SEMI-SUPERVISED COMMUNITY DETECTION VIA STRUCTURAL SIMILARITY METRICS

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ABSTRACT

Motivated by the interests of social network analysis and network-based recommendation systems, we consider a semi-supervised community detection problem, where the goal is to estimate the community label of a new node by leveraging on the network structure and partially observed community labels of existing nodes. We model the network with a degree-corrected stochastic block model, which allows for severe degree heterogeneity and potentially non-assortative communities. We propose a fast algorithm that computes a 'structural similarity metric' between the new node and each of the K communities, aggregating information in labeled and unlabeled data. The estimated label of the new node is equal to the value of k that maximizes this similarity metric. Our method is computationally fast and compares favorably with existing semi-supervised algorithms on numerical performance. In theory, we derive explicit bounds for the misclassification error and show the efficiency of our method by comparing it with an ideal classifier. To our best knowledge, our results provide the first semi-supervised community detection algorithm with theoretical guarantees.

1 INTRODUCTION

Nowadays, large network data are frequently observed on social media (such as Facebook, Twitter, and LinkedIn), science, and social science. Learning the latent community structure in a network is of particular interest. For example, community analysis is useful in designing recommendation systems Debnath et al. (2008), measuring scholarly impacts Ji et al. (2022), and re-constructing pseudo-dynamics in single-cell data Liu et al. (2018).

We consider a semi-supervised community detection setting: we are given a symmetric network with n nodes, and denote by $A \in \mathbb{R}^{n\times n}$ the adjacency matrix, where $A_{ij} \in \{0, 1\}$ indicates whether there is an edge between nodes i and j. Suppose the nodes partition into K non-overlapping communities $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_K$. For a subset $\mathcal{L} \subset \{1, 2, \ldots, n\}$, we observe the true community label $y_i \in \{1, 2, \ldots, K\}$ for each $i \in \mathcal{L}$. Write $m = |\mathcal{L}|$ and $Y_{\mathcal{L}} = (y_i)_{i\in\mathcal{L}}$. Under this setting, there are two sorts of community detection problems: in-sample classification problem, where one classifies all the existing unlabeled nodes, and prediction problem, where one classifies a new node joining the network. Remarkably, in-sample classification problem can be easily reduced to prediction problem: one can simply pick out existing unlabeled nodes one at a time, regarding it as the "new node", and then predict its label based on algorithms for prediction problem. Hence, in this paper, we would mainly focus on prediction problem and briefly discuss the in-sample classification in section 4. Formally, let $X \in \{0, 1\}^n$ denote the vector consisting of edges between the new node and each of the existing nodes. Given $(A, Y_{\mathcal{L}}, X)$, our goal is to estimate the community label of the new node.

This problem has several applications. Consider the news recommendation or advertising web push for a new Facebook user. Given a large Facebook network of the existing users, for a small fraction of nodes (e.g., those active users), we may have good knowledge of which communities they belong to, while for the majority of users, we only observe who they connect to. We are interested in estimating the community label of the new user, to help customize news or advertisement recommendation. As another example, in a co-citation network of researchers Ji et al. (2022), each community can be interpreted as a group of researchers working on the same research area. We often have good prior knowledge of the research areas of some authors (e.g., senior authors), and we hope to utilize such information in determining which community a new node (e.g., a junior author) belongs to.

The statistical literature on community detection has mainly focused on the *unsupervised* setting Bickel & Chen (2009); Rohe et al. (2011); Jin (2015); Gao et al. (2018); Li et al. (2021). The *semi-supervised* setting is less studied. Leng & Ma (2019) provides a nice literature review of semi-supervised community detection algorithms. Liu et al. (2014); Ji et al. (2016) derive systems of linear equations for the community labels through physics theory, and predict the labels by solving those equations. Zhou et al. (2018) leverages on the belief function to propagate labels across the network, so that one can estimate the label of a node through its belief. Yang et al. (2015) unifies a number of different community detection algorithms based on non-negative matrix factorization or spectral clustering under the unsupervised setting, and fits them into the semi-supervised scenario by adding various regularization terms to encourage the estimated labels for nodes in $\mathcal{L}$ to match with the clustering behavior of their observed labels.

However, the existing methods still face challenges. First, quite a few of them use the heuristic that a node tends to have more edges with nodes in the same community than those in other communities. This is true only when communities are *assortative*. But non-assortative communities are seen in many real networks Goldenberg et al. (2010); Betzel et al. (2018); for example, Facebook users who have similar restaurant preferences are not necessarily friends of each other. Second, real networks usually have severe degree heterogeneity, but most semi-supervised community detection algorithms do not handle degree heterogeneity. Third, the optimization-based algorithms (e.g., Yang et al. (2015)) solve non-convex problems and face the issue of local minima. Last, to our best knowledge, none of the existing methods have theoretical guarantees.

Attributed network clustering is a similar problem to community detection, for which a great many algorithms have been developed. Chunaev et al. (2019) provides a comprehensive review of the methods on attributed network clustering. Also, graph neural networks (GNN) have attained great success in this field. For instance, Kipf & Welling (2016); Jin et al. (2019) propose highly well-performed GNN-based algorithms on semi-supervised community classification. However, all those methods require enough attributes, which can be hard or costly to acquire in practice. For example, it is easy to construct a co-authorship network from bibtex files, but collecting features of authors can be quite difficult. Also, a number of existing graph datasets do not have attributes (e.g. Caltech (Red et al., 2011; Traud et al., 2012) , Simmons (Red et al., 2011; Traud et al., 2012) , and Polblogs (Adamic & Glance, 2005)), whereas people may still hope to gain information from them. Furthermore, most of these existing methods have no theoretical guarantee, and may need careful tuning.

To tackle these challenges, we propose a new semi-supervised community detection algorithm. We model the network with the DCBM model (Karrer & Newman, 2011), which models degree heterogeneity and allows for both assortative and non-assortative communities. Inspired by the viewpoint of Goldenberg et al. (2010) that a 'community' is a group of 'structurally equivalent' nodes, we design a *structural similar metric* between the new node and each of the K communities. This metric aggregates information in both labeled and unlabeled nodes. We then estimate the community label of the new node by the k that maximizes this similarity metric. Our method is easy to implement, computationally fast, and compares favorably with other methods in numerical experiments. In theory, we derive explicit bounds for the misclassification probability of our method under the DCBM model. We also study the efficiency of our method by comparing its misclassification probability with that of an ideal classifier having access to the community labels of all nodes.

2 SEMI-SUPERVISED COMMUNITY DETECTION

Recall that A is the $n \times n$ adjacency matrix on the existing nodes and $Y_{\mathcal{L}}$ contains the community labels of nodes in $\mathcal{L}$. Write $[n] = \{1, 2, \ldots, n\}$ and let $\mathcal{U} = [n] \setminus \mathcal{L}$ denote the set of unlabeled nodes. We index the new node by $n+1$ and let $X \in \mathbb{R}^n$ be the binary vector consisting of the edges between the new node and existing nodes. Denote by $\bar{A}$ the adjacency matrix for the network of $(n+1)$ nodes.

2.1 THE DCBM MODEL AND STRUCTURAL EQUIVALENCE OF COMMUNITIES

We model $\bar{A}$ with the degree-corrected block model (DCBM) (Karrer & Newman, 2011). Define a K-dimensional membership matrix $\pi_i \in \{e_1, e_2, \ldots, e_K\}$, where e_k's are the standard basis vectors of $\mathbb{R}^K$. We encode the community labels by π_i, where $\pi_i = e_k$ if and only if $y_i = k$. For a symmetric nonnegative matrix $P \in \mathbb{R}^{K \times K}$ and a degree parameter $\theta_i \in (0, 1]$ for each node i, we assume that

the upper triangle of $\bar{A}$ contains independent Bernoulli variables, where

$$\mathbb{P}(\bar{A}_{ij} = 1) = \theta_i \theta_j \cdot \pi_i' P \pi_j, \qquad \text{for all } \ 1 \le i \ne j \le n+1. \tag{1}$$

When θ_i are equal, the DCBM model reduces to the stochastic block model (SBM). Compared with SBM, DCBM is more flexible as it accommodates degree heterogeneity. For a matrix M or a vector v, let $\mathrm{diag}(M)$ and $\mathrm{diag}(v)$ denote the diagonal matrices whose diagonals are from the diagonal of M or the vector v, respectively. Write $\theta = (\theta_1, \theta_2, \ldots, \theta_{n+1})'$, $\Theta = \mathrm{diag}(\theta)$, and $\Pi = [\pi_1, \pi_2, \ldots, \pi_{n+1}]' \in \mathbb{R}^{n \times K}$. Model (1) yields that

$$\bar{A} = \Omega - \mathrm{diag}(\Omega) + W, \qquad \text{where } \ \Omega = \Theta \Pi P \Pi' \Theta \ \text{ and } \ \bar{W} = \bar{A} - \mathbb{E}\bar{A}. \tag{2}$$

Here, Ω is a low-rank matrix that captures the 'signal', W is a generalized Wigner matrix that captures 'noise', and $\mathrm{diag}(\Omega)$ yields a bias to the 'signal' but its effect is usually negligible.

The DCBM belongs to the family of block models for networks. In block models, it is not necessarily true that the edge densities within a community are higher than those between different communities. Such communities are called assortative communities. However, non-assortative communities also appear in many real networks Goldenberg et al. (2010); Betzel et al. (2018). For instance, in news and ad recommendation, we are interested in identifying a group of users who have similar behaviors, but they may not be densely connected to each other. Goldenberg et al. (2010) introduced an intuitive notion of *structural equivalence* - two nodes are structurally equivalent if their connectivity with similar nodes is similar. They argued that a 'community' in block models is a group of structurally equivalent nodes. This way of defining communities is more general than assortative communities.

We introduce a rigorous description of structural equivalence in the DCBM model. For two vectors u and v, define $\psi(u, v) = \arccos\langle \frac{u}{\|u\|}, \frac{v}{\|v\|} \rangle$, which is the angle between these two vectors. Let $\bar{A}_i$ be the ith column of $\bar{A}$. This vector describes the 'behavior' of node i in the network. Recall that Ω is as in (2). When the signal-to-noise ratio is sufficiently large, $\bar{A}_i \approx \Omega_i$, where Ω_i is the ith column of Ω. We approximate the angle between $\bar{A}_i$ and $\bar{A}_j$ by the angle between Ω_i and Ω_j. By DCBM model, for a node i in community k, $\Omega_i = \theta_i \Theta \Pi P e_k$, where e_k is the kth standard basis of $\mathbb{R}^K$. It follows that for $i \in \mathcal{C}_k$ and $j \in \mathcal{C}_\ell$,

$$\cos\psi(\Omega_i, \Omega_j) = \frac{\langle \cancel{\theta_i} \Theta \Pi P e_k, \ \cancel{\theta_j} \Theta \Pi P e_\ell \rangle}{\|\cancel{\theta_i} \Theta \Pi P e_k\| \cdot \|\cancel{\theta_j} \Theta \Pi P e_\ell\|} = \frac{M_{k\ell}}{\sqrt{M_{kk} M_{\ell\ell}}}, \quad \text{with } M := P \Pi' \Theta^2 \Pi P. \tag{3}$$

It is seen that $\cos\psi(\Omega_i, \Omega_j)$ does not depend on the degree parameters of nodes and is solely determined by community membership. When $k = \ell$ (i.e., i and j are in the same community), $\cos\psi(\Omega_i, \Omega_j) = 1$, which means the angle between these two vectors is zero. When $k \ne \ell$, as long as P is non-singular and Π has a full column rank, M is a positive-definite matrix. It follows that $\cos\psi(\Omega_i, \Omega_j) < 1$ and that the angle between Ω_i and Ω_j is nonzero.

Example 1. *Suppose $K = 2$, $P \in \mathbb{R}^{2\times 2}$ is such that the diagonal entries are 1 and off-diagonal entries are b, for some $b > 0$ and $b \ne 1$, and $\max_i\{\theta_i\} < \min\{1/b, 1\}$ (to guarantee that all entries of Ω are smaller than 1). For simplicity, we assume $\sum_{i\in\mathcal{C}_1} \theta_i^2 = \sum_{i\in\mathcal{C}_2} \theta_i^2$. It can be shown that M is proportional to the matrix whose diagonal entries are $(1 + b^2)$ and off-diagonal entries are $2b$. When $b < 1$, the communities are assortative, and when $b > 1$, the communities are non-assortative. However, regardless of the value of b, the off-diagonal entries of M are always strictly smaller than the diagonal entries, so that $\cos\psi(\Omega_i, \Omega_j) < 1$, for nodes in distinct communities.*

2.2 Semi-supervised community detection

Inspired by (3), we propose assigning a community label to the new node based on its 'similarity' to those labeled nodes. For each $1 \le k \le K$, assume that $\mathcal{L} \cap \mathcal{C}_k \ne \emptyset$ and define a vector $A^{(k)} \in \mathbb{R}^n$ by $A_j^{(k)} = \sum_{i\in\mathcal{L}\cap\mathcal{C}_k} A_{ij}$, for $1 \le j \le n$. The vector $A^{(k)}$ describes the 'aggregated behavior' of all labeled nodes in community k. Recall that $X \in \mathbb{R}^n$ contains the edges between the new node and all the existing nodes. We can estimate the community label of the new node by

$$\hat{y} = \arg\min_{1\le k\le K} \psi(A^{(k)}, \ X). \tag{4}$$

We call (4) the *AngleMin* estimate. Note that each $A^{(k)}$ is an n-dimensional vector, the construction of which uses both $A_{\mathcal{L}\mathcal{L}}$ and $A_{\mathcal{L}\mathcal{U}}$. Therefore, $A^{(k)}$ aggregates information from both labeled and unlabeled nodes, and so AngleMin is indeed a semi-supervised approach.

The estimate in (4) still has space to improve. First, $A^{(k)}$ and X are high-dimensional random vectors, each entry of which is a sum of independent Bernoulli variables. When the network is very sparse or communities are heavily imbalanced in size or degree, the large-deviation bound for $\psi(A^{(k)}, X)$ can be unsatisfactory. Second, recall that our observed data include A and X. Denote by $A_{\mathcal{L}\mathcal{L}}$ the submatrix of A restricted on $\mathcal{L} \times \mathcal{L}$ and $X_{\mathcal{L}}$ the subvector of X restricted on $\mathcal{L}$; other notations are similar. In (4), only $(A_{\mathcal{L}\mathcal{L}}, A_{\mathcal{L}\mathcal{U}}, X)$ are used, but the information in $A_{\mathcal{U}\mathcal{U}}$ is wasted. We now propose a variant of (4). For any vector $x \in \mathbb{R}^n$, let $x_{\mathcal{L}}$ and $x_{\mathcal{U}}$ be the sub-vectors restricted to indices in $\mathcal{L}$ and $\mathcal{U}$, respectively. Let $\mathbf{1}_{(k)}$ denote the $|\mathcal{L}|$-dimensional vector indicating whether each labeled node is in community k. Given any $|\mathcal{U}| \times K$ matrix $H = [h_1, h_2, \ldots, h_K]$, define

$$f(x; H) = [x'_{\mathcal{L}}\mathbf{1}_{(1)},\ \ldots,\ x'_{\mathcal{L}}\mathbf{1}_{(k)},\ x'_{\mathcal{U}}h_1, \ldots, x'_{\mathcal{U}}h_K]' \ \in\ \mathbb{R}^{2K}. \tag{5}$$

The mapping $f(\cdot; H)$ creates a low-dimensional projection of x. Suppose we now apply this mapping to $A^{(k)}$. In the projected vector, each entry is a weighted sum of a large number of entries of $A^{(k)}$. Since $A^{(k)}$ contains independent entries, it follows from large-deviation inequalities that each entry of $f(A^{(k)}, H)$ has a nice asymptotic tail behavior. This resolves the first issue above. We then modify the AngleMin estimate in (4) to the following estimate, which we call (4):

$$\hat{y}(H) = \arg\min_{1 \le k \le K} \psi\big(f(A^{(k)}; H),\ f(X; H)\big). \tag{6}$$

AngleMin+ requires an input of H. Our theory suggests that H has to satisfy two conditions: (a) The spectral norm of $H'H$ is $O(|\mathcal{U}|)$. In fact, given any H, we can always multiply it by a scalar so that $\|H'H\|$ is at the order of $|\mathcal{U}|$. Hence, this condition says that the scaling of H should be properly set to balance the contributions from labeled and unlabeled nodes. (b) The minimum singular value of $H'\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}$ has to be at least a constant times $\|H\|\|\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}\|$, where $\Theta_{\mathcal{U}\mathcal{U}}$ is the submatrix of Θ restricted to the $(\mathcal{U}, \mathcal{U})$ block and $\Pi_{\mathcal{U}}$ is the sub-matrix of Π restricted to the rows in $\mathcal{U}$. This condition prevents the columns of H from being orthogonal to the columns of $\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}$, and it guarantees that the last K entries of $f(x; H)$ retain enough information of the unlabeled nodes.

We construct a data-driven H from $\mathcal{A}_{\mathcal{U}\mathcal{U}}$, by taking advantage of the existing unsupervised community detection algorithms such as Gao et al. (2018); Jin et al. (2021). Let $\hat{\Pi}_{\mathcal{U}} = [\hat{\pi}_i]_{i \in \mathcal{U}}$ be the community labels obtained by applying a community detection algorithm on the sub-network restricted to unlabeled nodes, where $\hat{\pi}_i = e_k$ if and only if node k is clustered to community k. We propose using

$$H = \hat{\Pi}_{\mathcal{U}}. \tag{7}$$

This choice of H always satisfies the aforementioned condition (a). Furthermore, under mild regularity conditions, as long as the clustering error fraction is bounded by a constant, this H also satisfies the aforementioned condition (b). We note that the information in $\mathcal{A}_{\mathcal{U}\mathcal{U}}$ has been absorbed into H, so it resolves the second issue above. Combining (7) with (4) gives a two-stage algorithm for estimating y.

Remark 1: In AngleMin+, the projection H serves to reduce noise. To see the idea, let $X, Y \in \mathbb{R}^{2m}$ be two random Bernoulli vectors, where $\mathbb{E}X = \mathbb{E}Y = (.1, \ldots, .1, .4, \ldots, .4)'$. As $m \to \infty$, it can be shown that $\psi(X, Y) \to 0.34 \neq 1$ almost surely. If we project X and Y into $\mathbb{R}^2$ by summing the first m coordinates and last m coordinates separately, then as $m \to \infty$, $\psi(X, Y) \to 1$ almost surely.

Remark 2: A nice property of AngleMin+ is that it tolerates an arbitrary permutation of communities in $\hat{\Pi}_{\mathcal{U}}$. In other words, the communities output by the unsupervised community detection algorithm do not need to have a one-to-one correspondence with the communities on the labeled nodes. To see the reason, we consider an arbitrary permutation of columns of $\hat{\Pi}_{\mathcal{U}}$. By (15), this yields a permutation of the last K entries of $f(x; H)$, simultaneously for all x. However, the angle between $f(A^{(k)}; H)$ and $f(X; H))$ is still the same, and so $\hat{y}(H)$ is unchanged. This property brings a lot of practical conveniences. When K is large or the signals are weak, it is challenging (both computationally and statistically) to match the communities in $\hat{\Pi}_{\mathcal{U}}$ with those in $\Pi_{\mathcal{L}}$. Our method avoids this issue.

Remark 3: AngleMin+ is flexible to accommodate other choices of H. Some unsupervised community detection algorithms provide both $\hat{\Pi}_{\mathcal{U}}$ and $\hat{\Theta}_{\mathcal{U}\mathcal{U}}$ Jin et al. (2022). We may use $H \propto \hat{\Theta}_{\mathcal{U}\mathcal{U}}\hat{\Pi}_{\mathcal{U}}$, (subject to a re-scaling to satisfy the aforementioned condition (a)). Compared with (7), this H down-weights the contribution of low-degree unlabeled nodes in the last K entries of (15). This may be beneficial if the signals are extremely weak and the degree heterogeneity is severe. Another choice is $H \propto \hat{\Xi}_{(\mathcal{U})}\hat{\Lambda}^{-1}_{(\mathcal{U})}$, where $\hat{\Lambda}_{\mathcal{U}}$ is a diagonal matrix containing the K largest eigenvalues (in magnitude)

of $A_{\mathcal{U}\mathcal{U}}$ and $\hat{\Xi}_{(\mathcal{U})}$ is the associated matrix of eigenvectors. For this H, we do not even need to perform any community detection algorithm on $\mathcal{A}_{\mathcal{U}\mathcal{U}}$. Additionally, we may also use spectral embedding as in e.g. Rubin-Delanchy et al. (2017). We stick to the H in (7) for theoretical convenience, but other H may also be useful in practice.

Remark 4: Our method can be generalized to attributed networks. We can combine the adjacency matrix $\bar{A}$ of $n+1$ nodes (including the new node) with node attributes to obtain a weighted adjacency matrix $\bar{A}_{\text{fuse}}$, by the fusion methods in Chunaev et al. (2019). Let $A_{\text{fuse}} \in \mathbb{R}^{n\times n}$ and $X_{\text{fuse}} \in \mathbb{R}^n$ be the corresponding counterparts of A and X on the weighted network. We can apply AngleMin+ on A_{fuse} and X_{fuse}. Also, our method can be extended to multi-layer networks by applying the cluster ensembling techniques (see e.g. Boongoen & Iam-On (2018); Iam-On et al. (2011)).

2.3 THE CHOICE OF THE UNSUPERVISED COMMUNITY DETECTION ALGORITHM

We discuss how to obtain $\hat{\Pi}_{\mathcal{U}}$. In the statistical literature, there are several approaches to unsupervised community detection. The first is modularity maximization (Girvan & Newman, 2002; Bickel & Chen, 2009). It exhaustively searches for all cluster assignments and selects the one that maximizes an empirical modularity function. The second is spectral clustering Rohe et al. (2011); Jin (2015). It applies k-means clustering to rows of the matrix consisting of empirical eigenvectors. Other methods include post-processing the output of spectral clustering by majority vote (Gao et al., 2018). Not every method deals with degree heterogeneity and non-assortative communities as in the DCBM model. We use a recent spectral algorithm SCORE+ Jin et al. (2021), which allows for both severe degree heterogeneity and non-assortative communities. Jin et al. (2021) demonstrated that SCORE+ has especially good performance on real data when the community signals are weak.

SCORE+: We tentatively write $A_{\mathcal{U}\mathcal{U}}$=$A$ and $|\mathcal{U}|$=n and assume the network (on unlabeled nodes) is connected (otherwise consider its giant component). SCORE+ first computes L=$D_\tau^{-1/2}AD_\tau^{-1/2}$, where D_τ=$\mathrm{diag}(d_1,\ldots,d_n)$+$0.1d_{\max}I_n$, and d_i is degree of node i. Let $\hat{\lambda}_k$ be the kth eigenvalue (in magnitude) of L and let $\hat{\xi}_k$ be the associated eigenvector. Let r=K or r=K+1 (see Jin et al. (2021) for details). Let $\hat{R} \in \mathbb{R}^{n\times(r-1)}$ by $\hat{R}_{ik} = (\hat{\lambda}_{k+1}/\hat{\lambda}_1)\cdot[\hat{\xi}_{k+1}(i)/\hat{\xi}_1(i)]$. Run k-means on rows of $\hat{R}$.

3 THEORETICAL PROPERTIES

We assume that the observed adjacency matrix $\bar{A}$ follows the DCBM model in (1)-(2). From now on, let θ_* denote the degree parameter of the new node $n+1$. Suppose $k^* \in \{1,2,\ldots,K\}$ is its true community label, and the corresponding K-dimensional membership vector is $\pi^* = e_{k^*}$. In (2), θ and P are not identifiable. To have identifiability, we assume that all diagonal entries of P are equal to 1 (if this is not true, we replace P by $[\mathrm{diag}(P)]^{-\frac{1}{2}}P[\mathrm{diag}(P)]^{-\frac{1}{2}}$ and each θ_i in community k by $\theta_i\sqrt{P_{kk}}$, while keeping $\Omega = \Theta\Pi P\Pi'\Theta$ unchanged). In the asymptotic framework, we fix K and assume $n\to\infty$. We need some regularity conditions. For any symmetric matrix B, let $\|B\|_{\max}$ denote its entry-wise maximum norm and $\lambda_{\min}(B)$ denote its minimum eigenvalue (in magnitude). We assume for a constant $C_1 > 0$ and a positive sequence β_n (which may tend to 0),

$$\|P\|_{\max} \le C_1, \qquad |\lambda_{\min}(P)| \ge \beta_n. \tag{8}$$

For $1\le k\le K$, let $\theta^{(k)} \in \mathbb{R}^n$ be the vector with $\theta_i^{(k)} = \theta_i\cdot 1\{i\in\mathcal{C}_k\}$, and let $\theta_{\mathcal{L}}^{(k)}$ and $\theta_{\mathcal{U}}^{(k)}$ be the sub-vectors restricted to indices in $\mathcal{L}$ and $\mathcal{U}$, respectively. We assume for a constant $C_2 > 0$ and a properly small constant $c_3 > 0$,

$$\max_k \|\theta^{(k)}\|_1 \le C_2 \min_k \|\theta^{(k)}\|_1, \qquad \|\theta_{\mathcal{L}}^{(k)}\|^2 \le c_3\beta_n\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1, \quad \text{for all } 1\le k\le K. \tag{9}$$

These conditions are mild. Consider (9). For identifiability, P is already scaled to make $P_{kk} = 1$ for all k. It is thus a mild condition to assume $\|P\|_{\max} \le C_1$. The condition of $|\lambda_{\min}(P)| \ge \beta_n$ is also mild, because we allow $\beta_n \to 0$. Here, β_n captures the 'dissimilarity' of communities. To see this, consider a special P where the diagonals are 1 and the off-diagonals are all equal to b; in this example, $|1-b|$ captures the difference of within-community connectivity and between-community connectivity, and it can be shown that $|\lambda_{\min}(P)| = |1-b|$. Consider (10). The first condition requires that the total degree in different communities are balanced, which is mild. The second condition is about degree heterogeneity. Let $\theta_{\max}$ and $\bar{\theta}$ be the maximum and average of θ_i, respectively. In the

second inequality of (10), the left hand side is $O(n^{-1}\theta_{\max}/\bar{\theta})$, so this condition is satisfied as long as $\theta_{\max}/\bar{\theta} = O(n\beta_n)$. This is a very mild requirement.

3.1 THE MISCLASSIFICATION ERROR OF ANGLEMIN+

For any $|\mathcal{U}| \times K$ matrix H, let $\hat{\psi}_k(H) = \psi(f(A^{(k)}; H), f(X; H))$ be as in (4). AngleMin+ estimates the community label to the new node by finding the minimum of $\hat{\psi}_1(H), \ldots, \hat{\psi}_K(H)$, with $H = \hat{\Pi}_{\mathcal{U}}$. We first introduce a counterpart of $\hat{\psi}_k(H)$. Recall that Ω is as in (2), which is the 'signal' matrix. Let $\Omega^{(k)} \in \mathbb{R}^n$ by $\Omega^{(k)}_j = \sum_{i \in \mathcal{L} \cap \mathcal{C}_k} \Omega_{ij}$, for $1 \le j \le n$, and define

$$\psi_k(H) = \psi\big(f(\Omega^{(k)}; H),\ f(\mathbb{E}X; H)\big), \qquad \text{for } 1 \le k \le K. \tag{10}$$

The next lemma gives the explicit expression of $\psi_k(H)$ for an arbitrary H.

