On the Performance of Direct Loss Minimization for Bayesian Neural Networks

Anonymous Author(s)
Affiliation
Address
email

Abstract

Direct Loss Minimization (DLM) has been proposed as a pseudo-Bayesian method motivated as regularized loss minimization. Compared to variational inference, it replaces the loss term in the evidence lower bound (ELBO) with the predictive log loss, which is the same loss function used in evaluation. A number of theoretical and empirical results in prior work suggest that DLM can significantly improve over ELBO optimization for some models. However, as we point out in this paper, this is not the case for Bayesian neural networks (BNNs). The paper explores the practical performance of DLM for BNN, the reasons for its failure and its relationship to optimizing the ELBO, uncovering some interesting facts about both algorithms.

1 Introduction

In probabilistic machine learning tasks, we need to train our model to fit the distribution of data. From the frequentist view, we need to find a *single* set of parameters that best fits the data; while from a Bayesian view, we specify a prior distribution on the parameters, calculate the posterior, and then use the posterior to predict on new data. Let $D = \{(x^{(i)}, y^{(i)})\}_{i=1}^N$ be the dataset sampled i.i.d. from distribution $\mathcal{D}$. From now on, we use superscript with parentheses to denote the i-th instance. Let θ denote the parameters. The frequentist method chooses one best set of parameters θ^* and makes predictions on new data x^* such that $p(y^*|x^*) = p(y^*|\theta^*, x^*)$. Bayesian methods specify a prior $p(\theta)$, calculate the posterior

$$p(\theta|D) \propto p(\theta)p(D|\theta),$$

and make predictions on new data x^* as

$$p(y^*|x^*) = \mathbb{E}_{p(\theta|D)}[p(y^*|\theta, x^*)].$$

For simple models, the posterior can be computed analytically. But for complicated models, the posterior becomes intractable. One solution is to get samples from true posterior and calculate the objective, for example using MCMC. Another line of work aims to find a distribution q from an analytical distribution family $\mathcal{Q}$ that is closest to the posterior. When making prediction on new data x^*, we marginalize over q, that is, $p(y^*|x^*) = \mathbb{E}_{q(\theta)}[p(y^*|\theta, x^*)]$. A typical example is variational inference, which tries to minimize the KL divergence between the variational distribution and the true posterior:

$$\begin{aligned} q^*(\theta) &= \arg\min_{q\in\mathcal{Q}} \mathrm{KL}(q(\theta)||p(\theta|D)) \\ &= \arg\min_{q\in\mathcal{Q}} \mathbb{E}_{q(\theta)}[\log q(\theta) - \log p(\theta, D)] + \log p(D) \\ &= \arg\min_{q\in\mathcal{Q}} \sum_{(x,y)\in D} \mathbb{E}_{q(\theta)}[-\log p(y|\theta, x)] + \mathrm{KL}(q(\theta)||p(\theta)). \end{aligned} \tag{1}$$

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The last line is referred to as the Evidence Lower Bound (ELBO). The most common measure to evaluate the quality of predictions is negative log-loss (NLL):

$$l(q, (x^*, y^*)) = -\log \mathbb{E}_{q(\theta)}[p(y^*|\theta, x^*)]. \tag{2}$$

We use $l_{\text{test}}(q)$ to denote the averaged NLL on test set. We optimize eq (1) but hope to get lower NLL, i.e. eq (2). This suggests a discrepancy. If we care about the NLL, why not directly optimize NLL (2)? From this perspective the KL term in eq (1) is seen as a regularizer to prevent overfitting. This motivates the idea of Direct Loss Minimization(DLM) which has been studied by multiple authors:

$$q_{\text{DLM}}(\theta) = \arg\min_{q \in \mathcal{Q}} \sum_{(x,y)\in D} -\log \mathbb{E}_{q(\theta)}[p(y|\theta, x)] + \text{KL}(q(\theta)||p(\theta)). \tag{3}$$

Notice that the idea of DLM can be applied on various loss functions. We can replace the first term in eq (3) with any loss functions we are interested in. Similarly, the regularizer can be chosen among several options. However, in this paper we focus on the choice given above.

