1)^{\text{th}}$ iteration, and $|\boldsymbol{\mathcal{P}}^t|$ denotes the number of cutting planes in $\boldsymbol{\mathcal{P}}^t$. Then, the Lagrangian function of Eq. (39) can be written as,

$$L_p(\boldsymbol{x},\boldsymbol{y},\{\lambda_l\})=F(\boldsymbol{x},\boldsymbol{y})+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}\lambda_l(\boldsymbol{a}_l{}^{\top}\boldsymbol{x}+\boldsymbol{b}_l{}^{\top}\boldsymbol{y}+\kappa_l). \tag{40}$$

The proposed algorithm proceeds as follows in $(t+1)^{\text{th}}$ iteration:

If $t+1<T_1$, the variables are updated as follows,

$$\boldsymbol{x}^{t+1}=\boldsymbol{x}^t-\eta_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}L_p(\boldsymbol{x}^t,\boldsymbol{y}^t,\{\lambda_l^t\}), \tag{41}$$

$$\boldsymbol{y}^{t+1}=\boldsymbol{y}^t-\eta_{\boldsymbol{y}}\nabla_{\boldsymbol{y}}L_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^t,\{\lambda_l^t\}), \tag{42}$$

$$\lambda_l^{t+1}=\lambda_l^t+\eta_{\lambda_l}\nabla_{\lambda_l}L_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^{t+1},\{\lambda_l^t\}),\ l=1,\cdots,|\boldsymbol{\mathcal{P}}^t|, \tag{43}$$

where $\eta_{\boldsymbol{x}}$, $\eta_{\boldsymbol{y}}$ and η_{λ_l} represent the step-sizes.

And every k_{pre} iterations ($k_{\text{pre}}>0$ is a pre-set constant, which can be controlled flexibly) the cutting planes will be updated based on the following two steps:

(a) Removing the inactive cutting planes, that is,

$$\boldsymbol{\mathcal{P}}^{t+1}=\begin{cases} \text{Drop}(\boldsymbol{\mathcal{P}}^t,cp_l),\text{if } \lambda_l^{t+1} \text{ and } \lambda_l^t=0,l=1,\cdots,|\boldsymbol{\mathcal{P}}^t| \\ \boldsymbol{\mathcal{P}}^t,\text{if } \lambda_l^{t+1}\neq 0 \text{ or } \lambda_l^t\neq 0,\forall l=1,\cdots,|\boldsymbol{\mathcal{P}}^t| \end{cases}, \tag{44}$$

where cp_l represents the l-th cutting plane in $\boldsymbol{\mathcal{P}}^t$ and $\mathrm{Drop}(\boldsymbol{\mathcal{P}}^t, cp_l)$ represents removing the l-th cutting plane cp_l from $\boldsymbol{\mathcal{P}}^t$. And the dual variable set $\{\lambda^t\}$ will be updated as follows,

$$\{\lambda^{t+1}\}=\begin{cases} \mathrm{Drop}(\{\lambda^t\}, \lambda_l^{t+1}), \text{if } \lambda_l^{t+1}=0, l=1,\cdots,|\boldsymbol{\mathcal{P}}^t| \\ \{\lambda^t\}, \forall \lambda_l^{t+1} \neq 0, l=1,\cdots,|\boldsymbol{\mathcal{P}}^t| \end{cases}, \tag{45}$$

where $\{\lambda^{t+1}\}$ and $\{\lambda^t\}$ respectively represent the dual variable set in $(t+1)^{\mathrm{th}}$ and t^{th} iteration. And $\mathrm{Drop}(\{\lambda^t\}, \lambda_l^{t+1})$ represents remove λ_l^{t+1} from the dual variable set $\{\lambda^t\}$.

(b) Adding new cutting planes. Firstly, we investigate whether $(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})$ is a feasible solution to the original problem in Eq. (36). If $(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})$ is not a feasible solution to the original problem, that is $h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) > \varepsilon$, new cutting plane is generated to separate the point $(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})$ from $\boldsymbol{Z}^{relax}$, that is, the *valid* cutting plane $\boldsymbol{a}_l{}^\top \boldsymbol{x}+\boldsymbol{b}_l{}^\top \boldsymbol{y}+\kappa_l \leq 0$ must satisfy that,

$$\begin{cases} \boldsymbol{a}_l{}^\top \boldsymbol{x}+\boldsymbol{b}_l{}^\top \boldsymbol{y}+\kappa_l \leq 0, \forall (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{Z}^{relax} \\ \boldsymbol{a}_l{}^\top \boldsymbol{x}^{t+1}+\boldsymbol{b}_l{}^\top \boldsymbol{y}^{t+1}+\kappa_l > 0 \end{cases}, \tag{46}$$

Since $h(\boldsymbol{x}, \boldsymbol{y})$ is a convex function, we have that,

$$h(\boldsymbol{x}, \boldsymbol{y}) \geq h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) + \begin{bmatrix} \frac{\partial h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})}{\partial \boldsymbol{x}} \\ \frac{\partial h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})}{\partial \boldsymbol{y}} \end{bmatrix}^\top \left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} - \begin{bmatrix} \boldsymbol{x}^{t+1} \\ \boldsymbol{y}^{t+1} \end{bmatrix} \right). \tag{47}$$

According to Eq. (47), $h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) + \begin{bmatrix} \frac{\partial h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})}{\partial \boldsymbol{x}} \\ \frac{\partial h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})}{\partial \boldsymbol{y}} \end{bmatrix}^\top \left(\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} - \begin{bmatrix} \boldsymbol{x}^{t+1} \\ \boldsymbol{y}^{t+1} \end{bmatrix} \right) \leq \varepsilon$ is a valid cutting plane at point $(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1})$ which satisfies Eq. (46). For brevity, we utilize cp_{new}^{t+1} to denote this cutting plane. Thus, we have that,

$$\boldsymbol{\mathcal{P}}^{t+1} = \begin{cases} \mathrm{Add}(\boldsymbol{\mathcal{P}}^{t+1}, cp_{new}^{t+1}), \text{if } h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) > \varepsilon \\ \boldsymbol{\mathcal{P}}^{t+1}, \text{if } h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) \leq \varepsilon \end{cases}, \tag{48}$$

where $\mathrm{Add}(\boldsymbol{\mathcal{P}}^{t+1}, cp_{new}^{t+1})$ represents that new cutting plane cp_{new}^{t+1} is added to polytope $\boldsymbol{\mathcal{P}}^{t+1}$. And the dual variable set is updated as follows,

$$\{\lambda^{t+1}\} = \begin{cases} \mathrm{Add}(\{\lambda^{t+1}\}, \lambda_{|\boldsymbol{\mathcal{P}}^{t+1}|}^{t+1}), \text{if } h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) > \varepsilon \\ \{\lambda^{t+1}\}, \text{if } h(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) \leq \varepsilon \end{cases}, \tag{49}$$

Else if $t+1 \geq T_1$, the polytope $\boldsymbol{\mathcal{P}}^{T_1}$ and dual variables will be fixed. Variables $\boldsymbol{x}, \boldsymbol{y}$ will be updated as follows,

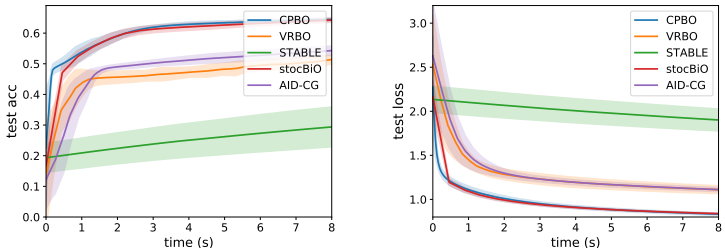
$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta_{\boldsymbol{x}} \nabla_{\boldsymbol{x}} \hat{L}_p(\boldsymbol{x}^t, \boldsymbol{y}^t), \tag{50}$$

$$\boldsymbol{y}^{t+1} = \boldsymbol{y}^t - \eta_{\boldsymbol{y}} \nabla_{\boldsymbol{y}} \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t), \tag{51}$$

where $\hat{L}_p(\boldsymbol{x}, \boldsymbol{y}) = F(\boldsymbol{x}, \boldsymbol{y}) + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^{T_1}|} \lambda_l \max\{0, \boldsymbol{a}_l{}^\top \boldsymbol{x}+\boldsymbol{b}_l{}^\top \boldsymbol{y}+\kappa_l\}$. And details of the algorithm are summarized in Algorithm 2. The comparison about the convergence results between the proposed method and state-of-the-art are summarized in Table 1.

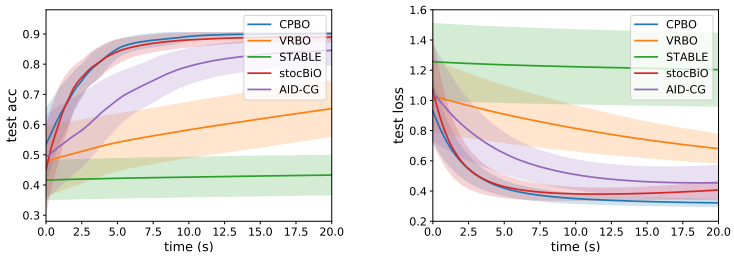
A.1 Experiment

To evaluate the performance of the proposed CPBO, we compare CPBO with baseline algorithms stocBio (Ji et al., 2021), STABLE (Chen et al., 2022a) and VRBO (Yang et al., 2021)), AID-CG (Grazzi et al., 2020) on the regularization coefficient optimization task (Chen et al., 2022a) on Covertype (Blackard & Dean, 1999) and IJCNN1 (Prokhorov, 2001) datasets. We compare the performance of the proposed CPBO with all competing algorithms in terms of both the test accuracy and the test loss in Figure 7 and 8. It is seen that the proposed CPBO can achieve relatively fast convergence rate among all competing algorithms since 1) the iteration complexity of the proposed method is not high 2) every step in CPBO is computationally efficient.



(a) test accuracy vs time (b) test loss vs time

Figure 7: Comparison of (a) test accuracy vs time , (b) test loss vs time on Covertype dataset.



(a) test accuracy vs time (b) test loss vs time

Figure 8: Comparison of (a) test accuracy vs time , (b) test loss vs time on IJCNN1 dataset.

A.2 Discussion

Definition A.1 *$(\boldsymbol{x}, \boldsymbol{y})$ is an ϵ-stationary point of a differentiable function $\hat{L}_p$, if $||\nabla_{\boldsymbol{x}} \hat{L}_p(\boldsymbol{x}, \boldsymbol{y})||^2 + ||\nabla_{\boldsymbol{y}} \hat{L}_p(\boldsymbol{x}, \boldsymbol{y})||^2 \leq \epsilon$.*

Assumption A.1 ***(Smoothness/Gradient Lipschitz)*** *Following (Ji et al., 2021), we assume that L_p has Lipschitz continuous gradients, i.e., for any $\boldsymbol{\omega}, \boldsymbol{\omega}'$, we assume that there exists $L > 0$ satisfying that,*

$$||\nabla \hat{L}_p(\boldsymbol{\omega}) - \nabla \hat{L}_p(\boldsymbol{\omega}')|| \leq L||\boldsymbol{\omega} - \boldsymbol{\omega}'||, \tag{52}$$

Assumption A.2 ***(Boundedness)*** *Following (Qian et al., 2019), we assume that variables have boundedness, i.e., $||\boldsymbol{x}||^2 \leq \beta_1$, $||\boldsymbol{y}||^2 \leq \beta_2$.*

Theorem 3 ***(Iteration Complexity)*** *Under Assumption1 and setting the stepsizes as $\eta_{\boldsymbol{x}} < \frac{2}{L}, \eta_{\boldsymbol{y}} < \frac{2}{L}$, the iteration complexity (also the gradient complexity) of the proposed algorithm to obtain ϵ-stationary point is bounded by $\mathcal{O}(\frac{1}{\epsilon})$.*

Proof of Theorem 3:

According to the Assumption A.1, when $t > T_1$, we have,

$$\begin{aligned} \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t) &\leq \hat{L}_p(\boldsymbol{x}^t, \boldsymbol{y}^t) + \left\langle \nabla_{\boldsymbol{x}} \hat{L}_p(\boldsymbol{x}^t, \boldsymbol{y}^t), \boldsymbol{x}^{t+1} - \boldsymbol{x}^t \right\rangle + \tfrac{L}{2}||\boldsymbol{x}^{t+1} - \boldsymbol{x}^t|| \\ &\leq \hat{L}_p(\boldsymbol{x}^t, \boldsymbol{y}^t) - \eta_{\boldsymbol{x}}||\nabla_{\boldsymbol{x}} \hat{L}_p(\boldsymbol{x}^t, \boldsymbol{y}^t)||^2 + \tfrac{L{\eta_{\boldsymbol{x}}}^2}{2}||\nabla_{\boldsymbol{x}} \hat{L}_p(\boldsymbol{x}^t, \boldsymbol{y}^t)||^2. \end{aligned} \tag{53}$$

Similarly, according to the Assumption A.1, we have,

$$\begin{aligned} \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^{t+1}) &\leq \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t) + \left\langle \nabla_{\boldsymbol{y}} \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t), \boldsymbol{y}^{t+1} - \boldsymbol{y}^t \right\rangle + \tfrac{L}{2}||\boldsymbol{y}^{t+1} - \boldsymbol{y}^t|| \\ &\leq \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t) - \eta_{\boldsymbol{y}}||\nabla_{\boldsymbol{y}} \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t)||^2 + \tfrac{L{\eta_{\boldsymbol{y}}}^2}{2}||\nabla_{\boldsymbol{y}} \hat{L}_p(\boldsymbol{x}^{t+1}, \boldsymbol{y}^t)||^2. \end{aligned} \tag{54}$$

Algorithm 2 Customized Cutting Plane Method for Bilevel Optimization

Initialization: iteration $t = 0$, variables $\boldsymbol{x}^0$, $\boldsymbol{y}^0$, $\{\lambda_l^0\}$ and polytope $\boldsymbol{\mathcal{P}}^0$.
repeat
  if $t < T_1$ **then**
    updating variable $\boldsymbol{x}^{t+1}$, $\boldsymbol{y}^{t+1}$ and λ_l^{t+1} according to Eq. (41), (42) and (43);
    if t mod $k_{\text{pre}} == 0$ **then**
      updating the polytope $\boldsymbol{\mathcal{P}}^{t+1}$ according to Eq.(44) and Eq.(48);
      updating the dual variable set $\{\lambda^{t+1}\}$ according to Eq.(45) and Eq.(49);
    end if
  else
    updating variable $\boldsymbol{x}^{t+1}$ and $\boldsymbol{y}^{t+1}$ according to Eq. (50) and (51);
  end if
  $t = t + 1$;
until termination.