Lemma 1. *Consider the DCBM model where (9)-(10) are satisfied. We define three $K \times K$ matrices: $G_{\mathcal{L}\mathcal{L}} = \Pi'_{\mathcal{L}}\Theta_{\mathcal{L}\mathcal{L}}\Pi_{\mathcal{L}}$, $G_{\mathcal{U}\mathcal{U}} = \Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}$, and $Q = G^{-1}_{\mathcal{U}\mathcal{U}}\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}H$. For $1 \le k \le K$, $\psi_k(H) = \arccos\Big(\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\Big)$, where $M = P(G^2_{\mathcal{L}\mathcal{L}} + G_{\mathcal{U}\mathcal{U}}QQ'G_{\mathcal{U}\mathcal{U}})P$.*

The choice of H is flexible. For convenience, we focus on the class of H that is an eligible community membership matrix, i.e., $H = \hat{\Pi}_{\mathcal{U}}$. Our theory can be easily extended to more general forms of H.

Definition 1. *For any $b_0 \in (0,1)$, we say that $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct if $\min_T\big(\sum_{i\in\mathcal{U}} \theta_i \cdot 1\{T\hat{\pi}_i \neq \pi_i\}\big) \le b_0\|\theta\|_1$, where the minimum is taken over all permutations of K columns of $\hat{\Pi}_{\mathcal{U}}$.*

The next two theorems study $\psi_k(H)$ and $\hat{\psi}_k(H)$, respectively, for $H = \hat{\Pi}_{\mathcal{U}}$.

Theorem 1. *Consider the DCBM model where (9)-(10) hold. Let k^* denote the true community label of the new node. Suppose $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct, for a constant $b_0 \in (0,1)$. When b_0 is properly small, there exists a constant $c_0 > 0$, which does not depend on b_0, such that $\psi_{k^*}(\hat{\Pi}_{\mathcal{U}}) = 0$ and $\min_{k \neq k^*}\{\psi_k(\hat{\Pi}_{\mathcal{U}})\} \ge c_0\beta_n$.*

Theorem 2. *Consider the DCBM model where (9)-(10) hold. There exists constant $C > 0$, such that for any $\delta \in (0, 1/2)$, with probability $1-\delta$, simultaneously for $1 \le k \le K$, $|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \le C\left(\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta^{(k)}_{\mathcal{L}}\|_1\}}} + \frac{\|\theta^{(k)}_{\mathcal{L}}\|^2}{\|\theta^{(k)}_{\mathcal{L}}\|_1\|\theta\|_1}\right)$.*

Write $\hat{\psi}_k = \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})$ and $\psi_k = \psi_k(\hat{\Pi}_{\mathcal{U}})$ for short. When $\max_k\{|\hat{\psi}_k - \psi_k|\} < (1/2)\min_{k\neq k^*}\{\psi_k\}$, the community label of the new node is correctly estimated. We can immediately translate the results in Theorems 1-2 to an upper bound for the misclassification probability.

Corollary 1. *Consider the DCBM model where (9)-(10) hold. Suppose for some constants $b_0 \in (0,1)$ and $\epsilon \in (0, 1/2)$, $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct with probability $1-\epsilon$. When b_0 is properly small, there exist constants $C_0 > 0$ and $\bar{C} > 0$, which do not depend on (b_0, ϵ), such that $\mathbb{P}(\hat{y} \neq k^*) \le \epsilon + \bar{C}\sum_{k=1}^K \exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta^{(k)}_{\mathcal{L}}\|_1\}\Big)$.*

Remark 5: When $\min_k \|\theta^{(k)}_{\mathcal{L}}\|_1 \ge O(\theta^*)$, the stochastic noise in X will dominate the error, and the misspecification probability in Corollary 1 will not improve with more label information. Typically, the error rate will be the same as in the ideal case that $\Pi_{\mathcal{U}}$ is known (except there is no ϵ in the ideal case). Hence, only little label information can make AngleMin+ perform almost as well as a fully supervised algorithm that possesses all the label information. We will formalize this in Section 3.2.

Remark 6: Notice that $\min_T\big(\sum_{i\in\mathcal{U}} \theta_i \cdot 1\{T\hat{\pi}_i \neq \pi_i\}\big) \le \frac{1}{K!}\sum_T\big(\sum_{i\in\mathcal{U}} \theta_i \cdot 1\{T\hat{\pi}_i \neq \pi_i\}\big) \le \frac{K-1}{K}\|\theta_{\mathcal{U}}\|_1$. Therefore, if $\|\theta_{\mathcal{L}}\|_1 \ge (1 - \frac{Kb_0}{K-1})\|\theta\|_1$, then $\min_T\big(\sum_{i\in\mathcal{U}} \theta_i \cdot 1\{T\hat{\pi}_i \neq \pi_i\}\big) \le b_0\|\theta\|_1$ is always true. In other words, as long as the information on the labels is strong enough, AngleMin+ would not require any assumption on the unsupervised community detection algorithm.

For AngleMin+ to be consistent, we need the bound in Corollary 1 to be $o(1)$. It then requires that for a small constant b_0, $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct with probability $1 - o(1)$. This is a mild requirement and can be achieved by several unsupervised community detection algorithms. The next corollary studies the specific version of AngleMin+, when $\hat{\Pi}_{\mathcal{U}}$ is from SCORE+:

Corollary 2. *Consider the DCBM model where (9)-(10) hold. We apply SCORE+ to obtain $\hat{\Pi}_{\mathcal{U}}$ and plug it into AngleMin+. As $n \to \infty$, suppose for some constant $q_0 > 0$, $\min_{i\in\mathcal{U}} \theta_i \geq q_0 \max_{i\in\mathcal{U}} \theta_i$, $\beta_n \|\theta_{\mathcal{U}}\| \geq q_0\sqrt{\log(n)}$, $\beta_n^2\|\theta\|_1\theta^* \to \infty$, and $\beta_n^2\|\theta\|_1 \min_k\{\|\theta_{\mathcal{L}}^{(k)}\|_1\} \to \infty$. Then, $\mathbb{P}(\hat{y} \neq k^*) \to 0$, so the AngleMin+ estimate is consistent.*

3.2 Comparison with an information theoretical lower bound

We compare the performance of AngleMin+ with an ideal estimate that has access to all model parameters, except for the community label k^* of the new node. For simplicity, we first consider the case of $K = 2$. For any label predictor $\tilde{y}$ for the new node, define $\mathrm{Risk}(\tilde{y}) = \sum_{k^*\in[K]} \mathbb{P}(\tilde{y} \neq k^*|\pi^* = e_{k^*})$.

Lemma 2. *Consider a DCBM with $K = 2$ and $P = (1-b)I_2 + b\mathbf{1}_2\mathbf{1}_2'$. Suppose $\theta^* = o(1)$, $\frac{\theta^*}{\min_k \|\theta_{\mathcal{L}}^{(k)}\|_1} = o(1)$, $1 - b = o(1)$, $\frac{\|\theta_{\mathcal{L}}^{(1)}\|_1}{\|\theta_{\mathcal{L}}^{(2)}\|_1} = \frac{\|\theta_{\mathcal{U}}^{(1)}\|_1}{\|\theta_{\mathcal{U}}^{(2)}\|_1} = 1$. There exists a constant $c_4 > 0$ such that $\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} \geq c_4 \exp\Big\{-2[1 + o(1)]\frac{(1-b)^2}{8} \cdot \theta^*(\|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1)\Big\}$, where the infimum is taken over all measurable functions of A, X, and parameters $\Pi_{\mathcal{L}}$, $\Pi_{\mathcal{U}}$, Θ, P, θ^*. In AngleMin+, suppose the second part of condition 10 holds with $c_3 = o(1)$, $\hat{\Pi}_{\mathcal{U}}$ is $\tilde{b}_0$-correct with $\tilde{b}_0 \overset{a.s.}{\to} 0$. There is a constant $C_4 > 0$ such that, $\mathrm{Risk}(\hat{y}) \leq C_4 \exp\Big\{-[1 - o(1)]\frac{(1-b)^2}{8} \cdot \theta^* \frac{(\|\theta_{\mathcal{L}}\|_1^2+\|\theta_{\mathcal{U}}\|_1^2)^2}{\|\theta_{\mathcal{L}}\|_1^3+\|\theta_{\mathcal{U}}\|_1^3}\Big\}$.*

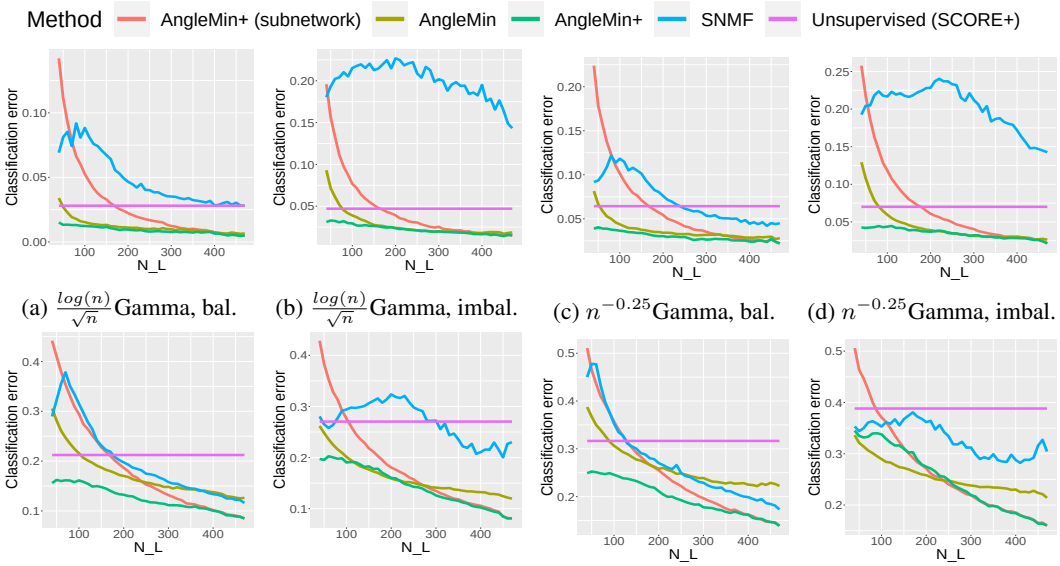
Lemma 2 indicates that the classification error of AngleMin+ is almost the same as the information theoretical lower bound of an algorithm that knows all the parameters except π^* apart from a mild difference of the exponents. This difference comes from two sources. The first is the extra "2" in the exponent of $\mathrm{Risk}(\hat{y})$, which is largely an artifact of proof techniques, because we bound the total variation distance by the Hellinger distance (the total variation distance is hard to analyze directly). The second is the difference of $\|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1$ in $\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\}$ and $\frac{(\|\theta_{\mathcal{L}}\|_1^2+\|\theta_{\mathcal{U}}\|_1^2)^2}{\|\theta_{\mathcal{L}}\|_1^3+\|\theta_{\mathcal{U}}\|_1^3}$ in $\mathrm{Risk}(\hat{y})$. Note that $\frac{(\|\theta_{\mathcal{L}}\|_1^2+\|\theta_{\mathcal{U}}\|_1^2)^2}{\|\theta_{\mathcal{L}}\|_1^3+\|\theta_{\mathcal{U}}\|_1^3} \leq \|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1 \leq 1.125\frac{(\|\theta_{\mathcal{L}}\|_1^2+\|\theta_{\mathcal{U}}\|_1^2)^2}{\|\theta_{\mathcal{L}}\|_1^3+\|\theta_{\mathcal{U}}\|_1^3}$, so this difference is quite mild. It arises from the fact that AngleMin+ does not aggregate the information in labeled and unlabeled data by adding the first and last K coordinates of $f(x; H)$ together. The reason we do not do this is that unsupervised community detection methods only provide class labels up to a permutation, and practically it is really hard to estimate this permutation, which will result in the algorithm being extremely unstable. To conclude, the difference of the error rate of our method and the information theoretical lower bound is either almost technically inevitable or mild and almost practically inevitable, demonstrating that our algorithm is nearly optimal. For a general K, we have a similar conclusion:

Theorem 3. *Suppose the conditions of Corollary 1 hold, where b_0 is properly small , and suppose that $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct. Furthermore, we assume for sufficiently large constant C_3, $\theta^* \leq \frac{1}{C_3}$, $\theta^* \leq \min_{k\in[K]} C_3\|\theta_{\mathcal{L}}^{(k)}\|_1$, and for a constant $r_0 > 0$, $\min_{k\neq\ell}\{P_{k\ell}\} \geq r_0$. Then, there is a constant $\tilde{c}_2 = \tilde{c}_2(K, C_1, C_2, C_3, c_3, r_0) > 0$ such that $[-\log(\tilde{c}_2\mathrm{Risk}(\hat{y}))]/[-\log(\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\})] \geq \tilde{c}_2$.*

4 In-sample Classification

In this section, we briefly discuss the in-sample classification problem. Formally, our goal is to estimate π_i for all $i \in \mathcal{U}$. As mentioned in section 1, an in-sample classification algorithm can be directly derived from AngleMin+: for each $i \in \mathcal{U}$, predict the label of i as $\hat{y}_i(H) = \arg\min_{1\leq k\leq K} \psi\big(f(A_{-i}^{(k)}; H_i),\ f(A_{-i,i}; H_i)\big)$, where $A_{-i}^{(k)}$ is the subvector of $A^{(k)}$ by removing the ith entry, $A_{-i,i}$ is the subvector of A_i by removing the ith entry, and H_i is a $(|\mathcal{U}| - 1) \times K$ projection matrix which may be different across distinct i. As discussed in subsection 2.2, the choices of H_i are quite flexible. For purely theoretical convenience, we would focus on the case that $H_i = \hat{\Pi}_{\mathcal{U}\setminus\{i\}}$.

For any in-sample classifier $\tilde{y} = (\tilde{y}_i)_{i\in\mathcal{U}} \in [K]^{|\mathcal{U}|}$, define the in-sample risk $\mathrm{Risk}_{ins}(\tilde{y}) = \frac{1}{|\mathcal{U}|}\sum_{i\in\mathcal{U}}\sum_{k^*\in[K]} \mathbb{P}(\tilde{y}_i \neq k^*|\pi_i = e_{k^*})$. For the above in-sample classification algorithm, we have similar theoretical results as in section 3 on consistency and efficiency under some very mild conditions:



(a) $\frac{log(n)}{\sqrt{n}}$Gamma, bal. (b) $\frac{log(n)}{\sqrt{n}}$Gamma, imbal. (c) $n^{-0.25}$Gamma, bal. (d) $n^{-0.25}$Gamma, imbal.

(e) $\frac{log(n)}{\sqrt{n}}$Pareto, bal. (f) $\frac{log(n)}{\sqrt{n}}$Pareto, imbal. (g) $n^{-0.25}$Pareto, bal. (h) $n^{-0.25}$Pareto, imbal.

Figure 1: Simulations ($n = 500$, $K = 3$; data are generated from DCBM). In each plot, the x-axis is the number of labeled nodes, and the y-axis is the average misclassification rate over 100 repetitions.

Theorem 4. *Consider the DCBM model where (9)-(10) hold. We apply SCORE+ to obtain* $\hat{\Pi}_{\mathcal{U}\setminus\{i\}}$ *and plug it into the above algorithm. As* $n \to \infty$*, suppose for some constant* $q_0 > 0$ *,* $\min_{i\in\mathcal{U}} \theta_i \geq q_0 \max_{i\in\mathcal{U}} \theta_i$*,* $\beta_n \|\theta_{\mathcal{U}}\| \geq q_0\sqrt{\log(n)}$*,* $\beta_n^2 \|\theta\|_1 \min_{i\in\mathcal{U}} \theta_i \to \infty$*, and* $\beta_n^2 \|\theta\|_1 \min_k\{\|\theta_{\mathcal{L}}^{(k)}\|_1\} \to \infty$*. Then,* $\frac{1}{|\mathcal{U}|}\sum_{i\in\mathcal{U}} \mathbb{P}(\hat{y}_i \neq k_i) \to 0$*, so the above in-sample classification algorithm is consistent.*

Theorem 5. *Suppose the conditions of Corollary 1 hold, where* b_0 *is properly small , and suppose that* $\hat{\Pi}_{\mathcal{U}\setminus\{i\}}$ *is* b_0*-correct for all* $i \in \mathcal{U}$*. Furthermore, we assume for sufficiently large constant* C_3*,* $\max_{i\in\mathcal{U}} \theta_i \leq \frac{1}{C_3}$*,* $\max_{i\in\mathcal{U}} \theta_i \leq \min_{k\in[K]} C_3 \|\theta_{\mathcal{L}}^{(k)}\|_1$*,* $\log(|\mathcal{U}|) \leq C_3 \beta_n^2 \|\theta\|_1 \min_{i\in\mathcal{U}} \theta_i$*, and for a constant* $r_0 > 0$*,* $\min_{k\neq\ell}\{P_{k\ell}\} \geq r_0$*. Then, there is a constant* $\tilde{c}_{21} = \tilde{c}_{21}(K, C_1, C_2, C_3, c_3, r_0) > 0$ *such that* $[-\log(\tilde{c}_{21}\mathrm{Risk}_{ins}(\hat{y}))]/[-\log(\inf_{\tilde{y}}\{\mathrm{Risk}_{ins}(\tilde{y})\})] \geq \tilde{c}_{21}$*, so the above in-sample classification algorithm is efficient.*

5 Empirical Study

We study the performance of AngelMin+, where $\hat{\Pi}_{\mathcal{U}}$ is from SCORE+ Jin et al. (2021). As mentioned in Section 1, not many semi-supervised community detection algorithms are available, where Yang et al. (2015) provides one of the state-of-art approaches, and the SNMF (symmetric non-negative matrix factorization) version of their method generally has the best empirical performance. We compare our methods with SNMF (a representative of semi-supervised approaches) and SCORE+ (a fully unsupervised approach). We also compare our algorithm to typical GNN methods (Kipf & Welling (2016)) in the real data part. The hyper-parameters of SNMF are set as in Yang et al. (2015). For SCORE+, we use code as in Jin et al. (2021).For GNN, we use code in Kipf & Welling (2016).

Simulations: To illustrate how information in $A_{\mathcal{U}\mathcal{U}}$ will improve the classification accuracy, we would consider AngleMin in (4) in simulations. Also, to cast light on how information on unlabeled data will ameliorate the classification accuracy, we consider a special version of AngleMin+ in simulations by feeding into the algorithm only $A_{\mathcal{L}\mathcal{L}}$ and $X_{\mathcal{L}}$. It ignores information on unlabeled data and only uses the subnetwork consisting of labeled nodes. We call it AngleMin+(subnetwork). This method is practically uninteresting, but it serves as a representative of the fully supervised approach that ignores unlabeled nodes. We simulate data from the DCBM with $(n, K) = (500, 3)$. To generate P, we draw its (off diagonal) entries from $\mathrm{Uniform}(0, 1)$, and then symmetrize it. We generate the degree heterogeneity parameters θ_i i.i.d. from one of the 4 following distributions:

$n^{-0.5}\sqrt{\log(n)}\mathrm{Gamma}(3.5)$, $n^{-0.25}\mathrm{Gamma}(3.5)$, $n^{-0.5}\sqrt{\log(n)}\mathrm{Pareto}(3.5)$, $n^{-0.25}\mathrm{Pareto}(3.5)$. They cover most scenarios: Gamma distributions have considerable mass near 0, so the network has severely low degree nodes; Pareto distributions have heavy tails, so the network has severely high degree nodes. The scaling $n^{-0.5}\sqrt{\log(n)}$ corresponds to the sparse regime, where the average node degree is $\asymp \log(n)^2$, and $n^{-0.25}$ corresponds to the dense regime, with average node degree $\asymp \sqrt{n}$. We consider two cases of Π: the balanced case (bal.) and the imbalanced case (inbal.). In the former, $\pi(i)$ are i.i.d. from $\mathrm{Multinomial}(1/3, 1/3, 1/3)$, and in the latter, $\pi(i)$ are i.i.d. from $\mathrm{Multinomial}(0.2, 0.2, 0.6)$. We repeat the simulation 100 times. Our results are presented in Figure 1, which shows the average classification error of each algorithm as the number of labeled nodes, N_L increases. The plots indicate that AngleMin+ outperforms other methods in all the cases. Furthermore, though AngleMin is not so good as AngleMin+ when N_L is small, it still surpasses all the other approaches except AngleMin+ in most scenarios. Compared to supervised and unsupervised methods which only use part of the data, we can see that AngleMin+ gains a great amount of accuracy by leveraging on both the labeled and unlabeled data.

Real data: We consider three benchmark datasets for community detection, Caltech Red et al. (2011); Traud et al. (2012) , Simmons Red et al. (2011); Traud et al. (2012) , and Polblogs Adamic & Glance (2005). For each data set, we separate nodes into 10 folds and treat each fold as the test data at a time, with the other 9 folds as training data. In the training network, we randomly choose n_L nodes as labeled nodes. We then estimate the label of each node in the test data and report the misclassification error rate. The error rate is further averaged over the 10 folds. We consider $n_L = 0.3n$, $n_L = 0.5n$, and $n_L = 0.7n$, where n is the number of nodes in training data. In this part, we also compare our method to the GNN algorithm in Kipf & Welling (2016), which performs very well on attributed datasets. Since the GNN approach requires the input of a feature vector for each node, but the datasets we considered do not contain node features. We tried i) adding a 50-dimensional constant feature vector for each node whose entries are all 1, ii) adding a 50-dimensional randomly generated feature vector for each node, and iii) using the n-dimensional adjacency vector of a node as the feature vector. The results are shown in Table 1 . In most cases, AngleMin+ significantly outperforms all the other unsupervised or semi-supervised methods, demonstrating the high accuracy of our algorithm. Furthermore, faced with Polblogs, SCORE+ is seriously unstable: the standard deviation of the SCORE+'s classification error is nearly the same as the mean. Since our algorithm is based on SCORE+, its performance should be deteriorated by the poor performance of SCORE+. However, remarkably, even under this substantially unfavorable circumstance, AngleMin+ still maintains rather low means and standard deviations for classification error, indicating that AngleMin+ is strongly robust confronting bad underlying unsupervised community detection algorithms. In total, all these results demonstrate our algorithms' high accuracy and stability in practice.

Table 1: Real data results on Caltech, Simons, and Polblogs networks. The quantities outside and inside the parentheses are the means and standard deviations of misclassification rates, respectively.

Dataset	n	K	SCORE+	n_L/n	AngleMin+	SNMF	GNN (constant)	GNN (random)	GNN (adjacency)
Caltech	590	8	0.237 (0.061)	0.3	**0.207** (0.059)	0.312 (0.049)	0.858 (0.038)	0.859 (0.035)	0.875 (0.038)
				0.5	**0.151** (0.040)	0.310 (0.042)	0.846 (0.054)	0.895 (0.026)	0.859 (0.037)
				0.7	**0.137** (0.046)	0.264 (0.051)	0.849 (0.043)	0.861 (0.034)	0.856 (0.031)
Simmons	1137	4	0.234 (0.084)	0.3	**0.128** (0.024)	0.266 (0.041)	0.691 (0.022)	0.702 (0.039)	0.702 (0.036)
				0.5	**0.096** (0.024)	0.233 (0.033)	0.691 (0.022)	0.711 (0.034)	0.685 (0.025)
				0.7	**0.092** (0.015)	0.220 (0.037)	0.691 (0.022)	0.692 (0.022)	0.691 (0.022)
Polblogs	1222	2	0.166 (0.165)	0.3	0.074 (0.036)	**0.073** (0.019)	0.499 (0.044)	0.502 (0.038)	0.439 (0.048)
				0.5	0.092 (0.041)	**0.068** (0.033)	0.517 (0.040)	0.516 (0.038)	0.453 (0.056)
				0.7	0.066 (0.026)	**0.063** (0.028)	0.485 (0.041)	0.492 (0.043)	0.430 (0.062)

Conclusion and discussions: In this paper, we propose a fast semi-supervised community detection algorithm AngleMin+ based on the structural similarity metric of DCBM. Our method is able to address degree heterogeneity and non-assortative network, is computationally fast, and possesses favorable theoretical properties on consistency and efficiency. Also, our algorithm performs well on both simulations and real data, indicating its strong usage in practice.

There are possible extensions for our method. Our method does not directly deal with soft label (a.k.a mixed membership) where the available label information is the probability of a certain node being in each community. We are currently endeavoring to solve this by fitting our algorithm into the degree-corrected mixed membership model (DCMM), and developing sharp theories for it.

ETHICS STATEMENT

This paper proposes a novel semi-supervised community detection algorithm, AngleMin+, based on the structural similarity metric of DCBM. To the best of our knowledge, we do not think that our work has any potential negative influence. Also, all the real data we use come from public datasets which we have clearly cited, and we do not think that they will raise any privacy issues or other potential problems.

REPRODUCIBILITY STATEMENT

We provide detailed theory on our algorithm AngleMin+. we derive explicit bounds for the misclassification probability of our method under DCBM, and show that it is consistent. We also study the efficiency of our method by comparing its misclassification probability with that of an ideal classifier having access to the community labels of all nodes. Additionally, we provide clear explanations and insights of our theory. All the proofs, together with some generalization of our theory, are available in the appendix. Also, we perform empirical study on our proposed algorithms under both simulations and real data settings, and we consider a large number of scenarios in both cases. All the codes are available in the supplementary materials.

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A GENERALIZATION OF LEMMA 2

In the main paper, for the smoothness and comprehensibility of the text, we do not present the most general form of Lemma 2. We have a more general version of the lemma by relaxing the condition $\frac{\|\theta_{\mathcal{L}}^{(1)}\|_1}{\|\theta_{\mathcal{L}}^{(2)}\|_1} = \frac{\|\theta_{\mathcal{U}}^{(1)}\|_1}{\|\theta_{\mathcal{U}}^{(2)}\|_1} = 1$. Please see Section G for more details.

B PRELIMINARIES

For any positive integer N, Define $[N] = \{1, 2, ..., N\}$.

For a matrix D and two index sets S_1, S_2, define $D_{S_1 S_2}$ to be the submatrix $(D_{ij})_{i \in S_1, j \in S_2}$, $D_{S_1 \cdot}$ to be the submatrix $(D_{ij})_{i \in S_1, j \in \mathcal{L} \cup \mathcal{U}}$, and $D_{\cdot S_2}$ to be the submatrix $(D_{ij})_{i \in \mathcal{L} \cup \mathcal{U}, | \in \mathcal{S}_\in}$.