In practice, it is common to use a hyperparameter η to tune the KL regularizer. In addition, when it is hard to compute the expectations analytically we can use Monte Carlo samples to approximate them. With these modifications the objectives in eq (1) and eq (3)) become:

$$\bar{q}^{(\eta,M)}_{\text{ELBO}}(\theta) = \arg\min_{q \in \mathcal{Q}} \sum_{(x,y)\in D} \frac{1}{M} \sum_{m=1}^{M} [-\log p(y|\theta^{(m)}, x)] + \eta \text{KL}(q(\theta)||p(\theta)), \quad \theta^{(i)} \sim q(\theta); \tag{4}$$

$$\bar{q}^{(\eta,M)}_{\text{DLM}}(\theta) = \arg\min_{q \in \mathcal{Q}} \sum_{(x,y)\in D} -\log \frac{1}{M} \sum_{m=1}^{M} p(y|\theta^{(m)}, x) + \eta \text{KL}(q(\theta)||p(\theta)), \quad \theta^{(m)} \sim q(\theta). \tag{5}$$

Notice that eq (4) is an unbiased estimate of eq (1) while eq (5) is a *biased* estimate of eq (3).

2 Theoretical Motivation of DLM

Prior work motivates the use of DLM from a theoretical perspective. In this section we review some of these results. Specifically, Sheth and Khardon [2019] provide risk bounds for several variants of DLM. Here we focus on one result (see their appendix B) that uses a bounded optimization view. Suppose we restrict our distribution family $\mathcal{Q}$ to $\mathcal{Q}_A = \{q \in \mathcal{Q} \text{ s.t. } \text{KL}(q, p) \leq A\}$, and we perform empirical risk minimization (ERM) on $\mathcal{Q}_A$:

$$q^{(A)}_{\text{ERM}}(\theta) = \arg\min_{q \in \mathcal{Q}_A} \sum_{(x,y)\in D} -\log \mathbb{E}_{q(\theta)}[p(y|\theta, x)]. \tag{6}$$

Then with probability $1 - \delta$ over the choice of S, for all $q \in \mathcal{Q}_A$,

$$\mathbb{E}_{(x,y)\sim\mathcal{D}}\left[-\log\left(\mathbb{E}_{q^{(A)}_{\text{ERM}}(\theta)} p(y|\theta, x)\right)\right] \leq \mathbb{E}_{(x,y)\sim\mathcal{D}}\left[-\log(\mathbb{E}_{q(\theta)}(p(y|\theta, x)))\right] + \mathcal{O}\left(\sqrt{\frac{A}{N}} + \sqrt{\frac{\log\frac{1}{\delta}}{N}}\right). \tag{7}$$

This result holds when the log-loss is bounded, which can be achieved by replacing the log function with

$$\log^{(a)} p = \log((1-a)p + a) \tag{8}$$

or by further bounding the parameter space of distribution family $\mathcal{Q}$. We next observe that while the above holds for eq (6) we can obtain similar risk bounds for eq (3).

Proposition 1. *Let $A_\eta = KL(q^{(\eta)}_{DLM}||p)$. Then the solution of eq* (3)*, i.e., $q^{(\eta)}_{DLM}$, is also the solution of eq* (6) *with $A = A_\eta$.*

Proof. We prove the claim by contradiction. If there exists $q' \in \mathcal{Q}_{A_\eta}$ such that $\sum_{(x,y)\in D} -\log \mathbb{E}_{q'(\theta)}[p(y|\theta, x)] < \sum_{(x,y)\in D} -\log \mathbb{E}_{q^{(\eta)}_{\text{DLM}}(\theta)}[p(y|\theta, x)]$. Then

$$\begin{aligned}\sum_{(x,y)\in D} -\log \mathbb{E}_{q'(\theta)}[p(y|\theta, x)] + \eta \text{KL}(q'||p) &\leq \sum_{(x,y)\in D} -\log \mathbb{E}_{q'(\theta)}[p(y|\theta, x)] + \eta A_\eta \\ &< \sum_{(x,y)\in D} -\log \mathbb{E}_{q^{(\eta)}_{\text{DLM}}(\theta)}[p(y|\theta, x)] + \eta \text{KL}(q^{(\eta)}_{\text{DLM}}||p).\end{aligned}$$

This contradicts the fact that $q^{(\eta)}_{\text{DLM}}$ minimizes the log-loss plus regularizer. Thus, $q^{(\eta)}_{\text{DLM}} = q^{(A_\eta)}_{\text{ERM}}$. □

One can further extend eq (7) into a data dependent bound as in Theorem 10 in Meir and Zhang [2003]. Therefore DLM as motivated above enjoys some theoretical support.