Combing Eq. (53) and Eq. (54), we have,

$$(\eta_{\boldsymbol{x}}-\frac{L{\eta_{\boldsymbol{x}}}^2}{2})||\nabla_{\boldsymbol{x}}\hat{L}_p(\boldsymbol{x}^t,\boldsymbol{y}^t)||^2+(\eta_{\boldsymbol{y}}-\frac{L{\eta_{\boldsymbol{y}}}^2}{2})||\nabla_{\boldsymbol{y}}\hat{L}_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^t)||^2 \leq \hat{L}_p(\boldsymbol{x}^t,\boldsymbol{y}^t)-\hat{L}_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^{t+1}). \tag{55}$$

According to the setting of $\eta_{\boldsymbol{x}}$, $\eta_{\boldsymbol{y}}$, we have $\eta_{\boldsymbol{x}} - \frac{L{\eta_{\boldsymbol{x}}}^2}{2} > 0$, $\eta_{\boldsymbol{y}}$, we have $\eta_{\boldsymbol{y}} - \frac{L{\eta_{\boldsymbol{y}}}^2}{2} > 0$. And we set that $d = \min\{\eta_{\boldsymbol{x}} - \frac{L{\eta_{\boldsymbol{x}}}^2}{2}, \eta_{\boldsymbol{y}} - \frac{L{\eta_{\boldsymbol{y}}}^2}{2}\}$, thus we can obtain that,

$$||\nabla_{\boldsymbol{x}}\hat{L}_p(\boldsymbol{x}^t,\boldsymbol{y}^t)||^2 + ||\nabla_{\boldsymbol{y}}\hat{L}_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^t)||^2 \leq \frac{\hat{L}_p(\boldsymbol{x}^t,\boldsymbol{y}^t) - \hat{L}_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^{t+1})}{d}. \tag{56}$$

Summing both sides of Eq. (56) for $t = \{T_1, \cdots, T-1\}$, we obtain that,

$$\frac{1}{T-T_1}\sum_{t=T_1}^{T-1}(||\nabla_{\boldsymbol{x}}\hat{L}_p(\boldsymbol{x}^t,\boldsymbol{y}^t)||^2 + ||\nabla_{\boldsymbol{y}}\hat{L}_p(\boldsymbol{x}^{t+1},\boldsymbol{y}^t)||^2) \leq \frac{\hat{L}_p(\boldsymbol{x}^{T_1},\boldsymbol{y}^{T_1}) - \hat{L}_p^*}{(T-T_1)d}, \tag{57}$$

where $\hat{L}_p^* = \min \hat{L}_p(\boldsymbol{x},\boldsymbol{y})$. This implies that the number of iterations required by Algorithm 2 to return an ϵ-stationary point is bounded by

$$\mathcal{O}(\frac{d}{\hat{L}_p(\boldsymbol{x}^{T_1},\boldsymbol{y}^{T_1}) - \hat{L}_p^*}\frac{1}{\epsilon} + T_1). \tag{58}$$

B Proof of Theorem 2

In this section, we provide complete proofs for Theorem 2. Firstly, we make some definitions about our problem.

Definition B.1 *The stationarity gap at t^{th} iteration is defined as:*

$$\nabla G^t = \begin{bmatrix} \{\nabla_{\boldsymbol{x}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \\ \{\nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \\ \nabla_{\boldsymbol{v}} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}) \\ \nabla_{\boldsymbol{z}} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}) \\ \{\nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \\ \{\nabla_{\boldsymbol{\theta}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\} \end{bmatrix}. \tag{59}$$

And we also define:

$$
\begin{aligned}
(\nabla G^t)_{\boldsymbol{x}_i} &= \nabla_{\boldsymbol{x}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla G^t)_{\boldsymbol{y}_i} &= \nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla G^t)_{\boldsymbol{v}} &= \nabla_{\boldsymbol{v}} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla G^t)_{\boldsymbol{z}} &= \nabla_{\boldsymbol{z}} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla G^t)_{\lambda_l} &= \nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla G^t)_{\boldsymbol{\theta}_i} &= \nabla_{\boldsymbol{\theta}_i} L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}).
\end{aligned}
\tag{60}
$$

It follows that,

$$
||\nabla G^t||^2 = \sum_{i=1}^{N}(||(\nabla G^t)_{\boldsymbol{x}_i}||^2 + ||(\nabla G^t)_{\boldsymbol{y}_i}||^2 + ||(\nabla G^t)_{\boldsymbol{\theta}_i}||^2) + ||(\nabla G^t)_{\boldsymbol{v}}||^2 + ||(\nabla G^t)_{\boldsymbol{z}}||^2 + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||(\nabla G^t)_{\lambda_l}||^2. \tag{61}
$$

Definition B.2 *At t^{th} iteration, the stationarity gap w.r.t $\widetilde{L}_p(\{\boldsymbol{x}_i\},\boldsymbol{z},h,\{\lambda_l\},\{\boldsymbol{\theta}_i\})$ is defined as:*

$$
\nabla \widetilde{G}^t = \begin{bmatrix}
\{\nabla_{\boldsymbol{x}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\}\\
\{\nabla_{\boldsymbol{y}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\}\\
\nabla_{\boldsymbol{v}} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
\nabla_{\boldsymbol{z}} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
\{\nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\}\\
\{\nabla_{\boldsymbol{\theta}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\}
\end{bmatrix}. \tag{62}
$$

We further define:

$$
\begin{aligned}
(\nabla \widetilde{G}^t)_{\boldsymbol{x}_i} &= \nabla_{\boldsymbol{x}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla \widetilde{G}^t)_{\boldsymbol{y}_i} &= \nabla_{\boldsymbol{y}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla \widetilde{G}^t)_{\boldsymbol{v}} &= \nabla_{\boldsymbol{v}} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla \widetilde{G}^t)_{\boldsymbol{z}} &= \nabla_{\boldsymbol{z}} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla \widetilde{G}^t)_{\lambda_l} &= \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\\
(\nabla \widetilde{G}^t)_{\boldsymbol{\theta}_i} &= \nabla_{\boldsymbol{\theta}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}).
\end{aligned}
\tag{63}
$$

It follows that,

$$
||\nabla \widetilde{G}^t||^2 = \sum_{i=1}^{N}(||(\nabla \widetilde{G}^t)_{\boldsymbol{x}_i}||^2 + ||(\nabla \widetilde{G}^t)_{\boldsymbol{y}_i}||^2 + ||(\nabla \widetilde{G}^t)_{\boldsymbol{\theta}_i}||^2) + ||(\nabla \widetilde{G}^t)_{\boldsymbol{v}}||^2 + ||(\nabla \widetilde{G}^t)_{\boldsymbol{z}}||^2 + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||(\nabla \widetilde{G}^t)_{\lambda_l}||^2. \tag{64}
$$

Definition B.3 *In our asynchronous algorithm, for the worker i in t^{th} iteration, we define the last iteration where worker i was active as $\hat{t}_i$. And we define the next iteration that worker i will be active as $\overline{t_i}$. For the iteration index set that worker i is active during $T+\tau$ iteration, we define it as $\mathcal{V}_i(T)$. And the jth element in $\mathcal{V}_i(T)$ is defined as $\hat{v}_i(j)$.*

Tnen, we provide some useful lemmas used for proving the main convergence results in Theorem 2.

Lemma 1 *Suppose Assumption 1 and 2 hold, we can obtain that,*

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq \sum_{i=1}^{N}(\tfrac{L+1}{2}-\tfrac{1}{\eta_{\boldsymbol{x}}})||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+\sum_{i=1}^{N}(\tfrac{L+1}{2}-\tfrac{1}{\eta_{\boldsymbol{y}}})||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2+3NL^2\tau k_1\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2\\
&+(\tfrac{L+6NL^2\tau k_1}{2}-\tfrac{1}{\eta_{\boldsymbol{v}}})||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+(\tfrac{L+6NL^2\tau k_1}{2}-\tfrac{1}{\eta_{\boldsymbol{z}}})||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2
\end{aligned} \tag{65}$$

Proof of Lemma 1:

Utilizing the Lipschitz properties in Assumption 1, we can obtain that,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_1^{t+1},\boldsymbol{x}_2^t,\cdots,\boldsymbol{x}_N^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq \left\langle\nabla_{\boldsymbol{x}_1}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{x}_1^{t+1}-\boldsymbol{x}_1^t\right\rangle+\tfrac{L}{2}||\boldsymbol{x}_1^{t+1}-\boldsymbol{x}_1^t||^2,\\
&L_p(\{\boldsymbol{x}_1^{t+1},\boldsymbol{x}_2^{t+1},\cdots,\boldsymbol{x}_N^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_1^{t+1},\boldsymbol{x}_2^t,\cdots,\boldsymbol{x}_N^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq \left\langle\nabla_{\boldsymbol{x}_2}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{x}_2^{t+1}-\boldsymbol{x}_2^t\right\rangle+\tfrac{L}{2}||\boldsymbol{x}_2^{t+1}-\boldsymbol{x}_2^t||^2,\\
&\qquad\qquad\vdots\\
&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_1^{t+1},\cdots,x_{N-1}^{t+1},\boldsymbol{x}_N^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq \left\langle\nabla_{\boldsymbol{x}_N}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{x}_N^{t+1}-\boldsymbol{x}_N^t\right\rangle+\tfrac{L}{2}||\boldsymbol{x}_N^{t+1}-\boldsymbol{x}_N^t||^2.
\end{aligned} \tag{66}$$

Summing up the above inequalities in Eq. (66), we can obtain that,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq \sum_{i=1}^{N}\left(\left\langle\nabla_{\boldsymbol{x}_i}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t\right\rangle+\tfrac{L}{2}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2\right).
\end{aligned} \tag{67}$$

Combing $\nabla_{\boldsymbol{x}_i}L_p(\{\boldsymbol{x}_i^{\hat{t}_i}\},\{\boldsymbol{y}_i^{\hat{t}_i}\},\boldsymbol{v}^{\hat{t}_i},\boldsymbol{z}^{\hat{t}_i},\{\lambda_l^{\hat{t}_i}\},\{\boldsymbol{\theta}_i^{\hat{t}_i}\})=\nabla_{\boldsymbol{x}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{\hat{t}_i}\},\{\boldsymbol{y}_i^{\hat{t}_i}\},\boldsymbol{v}^{\hat{t}_i},\boldsymbol{z}^{\hat{t}_i},\{\lambda_l^{\hat{t}_i}\},\{\boldsymbol{\theta}_i^{\hat{t}_i}\})$ with Eq. (15), we have that,

$$\left\langle\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t,\eta_{\boldsymbol{x}}\nabla_{\boldsymbol{x}_i}L_p(\{\boldsymbol{x}_i^{\hat{t}_i}\},\{\boldsymbol{y}_i^{\hat{t}_i}\},\boldsymbol{v}^{\hat{t}_i},\boldsymbol{z}^{\hat{t}_i},\{\lambda_l^{\hat{t}_i}\},\{\boldsymbol{\theta}_i^{\hat{t}_i}\})\right\rangle=-\frac{1}{\eta_{\boldsymbol{x}}}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2. \tag{68}$$

Combing the Cauchy-Schwarz inequality with Assumption 1, 2, we can get,

$$\begin{aligned}
&\left\langle\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t,\nabla_{\boldsymbol{x}_i}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{x}_i}L_p(\{\boldsymbol{x}_i^{\hat{t}_i}\},\{\boldsymbol{y}_i^{\hat{t}_i}\},\boldsymbol{v}^{\hat{t}_i},\boldsymbol{z}^{\hat{t}_i},\{\lambda_l^{\hat{t}_i}\},\{\boldsymbol{\theta}_i^{\hat{t}_i}\})\right\rangle\\
&\leq\tfrac{1}{2}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+\tfrac{L^2}{2}(||\boldsymbol{v}^t-\boldsymbol{v}^{\hat{t}_j}||^2+||\boldsymbol{z}^t-\boldsymbol{z}^{\hat{t}_j}||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t-\lambda_l^{\hat{t}_j}||^2)\\
&\leq\tfrac{1}{2}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+\tfrac{3L^2\tau k_1}{2}(||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2).
\end{aligned} \tag{69}$$