The two main assumptions (9), (10) in the main paper are presented below for convenience.

$$\|P\|_{\max} \leq C_1, \qquad |\lambda_{\min}(P)| \geq \beta_n. \tag{9}$$

$$\frac{\max_{1 \leq k \leq K}\{\|\theta^{(k)}\|_1\}}{\min_{1 \leq k \leq K}\{\|\theta^{(k)}\|_1\}} \leq C_2, \qquad \max_{1 \leq k \leq K}\left\{\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1 \|\theta\|_1}\right\} \leq c_3 \beta_n. \tag{10}$$

, where constant c_3 is properly small. We would specify this precisely in our proofs.

A number of lemmas used in our proofs will be presented as follows.

The following lemma shows that $\sin x$ and x have the same order.

Lemma 3. *Let $x \in \mathbb{R}$. When $x \geq 0$, $\sin x \leq x$; when $x \in [0, \frac{\pi}{2}]$, $\sin x \geq \frac{2}{\pi} x$.*

Lemma 3 is quite obvious, but for the completeness of our work, we provide a proof for it.

Proof. Let $g_1(x) = \sin x - x$.

Then

$$\frac{d}{dx} g_1(x) = \cos x - 1 \leq 0$$

Hence $g_1(x)$ is monotonously decreasing on $\mathbb{R}$. As a result, when $x \geq 0$, $g_1(x) \geq g_1(0) = 0$. Therefore, when $x \geq 0$, $\sin x \leq x$.

Let $g_2(x) = \sin x - \frac{2}{\pi} x$. Then

$$\frac{d}{dx} g_2(x) = \cos x - \frac{2}{\pi}$$

Since $\cos x$ is monotonously decreasing on $[0, \frac{\pi}{2}]$, $\frac{d}{dx} g_2(x) \geq 0$ when $x \in [0, \arccos \frac{2}{\pi}]$ and $\frac{d}{dx} g_2(x) \leq 0$ when $x \in [\arccos \frac{2}{\pi}, \frac{\pi}{2}]$. Hence, $g_2(x)$ is monotonously increasing on $[0, \arccos \frac{2}{\pi}]$ and is monotonously decreasing on $[\arccos \frac{2}{\pi}, \frac{\pi}{2}]$. As a result, when $x \in [0, \frac{\pi}{2}]$,

$$g_2(x) \geq \min\{g_2(0), g_2(\frac{2}{\pi})\} = 0$$

Therefore, when $x \in [0, \frac{\pi}{2}]$, $\sin x \geq \frac{2}{\pi} x$.

□

The following lemma demonstrates that the angle $\psi(u, v)$ in Definition 1 satisfies the triangle inequality, so that it can be regarded as a sort of "metric".

Lemma 4 (Angle Inequality)**.** *Let x, y, z be three real vectors. Then,*

$$\psi(x, z) \leq \psi(x, y) + \psi(y, z)$$

The proof of Lemma 4 can be seen in Gustafson & Rao (1997), pg 56. Also, for completeness of our work, we provide a proof of Lemma 4.

Proof. Let $\tilde{x} = \frac{x}{\|x\|}$, $\tilde{y} = \frac{y}{\|y\|}$, $\tilde{z} = \frac{z}{\|z\|}$, then $\cos\psi(x,y) = \langle \tilde{x}, \tilde{y} \rangle$, $\cos\psi(y,z) = \langle \tilde{y}, \tilde{z} \rangle$, $\cos\psi(x,z) = \langle \tilde{x}, \tilde{z} \rangle$. Consider the following matrix

$$G = \begin{pmatrix} 1 & \cos\psi(x,y) & \cos\psi(x,z) \\ \cos\psi(x,y) & 1 & \cos\psi(y,z) \\ \cos\psi(x,z) & \cos\psi(y,z) & 1 \end{pmatrix} = \begin{pmatrix} \langle \tilde{x}, \tilde{x} \rangle & \langle \tilde{x}, \tilde{y} \rangle & \langle \tilde{x}, \tilde{z} \rangle \\ \langle \tilde{y}, \tilde{x} \rangle & \langle \tilde{y}, \tilde{y} \rangle & \langle \tilde{y}, \tilde{z} \rangle \\ \langle \tilde{z}, \tilde{x} \rangle & \langle \tilde{z}, \tilde{y} \rangle & \langle \tilde{z}, \tilde{z} \rangle \end{pmatrix}$$

For any vector $c = (c_1, c_2, c_3)^T \in \mathbb{R}^3$,

$$c^T G c = \langle c_1\tilde{x} + c_2\tilde{y} + c_3\tilde{z}, c_1\tilde{x} + c_2\tilde{y} + c_3\tilde{z} \rangle \geq 0$$

Also, G is symmetric. Therefore, G is positive semi-definite. As a result, $det(G) \geq 0$. In other word,

$$1 - \cos^2\psi(x,y) - \cos^2\psi(y,z) - \cos^2\psi(x,z) + 2\cos\psi(x,y)\cos\psi(y,z)\cos\psi(x,z) \geq 0$$

The above inequality can be rewritten as

$$(1 - \cos^2\psi(x,y))(1 - \cos^2\psi(y,z)) \geq (\cos\psi(x,y)\cos\psi(y,z) - \cos\psi(x,z))^2$$

or

$$(\sin\psi(x,y)\sin\psi(y,z))^2 \geq (\cos\psi(x,y)\cos\psi(y,z) - \cos\psi(x,z))^2$$

By definition of $\arccos$, $\psi(x,y), \psi(y,z) \in [0,\pi]$, so $\sin\psi(x,y)\sin\psi(y,z) \geq 0$. Therefore,

$$-\sin\psi(x,y)\sin\psi(y,z) \leq \cos\psi(x,y)\cos\psi(y,z) - \cos\psi(x,z) \leq \sin\psi(x,y)\sin\psi(y,z)$$

$$\cos\psi(x,z) \geq \cos\psi(x,y)\cos\psi(y,z) - \sin\psi(x,y)\sin\psi(y,z)$$
$$\cos\psi(x,z) \geq \cos(\psi(x,y) + \psi(y,z))$$

If $\psi(x,y) + \psi(y,z) > \pi$, because by definition of $\arccos$, $\psi(x,z) \in [0,\pi]$, it is immediate that

$$\psi(x,z) \leq \psi(x,y) + \psi(y,z)$$

If $\psi(x,y) + \psi(y,z) \leq \pi$, recall that $\psi(x,y), \psi(y,z) \in [0,\pi]$, hence $\psi(x,y) + \psi(y,z) \in [0,\pi]$. Also, $\psi(x,z) \in [0,\pi]$. Since $\cos$ is monotone decreasing on $[0,\pi]$, we obtain

$$\psi(x,z) \leq \psi(x,y) + \psi(y,z)$$

In all,

$$\psi(x,z) \leq \psi(x,y) + \psi(y,z)$$

□

The following lemma relates angle to Euclidean distance.

Lemma 5. *Suppose that $x, y \in \mathbb{R}^m$, $\|y\| < \|x\|$. Then,*

$$\psi(x, x+y) \leq \arcsin\left(\frac{\|y\|}{\|x\|}\right)$$

The equality holds if and only if $\langle y, x+y \rangle = 0$

Proof. Let $\rho = \frac{\|y\|}{\|x\|}$, $\psi_0 = \psi(x, y)$. Then $\langle x, y\rangle = \|x\|\|y\|\cos\psi_0$. Notice that

$$\rho^2(\rho + \cos\psi_0)^2 \geq 0$$

This can be rewritten as

$$(1 + \rho\cos\psi_0)^2 \geq (1 + \rho^2 + 2\rho\cos\psi_0)(1 - \rho^2)$$

Since $\|y\| < \|x\|$, so $\rho < 1$, $1 + \rho\cos\psi_0 > 0$. Hence

$$\frac{1 + \rho\cos\psi_0}{\sqrt{1 + \rho^2 + 2\rho\cos\psi_0}} \geq \sqrt{1 - \rho^2}$$

Plugging in $\rho = \frac{\|y\|}{\|x\|}$, we have

$$\frac{\|x\|^2 + \|x\|\|y\|\cos\psi_0}{\sqrt{\|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\cos\psi_0}} \geq \sqrt{1 - \frac{\|y\|^2}{\|x\|^2}}$$

Since $\langle x, y\rangle = \|x\|\|y\|\cos\psi_0$,

$$\frac{\|x\|^2 + \langle x, y\rangle}{\|x\|\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, y\rangle}} \geq \sqrt{1 - \frac{\|y\|^2}{\|x\|^2}}$$

$$\frac{\langle x, x + y\rangle}{\|x\|\sqrt{\langle x + y, x + y\rangle}} \geq \sqrt{1 - \frac{\|y\|^2}{\|x\|^2}}$$

In other words,

$$\cos\psi(x, x + y) \geq \sqrt{1 - \frac{\|y\|^2}{\|x\|^2}} = \cos\arcsin\left(\frac{\|y\|}{\|x\|}\right)$$

Since $\frac{\|y\|}{\|x\|} \geq 0$, $\arcsin\left(\frac{\|y\|}{\|x\|}\right) \in [0, \frac{\pi}{2}]$. Therefore, by monotonicity of $\cos$ on $[0, \frac{\pi}{2}]$,

$$\psi(x, x + y) \leq \arcsin\left(\frac{\|y\|}{\|x\|}\right)$$

The equality holds if and only if $\rho^2(\rho + \cos\psi_0)^2 \geq 0$, or equivalently,

$$(\|y\|^2 + \|x\|\|y\|\cos\psi_0)^2 = 0$$

This can be reduced to

$$\langle y, x + y\rangle = 0$$

□

C PROOF OF LEMMA 1

Lemma 1. *Consider the DCBM model where (9)-(10) are satisfied. We define three $K \times K$ matrices: $G_{\mathcal{L}\mathcal{L}} = \Pi_{\mathcal{L}}'\Theta_{\mathcal{L}\mathcal{L}}\Pi_{\mathcal{L}}$, $G_{\mathcal{U}\mathcal{U}} = \Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}$, and $Q = G_{\mathcal{U}\mathcal{U}}^{-1}\Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}H$. For $1 \leq k \leq K$,*

$$\psi_k(H) = \arccos\Big(\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\Big), \qquad \textit{where} \;\; M = P\Big(G_{\mathcal{L}\mathcal{L}}^2 + G_{\mathcal{U}\mathcal{U}}QQ'G_{\mathcal{U}\mathcal{U}}\Big)P.$$

Proof. Recall that

$$\psi_k(H) = \psi\Big(f(\Omega^{(k)}; H),\ f(\mathbb{E}X; H)\Big), \qquad \text{for } 1 \le k \le K. \tag{14}$$

where

$$f(x; H) = \Big[x'_{\mathcal{L}}\mathbf{1}_{(1)},\ \ldots,\ x'_{\mathcal{L}}\mathbf{1}_{(k)},\ x'_{\mathcal{U}}h_1, \ldots, x'_{\mathcal{U}}h_K\Big]' = [x'_{\mathcal{L}}\Pi_{\mathcal{L}}, x'_{\mathcal{U}}H]'. \tag{15}$$

and

$$\Omega_j^{(k)} = \sum_{i \in \mathcal{L} \cap \mathcal{C}_k} \Omega_{ij} = e'_k \Pi'_{\mathcal{L}} \Omega_{\mathcal{L}\cdot} e_j$$

which indicates

$$\Omega^{(k)} = (e'_k \Pi'_{\mathcal{L}} \Omega_{\mathcal{L}\cdot})'$$

Hence

$$\begin{aligned}
f(\Omega^{(k)}; H) &= f((e'_k \Pi'_{\mathcal{L}} \Omega_{\mathcal{L}\cdot})'; H) \\
&= [e'_k \Pi'_{\mathcal{L}} \Omega_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}}, e'_k \Pi'_{\mathcal{L}} \Omega_{\mathcal{L}\mathcal{U}} H]' \\
&= [e'_k \Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}} P \Pi_{\mathcal{L}}^T \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}}, e'_k \Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}} P \Pi_{\mathcal{U}}^T \Theta_{\mathcal{U}\mathcal{U}} H]' \\
&= [\Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}}, H' \Pi'_{\mathcal{U}} \Theta_{\mathcal{U}\mathcal{U}}]' P \Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}} e_k \\
&= [G_{\mathcal{L}\mathcal{L}}, Q' G_{\mathcal{U}\mathcal{U}}]' P \Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}} e_k
\end{aligned} \tag{16}$$

Notice that

$$(\Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}})_{kl} = \mathbf{1}_{(k)} \Theta_{\mathcal{L}\mathcal{L}} \mathbf{1}_{(l)} = \begin{cases} \|\theta_{\mathcal{L}}^{(k)}\|_1, & k = l \\ 0, & k \neq l \end{cases}$$

In other words,

$$\Pi'_{\mathcal{L}} \Theta_{\mathcal{L}\mathcal{L}} \Pi_{\mathcal{L}} = diag\left(\|\theta_{\mathcal{L}}^{(1)}\|_1, ..., \|\theta_{\mathcal{L}}^{(K)}\|_1\right)$$

Hence

$$f(\Omega^{(k)}; H) = \|\theta_{\mathcal{L}}^{(k)}\|_1 [G_{\mathcal{L}\mathcal{L}}, Q' G_{\mathcal{U}\mathcal{U}}]' P e_k$$

Similarly,

$$f(\mathbb{E}X; H) = \theta^* [G_{\mathcal{L}\mathcal{L}}, Q' G_{\mathcal{U}\mathcal{U}}]' P e_{k^*}$$

Therefore,

$$\begin{aligned}
\langle f(\Omega^{(k)}; H), f(\mathbb{E}X; H) \rangle &= (\|\theta_{\mathcal{L}}^{(k)}\|_1 [G_{\mathcal{L}\mathcal{L}}, Q' G_{\mathcal{U}\mathcal{U}}]' P e_k)' \theta^* [G_{\mathcal{L}\mathcal{L}}, Q' G_{\mathcal{U}\mathcal{U}}]' P e_{k^*} \\
&= \theta^* \|\theta_{\mathcal{L}}^{(k)}\|_1 e'_k P [G_{\mathcal{L}\mathcal{L}}, G_{\mathcal{U}\mathcal{U}} Q][G_{\mathcal{L}\mathcal{L}}, Q' G_{\mathcal{U}\mathcal{U}}]' P e_{k^*} \\
&= \theta^* \|\theta_{\mathcal{L}}^{(k)}\|_1 e'_k P \Big(G_{\mathcal{L}\mathcal{L}}^2 + G_{\mathcal{U}\mathcal{U}} Q Q' G_{\mathcal{U}\mathcal{U}}\Big) P e_{k^*} \\
&= \theta^* \|\theta_{\mathcal{L}}^{(k)}\|_1 e'_k M e_{k^*} \\
&= \theta^* \|\theta_{\mathcal{L}}^{(k)}\|_1 M_{kk^*}
\end{aligned} \tag{17}$$

Similarly,

$$\|f(\Omega^{(k)}; H)\| = \sqrt{\langle f(\Omega^{(k)}; H), f(\Omega^{(k)}; H) \rangle} = \|\theta_{\mathcal{L}}^{(k)}\|_1 \sqrt{M_{kk}} \tag{18}$$

$$\|f(\mathbb{E}X; H)\| = \sqrt{\langle f(\mathbb{E}X; H), f(\mathbb{E}X; H) \rangle} = \theta^* \sqrt{M_{k^*k^*}} \tag{19}$$

Hence,

$$\begin{aligned}\psi_k(H) &= \psi\Big(f(\Omega^{(k)}; H),\ f(\mathbb{E}X; H)\Big)\\ &= \arccos\left(\frac{\langle f(\Omega^{(k)}; H), f(\mathbb{E}X; H)\rangle}{\|f(\Omega^{(k)}; H)\|\|f(\mathbb{E}X; H)\|}\right)\\ &= \arccos\left(\frac{\theta^*\|\theta^{(k)}_{\mathcal{L}}\|_1 M_{kk^*}}{\|\theta^{(k)}_{\mathcal{L}}\|_1\sqrt{M_{kk}}\theta^*\sqrt{M_{k^*k^*}}}\right)\\ &= \arccos\left(\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\right) \qquad (20)\end{aligned}$$

□

D PROOF OF THEOREM 1

Theorem 1. *Consider the DCBM model where (9)-(10) hold. Let k^* denote the true community label of the new node. Suppose $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct, for a constant $b_0 \in (0, 1)$. When b_0 is properly small, there exists a constant $c_0 > 0$, which does not depend on b_0, such that $\psi_{k^*}(\hat{\Pi}_{\mathcal{U}}) = 0$ and $\min_{k\neq k^*}\{\psi_k(\hat{\Pi}_{\mathcal{U}})\} \geq c_0\beta_n$.*

Proof. Define $G_{\mathcal{L}\mathcal{L}} = \Pi'_{\mathcal{L}}\Theta_{\mathcal{L}\mathcal{L}}\Pi_{\mathcal{L}}$, $G_{\mathcal{U}\mathcal{U}} = \Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}$, $Q = G^{-1}_{\mathcal{U}\mathcal{U}}\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\hat{\Pi}_{\mathcal{U}}$, and $M = P\Big(G^2_{\mathcal{L}\mathcal{L}} + G_{\mathcal{U}\mathcal{U}}QQ'G_{\mathcal{U}\mathcal{U}}\Big)P$ as in Lemma 1. According to Lemma 1,

$$\psi_k(\hat{\Pi}_{\mathcal{U}}) = \arccos\Big(\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\Big)$$

Hence,

$$\psi_{k^*}(\hat{\Pi}_{\mathcal{U}}) = \arccos\Big(\frac{M_{k^*k^*}}{\sqrt{M_{k^*k^*}}\sqrt{M_{k^*k^*}}}\Big) = \arccos 1 = 0$$

When $k \neq k^*$, according to Lemma 3,

$$\begin{aligned}\psi_k(\hat{\Pi}_{\mathcal{U}}) &= 2\cdot\frac{1}{2}\psi_k(\hat{\Pi}_{\mathcal{U}})\\ &\geq 2\sin\frac{1}{2}\psi_k(\hat{\Pi}_{\mathcal{U}})\\ &= 2\sqrt{\frac{1-\cos\psi_k(\hat{\Pi}_{\mathcal{U}})}{2}}\\ &= \sqrt{2\Big(1-\cos\arccos\Big(\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\Big)\Big)}\\ &= \sqrt{2\Big(1-\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\Big)} \qquad (21)\end{aligned}$$

Let $D_M = \mathrm{diag}(M_{11}, ..., M_{KK})$, $\tilde{M} = D_M^{-\frac{1}{2}}MD_M^{-\frac{1}{2}}$. Then

$$\begin{aligned}&(e_k - e_{k^*})'\tilde{M}(e_k - e_{k^*})\\ &= \tilde{M}_{kk} + \tilde{M}_{k^*k^*} - \tilde{M}_{kk^*} - \tilde{M}_{k^*k}\\ &= \frac{M_{kk}}{\sqrt{M_{kk}}\sqrt{M_{kk}}} + \frac{M_{k^*k^*}}{\sqrt{M_{k^*k^*}}\sqrt{M_{k^*k^*}}} - \frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}} - \frac{M_{k^*k}}{\sqrt{M_{k^*k^*}\sqrt{M_{kk}}}}\\ &= 2\Big(1-\frac{M_{kk^*}}{\sqrt{M_{kk}}\sqrt{M_{k^*k^*}}}\Big) \qquad (22)\end{aligned}$$

Hence,

$$\psi_k(\hat{\Pi}_{\mathcal{U}}) \geq \sqrt{(e_k - e_{k^*})'\tilde{M}(e_k - e_{k^*})} \tag{23}$$

$\tilde{M}$ is affected by $\hat{\Pi}_{\mathcal{U}}$ and is complicated to evaluate directly. Hence, we would first evaluate its oracle version and then reduce the noisy version to the oracle version.

Define the oracle version of M as follow, where $\hat{\Pi}_{\mathcal{U}}$ is replaced by $\Pi_{\mathcal{U}}$

$$M^{(0)} = P\Big(G^2_{\mathcal{L}\mathcal{L}} + G^2_{\mathcal{U}\mathcal{U}}\Big)P$$

Similarly, define the oracle version of D_M, $D_{M^{(0)}} = \text{diag}(M^{(0)}_{11}, ..., M^{(0)}_{KK})$, and the oracle version of $\tilde{M}$, $\tilde{M}^{(0)} = D^{-\frac{1}{2}}_{M^{(0)}} M^{(0)} D^{-\frac{1}{2}}_{M^{(0)}}$

Oracle Case We first study the oracle case $|\alpha'\tilde{M}^{(0)}\alpha|$.

Since $G_{\mathcal{L}\mathcal{L}} = \Pi'_{\mathcal{L}}\Theta_{\mathcal{L}\mathcal{L}}\Pi_{\mathcal{L}} = \text{diag}\Big(\|\theta^{(1)}_{\mathcal{L}}\|_1, ..., \|\theta^{(K)}_{\mathcal{L}}\|_1\Big)$, $G_{\mathcal{U}\mathcal{U}} = \Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}} = \text{diag}\Big(\|\theta^{(1)}_{\mathcal{U}}\|_1, ..., \|\theta^{(K)}_{\mathcal{U}}\|_1\Big)$, which indicates that $G^2_{\mathcal{L}\mathcal{L}} + G^2_{\mathcal{U}\mathcal{U}} = \text{diag}\Big(\Big(\|\theta^{(k)}_{\mathcal{L}}\|^2_1 + \|\theta^{(k)}_{\mathcal{U}}\|^2_1\Big)^K_{k=1}\Big)$, for any vector $\alpha \in \mathbb{R}^k$,

$$\begin{aligned}
|\alpha'\tilde{M}^{(0)}\alpha| &= |\alpha' D^{-\frac{1}{2}}_{M^{(0)}} M^{(0)} D^{-\frac{1}{2}}_{M^{(0)}}\alpha| \\
&= |\alpha' D^{-\frac{1}{2}}_{M^{(0)}} P\Big(G^2_{\mathcal{L}\mathcal{L}} + G^2_{\mathcal{U}\mathcal{U}}\Big)P D^{-\frac{1}{2}}_{M^{(0)}}\alpha| \\
&\geq \|P D^{-\frac{1}{2}}_{M^{(0)}}\alpha\|^2 \min_k(\|\theta^{(k)}_{\mathcal{L}}\|^2_1 + \|\theta^{(k)}_{\mathcal{U}}\|^2_1) \\
\text{(Cauchy-Schwartz Inequality)} &\geq |\alpha' D^{-\frac{1}{2}}_{M^{(0)}} P P' D^{-\frac{1}{2}}_{M^{(0)}}\alpha| \min_k \frac{1}{2}(\|\theta^{(k)}_{\mathcal{L}}\|_1 + \|\theta^{(k)}_{\mathcal{U}}\|_1)^2 \\
&\geq \frac{1}{2}\lambda_{\min}(P)^2 \|D^{-\frac{1}{2}}_{M^{(0)}}\alpha\|^2 \min_k(\|\theta^{(k)}\|_1)^2 \\
\text{(Condition (9))} &\geq \frac{1}{2}\beta_n^2 |\alpha D^{-1}_{M^{(0)}}\alpha| \min_k(\|\theta^{(k)}\|_1)^2 \\
&\geq \frac{1}{2}\beta_n^2 \|\alpha\|^2 \min_k(\|\theta^{(k)}_{\mathcal{L}}\|^2_1 + \|\theta^{(k)}_{\mathcal{U}}\|^2_1)^{-1} \min_k(\|\theta^{(k)}\|_1)^2 \\
&\geq \frac{1}{2}\beta_n^2 \|\alpha\|^2 \min_k\Big[(\|\theta^{(k)}_{\mathcal{L}}\|_1 + \|\theta^{(k)}_{\mathcal{U}}\|_1)^2\Big]^{-1} (\min_k \|\theta^{(k)}\|_1)^2 \\
&= \frac{1}{2}\beta_n^2 \|\alpha\|^2 \left(\frac{\min_k \|\theta^{(k)}\|_1}{\max_k \|\theta^{(k)}\|_1}\right)^2 \\
\text{(Condition (10))} &\geq \frac{\beta_n^2 \|\alpha\|^2}{2C_2^2}
\end{aligned} \tag{24}$$

It remains to study the noisy case. We reduce the noisy case to the oracle case through the following lemma.

Lemma 6. *Denote*

$$C_5 = 8K^2\sqrt{K}C_2^2 b_0 \frac{\|\theta_{\mathcal{U}}\|_1}{\|\theta\|_1} \tag{25}$$

Suppose that $C_5 \leq \frac{1}{4}$. Then, for any vector $\alpha \in \mathbb{R}^k$,

$$|\alpha'\tilde{M}\alpha| \geq \frac{1 - 3C_5}{1 - C_5}|\alpha'\tilde{M}^{(0)}\alpha| \geq \frac{1}{3}|\alpha'\tilde{M}^{(0)}\alpha| \tag{26}$$

The proof of Lemma 6 is quite tedious and we would defer it to the end of this section.

