ProxSkip for Stochastic Variational Inequalities: A Federated Learning Algorithm for Provable Communication Acceleration

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Abstract

Recently Mishchenko et al. [12] proposed and analyzed ProxSkip, a provably efficient method for minimizing the sum of a smooth (f) and an expensive nonsmooth proximable (R) function (i.e. $\min_{x\in\mathbb{R}^d} f(x) + R(x)$). The main advantage of ProxSkip, is that in the federated learning (FL) setting, offers provably an effective acceleration of communication complexity.

This work extends this approach to the more general regularized variational inequality problems (VIP). In particular, we propose ProxSkip-VIP algorithm, which generalizes the original ProxSkip framework of [12] to VIP, and we provide convergence guarantees for a class of structured non-monotone problems. In the federated learning setting, we explain how our approach achieves acceleration in terms of the communication complexity over existing state-of-the-art FL algorithms.

1. Introduction

Min-max optimization and, more generally, variational inequality problems (VIPs) appear in a wide range of applications in machine learning including Generative Adversarial Networks (GANs) [7], adversarial training of neural networks [10, 15] and distributionally robust learning [16]. Motivated by these applications, in this work, we consider the following regularized variational inequality problem (VIP): find $x^* \in \mathbb{R}^d$, such that

$$\langle F(x^*), x - x^*\rangle + R(x) - R(x^*) \geq 0, \quad \forall x \in \mathbb{R}^d, \tag{1}$$

where $F : \mathbb{R}^d \to \mathbb{R}$ and $R : \mathbb{R}^d \to \mathbb{R}$ is the regularizer (a proper lower semicontinuous convex function). This problem is quite general and covers a wide range of possible problem formulations. For example, special cases of (1) are the regularized minimization problems [11] and minimax problems [1]:

$$\min_{x\in\mathbb{R}^d} f(x) + R(x) \quad \text{and} \quad \min_{x_1\in\mathbb{R}^{d_1}} \max_{x_2\in\mathbb{R}^{d_2}} f(x_1, x_2) + R(x_1, x_2). \tag{2}$$

In this work, we are interested in the situations when operator F is accessible through the calls of unbiased stochastic oracle. This is natural when F has an expectation form $F(x) = \mathbb{E}_{\xi\sim\mathcal{D}}[F_\xi(x)]$ or a finite-sum form $F(x) = \frac{1}{n}\sum_{i=1}^n F_i(x)$. In this scenario, one of the most popular algorithms for solving (1) is the stochastic proximal method[13]

$$x_{t+1} = \mathbf{prox}_{\gamma R}(x_t - \gamma g_t),$$

where $\mathbf{prox}_{\gamma R}(x) \triangleq \operatorname{argmin}_{z\in\mathbb{R}^d} \left\{R(z) + \frac{1}{2\gamma}\|z - x\|^2\right\}$, g_t is an unbiased estimator of $F(x_t)$ (i.e. $\mathbb{E}[g_t] = F(x_t)$) and $\gamma > 0$ is the step-size of the method.

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Typically computing the proximal operator is easy and cheap. However, in our work, following the approach of Mishchenko et al. [12], we are interested in the situation when the evaluation of the proximity operator is expensive. That is, we assume that the computation of $\mathbf{prox}_{\gamma R}$ is costly relative to the evaluation of the unbiased estimator g_t. It is in this scenario that ProxSkip-VIP (Alg. 1) thrives, as it skips the evaluation of the proximity operator and it requires its computation only once every few iterations.

In the federated learning setting (see Sec. 4 for more details) ProxSkip-VIP can be interpreted as a new distributed method performing local steps. In that scenario, ProxSkip-VIP becomes equivalent to the update rule of Algorithm (2) (ProxSkip-VIP-FL) where the computation of the proximity operator becomes equivalent to communications between workers. Thus, skipping proximity operator's computation means that the algorithm performs local updates (it skips communication). See [12] for the full exposition of this connection.