Thus, According to Eq. (67), (68) and (69), we can obtain that,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq\sum_{i=1}^{N}(\tfrac{L+1}{2}-\tfrac{1}{\eta_{\boldsymbol{x}}})||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+\tfrac{3NL^2\tau k_1}{2}(||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2).
\end{aligned} \tag{70}$$

Similarly, using the Lipschitz properties in Assumption 1, we have,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\
&\leq \sum_{i=1}^{N}\left(\left\langle\nabla_{\boldsymbol{y}_i}L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t\right\rangle+\tfrac{L}{2}||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2\right).
\end{aligned} \tag{71}$$

Combing $\nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{\hat{t}_i}\}, \boldsymbol{v}^{\hat{t}_i}, \boldsymbol{z}^{\hat{t}_i}, \{\lambda_l^{\hat{t}_i}\}, \{\boldsymbol{\theta}_i^{\hat{t}_i}\}) = \nabla_{\boldsymbol{x}_i} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{\hat{t}_i}\}, \boldsymbol{v}^{\hat{t}_i}, \boldsymbol{z}^{\hat{t}_i}, \{\lambda_l^{\hat{t}_i}\}, \{\boldsymbol{\theta}_i^{\hat{t}_i}\})$ with Eq. (16), we can obtain that,

$$\left\langle \boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t, \eta_{\boldsymbol{y}} \nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{\hat{t}_i}\}, \boldsymbol{v}^{\hat{t}_i}, \boldsymbol{z}^{\hat{t}_i}, \{\lambda_l^{\hat{t}_i}\}, \{\boldsymbol{\theta}_i^{\hat{t}_i}\}) \right\rangle = -\frac{1}{\eta_{\boldsymbol{y}}} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2. \tag{72}$$

Combing the Cauchy-Schwarz inequality with Assumption 1, 2, we can get,

$$\begin{aligned}
&\left\langle \boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t, \nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - \nabla_{\boldsymbol{y}_i} L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{\hat{t}_i}\}, \boldsymbol{v}^{\hat{t}_i}, \boldsymbol{z}^{\hat{t}_i}, \{\lambda_l^{\hat{t}_i}\}, \{\boldsymbol{\theta}_i^{\hat{t}_i}\}) \right\rangle \\
&\leq \tfrac{1}{2} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 + \tfrac{L^2}{2} (||\boldsymbol{v}^t - \boldsymbol{v}^{\hat{t}_j}||^2 + ||\boldsymbol{z}^t - \boldsymbol{z}^{\hat{t}_j}||^2 + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^t - \lambda_l^{\hat{t}_j}||^2) \\
&\leq \tfrac{1}{2} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 + \tfrac{3L^2 \tau k_1}{2} (||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^{t+1} - \lambda_l^t||^2).
\end{aligned} \tag{73}$$

Thus, combing Eq. (71), (72) and (73), we have,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) \\
&\leq \sum_{i=1}^{N} (\tfrac{L+1}{2} - \tfrac{1}{\eta_{\boldsymbol{y}}}) ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 + \tfrac{3NL^2 \tau k_1}{2} (||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 + \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^{t+1} - \lambda_l^t||^2).
\end{aligned} \tag{74}$$

Combing the Lipschitz properties in Assumption 1 with Eq. (17), we have,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) \\
&\leq \left\langle \nabla_{\boldsymbol{v}} L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}), \boldsymbol{v}^{t+1} - \boldsymbol{v}^t \right\rangle + \tfrac{L}{2} ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 \\
&= (\tfrac{L}{2} - \tfrac{1}{\eta_{\boldsymbol{v}}}) ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2
\end{aligned} \tag{75}$$

Similarly, combing the Lipschitz properties in Assumption 1 with Eq. (18), we have,

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) \\
&\leq \left\langle \nabla_{\boldsymbol{z}} L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}), \boldsymbol{z}^{t+1} - \boldsymbol{z}^t \right\rangle + \tfrac{L}{2} ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 \\
&= (\tfrac{L}{2} - \tfrac{1}{\eta_{\boldsymbol{z}}}) ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2
\end{aligned} \tag{76}$$

By combining Eq. (70), (74), (75), (76), we conclude the proof of Lemma 1.

Lemma 2 *Suppose the Assumption 1 and 2 hold, we have:*

$$\begin{aligned}
&L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^{t+1}\}, \{\boldsymbol{\theta}_i^{t+1}\}) - L_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) \\
&\leq (\tfrac{L+1}{2} - \tfrac{1}{\eta_{\boldsymbol{x}}} + \tfrac{|\boldsymbol{\mathcal{P}}^t| L^2}{2a_1} + \tfrac{|\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2a_3}) \sum_{i=1}^{N} ||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 + (\tfrac{L+1}{2} - \tfrac{1}{\eta_{\boldsymbol{y}}} + \tfrac{|\boldsymbol{\mathcal{P}}^t| L^2}{2a_1} + \tfrac{|\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2a_3}) \sum_{i=1}^{N} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 \\
&+ (\tfrac{L+6\tau k_1 N L^2}{2} - \tfrac{1}{\eta_{\boldsymbol{v}}} + \tfrac{|\boldsymbol{\mathcal{P}}^t| L^2}{2a_1} + \tfrac{|\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2a_3}) ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + (\tfrac{L+6\tau k_1 N L^2}{2} - \tfrac{1}{\eta_{\boldsymbol{z}}} + \tfrac{|\boldsymbol{\mathcal{P}}^t| L^2}{2a_1} + \tfrac{|\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2a_3}) ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 \\
&+ (\tfrac{a_1 + 6\tau k_1 N L^2}{2} - \tfrac{c_1^{t-1} - c_1^t}{2} + \tfrac{1}{2\eta_\lambda}) \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^{t+1} - \lambda_l^t||^2 + (\tfrac{a_3}{2} - \tfrac{c_2^{t-1} - c_2^t}{2} + \tfrac{1}{2\eta_{\boldsymbol{\theta}}}) \sum_{i=1}^{N} ||\boldsymbol{\theta}_i^{t+1} - \boldsymbol{\theta}_i^t||^2 \\
&+ \tfrac{c_1^{t-1}}{2} \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} (||\lambda_l^{t+1}||^2 - ||\lambda_l^t||^2) + \tfrac{1}{2\eta_\lambda} \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^t - \lambda_l^{t-1}||^2) + \tfrac{c_2^{t-1}}{2} \sum_{i=1}^{N} (||\boldsymbol{\theta}_i^{t+1}||^2 - ||\boldsymbol{\theta}_i^t||^2) + \tfrac{1}{2\eta_{\boldsymbol{\theta}}} \sum_{i=1}^{N} ||\boldsymbol{\theta}_i^t - \boldsymbol{\theta}_i^{t-1}||^2,
\end{aligned} \tag{77}$$

where $a_1 > 0$ and $a_3 > 0$ are constants.

Proof of Lemma 2:

According to Eq. (19), $\forall \lambda \in \boldsymbol{\Lambda}$, it follows that:

$$\left\langle \lambda_l^{t+1} - \lambda_l^t - \eta_\lambda \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}), \lambda - \lambda_l^{t+1} \right\rangle = 0. \tag{78}$$

Let $\lambda = \lambda_l^t$, we can obtain:

$$\left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}) - \frac{1}{\eta_\lambda}(\lambda_l^{t+1}-\lambda_l^t), \lambda_l^t - \lambda_l^{t+1} \right\rangle = 0. \tag{79}$$

Likewise, we can obtain:

$$\left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}) - \frac{1}{\eta_\lambda}(\lambda_l^{t}-\lambda_l^{t-1}), \lambda_l^{t+1} - \lambda_l^{t} \right\rangle = 0. \tag{80}$$

Since $\widetilde{L}_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\})$ is concave with respect to λ_l and follows from Eq. (79) and Eq. (80),

$$\begin{aligned}
&\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\}) - \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) \\
&\leq \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&\leq \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} (\left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&+ \tfrac{1}{\eta_\lambda} \left\langle \lambda_l^t - \lambda_l^{t-1}, \lambda_l^{t+1} - \lambda_l^t \right\rangle).
\end{aligned} \tag{81}$$

Denoting $\boldsymbol{v}_{1,l}^{t+1} = \lambda_l^{t+1} - \lambda_l^t - (\lambda_l^t - \lambda_l^{t-1})$, we have,

$$\begin{aligned}
&\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&= \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\})), \lambda_l^{t+1} - \lambda_l^t \right\rangle (1a) \\
&+ \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}), \boldsymbol{v}_{1,l}^{t+1} \right\rangle (1b) \\
&+ \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}), \lambda_l^{t} - \lambda_l^{t-1} \right\rangle (1c).
\end{aligned} \tag{82}$$

Firstly, we focus on the $(1a)$ in Eq. (82), $(1a)$ can be expressed as follows,

$$\begin{aligned}
&\left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&= \left\langle \nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&+ \tfrac{c_1^{t-1}-c_1^t}{2}(||\lambda_l^{t+1}||^2 - ||\lambda_l^t||^2) - \tfrac{c_1^{t-1}-c_1^t}{2}||\lambda_l^{t+1} - \lambda_l^t||^2.
\end{aligned} \tag{83}$$

Combing Cauchy-Schwarz inequality with Assumption 1, we can obtain,

$$\begin{aligned}
&\left\langle \nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} L_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&\leq \tfrac{L^2}{2a_1}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 + ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2) + ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2) + \tfrac{a_1}{2}||\lambda_l^{t+1} - \lambda_l^t||^2,
\end{aligned} \tag{84}$$

where $a_1 > 0$ is a constant. Combining Eq. (83) with Eq. (84), we can obtain that,

$$\begin{aligned}
&\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^t\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
&\leq \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} (\tfrac{L^2}{2a_1}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 + ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2) + ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2) + \tfrac{a_1}{2}||\lambda_l^{t+1} - \lambda_l^t||^2 \\
&\qquad + \tfrac{c_1^{t-1}-c_1^t}{2}(||\lambda_l^{t+1}||^2 - ||\lambda_l^t||^2) - \tfrac{c_1^{t-1}-c_1^t}{2}||\lambda_l^{t+1} - \lambda_l^t||^2).
\end{aligned} \tag{85}$$

Then, we put attention on the $(1b)$ in Eq. (82). According to Cauchy-Schwarz inequality, $(1b)$ can be expressed as follows,

$$\begin{aligned}&\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}\Big\langle\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{v}_{1,l}^{t+1}\Big\rangle\\&\leq\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}(\tfrac{a_2}{2}||\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\})||^2+\tfrac{1}{2a_2}||\boldsymbol{v}_{1,l}^{t+1}||^2).\end{aligned}\tag{86}$$

where $a_2>0$ is a constant. Then, we focus on the $(1c)$ in Eq. (82). Defining $L_1{}'=L+c_1^0$, according to the Assumption 1 and the trigonometric inequality, $\forall\lambda_l$, we have,

$$\begin{aligned}&||\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\})||\\&=||\nabla_{\lambda_l}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}L_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\})||\\&\leq(L+c_1^{t-1})||\lambda_l^t-\lambda_l^{t-1}||\\&\leq L_1{}'||\lambda_l^t-\lambda_l^{t-1}||.\end{aligned}\tag{87}$$

Following from Eq. (87) and the strong concavity of $\widetilde{L}_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\})$ *w.r.t* λ_l (Nesterov, 2003), we can obtain that,

$$\begin{aligned}&\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}\Big\langle\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\})\Big\rangle\\&\leq\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}(-\tfrac{1}{L_1{}'+c_1^{t-1}}||\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\})||^2\\&\qquad-\tfrac{c_1^{t-1}L_1{}'}{L_1{}'+c_1^{t-1}}||\lambda_l^t-\lambda_l^{t-1}||^2).\end{aligned}\tag{88}$$

In addition, the following inequality can be obtained,

$$\tfrac{1}{\eta_\lambda}\left\langle\lambda_l^t-\lambda_l^{t-1},\lambda_l^{t+1}-\lambda_l^t\right\rangle\leq\tfrac{1}{2\eta_\lambda}||\lambda_l^{t+1}-\lambda_l^t||^2-\tfrac{1}{2\eta_\lambda}||\boldsymbol{v}_{1,l}^{t+1}||^2+\tfrac{1}{2\eta_\lambda}||\lambda_l^t-\lambda_l^{t-1}||^2.\tag{89}$$

Combining Eq. (81), (82), (85), (86), (88), (89), $\frac{\eta_\lambda}{2}\leq\frac{1}{L_1{}'+c_1^0}$, and setting $a_2=\eta_\lambda$, we have:

$$\begin{aligned}&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\})-L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})\\&\leq\tfrac{|\boldsymbol{\mathcal{P}}^t|L^2}{2a_1}(\sum_{i=1}^N(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2)\\&+(\tfrac{a_1}{2}-\tfrac{c_1^{t-1}-c_1^t}{2}+\tfrac{1}{2\eta_\lambda})\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2+\tfrac{c_1^{t-1}}{2}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}(||\lambda_l^{t+1}||^2-||\lambda_l^t||^2)+\tfrac{1}{2\eta_\lambda}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t-\lambda_l^{t-1}||^2.\end{aligned}\tag{90}$$