Set $b_0 \leq \frac{1}{32K^2\sqrt{K}C_2^2}$, then $C_5 \leq \frac{\|\theta_{\mathcal{U}}\|_1}{4\|\theta\|_1} \leq \frac{1}{4}$. As a result, combining (24) with Lemma 6, we have for any vector $\alpha \in \mathbb{R}^k$,

$$|\alpha'\tilde{M}\alpha| \geq \frac{1}{3}|\alpha'\tilde{M}^{(0)}\alpha| \geq \frac{\beta_n^2\|\alpha\|^2}{6C_2^2} \tag{27}$$

Hence, take $\alpha = e_k - e_{k^*}$ in (27) and combine it with (23), we obtain that for any $k \in [K]$

$$\psi_k(\hat{\Pi}_{\mathcal{U}}) \geq \sqrt{(e_k - e_{k^*})'\tilde{M}(e_k - e_{k^*})} \geq \sqrt{\frac{1}{6C_2^2}\beta_n^2\|e_k - e_{k^*}\|^2} = \sqrt{\frac{1}{3C_2^2}}\beta_n \tag{28}$$

Therefore, set $c_0 = \sqrt{\frac{1}{3C_2^2}}$, we have

$$\min_{k \neq k^*}\{\psi_k(\hat{\Pi}_{\mathcal{U}})\} \geq c_0\beta_n \tag{29}$$

In all, when b_0 is properly small such that $b_0 \leq \frac{1}{32K^2\sqrt{K}C_2^2}$, there exists constant $c_0 = \sqrt{\frac{1}{3C_2^2}} > 0$ not depending on b_0 such that $\psi_{k^*}(\hat{\Pi}_{\mathcal{U}}) = 0$ and $\min_{k \neq k^*}\{\psi_k(\hat{\Pi}_{\mathcal{U}})\} \geq c_0\beta_n$. □

D.1 Proof of Lemma 6

Proof. For any vector $\alpha \in \mathbb{R}^k$,

$$\begin{aligned}
|\alpha'\tilde{M}\alpha - \alpha'\tilde{M}^{(0)}\alpha| &= |\alpha' D_M^{-\frac{1}{2}} M D_M^{-\frac{1}{2}}\alpha - \alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}}\alpha| \\
&= |\alpha' D_M^{-\frac{1}{2}}(M - M^{(0)}) D_M^{-\frac{1}{2}}\alpha \\
&\quad + \alpha' D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}}\alpha - \alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}}\alpha| \\
&\leq |\alpha' D_M^{-\frac{1}{2}}(M - M^{(0)}) D_M^{-\frac{1}{2}}\alpha| \\
&\quad + |\alpha'\Big(D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}}\Big)\alpha|
\end{aligned} \tag{30}$$

The first part on the RHS of (30),

$$\begin{aligned}
|\alpha' D_M^{-\frac{1}{2}}(M - M^{(0)}) D_M^{-\frac{1}{2}}\alpha| &= |\alpha' D_M^{-\frac{1}{2}} P\Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}(\hat{\Pi}_{\mathcal{U}}\hat{\Pi}_{\mathcal{U}}' - \Pi_{\mathcal{U}}\Pi_{\mathcal{U}}')\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}} P D_M^{-\frac{1}{2}}\alpha| \\
&\leq \|P D_M^{-\frac{1}{2}}\alpha\|^2 \|\Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}(\hat{\Pi}_{\mathcal{U}}\hat{\Pi}_{\mathcal{U}}' - \Pi_{\mathcal{U}}\Pi_{\mathcal{U}}')\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}\|_2 \\
&\leq \sqrt{K}\|P D_M^{-\frac{1}{2}}\alpha\|^2 \|\Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}(\hat{\Pi}_{\mathcal{U}}\hat{\Pi}_{\mathcal{U}}' - \Pi_{\mathcal{U}}\Pi_{\mathcal{U}}')\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}\|_\infty
\end{aligned} \tag{31}$$

Denote $G^{(d)} = \Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}(\hat{\Pi}_{\mathcal{U}}\hat{\Pi}_{\mathcal{U}}' - \Pi_{\mathcal{U}}\Pi_{\mathcal{U}}')\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}}$. Define $\eta_{l\tilde{l}} = \sum_{i \in \mathcal{U}, \pi_i = e_l, \hat{\pi}_i = e_{\tilde{l}}} \theta_i$. In other words, $\eta_{l\tilde{l}}$ is the sum of the degree heterogeneity parameters of all the nodes in $\mathcal{U}$ with true label l and estimated label $\tilde{l}$.

Then,

$$(\Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}\hat{\Pi}_{\mathcal{U}})_{l\tilde{l}} = \eta_{l\tilde{l}}$$

$$(\Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}})_{l\tilde{l}} = \sum_{i \in \mathcal{U}, \pi_i = e_l, \pi_i = e_{\tilde{l}}} \theta_i = I_{l=\tilde{l}} \sum_{s \in [K]} \eta_{ls}$$

where $I_{l=\tilde{l}}$ is the indicator function of event $\{l = \tilde{l}\}$.

Hence,

$$
\begin{aligned}
G^{(d)}_{l\tilde{l}} &= (\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}(\hat{\Pi}_{\mathcal{U}}\hat{\Pi}'_{\mathcal{U}} - \Pi_{\mathcal{U}}\Pi'_{\mathcal{U}})\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}})_{l\tilde{l}} \\
&= ((\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\hat{\Pi}_{\mathcal{U}})(\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\hat{\Pi}_{\mathcal{U}})')_{l\tilde{l}} - ((\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}})(\Pi'_{\mathcal{U}}\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}})')_{l\tilde{l}} \\
&= \sum_{s\in[K]} \eta_{ls}\eta_{\tilde{l}s} - I_{l=\tilde{l}}(\sum_{s\in[K]} \eta_{ls})^2
\end{aligned} \tag{32}
$$

Since $\hat{\Pi}_{\mathcal{U}}$ is b_0 correct, there exists permutation T of K columns of $\hat{\Pi}_{\mathcal{U}}$ such that $\big(\sum_{i\in\mathcal{U}} \theta_i \cdot 1\{T\hat{\pi}_i \neq \pi_i\}\big) \leq b_0\|\theta\|_1$.

Let $r = r(l)$ satisfies $e_r = T^{-1}e_l$.

When $l = \tilde{l}$, we have

$$
\begin{aligned}
|G^{(d)}_{l\tilde{l}}| &= |\sum_{s\in[K]} \eta^2_{ls} - (\sum_{s\in[K]} \eta_{ls})^2| \\
&= (\sum_{s\in[K]} \eta_{ls})^2 - \sum_{s\in[K]} \eta^2_{ls} \\
&\leq (\sum_{s\in[K]} \eta_{ls})^2 - \eta^2_{lr} \\
&= (\sum_{s\neq r} \eta_{ls})(\eta_{lr} + \sum_{s\in[K]} \eta_{ls}) \\
&\leq 2(\sum_{s\neq r} \eta_{ls})(\sum_{s\in[K]} \eta_{ls})
\end{aligned} \tag{33}
$$

When $l \neq \tilde{l}$, we have

$$
\begin{aligned}
|G^{(d)}_{l\tilde{l}}| &= |\sum_{s\in[K]} \eta_{ls}\eta_{\tilde{l}s}| \\
&= \sum_{s\in[K]} \eta_{ls}\eta_{\tilde{l}s}
\end{aligned} \tag{34}
$$

Therefore,

$$
\begin{aligned}
\|G^{(d)}\|_\infty &\leq \sum_{l,\tilde{l}\in[K]} |G^{(d)}_{l\tilde{l}}| \\
&= \sum_{l\in[K]} |G^{(d)}_{ll}| + \sum_{l\neq\tilde{l}} |G^{(d)}_{ll}| \\
&\leq \sum_{l\in[K]} \Big(2(\sum_{s\neq r}\eta_{ls})(\sum_{s\in[K]}\eta_{ls})\Big) + \sum_{l\neq\tilde{l}}\sum_{s\in[K]} \eta_{ls}\eta_{\tilde{l}s} \\
&\leq 2\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big)\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) + \sum_{s\in[K]}\sum_{l\neq\tilde{l}}\eta_{ls}\eta_{\tilde{l}s} \\
&= 2\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big)\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) + \sum_{s\in[K]}\big[\big(\sum_{l\in[k]}\eta_{ls}\big)^2 - \sum_{l\in[k]}\eta_{ls}^2\big] \\
&\leq 2\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big)\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) + \sum_{s\in[K]}\big[\big(\sum_{l\in[k]}\eta_{ls}\big)^2 - \sum_{r(l)=s}\eta_{ls}^2\big] \\
&= 2\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big)\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) + \sum_{s\in[K]}\big[\big(\sum_{r(l)\neq s}\eta_{ls}\big)\big(\sum_{l\in[K]}\eta_{ls} + \sum_{r(l)=s}\eta_{ls}\big)\big] \\
&\leq 2\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big)\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) + 2\sum_{s\in[K]}\big[\big(\sum_{r(l)\neq s}\eta_{ls}\big)\big(\sum_{l\in[K]}\eta_{ls}\big)\big] \\
&\leq 2\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big)\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) + 2\max_s\Big(\sum_{l\in[K]}\eta_{ls}\Big)\sum_{s\in[K]}\sum_{r(l)\neq s}\eta_{ls} \\
&= 2\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big)\Big(\max_l\Big(\sum_{s\in[K]}\eta_{ls}\Big) + \max_s\Big(\sum_{l\in[K]}\eta_{ls}\Big)\Big) \\
&\leq 2\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big)\Big(\sum_{l\in[K]}\sum_{s\in[K]}\eta_{ls} + \sum_{s\in[K]}\sum_{l\in[K]}\eta_{ls}\Big) \\
&= 4\|\theta_{\mathcal{U}}\|_1\big(\sum_{l\in[K],s\neq r(l)}\eta_{ls}\big) \qquad (35)
\end{aligned}
$$

Recall that T satisfies $\big(\sum_{i\in\mathcal{U}}\theta_i\cdot 1\{T\hat{\pi}_i\neq\pi_i\}\big)\leq b_0\|\theta\|_1$, hence

$$
\sum_{l\in[K],s\neq r(l)}\eta_{ls} = \sum_{\substack{l\in[K],s\neq r(l),\\ i\in\mathcal{U},\pi_i=e_l,\hat{\pi}_i=e_s}}\theta_i = \sum_{i\in\mathcal{U}}\theta_i\cdot 1\{T\hat{\pi}_i\neq\pi_i\}\leq b_0\|\theta\|_1
$$

Therefore,

$$
\|G^{(d)}\|_\infty \leq 4b_0\|\theta_{\mathcal{U}}\|_1\|\theta\|_1 \qquad (36)
$$

Plugging (36) into (31), we obtain

$$
|\alpha' D_M^{-\frac{1}{2}}(M-M^{(0)})D_M^{-\frac{1}{2}}\alpha| \leq 4\sqrt{K}b_0\|\theta_{\mathcal{U}}\|_1\|\theta\|_1\|PD_M^{-\frac{1}{2}}\alpha\|^2 \qquad (37)
$$

On the other hand,

$$
|\alpha' D_M^{-\frac{1}{2}}M^{(0)}D_M^{-\frac{1}{2}}\alpha| = |\alpha' D_M^{-\frac{1}{2}}P\Big(G^2_{\mathcal{L}\mathcal{L}} + G^2_{\mathcal{U}\mathcal{U}}\Big)PD_M^{-\frac{1}{2}}\alpha|
$$

Since $G_{\mathcal{L}\mathcal{L}} = \Pi_{\mathcal{L}}'\Theta_{\mathcal{L}\mathcal{L}}\Pi_{\mathcal{L}} = \text{diag}(\|\theta_{\mathcal{L}}^{(1)}\|_1, ..., \|\theta_{\mathcal{L}}^{(K)}\|_1)$, $G_{\mathcal{U}\mathcal{U}} = \Pi_{\mathcal{U}}'\Theta_{\mathcal{U}\mathcal{U}}\Pi_{\mathcal{U}} = \text{diag}(\|\theta_{\mathcal{U}}^{(1)}\|_1, ..., \|\theta_{\mathcal{U}}^{(K)}\|_1)$,

$$
\begin{aligned}
|\alpha' D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} \alpha| &\geq \|P D_M^{-\frac{1}{2}} \alpha\|^2 \min_k (\|\theta_{\mathcal{L}}^{(k)}\|_1^2 + \|\theta_{\mathcal{U}}^{(k)}\|_1^2) \\
\text{(Cauchy-Schwartz Inequality)} &\geq \|P D_M^{-\frac{1}{2}} \alpha\|^2 \min_k \frac{1}{2} (\|\theta_{\mathcal{L}}^{(k)}\|_1 + \|\theta_{\mathcal{U}}^{(k)}\|_1)^2 \\
&= \frac{1}{2} \|P D_M^{-\frac{1}{2}} \alpha\|^2 \min_k \|\theta^{(k)}\|_1^2
\end{aligned}
$$

Recall condition (10) in the main paper,

$$
\frac{\max_{1 \leq k \leq K} \{\|\theta^{(k)}\|_1\}}{\min_{1 \leq k \leq K} \{\|\theta^{(k)}\|_1\}} \leq C_2
$$

Hence

$$
\begin{aligned}
|\alpha' D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} \alpha| &\geq \frac{1}{2} \|P D_M^{-\frac{1}{2}} \alpha\|^2 (\frac{1}{C_2} \max_k \|\theta^{(k)}\|_1)^2 \\
&\geq \frac{1}{2} \|P D_M^{-\frac{1}{2}} \alpha\|^2 (\frac{1}{K C_2} \|\theta\|_1)^2 \\
&= \frac{1}{2K^2 C_2^2} \|P D_M^{-\frac{1}{2}} \alpha\|^2 \|\theta\|_1^2
\end{aligned} \tag{38}
$$

Comparing (38) with (37), we obtain

$$
\begin{aligned}
&|\alpha' D_M^{-\frac{1}{2}} (M - M^{(0)}) D_M^{-\frac{1}{2}} \alpha| \\
&\leq 8K^2 \sqrt{K} C_2^2 b_0 \frac{\|\theta_{\mathcal{U}}\|_1}{\|\theta\|_1} |\alpha' D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} \alpha| \\
&\leq C_5 |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \alpha| \\
&\quad + C_5 |\alpha' \Big(D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \Big) \alpha|
\end{aligned} \tag{39}
$$

Consequently, we bound the first part of (30) by the second part of (30). It remains to bound the second part of (30).

Since D_M, $D_{M^{(0)}}$, $M^{(0)}$ are all diagonal matrices, we can rewrite the second part on the LHS of (30) as follows:

$$
\begin{aligned}
&|\alpha' \Big(D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \Big) \alpha| \\
&= |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} (M^{(0)})^{\frac{1}{2}} \Big((M^{(0)})^{-\frac{1}{2}} D_{M^{(0)}}^{\frac{1}{2}} D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} D_{M^{(0)}}^{\frac{1}{2}} (M^{(0)})^{-\frac{1}{2}} - 1 \Big) (M^{(0)})^{\frac{1}{2}} D_{M^{(0)}}^{-\frac{1}{2}} \alpha| \\
&= |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} (M^{(0)})^{\frac{1}{2}} \Big(D_{M^{(0)}} D_M^{-1} - 1 \Big) (M^{(0)})^{\frac{1}{2}} D_{M^{(0)}}^{-\frac{1}{2}} \alpha| \\
&\leq \lambda_{\max} \Big(D_{M^{(0)}} D_M^{-1} - 1 \Big) \|(M^{(0)})^{\frac{1}{2}} D_{M^{(0)}}^{-\frac{1}{2}} \alpha\|^2 \\
&= \max_{k \in [K]} \left| \frac{M_{kk}^{(0)}}{M_{kk}} - 1 \right| \cdot \|(M^{(0)})^{\frac{1}{2}} D_{M^{(0)}}^{-\frac{1}{2}} \alpha\|^2 \\
&= \max_{k \in [K]} \left| \frac{1}{\frac{M_{kk}^{(0)}}{(M^{(0)} - M)_{kk}} - 1} \right| \cdot |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \alpha|
\end{aligned} \tag{40}
$$

Notice that for any $k \in [K]$

$$
\begin{aligned}
\left|\frac{M_{kk}^{(0)}}{(M^{(0)}-M)_{kk}}\right| &= \frac{\left|e_k' M^{(0)} e_k\right|}{\left|e_k'(M^{(0)}-M)e_k\right|} \\
&= \frac{\left|e_k' P\Big(G_{\mathcal{L}\mathcal{L}}^2 + G_{\mathcal{U}\mathcal{U}}^2\Big) P e_k\right|}{\left|e_k' P G^{(d)} P e_k\right|} \\
&\geq \frac{\|Pe_k\|^2 \min_k(\|\theta_{\mathcal{L}}^{(k)}\|_1^2 + \|\theta_{\mathcal{U}}^{(k)}\|_1^2)}{\|Pe_k\|^2 \|G^{(d)}\|_2} \\
\text{(Cauchy-Schwartz Inequality)} &\geq \frac{\min_k \frac{1}{2}(\|\theta_{\mathcal{L}}^{(k)}\|_1 + \|\theta_{\mathcal{U}}^{(k)}\|_1)^2}{\sqrt{K}\|G^{(d)}\|_\infty} \\
\text{(Plugging in (36))} &\geq \frac{\min_k(\|\theta^{(k)}\|_1)^2}{8\sqrt{K} b_0 \|\theta_{\mathcal{U}}\|_1 \|\theta\|_1} \\
\text{(Condition (10))} &\geq \frac{(\max_k \frac{1}{C_2}\|\theta^{(k)}\|_1)^2}{8\sqrt{K} b_0 \|\theta_{\mathcal{U}}\|_1 \|\theta\|_1} \\
&\geq \frac{(\frac{1}{C_2 K}\|\theta\|_1)^2}{8\sqrt{K} b_0 \|\theta_{\mathcal{U}}\|_1 \|\theta\|_1} \\
&= \frac{1}{C_5} \\
&\geq 4 > 1
\end{aligned} \tag{41}
$$

Plugging (41) into (40), we have

$$
\begin{aligned}
&|\alpha'\Big(D_M^{-\frac{1}{2}} M^{(0)} D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}}\Big)\alpha| \\
&\leq \max_{k\in[K]} \frac{1}{\left|\frac{M_{kk}^{(0)}}{(M^{(0)}-M)_{kk}}\right| - 1} \cdot |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \alpha| \\
&\leq \max_{k\in[K]} \frac{1}{\frac{1}{C_5} - 1} \cdot |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \alpha| \\
&= \frac{C_5}{1-C_5} |\alpha' D_{M^{(0)}}^{-\frac{1}{2}} M^{(0)} D_{M^{(0)}}^{-\frac{1}{2}} \alpha|
\end{aligned} \tag{42}
$$

Combining (30), (39), and (42), we have

$$
\begin{aligned}
|\alpha'\tilde{M}\alpha - \alpha'\tilde{M}^{(0)}\alpha| &\leq |\alpha' D_M^{-\frac{1}{2}}(M - M^{(0)})D_M^{-\frac{1}{2}}\alpha| \\
&\quad + |\alpha'\Big(D_M^{-\frac{1}{2}}M^{(0)}D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\Big)\alpha| \\
&\leq C_5|\alpha' D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\alpha| \\
&\quad + C_5|\alpha'\Big(D_M^{-\frac{1}{2}}M^{(0)}D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\Big)\alpha| \\
&\quad + |\alpha'\Big(D_M^{-\frac{1}{2}}M^{(0)}D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\Big)\alpha| \\
&= C_5|\alpha' D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\alpha| \\
&\quad + (C_5+1)|\alpha'\Big(D_M^{-\frac{1}{2}}M^{(0)}D_M^{-\frac{1}{2}} - D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\Big)\alpha| \\
&\leq C_5|\alpha' D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\alpha| \\
&\quad + (C_5+1)\frac{C_5}{1-C_5}|\alpha' D_{M^{(0)}}^{-\frac{1}{2}}M^{(0)}D_{M^{(0)}}^{-\frac{1}{2}}\alpha| \\
&= \frac{2C_5}{1-C_5}|\alpha'\tilde{M}^{(0)}\alpha| \qquad (43)
\end{aligned}
$$

Hence for any vector $\alpha \in \mathbb{R}^k$,

$$
|\alpha'\tilde{M}\alpha| \geq |\alpha'\tilde{M}^{(0)}\alpha| - |\alpha'\tilde{M}\alpha - \alpha'\tilde{M}^{(0)}\alpha| \geq \frac{1-3C_5}{1-C_5}|\alpha'\tilde{M}^{(0)}\alpha| \qquad (44)
$$

To conclude, in this subsection, we successfully reduce the noisy case $|\alpha'\tilde{M}\alpha|$ to the oracle case $|\alpha'\tilde{M}^{(0)}\alpha|$. Result (44) will also be used in the proof of other claims.

□

E PROOF OF THEOREM 2

Theorem 2. *Consider the DCBM model where (9)-(10) hold. There exists constant $C > 0$, such that for any $\delta \in (0, 1/2)$, with probability $1 - \delta$, simultaneously for $1 \leq k \leq K$,*

$$
|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq C\left(\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} + \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\right).
$$

To prove Theorem 2, we need a famous concentration inequality, Bernstein inequality:

Lemma 7 (Bernstein inequality)**.** *Suppose $X_1, ..., X_n$ are independent random variables such that $\mathbb{E}X_i = 0$, $|X_i| \leq b$ and $Var(X_i) \leq \sigma_i^2$ for all i. Let $\sigma^2 = n^{-1}\sum_{i=1}^n \sigma_i^2$. Then, for any $t > 0$,*

$$
\mathbb{P}\Big(n^{-1}|\sum_{i=1}^n X_i| \geq t\Big) \leq 2\exp\left(-\frac{nt^2/2}{\sigma^2 + bt/3}\right)
$$

The proof of Lemma 7, Bernstein inequality, can be seen in most probability textbooks such as Uspensky (1937).

Proof. Recall that for $k \in [K]$,

$$
\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) = \psi(f(A^{(k)}; \hat{\Pi}_{\mathcal{U}}), f(X; H)),
$$

$$
\psi_k(\hat{\Pi}_{\mathcal{U}}) = \psi\Big(f(\Omega^{(k)}; \hat{\Pi}_{\mathcal{U}}),\ f(\mathbb{E}X; \hat{\Pi}_{\mathcal{U}})\Big).
$$

Denote $v_k = f(A^{(k)}; \hat{\Pi}_{\mathcal{U}})$, $v^* = f(X; \hat{\Pi}_{\mathcal{U}})$, $\tilde{v}_k = f(\Omega^{(k)}; \hat{\Pi}_{\mathcal{U}})$, $\tilde{v}^* = f(EX; \hat{\Pi}_{\mathcal{U}})$, $k \in [K]$. Then, by Lemma 4,

$$\begin{aligned}\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) = \psi(v_k, v^*) &\leq \psi(v_k, \tilde{v}_k) + \psi(\tilde{v}_k, v^*) \\ &\leq \psi(v_k, \tilde{v}_k) + \psi(\tilde{v}_k, \tilde{v}^*) + \psi(\tilde{v}^*, v^*) \\ &= \psi_k(\hat{\Pi}_{\mathcal{U}}) + \psi(v_k, \tilde{v}_k) + \psi(\tilde{v}^*, v^*)\end{aligned}$$

Similarly,

$$\psi_k(\hat{\Pi}_{\mathcal{U}}) \leq \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) + \psi(v_k, \tilde{v}_k) + \psi(\tilde{v}^*, v^*)$$

Therefore,

$$|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq \psi(v_k, \tilde{v}_k) + \psi(\tilde{v}^*, v^*). \tag{45}$$

For any $\phi_1, ..., \phi_K \geq 0$.

$$\begin{aligned}&\mathbb{P}\Big(\forall k \in [K], |\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq \phi_k\Big) \\ &= 1 - \mathbb{P}\Big(\exists k \in [K], |\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) \\ &\geq 1 - \sum_{k=1}^{K} \mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big)\end{aligned} \tag{46}$$

By definition of ψ, $\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}), \psi_k(\hat{\Pi}_{\mathcal{U}}) \in [0, \pi]$. Hence, $|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \in [0, \pi]$. As a result, when $\phi_k \geq \pi$,

$$\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) = 0$$

When $\phi_k < \pi$, by (45),

$$\begin{aligned}\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) &\leq \mathbb{P}\Big(\psi(v_k, \tilde{v}_k) + \psi(\tilde{v}^*, v^*) > \phi_k\Big) \\ &\leq \mathbb{P}\Big(\psi(v_k, \tilde{v}_k) > \frac{1}{2}\phi_k \text{ or } \psi(\tilde{v}^*, v^*) > \frac{1}{2}\phi_k\Big) \\ &\leq \mathbb{P}\Big(\psi(v_k, \tilde{v}_k) > \frac{1}{2}\phi_k\Big) + \mathbb{P}\Big(\psi(\tilde{v}^*, v^*) > \frac{1}{2}\phi_k\Big)\end{aligned} \tag{47}$$

By lemma 5, when $\|v_k - \tilde{v}_k\| < \|\tilde{v}_k\|$

$$\psi(v_k, \tilde{v}_k) \leq \arcsin \frac{\|v_k - \tilde{v}_k\|}{\|\tilde{v}_k\|}$$

Hence, for any $\phi \in [0, \frac{\pi}{2})$, $\|v_k - \tilde{v}_k\| \leq \sin(\phi)\|\tilde{v}_k\|$ implies $\psi(v_k, \tilde{v}_k) \leq \phi$.

As a result, for any $\phi \in [0, \frac{\pi}{2})$, $\psi(v_k, \tilde{v}_k) > \phi$ implies $\|v_k - \tilde{v}_k\| > \sin(\phi)\|\tilde{v}_k\|$.

Similarly, for any $\phi \in [0, \frac{\pi}{2})$, $\psi(v^*, \tilde{v}^*) > \phi$ implies $\|v^* - \tilde{v}^*\| \geq \sin(\phi)\|\tilde{v}^*\|$.

By definition of ϕ_k, $\phi_k \geq 0$. Hence, when $\phi_k < \pi$, $\frac{1}{2}\phi_k \in [0, \frac{\pi}{2})$. Plugging the above results into (47), we have

$$
\begin{aligned}
&\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) \\
&\leq \mathbb{P}\Big(\psi(v_k, \tilde{v}_k) > \frac{1}{2}\phi_k\Big) + \mathbb{P}\Big(\psi(\tilde{v}^*, v^*) > \frac{1}{2}\phi_k\Big) \\
&\leq \mathbb{P}\Big(\|v_k - \tilde{v}_k\| \geq \sin(\frac{1}{2}\phi_k)\|\tilde{v}_k\|\Big) + \mathbb{P}\Big(\|v^* - \tilde{v}^*\| \geq \sin(\frac{1}{2}\phi_k)\|\tilde{v}^*\|\Big) \\
&\leq \mathbb{P}\Big(\exists l \in [2K], |(v_k - \tilde{v}_k)_l| \geq \frac{1}{\sqrt{K}}\sin(\frac{1}{2}\phi_k)\|\tilde{v}_k\|\Big) \\
&\quad + \mathbb{P}\Big(\exists l \in [2K], |(v^* - \tilde{v}^*)_l| \geq \frac{1}{\sqrt{K}}\sin(\frac{1}{2}\phi_k)\|\tilde{v}^*\|\Big) \\
&\leq \sum_{l=1}^{2K}\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\sqrt{K}}\sin(\frac{1}{2}\phi_k)\|\tilde{v}_k\|\Big) \\
&\quad + \sum_{l=1}^{2K}\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\sqrt{K}}\sin(\frac{1}{2}\phi_k)\|\tilde{v}^*\|\Big)
\end{aligned}
\tag{48}
$$

Since when $\phi_k < \pi$, $\frac{1}{2}\phi_k \in [0, \frac{\pi}{2}]$, by Lemma 3, $\sin(\frac{1}{2}\phi_k) \geq \frac{2}{\pi}\frac{1}{2}\phi_k = \frac{1}{\pi}\phi_k$. Plugging back to (48), we have when $\phi_k < \pi$,

$$
\begin{aligned}
&\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) \\
&\leq \sum_{l=1}^{2K}\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) + \sum_{l=1}^{2K}\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big)
\end{aligned}
\tag{49}
$$

It remains to evaluate $\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big)$ and $\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big)$, which are illustrated in the following two lemmas.

Lemma 8. *Define* $C_6 = \frac{C^2}{16\sqrt{2}\pi^2 C_2 K^2(\sqrt{K}+\frac{1}{3})}$. *When* $\phi_k \geq 2\sqrt{2}\pi C_2 K^2 \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}$,

$$
\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) \leq 2\exp\left(-\frac{C_6}{C^2}\phi_k^2\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1\right)
$$

Lemma 9. *Define* $C_7 = \frac{C^2}{2\sqrt{2}\pi^2 C_2 K^2(\sqrt{K}+\frac{1}{3})}$. *Then,*

$$
\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big) \leq 2\exp\left(-\frac{C_7}{C^2}\phi_k^2\theta^*\|\theta\|_1\right)
$$

The proof of Lemma 8 and 9 are quite tedious. We would defer their proofs to the end of this section.