Main Contributions Our main contributions are summarized below:

- We generalize the ProxSkip framework proposed in [12] for minimization problems into the VIP regime, and proposed the ProxSkip-VIP algorithm.
- We prove that ProxSkip-VIP converges linearly to a neighborhood of the optimal set when problem (1), has μ-quasi-strongly monotone and L-star-cocoercive operator F. This is a class of structure non-monotone problems. As a corollary of our results, for the deterministic regime where $g_t = F(x_t)$, ProxSkip-VIP converges linearly to the exact solution.
- We extend the ProxSkip-VIP method into the federated learning setting, and propose ProxSkip-VIP-FL; we show that the algorithm enjoys an improved communication complexity over existing literature of local Stochastic Methods for solving VIPs.

2. Preliminaries

First, we introduce the main settings throughout the paper.

Assumption 1 (Main Settings) *We assume that problem (1) has a unique[1] solution x^* and that:*

1. *The operator F is μ-quasi-strongly monotone and ℓ-star-cocoercive with $\mu, L > 0$, i.e.,*

$$\langle F(x) - F(x^*), x - x^* \rangle \geq \mu \|x - x^*\|^2, \quad \langle F(x) - F(x^*), x - x^* \rangle \geq \frac{1}{\ell} \|F(x) - F(x^*)\|^2. \quad (3)$$

2. *The function R is a proper lower semicontinuous convex function.*

Note that given that an operator F is L'-Lipschitz continuous and μ'-strongly monotone, it can be shown that the operator F is $(\kappa L')$-star-cocoercive with $\kappa = L'/\mu'$ [9].

The convergence results in this paper will depend on the following operator noise at x^* that is finite for any reasonable sampling: $\sigma^2 \triangleq \mathrm{Var}(g(x^*; \xi)) < +\infty$. Regarding the inherent stochasticity, we further use the following expected cocoercivity assumption [5, 9] to characterize the behavior of the operator estimation

[1] This assumption can be relaxed; but for simplicity of exposition we enforce it.

Assumption 2 (Expected Cocoercivity) *We assume that stochastic operator $g(x;\xi)$ is such that for all $x \in \mathbb{R}^d$ there is $L > 0$:*

$$\mathbb{E}\|g(x;\xi) - g(x^*;\xi)\|^2 \leq L\langle F(x) - F(x^*), x - x^*\rangle.$$

See [5, 9] for more details about this assumption and why is weaker among other bounds on he noise.

3. Algorithm: ProxSkip-VIP

In this section, we incorporate the ProxSkip algorithm [12] into our problem (1), and propose the following ProxSkip-VIP algorithm.

Algorithm 1 ProxSkip-VIP

Input: Initial point x_0, parameters $\gamma_1, \gamma_2, \gamma_3, p$, initial control variate h_0, number of iterations T

for all $t = 0, 1, ..., T$ **do**
  $\hat{x}_{t+1} = x_t - \gamma_1(g(x_t;\xi_t) - h_t)$
  Flip a coin θ_t, $\theta_t = 1$ w.p. p, otherwise 0
  if $\theta_t = 1$ **then**
    $x_{t+1} = \mathbf{prox}_{\gamma_2 R}(\hat{x}_{t+1} - \gamma_2 h_t)$
  else
    $x_{t+1} = \hat{x}_{t+1}$
  end if
  $h_{t+1} = h_t + \gamma_3(x_{t+1} - \hat{x}_{t+1})$
end for

Output: x_T

Here the key step is the randomized prox-skipping and the control variate h_t. The proximal oracle is called very rarely if p is small, which helps to reduce the computational cost if the proximal oracle is expensive; the introducing of h_t helps to stabilize the iterations toward the optimal point.

3.1. Convergence Analysis

Our main theorem on the convergence guarantee is presented below.

Theorem 1 (Convergence of ProxSkip-VIP) *Let Assumption 1 and 2 hold, and let $\gamma_1 = \gamma \in \left(0, \min\left\{\frac{1}{\mu}, \frac{1}{2L}\right\}\right)$, $\gamma_1 = \gamma_2 p$, $\gamma_3 = \frac{1}{\gamma_2}$. Then the iterates of ProxSkip (Alg. 1) satisfy*

$$\mathbb{E}[V_T] \leq \left(1 - \min\left\{\gamma\mu, p^2\right\}\right)^T V_0 + \frac{2\gamma^2\sigma^2}{\min\left\{\gamma\mu, p^2\right\}}. \tag{4}$$

where $V_t \triangleq \|x_t - x_t^*\|^2 + \gamma_2^2\|h_t - F(x_t^*)\|^2$.