A second theoretical perspective is given by the recent work of Morningstar et al. [2022]. This work presents a PAC-Bayes bound for $\bar{q}^{(\eta)}_{\text{DLM}}$ called PAC^m bound. With probability $1-\delta$, for any $q \in \mathcal{Q}$,

$$\begin{aligned}
&\mathbb{E}_{(x,y)\sim\mathcal{D}}[-\log \mathbb{E}_{q(\theta)}[p(y|\theta,x)]] \\
&\leq -\frac{1}{N}\sum_i \mathbb{E}_{q(\theta^M)}\left[\log\left(\frac{1}{M}\sum_m p\left(y^{(i)}|x^{(i)},\theta^{(m)}\right)\right)\right] + \frac{1}{\eta N}\text{KL}(q||p) \\
&\quad + \psi(\mathcal{D},\eta,M,N,p,\delta) + \frac{1}{\eta M N}\log\frac{1}{\delta},
\end{aligned} \tag{9}$$

where

$$\psi(\mathcal{D},\eta,M,N,p,\delta) = \frac{1}{\eta M N}\log \mathbb{E}_{D\sim\mathcal{D}^N}\mathbb{E}_{p(\theta^{(1:m)})}\left[\exp\left(\eta N M\cdot\Delta\left(D,\theta^{(1:M)}\right)\right)\right],$$

$$\Delta(D,\theta^{(1:M)}) = \frac{1}{N}\sum_i \log\left(\frac{1}{M}\sum_m p(y^{(i)}|x^{(i)},\theta^{(m)})\right) - \mathbb{E}_{(x,y)\sim\mathcal{D}}\left[\log\left(\frac{1}{M}\sum_m p(y^{(i)}|x^{(i)},\theta^{(m)})\right)\right].$$

The right-hand-side is minimized at $\bar{q}^{(\eta)}_{\text{DLM}}$. For the corresponding algorithm note that the terms on the last line of (9) do not depend on $\bar{q}^{(\eta)}_{\text{DLM}}$ and can be omitted in the optimization. This is different from previous analysis Sheth and Khardon [2017, 2019], that uses the predictive loss

$$\begin{aligned}
-\log \mathbb{E}_{q(\theta)}[p(y|x,\theta)] &= -\log \mathbb{E}_{q(\theta^{(1:M)})}\left[\frac{1}{M}\sum_m p(y|x,\theta^{(m)})\right] \\
&\leq -\mathbb{E}_{q(\theta^{(1:M)})}\left[\log\left(\frac{1}{M}\sum_m p\left(y|x,\theta^{(m)}\right)\right)\right]
\end{aligned} \tag{10}$$

for the data-dependent upper bound. When the outside expectation in (9) is implemented with a single multi-sample from $q()$, which is the case in Morningstar et al. [2022], both bounds (eq (7), eq (9)) lead to the same implementation. Theoretically, loss term in bound (7) is lower but it requires $\bar{q}^{(\eta)}_{\text{DLM}}$ to converge to $q^{(\eta)}_{\text{DLM}}$ to guarantee the performance Wei et al. [2021], while the bound (9) directly guarantees the performance of $\bar{q}^{(\eta)}_{\text{DLM}}$ with a higher loss as shown in eq (10).

Morningstar et al. [2022] also establishes the relationship between ELBO and DLM. Notice that with $M=1$, the right hand sides of eq (4) and eq (5) are the same. They also show that as M becomes larger, the data dependent bound when $M>1$ (i.e., the right hand side of eq (9)) is tighter than that when $M=1$, corresponding to the data dependent bound for ELBO.

3 Applications of DLM

DLM has already been applied in practice and is shown to yield good results. Sheth and Khardon [2017] apply DLM to the correlated topic model and it achieves lower predictive loss than ELBO. Jankowiak et al. [2020] explores the application of DLM in conjugate sparse Gaussian processes and Wei et al. [2021] extends this to non-conjugate sparse Gaussian processes. In those experiments, η is chosen appropriately through cross validation. For conjugate cases, where both the ELBO objective (1) and the DLM objective (3) can be computed exactly without approximation, DLM significantly outperforms ELBO; while for non-conjugate cases, Monte Carlo sampling is needed and DLM is better than or comparable to ELBO.

DLM has also been applied to BNNs. Dusenberry et al. [2020] apply both ELBO and DLM on a rank-1 parametrization of BNNs that they introduce. Their experiments show that DLM has higher NLL than ELBO. Morningstar et al. [2022] also compare ELBO and DLM on BNNs and conclude that DLM performs better than ELBO when data is misspecified, i.e. the data generating distribution is not inside the model space. To see this, they apply ELBO and DLM to independently predict pixels of the bottom half of an image given the top half, which encounters data misspecification as the pixels are not independent. When there is no evident data misspecification they show that, under the same

KL value, DLM performs slightly better than ELBO. However, the converged ELBO solution may not have the same KL value as the converged DLM solution. So the experiment is not sufficient to show that the converged DLM solution performs better than the converged ELBO solution.