Choose

$$
\phi_k = \phi_k(C, \delta) = C\left(\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} + \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\right).
$$

Then leveraging on Lemma 8 and 9, we have when $C \geq 2\sqrt{2}\pi K^2$,

$$
\begin{aligned}
\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) &\leq 2\exp\left(-\frac{C_6 C^2\log(1/\delta)\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}{C^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}\right) \\
&\leq 2\exp\left(-C_6\log(1/\delta)\right) \\
&= 2\delta^{C_6}
\end{aligned}
\tag{50}
$$

$$\begin{aligned}\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big) &\leq 2\exp\left(-\frac{C_7 C^2\log(1/\delta)\theta^*\|\theta\|_1}{C^2\|\theta\|_1\cdot\min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}\right)\\ &\leq 2\exp\left(-C_7\log(1/\delta)\right)\\ &= 2\delta^{C_7}\end{aligned} \tag{51}$$

Plugging (50) and (51) back to (49), leveraging on the fact that $\delta \leq \frac{1}{2} < 1$, we obtain when $\phi_k < \pi$, and $C \geq 2\sqrt{2}\pi K^2$,

$$\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) \leq 4K\delta^{C_6} + 4K\delta^{C_7} \leq 8K\delta^{C_6}$$

Recall that when $\phi_k \geq \pi$,

$$\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) = 0$$

.

In all, we have that when $C \geq 2\sqrt{2}\pi K^2$,

$$\mathbb{P}\Big(|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| > \phi_k\Big) \leq 8K\delta^{C_6} \tag{52}$$

Substituting (52) into (46), we obtain that when $C \geq 2\sqrt{2}\pi K^2$,

$$\mathbb{P}\Big(\forall k \in [K], |\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq \phi_k\Big) \geq 1 - 8K^2\delta^{C_6} \tag{53}$$

Hence, it suffices to make $8K^2\delta^{C_6} \leq \delta$. Choose

$$C = \max\{2\sqrt{2}\pi K^2, \sqrt{16\sqrt{2}\pi^2 C_2 K^2(\sqrt{K} + \frac{1}{3})(1 + \frac{\log(8K^2)}{\log 2})}\}, \tag{54}$$

Then $C_6 - 1 \geq \frac{\log(8K^2)}{\log 2} \geq 1$. Since $\delta \leq \frac{1}{2}$,

$$\delta^{C_6-1} \leq \left(\frac{1}{2}\right)^{C_6-1} \leq \frac{1}{2^{\frac{\log(8K^2)}{\log 2}}} = \frac{1}{8K^2}$$

As a result, $8K^2\delta^{C_6} \leq \delta$.

Hence, choose C as in (54), then $C >)$, and for any $\delta \in (0, 1/2)$,

$$\mathbb{P}\Big(\forall k \in [K], |\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq \phi_k\Big) \geq 1 - \delta \tag{55}$$

To conclude, there exists constant $C > 0$, such that for any $\delta \in (0, 1/2)$, with probability $1 - \delta$, simultaneously for $1 \leq k \leq K$,

$$|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq C\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1\cdot\min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}}.$$

□

E.1 Proof of Lemma 8

Proof. When $l \in [K]$,

$$(\tilde{v}_k)_l = (f(\Omega^{(k)}; \hat{\Pi}_{\mathcal{U}}))_l = \Omega^{(k)}\mathbf{1}_{(l)} = \sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\mathcal{C}_l\cap\mathcal{L}}\Omega_{ij}$$

When $l \in \{K+1, ..., 2K\}$, define

$$\hat{\mathcal{C}}_l = \{i \in \mathcal{U} : \hat{\pi}_i = e_{l-K}\}$$

, then

$$(\tilde{v}_k)_l = (f(\Omega^{(k)}; \hat{\Pi}_{\mathcal{U}}))_l = \Omega^{(k)}\left(\hat{\Pi}_{\mathcal{U}}\right)_l = \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \hat{\mathcal{C}}_l} \Omega_{ij}$$

Hence

$$\begin{aligned}
\|\tilde{v}_k\| &= \sqrt{\sum_{l \in [2K]} (\tilde{v}_k)_l^2} \\
\text{(Cauchy-Schwartz)} &\geq \sqrt{\frac{1}{2K}(\sum_{l \in [2K]} (\tilde{v}_k)_l)^2} \\
&= \frac{1}{\sqrt{2K}} |\sum_{l \in [2K]} (\tilde{v}_k)_l| \\
&= \frac{1}{\sqrt{2K}} |\sum_{l=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} \Omega_{ij} + \sum_{l=K+1}^{2K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \hat{\mathcal{C}}_l} \Omega_{ij}| \\
&= \frac{1}{\sqrt{2K}} |\sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{L}} \Omega_{ij} + \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \hat{\mathcal{U}}} \Omega_{ij}| \\
&= \frac{1}{\sqrt{2K}} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in [n]} \Omega_{ij} \\
&\geq \frac{1}{\sqrt{2K}} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_k} \Omega_{ij} \\
&= \frac{1}{\sqrt{2K}} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_k} \theta_i \theta_j P_{kk} \\
\text{(Identifiability condition)} &= \frac{1}{\sqrt{2K}} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_k} \theta_i \theta_j \\
&= \frac{1}{\sqrt{2K}} \|\theta_{\mathcal{L}}^{(k)}\|_1 \|\theta^{(k)}\|_1 \\
&\geq \frac{1}{\sqrt{2K}} \|\theta_{\mathcal{L}}^{(k)}\|_1 \min_{l \in [K]} \|\theta^{(l)}\|_1 \\
\text{(Condition (10))} &\geq \frac{1}{C_2\sqrt{2K}} \|\theta_{\mathcal{L}}^{(k)}\|_1 \max_{l \in [K]} \|\theta^{(l)}\|_1 \\
&\geq \frac{1}{C_2 K\sqrt{2K}} \|\theta_{\mathcal{L}}^{(k)}\|_1 \|\theta\|_1 \qquad (56)
\end{aligned}$$

When $l \in [K]$,

$$(v_k)_l = (f(A^{(k)}; \hat{\Pi}_{\mathcal{U}}))_l = A^{(k)}\mathbf{1}_{(l)} = \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} A_{ij}$$

Recall that,

$$(\tilde{v}_k)_l = (f(\Omega^{(k)}; \hat{\Pi}_{\mathcal{U}}))_l = \Omega^{(k)}\mathbf{1}_{(l)} = \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} \Omega_{ij}$$

So

$$|(v_k - \tilde{v}_k)_l| = \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} (A_{ij} - \Omega_{ij})$$

When $l \in [K]\backslash\{k\}$, since $\hat{\Pi}_{\mathcal{U}}$ only depends on $A_{\mathcal{U}\mathcal{U}}$, it is independent of $A_{\mathcal{L}\mathcal{L}}$. Hence, given $\hat{\Pi}_{\mathcal{U}}$, $\{A_{ij} - \Omega_{ij} : i \in \mathcal{C}_k \cap \mathcal{L}, j \in \mathcal{C}_l \cap \mathcal{L}\}$ are a collection of $|\mathcal{C}_k \cap \mathcal{L}||\mathcal{C}_l \cap \mathcal{L}|$ independent random variables. Furthermore, given $\hat{\Pi}_{\mathcal{U}}$, for any $i \in \mathcal{C}_k \cap \mathcal{L}, j \in \mathcal{C}_l \cap \mathcal{L}$,

$$\mathbb{E}\Big[A_{ij} - \Omega_{ij}|\hat{\Pi}_{\mathcal{U}}\Big] = \mathbb{E}\Big[A_{ij}|\hat{\Pi}_{\mathcal{U}}\Big] - \Omega_{ij} = \Omega_{ij} - \Omega_{ij} = 0$$

Also,

$$-1 \leq -\Omega_{ij} \leq A_{ij} - \Omega_{ij} \leq A_{ij} \leq 1$$

So $|A_{ij} - \Omega_{ij}| \leq 1$. Additionally,

$$var\Big(A_{ij} - \Omega_{ij}\Big) = var\Big(A_{ij}|\hat{\Pi}_{\mathcal{U}}\Big) = \Omega_{ij}(1 - \Omega_{ij}) \leq \Omega_{ij}$$

Therefore, denote $n_{kl} = |\mathcal{C}_k \cap \mathcal{L}||\mathcal{C}_l \cap \mathcal{L}|$, by Lemma 7,

$$\begin{aligned}
&\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) \\
&= \mathbb{E}\left[\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&= \mathbb{E}\left[\mathbb{P}\Big(\frac{1}{n_{kl}}|\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\mathcal{C}_l\cap\mathcal{L}}(A_{ij} - \Omega_{ij})| \geq \frac{1}{\pi\sqrt{K}n_{kl}}\phi_k\|\tilde{v}_k\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&\leq 2\mathbb{E}\exp\left(-\frac{\frac{1}{2}n_{kl}\Big(\frac{1}{\pi\sqrt{K}n_{kl}}\phi_k\|\tilde{v}_k\|\Big)^2}{\frac{1}{n_{kl}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\mathcal{C}_l\cap\mathcal{L}}\Omega_{ij} + \frac{1}{3}\frac{1}{\pi\sqrt{K}n_{kl}}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}}\frac{\|\tilde{v}_k\|^2}{\pi\sqrt{K}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\mathcal{C}_l\cap\mathcal{L}}\Omega_{ij} + \frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}}\frac{\|\tilde{v}_k\|^2}{\pi\sqrt{K}|(\tilde{v}_k)_l| + \frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{2\pi\sqrt{K}}\frac{1}{\pi\sqrt{K}\frac{|(\tilde{v}_k)_l|}{\|\tilde{v}_k\|} + \frac{1}{3}\phi_k}\right) \\
&\leq 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{2\pi\sqrt{K}(\pi\sqrt{K} + \frac{\pi}{3})}\right) \qquad (57)
\end{aligned}$$

When $l = k$, $\{A_{ij} - \Omega_{ij} : i, j \in \mathcal{C}_k \cap \mathcal{L}, i < j\}$ are a collection of $\frac{1}{2}|\mathcal{C}_k \cap \mathcal{L}|(|\mathcal{C}_k \cap \mathcal{L}| - 1)$ independent random variables. Furthermore, for any $i, j \in \mathcal{C}_k \cap \mathcal{L}, i < j$,

$$\mathbb{E}(A_{ij} - \Omega_{ij}) = \mathbb{E}A_{ij} - \Omega_{ij} = \Omega_{ij} - \Omega_{ij} = 0$$

Also,

$$-1 \leq -\Omega_{ij} \leq A_{ij} - \Omega_{ij} \leq A_{ij} \leq 1$$

So $|A_{ij} - \Omega_{ij}| \leq 1$.

Additionally,

$$var(A_{ij} - \Omega_{ij}) = var(A_{ij}) = \Omega_{ij}(1 - \Omega_{ij}) \leq \Omega_{ij}$$

Denote $n_{kk} = \frac{1}{2}|\mathcal{C}_k \cap \mathcal{L}|(|\mathcal{C}_k \cap \mathcal{L}| - 1)$, we have

$$
\begin{aligned}
&\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) \\
&= \mathbb{P}\Big(\frac{1}{n_{kk}}|\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\mathcal{C}_k\cap\mathcal{L}}(A_{ij} - \Omega_{ij})| \geq \frac{1}{\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\|\Big) \\
&= \mathbb{P}\Big(\frac{1}{n_{kk}}|2\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}(A_{ij} - \Omega_{ij}) + \sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ii}| \geq \frac{1}{\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\|\Big) \\
&\leq \mathbb{P}\Big(\frac{1}{n_{kk}}|2\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}(A_{ij} - \Omega_{ij})| \geq \frac{1}{\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\| - \frac{1}{n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ii}\Big) \\
&= \mathbb{P}\Big(\frac{1}{n_{kk}}|\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}(A_{ij} - \Omega_{ij})| \geq \frac{1}{2\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\| - \frac{1}{2n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ii}\Big) \\
&= \mathbb{E}\left[\mathbb{P}\Big(\frac{1}{n_{kk}}|\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}(A_{ij} - \Omega_{ij})| \geq \frac{1}{2\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\| - \frac{1}{2n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ii}\Big)|\hat{\Pi}_{\mathcal{U}}\right] \qquad (58)
\end{aligned}
$$

Notice that

$$
\phi_k \geq C\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}.
$$

Since $\phi_k \geq 2\sqrt{2}\pi C_2 K^2 \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\|\tilde{v}_k\|$,

$$
\begin{aligned}
\frac{1}{2\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\| &\geq \frac{2\sqrt{2}\pi K^2}{2\pi\sqrt{K}n_{kk}}\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\|\tilde{v}_k\| \\
\text{((By (56))} &\geq \frac{C_2K\sqrt{2K}}{n_{kk}}\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\frac{1}{C_2K\sqrt{2K}}\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1 \\
&= 2\|\frac{1}{2n_{kk}}\theta_{\mathcal{L}}^{(k)}\|^2 \\
&= 2\frac{1}{2n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\theta_i^2 \\
\text{(Identifiability condition)} &= \frac{C}{\sqrt{2}\pi K^2}\frac{1}{2n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\theta_i^2 P_{kk} \\
&= 2\frac{1}{2n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ii} \qquad (59)
\end{aligned}
$$

Therefore, by Lemma 7,

$$
\begin{aligned}
&\mathbb{P}\Big(|(v_k-\tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) \\
&= \mathbb{E}\left[\mathbb{P}\Big(\frac{1}{n_{kk}}|\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}(A_{ij}-\Omega_{ij})| \geq \frac{1}{2\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\| - \frac{1}{2}\frac{1}{2\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&\leq 2\mathbb{E}\exp\left(-\frac{\frac{1}{2}n_{kk}\Big(\frac{1}{4\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\|\Big)^2}{\frac{1}{n_{kk}}\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ij}+\frac{1}{3}\Big(\frac{1}{2\pi\sqrt{K}n_{kk}}\phi_k\|\tilde{v}_k\| - \frac{1}{2n_{kk}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ii}\Big)}\right) \\
&\leq 2\mathbb{E}\exp\left(-\frac{1}{16\pi\sqrt{K}}\frac{\Big(\phi_k\|\tilde{v}_k\|\Big)^2}{\pi\sqrt{K}\cdot 2\sum_{i<j\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ij}+\frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&\leq 2\mathbb{E}\exp\left(-\frac{1}{16\pi\sqrt{K}}\frac{\Big(\phi_k\|\tilde{v}_k\|\Big)^2}{\pi\sqrt{K}\sum_{i,j\in\mathcal{C}_k\cap\mathcal{L}}\Omega_{ij}+\frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{16\pi\sqrt{K}}\frac{\|\tilde{v}_k\|^2}{\pi\sqrt{K}|(\tilde{v}_k)_k|+\frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{16\pi\sqrt{K}}\frac{1}{\pi\sqrt{K}\frac{|(\tilde{v}_k)_k|}{\|\tilde{v}_k\|}+\frac{1}{3}\phi_k}\right) \\
&\leq 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{16\pi\sqrt{K}(\pi\sqrt{K}+\frac{\pi}{3})}\right) \qquad (60)
\end{aligned}
$$

When $l \in \{K+1, ..., 2K\}$, recall

$$\hat{\mathcal{C}}_l = \{i \in \mathcal{U} : \hat{\pi}_i = e_{l-K}\}$$

So

$$(v_k)_l = (f(A^{(k)}; \hat{\Pi}_{\mathcal{U}}))_l = A^{(k)}\Big(\hat{\Pi}_{\mathcal{U}}\Big)_l = \sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\hat{\mathcal{C}}_l}A_{ij}$$

Recall that

$$(\tilde{v}_k)_l = (f(\Omega^{(k)}; \hat{\Pi}_{\mathcal{U}}))_l = \Omega^{(k)}\Big(\hat{\Pi}_{\mathcal{U}}\Big)_l = \sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\hat{\mathcal{C}}_l}\Omega_{ij}$$

So

$$|(v_k-\tilde{v}_k)_l| = \sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\hat{\mathcal{C}}_l}(A_{ij}-\Omega_{ij})$$

Since $\hat{\Pi}_{\mathcal{U}}$ only depends on $A_{\mathcal{U}\mathcal{U}}$, it is independent of $A_{\mathcal{L}\mathcal{U}}$. Hence, given $\hat{\Pi}_{\mathcal{U}}$, $\{A_{ij}-\Omega_{ij} : i \in \mathcal{C}_k\cap\mathcal{L}, j \in \hat{\mathcal{C}}_l\}$ are a collection of $|\mathcal{C}_k\cap\mathcal{L}||\hat{\mathcal{C}}_l|$ independent random variables. Furthermore, given $\hat{\Pi}_{\mathcal{U}}$, for any $i \in \mathcal{C}_k\cap\mathcal{L}, j \in \hat{\mathcal{C}}_l$,

$$\mathbb{E}\Big[A_{ij}-\Omega_{ij}|\hat{\Pi}_{\mathcal{U}}\Big] = \mathbb{E}\Big[A_{ij}|\hat{\Pi}_{\mathcal{U}}\Big] - \Omega_{ij} = \Omega_{ij}-\Omega_{ij} = 0$$

Also,

$$-1 \leq -\Omega_{ij} \leq A_{ij}-\Omega_{ij} \leq A_{ij} \leq 1$$

So $|A_{ij}-\Omega_{ij}| \leq 1$. Additionally,

$$var\Big(A_{ij}-\Omega_{ij}\Big) = var\Big(A_{ij}|\hat{\Pi}_{\mathcal{U}}\Big) = \Omega_{ij}(1-\Omega_{ij}) \leq \Omega_{ij}$$

Therefore, denote $\hat{n}_{kl} = |\mathcal{C}_k \cap \mathcal{L}||\hat{\mathcal{C}}_l|$, by Lemma 7,

$$\begin{aligned}
&\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) \\
&= \mathbb{E}\left[\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&= \mathbb{E}\left[\mathbb{P}\Big(\frac{1}{\hat{n}_{kl}}|\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\hat{\mathcal{C}}_l}(A_{ij} - \Omega_{ij})| \geq \frac{1}{\pi\sqrt{K}\hat{n}_{kl}}\phi_k\|\tilde{v}_k\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&\leq 2\mathbb{E}\exp\left(-\frac{\frac{1}{2}\hat{n}_{kl}\Big(\frac{1}{\pi\sqrt{K}\hat{n}_{kl}}\phi_k\|\tilde{v}_k\|\Big)^2}{\frac{1}{\hat{n}_{kl}}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\hat{\mathcal{C}}_l}\Omega_{ij} + \frac{1}{3}\frac{1}{\pi\sqrt{K}\hat{n}_{kl}}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}}\frac{\|\tilde{v}_k\|^2}{\pi\sqrt{K}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}}\sum_{j\in\hat{\mathcal{C}}_l}\Omega_{ij} + \frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}}\frac{\|\tilde{v}_k\|^2}{\pi\sqrt{K}|(\tilde{v}_k)_l| + \frac{1}{3}\phi_k\|\tilde{v}_k\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{2\pi\sqrt{K}}\frac{1}{\pi\sqrt{K}\frac{|(\tilde{v}_k)_l|}{\|\tilde{v}_k\|} + \frac{1}{3}\phi_k}\right) \\
&\leq 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{2\pi\sqrt{K}(\pi\sqrt{K} + \frac{\pi}{3})}\right) \qquad (61)
\end{aligned}$$

In all, for any $l \in [2K]$,

$$\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) \leq 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}_k\|}{16\pi\sqrt{K}(\pi\sqrt{K} + \frac{\pi}{3})}\right) \qquad (62)$$

Plugging (56) into (62), we obtain

$$\begin{aligned}
\mathbb{P}\Big(|(v_k - \tilde{v}_k)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}_k\|\Big) &\leq 2\exp\left(-\frac{\phi_k^2\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}{16\sqrt{2}\pi^2 C_2 K^2(\sqrt{K} + \frac{1}{3})}\right) \\
&= 2\exp\left(-\frac{C_6}{C^2}\phi_k^2\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1\right) \qquad (63)
\end{aligned}$$

That concludes the proof.

□

E.2 Proof of Lemma 9

The proof of Lemma 9 is nearly the same as Lemma 8. For the completeness of our paper, we will present a proof of Lemma 9 as follows.

Proof. When $l \in [K]$,

$$(v^*)_l = (f(X; \hat{\Pi}_{\mathcal{U}}))_l = X\mathbf{1}_{(l)} = \sum_{j\in\mathcal{C}_l\cap\mathcal{L}} X_j$$

Similarly,

$$(\tilde{v}^*)_l = (f(\mathbb{E}[X]; \hat{\Pi}_{\mathcal{U}}))_l = \mathbb{E}[X]\mathbf{1}_{(l)} = \sum_{j\in\mathcal{C}_l\cap\mathcal{L}} \mathbb{E}[X_j]$$

So

$$|(v^* - \tilde{v}^*)_l| = \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} (X_j - \mathbb{E}[X_j])$$

When $l \in [K]$, since $\hat{\Pi}_{\mathcal{U}}$ only depends on $A_{\mathcal{U}\mathcal{U}}$, it is independent of X. Hence, given $\hat{\Pi}_{\mathcal{U}}$, $\{X_j - \mathbb{E}[X_j] : j \in \mathcal{C}_l \cap \mathcal{L}\}$ are a collection of $|\mathcal{C}_l \cap \mathcal{L}|$ independent random variables. Furthermore, given $\hat{\Pi}_{\mathcal{U}}$, for any $j \in \mathcal{C}_l \cap \mathcal{L}$,

$$\mathbb{E}\Big[X_j - \mathbb{E}[X_j] | \hat{\Pi}_{\mathcal{U}}\Big] = \mathbb{E}\Big[X_j | \hat{\Pi}_{\mathcal{U}}\Big] - \mathbb{E}[X_j] = \mathbb{E}[X_j] - \mathbb{E}[X_j] = 0$$

Also,

$$-1 \leq -\mathbb{E}[X_j] \leq X_j - \mathbb{E}[X_j] \leq X_j \leq 1$$

So $|X_j - \mathbb{E}[X_j]| \leq 1$. Additionally,

$$var\Big(X_j - \mathbb{E}[X_j]\Big) = var\Big(X_j | \hat{\Pi}_{\mathcal{U}}\Big) = \mathbb{E}[X_j](1 - \mathbb{E}[X_j]) \leq \mathbb{E}[X_j]$$

Therefore, denote $n_l = |\mathcal{C}_l \cap \mathcal{L}|$, by Lemma 7,

$$\begin{aligned}
&\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}} \phi_k \|\tilde{v}^*\|\Big) \\
&= \mathbb{E}\left[\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}} \phi_k \|\tilde{v}^*\|\Big) | \hat{\Pi}_{\mathcal{U}}\right] \\
&= \mathbb{E}\left[\mathbb{P}\Big(\frac{1}{n_l} |\sum_{j \in \mathcal{C}_l \cap \mathcal{L}} (X_j - \mathbb{E}[X_j])| \geq \frac{1}{\pi\sqrt{K} n_l} \phi_k \|\tilde{v}^*\|\Big) | \hat{\Pi}_{\mathcal{U}}\right] \\
&\leq 2\mathbb{E} \exp\left(-\frac{\frac{1}{2} n_l \Big(\frac{1}{\pi\sqrt{K} n_l} \phi_k \|\tilde{v}^*\|\Big)^2}{\frac{1}{n_l} \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} \mathbb{E}[X_j] + \frac{1}{3} \frac{1}{\pi\sqrt{K} n_l} \phi_k \|\tilde{v}^*\|}\right) \\
&= 2\mathbb{E} \exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}} \frac{\|\tilde{v}^*\|^2}{\pi\sqrt{K} \sum_{j \in \mathcal{C}_l \cap \mathcal{L}} \mathbb{E}[X_j] + \frac{1}{3} \phi_k \|\tilde{v}^*\|}\right) \\
&= 2\mathbb{E} \exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}} \frac{\|\tilde{v}^*\|^2}{\pi\sqrt{K} |(\tilde{v}^*)_l| + \frac{1}{3} \phi_k \|\tilde{v}^*\|}\right) \\
&= 2\mathbb{E} \exp\left(-\frac{\phi_k^2 \|\tilde{v}^*\|}{2\pi\sqrt{K}} \frac{1}{\pi\sqrt{K} \frac{|(\tilde{v}^*)_l|}{\|\tilde{v}^*\|} + \frac{1}{3} \phi_k}\right) \\
&\leq 2\mathbb{E} \exp\left(-\frac{\phi_k^2 \|\tilde{v}^*\|}{2\pi\sqrt{K}(\pi\sqrt{K} + \frac{\pi}{3})}\right) \qquad (64)
\end{aligned}$$

When $l \in \{K+1, ..., 2K\}$, define

$$\hat{\mathcal{C}}_l = \{i \in \mathcal{U} : \hat{\pi}_i = e_{l-K}\}$$

Then

$$(v^*)_l = (f(X; \hat{\Pi}_{\mathcal{U}}))_l = X \mathbf{1}_{(l)} = \sum_{j \in \hat{\mathcal{C}}_l} X_j$$

Similarly,

$$(\tilde{v}^*)_l = (f(\mathbb{E}[X]; \hat{\Pi}_{\mathcal{U}}))_l = \mathbb{E}[X] \mathbf{1}_{(l)} = \sum_{j \in \hat{\mathcal{C}}_l} \mathbb{E}[X_j]$$

So

$$|(v^* - \tilde{v}^*)_l| = \sum_{j \in \hat{\mathcal{C}}_l} (X_j - \mathbb{E}[X_j])$$

When $l \in [K]$, since $\hat{\Pi}_{\mathcal{U}}$ only depends on $A_{\mathcal{U}\mathcal{U}}$, it is independent of X. Hence, given $\hat{\Pi}_{\mathcal{U}}$, $\{X_j - \mathbb{E}[X_j] : j \in \hat{\mathcal{C}}_l\}$ are a collection of $|\mathcal{C}_l \cap \mathcal{L}|$ independent random variables. Furthermore, given $\hat{\Pi}_{\mathcal{U}}$, for any $j \in \hat{\mathcal{C}}_l$,

$$\mathbb{E}\Big[X_j - \mathbb{E}[X_j]|\hat{\Pi}_{\mathcal{U}}\Big] = \mathbb{E}\Big[X_j|\hat{\Pi}_{\mathcal{U}}\Big] - \mathbb{E}[X_j] = \mathbb{E}[X_j] - \mathbb{E}[X_j] = 0$$

Also,

$$-1 \leq -\mathbb{E}[X_j] \leq X_j - \mathbb{E}[X_j] \leq X_j \leq 1$$

So $|X_j - \mathbb{E}[X_j]| \leq 1$. Additionally,

$$var\Big(X_j - \mathbb{E}[X_j]\Big) = var\Big(X_j|\hat{\Pi}_{\mathcal{U}}\Big) = \mathbb{E}[X_j](1 - \mathbb{E}[X_j]) \leq \mathbb{E}[X_j]$$