We defer the proof to Appendix B. With the above theorem, we can get the corresponding complexity results in the following, whose proof is deferred to Appendix C.

Corollary 2 *With the setting in Theorem 1, if we further set $\gamma \leq \frac{\mu\epsilon}{2\sigma^2}$ and $p = \sqrt{\gamma\mu}$, we have $\mathbb{E}[V_T] \leq \epsilon$ with iteration complexity and the number of calls of the proximal oracle $\mathbf{prox}(\cdot)$ as*

$$\mathcal{O}\left(\max\left\{1, \frac{L}{\mu}, \frac{\sigma^2}{\mu^2\epsilon}\right\}\ln\left(\frac{1}{\epsilon}\right)\right) \quad \textit{and} \quad \mathcal{O}\left(\sqrt{\max\left\{1, \frac{L}{\mu}, \frac{\sigma^2}{\mu^2\epsilon}\right\}}\ln\left(\frac{1}{\epsilon}\right)\right). \tag{5}$$

Deterministic ProxSkip: As a corollary of our results, for the deterministic regime where $g_t = F(x_t)$, ProxSkip-VIP converges linearly to the exact solution since $\sigma^2 = 0$. In this scenario, under the same assumptions with Theorem 1, the iterates of ProxSkip (Alg. 1) satisfy:

$$\mathbb{E}[V_T] \leq \left(1 - \min\left\{\gamma\mu, p^2\right\}\right)^T V_0.$$

In this setting, we get $\mathbb{E}[V_T] \leq \epsilon$ with iteration complexity and number of calls of the proximal oracle $\mathbf{prox}(\cdot)$ as

$$\mathcal{O}\left(\frac{L}{\mu}\ln\left(\frac{1}{\epsilon}\right)\right) \quad \text{and} \quad \mathcal{O}\left(\sqrt{\frac{L}{\mu}}\ln\left(\frac{1}{\epsilon}\right)\right) \tag{6}$$

4. Connection with Federated Learning

Now we specialize the discussion into the federated learning setting, i.e., find $x^* \in \mathbb{R}^d$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathbb{R}^d, \tag{7}$$

where $F(x) \triangleq \frac{1}{n}\sum_{i=1}^n f_i(x)$ and $f_i(x) \triangleq \mathbb{E}_{\xi_i \sim \mathcal{D}_i}[f_i(x;\xi_i)]$, the data ξ_i follows an unknown distribution $\mathcal{D}_i$ $(i = 1, 2, \cdots, n)$. For example, the federated learning minimization or minimax problems [12, 14] are special cases of (7). As mentioned in [13], the problem (7) can be recast into the problem (1) while

$$F(x) \triangleq \frac{1}{n}\sum_{i=1}^n f_i(x_i), \quad f_i(x) \triangleq \mathbb{E}_{\xi_i \sim \mathcal{D}_i}[f_i(x;\xi_i)] \tag{8}$$

where $x_i \in \mathbb{R}^d$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^{nd}$, and

$$R(x) = R((x_1, x_2, \cdots, x_n)) \triangleq \begin{cases} 0 & \text{if } x_1 = x_2 = \cdots = x_n \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

Note that $\mathbf{prox}_{\gamma R}(x) = (\bar{x}, \bar{x}, \cdots, \bar{x})$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$, which is easy to compute [13].

4.1. Algorithm: ProxSkip-VIP-FL

With the above reformulation (8), we propose the following ProxSkip-VIP-FL algorithm based on Algorithm 1 for the federated learning problem 7, which is presented below.

Our algorithm is a unified framework which can attain many variants by applying different (local) operator estimator $g_i(x_{i,t};\xi_{i,t})$. Different from the centralized setting we discussed in Section 3, the federated learning framework often incurs the heterogeneous setting, i.e., the distributions $\{\mathcal{D}_i\}_i$ are not identical.

4.2. Convergence Analysis

Let us now present the convergence guarantees of ProxSkip-VIP-FL (Alg. 2).