4 DLM in Bayesian Neural Networks

In contrast with the positive evidence, our work found that DLM does not perform as well for Bayesian neural networks (BNNs). In BNN, θ represents the weights of the neural network and we use eq (4) and eq (5) with a mean-field diagonal Gaussian variational distribution $q(\theta_j) = \mathcal{N}(\mu_j, \sigma_j^2)$ and μ and σ^2 are vectors of the same dimension as the parameter space. BNNs normally have millions of parameters, so the KL-divergence term can be very high and using a high value of $\eta = 1$ leads to poor performance, hence we fix $\eta = 0.1$. Following Wilson et al. [2022], we set the prior variance to 0.05. We experimented with two neural network structures, AlexNet Krizhevsky et al. [2012] and PreResNet He et al. [2016] with depth 20 on four datasets, CIFAR10, CIFAR100, STL10, SVHN. For all experiments, we set the batch size to 512, use Adam optimizer with learning rate 0.001 and train for 500 epochs. In training, we set $M = 5$ in eq (4) and eq (5) for training and $M = 10$ for evaluation. Most combinations of datasets and structures have similar results (Figure 1) and behave similarly during exploration, so we only show the detailed exploration of using AlexNet on CIFAR10 and discuss exceptions in the Appendix. To simplify the presentation, we use upper case "ELBO" and "DLM" to indicate the solution trained with ELBO and DLM loss respectively, and use lower case "elbo" and "dlm" to denote the corresponding loss functions.

Observation 1. *ELBO performs better than DLM.*

DLM is worse than ELBO in most experiments as in Figure 2a. In a few cases, DLM has similar performance but as shown in Figure 1, it does not outperform ELBO in any of the experiments .

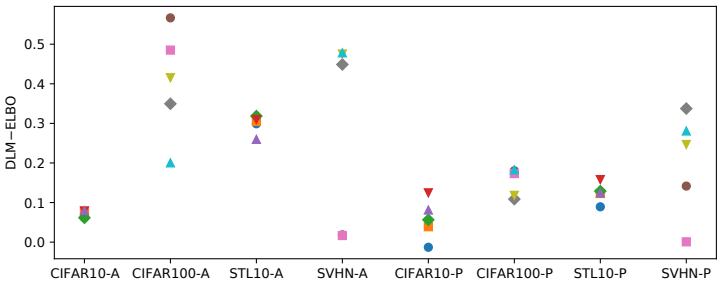


Figure 1: Comparison of ELBO and DLM in all experiments, "-A" means on AlexNet and "-P" means on PreResNet20. y-axis is $l_{\text{test}}(q_{\text{DLM}}) - l_{\text{test}}(q_{\text{ELBO}})$, and a positive value indicates that DLM performs worse than ELBO. Each point represents an independent run with random initialization.

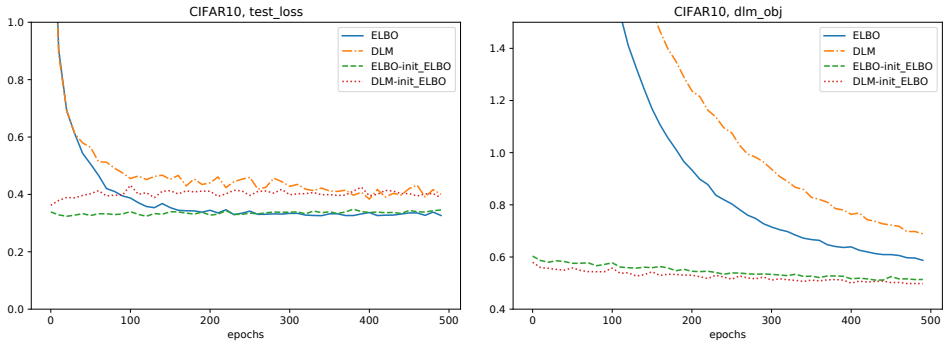
To better understand the behavior of ELBO and DLM, we also compute some quantities during the training, including elbo objective, dlm objective and KL divergence, and see how they change. To our surprise, we observe:

Observation 2. *ELBO appears to optimize the dlm objective better than the DLM algorithm.*

As shown in Figure 2b, ELBO (blue solid line) is below DLM (orange dashdot line). One might suspect that the reason for this is that the dlm objective is likely to get stuck in local optima. Thus if we initialize DLM with a good starting point, the DLM may be improved. To test this, we initialize DLM with ELBO solution and continue to train with dlm objective and we denote this solution as DLM-init_ELBO. For comparison we also have ELBO-init_ELBO which is initialized with ELBO and then continue to train with elbo objective. However, DLM is not improved with ELBO initialization and it even makes ELBO worse, as shown in Figure 2a. The increase of test loss for DLM shows that DLM is not stuck in local optima by accident, but is inherently worse than ELBO.