Therefore, denote $\hat{n}_l = |\hat{\mathcal{C}}_l|$, by Lemma 7,

$$\begin{aligned}
&\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big) \\
&= \mathbb{E}\left[\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&= \mathbb{E}\left[\mathbb{P}\Big(\frac{1}{\hat{n}_l}|\sum_{j\in\hat{\mathcal{C}}_l}(X_j - \mathbb{E}[X_j])| \geq \frac{1}{\pi\sqrt{K}\hat{n}_l}\phi_k\|\tilde{v}^*\|\Big)|\hat{\Pi}_{\mathcal{U}}\right] \\
&\leq 2\mathbb{E}\exp\left(-\frac{\frac{1}{2}\hat{n}_l\Big(\frac{1}{\pi\sqrt{K}\hat{n}_l}\phi_k\|\tilde{v}^*\|\Big)^2}{\frac{1}{\hat{n}_l}\sum_{j\in\hat{\mathcal{C}}_l}\mathbb{E}[X_j] + \frac{1}{3}\frac{1}{\pi\sqrt{K}\hat{n}_l}\phi_k\|\tilde{v}^*\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}}\frac{\|\tilde{v}^*\|^2}{\pi\sqrt{K}\sum_{j\in\hat{\mathcal{C}}_l}\mathbb{E}[X_j] + \frac{1}{3}\phi_k\|\tilde{v}^*\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2}{2\pi\sqrt{K}}\frac{\|\tilde{v}^*\|^2}{\pi\sqrt{K}|(\tilde{v}^*)_l| + \frac{1}{3}\phi_k\|\tilde{v}^*\|}\right) \\
&= 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}^*\|}{2\pi\sqrt{K}}\frac{1}{\pi\sqrt{K}\frac{|(\tilde{v}^*)_l|}{\|\tilde{v}^*\|} + \frac{1}{3}\phi_k}\right) \\
&\leq 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}^*\|}{2\pi\sqrt{K}(\pi\sqrt{K} + \frac{\pi}{3})}\right) \qquad (65)
\end{aligned}$$

In all, for any $l \in [2K]$,

$$\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big) \leq 2\mathbb{E}\exp\left(-\frac{\phi_k^2\|\tilde{v}^*\|}{2\pi\sqrt{K}(\pi\sqrt{K} + \frac{\pi}{3})}\right) \qquad (66)$$

Notice that

$$
\begin{aligned}
\|\tilde{v}^*\| &= \sqrt{\sum_{l\in[2K]} (\tilde{v}^*)_l^2} \\
\text{(Cauchy-Schwartz)} &\geq \sqrt{\frac{1}{2K}(\sum_{l\in[2K]} (\tilde{v}^*)_l)^2} \\
&= \frac{1}{\sqrt{2K}}|\sum_{l\in[2K]} (\tilde{v}^*)_l| \\
&= \frac{1}{\sqrt{2K}}|\sum_{l=1}^{K}\sum_{j\in\hat{\mathcal{C}}_l} \mathbb{E}[X_j] + \sum_{l=K+1}^{2K}\sum_{j\in\hat{\mathcal{C}}_l} \mathbb{E}[X_j]| \\
&= \frac{1}{\sqrt{2K}}|\sum_{j\in\mathcal{L}} \mathbb{E}[X_j] + \sum_{j\in\hat{\mathcal{U}}} \mathbb{E}[X_j]| \\
&= \frac{1}{\sqrt{2K}}\sum_{j\in[n]} \mathbb{E}[X_j] \\
&\geq \frac{1}{\sqrt{2K}}\sum_{j\in\mathcal{C}_{k^*}} \mathbb{E}[X_j] \\
&= \frac{1}{\sqrt{2K}}\sum_{j\in\mathcal{C}_k} \theta^*\theta_j P_{k^*k^*} \\
\text{(Identifiability condition)} &= \frac{1}{\sqrt{2K}}\sum_{j\in\mathcal{C}_k} \theta^*\theta_j \\
&= \frac{1}{\sqrt{2K}}\theta^*\|\theta^{(k)}\|_1 \\
&\geq \frac{1}{\sqrt{2K}}\theta^* \min_{l\in[K]} \|\theta^{(l)}\|_1 \\
\text{(Condition (10))} &\geq \frac{1}{C_2\sqrt{2K}}\theta^* \max_{l\in[K]} \|\theta^{(l)}\|_1 \\
&\geq \frac{1}{C_2K\sqrt{2K}}\theta^*\|\theta\|_1 \qquad (67)
\end{aligned}
$$

Plugging (67) into (66), we obtain

$$
\begin{aligned}
\mathbb{P}\Big(|(v^* - \tilde{v}^*)_l| \geq \frac{1}{\pi\sqrt{K}}\phi_k\|\tilde{v}^*\|\Big) &\leq 2\exp\left(-\frac{\phi_k^2\theta^*\|\theta\|_1}{2\sqrt{2}\pi^2C_2K^2(\sqrt{K}+\frac{1}{3})}\right) \\
&= 2\exp\left(-\frac{C_7}{C^2}\phi_k^2\theta^*\|\theta\|_1\right) \qquad (68)
\end{aligned}
$$

That concludes the proof.

□

F PROOF OF COROLLARY 1, 2

F.1 PROOF OF COROLLARY 1

Corollary 1. *Consider the DCBM model where (9)-(10) hold. Suppose for some constants $b_0 \in (0,1)$ and $\epsilon \in (0, 1/2)$, $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct with probability $1-\epsilon$. When b_0 is properly small, there exist constants $C_0 > 0$ and $\bar{C} > 0$, which do not depend on (b_0, ϵ), such that $\mathbb{P}(\hat{y} \neq k^*) \leq \epsilon + \bar{C}\sum_{k=1}^{K}\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}\Big)$.*

Proof. Let B_0 be the event that $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct. Then,

$$\begin{aligned}
&\mathbb{P}(\hat{y} \neq k^*) \\
&= \mathbb{P}(\hat{y} \neq k^*, B_0^C) + \mathbb{P}\Big(\hat{y} \neq k^*, B_0\Big) \\
&\leq \mathbb{P}(B_0^C) + \mathbb{P}\Big(\Big\{\exists k \neq k^*, \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) \leq \hat{\psi}_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big\}, B_0\Big) \\
&\leq \epsilon + \mathbb{P}\Big(\Big\{\exists k \neq k^*, \Big(\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big) \\
&\qquad + \Big(\hat{\psi}_{k^*}(\hat{\Pi}_{\mathcal{U}}) - \psi_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big) \geq \Big(\psi_k(\hat{\Pi}_{\mathcal{U}}) - \psi_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big)\Big\}, B_0\Big) \\
&\leq \epsilon + \mathbb{P}\Big(\Big\{\exists k \neq k^*, \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| \\
&\qquad + \Big|\hat{\psi}_{k^*}(\hat{\Pi}_{\mathcal{U}}) - \psi_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big| \geq \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \psi_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big|\Big\}, B_0\Big)
\end{aligned} \tag{69}$$

By Theorem 1, when b_0 is properly small, B_0 implies that there exists a constant $c_0 > 0$, which does not depend on b_0, such that $\psi_{k^*}(\hat{\Pi}_{\mathcal{U}}) = 0$ and $\min_{k \neq k^*}\{\psi_k(\hat{\Pi}_{\mathcal{U}})\} \geq c_0\beta_n$.

Substituting this result into (69), we have

$$\begin{aligned}
&\mathbb{P}(\hat{y} \neq k^*) \\
&\leq \epsilon + \mathbb{P}\Big(\Big\{\exists k \neq k^*, \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| + \Big|\hat{\psi}_{k^*}(\hat{\Pi}_{\mathcal{U}}) - \psi_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big| \geq c_0\beta_n\Big\}, B_0\Big) \\
&\leq \epsilon + \mathbb{P}\Big(\exists k \neq k^*, \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| + \Big|\hat{\psi}_{k^*}(\hat{\Pi}_{\mathcal{U}}) - \psi_{k^*}(\hat{\Pi}_{\mathcal{U}})\Big| \geq c_0\beta_n\Big) \\
&\leq \epsilon + \mathbb{P}\Big(\exists k \in [K], \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| \geq \frac{1}{2}c_0\beta_n\Big) \\
&\leq \epsilon + \mathbb{P}\Big(\exists k \in [K], \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| > \frac{1}{3}c_0\beta_n\Big)
\end{aligned} \tag{70}$$

According to Theorem 2, there exists a constant $C > 0$, such that for any $\delta \in (0, 1/2)$, with probability $1 - \delta$, simultaneously for $1 \leq k \leq K$,

$$|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq C\left(\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} + \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\right).$$

Take $C_0 = \frac{c_0^2}{36C^2}$,

$$\delta = \exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\Big\{\theta^*, \min_{k\in[K]} \|\theta_{\mathcal{L}}^{(k)}\|_1\Big\}\Big)$$

Then

$$\begin{aligned}
C\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} &= C\sqrt{\frac{\log\Big(1/\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\Big\{\theta^*, \min_{k\in[K]} \|\theta_{\mathcal{L}}^{(k)}\|_1\Big\}\Big)\Big)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} \\
&= C\sqrt{\frac{C_0\beta_n^2\|\theta\|_1 \cdot \min\Big\{\theta^*, \min_{k\in[K]} \|\theta_{\mathcal{L}}^{(k)}\|_1\Big\}}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} \\
&\leq \frac{1}{6}c_0\beta_n
\end{aligned} \tag{71}$$

On the other hand, take c_3 in condition (10) properly small such that $c_3 \leq \frac{c_0}{6C}$, then according to condition (10),

$$C\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1} \leq C \cdot c_3\beta_n \leq \frac{1}{6}c_0\beta_n \tag{72}$$

Combining (71) and (72), we have

$$C\left(\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}}} + \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1 \|\theta\|_1}\right) \leq \frac{1}{3}c_0\beta_n$$

Therefore, when $\delta < \frac{1}{2}$, by Theorem 2, with probability $1-\delta$, simultaneously for $1 \leq k \leq K$,

$$|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq \frac{1}{3}c_0\beta_n.$$

As a result, when $\delta < \frac{1}{2}$,

$$\mathbb{P}\Big(\exists k \in [K], \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| > \frac{1}{3}c_0\beta_n\Big) \leq \delta$$

When $\delta \geq \frac{1}{2}$,

$$\mathbb{P}\Big(\exists k \in [K], \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| > \frac{1}{3}c_0\beta_n\Big) \leq 1 \leq 2\delta$$

Hence in total, we have

$$\mathbb{P}\Big(\exists k \in [K], \Big|\psi_k(\hat{\Pi}_{\mathcal{U}}) - \hat{\psi}_k(\hat{\Pi}_{\mathcal{U}})\Big| > \frac{1}{3}c_0\beta_n\Big) \leq 2\delta \tag{73}$$

Plugging (73) into (70), we obtain

$$\begin{aligned}\mathbb{P}(\hat{y} \neq k^*) &\leq \epsilon + 2\delta \\ &= \epsilon + 2\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\Big\{\theta^*, \min_{k\in[K]} \|\theta_{\mathcal{L}}^{(k)}\|_1\Big\}\Big) \\ &\leq \epsilon + 2\sum_{k=1}^{K}\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}\Big) \end{aligned} \tag{74}$$

Choose $\bar{C} = 2$, we obtain

$$\mathbb{P}(\hat{y} \neq k^*) \leq \epsilon + \bar{C}\sum_{k=1}^{K}\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}\Big)$$

To conclude, when b_0 is properly small, there exist constants $C_0 = \frac{c_0^2}{36C^2} > 0$ and $\bar{C} = 2 > 0$, which do not depend on (b_0, ϵ), such that $\mathbb{P}(\hat{y} \neq k^*) \leq \epsilon + \bar{C}\sum_{k=1}^{K}\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}\Big)$.

□

F.2 Proof of Corollary 2

Corollary 2. *Consider the DCBM model where (9)-(10) hold. We apply SCORE+ to obtain $\hat{\Pi}_{\mathcal{U}}$ and plug it into AngleMin+. As $n \to \infty$, suppose for some constant $q_0 > 0$, $\min_{i\in\mathcal{U}} \theta_i \geq q_0 \max_{i\in\mathcal{U}} \theta_i$, $\beta_n\|\theta_{\mathcal{U}}\| \geq q_0\sqrt{\log(n)}$, $\beta_n^2\|\theta\|_1\theta^* \to \infty$, and $\beta_n^2\|\theta\|_1 \min_k\{\|\theta_{\mathcal{L}}^{(k)}\|_1\} \to \infty$. Then, $\mathbb{P}(\hat{y} \neq k^*) \to 0$, so the AngleMin+ estimate is consistent.*

Proof. By Corollary 2, let ϵ be the probability that $\hat{\Pi}_{\mathcal{U}}$ obtained through SCORE+ is not b_0-correct, then

$$\mathbb{P}(\hat{y} \neq k^*) \leq \epsilon + \bar{C}\sum_{k=1}^{K}\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}\Big)$$

Since $\beta_n^2\|\theta\|_1\theta^* \to \infty$, and $\beta_n^2\|\theta\|_1 \min_k\{\|\theta_{\mathcal{L}}^{(k)}\|_1\} \to \infty$,

$$\bar{C}\sum_{k=1}^{K}\exp\Big(-C_0\beta_n^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta_{\mathcal{L}}^{(k)}\|_1\}\Big) \to 0$$

By Theorem 2.2 in Jin et al. (2021), when q_0 is sufficiently large, $\min_{i\in\mathcal{U}}\theta_i \geq q_0 \max_{i\in\mathcal{U}}\theta_i$ and $\beta_n\|\theta_{\mathcal{U}}\| \geq q_0\sqrt{\log(n)}$ imply that $\epsilon \to 0$.

Hence, in all, we have $\mathbb{P}(\hat{y} \neq k^*) \to 0$, so the AngleMin+ estimate is consistent. □

G PROOF OF LEMMA 2

As mentioned in section A, in the main paper, for the smoothness and comprehensibility of the text, we do not present the most general form of Lemma 2. Here, we present both the original version, Lemma 2, and the generalized version, Lemma 2' below, where we relax the assumption that $\frac{\|\theta^{(1)}_{\mathcal{L}}\|_1}{\|\theta^{(2)}_{\mathcal{L}}\|_1} = \frac{\|\theta^{(1)}_{\mathcal{U}}\|_1}{\|\theta^{(2)}_{\mathcal{U}}\|_1} = 1$ to the much weaker assumption: the first part of condition (10) in the main text, which only assumes that $\|\theta^{(1)}\|_1$ and $\|\theta^{(2)}\|_1$ are of the same order.

Lemma 2. *Consider a DCBM with $K = 2$ and $P = (1-b)I_2 + b\mathbf{1}_2\mathbf{1}_2'$. Suppose $\theta^* = o(1)$, $\frac{\theta^*}{\min_k \|\theta^{(k)}_{\mathcal{L}}\|_1} = o(1)$, $1-b = o(1)$, $\frac{\|\theta^{(1)}_{\mathcal{L}}\|_1}{\|\theta^{(2)}_{\mathcal{L}}\|_1} = \frac{\|\theta^{(1)}_{\mathcal{U}}\|_1}{\|\theta^{(2)}_{\mathcal{U}}\|_1} = 1$. There exists a constant $c_4 > 0$ such that*

$$\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} \geq c_4 \exp\Big\{-2[1+o(1)]\frac{(1-b)^2}{8}\cdot\theta^*(\|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1)\Big\}, \tag{12}$$

where the infimum is taken over all measurable functions of A, X, and parameters $\Pi_{\mathcal{L}}$, $\Pi_{\mathcal{U}}$, Θ, P, θ^*. In AngleMin+, suppose the second part of condition 10 holds with $c_3 = o(1)$, $\hat{\Pi}_{\mathcal{U}}$ is $\tilde{b}_0$-correct with $\tilde{b}_0 \overset{a.s.}{\to} 0$. There is a constant $C_4 > 0$ such that,

$$\mathrm{Risk}(\hat{y}) \leq C_4 \exp\Big\{-[1-o(1)]\frac{(1-b)^2}{8}\cdot\theta^*\frac{(\|\theta_{\mathcal{L}}\|_1^2 + \|\theta_{\mathcal{U}}\|_1^2)^2}{\|\theta_{\mathcal{L}}\|_1^3 + \|\theta_{\mathcal{U}}\|_1^3}\Big\}. \tag{13}$$

Lemma 2'. *Consider a DCBM with $K = 2$ and $P = (1-b)I_2 + b\mathbf{1}_2\mathbf{1}_2'$. Suppose $1-b = o(1)$. There exists a constant $c_4 > 0$ such that*

$$\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} \geq c_4 \exp\Big\{-2[1+o(1)]\frac{(1-b)^2}{8}\cdot\theta^*(\|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1)\Big\}, \tag{75}$$

where the infimum is taken over all measurable functions of A, X, and parameters $\Pi_{\mathcal{L}}$, $\Pi_{\mathcal{U}}$, Θ, P, θ^*. In AngleMin+, suppose condition 10 holds with $c_3 = o(1)$, $\theta^* = o(1)$, $\frac{\theta^*}{\min_k \|\theta^{(k)}_{\mathcal{L}}\|_1} = o(1)$, $\hat{\Pi}_{\mathcal{U}}$ is $\tilde{b}_0$-correct with $\tilde{b}_0 \overset{a.s.}{\to} 0$. There is a constant $C_4 > 0$ such that,

$$\mathrm{Risk}(\hat{y}) \leq C_4 \exp\left(-[1-o(1)]\frac{(1-b)^2}{8}\cdot\theta^*\frac{4}{\frac{\|\theta^{(1)}_{\mathcal{L}}\|_1^3+\|\theta^{(1)}_{\mathcal{U}}\|_1^3}{(\|\theta^{(1)}_{\mathcal{L}}\|_1^2+\|\theta^{(1)}_{\mathcal{U}}\|_1^2)^2} + \frac{\|\theta^{(2)}_{\mathcal{L}}\|_1^3+\|\theta^{(2)}_{\mathcal{U}}\|_1^3}{(\|\theta^{(2)}_{\mathcal{L}}\|_1^2+\|\theta^{(2)}_{\mathcal{U}}\|_1^2)^2}}\right). \tag{76'}$$

When conditions of Lemma 2 hold, conditions of Lemma 2' hold. Also, with $\frac{\|\theta^{(1)}_{\mathcal{L}}\|_1}{\|\theta^{(2)}_{\mathcal{L}}\|_1} = \frac{\|\theta^{(1)}_{\mathcal{U}}\|_1}{\|\theta^{(2)}_{\mathcal{U}}\|_1} = 1$ assumed in Lemma 2, the results of Lemma 2' imply the results of 2. Therefore, it suffices to prove the generalized version, Lemma 2'.

We prove the lower bound (75) and upper bound (76') separately.

G.1 PROOF OF LOWER BOUND (75)

Proof. Let $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ be the joint distribution of A and X given $\pi^* = e_1$ and $\pi^* = e_2$, respectively. For a random variable or vector or matrix Y, let $\mathbb{P}^{(1)}_Y$ and $\mathbb{P}^{(2)}_Y$ be the distribution of Y given $\pi^* = e_1$ and $\pi^* = e_2$, respectively.

According to Theorem 2.2 in Section 2.4.2 of Tsybakov (2009),

$$\begin{aligned}\inf_{\tilde{y}}\{\text{Risk}(\tilde{y})\} &\geq 2\cdot\frac{1}{2}(1-\sqrt{H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)})(1-H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)})/4)})\\ &= 1-\sqrt{1-\Big(1-\frac{1}{2}H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)})\Big)^2}\\ &\geq 1-\Big(1-\frac{1}{2}\Big(1-\frac{1}{2}H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)})\Big)^2\Big)\\ &= \frac{1}{2}\Big(1-\frac{1}{2}H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)})\Big)^2 \end{aligned} \tag{76}$$

where

$$H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)}) = \int\Big(\sqrt{d\mathbb{P}^{(1)}}-\sqrt{d\mathbb{P}^{(2)}}\Big)^2$$

is the Hellinger distance between $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$.

As in Section 2.4 of Tsybakov (2009), one key property of Hellinger distance is that if $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(2)}$ are product measures, $\mathbb{Q}^{(1)} = \otimes_{i=1}^N \mathbb{Q}_i^{(1)}$, $\mathbb{Q}^{(2)} = \otimes_{i=1}^N \mathbb{Q}_i^{(2)}$, then

$$H^2(\mathbb{Q}^{(1)},\mathbb{Q}^{(2)}) = 2\Big(1-\prod_{i=1}^N\Big(1-\frac{H^2(\mathbb{Q}_i^{(1)},\mathbb{Q}_i^{(2)})}{2}\Big)\Big) \tag{77}$$

Notice that for $k=1,2$, since according to DCBM, $A, X_1, ..., X_n$ are independent,

$$\mathbb{P}^{(k)} = \mathbb{P}_A^{(k)} \times \otimes_{i=1}^n \mathbb{P}_{X_i}^{(k)} \tag{78}$$

Combining (77) and (78), we obtain

$$H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)}) = 2\Big(1-\Big(1-\frac{H^2(\mathbb{P}_A^{(1)},\mathbb{P}_A^{(2)})}{2}\Big)\prod_{i=1}^n\Big(1-\frac{H^2(\mathbb{P}_{X_i}^{(1)},\mathbb{P}_{X_i}^{(2)})}{2}\Big)\Big) \tag{79}$$

Given $\pi^* = e_1$ and $\pi^* = e_2$, according to DCBM, the distribution of A remains the same. As a result, $H^2(\mathbb{P}_A^{(1)},\mathbb{P}_A^{(2)}) = 0$.

On the other hand, for $k=1,2$ and $i\in[n]$, according to DCBM model

$$\mathbb{P}_{X_i}^{(k)} \sim Bern(\theta^*\theta_i P_{kk_i})$$

where k_i is the true label of node i.

As a result,

$$\begin{aligned}&1-\frac{H^2(\mathbb{P}_{X_i}^{(1)},\mathbb{P}_{X_i}^{(2)})}{2}\\ &= 1-\frac{1}{2}\Big[\Big(\sqrt{\theta^*\theta_i P_{1k_i}}-\sqrt{\theta^*\theta_i P_{2k_i}}\Big)^2+\Big(\sqrt{1-\theta^*\theta_i P_{1k_i}}-\sqrt{1-\theta^*\theta_i P_{2k_i}}\Big)^2\Big]\\ &= \sqrt{\theta^*\theta_i P_{1k_i}\theta^*\theta_i P_{2k_i}}+\sqrt{(1-\theta^*\theta_i P_{1k_i})(1-\theta^*\theta_i P_{2k_i})}\end{aligned} \tag{80}$$

For $x, y\in\mathbb{R}$, denote $g(x,y) = \log(xy+\sqrt{(1-x^2)(1-y^2)})$.

Then, $1-\frac{H^2(\mathbb{P}_{X_i}^{(1)},\mathbb{P}_{X_i}^{(2)})}{2} = \exp g(\sqrt{\theta^*\theta_i P_{1k_i}}, \sqrt{\theta^*\theta_i P_{2k_i}})$

Hence, by (79)

$$1-\frac{1}{2}H^2(\mathbb{P}^{(1)},\mathbb{P}^{(2)}) = \Big(1-\frac{H^2(\mathbb{P}_A^{(1)},\mathbb{P}_A^{(2)})}{2}\Big)\prod_{i=1}^{n}\Big(1-\frac{H^2(\mathbb{P}_{X_i}^{(1)},\mathbb{P}_{X_i}^{(2)})}{2}\Big)$$
$$= \exp\sum_{i=1}^{n} g(\sqrt{\theta^*\theta_i P_{1k_i}}, \sqrt{\theta^*\theta_i P_{2k_i}}) \tag{81}$$

Substituting (81) into (76), we obtain

$$\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} \geq \frac{1}{2}\exp\Big(2\sum_{i=1}^{n} g(\sqrt{\theta^*\theta_i P_{1k_i}}, \sqrt{\theta^*\theta_i P_{2k_i}})\Big) \tag{82}$$

To reduce the RHS of (82), we need to evaluate g. The following lemma shows that $g(x,y) \approx -\frac{1}{2}(x-y)^2$.

Lemma 10. *Suppose that* $0 \leq x, y \leq a < 1$. *Then,*

$$g(x,y) \geq -\frac{1}{2(1-a^2)^{\frac{3}{2}}}(x-y)^2$$

Proof. We first prove a short inequality on log. For $z > 0$, define $g_3(z) = \log(z) - \frac{z-1}{z}$. Then

$$\frac{d}{dz}g_3(z) = \frac{1}{z} - \frac{1}{z^2} = \frac{z-1}{z^2}$$

Therefore, when $z \leq 1$, $\frac{d}{dz}g_3(z) \leq 0$, $g_3(z)$ is monotonously deceasing, hence $g_3(z) \geq g_3(1) = 0$; when $z \geq 1$, $\frac{d}{dz}g_3(z) \geq 0$, $g_3(z)$ is monotonously increasing, hence $g_3(z) \geq g_3(1) = 0$.