Theorem 3 (Complexity of ProxSkip-VIP-FL) *Lets assume the same setting as in Corollary 2. Then ProxSkip-VIP-FL achieves $\mathbb{E}[V_T] \leq \epsilon$ (where V_T is defined in Theorem 1),*

Algorithm 2 ProxSkip-VIP-FL

Input: Initial point $x_{1,0} = x_{2,0} = \cdots = x_{n,0} \in \mathbb{R}^d$, parameters $\gamma_1, \gamma_2, \gamma_3, p \in \mathbb{R}$, initial control variate for each client $h_{1,0}, h_{2,0}, \cdots, h_{n,0} \in \mathbb{R}^d$, number of iterations T

for all $t = 0, 1, ..., T$ **do**
  Server: Flip a coin θ_t, $\theta_t = 1$ w.p. p, otherwise 0. Send θ_t to all workers
  for each workers $i \in [n]$ **in parallel do**
    $\hat{x}_{i,t+1} = x_{i,t} - \gamma_1(g_i(x_{i,t}; \xi_{i,t}) - h_{i,t})$ // Local update with control variate
    if $\theta_t = 1$ **then**
      Worker: $x'_{i,t+1} = \hat{x}_{i,t+1} - \gamma_2 h_{i,t}$, sends $x'_{i,t+1}$ to the server
      Server: computes $x_{i,t+1} = \frac{1}{n}\sum_{i=1}^n x'_{i,t+1}$ and send to workers // Communication
    else
      $x_{i,t+1} = \hat{x}_{i,t+1}$ // Otherwise skip the communication step
    end if
    $h_{i,t+1} = h_{i,t} + \gamma_3(x_{i,t+1} - \hat{x}_{i,t+1})$
  end for
end for

Output: x_T

- *with iteration complexity*

$$\mathcal{O}\left(\max\left\{\frac{L}{\mu}, \frac{\sigma^2}{\mu^2\epsilon}\right\}\ln\left(\frac{1}{\epsilon}\right)\right)$$

- *and communication complexity as*

$$\mathcal{O}\left(\sqrt{\max\left\{\frac{L}{\mu}, \frac{\sigma^2}{\mu^2\epsilon}\right\}}\ln\left(\frac{1}{\epsilon}\right)\right).$$

We further extend our discussion on the complexity of ProxSkip-VIP-FL regarding the heterogeneous setting in Appendix D.

Comparison with Existing Literature The above theorem provided the complexity results of Algorithm 2 in the federated learning setting. Note that many algorithms in federated learning can be cast into our framework. For example, in federated minimax problems, a well-known algorithm is the Local SGDA algorithm [6], if we apply their (mini-batch) gradient estimator into our Algorithm 2, our result will avoid the dependence on the condition number κ (when ϵ is small enough), and we can attain an improved communication and iteration complexity for solving (7).

We also compared our Algorithm 2 with existing literature in federated learning, as shown in Table 1. We can find that our algorithm extended the problem settings, also outperforms over other algorithms in terms of both the iteration and communication complexities (when ϵ is small enough).

Algorithm	**Setting**[1]	**# Communication**[2]	**# Iteration**
Local SEG [3, 4]	SM, LS	$\mathcal{O}\left(\max\left(\kappa \ln \frac{1}{\epsilon}, \frac{p\sigma^2}{\mu^2 n\epsilon}, \frac{\kappa\xi}{\mu\sqrt{\epsilon}}, \frac{\sqrt{p}\kappa\sigma}{\mu\sqrt{\epsilon}}\right)\right)$	$\mathcal{O}\left(\max\left(\frac{\kappa}{p} \ln \frac{1}{\epsilon}, \frac{\sigma^2}{\mu^2 n\epsilon}, \frac{\kappa\xi}{p\mu\sqrt{\epsilon}}, \frac{\kappa\sigma}{\mu\sqrt{p\epsilon}}\right)\right)$
Local SGDA [6]	SM, LS	$\mathcal{O}\left(\sqrt{\frac{\kappa^2(\xi^2+\sigma^2)}{\mu\epsilon}}\right)$	$\mathcal{O}\left(\frac{\kappa^2(\xi^2+\sigma^2)}{\mu\epsilon}\right)$
FedAvg-S [8]	SM, LS	$\tilde{\mathcal{O}}\left(\frac{p\sigma^2}{n\mu^2\epsilon} + \frac{\sqrt{p}\kappa\sigma}{\mu\sqrt{\epsilon}} + \frac{\kappa\xi}{\mu\sqrt{\epsilon}}\right)$	$\tilde{\mathcal{O}}\left(\frac{\sigma^2}{n\mu^2\epsilon} + \frac{\kappa\sigma}{\mu\sqrt{p\epsilon}} + \frac{\kappa\xi}{p\mu\sqrt{\epsilon}}\right)$
Ours (Theorem 3)	SM, LS[3]	$\tilde{\mathcal{O}}\left(\sqrt{\max\left\{\kappa^2, \frac{\sigma^2}{\mu^2\epsilon}\right\}}\right)$	$\tilde{\mathcal{O}}\left(\max\left\{\kappa^2, \frac{\sigma^2}{\mu^2\epsilon}\right\}\right)$