Observation 3. *The failure of DLM is not due to local optima.*

The good news is that with ELBO initialization, DLM optimizes the dlm objective better than ELBO, as shown in Figure 2b. Notice that this does not happen for every experiment. In Figure A.1b in the Appendix, DLM still optimizes dlm loss worse than ELBO even with ELBO initialization.



(a) test loss(↓)

(b) Trajectory of dlm loss

Figure 2: Comparison of ELBO and DLM with/without initialization

All these abnormalities lead us to further explore the structure of elbo and dlm losses. Motivated by Garipov et al. [2018], we create a path from ELBO to DLM, i.e. $\mu = (1-\alpha)\mu_{\text{ELBO}} + \alpha\mu_{\text{DLM}}, \sigma^2 = (1-\alpha)\sigma^2_{\text{ELBO}} + \alpha\sigma^2_{\text{DLM}}$, and then evaluate the elbo objective (with/without KL), dlm objective (with/without KL) and test loss on (μ, σ^2). It is clear that ELBO corresponds to $\alpha = 0$ and DLM corresponds to $\alpha = 1$. From the loss surface plotted in Figure 3a, we confirm Observation 1 and 2 (note that the dlm objective is lower at $\alpha = 0$). Figure 3b plots the path from ELBO to DLM-init_ELBO, and we can see how optimizing with dlm loss will change these loss functions.

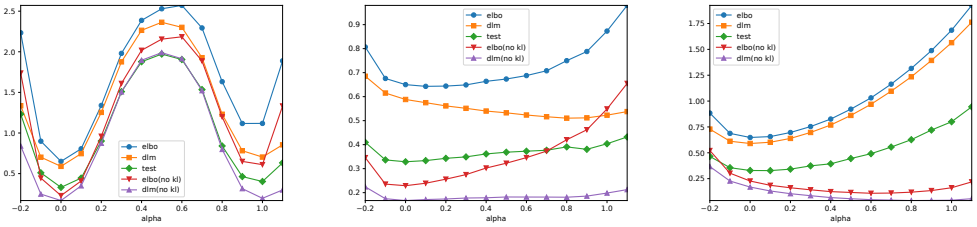
Observation 4. *The elbo objective with $\eta = 0.1$ is better aligned with test loss than the dlm objective, which indicates that the elbo objective generalizes better.*

Figure 3b shows that as α increases, the dlm objective decreases, but both the elbo objective and the test loss decreases. In our experiments, Figure A.2b shows a different situation where the dlm objective also increases on CIFAR100 with AlexNet but the elbo objective and the test loss are always aligned. Observation 4 also explains the abnormality in Figure A.1b, in which DLM-init_ELBO significantly increases the dlm objective in first few epochs. This is because we optimize the dlm objective within a batch but plot the average dlm objective value among all batches. The poor generalization of the dlm objective may cause the value evaluated on other batches to increase and the overall value increases.

We also observe from Figure 3b that the dlm objective goes down but its loss term goes up, implying that the reduction in objective is due to the KL term. The same sensitivity regarding the *tradeoff between the loss term and regularizer* appears in other cases as well. To explore this we reduce η to 0 after initializing with ELBO. Figure 4a shows that reducing η to 0 makes both ELBO and DLM perform worse than their original version but the relationship of ELBO-init-no_kl and DLM-init-no_kl still follows Observation 1. Figure 4b again shows Observation 2, that the dlm objective ($\eta = 0$) achieves lower value at ELBO-init-no_kl than at DLM-init-no_kl.