According to Eq. (20), $\forall\boldsymbol{\theta}\in\boldsymbol{\Theta}$, it follows that,

$$\left\langle\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t-\eta_{\boldsymbol{\theta}}\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{\theta}-\boldsymbol{\theta}_i^{t+1}\right\rangle=0.\tag{91}$$

Choosing $\boldsymbol{\theta}=\boldsymbol{\theta}_i^t$, we can obtain,

$$\left\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\})-\frac{1}{\eta_{\boldsymbol{\theta}}}(\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t),\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t+1}\right\rangle\leq0.\tag{92}$$

Likewise, we have,

$$\left\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\})-\frac{1}{\eta_{\boldsymbol{\theta}}}(\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t\right\rangle\leq0.\tag{93}$$

Since $\widetilde{L}_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\})$ is concave with respect to $\boldsymbol{\theta}_i$ and follows from Eq. (93):

$$\begin{aligned}
&\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t+1}\})-\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})\\
&\leq\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\boldsymbol{\theta}_i^{t}\Big\rangle\\
&\leq\sum_{i=1}^{N}(\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\Big\rangle\\
&+\tfrac{1}{\eta_{\boldsymbol{\theta}}}\left\langle\boldsymbol{\theta}_i^{t}-\boldsymbol{\theta}_i^{t-1},\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\right\rangle).
\end{aligned}\tag{94}$$

Denoting $\boldsymbol{v}_{2,l}^{t+1}=\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}-(\boldsymbol{\theta}_i^{t}-\boldsymbol{\theta}_i^{t-1})$, we have that,

$$\begin{aligned}
&\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\Big\rangle\\
&=\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\Big\rangle(2a)\\
&+\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{v}_{2,l}^{t+1}\Big\rangle(2b)\\
&+\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{\theta}_i^{t}-\boldsymbol{\theta}_i^{t-1}\Big\rangle(2c).
\end{aligned}\tag{95}$$

We firstly focus on the $(2a)$ in Eq. (95), we can write the $(2a)$ as,

$$\begin{aligned}
&\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\Big\rangle\\
&=\left\langle\nabla_{\boldsymbol{\theta}_i}L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}L_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\right\rangle\\
&+\tfrac{c_2^{t-1}-c_2^{t}}{2}(||\boldsymbol{\theta}_i^{t+1}||^2-||\boldsymbol{\theta}_i^{t}||^2)-\tfrac{c_2^{t-1}-c_2^{t}}{2}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}||^2).
\end{aligned}\tag{96}$$

And combing the Cauchy-Schwarz inequality with Assumption 1, we can obtain,

$$\begin{aligned}
&\left\langle\nabla_{\boldsymbol{\theta}_i}L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}L_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\right\rangle\\
&=\left\langle\nabla_{\boldsymbol{\theta}_i}L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}L_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\right\rangle\\
&\leq\tfrac{L^2}{2a_3}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^{t}||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^{t}||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^{t}||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^{t}||^2)+\tfrac{a_3}{2}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}||^2,
\end{aligned}\tag{97}$$

where $a_3>0$ is a constant. Thus, we can obtain the upper bound of $(2a)$ by combining Eq. (96) with Eq. (97), that is,

$$\begin{aligned}
&\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t}\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t}\},\{\boldsymbol{y}_i^{t}\},\boldsymbol{v}^{t},\boldsymbol{z}^{t},\{\lambda_l^{t}\},\{\boldsymbol{\theta}_i^{t}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}\Big\rangle\\
&\leq\sum_{j\in\boldsymbol{\mathcal{Q}}^{t+1}}(\tfrac{L^2}{2a_3}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^{t}||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^{t}||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^{t}||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^{t}||^2)+\tfrac{a_3}{2}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}||^2\\
&\qquad+\tfrac{c_2^{t-1}-c_2^{t}}{2}(||\boldsymbol{\theta}_i^{t+1}||^2-||\boldsymbol{\theta}_i^{t}||^2)-\tfrac{c_2^{t-1}-c_2^{t}}{2}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^{t}||^2).
\end{aligned}\tag{98}$$

Next we focus on the $(2b)$ in Eq. (95). According to Cauchy-Schwarz inequality we can write the $(2b)$ as,

$$\begin{aligned}&\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{v}_{2,l}^{t+1}\Big\rangle\\&\leq\sum_{i=1}^{N}(\tfrac{a_4}{2}||\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\})||^2\\&\qquad+\tfrac{1}{2a_4}||\boldsymbol{v}_{2,l}^{t+1}||^2),\end{aligned}\tag{99}$$

where $a_4>0$ is a constant. Then, we focus on the $(2c)$ in Eq. (95). Defining $L_2{}'=L+c_2^0$, according to Assumption 1 and the trigonometric inequality, we have,

$$||\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\})||\leq L_2{}'||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||.\tag{100}$$

Following Eq. (100) and the strong concavity of $\widetilde{L}_p(\{\boldsymbol{x}_i\},\{\boldsymbol{y}_i\},\boldsymbol{v},\boldsymbol{z},\{\lambda_l\},\{\boldsymbol{\theta}_i\})$ *w.r.t* $\boldsymbol{\theta}_i$, we can obtain the upper bound of $(2c)$,

$$\begin{aligned}&\sum_{i=1}^{N}\Big\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}\Big\rangle\\&\leq\sum_{i=1}^{N}(-\tfrac{1}{L_2{}'+c_2^{t-1}}||\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\})||^2\\&\qquad-\tfrac{c_2^{t-1}L_2{}'}{L_2{}'+c_2^{t-1}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2).\end{aligned}\tag{101}$$

In addition, the following inequality can also be obtained,

$$\sum_{i=1}^{N}\tfrac{1}{\eta_{\boldsymbol{\theta}}}\left\langle\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1},\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t\right\rangle\leq\sum_{i=1}^{N}(\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2-\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{v}_{2,l}^{t+1}||^2+\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2).\tag{102}$$

Combining Eq. (94), (95), (98), (99), (101), (102), $\frac{\eta_{\boldsymbol{\theta}}}{2}\leq\frac{1}{L_2{}'+c_2(0)}$, and setting $a_4=\eta_{\boldsymbol{\theta}}$, we have,

$$\begin{aligned}&L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t+1}\})-L_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\})\\&\leq\tfrac{|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2a_3}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2)\\&+(\tfrac{a_3}{2}-\tfrac{c_2^{t-1}-c_2^t}{2}+\tfrac{1}{2\eta_{\boldsymbol{\theta}}})\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2+\tfrac{c_2^{t-1}}{2}\sum_{i=1}^{N}(||\boldsymbol{\theta}_i^{t+1}||^2-||\boldsymbol{\theta}_i^t||^2)+\tfrac{1}{2\eta_{\boldsymbol{\theta}}}\sum_{i=1}^{N}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2.\end{aligned}\tag{103}$$

By combining the Lemma 2 with Eq. (90) with Eq. (103), we conclude the proof of Lemma 2.

Lemma 3 *Firstly, we denote S_1^{t+1}, S_2^{t+1} and F^{t+1} as,*

$$S_1^{t+1}=\frac{4}{{\eta_\lambda}^2c_1^{t+1}}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2-\frac{4}{\eta_\lambda}(\frac{c_1^{t-1}}{c_1^t}-1)\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}||^2,\tag{104}$$

$$S_2^{t+1}=\frac{4}{{\eta_{\boldsymbol{\theta}}}^2c_2^{t+1}}\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2-\frac{4}{\eta_{\boldsymbol{\theta}}}(\frac{c_2^{t-1}}{c_2^t}-1)\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}||^2,\tag{105}$$

$$\begin{aligned}F^{t+1}&=L_p(\{\boldsymbol{x}_i^{t+1}\},\boldsymbol{z}^{t+1},h^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^{t+1}\})+S_1^{t+1}+S_2^{t+1}\\&\quad-\tfrac{7}{2\eta_\lambda}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2-\tfrac{c_1^t}{2}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}||^2-\tfrac{7}{2\eta_{\boldsymbol{\theta}}}\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2-\tfrac{c_2^t}{2}\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}||^2.\end{aligned}\tag{106}$$

Defining $a_5 = \max\{1, 6\tau k_1 N L^2\}$, $\forall t \geq 0$, *we have,*

$$
\begin{aligned}
& F^{t+1} - F^t \\
& \leq \left(\tfrac{L+a_5}{2} - \tfrac{1}{\eta_{\boldsymbol{x}}} + \tfrac{\eta_\lambda |\boldsymbol{\mathcal{P}}^t| L^2}{2} + \tfrac{\eta_{\boldsymbol{\theta}} |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2} + \tfrac{8|\boldsymbol{\mathcal{P}}^t| L^2}{\eta_\lambda (c_1^t)^2} + \tfrac{8NL^2}{\eta_{\boldsymbol{\theta}} (c_2^t)^2}\right) \sum_{i=1}^N ||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 \\
& + \left(\tfrac{L+a_5}{2} - \tfrac{1}{\eta_{\boldsymbol{y}}} + \tfrac{\eta_\lambda |\boldsymbol{\mathcal{P}}^t| L^2}{2} + \tfrac{\eta_{\boldsymbol{\theta}} |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2} + \tfrac{8|\boldsymbol{\mathcal{P}}^t| L^2}{\eta_\lambda (c_1^t)^2} + \tfrac{8NL^2}{\eta_{\boldsymbol{\theta}} (c_2^t)^2}\right) \sum_{i=1}^N ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 \\
& + \left(\tfrac{L+a_5}{2} - \tfrac{1}{\eta_{\boldsymbol{v}}} + \tfrac{\eta_\lambda |\boldsymbol{\mathcal{P}}^t| L^2}{2} + \tfrac{\eta_{\boldsymbol{\theta}} |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2} + \tfrac{8|\boldsymbol{\mathcal{P}}^t| L^2}{\eta_\lambda (c_1^t)^2} + \tfrac{8NL^2}{\eta_{\boldsymbol{\theta}} (c_2^t)^2}\right) ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 \\
& + \left(\tfrac{L+a_5}{2} - \tfrac{1}{\eta_{\boldsymbol{z}}} + \tfrac{\eta_\lambda |\boldsymbol{\mathcal{P}}^t| L^2}{2} + \tfrac{\eta_{\boldsymbol{\theta}} |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2} + \tfrac{8|\boldsymbol{\mathcal{P}}^t| L^2}{\eta_\lambda (c_1^t)^2} + \tfrac{8NL^2}{\eta_{\boldsymbol{\theta}} (c_2^t)^2}\right) ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 \\
& - \left(\tfrac{1}{10\eta_\lambda} - \tfrac{6\tau k_1 N L^2}{2}\right) \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^{t+1} - \lambda_l^t||^2 - \tfrac{1}{10\eta_{\boldsymbol{\theta}}} \sum_{i=1}^N ||\boldsymbol{\theta}_i^{t+1} - \boldsymbol{\theta}_i^t||^2 + \tfrac{c_1^{t-1} - c_1^t}{2} \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^{t+1}||^2 \\
& + \tfrac{c_2^{t-1} - c_2^t}{2} \sum_{i=1}^N ||\boldsymbol{\theta}_i^{t+1}||^2 + \tfrac{4}{\eta_\lambda}\left(\tfrac{c_1^{t-2}}{c_1^{t-1}} - \tfrac{c_1^{t-1}}{c_1^t}\right) \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^t||^2 + \tfrac{4}{\eta_{\boldsymbol{\theta}}}\left(\tfrac{c_2^{t-2}}{c_2^{t-1}} - \tfrac{c_2^{t-1}}{c_2^t}\right) \sum_{i=1}^N ||\boldsymbol{\theta}_i^t||^2,
\end{aligned}
\tag{107}
$$