[1] SM: strongly monotone, LS: (Lipschitz) smooth. $\kappa \triangleq L/\mu$, L and μ are the modulus of SM and LS. The definition of ξ^2 follows that in Appendix D, and $\sigma^2 \triangleq \mathrm{Var}(g(x^*;\xi)) < +\infty$ (or uniform bound on $\mathrm{Var}(g(\cdot;\xi))$).

[2] p is the probability of synchronization, by setting ϵ is small enough, we can take $p = \mathcal{O}(\sqrt{\epsilon})$, which recovers $\mathcal{O}(1/\sqrt{\epsilon})$ communication complexity dependence on ϵ in our result. $\tilde{\mathcal{O}}(\cdot)$ hides the logarithmic terms.

[3] Our algorithm works for quasi-strongly monotone and star-cocoercive, which is more general than the SM and LS setting, note that an L-LS and μ-SM operator can be shown to be (κL)-star-cocoercive [9].

Table 1: Comparison of Different Algorithms

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Supplementary Material

Appendix A. Useful Lemmas

Lemma 4 *For any optimal solution $x^* \in X^*$ of (1), we have*

$$x^* = \mathbf{prox}_{\gamma_2 R}(x^* - \gamma_2 F(x^*)). \tag{10}$$

Proof Note that

$$\langle F(x^*), x - x^* \rangle + R(x) - R(x^*) \geq 0, \tag{11}$$

next for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
& R(x^*) + \frac{1}{2\gamma_2}\|x^* - x^* + \gamma_2 F(x^*)\|^2 \leq R(x) + \frac{1}{2\gamma_2}\|x - x^* + \gamma_2 F(x^*)\|^2 \\
\iff \quad & R(x^*) + \frac{1}{2\gamma_2}\|\gamma_2 F(x^*)\|^2 \leq R(x) + \frac{1}{2\gamma_2}\|x - x^*\|^2 + \frac{1}{2\gamma_2}\|\gamma_2 F(x^*)\|^2 + \langle F(x^*), x - x^* \rangle \\
\iff \quad & R(x^*) \leq R(x) + \frac{1}{2\gamma_2}\|x - x^*\|^2 + \langle F(x^*), x - x^* \rangle \\
\Longleftarrow \quad & R(x^*) \leq R(x) + \langle F(x^*), x - x^* \rangle,
\end{aligned}$$

which concludes the proof, note that this conclusion indicates that the two parameters in the RHS above should be identical. ■

Lemma 5 (Firm Nonexpansivity of the Proximal Operator [2]) *Let f be a proper closed and convex function, then for any $x, y \in \mathbb{R}^d$ we have*

$$\big\langle x - y, \mathbf{prox}_f(x) - \mathbf{prox}_f(y) \big\rangle \geq \big\|\mathbf{prox}_f(x) - \mathbf{prox}_f(y)\big\|^2, \tag{12}$$

or equivalently,

$$\big\|\big(x - \mathbf{prox}_f(x)\big) - \big(y - \mathbf{prox}_f(y)\big)\big\|^2 + \big\|\mathbf{prox}_f(x) - \mathbf{prox}_f(y)\big\|^2 \leq \|x - y\|^2. \tag{13}$$

The following result is helpful in the proof.