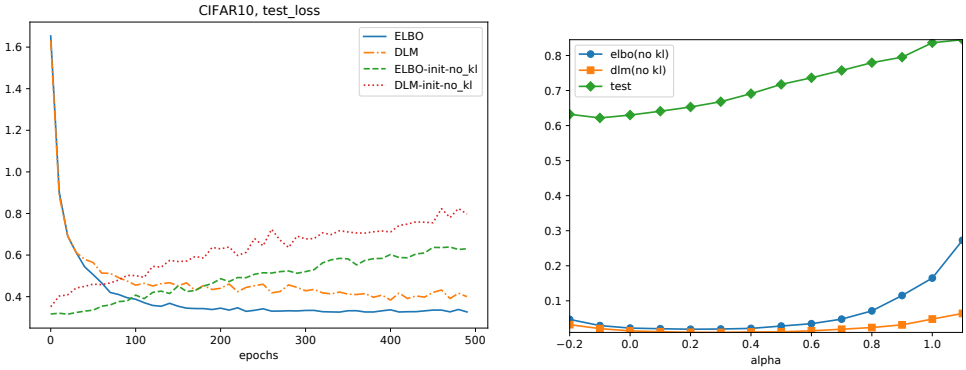
In contrast with Figure 4b, Figure 3c depicts the path between DLM-init-no_kl and the original ELBO solution (which is the best), instead of ELBO-init-no_kl. Then we can see that neither the elbo loss nor the dlm loss without KL is aligned with the test loss, indicating overfitting. From another view, the three plots in Figure 3 depict the change of loss functions along three directions from ELBO. The elbo objective with $\eta = 0.1$ is aligned with the test loss in all three cases. But we cannot find such proper η for dlm. In (a) and (c), the dlm objective with $\eta = 0.1$ is aligned with the test loss, but in (b) the dlm objective with $\eta = 0$ is aligned with the test loss. All these results support Observation 4.

In addition to the work mentioned above, we have explored bounded optimization, smoothed loss, collapsed variational inference Tomczak et al. [2021] and empirical Bayes Wu et al. [2019]. The first two measures aim to close the gap between theoretical analysis and real applications so that we can utilize the upper bounds to guarantee the performance of DLM. The latter two measures



(a) ELBO and DLM (b) ELBO and DLM-init_ELBO (c) ELBO and DLM-init-no_kl

Figure 3: Loss Surface



(a) test loss (b) ELBO-init-no_kl and DLM-init-no_kl

Figure 4: Test Performance and Loss Surface for $\eta = 0$

define a hierarchical model and performs inference on the prior parameters as well, which results in a different regularizer to replace the original KL divergence. Although these measures can sometimes improve the performance of DLM, they do not help DLM outperform ELBO. We leave the details in Appendix.

Overall, we found that at least one of Observation 2 and 4 appears in all experiments. In cases where ELBO does not optimize the dlm objective better than DLM, Observation 4 kicks in and shows that optimizing the dlm objective cannot make the performance better; In cases where the dlm objective is aligned with the test loss, we find that ELBO optimizes the dlm objective better. Thus, none of the variants of DLM mentioned in this paper outperforms ELBO.

5 Conclusion and Future Work

Direct loss minimization has a strong motivation that we should use the same loss function in both training and testing. During training, we add a regularizer to prevent overfitting. Many theoretical results guarantee the performance of DLM optimizers. Despite its empirical success in sparse Gaussian processes, we observe that such success does not appear for Bayesian neural networks. In empirical exploration, we found that the goal of DLM is also severely challenged as ELBO optimizes dlm objective better than DLM itself. The most likely reason for this is that the dlm objective is hard to optimize. Besides, DLM generalizes worse than ELBO, because elbo loss is more consistent with test loss, pointing out overfitting of the dlm objective. This relates to data misspecification as suggested in Morningstar et al. [2022] but how to test the notion of misspecification in image classification remains unclear as the neural networks are expressive.

The failure of DLM in BNNs may be related to the notion of surrogate loss. With Jensen's inequality we can see that the elbo loss is an upper bound of the dlm loss, and thus the elbo loss can be considered as the surrogate loss of dlm. Thus, using a surrogate loss may not only make the computation easy (original losses are often non-differentiable), but also make the performance better.

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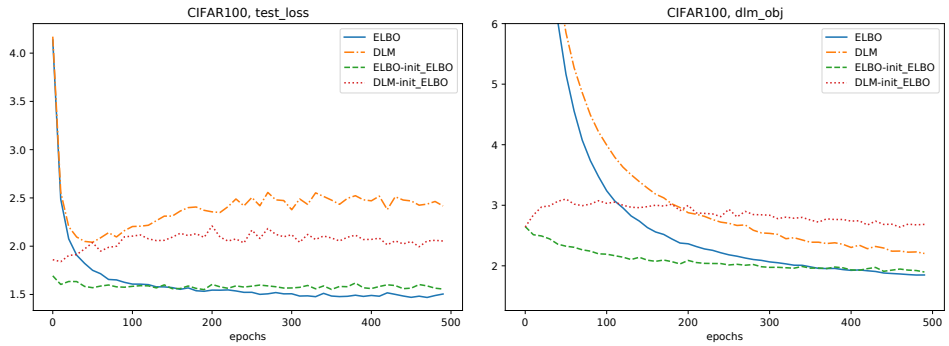
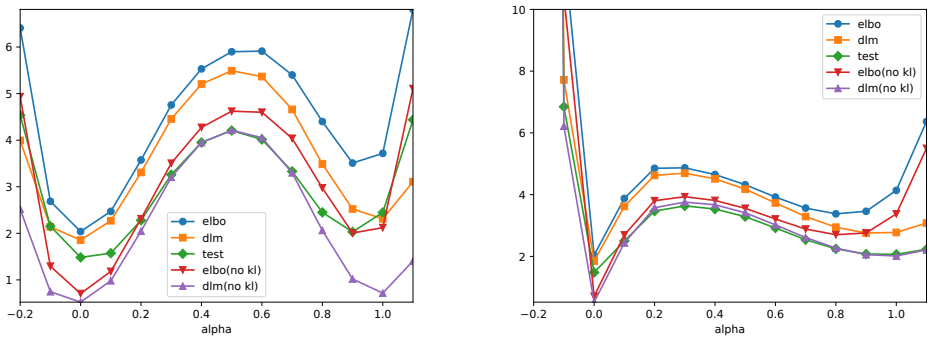
A Additional Results