Proof of Lemma 3*:*

Let $a_1 = \frac{1}{\eta_\lambda}$, $a_3 = \frac{1}{\eta_{\boldsymbol{\theta}}}$ and substitute them into the Lemma 2, we have,

$$
\begin{aligned}
& L_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^{t+1}\}, \{\boldsymbol{\theta}_i^{t+1}\}) - L_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) \\
& \leq \left(\tfrac{L+1}{2} - \tfrac{1}{\eta_{\boldsymbol{x}}} + \tfrac{\rho_1 |\boldsymbol{\mathcal{P}}^t| L^2 + \rho_2 |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2}\right) \sum_{i=1}^N ||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 + \left(\tfrac{L+1}{2} - \tfrac{1}{\eta_{\boldsymbol{y}}} + \tfrac{\rho_1 |\boldsymbol{\mathcal{P}}^t| L^2 + \rho_2 |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2}\right) \sum_{i=1}^N ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 \\
& + \left(\tfrac{L + 6\tau k_1 N L^2}{2} - \tfrac{1}{\eta_{\boldsymbol{v}}} + \tfrac{\rho_1 |\boldsymbol{\mathcal{P}}^t| L^2 + \rho_2 |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2}\right) ||\boldsymbol{v}^{t+1} - \boldsymbol{v}^t||^2 + \tfrac{1}{2\eta_\lambda} \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^t - \lambda_l^{t-1}||^2 \\
& + \left(\tfrac{L + 6\tau k_1 N L^2}{2} - \tfrac{1}{\eta_{\boldsymbol{z}}} + \tfrac{\rho_1 |\boldsymbol{\mathcal{P}}^t| L^2 + \rho_2 |\boldsymbol{\mathcal{Q}}^{t+1}| L^2}{2}\right) ||\boldsymbol{z}^{t+1} - \boldsymbol{z}^t||^2 + \tfrac{1}{2\eta_{\boldsymbol{\theta}}} \sum_{i=1}^N ||\boldsymbol{\theta}_i^t - \boldsymbol{\theta}_i^{t-1}||^2 \\
& + \left(\tfrac{6\tau k_1 N L^2}{2} - \tfrac{c_1^{t-1} - c_1^t}{2} + \tfrac{1}{\eta_\lambda}\right) \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||\lambda_l^{t+1} - \lambda_l^t||^2 + \left(\tfrac{1}{\eta_{\boldsymbol{\theta}}} - \tfrac{c_2^{t-1} - c_2^t}{2}\right) \sum_{i=1}^N ||\boldsymbol{\theta}_i^{t+1} - \boldsymbol{\theta}_i^t||^2 \\
& + \tfrac{c_1^{t-1}}{2} \sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} (||\lambda_l^{t+1}||^2 - ||\lambda_l^t||^2) + \tfrac{c_2^{t-1}}{2} \sum_{i=1}^N (||\boldsymbol{\theta}_i^{t+1}||^2 - ||\boldsymbol{\theta}_i^t||^2).
\end{aligned}
\tag{108}
$$

According to Eq. (19), it follows that:

$$
\left\langle \lambda_l^{t+1} - \lambda_l^t - \eta_\lambda \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}), \lambda_l^t - \lambda_l^{t+1} \right\rangle = 0, \tag{109}
$$

Similar to Eq. (109), we have,

$$
\left\langle \lambda_l^t - \lambda_l^{t-1} - \eta_\lambda \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^{t-1}\}, \{\boldsymbol{\theta}_i^{t-1}\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle = 0. \tag{110}
$$

Thus, we can obtain the following inequality,

$$
\begin{aligned}
& \tfrac{1}{\eta_\lambda} \left\langle \boldsymbol{v}_{1,l}^{t+1}, \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
& = \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^{t-1}\}, \{\boldsymbol{\theta}_i^{t-1}\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
& = \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\}, \{\boldsymbol{y}_i^{t+1}\}, \boldsymbol{v}^{t+1}, \boldsymbol{z}^{t+1}, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}), \lambda_l^{t+1} - \lambda_l^t \right\rangle \\
& + \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^{t-1}\}, \{\boldsymbol{\theta}_i^{t-1}\}), \boldsymbol{v}_{1,l}^{t+1} \right\rangle \\
& + \left\langle \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\}) - \nabla_{\lambda_l} \widetilde{L}_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^{t-1}\}, \{\boldsymbol{\theta}_i^{t-1}\}), \lambda_l^t - \lambda_l^{t-1} \right\rangle.
\end{aligned}
\tag{111}
$$

Since we have the following inequality,

$$
\tfrac{1}{\eta_\lambda} \left\langle \boldsymbol{v}_{1,l}^{t+1}, \lambda_l^{t+1} - \lambda_l^t \right\rangle = \tfrac{1}{2\eta_\lambda} ||\lambda_l^{t+1} - \lambda_l^t||^2 + \tfrac{1}{2\eta_\lambda} ||\boldsymbol{v}_{1,l}^{t+1}||^2 - \tfrac{1}{2\eta_\lambda} ||\lambda_l^t - \lambda_l^{t-1}||^2, \tag{112}
$$

it follows that,

$$\begin{aligned}
&\tfrac{1}{2\eta_\lambda}||\lambda_l^{t+1}-\lambda_l^t||^2+\tfrac{1}{2\eta_\lambda}||\boldsymbol{v}_{1,l}^{t+1}||^2-\tfrac{1}{2\eta_\lambda}||\lambda_l^t-\lambda_l^{t-1}||^2\\
&\leq \tfrac{L^2}{2b_1^t}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2)+\tfrac{b_1^t}{2}||\lambda_l^{t+1}-\lambda_l^t||^2\\
&+\tfrac{c_1^{t-1}-c_1^t}{2}(||\lambda_l^{t+1}||^2-||\lambda_l^t||^2)-\tfrac{c_1^{t-1}-c_1^t}{2}||\lambda_l^{t+1}-\lambda_l^t||^2\\
&+\tfrac{\eta_\lambda}{2}||\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}))||^2+\tfrac{1}{2\eta_\lambda}||\boldsymbol{v}_{1,l}^{t+1}||^2\\
&-\tfrac{(c_1^{t-1})^2}{(L_1{}'+c_1^{t-1})L_1{}'^2}||\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\lambda_l}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^{t-1}\},\{\boldsymbol{\theta}_i^{t-1}\}))||^2\\
&-\tfrac{c_1^{t-1}L_1{}'}{L_1{}'+c_1^{t-1}}||\lambda_l^t-\lambda_l^{t-1}||^2,
\end{aligned}\tag{113}$$

where $b_1^t>0$. According to the setting that $c_1^0\leq L_1{}'$, we have $-\frac{c_1^{t-1}L_1{}'}{L_1{}'+c_1^{t-1}}\leq-\frac{c_1^{t-1}L_1{}'}{2L_1{}'}=-\frac{c_1^{t-1}}{2}\leq-\frac{c_1^t}{2}$. Multiplying both sides of the inequality Eq. (113) by $\frac{8}{\eta_\lambda c_1^t}$, we have,

$$\begin{aligned}
&\tfrac{4}{\eta_\lambda{}^2c_1^t}||\lambda_l^{t+1}-\lambda_l^t||^2-\tfrac{4}{\eta_\lambda}(\tfrac{c_1^{t-1}-c_1^t}{c_1^t})||\lambda_l^{t+1}||^2\\
&\leq\tfrac{4}{\eta_\lambda{}^2c_1^t}||\lambda_l^t-\lambda_l^{t-1}||^2-\tfrac{4}{\eta_\lambda}(\tfrac{c_1^{t-1}-c_1^t}{c_1^t})||\lambda_l^t||^2+\tfrac{4b_1^t}{\eta_\lambda c_1^t}||\lambda_l^{t+1}-\lambda_l^t||^2-\tfrac{4}{\eta_\lambda}||\lambda_l^t-\lambda_l^{t-1}||^2\\
&+\tfrac{4L^2}{\eta_\lambda c_1^tb_1^t}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2).
\end{aligned}\tag{114}$$

It is seen that $\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t-\lambda_l^{t-1}||^2$ and $\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t||^2$ in $(t+1)^{\text{th}}$ iteration respectively equal to $\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^{t-1}|}||\lambda_l^t-\lambda_l^{t-1}||^2$ and $\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^{t-1}|}||\lambda_l^t||^2$ in t^{th} iteration. Setting $b_1^t=\frac{c_1^t}{2}$ in Eq. (114) and using the definition of S_1^t, we have,

$$\begin{aligned}
&S_1^{t+1}-S_1^t\\
&\leq\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}\tfrac{4}{\eta_\lambda}(\tfrac{c_1^{t-2}}{c_1^{t-1}}-\tfrac{c_1^{t-1}}{c_1^t})||\lambda_l^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}(\tfrac{2}{\eta_\lambda}+\tfrac{4}{\rho_1{}^2}(\tfrac{1}{c_1^{t+1}}-\tfrac{1}{c_1^t}))||\lambda_l^{t+1}-\lambda_l^t||^2\\
&-\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}\tfrac{4}{\eta_\lambda}||\lambda_l^t-\lambda_l^{t-1}||^2+\tfrac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2).
\end{aligned}\tag{115}$$

According to Eq. (20), it follows that,

$$\begin{aligned}
&\tfrac{1}{\eta_{\boldsymbol{\theta}}}\left\langle\boldsymbol{v}_{2,l}^{t+1},\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t\right\rangle\\
&\leq\left\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t\right\rangle\\
&=\left\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^{t+1}\},\{\boldsymbol{y}_i^{t+1}\},\boldsymbol{v}^{t+1},\boldsymbol{z}^{t+1},\{\lambda_l^{t+1}\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\}),\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t\right\rangle\\
&+\left\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{v}_{2,l}^{t+1}\right\rangle\\
&+\left\langle\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\}),\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}\right\rangle.
\end{aligned}\tag{116}$$

In addition, since

$$\tfrac{1}{\eta_{\boldsymbol{\theta}}}\left\langle\boldsymbol{v}_{2,l}^{t+1},\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t\right\rangle=\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2+\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{v}_{2,l}^{t+1}||^2-\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2,\tag{117}$$

it follows that,

$$\begin{aligned}
&\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2+\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{v}_{2,l}^{t+1}||^2-\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2\\
&\leq \tfrac{L^2}{2b_2^t}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2)+\tfrac{b_2^t}{2}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2\\
&+\tfrac{c_2^{t-1}-c_2^t}{2}(||\boldsymbol{\theta}_i^{t+1}||^2-||\boldsymbol{\theta}_i^t||^2)-\tfrac{c_2^{t-1}-c_2^t}{2}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2-\tfrac{c_2^{t-1}L_2'}{L_2'+c_2^{t-1}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2+\tfrac{1}{2\eta_{\boldsymbol{\theta}}}||\boldsymbol{v}_{2,l}^{t+1}||^2\\
&+\tfrac{\eta_{\boldsymbol{\theta}}}{2}||\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\}))||^2\\
&-\tfrac{(c_2^{t-1})^2}{(L_2'+c_2^{t-1})L_2'^2}||\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^t\})-\nabla_{\boldsymbol{\theta}_i}\widetilde{L}_p(\{\boldsymbol{x}_i^t\},\{\boldsymbol{y}_i^t\},\boldsymbol{v}^t,\boldsymbol{z}^t,\{\lambda_l^t\},\{\boldsymbol{\theta}_i^{t-1}\})||^2.
\end{aligned} \tag{118}$$

According to the setting $c_2^0 \leq L_2{}'$, we have $-\frac{c_2^{t-1}L_2{}'}{L_2{}'+c_2^{t-1}} \leq -\frac{c_2^{t-1}L_2{}'}{2L_2{}'} = -\frac{c_2^{t-1}}{2} \leq -\frac{c_2^t}{2}$. Multiplying both sides of the inequality Eq. (118) by $\frac{8}{\eta_{\boldsymbol{\theta}}c_2^t}$, we have,

$$\begin{aligned}
&\tfrac{4}{\eta_{\boldsymbol{\theta}}{}^2c_2^t}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2-\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_2^{t-1}-c_2^t}{c_2^t})||\boldsymbol{\theta}_i^{t+1}||^2\\
&\leq \tfrac{4}{\eta_{\boldsymbol{\theta}}{}^2c_2^t}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2-\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_2^{t-1}-c_2^t}{c_2^t})||\boldsymbol{\theta}_i^t||^2+\tfrac{4b_2^t}{\eta_{\boldsymbol{\theta}}c_2^t}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2-\tfrac{4}{\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2\\
&+\tfrac{4L^2}{\eta_{\boldsymbol{\theta}}c_2^tb_2^t}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2).
\end{aligned} \tag{119}$$

Setting $b_2^t = \frac{c_2^t}{2}$ in Eq. (119) and using the definition of S_2^t, we can obtain,

$$\begin{aligned}
&S_2^{t+1}-S_2^t\\
&\leq \sum_{i=1}^{N}\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_2^{t-2}}{c_2^{t-1}}-\tfrac{c_2^{t-1}}{c_2^t})||\boldsymbol{\theta}_i^t||^2+\sum_{i=1}^{N}(\tfrac{2}{\eta_{\boldsymbol{\theta}}}+\tfrac{4}{\eta_{\boldsymbol{\theta}}{}^2}(\tfrac{1}{c_2^{t+1}}-\tfrac{1}{c_2^t}))||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2\\
&-\sum_{i=1}^{N}\tfrac{4}{\eta_{\boldsymbol{\theta}}}||\boldsymbol{\theta}_i^t-\boldsymbol{\theta}_i^{t-1}||^2+\tfrac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2}(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2).
\end{aligned} \tag{120}$$