Lemma 6 *With Assumption 1 and 2, we have*

$$\mathbb{E}\|g(x;\xi) - F(x^*)\|^2 \leq 2L\langle F(x) - F(x^*), x - x^* \rangle + 2\sigma^2. \tag{14}$$

Proof

$$\begin{aligned}
\mathbb{E}\|g(x;\xi) - F(x^*)\|^2 &\leq 2\mathbb{E}\Big[\|g(x;\xi) - g(x^*;\xi)\|^2 + \|g(x^*;\xi) - F(x^*)\|^2\Big] \\
&\leq 2L\langle F(x) - F(x^*), x - x^* \rangle + 2\sigma^2,
\end{aligned} \tag{15}$$

which concludes the proof. ■

Appendix B. Proof of Theorem 1

Proof Note that

$$x_{t+1} = \begin{cases} \mathbf{prox}_{\gamma_2 R}(\hat{x}_{t+1} - \gamma_2 h_t) & \text{with probability } p \\ \hat{x}_{t+1} & \text{with probability } 1-p, \end{cases} \tag{16}$$

and

$$h_{t+1} = h_t + \gamma_3(x_{t+1} - \hat{x}_{t+1}) = \begin{cases} h_t + \gamma_3\big(\mathbf{prox}_{\gamma_2 R}(\hat{x}_{t+1} - \gamma_2 h_t) - \hat{x}_{t+1}\big) & \text{with probability } p \\ h_t & \text{with probability } 1-p. \end{cases} \tag{17}$$

For simplicity, we denote $P(x_t) \triangleq \mathbf{prox}_{\gamma_2 R}(\hat{x}_{t+1} - \gamma_2 h_t)$, so we have

$$\begin{aligned} &\mathbb{E}_{\xi_t}[V_{t+1}] \\ &= p\Big(\big\|P(x_t) - x^*_{t+1}\big\|^2 + \gamma_2^2\big\|h_t + \gamma_3(P(x_t) - \hat{x}_{t+1}) - F(x^*_{t+1})\big\|^2\Big) \\ &\qquad + (1-p)\Big(\big\|\hat{x}_{t+1} - x^*_{t+1}\big\|^2 + \gamma_2^2\big\|h_t - F(x^*_{t+1})\big\|^2\Big) \\ &= p\Big(\big\|P(x_t) - x^*_{t+1}\big\|^2 + \big\|P(x_t) - (\hat{x}_{t+1} - \gamma_2 h_t) - \gamma_2 F(x^*_{t+1})\big\|^2\Big) \\ &\qquad + (1-p)\Big(\big\|\hat{x}_{t+1} - x^*_{t+1}\big\|^2 + \gamma_2^2\big\|h_t - F(x^*_{t+1})\big\|^2\Big) \end{aligned} \tag{18}$$

next note that $x^*_t = \mathbf{prox}_{\gamma_2 R}(x^*_t - \gamma_2 F(x^*_t))$, we have

$$\begin{aligned} &\big\|P(x_t) - (\hat{x}_{t+1} - \gamma_2 h_t) - \gamma_2 F(x^*_{t+1})\big\|^2 \\ &= \big\|P(x_t) - (\hat{x}_{t+1} - \gamma_2 h_t) - \big(\mathbf{prox}_{\gamma_2 R}\big(x^*_{t+1} - \gamma_2 F(x^*_{t+1})\big) - (x^*_{t+1} - \gamma_2 F(x^*_{t+1}))\big)\big\|^2 \end{aligned} \tag{19}$$

so by Lemma 5, we have

$$\begin{aligned} &\mathbb{E}_{\xi_t}[V_{t+1}] \\ &\le p\big\|\hat{x}_{t+1} - \gamma_2 h_t - x^*_{t+1} + \gamma_2 F(x^*_{t+1})\big\|^2 + (1-p)\Big(\big\|\hat{x}_{t+1} - x^*_{t+1}\big\|^2 + \gamma_2^2\big\|h_t - F(x^*_{t+1})\big\|^2\Big) \\ &= \big\|\hat{x}_{t+1} - x^*_{t+1}\big\|^2 + \gamma_2^2\big\|h_t - F(x^*_{t+1})\big\|^2 - 2\gamma_2 p\big\langle \hat{x}_{t+1} - x^*_{t+1}, h_t - F(x^*_{t+1})\big\rangle, \end{aligned} \tag{20}$$