A.1 ELBO initialization on CIFAR100 using AlexNet



(a) test loss(↓) (b) Trajectory of dlm loss

Figure A.1: Comparison of ELBO and DLM with/without initialization, on CIFAR100 with AlexNet



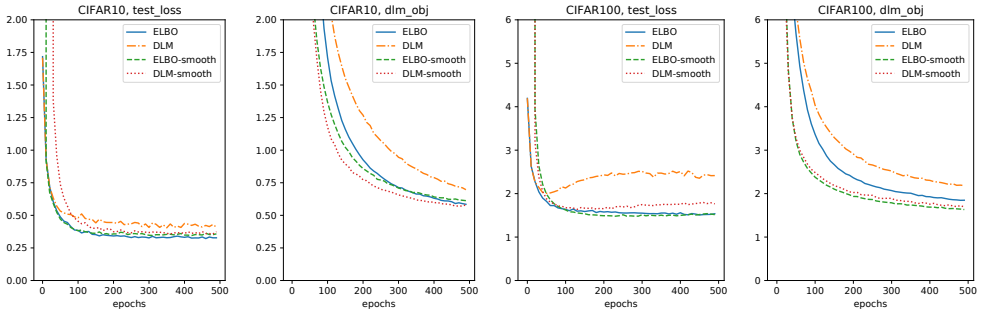
(a) ELBO and DLM (b) ELBO and DLM-init_ELBO

Figure A.2: Loss Surface, on CIFAR100 with AlexNet

Results for AlexNet on CIFAR100 are shown in Figure A.1 and A.2. The behavior of ELBO and DLM on CIFAR100 is similar to that on CIFAR10, except in Figure A.2b, where the dlm loss is also consistent with test loss. Experiments on CIFAR100 do not break the two major observations we make in Section 4.

A.2 Smoothed Loss

We use the smoothing factor $a = 0.001$ in $\log^{(a)} p = \log((1-a)p + a)$. We use the smoothed loss only in elbo loss and dlm loss in training not in testing. Figure A.3 shows that adding a smoothing factor does improve the performance of DLM, as in (a) and (c) DLM-smooth achieves lower test loss than DLM. However, DLM-smooth is still worse than ELBO. In Figure A.3b and A.3d we plot the dlm objectives during training. We observe that in Figure A.3b, ELBO does not optimize dlm loss better than DLM. However, since elbo loss generalizes better, DLM still perform worse than ELBO. This may indicate that the generalizability plays a more important role in explaining the failure of DLM.

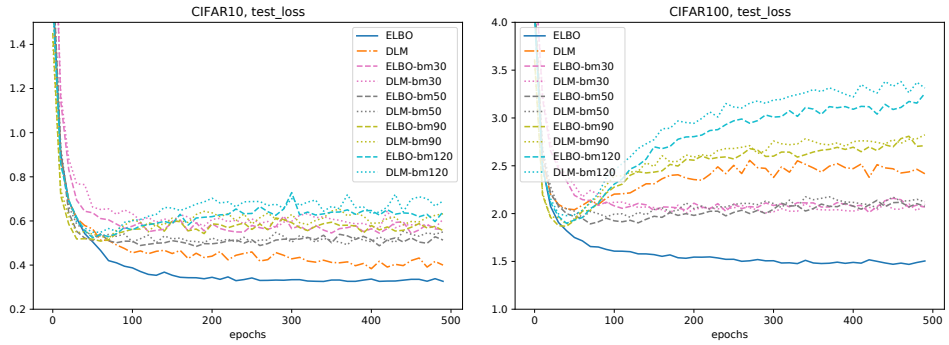


(a) test loss, CIFAR10 (b) dlm loss, CIFAR10 (c) test loss, CIFAR100 (d) dlm loss, CIFAR100