According to the setting about c_1^t and c_2^t, we have $\frac{\eta_\lambda}{10} \geq \frac{1}{c_1^{t+1}}-\frac{1}{c_1^t}$, $\frac{\eta_{\boldsymbol{\theta}}}{10} \geq \frac{1}{c_2^{t+1}}-\frac{1}{c_2^t}, \forall t \geq 0$. Defining $a_5 = \max\{1, 6\tau k_1 NL^2\}$. Using the definition of F^{t+1} and combining it with Eq. (115) and Eq. (120), $\forall t \geq 0$, we have,

$$\begin{aligned}
&F^{t+1}-F^t\\
&\leq(\tfrac{L+a_5}{2}-\tfrac{1}{\eta_{\boldsymbol{x}}}+\tfrac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\tfrac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\tfrac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\tfrac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2})\sum_{i=1}^{N}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2\\
&+(\tfrac{L+a_5}{2}-\tfrac{1}{\eta_{\boldsymbol{y}}}+\tfrac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\tfrac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\tfrac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\tfrac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2})\sum_{i=1}^{N}||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2\\
&+(\tfrac{L+a_5}{2}-\tfrac{1}{\eta_{\boldsymbol{v}}}+\tfrac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\tfrac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\tfrac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\tfrac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2})||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2\\
&+(\tfrac{L+a_5}{2}-\tfrac{1}{\eta_{\boldsymbol{z}}}+\tfrac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\tfrac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\tfrac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\tfrac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2})||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2\\
&-(\tfrac{1}{10\eta_\lambda}-\tfrac{6\tau k_1NL^2}{2})\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2-\tfrac{1}{10\eta_{\boldsymbol{\theta}}}\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2+\tfrac{c_1^{t-1}-c_1^t}{2}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}||^2\\
&+\tfrac{c_2^{t-1}-c_2^t}{2}\sum_{i=1}^{N}||\boldsymbol{\theta}_i^{t+1}||^2+\tfrac{4}{\eta_\lambda}(\tfrac{c_1^{t-2}}{c_1^{t-1}}-\tfrac{c_1^{t-1}}{c_1^t})\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t||^2+\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_2^{t-2}}{c_2^{t-1}}-\tfrac{c_2^{t-1}}{c_2^t})\sum_{i=1}^{N}||\boldsymbol{\theta}_i^t||^2,
\end{aligned} \tag{121}$$

Proof of Theorem 1:

According to the setting of $\eta_{\boldsymbol{x}}$, $\eta_{\boldsymbol{y}}$, $\eta_{\boldsymbol{v}}$, $\eta_{\boldsymbol{z}}$ and c_1^t, c_2^t, we have,

$$\frac{L+a_5}{2}-\frac{1}{\eta_{\boldsymbol{x}}}+\frac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\frac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\frac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\frac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2}=-a_6^t, \tag{122}$$

$$\frac{L+a_5}{2}-\frac{1}{\eta_{\boldsymbol{y}}}+\frac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\frac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\frac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\frac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2}=-a_6^t, \tag{123}$$

$$\frac{L+a_5}{2}-\frac{1}{\eta_{\boldsymbol{v}}}+\frac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\frac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\frac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\frac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2}=-a_6^t, \tag{124}$$

$$\frac{L+a_5}{2}-\frac{1}{\eta_{\boldsymbol{z}}}+\frac{\eta_\lambda|\boldsymbol{\mathcal{P}}^t|L^2}{2}+\frac{\eta_{\boldsymbol{\theta}}|\boldsymbol{\mathcal{Q}}^{t+1}|L^2}{2}+\frac{8|\boldsymbol{\mathcal{P}}^t|L^2}{\eta_\lambda(c_1^t)^2}+\frac{8NL^2}{\eta_{\boldsymbol{\theta}}(c_2^t)^2}=-a_6^t, \tag{125}$$

Combining with Lemma 3, $\forall t\geq 0$, it follows that,

$$\begin{aligned}
&a_6^t\sum_{i=1}^N(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+a_6^t||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+a_6^t||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2\\
&+(\tfrac{1}{10\eta_\lambda}-\tfrac{3\tau k_1NL^2}{2})\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2+\tfrac{1}{10\eta_{\boldsymbol{\theta}}}\sum_{i=1}^N||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2\\
&\leq F^t-F^{t+1}+\tfrac{c_1^{t-1}-c_1^t}{2}\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}||^2+\tfrac{c_2^{t-1}-c_2^t}{2}\sum_{i=1}^N||\boldsymbol{\theta}_i^{t+1}||^2\\
&+\tfrac{4}{\eta_\lambda}(\tfrac{c_1^{t-2}}{c_1^{t-1}}-\tfrac{c_1^{t-1}}{c_1^t})\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t||^2+\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_2^{t-2}}{c_2^{t-1}}-\tfrac{c_2^{t-1}}{c_2^t})\sum_{i=1}^N||\boldsymbol{\theta}_i^t||^2.
\end{aligned} \tag{126}$$

Combing the definition of $(\nabla\widetilde{G}^t)_{\boldsymbol{x}_i}$ with trigonometric inequality, Cauchy-Schwarz inequality and the Assumption 1 and 2, we have,

$$||(\nabla\widetilde{G}^t)_{\boldsymbol{x}_i}||^2\leq\tfrac{2}{\eta_{\boldsymbol{x}}{}^2}||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2+6L^2\tau k_1(||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2). \tag{127}$$

Combing the definition of $(\nabla\widetilde{G}^t)_{\boldsymbol{x}_i}$ with trigonometric inequality, Cauchy-Schwarz inequality, we have,

$$\begin{aligned}
&||(\nabla\widetilde{G}^t)_{\boldsymbol{y}_i}||^2\\
&\leq\tfrac{2}{\eta_{\boldsymbol{y}}{}^2}||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2+2L^2\sum_{i=1}^N||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2+6L^2\tau k_1(||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2).
\end{aligned} \tag{128}$$

Combining the definition of $(\nabla\widetilde{G}^t)_{\boldsymbol{v}}$ with trigonometric inequality and Cauchy-Schwarz inequality, we have,

$$||(\nabla\widetilde{G}^t)_{\boldsymbol{v}}||^2\leq 2L^2\sum_{i=1}^N(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+\tfrac{2}{\eta_{\boldsymbol{v}}{}^2}||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2. \tag{129}$$

Combining the definition of $(\nabla\widetilde{G}^t)_{\boldsymbol{z}}$ with trigonometric inequality and Cauchy-Schwarz inequality, we have,

$$||(\nabla\widetilde{G}^t)_{\boldsymbol{z}}||^2\leq 2L^2\left(\sum_{i=1}^N(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2\right)+\tfrac{2}{\eta_{\boldsymbol{z}}{}^2}||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2. \tag{130}$$

Combining the definition of $(\nabla\widetilde{G}^t)_{\lambda_l}$ with trigonometric inequality and Cauchy-Schwarz inequality,

$$\begin{aligned}
||(\nabla\widetilde{G}^t)_{\lambda_l}||^2\leq&\tfrac{3}{\rho_1{}^2}||\lambda_l^{t+1}-\lambda_l^t||^2+3((c_1^{t-1})^2-(c_1^t)^2)||\lambda_l^t||^2\\
&+3L^2\left(\sum_{i=1}^N(||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2\right).
\end{aligned} \tag{131}$$

Combining the definition of $(\nabla\widetilde{G}^t)_{\boldsymbol{\theta}_i}$ with Cauchy-Schwarz inequality and Assumption 2, we have,

$$\begin{aligned}
&||(\nabla\widetilde{G}^t)_{\boldsymbol{\theta}_i}||^2\\
&\leq \tfrac{3}{{\rho_2}^2}||{\boldsymbol{\theta}_i}^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2+3L^2\left(\sum_{i=1}^{N}(||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2)+||\boldsymbol{v}^{\overline{t_i}}-\boldsymbol{v}^t||^2\right)+3(c_2^{\hat{t}_i-1}-c_2^{\overline{t_i}-1})^2||\boldsymbol{\theta}_i^t||^2\\
&\leq \tfrac{3}{{\rho_2}^2}||{\boldsymbol{\theta}_i}^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2+3L^2\sum_{i=1}^{N}(||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2)\\
&+3L^2\tau k_1(||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2)+3((c_2^{\hat{t}_i-1})^2-(c_2^{\overline{t_i}-1})^2)||\boldsymbol{\theta}_i^t||^2.
\end{aligned}\tag{132}$$

According to the Definition B.2 as well as Eq. (127), (128), (129), (130), (131) and Eq. (132), we can obtain,

$$\begin{aligned}
||\nabla\widetilde{G}^t||^2=&\sum_{i=1}^{N}(||(\nabla\widetilde{G}^t)_{\boldsymbol{x}_i}||^2+||(\nabla\widetilde{G}^t)_{\boldsymbol{y}_i}||^2+||(\nabla\widetilde{G}^t)_{\boldsymbol{\theta}_i}||^2)+||(\nabla\widetilde{G}^t)_{\boldsymbol{v}}||^2+||(\nabla\widetilde{G}^t)_{\boldsymbol{z}}||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||(\nabla\widetilde{G}^t)_{\lambda_l}||^2\\
&\leq(\tfrac{2}{{\eta_{\boldsymbol{x}}}^2}+5NL^2)\sum_{i=1}^{N}||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2+(\tfrac{2}{{\eta_{\boldsymbol{y}}}^2}+3NL^2)\sum_{i=1}^{N}||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2\\
&+(4+3|\boldsymbol{\mathcal{P}}^t|L^2)\sum_{i=1}^{N}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+(4+3|\boldsymbol{\mathcal{P}}^t|L^2)\sum_{i=1}^{N}||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2\\
&+(\tfrac{2}{{\eta_{\boldsymbol{v}}}^2}+(2+15\tau k_1N+3|\boldsymbol{\mathcal{P}}^t|)L^2)||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+(\tfrac{2}{{\eta_{\boldsymbol{z}}}^2}+(15\tau k_1N+3|\boldsymbol{\mathcal{P}}^t|)L^2)||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2\\
&+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}(\tfrac{3}{{\rho_1}^2}+15\tau k_1NL^2)||\lambda_l^{t+1}-\lambda_l^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}3((c_1^{t-1})^2-(c_1^t)^2)||\lambda_l^t||^2\\
&+\sum_{i=1}^{N}\tfrac{3}{{\rho_2}^2}||{\boldsymbol{\theta}_i}^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2+\sum_{i=1}^{N}3((c_2^{\hat{t}_i-1})^2-(c_2^{\overline{t_i}-1})^2)||\boldsymbol{\theta}_i^t||^2.
\end{aligned}\tag{133}$$

We set that,

$$d_1=\frac{2k_\tau\tau+(4+3M+5k_\tau\tau N)L^2{\underline{\eta_{\boldsymbol{x}}}}^2}{{\underline{\eta_{\boldsymbol{x}}}}^2(a_6^0)^2}\geq\frac{2k_\tau\tau+(4+3|\boldsymbol{\mathcal{P}}^t|+5k_\tau\tau N)L^2\eta_{\boldsymbol{x}}^2}{\eta_{\boldsymbol{x}}^2(a_6^t)^2},\tag{134}$$

$$d_2=\frac{2k_\tau\tau+(4+3M+3k_\tau\tau N)L^2{\underline{\eta_{\boldsymbol{y}}}}^2}{{\underline{\eta_{\boldsymbol{y}}}}^2(a_6^0)^2}\geq\frac{2k_\tau\tau+(4+3|\boldsymbol{\mathcal{P}}^t|+3k_\tau\tau N)L^2\eta_{\boldsymbol{y}}^2}{\eta_{\boldsymbol{y}}^2(a_6^t)^2},\tag{135}$$

$$d_3=\frac{2+(2+15\tau k_1N+3M)L^2{\underline{\eta_{\boldsymbol{v}}}}^2}{{\underline{\eta_{\boldsymbol{v}}}}^2(a_6^0)^2}\geq\frac{2+(2+15\tau k_1N+3|\boldsymbol{\mathcal{P}}^t|)L^2\eta_{\boldsymbol{v}}^2}{\eta_{\boldsymbol{v}}^2(a_6^t)^2},\tag{136}$$

$$d_4=\frac{2+(2+15\tau k_1N+3M)L^2{\underline{\eta_{\boldsymbol{z}}}}^2}{{\underline{\eta_{\boldsymbol{z}}}}^2(a_6^0)^2}\geq\frac{2+(2+15\tau k_1N+3|\boldsymbol{\mathcal{P}}^t|)L^2\eta_{\boldsymbol{z}}^2}{\eta_{\boldsymbol{z}}^2(a_6^t)^2},\tag{137}$$

where k_τ, $\underline{\eta_{\boldsymbol{x}}}$, $\underline{\eta_{\boldsymbol{y}}}$, $\underline{\eta_{\boldsymbol{v}}}$ and $\underline{\eta_{\boldsymbol{z}}}$ are positive constants. Thus, combining Eq. (133) with Eq. (134), Eq. (135), (136), (137), we can obtain,

$$\begin{aligned}
||\nabla\widetilde{G}^t||^2\leq&\sum_{i=1}^{N}d_1(a_6^t)^2||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+\sum_{i=1}^{N}d_2(a_6^t)^2||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2\\
&+d_3(a_6^t)^2||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+d_4(a_6^t)^2||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2+\sum_{i=1}^{N}\tfrac{3}{{\rho_2}^2}||{\boldsymbol{\theta}_i}^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2\\
&+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}(\tfrac{3}{{\rho_1}^2}+15\tau k_1NL^2)||\lambda_l^{t+1}-\lambda_l^t||^2+\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}3((c_1^{t-1})^2-(c_1^t)^2)||\lambda_l^t||^2\\
&+\sum_{i=1}^{N}3((c_2^{\hat{t}_i-1})^2-(c_2^{\overline{t_i}-1})^2)||\boldsymbol{\theta}_i^t||^2+(\tfrac{2}{{\eta_{\boldsymbol{x}}}^2}+5NL^2)\sum_{i=1}^{N}||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2\\
&-(\tfrac{2k_\tau\tau}{{\eta_{\boldsymbol{x}}}^2}+5k_\tau\tau NL^2)\sum_{i=1}^{N}||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+(\tfrac{2}{{\eta_{\boldsymbol{y}}}^2}+3NL^2)\sum_{i=1}^{N}||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2\\
&-(\tfrac{2k_\tau\tau}{{\eta_{\boldsymbol{y}}}^2}+3k_\tau\tau NL^2)\sum_{i=1}^{N}||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2.
\end{aligned}\tag{138}$$