In all, $g_3(z) \geq 0$, so

$$log(z) \geq \frac{z-1}{z}. \tag{83}$$

Since $x, y \in [0,1]$, we could define $\phi_x = \arcsin x \in [0, \frac{\pi}{2}]$, $\phi_y = \arcsin y \in [0, \frac{\pi}{2}]$. Then,

$$\begin{aligned}
g(x,y) &= g(\sin\phi_x, \sin\phi_y) \\
&= \log\Big(\sin\phi_x\sin\phi_y + \sqrt{(1-\sin^2\phi_x)(1-\sin^2\phi_y)}\Big) \\
&= \log(\sin\phi_x\sin\phi_y + \cos\phi_x\cos\phi_y) \\
&= \log(\cos(\phi_x-\phi_y)) \\
\text{(By (83))} &\geq \frac{\cos(\phi_x-\phi_y)-1}{\cos(\phi_x-\phi_y)} \\
&= \frac{-2\sin^2\frac{(\phi_x-\phi_y)}{2}}{\cos(\phi_x-\phi_y)} \\
&= -\frac{1}{2}\frac{4\sin^2\frac{(\phi_x-\phi_y)}{2}}{(\sin\phi_x-\sin\phi_y)^2}\frac{1}{\cos(\phi_x-\phi_y)}(\sin\phi_x-\sin\phi_y)^2 \\
&= -\frac{1}{2}\frac{4\sin^2\frac{(\phi_x-\phi_y)}{2}}{(2\sin\frac{(\phi_x-\phi_y)}{2}\cos\frac{(\phi_x+\phi_y)}{2})^2}\frac{1}{\cos(\phi_x-\phi_y)}(x-y)^2 \\
&= -\frac{1}{2}(x-y)^2\frac{1}{\cos(|\phi_x-\phi_y|)\cos^2\frac{(\phi_x+\phi_y)}{2}}
\end{aligned} \tag{84}$$

Since $x, y \in [0,a]$, $\phi_x, \phi_y \in [0, \arcsin a]$.As a result, $|\phi_x-\phi_y|$, $\frac{(\phi_x+\phi_y)}{2} \in [0, \arcsin a]$. Plugging this result back into (84), we have

$$g(x,y) \geq -\frac{1}{2}(x-y)^2\frac{1}{\cos(\arcsin a)\cos^2\arcsin a} = -\frac{1}{2(1-a^2)^{\frac{3}{2}}}(x-y)^2 \tag{85}$$

This concludes the proof. □

Back to the proof of lower bound (75). Define $\theta^a = \sqrt{\theta^*\Big(\max_{i\in[n]}\theta_i\Big)(\max\{1,b\})}$, then for any $k \in \{1,2\}$, $i \in [n]$, $\sqrt{\theta^*\theta_i P_{kk_i}} \le \theta^a$. Therefore, applying Lemma 10 in (82), we have when $\theta^a < 1$,

$$\begin{aligned}
\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} &\ge \frac{1}{2}\exp\Big(2\sum_{i=1}^n -\frac{1}{2(1-(\theta^a)^2)^{\frac{3}{2}}}\Big(\sqrt{\theta^*\theta_i P_{1k_i}} - \sqrt{\theta^*\theta_i P_{2k_i}}\Big)^2\Big) \\
&= \frac{1}{2}\exp\Big(-\frac{1}{(1-(\theta^a)^2)^{\frac{3}{2}}}\sum_{i=1}^n \theta^*\theta_i\Big(\sqrt{P_{1k_i}} - \sqrt{P_{2k_i}}\Big)^2\Big) \\
&= \frac{1}{2}\exp\Big(-\frac{1}{(1-(\theta^a)^2)^{\frac{3}{2}}}\sum_{i=1}^n \theta^*\theta_i(1-\sqrt{b})^2\Big) \\
&= \frac{1}{2}\exp\Big(-\frac{1}{(1-(\theta^a)^2)^{\frac{3}{2}}}\theta^*\|\theta\|_1\frac{(1-b)^2}{(1+\sqrt{b})^2}\Big) \\
&= \frac{1}{2}\exp\Big(-2\frac{4}{(1+\sqrt{b})^2(1-(\theta^a)^2)^{\frac{3}{2}}}\frac{(1-b)^2}{8}\cdot\theta^*(\|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1)\Big) \qquad (86)
\end{aligned}$$

Since $\theta^* = o(1)$, $b = 1 - o(1)$, and by DCBM model, $\max_{i\in[n]}\theta_i \le$, we have $\theta^a = \sqrt{\theta^*\Big(\max_{i\in[n]}\theta_i\Big)(\max\{1,b\})} = o(1)$. Since $b = 1 - o(1)$, $(1+\sqrt{b})^2 \to 4$. Therefore, $\frac{4}{(1+\sqrt{b})^2(1-(\theta^a)^2)^{\frac{3}{2}}} = 1 - o(1)$. Substituting these results into (86), we obtain

$$\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} \ge \frac{1}{2}\exp\Big\{-2[1+o(1)]\frac{(1-b)^2}{8}\cdot\theta^*(\|\theta_{\mathcal{L}}\|_1 + \|\theta_{\mathcal{U}}\|_1)\Big\}, \qquad (87)$$

This concludes our proof of lower bound (75), with $c_4 = \frac{1}{2}$.

□

G.2 Proof of Upper Bound (76')

Proof. When $K = 2$, $\mathrm{Risk}(\hat{y}) = \mathbb{P}(\hat{y} = 2|\pi^* = e_1) + \mathbb{P}(\hat{y} = 1|\pi^* = e_2)$. The evaluation of $\mathbb{P}(\hat{y} = 2|\pi^* = e_1)$ and $\mathbb{P}(\hat{y} = 1|\pi^* = e_2)$ are exactly the same. Without the loss of generosity, we would focus on $\mathbb{P}(\hat{y} = 2|\pi^* = e_1)$.

Recall that in the proof of 2, we define $v_k = f(A^{(k)};\hat{\Pi}_{\mathcal{U}})$, $v^* = f(X;\hat{\Pi}_{\mathcal{U}})$, $\tilde{v}_k = f(\Omega^{(k)};\hat{\Pi}_{\mathcal{U}})$, $\tilde{v}^* = f(EX;\hat{\Pi}_{\mathcal{U}})$, $k \in [K]$.

We have

$$\begin{aligned}
\mathbb{P}(\hat{y} = 2|\pi^* = e_1) &= \mathbb{P}(\psi(v_2, v^*) \ge \psi(v_1, v^*)|\pi^* = e_1) \\
&= \mathbb{P}\Big(\frac{\langle v^*, v_2\rangle}{\|v^*\|\|v_2\|} \ge \frac{\langle v^*, v_1\rangle}{\|v^*\|\|v_1\|}\Big|\pi^* = e_1\Big) \\
&= \mathbb{P}\Big(\langle v^*, \Big(\frac{v_2}{\|v_2\|} - \frac{v_1}{\|v_1\|}\Big)\rangle \ge 0\Big|\pi^* = e_1\Big)
\end{aligned} \qquad (88)$$

Recall that in proof of Lemma 8 9, we define $\hat{\mathcal{C}}_k = \{i \in \mathcal{U} : \hat{\pi}_i = e_{k-K}\}, k = K+1, ..., 2K$. Let $w = \frac{v_2}{\|v_2\|} - \frac{v_1}{\|v_1\|}$. Since when $l \in [K]$,

$$(v^*)_l = (f(X;\hat{\Pi}_{\mathcal{U}}))_l = X\mathbf{1}_{(l)} = \sum_{j\in\mathcal{C}_l\cap\mathcal{L}} X_j$$

; when $l \in \{K+1, ..., 2K\}$,

$$(v^*)_l = (f(X; \hat{\Pi}_{\mathcal{U}}))_l = X\mathbf{1}_{(l)} = \sum_{j \in \hat{\mathcal{C}}_l} X_j$$

we have

$$\begin{aligned}
\mathbb{P}(\hat{y} = 2|\pi^* = e_1) &= \mathbb{P}\Big(\sum_{k=1}^{K} w_k \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} X_i + \sum_{k=K+1}^{2K} w_k \sum_{i \in \hat{\mathcal{C}}_k} X_i \geq \Big|\pi^* = e_1\Big) \\
&= \mathbb{P}\Big(\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k(X_i - \mathbb{E}X_i) + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k(X_i - \mathbb{E}X_i) \geq \\
&\quad - \sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k \mathbb{E}X_i - \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k \mathbb{E}X_i \Big|\pi^* = e_1\Big) \\
&\leq \mathbb{P}\Big(\Big|\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k(X_i - \mathbb{E}X_i) + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k(X_i - \mathbb{E}X_i)\Big| \geq \\
&\quad \Big|\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k \mathbb{E}X_i + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k \mathbb{E}X_i\Big| \Big|\pi^* = e_1\Big) \\
&= \mathbb{E}\Big[\mathbb{P}\Big(\Big|\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k(X_i - \mathbb{E}X_i) + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k(X_i - \mathbb{E}X_i)\Big| \geq \\
&\quad \Big|\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k \mathbb{E}X_i + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k \mathbb{E}X_i\Big| \Big|A, \pi^* = e_1\Big)\Big] \qquad (89)
\end{aligned}$$

Since A and X are independent and for $k \in [2K]$, w_k is measurable with respect to A, given A, $\{w_k(X_i - \mathbb{E}X_i) : k \in [K], i \in \mathcal{C}_k \cap \mathcal{L}\} \cup \{w_k(X_i - \mathbb{E}X_i) : k \in [2K] \backslash [K], i \in \hat{\mathcal{C}}_k\}$ are a collection of independent random variables. Also, for any $k \in [2K]$, $i \in [n]$,

$$\mathbb{E}[w_k(X_i - \mathbb{E}X_i)|A] = w_k(\mathbb{E}[X_i|A] - \mathbb{E}X_i) = w_k(\mathbb{E}X_i - \mathbb{E}X_i) = 0$$

Furthermore,

$$-1 \leq -\mathbb{E}X_i \leq X_i - \mathbb{E}X_i \leq X_i \leq 1$$

So $|w_k(X_i - \mathbb{E}X_i)| \leq \max_{k \in [2K]} |w_k|$.

Additionally,

$$var(w_k(X_i - \mathbb{E}X_i)|A) = w_k^2 var(X_i) = w_k^2 \mathbb{E}X_i(1 - \mathbb{E}X_i) \leq w_k^2 \mathbb{E}X_i$$

Let

$$t = \frac{1}{n}\Big|\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k \mathbb{E}X_i + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k \mathbb{E}X_i\Big| = \frac{1}{n}|\langle w, \tilde{v}^* \rangle|$$

$$\sigma^2 = \frac{1}{n}\Big(\sum_{k=1}^{K} \sum_{i \in \mathcal{C}_k \cap \mathcal{L}} w_k^2 \mathbb{E}X_i + \sum_{k=K+1}^{2K} \sum_{i \in \hat{\mathcal{C}}_k} w_k^2 \mathbb{E}X_i\Big) = \frac{1}{n}\langle w \circ w, \tilde{v}^* \rangle$$

where $w \circ w$ is defined as $(w_1^2, ..., w_{2K}^2)$.

Then by Lemma 7,

$$\begin{aligned}\mathbb{P}(\hat{y}=2|\pi^*=e_1) &\le \mathbb{E}\Big[\mathbb{P}\Big(\frac{1}{n}\Big|\sum_{k=1}^{K}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}} w_k(X_i-\mathbb{E}X_i)+\sum_{k=K+1}^{2K}\sum_{i\in\hat{\mathcal{C}}_k} w_k(X_i-\mathbb{E}X_i)\Big|\ge \\ &\qquad \frac{1}{n}\Big(\sum_{k=1}^{K}\sum_{i\in\mathcal{C}_k\cap\mathcal{L}} w_k\mathbb{E}X_i+\sum_{k=K+1}^{2K}\sum_{i\in\hat{\mathcal{C}}_k} w_k\mathbb{E}X_i\Big)\Big|A,\pi^*=e_1\Big)\Big] \\ &\le 2\mathbb{E}\exp\left(-\frac{\frac{1}{2}nt^2}{\sigma^2+\frac{1}{3}\max_{k\in[2K]}|w_k|t}\right) \\ &= 2\mathbb{E}\exp\left(-\frac{\frac{1}{2}\langle w,\tilde{v}^*\rangle^2}{\langle w\circ w,\tilde{v}^*\rangle+\frac{1}{3}\max_{k\in[2K]}|w_k||\langle w,\tilde{v}^*\rangle|}\right) \qquad (90)\end{aligned}$$

By Lemma 8, when $\phi \ge 2\sqrt{2}\pi C_2 K^2 \frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}$,

$$\mathbb{P}\Big(|(v_k-\tilde{v}_k)_l|\ge\frac{1}{\pi\sqrt{K}}\phi\|\tilde{v}_k\|\Big)\le 2\exp\left(-\frac{C_6}{C^2}\phi^2\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1\right)$$

Take

$$\phi=\max\{2\sqrt{2}\pi C_2K^2\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1},|1-b|\Big(\frac{\theta^*}{\min_{k\in[K]}\|\theta_{\mathcal{L}}^{(k)}\|_1}\Big)^{0.25}\}$$

Then because $\frac{\theta^*}{\min_{k\in[K]}\|\theta_{\mathcal{L}}^{(k)}\|_1}=o(1)$ and by condition 10, $\frac{\|\theta_{\mathcal{L}}^{(k)}\|^2}{\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1}\le c_3\beta_n=o(|1-b|)$, we have $\phi=o(|1-b|)$. Also,

$$\begin{aligned}&\mathbb{P}\Big(\exists k\in[K],l\in[2K],|(v_k-\tilde{v}_k)_l|\ge\frac{1}{\pi\sqrt{K}}\phi\|\tilde{v}_k\|\Big) \\ &\le\sum_{k\in[K]}\sum_{l\in[2K]}\mathbb{P}\Big(|(v_k-\tilde{v}_k)_l|\ge\frac{1}{\pi\sqrt{K}}\phi\|\tilde{v}_k\|\Big) \\ &\le\sum_{k\in[K]}\sum_{l\in[2K]}2\exp\left(-\frac{C_6}{C^2}\phi^2\|\theta_{\mathcal{L}}^{(k)}\|_1\|\theta\|_1\right) \\ &=\sum_{k\in[K]}\sum_{l\in[2K]}2\exp\left(-\frac{C_6}{C^2}(1-b)^2\Big(\frac{\theta^*}{\min_{k\in[K]}\|\theta_{\mathcal{L}}^{(k)}\|_1}\Big)^{-0.5}\frac{\|\theta_{\mathcal{L}}^{(k)}\|_1}{\min_{k\in[K]}\|\theta_{\mathcal{L}}^{(k)}\|_1}\theta^*\|\theta\|_1\right) \\ &\le 4K^2\exp\Big(-\frac{1}{o(1)}(1-b)^2\theta^*\|\theta\|_1\Big) \\ &\ll\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} \qquad (91)\end{aligned}$$

Hence, we can focus on the case where for all $k\in[K]$, $l\in[2K]$, $|(v_k-\tilde{v}_k)_l|=o(|1-b|)\|\tilde{v}_k\|$. Until the end of the proof, we assume that we are under this case.

We first evaluate $\langle w, \tilde{v}^* \rangle$. Let $\tilde{w} = \frac{\tilde{v}_2}{\|\tilde{v}_2\|} - \frac{\tilde{v}_1}{\|\tilde{v}_1\|}$. Then,

$$
\begin{aligned}
|\langle w, \tilde{v}^* \rangle - \langle \tilde{w}, \tilde{v}^* \rangle| &= |\langle (\frac{v_2}{\|v_2\|} - \frac{v_1}{\|v_1\|}) - (\frac{\tilde{v}_2}{\|\tilde{v}_2\|} - \frac{\tilde{v}_1}{\|\tilde{v}_1\|}), \tilde{v}^* \rangle| \\
&= \|\tilde{v}^*\| \cdot |\langle (\frac{v_2}{\|v_2\|} - \frac{\tilde{v}_2}{\|\tilde{v}_2\|}) - (\frac{v_1}{\|v_1\|} - \frac{\tilde{v}_1}{\|\tilde{v}_1\|}), \frac{\tilde{v}^*}{\|\tilde{v}^*\|} \rangle| \\
&= \|\tilde{v}^*\| \cdot |(\cos\psi(v_2, \tilde{v}^*) - \cos\psi(\tilde{v}_2, \tilde{v}^*)) - (\cos\psi(v_1, \tilde{v}^*) - \cos\psi(\tilde{v}_1, \tilde{v}^*))| \\
&= \|\tilde{v}^*\| \cdot | - 2\sin\frac{\psi(v_2, \tilde{v}^*) - \psi(\tilde{v}_2, \tilde{v}^*)}{2} \sin\frac{\psi(v_2, \tilde{v}^*) + \psi(\tilde{v}_2, \tilde{v}^*)}{2} \\
&\qquad + 2\sin\frac{\psi(v_1, \tilde{v}^*) - \psi(\tilde{v}_1, \tilde{v}^*)}{2} \sin\frac{\psi(v_1, \tilde{v}^*) + \psi(\tilde{v}_1, \tilde{v}^*)}{2}| \\
&\leq 2\|\tilde{v}^*\| \cdot \sin\frac{|\psi(v_2, \tilde{v}^*) - \psi(\tilde{v}_2, \tilde{v}^*)|}{2} \sin\frac{\psi(v_2, \tilde{v}^*) + \psi(\tilde{v}_2, \tilde{v}^*)}{2} \\
&\quad + 2\|\tilde{v}^*\| \cdot \sin\frac{|\psi(v_1, \tilde{v}^*) - \psi(\tilde{v}_1, \tilde{v}^*)|}{2} \sin\frac{\psi(v_1, \tilde{v}^*) + \psi(\tilde{v}_1, \tilde{v}^*)}{2} \\
\text{(By Lemma 4)} &\leq 2\|\tilde{v}^*\| \cdot \sin\frac{\psi(v_2, \tilde{v}_2)}{2} \sin\frac{2\psi(\tilde{v}_2, \tilde{v}^*) + \psi(v_2, \tilde{v}_2)}{2} \\
&\quad + 2\|\tilde{v}^*\| \cdot \sin\frac{\psi(v_1, \tilde{v}_1)}{2} \sin\frac{2\psi(\tilde{v}_1, \tilde{v}^*) + \psi(v_1, \tilde{v}_1)}{2}
\end{aligned} \tag{92}
$$

Since $\pi^* = 1$, $\tilde{b}_0 \to 0$, by Theorem 1, $\psi(\tilde{v}_1, \tilde{v}^*) = \psi_1 = 0$, $\psi(\tilde{v}_2, \tilde{v}^*) = \psi_2 \geq c_0\beta_n = c_0|1-b|$. On the other hand, that for all $k \in [K]$, $l \in [2K]$, $|(v_k - \tilde{v}_k)_l| = o(|1-b|)\|\tilde{v}_k\|$ indicates that $\|v_k - \tilde{v}_k\| = o(|1-b|)\|\tilde{v}_k\|$. By lemma 5, this implies that $\psi(v_1, \tilde{v}_1) = o(|1-b|), k = 1, 2$. Therefore,

$$
\begin{aligned}
|\langle w, \tilde{v}^* \rangle - \langle \tilde{w}, \tilde{v}^* \rangle| &\leq o(1) \cdot 2\|\tilde{v}^*\| \sin\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} \sin\psi(\tilde{v}_2, \tilde{v}^*) \\
&= o(1) \cdot 2\|\tilde{v}^*\| \sin\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} 2\sin\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} \cos\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} \\
&= o(1) \cdot \|\tilde{v}^*\| \sin^2\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} \cos\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} \\
&\leq o(1) \cdot \|\tilde{v}^*\| \sin^2\frac{\psi(\tilde{v}_2, \tilde{v}^*)}{2} \\
&\leq o(1) \cdot \|\tilde{v}^*\|(1 - \cos\psi(\tilde{v}_2, \tilde{v}^*)) \\
&= o(1) \cdot \|\tilde{v}^*\|(\cos\psi(\tilde{v}_1, \tilde{v}^*) - \cos\psi(\tilde{v}_2, \tilde{v}^*)) \\
&= o(1) \cdot \|\tilde{v}^*\|\langle \frac{\tilde{v}_1}{\|\tilde{v}_1\|} - \frac{\tilde{v}_2}{\|\tilde{v}_2\|}, \frac{\tilde{v}^*}{\|\tilde{v}^*\|} \rangle \\
&= o(1) \cdot (-\langle \tilde{w}, \tilde{v}^* \rangle) \\
&\leq o(1) \cdot |\langle \tilde{w}, \tilde{v}^* \rangle|
\end{aligned} \tag{93}
$$

Therefore, $\langle w, \tilde{v}^* \rangle = (1 + o(1))\langle \tilde{w}, \tilde{v}^* \rangle$.

Let $\eta_{kl} = \sum_{\pi_i = e_k, \hat{\pi}_i = e_l} \theta_i$. Let $\mu_a^{(k)} = \|\theta_a^{(k)}\|_1, a \in \{\mathcal{L}, \mathcal{U}\}, k \in [K]$. Then,

$$
\begin{aligned}
\tilde{v}_1 &= \mu_{\mathcal{L}}^{(1)}(\mu_{\mathcal{L}}^{(1)}, b\mu_{\mathcal{L}}^{(2)}, \mu_{\mathcal{U}}^{(1)} - \eta_{12} + b\eta_{21}, b\mu_{\mathcal{U}}^{(2)} + \eta_{12} - b\eta_{21}) \\
\tilde{v}_2 &= \mu_{\mathcal{L}}^{(2)}(b\mu_{\mathcal{L}}^{(1)}, \mu_{\mathcal{L}}^{(2)}, b\mu_{\mathcal{U}}^{(1)} - b\eta_{12} + \eta_{21}, \mu_{\mathcal{U}}^{(2)} + b\eta_{12} - \eta_{21}) \\
\tilde{v}^* &= \theta^*(\mu_{\mathcal{L}}^{(1)}, b\mu_{\mathcal{L}}^{(2)}, \mu_{\mathcal{U}}^{(1)} - \eta_{12} + b\eta_{21}, b\mu_{\mathcal{U}}^{(2)} + \eta_{12} - b\eta_{21})
\end{aligned}
$$

Hence,

$$
\begin{aligned}
-\langle \tilde{w}, \tilde{v}^* \rangle &= \|\tilde{v}^*\|\Big(\frac{\langle \tilde{v}_1 \tilde{v}^* \rangle}{\|\tilde{v}_1\|\|\tilde{v}^*\|} - \frac{\langle \tilde{v}_2 \tilde{v}^* \rangle}{\|\tilde{v}_2\|\|\tilde{v}^*\|}\Big) \\
&= \|\tilde{v}^*\| \left(1 - \sqrt{1 - (1-b^2)^2(\mu_{\mathcal{L}}^{(1)})^2(\mu_{\mathcal{L}}^{(2)})^2 \frac{\mathcal{I}_1\mathcal{I}_2 - 4\mathcal{I}_3^2}{\|\tilde{v}_1\|^2\|\tilde{v}_2\|^2}}\right)
\end{aligned} \tag{94}
$$

where

$$\mathcal{I}_1 = (\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)} - \eta_{12})^2 + \eta_{21}^2$$

$$\mathcal{I}_2 = (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)} - \eta_{21})^2 + \eta_{12}^2$$

$$\mathcal{I}_3 = (\mu_{\mathcal{U}}^{(1)} - \eta_{12})\eta_{21} + (\mu_{\mathcal{U}}^{(2)} - \eta_{21})\eta_{12}$$

Notice that

$$\begin{aligned}
&|\mathcal{I}_1\mathcal{I}_2 - 4\mathcal{I}_3^2 - ((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)| \\
&\leq ((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)(\eta_{12}^2 + \eta_{21}^2) \\
&\quad + 2\eta_{12}\mu_{\mathcal{U}}^{(1)}((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2) + 2\eta_{21}\mu_{\mathcal{U}}^{(2)}((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2) \\
&\quad + 4\mathcal{I}_3^2 \\
&\leq ((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)(\eta_{12} + \eta_{21})^2 \\
&\quad + 2\eta_{12}\mu_{\mathcal{U}}^{(1)}((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2) + 2\eta_{21}\mu_{\mathcal{U}}^{(2)}((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2) \\
&\quad + 8(\mu_{\mathcal{U}}^{(1)})^2\eta_{21}^2 + 8(\mu_{\mathcal{U}}^{(2)})^2\eta_{12}^2 \\
&\leq \|\theta\|_1^2(\eta_{12} + \eta_{21})^2 + 2\|\theta\|_1^3(\eta_{12} + \eta_{21}) + 8\|\theta\|_1^2(\eta_{12} + \eta_{21})^2 \\
&\leq (9\tilde{b}_0^2 + 2\tilde{b}_0)\|\theta\|_1^4 \\
&= o(1) \cdot \|\theta\|_1^4
\end{aligned} \tag{95}$$

On the other hand,

$$\begin{aligned}
((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2) &\geq \frac{1}{4}(\mu_{\mathcal{L}}^{(1)} + \mu_{\mathcal{U}}^{(1)})^2(\mu_{\mathcal{L}}^{(2)} + \mu_{\mathcal{U}}^{(2)})^2 \\
&= \frac{1}{4}\|\theta^{(1)}\|_1^2\|\theta^{(2)}\|_1^2 \\
&= \frac{1}{4}\min_{k\in[K]}\|\theta^{(k)}\|_1^2 \max_{k\in[K]}\|\theta^{(k)}\|_1^2 \\
\text{(By condition (10))} &\geq \frac{1}{4C_2}\max_{k\in[K]}\|\theta^{(k)}\|_1^4 \\
&\geq \frac{1}{64C_2}\|\theta\|_1^4
\end{aligned} \tag{96}$$

Therefore,

$$\mathcal{I}_1\mathcal{I}_2 - 4\mathcal{I}_3^2 = (1 + o(1))((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)$$

Similarly,

$$\frac{\|\tilde{v}_1\|^2\|\tilde{v}_2\|^2}{(\mu_{\mathcal{L}}^{(1)})^2(\mu_{\mathcal{L}}^{(2)})^2} = (1 + o(1))((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)^2$$

$$\|\tilde{v}^*\| = (1 + o(1))\theta^*\sqrt{(\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2}$$

Therefore,

$$
\begin{aligned}
-\langle \tilde{w}, \tilde{v}^* \rangle &= \|\tilde{v}^*\| \left(1 - \sqrt{1 - (1+o(1))(1-b^2)^2 \frac{((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)}{((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)^2}}\right) \\
&= \frac{1}{2}(1+o(1))\theta^*(1-b^2)^2 \frac{((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)}{((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)^{1.5}} \\
&= 2(1+o(1))\theta^*(1-b)^2 \frac{((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)}{((\mu_{\mathcal{L}}^{(1)})^2 + (\mu_{\mathcal{U}}^{(1)})^2 + (\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2)^{1.5}} \qquad (97)
\end{aligned}
$$

We turn to $\langle w \circ w, \tilde{v}^* \rangle$.