let

$$w_t \triangleq x_t - \gamma_1 g(x_t;\xi_t), \quad w^*_t \triangleq x^*_t - \gamma_1 F(x^*_t), \tag{21}$$

recall that $\gamma = \gamma_1 = \gamma_2 p$, so we have

$$\begin{aligned} &\big\|\hat{x}_{t+1} - x^*_{t+1}\big\|^2 - 2\gamma_2 p\big\langle \hat{x}_{t+1} - x^*_{t+1}, h_t - F(x^*_{t+1})\big\rangle \\ &= \big\|w_t - w^*_{t+1} + \gamma\big(h_t - F(x^*_{t+1})\big)\big\|^2 - 2\gamma\big\langle w_t - w^*_{t+1} + \gamma\big(h_t - F(x^*_{t+1})\big), h_t - F(x^*_{t+1})\big\rangle \\ &= \big\|w_t - w^*_{t+1}\big\|^2 - \gamma^2\big\|h_t - F(x^*_{t+1})\big\|^2, \end{aligned} \tag{22}$$

so we have

$$\mathbb{E}_{\xi_t}[V_{t+1}] \le \big\|w_t - w^*_{t+1}\big\|^2 + \big(1-p^2\big)\gamma_2^2\big\|h_t - F(x^*_{t+1})\big\|^2, \tag{23}$$

so we also have $w_t^* \equiv x^* - \gamma F(x^*) \triangleq w^*$.

Then by the standard analysis on GDA, we have

$$\begin{aligned} \left\|w_t - w_{t+1}^*\right\|^2 &= \|w_t - w^*\|^2 = \|x_t - x^* - \gamma(g(x_t;\xi_t) - F(x^*))\|^2 \\ &= \|x_t - x^*\|^2 - 2\gamma\langle g(x_t;\xi_t) - F(x^*), x_t - x^*\rangle + \gamma^2\|g(x_t;\xi_t) - F(x^*)\|^2, \end{aligned} \tag{24}$$

take the expectation, we have

$$\begin{aligned} \mathbb{E}_{\xi_t}\left[\|w_t - w^*\|^2\right] &= \|x_t - x^*\|^2 - 2\gamma\langle F(x_t) - F(x^*), x_t - x^*\rangle + \gamma^2\mathbb{E}_{\xi_t}\left[\|g(x_t;\xi_t) - F(x^*)\|^2\right] \\ &\leq \|x_t - x^*\|^2 - 2\gamma(1-\gamma L)\langle F(x_t) - F(x^*), x_t - x^*\rangle + 2\gamma^2\sigma^2, \end{aligned} \tag{25}$$

so we have

$$\begin{aligned} &\mathbb{E}_{\xi_t}[V_{t+1}] \\ &\leq \mathbb{E}_{\xi_t}\left[\|w_t - w^*\|^2 + \left(1-p^2\right)\gamma_2^2\|h_t - F(x^*)\|^2\right] \\ &\leq \mathbb{E}_{\xi_t}\left[\|x_t - x^*\|^2 - 2\gamma(1-\gamma L)\langle F(x_t) - F(x^*), x_t - x^*\rangle + 2\gamma^2\sigma^2 + \left(1-p^2\right)\gamma_2^2\|h_t - F(x^*)\|^2\right] \\ &\leq \mathbb{E}_{\xi_t}\left[(1 - 2\gamma\mu(1-\gamma L))\|x_t - x^*\|^2 + \left(1-p^2\right)\gamma_2^2\|h_t - F(x^*)\|^2 + 2\gamma^2\sigma^2\right], \end{aligned} \tag{26}$$

recall that $\gamma \leq \frac{1}{2L}$, we have

$$\begin{aligned} \mathbb{E}_{\xi_t}[V_{t+1}] &\leq \mathbb{E}_{\xi_t}\left[(1-\gamma\mu)\|x_t - x^*\|^2 + \left(1-p^2\right)\gamma_2^2\|h_t - F(x^*)\|^2 + 2\gamma^2\sigma^2\right] \\ &\leq \mathbb{E}_{\xi_t}\left[\left(1 - \min\left\{\gamma\mu, p^2\right\}\right)V_t\right] + 2\gamma^2\sigma^2, \end{aligned} \tag{27}$$