Let d_5^t denote a nonnegative sequence $d_5^t = \frac{1}{\max\{d_1 a_6^t, d_2 a_6^t, d_3 a_6^t, d_4 a_6^t, \frac{\frac{30}{\eta_\lambda}+150\eta_\lambda \tau k_1 NL^2}{1-30\eta_\lambda \tau k_1 NL^2}, \frac{30\tau}{\eta_{\boldsymbol{\theta}}}\}}$. It is seen that $d_5^0 \geq d_5^t, \forall t \geq 0$. And we denote the lower bound of d_5^t as $\underline{d_5}$, it appears that $d_5^t \geq \underline{d_5} \geq 0, \forall t \geq 0$. And we set the constant k_τ satisfies $k_\tau \geq \max\{\frac{d_5^0(\frac{2}{\eta_{\boldsymbol{x}}{}^2}+5NL^2)}{\underline{d_5}(\frac{2}{\overline{\eta_{\boldsymbol{x}}}^2}+5NL^2)}, \frac{d_5^0(\frac{2}{\eta_{\boldsymbol{y}}{}^2}+3NL^2)}{\underline{d_5}(\frac{2}{\overline{\eta_{\boldsymbol{y}}}^2}+3NL^2)}\}$, where $\overline{\eta_{\boldsymbol{x}}}$ and $\overline{\eta_{\boldsymbol{y}}}$ are the step-sizes in the first iteration (it is seen that $\overline{\eta_{\boldsymbol{x}}} \geq \eta_{\boldsymbol{x}}, \overline{\eta_{\boldsymbol{y}}} \geq \eta_{\boldsymbol{y}}$). We can obtain the following inequality from Eq. (138) and the definition of d_5^t:

$$
\begin{aligned}
d_5^t||\nabla \widetilde{G}^t||^2 \leq & a_6^t \sum_{i=1}^N (||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2+||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2)+a_6^t||\boldsymbol{v}^{t+1}-\boldsymbol{v}^t||^2+a_6^t||\boldsymbol{z}^{t+1}-\boldsymbol{z}^t||^2 \\
& +(\tfrac{1}{10\eta_\lambda}-\tfrac{6\tau k_1 NL^2}{2})\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^{t+1}-\lambda_l^t||^2+\tfrac{1}{10\tau\eta_{\boldsymbol{\theta}}}\sum_{i=1}^N||\boldsymbol{\theta}_i^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2 \\
& +3d_5^t((c_1^{t-1})^2-(c_1^t)^2)\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|}||\lambda_l^t||^2+3d_5^t((c_2^{\hat{t}_i-1})^2-(c_2^{\overline{t_i}-1})^2)\sum_{i=1}^N||\boldsymbol{\theta}_i^t||^2 \\
& +d_5^t(\tfrac{2}{\eta_{\boldsymbol{x}}{}^2}+5NL^2)\sum_{i=1}^N||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2-d_5^t(\tfrac{2k_\tau\tau}{\eta_{\boldsymbol{x}}{}^2}+5k_\tau\tau NL^2)\sum_{i=1}^N||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2 \\
& +d_5^t(\tfrac{2}{\eta_{\boldsymbol{y}}{}^2}+3NL^2)\sum_{i=1}^N||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2-d_5^t(\tfrac{2k_\tau\tau}{\eta_{\boldsymbol{y}}{}^2}+3k_\tau\tau NL^2)\sum_{i=1}^N||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2.
\end{aligned}
\tag{139}
$$

Combining the definition of d_5^t with Eq. (126) and according to the setting $||\lambda_l^t||^2 \leq \alpha_3$, $||\boldsymbol{\theta}_i^t||^2 \leq \alpha_4$ and $d_5^0 \geq d_5^t \geq \underline{d_5}, \forall t \geq 0$, thus, we have,

$$
\begin{aligned}
& d_5^t||\nabla \widetilde{G}^t||^2 \\
& \leq F^t - F^{t+1} + \tfrac{c_1^{t-1}-c_1^t}{2}M\alpha_3 + \tfrac{c_2^{t-1}-c_2^t}{2}N\alpha_4 + \tfrac{4}{\eta_\lambda}(\tfrac{c_1^{t-2}}{c_1^{t-1}}-\tfrac{c_1^{t-1}}{c_1^t})M\alpha_3 \\
& +\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_2^{t-2}}{c_2^{t-1}}-\tfrac{c_2^{t-1}}{c_2^t})N\alpha_4+3d_4^0((c_1^{t-1})^2-(c_1^t)^2)M\alpha_3+3d_4^0((c_2^{\hat{t}_i-1})^2-(c_2^{\overline{t_i}-1})^2)N\alpha_4 \\
& +\tfrac{1}{10\tau\eta_{\boldsymbol{\theta}}}\sum_{i=1}^N||\boldsymbol{\theta}_i^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2-\tfrac{1}{10\eta_{\boldsymbol{\theta}}}\sum_{i=1}^N||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2 \\
& +d_5^0(\tfrac{2}{\eta_{\boldsymbol{x}}{}^2}+5NL^2)\sum_{i=1}^N||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2-\underline{d_5}(\tfrac{2k_\tau\tau}{\eta_{\boldsymbol{x}}{}^2}+5k_\tau\tau NL^2)\sum_{i=1}^N||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2 \\
& +d_5^0(\tfrac{2}{\eta_{\boldsymbol{y}}{}^2}+3NL^2)\sum_{i=1}^N||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2-\underline{d_5}(\tfrac{2k_\tau\tau}{\eta_{\boldsymbol{y}}{}^2}+3k_\tau\tau NL^2)\sum_{i=1}^N||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2.
\end{aligned}
\tag{140}
$$

Denoting $\widetilde{T}(\epsilon)$ as $\widetilde{T}(\epsilon) = \min\{t \mid ||\nabla \widetilde{G}^t|| \leq \frac{\epsilon}{4}, t \geq 2\}$. Summing up Eq. (140) from $t = 2$ to $t = \widetilde{T}(\epsilon)$, we have,

$$
\begin{aligned}
& \sum_{t=2}^{\widetilde{T}(\epsilon)} d_5^t||\nabla \widetilde{G}^t||^2 \\
& \leq F^2 - \underline{L} + \tfrac{4}{\eta_\lambda}(\tfrac{c_1^0}{c_1^1}+\tfrac{c_1^1}{c_1^2})M\alpha_3+\tfrac{c_1^1}{2}M\alpha_3+\tfrac{7}{2\eta_\lambda}M\sigma_3{}^2+3d_4^0(c_1^1)^2M\alpha_3 \\
& +\tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_1^0}{c_1^1}+\tfrac{c_1^1}{c_1^2})N\alpha_4+\tfrac{c_2^1}{2}N\alpha_4+\tfrac{7}{2\eta_{\boldsymbol{\theta}}}N\sigma_4{}^2+\sum_{t=2}^{\widetilde{T}(\epsilon)}3d_4^0((c_2^{\hat{t}_i-1})^2-(c_2^{\overline{t_i}-1})^2)N\alpha_4 \\
& +\tfrac{1}{10\tau\eta_{\boldsymbol{\theta}}}\sum_{t=2}^{\widetilde{T}(\epsilon)}\sum_{i=1}^N||\boldsymbol{\theta}_i^{\overline{t_i}}-\boldsymbol{\theta}_i^t||^2-\tfrac{1}{10\eta_{\boldsymbol{\theta}}}\sum_{t=2}^{\widetilde{T}(\epsilon)}\sum_{i=1}^N||\boldsymbol{\theta}_i^{t+1}-\boldsymbol{\theta}_i^t||^2 \\
& +d_5^0(\tfrac{2}{\underline{\eta_{\boldsymbol{x}}}^2}+5NL^2)\sum_{t=2}^{\widetilde{T}(\epsilon)}\sum_{i=1}^N||\boldsymbol{x}_i^{\overline{t_i}}-\boldsymbol{x}_i^t||^2-\underline{d_5}(\tfrac{2k_\tau\tau}{\overline{\eta_{\boldsymbol{x}}}^2}+5k_\tau\tau NL^2)\sum_{t=2}^{\widetilde{T}(\epsilon)}\sum_{i=1}^N||\boldsymbol{x}_i^{t+1}-\boldsymbol{x}_i^t||^2 \\
& +d_5^0(\tfrac{2}{\underline{\eta_{\boldsymbol{y}}}^2}+3NL^2)\sum_{t=2}^{\widetilde{T}(\epsilon)}\sum_{i=1}^N||\boldsymbol{y}_i^{\overline{t_i}}-\boldsymbol{y}_i^t||^2-\underline{d_5}(\tfrac{2k_\tau\tau}{\overline{\eta_{\boldsymbol{y}}}^2}+3k_\tau\tau NL^2)\sum_{t=2}^{\widetilde{T}(\epsilon)}\sum_{i=1}^N||\boldsymbol{y}_i^{t+1}-\boldsymbol{y}_i^t||^2.
\end{aligned}
\tag{141}
$$

where $\sigma_3 = \max\{||\lambda_1 - \lambda_2||\}$, $\sigma_4 = \max\{||\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2||\}$ and $\underline{L} = \min L_p(\{\boldsymbol{x}_i^t\}, \{\boldsymbol{y}_i^t\}, \boldsymbol{v}^t, \boldsymbol{z}^t, \{\lambda_l^t\}, \{\boldsymbol{\theta}_i^t\})$, which satisfy that,

$$F^t \geq \underline{L} - \frac{4}{\eta_\lambda}\frac{c_1^1}{c_1^2} M\alpha_3 - \frac{4}{\eta_{\boldsymbol{\theta}}}\frac{c_2^1}{c_2^2} N\alpha_4 - \frac{7}{2\eta_\lambda} M{\sigma_3}^2 - \frac{7}{2\eta_{\boldsymbol{\theta}}} N{\sigma_4}^2, \forall t \geq 2. \tag{142}$$

For each worker j, we have that $\overline{t_i} - \hat{t}_i \leq \tau$, thus,

$$\begin{aligned}
&\sum_{t=2}^{\widetilde{T}(\epsilon)} 3d_4^0((c_2^{\hat{t}_i - 1})^2 - (c_2^{\overline{t_i} - 1})^2) N\alpha_4 \\
&\leq \tau \sum_{\substack{\hat{v}_i(j) \in \mathcal{V}_i(\widetilde{T}(\epsilon)), \\ 2 \leq \hat{v}_i(j) \leq \widetilde{T}(\epsilon)}} 3d_4^0((c_2^{\hat{v}_i(j) - 1})^2 - (c_2^{\hat{v}_i(j+1) - 1})^2) N\alpha_4 \\
&\leq 3\tau d_4^0 (c_2^1)^2 N\alpha_4.
\end{aligned} \tag{143}$$

Since the idle workers do not update their variables in each master iteration, for any t that satisfies $\hat{v}_i(j-1) \leq t < \hat{v}_i(j)$, we have $\boldsymbol{\theta}_i^t = \boldsymbol{\theta}_i^{\hat{v}_i(j)-1}$. And for $t \notin \mathcal{V}_i(T)$, we have $||\boldsymbol{\theta}_i^t - \boldsymbol{\theta}_i^{t-1}||^2 = 0$. Combing with $\hat{v}_i(j) - \hat{v}_i(j-1) \leq \tau$, we can obtain that,

$$\begin{aligned}
\sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||{\boldsymbol{\theta}_i}^{\overline{t_j}} - \boldsymbol{\theta}_i^t||^2 &\leq \tau \sum_{\substack{\hat{v}_i(j) \in \mathcal{V}_i(\widetilde{T}(\epsilon)), \\ 3 \leq \hat{v}_i(j)}} \sum_{i=1}^{N} ||\boldsymbol{\theta}_i^{\hat{v}_i(j)} - \boldsymbol{\theta}_i^{\hat{v}_i(j)-1}||^2 \\
&= \tau \sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||{\boldsymbol{\theta}_i}^{t+1} - \boldsymbol{\theta}_i^t||^2 + \tau \sum_{t=\widetilde{T}(\epsilon)+1}^{\widetilde{T}(\epsilon)+\tau-1} \sum_{i=1}^{N} ||{\boldsymbol{\theta}_i}^{t+1} - \boldsymbol{\theta}_i^t||^2 \\
&\leq \tau \sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||{\boldsymbol{\theta}_i}^{t+1} - \boldsymbol{\theta}_i^t||^2 + 4\tau(\tau-1) N\alpha_4.
\end{aligned} \tag{144}$$