Figure A.3: Comparison on losses(↓) of ELBO and DLM with/without smoothing

A.3 Bounded Optimization

It is hard to directly bound the KL divergence, so we instead bound the $\|\mu\|_2 \leq b_m$ and $\|\sigma^2\|_\infty \leq b_v$. By analyzing the original solution of ELBO and DLM, we found $\|\mu_{\text{ELBO}}\|_2$ and $\|\mu_{\text{DLM}}\|_2$ are around 80 to 90 and $\|\sigma^2\|_\infty$ is around 0.25. So we fix $b_v = 0.25$ and choose $b_m \in \{30, 50, 90, 120\}$.



(a) CIFAR10, test loss(↓) (b) CIFAR100, test loss(↓)

Figure A.4: Bounded Optimization

From Figure A.4 we can see that the original ELBO still outperforms all other methods, including the original DLM and the solutions with bounded optimization. One observation is that with bounded optimization, DLM is closer to ELBO.

A.4 Collapsed Varitional Inference

Tomczak et al. [2021] proposed a hierarchical model and performed variational inference on the prior parameters as well. The variational distribution of the prior parameters can be optimized and by marginalizing over prior parameters, a new evidence lower bound is derived. Let D be the total dimension of the weights. The new evidence lower bound replaces the original regularizer, the KL divergence, with a new one:

$$\frac{1}{2\gamma}[1^\top \sigma^2 + \alpha_{reg}\mu^\top \mu] - \frac{1}{2}1^\top \log \sigma^2 - \frac{D}{2}\log \alpha_{reg}, \tag{11}$$

if we fix prior variance but learn prior mean (we call this "mean"), or

$$(\alpha + \frac{1}{2})1^\top \log\left[\beta 1 + \frac{\delta}{2}\mu^2 + \frac{1}{2}\sigma^2\right] - \frac{1}{2}1^\top \log \sigma^2,$$

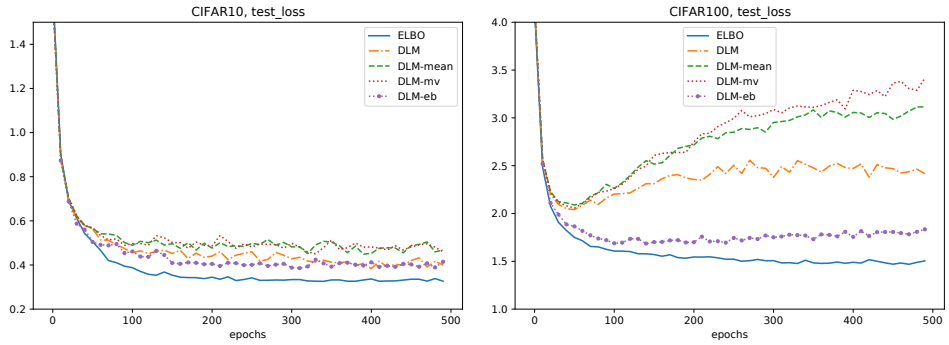
if we learn both mean and variance (we call this "mv"). We choose $\gamma = 0.3, \alpha_{reg} = 0.05$ in eq (11), $\alpha = 0.5, \beta = 0.01, \delta = 0.1$, following the standard settings in Tomczak et al. [2021].

Empirical Bayes Wu et al. [2019] can be viewed as a special case of collapsed variational inference by restricting the variational distribution of σ^2 to be a delta distribution and the new regularizer it derives is (we call this "eb"):

$$\frac{1}{2}\left[D\log\frac{\mu^\top\mu+1^\top\sigma^2+2\beta}{D+2\alpha+2}-1^\top\log|\sigma^2|-D\right]+\frac{1}{2}\frac{D+2\alpha+2}{\mu^\top\mu+1^\top\sigma^2+2\beta}(\mu^\top\mu+1^\top\sigma^2), \quad (12)$$

where we set $\alpha = 4.4798$ and $\beta = 10$ following Wu et al. [2019].

Figure A.5 shows the result of applying the new regularizers to DLM. Unlike ELBO, the new regularizers "mean" and "mv" do not improve the performance of DLM and even make it worse. In contrast, "eb" does improve the performance of DLM but this variant of DLM is still worse than ELBO.



(a) CIFAR10, test loss(↓)

(b) CIFAR100, test loss(↓)

Figure A.5: Collapsed Variational Inference