by taking the full expectation, we have

$$\begin{aligned} \mathbb{E}[V_T] &\leq \left(1 - \min\left\{\gamma\mu, p^2\right\}\right)\mathbb{E}[V_{T-1}] + 2\gamma^2\sigma^2 \\ &\leq \left(1 - \min\left\{\gamma\mu, p^2\right\}\right)^T V_0 + 2\gamma^2\sigma^2\sum_{i=0}^{T-1}\left(1 - \min\left\{\gamma\mu, p^2\right\}\right)^i \\ &\leq \left(1 - \min\left\{\gamma\mu, p^2\right\}\right)^T V_0 + \frac{2\gamma^2\sigma^2}{\min\left\{\gamma\mu, p^2\right\}}, \end{aligned} \tag{28}$$

which concludes the proof. ■

Appendix C. Proof of Corollary 2

Proof With the above setting, we know that

$$\min\left\{\gamma\mu, p^2\right\} = \gamma\mu, \tag{29}$$

and

$$\mathbb{E}[V_T] \leq (1-\gamma\mu)^T V_0 + \frac{2\gamma\sigma^2}{\mu}, \tag{30}$$

so it is easy to see that by setting

$$T \geq \frac{1}{\gamma\mu} \ln\left(\frac{2V_0}{\epsilon}\right), \quad \gamma \leq \frac{\mu\epsilon}{4\sigma^2}, \tag{31}$$

we have

$$\mathbb{E}[V_T] \leq \epsilon, \tag{32}$$

which induces the iteration complexity to be

$$T \geq \max\left\{1, \frac{2L}{\mu}, \frac{4\sigma^2}{\mu^2\epsilon}\right\} \ln\left(\frac{2V_0}{\epsilon}\right) \tag{33}$$

and the corresponding number of calls to the proximal oracle is

$$pT \geq \sqrt{\max\left\{1, \frac{2L}{\mu}, \frac{4\sigma^2}{\mu^2\epsilon}\right\}} \ln\left(\frac{2V_0}{\epsilon}\right) \tag{34}$$

which concludes the proof. ■

Appendix D. Additional Discussion of Section 4

In the heterogeneous setting, we often use the following assumption to characterize the difference in heterogeneity.

Assumption 3 (Bounded Heterogeneity) *We assume that $g_i(x;\xi_i)$ is an unbiased estimator of $f_i(x)$ for any $i \in \{1, 2, \cdots, n\}$, and $\xi_i^2(x) \triangleq \sup_{x\in\mathbb{R}^d} \|f_i(x) - F(x)\|^2 \leq \xi^2 \leq +\infty$.*

With the Assumption 3, we can easily get the following result for the stochastic operator

$$g(x;\xi) \triangleq (g_1(x;\xi_1), g_2(x;\xi_2), \cdots, g_n(x;\xi_n)).$$

Theorem 7 (Local Operator Estimator) *Let Assumptions 1 and 2 hold. Then if the local operator estimator $g_i(x;\xi_i)$ satisfies Assumption 3, we have*

$$\mathbb{E}_\xi\left[\|g(x;\xi) - F(x^*)\|^2\right] \leq 2L\langle F(x) - F(x^*), x - x^*\rangle + D$$

where $D \triangleq \frac{4}{n}\sum_{i=1}^n \left(Var(g_i(x^*;\xi_i)) + \xi_i^2(x^*)\right) \leq 4(\sigma^2 + \xi^2)$.

With the above result, it can be easily shown that, if we have the same setting in Theorem 3 and additional Assumption 3, the iteration complexity and communication complexity of ProxSkip-VIP-FL are

$$\mathcal{O}\left(\max\left\{\frac{L}{\mu}, \frac{\sigma^2+\xi^2}{\mu^2\epsilon}\right\} \ln\left(\frac{1}{\epsilon}\right)\right) \quad \text{and} \quad \mathcal{O}\left(\sqrt{\max\left\{\frac{L}{\mu}, \frac{\sigma^2+\xi^2}{\mu^2\epsilon}\right\}} \ln\left(\frac{1}{\epsilon}\right)\right),$$

the proof is similar to that of Corollary 2, which we do not repeat here.