Similarly, for any t that satisfies $\hat{v}_i(j-1) \leq t < \hat{v}_i(j)$, we have $\boldsymbol{x}_i^t = \boldsymbol{x}_i^{\hat{v}_i(j)-1}$, $\boldsymbol{y}_i^t = \boldsymbol{y}_i^{\hat{v}_i(j)-1}$. And for $t \notin \mathcal{V}_i(T)$, we have $||\boldsymbol{x}_i^t - \boldsymbol{x}_i^{t-1}||^2 = 0$, $||\boldsymbol{y}_i^t - \boldsymbol{y}_i^{t-1}||^2 = 0$. Combing with $\hat{v}_i(j) - \hat{v}_i(j-1) \leq \tau$, we can obtain,

$$\begin{aligned}
\sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||\boldsymbol{x}_i^{\overline{t_j}} - \boldsymbol{x}_i^t||^2 &\leq \tau \sum_{\substack{\hat{v}_i(j) \in \mathcal{V}_i(\widetilde{T}(\epsilon)), \\ 3 \leq \hat{v}_i(j)}} \sum_{i=1}^{N} ||\boldsymbol{x}_i^{\hat{v}_i(j)} - \boldsymbol{x}_i^{\hat{v}_i(j)-1}||^2 \\
&= \tau \sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 + \tau \sum_{t=\widetilde{T}(\epsilon)+1}^{\widetilde{T}(\epsilon)+\tau-1} \sum_{i=1}^{N} ||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 \\
&\leq \tau \sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||\boldsymbol{x}_i^{t+1} - \boldsymbol{x}_i^t||^2 + 4\tau(\tau-1) N{\alpha_1}^2.
\end{aligned} \tag{145}$$

$$\begin{aligned}
\sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||\boldsymbol{y}_i^{\overline{t_j}} - \boldsymbol{y}_i^t||^2 &\leq \tau \sum_{\substack{\hat{v}_i(j) \in \mathcal{V}_i(\widetilde{T}(\epsilon)), \\ 3 \leq \hat{v}_i(j)}} \sum_{i=1}^{N} ||\boldsymbol{y}_i^{\hat{v}_i(j)} - \boldsymbol{y}_i^{\hat{v}_i(j)-1}||^2 \\
&= \tau \sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 + \tau \sum_{t=\widetilde{T}(\epsilon)+1}^{\widetilde{T}(\epsilon)+\tau-1} \sum_{i=1}^{N} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 \\
&\leq \tau \sum_{t=2}^{\widetilde{T}(\epsilon)} \sum_{i=1}^{N} ||\boldsymbol{y}_i^{t+1} - \boldsymbol{y}_i^t||^2 + 4\tau(\tau-1) N{\alpha_2}^2.
\end{aligned} \tag{146}$$

It follows from Eq. (141), (143), (144), (145) that,

$$\begin{aligned}
&\sum_{t=2}^{\widetilde{T}(\epsilon)} d_5^t ||\nabla \widetilde{G}^t||^2 \\
&\leq F^2 - \underline{L} + \tfrac{4}{\eta_\lambda}(\tfrac{c_1^0}{c_1^1} + \tfrac{c_1^1}{c_1^2})M\alpha_3 + \tfrac{c_1^1}{2}M\alpha_3 + \tfrac{7}{2\eta_\lambda}M{\sigma_3}^2 + 3d_4^0(c_2^1)^2 M\alpha_3 \\
&+ \tfrac{4}{\eta_{\boldsymbol{\theta}}}(\tfrac{c_1^0}{c_1^1} + \tfrac{c_1^1}{c_1^2})N\alpha_4 + \tfrac{c_2^1}{2}N\alpha_4 + \tfrac{7}{2\eta_{\boldsymbol{\theta}}}N{\sigma_4}^2 + 3\tau d_4^0(c_2^1)^2 N\alpha_4 \\
&+ (\tfrac{2N\alpha_4}{5\eta_{\boldsymbol{\theta}}} + 4d_5^0(\tfrac{2}{\underline{\eta_{\boldsymbol{x}}}^2} + 5NL^2)N{\alpha_1}^2\tau + 4d_5^0(\tfrac{2}{\underline{\eta_{\boldsymbol{y}}}^2} + 3NL^2)N{\alpha_1}^2\tau)(\tau - 1) \\
&= \bar{d} + k_d\tau(\tau - 1),
\end{aligned} \tag{147}$$

where $\bar{d}$ and k_d are constants. d_6 is given by,

$$\begin{aligned}
d_6 &= \max\{d_1, d_2, d_3, d_4, \tfrac{\frac{30}{\eta_\lambda} + 90\eta_\lambda\tau k_1 NL^2}{(1 - 15\eta_\lambda\tau k_1 NL^2)a_6^0}, \tfrac{30\tau}{\eta_{\boldsymbol{\theta}} a_6^0}\} \\
&\geq \max\{d_1, d_2, d_3, d_4, \tfrac{\frac{30}{\eta_\lambda} + 90\eta_\lambda\tau k_1 NL^2}{(1 - 15\eta_\lambda\tau k_1 NL^2)a_6^t}, \tfrac{30\tau}{\eta_{\boldsymbol{\theta}} a_6^t}\} \\
&= \tfrac{1}{d_5^t a_6^t}
\end{aligned} \tag{148}$$

Thus, we can obtain that,

$$\sum_{t=2}^{\widetilde{T}(\epsilon)} \frac{1}{d_6 a_6^t}||\nabla \widetilde{G}^{\widetilde{T}(\epsilon)}||^2 \leq \sum_{t=2}^{\widetilde{T}(\epsilon)} \frac{1}{d_6 a_6^t}||\nabla \widetilde{G}^t||^2 \leq \sum_{t=2}^{\widetilde{T}(\epsilon)} d_5^t ||\nabla \widetilde{G}^t||^2 \leq \bar{d} + k_d\tau(\tau - 1). \tag{149}$$

And it follows from Eq. (149) that,

$$||\nabla \widetilde{G}^{\widetilde{T}(\epsilon)}||^2 \leq \frac{(\bar{d} + k_d\tau(\tau - 1))d_6}{\sum_{t=2}^{\widetilde{T}(\epsilon)} \frac{1}{a_6^t}}. \tag{150}$$

According to the setting of c_1^t, c_2^t, we have,

$$\frac{1}{a_6^t} \geq \frac{1}{4(\gamma - 2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}})(t+1)^{\frac{1}{2}} + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2}}. \tag{151}$$

Summing up $\frac{1}{a_6^t}$ from $t = 2$ to $t = \widetilde{T}(\epsilon)$, it follows that,

$$\begin{aligned}
\sum_{t=2}^{\widetilde{T}(\epsilon)} \tfrac{1}{a_6^t} &\geq \sum_{t=2}^{\widetilde{T}(\epsilon)} \tfrac{1}{4(\gamma-2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}})(t+1)^{\frac{1}{2}} + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2}} \\
&\geq \sum_{t=2}^{\widetilde{T}(\epsilon)} \tfrac{1}{4(\gamma-2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}})(t+1)^{\frac{1}{2}} + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2}(t+1)^{\frac{1}{2}}} \\
&\geq \tfrac{\widetilde{T}(\epsilon)^{\frac{1}{2}} - 2}{4(\gamma-2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}}) + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2}}.
\end{aligned} \tag{152}$$

The second inequality in Eq. (152) is due to that $\forall t \geq 2$, we have,

$$4(\gamma - 2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}})(t+1)^{\frac{1}{2}} + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2} \leq (4(\gamma - 2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}}) + \frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2})(t+1)^{\frac{1}{2}}. \tag{153}$$

The last inequality in Eq. (152) follows from the fact that $\sum_{t=2}^{\widetilde{T}(\epsilon)} \frac{1}{(t+1)^{\frac{1}{2}}} \geq \widetilde{T}(\epsilon)^{\frac{1}{2}} - 2$.

Thus, plugging Eq. (152) into Eq. (150), we can obtain:

$$||\nabla \widetilde{G}^{\widetilde{T}(\epsilon)}||^2 \leq \frac{(\bar{d}+k_d\tau(\tau-1))d_6}{\sum_{t=2}^{\widetilde{T}(\epsilon)} \frac{1}{a_6^t}} \leq \frac{(4(\gamma-2)L^2(M\eta_\lambda+N\eta_{\boldsymbol{\theta}})+\frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2})(\bar{d}+k_d\tau(\tau-1))d_6}{\widetilde{T}(\epsilon)^{\frac{1}{2}}-2}. \tag{154}$$

Let $d_7 = 4(\gamma-2)L^2(M\eta_\lambda + N\eta_{\boldsymbol{\theta}})$, and according to the definition of $\widetilde{T}(\epsilon)$, we have:

$$\widetilde{T}(\epsilon) \geq (\frac{4(d_7+\frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2})^2(\bar{d}+k_d\tau(\tau-1))d_6}{\epsilon}+2)^2. \tag{155}$$

Combining the definition of ∇G^t and $\nabla \widetilde{G}^t$ with trigonometric inequality, we then get:

$$||\nabla G^t|| - ||\nabla \widetilde{G}^t|| \leq ||\nabla G^t - \nabla \widetilde{G}^t|| \leq \sqrt{\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||c_1^{t-1}\lambda_l^t||^2 + \sum_{j=1}^{N} ||c_2^{t-1}\boldsymbol{\theta}_i^t||^2}. \tag{156}$$

If $t > (\frac{4M\alpha_3}{\eta_\lambda{}^2}+\frac{4N\alpha_4}{\eta_{\boldsymbol{\theta}}{}^2})^2\frac{1}{\epsilon^2}$, then we have $\sqrt{\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||c_1^{t-1}\lambda_l^t||^2 + \sum_{j=1}^{N} ||c_2^{t-1}\boldsymbol{\theta}_i^t||^2} \leq \frac{\epsilon}{2}$. Combining it with Eq. (155), we can conclude that there exists a

$$T(\epsilon) \sim \mathcal{O}(\max\{(\frac{4M\alpha_3}{\eta_\lambda{}^2}+\frac{4N\alpha_4}{\eta_{\boldsymbol{\theta}}{}^2})^2\frac{1}{\epsilon^2}, (\frac{4(d_7+\frac{\eta_{\boldsymbol{\theta}}(N-S)L^2}{2})^2(\bar{d}+k_d\tau(\tau-1))d_6}{\epsilon}+2)^2\}), \tag{157}$$

such that $||\nabla G^t|| \leq ||\nabla \widetilde{G}^t|| + \sqrt{\sum_{l=1}^{|\boldsymbol{\mathcal{P}}^t|} ||c_1^{t-1}\lambda_l^t||^2 + \sum_{j=1}^{N} ||c_2^{t-1}\boldsymbol{\theta}_i^t||^2} \leq \epsilon$, which concludes our proof.

C Proof of Theorem 1

Assuming that there are cutting planes added every k iteration, i.e.,

$$\boldsymbol{\mathcal{P}}^0 \supseteq \boldsymbol{\mathcal{P}}^k \supseteq \cdots \supseteq \boldsymbol{\mathcal{P}}^{nk}. \tag{158}$$

Let $\mathcal{R}^k$ denote the feasible region of problem in Eq. (12) in k^{th} iteration, and let $\mathcal{R}'$ denote the feasible region of problem in Eq. (10), we have that,

$$\mathcal{R}^0 \supseteq \mathcal{R}^k \supseteq \cdots \supseteq \mathcal{R}^{nk} \supseteq \mathcal{R}'. \tag{159}$$

Let $F(\{\boldsymbol{x}_i^{k*}\}, \{\boldsymbol{y}_i^{k*}\}, \boldsymbol{v}^{k*}, \boldsymbol{z}^{k*})$ denote the optimal objective value of the problem in Eq. (12) in k^{th} iteration and let F^* denotes the optimal objective value of the problem in Eq. (10). According to Eq. (159), we have that,

$$F(\{\boldsymbol{x}_i^{0*}\},\{\boldsymbol{y}_i^{0*}\},\boldsymbol{v}^{0*},\boldsymbol{z}^{0*}) \leq F(\{\boldsymbol{x}_i^{k*}\},\{\boldsymbol{y}_i^{k*}\},\boldsymbol{v}^{k*},\boldsymbol{z}^{k*}) \leq \cdots \leq F(\{\boldsymbol{x}_i^{nk*}\},\{\boldsymbol{y}_i^{nk*}\},\boldsymbol{v}^{nk*},\boldsymbol{z}^{nk*}). \tag{160}$$

And we can obtain that,

$$\frac{F^*}{F(\{\boldsymbol{x}_i^{0*}\},\{\boldsymbol{y}_i^{0*}\},\boldsymbol{v}^{0*},\boldsymbol{z}^{0*})} \geq \frac{F^*}{F(\{\boldsymbol{x}_i^{k*}\},\{\boldsymbol{y}_i^{k*}\},\boldsymbol{v}^{k*},\boldsymbol{z}^{k*})} \geq \cdots \geq \frac{F^*}{F(\{\boldsymbol{x}_i^{nk*}\},\{\boldsymbol{y}_i^{nk*}\},\boldsymbol{v}^{nk*},\boldsymbol{z}^{nk*})} \geq \beta. \tag{161}$$

It is seen from Eq. (161) that the sequence $\{\frac{F^*}{F(\{\boldsymbol{x}_i^{k*}\},\{\boldsymbol{y}_i^{k*}\},\boldsymbol{v}^{k*},\boldsymbol{z}^{k*})}\}$ is monotonically non-increasing. When $nk \to \infty$, the optimal objective value of the problem in Eq. (12) monotonically converges to β ($\beta \geq 1$).