Denote that

$$
\tilde{\nu}_1 = \frac{1}{\mu_{\mathcal{L}}^{(1)}} \tilde{v}_1 = (\mu_{\mathcal{L}}^{(1)}, b\mu_{\mathcal{L}}^{(2)}, \mu_{\mathcal{U}}^{(1)} - \eta_{12} + b\eta_{21}, b\mu_{\mathcal{U}}^{(2)} + \eta_{12} - b\eta_{21})
$$

$$
\tilde{\nu}_2 = \frac{1}{\mu_{\mathcal{L}}^{(2)}} \tilde{v}_2 = (b\mu_{\mathcal{L}}^{(1)}, \mu_{\mathcal{L}}^{(2)}, b\mu_{\mathcal{U}}^{(1)} - b\eta_{12} + \eta_{21}, \mu_{\mathcal{U}}^{(2)} + b\eta_{12} - \eta_{21})
$$

One simple but useful fact is that $\|\tilde{\nu}_1\|, \|\tilde{\nu}_2\| = O(\|\theta\|_1)$. The reason is that for $k \in [K]$

$$
\|\tilde{\nu}_k\| = O(\|\tilde{\nu}_k\|_1) = O(\mu_{\mathcal{L}}^{(1)} + b\mu_{\mathcal{L}}^{(2)} + \mu_{\mathcal{U}}^{(1)} + b\mu_{\mathcal{U}}^{(2)}) = O(\|\theta\|_1)
$$

Notice that

$$
\begin{aligned}
\tilde{w}_3 &= \frac{b\mu_{\mathcal{U}}^{(1)} - b\eta_{12} + \eta_{21}}{\|\tilde{\nu}_2\|} - \frac{\mu_{\mathcal{U}}^{(1)} - \eta_{12} + b\eta_{21}}{\|\tilde{\nu}_1\|} \\
&= (b\mu_{\mathcal{U}}^{(1)} - b\eta_{12} + \eta_{21})(\frac{1}{\|\tilde{\nu}_2\|} - \frac{1}{\|\tilde{\nu}_1\|}) - \frac{1-b}{\|\tilde{\nu}_1\|}(\mu_{\mathcal{U}}^{(1)} - \eta_{12} - \eta_{21}) \\
&= -\frac{b\mu_{\mathcal{U}}^{(1)} - b\eta_{12} + \eta_{21}}{\|\tilde{\nu}_1\|}(1 - \frac{\|\tilde{\nu}_1\|}{\|\tilde{\nu}_2\|}) - \frac{1-b}{\|\tilde{\nu}_1\|}\mu_{\mathcal{U}}^{(1)} + o(1-b) \\
&= -\frac{b\mu_{\mathcal{U}}^{(1)} - b\eta_{12} + \eta_{21}}{\|\tilde{\nu}_1\|}\left(1 - \sqrt{1 - \frac{\|\tilde{\nu}_2\|^2 - \|\tilde{\nu}_1\|^2}{\|\tilde{\nu}_2\|^2}}\right) - \frac{1-b}{\|\tilde{v}_1\|}\mu_{\mathcal{U}}^{(1)} + o(1-b) \qquad (98)
\end{aligned}
$$

Since

$$
\begin{aligned}
\|\tilde{\nu}_2\|^2 - \|\tilde{\nu}_1\|^2 &= (1-b^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)} - \eta_{21})^2 - (\mu_{\mathcal{L}}^{(1)})^2 - (\mu_{\mathcal{U}}^{(1)} - \eta_{12})^2) \\
&= (1-b^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2 - (\mu_{\mathcal{L}}^{(1)})^2 - (\mu_{\mathcal{U}}^{(1)})^2 + o(1) \cdot \|\theta\|_1^2) \qquad (99)
\end{aligned}
$$

We have

$$
\begin{aligned}
\tilde{w}_3 &= -\frac{b\mu_{\mathcal{U}}^{(1)} + o(1) \cdot \|\theta\|_1}{\|\tilde{\nu}_1\|} \frac{1}{2\|\tilde{\nu}_2\|^2}(1-b^2)((\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2 - (\mu_{\mathcal{L}}^{(1)})^2 - (\mu_{\mathcal{U}}^{(1)})^2 + o(1) \cdot \|\theta\|_1^2) \\
&\quad - \frac{1-b}{\|\tilde{\nu}_1\|}\mu_{\mathcal{U}}^{(1)} + o(1-b) \\
&= -\frac{1-b}{\|\tilde{\nu}_1\|}\mu_{\mathcal{U}}^{(1)}\left(1 + \frac{(\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2 - (\mu_{\mathcal{L}}^{(1)})^2 - (\mu_{\mathcal{U}}^{(1)})^2}{\|\tilde{\nu}_2\|^2}\right) + o(1-b) \qquad (100)
\end{aligned}
$$

Since for all $k \in [K]$, $l \in [2K]$, $|(v_k - \tilde{v}_k)_l| = o(|1-b|)\|\tilde{v}_k\|$, we have $|w_k - \tilde{w}_k| = o(|1-b|)$, $k = 1, 2, ..., 2K$. Hence, denote $\gamma = \frac{(\mu_{\mathcal{L}}^{(2)})^2 + (\mu_{\mathcal{U}}^{(2)})^2 - (\mu_{\mathcal{L}}^{(1)})^2 - (\mu_{\mathcal{U}}^{(1)})^2}{\|\tilde{\nu}_2\|^2}$, we obtain

$$w_3 = -\frac{1-b}{\|\tilde{\nu}_1\|}\mu_{\mathcal{U}}^{(1)}(1+\gamma) + o(1-b) \tag{101}$$

Similarly, we can show that

$$w_1 = -\frac{1-b}{\|\tilde{\nu}_1\|}\mu_{\mathcal{L}}^{(1)}(1+\gamma) + o(1-b) \tag{102}$$

$$w_2 = \frac{1-b}{\|\tilde{\nu}_2\|}\mu_{\mathcal{L}}^{(2)}(1-\gamma) + o(1-b) \tag{103}$$

$$w_4 = \frac{1-b}{\|\tilde{\nu}_2\|}\mu_{\mathcal{U}}^{(2)}(1-\gamma) + o(1-b) \tag{104}$$

As a result,

$$\begin{aligned}
&\langle w \circ w, \tilde{\nu}^* \rangle \\
&= \sum_{a \in \{\mathcal{L}, \mathcal{U}\}} \sum_{k=1}^{K} \left(\frac{1-b}{\|\tilde{\nu}_k\|}\mu_a^{(k)}(1-(-1)^k\gamma) + o(1-b)\right)^2 \theta^* \mu_a^{(k)} \\
&= \theta^*(1-b)^2 \sum_{a \in \{\mathcal{L}, \mathcal{U}\}} \sum_{k=1}^{K} \left(\frac{(\mu_a^{(k)})^3(1-(-1)^k\gamma)^2}{\|\tilde{\nu}_k\|^2} + 2o(1)\frac{(\mu_a^{(k)})^2(1-(-1)^k\gamma)}{\|\tilde{\nu}_k\|^2} + o(1)^2\frac{\mu_a^{(k)}}{\|\tilde{\nu}_k\|^2} \right)
\end{aligned} \tag{105}$$

When $\|\theta\|_1 = o(1)$, the bounds (75) and (76′) both become trivial, so we can focus on the case where $\|\theta\|_1 \geq O(1)$.

In this case, $\frac{(\mu_a^{(k)})^2(1-(-1)^k\gamma)}{\|\tilde{\nu}_k\|^2} \leq O(1)$, $\frac{\mu_a^{(k)}}{\|\tilde{\nu}_k\|^2} \leq O(1)$. On the other hand, for $k \in [K]$,

$$\begin{aligned}
\sum_{a \in \{\mathcal{L}, \mathcal{U}\}} \frac{(\mu_a^{(k)})^3}{\|\tilde{\nu}_k\|^2} &= \frac{(\mu_{\mathcal{L}}^{(k)})^3 + (\mu_{\mathcal{U}}^{(k)})^3}{\|\tilde{\nu}_k\|^2} \\
\text{(Holder Inequality)} &\geq \frac{(\mu_{\mathcal{L}}^{(k)} + \mu_{\mathcal{U}}^{(k)})^3}{4\|\tilde{\nu}_k\|^2} \\
&= \frac{\|\theta^{(k)}\|_1^3}{4\|\tilde{\nu}_k\|^2} \\
&\geq \frac{\min_{k \in [K]} \|\theta^{(k)}\|_1^3}{4\|\tilde{\nu}_k\|^2} \\
\text{(By (10))} &\geq \frac{\max_{k \in [K]} \|\theta^{(k)}\|_1^3}{4C_2^3\|\tilde{\nu}_k\|^2} \\
&\geq \frac{\|\theta\|_1^3}{4K^3C_2^3\|\tilde{\nu}_k\|^2} \\
&\geq O(1)
\end{aligned} \tag{106}$$

Since $1-\gamma$ and $1+\gamma$ cannot be both $o(1)$, we have

$$\sum_{a \in \{\mathcal{L}, \mathcal{U}\}} \sum_{k=1}^{K} \frac{(\mu_a^{(k)})^3(1-(-1)^k\gamma)^2}{\|\tilde{\nu}_k\|^2} \geq O(1)$$

Therefore,

$$\begin{aligned}\langle w \circ w, \tilde{\nu}^* \rangle &= (1+o(1))\theta^*(1-b)^2 \sum_{a\in\{\mathcal{L},\mathcal{U}\}} \sum_{k=1}^{K} \frac{(\mu_a^{(k)})^3(1-(-1)^k\gamma)^2}{\|\tilde{\nu}_k\|^2} \\ &= (1+o(1))4\theta^*(1-b)^2 \\ &\quad \cdot \frac{((\mu_{\mathcal{L}}^{(1)})^3+(\mu_{\mathcal{U}}^{(1)})^3)((\mu_{\mathcal{L}}^{(2)})^2+(\mu_{\mathcal{U}}^{(2)})^2)^2+((\mu_{\mathcal{L}}^{(2)})^3+(\mu_{\mathcal{U}}^{(2)})^3)((\mu_{\mathcal{L}}^{(1)})^2+(\mu_{\mathcal{U}}^{(1)})^2)^2}{((\mu_{\mathcal{L}}^{(1)})^2+(\mu_{\mathcal{U}}^{(1)})^2+(\mu_{\mathcal{L}}^{(2)})^2+(\mu_{\mathcal{U}}^{(2)})^2)^3}\end{aligned} \tag{107}$$

By (101) (102) (103) (104), $\max_{k\in[2K]}|w_k| = O(1-b) = o(1)$. Substituting this result together with (97) and (107) into (90), we obtain

$$\begin{aligned}&\mathbb{P}(\hat{y}=2|\pi^*=e_1) \\ &= 2\exp\left(-(1-o(1))\frac{(1-b)^2}{8}\cdot\theta^*\frac{4}{\frac{\|\theta_{\mathcal{L}}^{(1)}\|_1^3+\|\theta_{\mathcal{U}}^{(1)}\|_1^3}{(\|\theta_{\mathcal{L}}^{(1)}\|_1^2+\|\theta_{\mathcal{U}}^{(1)}\|_1^2)^2}+\frac{\|\theta_{\mathcal{L}}^{(2)}\|_1^3+\|\theta_{\mathcal{U}}^{(2)}\|_1^3}{(\|\theta_{\mathcal{L}}^{(2)}\|_1^2+\|\theta_{\mathcal{U}}^{(2)}\|_1^2)^2}}\right)\end{aligned} \tag{108}$$

Similarly,

$$\begin{aligned}&\mathbb{P}(\hat{y}=1|\pi^*=e_2) \\ &\le 2\exp\left(-(1-o(1))\frac{(1-b)^2}{8}\cdot\theta^*\frac{4}{\frac{\|\theta_{\mathcal{L}}^{(1)}\|_1^3+\|\theta_{\mathcal{U}}^{(1)}\|_1^3}{(\|\theta_{\mathcal{L}}^{(1)}\|_1^2+\|\theta_{\mathcal{U}}^{(1)}\|_1^2)^2}+\frac{\|\theta_{\mathcal{L}}^{(2)}\|_1^3+\|\theta_{\mathcal{U}}^{(2)}\|_1^3}{(\|\theta_{\mathcal{L}}^{(2)}\|_1^2+\|\theta_{\mathcal{U}}^{(2)}\|_1^2)^2}}\right)\end{aligned} \tag{109}$$

Therefore,

$$\mathrm{Risk}(\hat{y}) \le 4\exp\left(-(1-o(1))\frac{(1-b)^2}{8}\cdot\theta^*\frac{4}{\frac{\|\theta_{\mathcal{L}}^{(1)}\|_1^3+\|\theta_{\mathcal{U}}^{(1)}\|_1^3}{(\|\theta_{\mathcal{L}}^{(1)}\|_1^2+\|\theta_{\mathcal{U}}^{(1)}\|_1^2)^2}+\frac{\|\theta_{\mathcal{L}}^{(2)}\|_1^3+\|\theta_{\mathcal{U}}^{(2)}\|_1^3}{(\|\theta_{\mathcal{L}}^{(2)}\|_1^2+\|\theta_{\mathcal{U}}^{(2)}\|_1^2)^2}}\right). \tag{110}$$

Taking $C_4 = 4$, we conclude the proof. □

H PROOF OF THEOREM 3

Theorem 3. *Suppose the conditions of Corollary 1 hold, where b_0 is properly small , and suppose that $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct. Furthermore, we assume for sufficiently large constant C_3, $\theta^* \le \frac{1}{C_3}$, $\theta^* \le \min_{k\in[K]} C_3\|\theta_{\mathcal{L}}^{(k)}\|_1$, and for a constant $r_0 > 0$, $\min_{k\neq\ell}\{P_{k\ell}\} \ge r_0$. Then, there is a constant $\tilde{c}_2 = \tilde{c}_2(K, C_1, C_2, C_3, c_3, r_0) > 0$ such that $[-\log(\tilde{c}_2\mathrm{Risk}(\hat{y}))]/[-\log(\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\})] \ge \tilde{c}_2$.*

Proof. On one hand, for any $k, k^* \in [K], k \neq k^*$, using exactly the same proof as in Section G.1, we can show that when $C_3 > C_1$,

$$\begin{aligned}&\inf_{\tilde{y}}(\mathbb{P}(\hat{y}=k|\pi^*=e_{k^*})+\mathbb{P}(\hat{y}=k^*|\pi^*=k)) \\ &\ge \frac{1}{2}\exp\Big(2\sum_{i=1}^{n} -\frac{1}{2(1-(\theta^a)^2)^{\frac{3}{2}}}\Big(\sqrt{\theta^*\theta_i P_{kk_i}}-\sqrt{\theta^*\theta_i P_{k^*k_i}}\Big)^2\Big)\end{aligned} \tag{111}$$

where k_i is the true label of node i, $\theta^a = \max_{i\in[n]} \max_{k\in[K]} \sqrt{\theta^*\theta_i P_{kk_i}}$. According to DCBM model and condition (9), $\theta^a \leq \frac{C}{C_3}$. Hence take $C_3 \geq \sqrt{2}C_1$, then

$$\begin{aligned}\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\} &\geq \max_{k\neq k^*\in[K]} \inf_{\tilde{y}}(\mathbb{P}(\hat{y}=k|\pi^*=e_{k^*}) + \mathbb{P}(\hat{y}=k^*|\pi^*=e_k)) \\ &\geq \max_{k\neq k^*\in[K]} \frac{1}{2}\exp\Big(-2\sqrt{2}\sum_{i=1}^n\Big(\sqrt{\theta^*\theta_i P_{kk_i}} - \sqrt{\theta^*\theta_i P_{k^*k_i}}\Big)^2\Big) \\ &= \frac{1}{2}\exp\Big(-2\sqrt{2}\theta^* \min_{k\neq k^*\in[K]} \sum_{l=1}^K \|\theta^{(l)}\|_1\Big(\sqrt{P_{kl}} - \sqrt{P_{k^*l}}\Big)^2\Big) \end{aligned} \tag{112}$$

Let B_0 be the event that $\hat{\Pi}_{\mathcal{U}}$ is b_0-correct. When $\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\}$ is replaced by the version conditioning on B_0, since X and A are independent, and $B_0 \in \sigma(A)$, conditioning on B_0 or not does not affect the distribution of X. On the other hand, for any k, k^*, since π^* does not affect the distribution of A, the distribution of $A|B_0, \pi^* = e_k$ and $A|B_0, \pi^* = e_{k^*}$ are the same, so their Hellinger distance is still 0. Hence, all the proofs in Section G and above remain unaffected. In other words, one does not gain a lot of information from B_0.

On the other hand, notice that proof of Theorem 2 still works conditioning on B_0. In other words, there exists constant $C > 0$, such that given B_0, for any $\delta \in (0, 1/2)$, with probability $1-\delta$, simultaneously for $1 \leq k \leq K$,

$$|\hat{\psi}_k(\hat{\Pi}_{\mathcal{U}}) - \psi_k(\hat{\Pi}_{\mathcal{U}})| \leq C\left(\sqrt{\frac{\log(1/\delta)}{\|\theta\|_1 \cdot \min\{\theta^*, \|\theta^{(k)}_{\mathcal{L}}\|_1\}}} + \frac{\|\theta^{(k)}_{\mathcal{L}}\|^2}{\|\theta^{(k)}_{\mathcal{L}}\|_1\|\theta\|_1}\right).$$

Define $\tilde{\beta}_k = \psi_k(\hat{\Pi}_{\mathcal{U}})$. Replacing $c_0\beta_n$ by $\tilde{\beta}_k$ and replicating the proof of Corollary 1 , we can show that

$$\mathbb{P}(\hat{y} \neq k^*|B_0, \pi^* = e_{k^*}) \leq \bar{C}\sum_{k=1}^K \exp\Big(-\frac{C_0}{c_0^2}\tilde{\beta}_k^2\|\theta\|_1 \cdot \min\{\theta^*, \|\theta^{(k)}_{\mathcal{L}}\|_1\}\Big)$$

Since $\theta^* \leq \min_{k\in[K]} C_3\|\theta^{(k)}_{\mathcal{L}}\|_1$, $\min\{\theta^*, \|\theta^{(k)}_{\mathcal{L}}\|_1\} \geq \min\{1, \frac{1}{C_3}\}\theta^*$, therefore,

$$\mathbb{P}(\hat{y} \neq k^*|B_0, \pi^* = e_{k^*}) \leq \bar{C}\sum_{k=1}^K \exp\Big(-\frac{C_0}{c_0^2}\min\{1, \frac{1}{C_3}\}\tilde{\beta}_k^2\theta^*\|\theta\|_1\Big) \tag{113}$$

Recall that in (23) of Section D, we show that

$$\tilde{\beta}_k = \psi_k(\hat{\Pi}_{\mathcal{U}}) \geq \sqrt{(e_k - e_{k^*})'\tilde{M}(e_k - e_{k^*})} \tag{114}$$

By Lemma 6, denote

$$C_5 = 8K^2\sqrt{K}C_2^2 b_0 \frac{\|\theta_{\mathcal{U}}\|_1}{\|\theta\|_1} \tag{115}$$

Suppose that $C_5 \leq \frac{1}{4}$. Then, for any vector $\alpha \in \mathbb{R}^k$,

$$|\alpha'\tilde{M}\alpha| \geq \frac{1-3C_5}{1-C_5}|\alpha'\tilde{M}^{(0)}\alpha| \geq \frac{1}{3}|\alpha'\tilde{M}^{(0)}\alpha| \tag{116}$$

where recall that in Section D, we define

$$M^{(0)} = P\Big(G^2_{\mathcal{L}\mathcal{L}} + G^2_{\mathcal{U}\mathcal{U}}\Big)P$$

$D_{M^{(0)}} = \mathrm{diag}(M^{(0)}_{11}, ..., M^{(0)}_{KK})$, and $\tilde{M}$, $\tilde{M}^{(0)} = D^{-\frac{1}{2}}_{M^{(0)}} M^{(0)} D^{-\frac{1}{2}}_{M^{(0)}}$

Hence, when b_0 is sufficiently small,

$$\tilde{\beta}_k^2 \geq (e_k - e_{k^*})'\tilde{M}(e_k - e_{k^*}) \geq \frac{1}{3}|(e_k - e_{k^*})'\tilde{M}^{(0)}(e_k - e_{k^*})| = \frac{2}{3}\left(1 - \frac{M_{kk^*}^{(0)}}{\sqrt{M_{kk}^{(0)}}\sqrt{M_{k^*k^*}^{(0)}}}\right) \tag{117}$$

Define $\lambda_l = \sqrt{\|\theta_{\mathcal{L}}^{(l)}\|_1^2 + \|\theta_{\mathcal{U}}^{(l)}\|_1^2}, l \in [K]$. Then,

$$M_{kk^*}^{(0)} = \sum_{l=1}^K \lambda_l^2 P_{kl}P_{lk^*} = \sum_{l=1}^K \lambda_l^2 P_{kl}P_{k^*l}$$
$$M_{kk}^{(0)} = \sum_{l=1}^K \lambda_l^2 P_{kl}P_{lk} = \sum_{l=1}^K \lambda_l^2 P_{kl}^2$$
$$M_{k^*k^*}^{(0)} = \sum_{l=1}^K \lambda_l^2 P_{k^*l}P_{lk^*} = \sum_{l=1}^K \lambda_l^2 P_{k^*l}^2$$

Therefore, plugging the above result into (117), we have

$$\begin{aligned}
\tilde{\beta}_k^2 &\geq \frac{2}{3}\left(1 - \frac{M_{kk^*}^{(0)}}{\sqrt{M_{kk}^{(0)}}\sqrt{M_{k^*k^*}^{(0)}}}\right) \\
&= \frac{2}{3}\left(1 - \frac{\sum_{l=1}^K \lambda_l^2 P_{kl}P_{k^*l}}{\sqrt{\sum_{l=1}^K \lambda_l^2 P_{kl}^2}\sqrt{\sum_{l=1}^K \lambda_l^2 P_{k^*l}^2}}\right) \\
&= \frac{2}{3}\left(1 - \sqrt{1 - \frac{(\sum_{l=1}^K \lambda_l^2 P_{kl}^2)(\sum_{l=1}^K \lambda_l^2 P_{k^*l}^2) - (\sum_{l=1}^K \lambda_l^2 P_{kl}P_{k^*l})^2}{(\sum_{l=1}^K \lambda_l^2 P_{kl}^2)(\sum_{l=1}^K \lambda_l^2 P_{k^*l}^2)}}\right) \\
&\geq \frac{1}{3}\frac{(\sum_{l=1}^K \lambda_l^2 P_{kl}^2)(\sum_{l=1}^K \lambda_l^2 P_{k^*l}^2) - (\sum_{l=1}^K \lambda_l^2 P_{kl}P_{k^*l})^2}{(\sum_{l=1}^K \lambda_l^2 P_{kl}^2)(\sum_{l=1}^K \lambda_l^2 P_{k^*l}^2)}
\end{aligned} \tag{118}$$

Notice that

$$\begin{aligned}
&(\sum_{l=1}^K \lambda_l^2 P_{kl}^2)(\sum_{l=1}^K \lambda_l^2 P_{k^*l}^2) - (\sum_{l=1}^K \lambda_l^2 P_{kl}P_{k^*l})^2 \\
&= \sum_{l,\tilde{l}\in[K]} \lambda_l^2\lambda_{\tilde{l}}^2(P_{kl}P_{k^*\tilde{l}} - P_{k\tilde{l}}P_{k^*l})^2 \\
&\geq \sum_{l\in[K]} \lambda_l^2\lambda_k^2(P_{kl}P_{k^*k} - P_{kk}P_{k^*l})^2 + \sum_{l\in[K]} \lambda_l^2\lambda_{k^*}^2(P_{kl}P_{k^*k^*} - P_{kk^*}P_{k^*l})^2 \\
\text{(Identification Condition)} &= \sum_{l\in[K]} \lambda_l^2\Big(\lambda_k^2(P_{kk^*}P_{kl} - P_{k^*l})^2 + \lambda_{k^*}^2(P_{kl} - P_{kk^*}P_{k^*l})^2\Big) \\
\text{(Cauchy-Schwartz)} &\geq \sum_{l\in[K]} \lambda_l^2 \frac{1}{\frac{1}{\lambda_k^2} + \frac{1}{\lambda_{k^*}^2}}\Big(P_{kk^*}P_{kl} - P_{k^*l} + P_{kl} - P_{kk^*}P_{k^*l}\Big)^2 \\
&= \sum_{l\in[K]} \lambda_l^2 \frac{1}{\frac{1}{\lambda_k^2} + \frac{1}{\lambda_{k^*}^2}}(1 + P_{kk^*})^2(P_{kl} - P_{k^*l})^2 \\
&\geq \frac{1}{\frac{1}{\min_{l\in[K]}\lambda_l^2} + \frac{1}{\min_{l\in[K]}\lambda_l^2}} \sum_{l\in[K]} \lambda_l^2(P_{kl} - P_{k^*l})^2 \\
&= \frac{1}{2}\min_{l\in[K]} \lambda_l^2 \sum_{l\in[K]} \lambda_l^2(P_{kl} - P_{k^*l})^2
\end{aligned} \tag{119}$$

Substituting (119) into (118), we have

$$\begin{aligned}\tilde{\beta}_k^2 &\geq \frac{1}{6}\frac{\min_{l\in[K]}\lambda_l^2\sum_{l\in[K]}\lambda_l^2(P_{kl}-P_{k^*l})^2}{(\sum_{l=1}^K\lambda_l^2P_{kl}^2)(\sum_{l=1}^K\lambda_l^2P_{k^*l}^2)}\\ \text{(By Condition (9))} &\geq \frac{1}{6C_1^4}\frac{\min_{l\in[K]}\lambda_l^2\sum_{l\in[K]}\lambda_l^2(P_{kl}-P_{k^*l})^2}{(\sum_{l=1}^K\lambda_l^2)^2} \qquad (120)\end{aligned}$$

On one hand, by Cauchy-Schwartz inequality,

$$\begin{aligned}\lambda_l^2 &= \|\theta_{\mathcal{L}}^{(l)}\|_1^2+\|\theta_{\mathcal{U}}^{(l)}\|_1^2\\ &\geq \frac{1}{2}(\|\theta_{\mathcal{L}}^{(l)}\|_1+\|\theta_{\mathcal{U}}^{(l)}\|_1)^2\\ &= \frac{1}{2}\|\theta^{(l)}\|_1^2 \qquad (121)\end{aligned}$$

So

$$\begin{aligned}\min_{l\in[K]}\lambda_l^2 &\geq \frac{1}{2}(\min_{l\in[K]}\|\theta^{(l)}\|_1)^2\\ \text{(By Condition (10))} &\geq \frac{1}{2}(\frac{1}{C_2}\max_{l\in[K]}\|\theta^{(l)}\|_1)^2\\ &\geq \frac{1}{2}(\frac{1}{KC_2}\|\theta\|_1)^2\\ &= \frac{1}{2K^2C_2^2}\|\theta\|_1^2 \qquad (122)\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{l=1}^K\lambda_l^2 &= \sum_{l=1}^K(\|\theta_{\mathcal{L}}^{(l)}\|_1^2+\|\theta_{\mathcal{U}}^{(l)}\|_1^2)\\ &\leq (\sum_{l=1}^K\|\theta_{\mathcal{L}}^{(l)}\|_1+\sum_{l=1}^K\|\theta_{\mathcal{U}}^{(l)}\|_1)^2\\ &= \|\theta\|_1^2 \qquad (123)\end{aligned}$$

Plugging (121), (122), (123) into (120), we obtain

$$\begin{aligned}\tilde{\beta}_k^2 &\geq \frac{1}{6C_1^4}\frac{\min_{l\in[K]}\lambda_l^2\sum_{l\in[K]}\lambda_l^2(P_{kl}-P_{k^*l})^2}{(\sum_{l=1}^K\lambda_l^2)^2}\\ &\geq \frac{1}{6C_1^4}\frac{\frac{1}{2K^2C_2^2}\|\theta\|_1^2\sum_{l\in[K]}\frac{1}{2}\|\theta^{(l)}\|_1^2(P_{kl}-P_{k^*l})^2}{(\|\theta\|_1^2)^2}\\ &= \frac{1}{24K^2C_1^4C_2^2}\frac{\sum_{l\in[K]}\|\theta^{(l)}\|_1^2(P_{kl}-P_{k^*l})^2}{\|\theta\|_1^2}\\ &\geq \frac{1}{24K^2C_1^4C_2^2}\frac{\min_{l\in[K]}\|\theta^{(l)}\|_1\sum_{l\in[K]}\|\theta^{(l)}\|_1(\sqrt{P_{kl}}-\sqrt{P_{k^*l}})^2}{\|\theta\|_1^2(\sqrt{P_{kl}}+\sqrt{P_{k^*l}})}\\ \text{(By (10), }\min_{k\neq\ell}\{P_{k\ell}\}\geq r_0) &\geq \frac{1}{24K^2C_1^4C_2^2}\frac{\frac{1}{C_2}\max_{l\in[K]}\|\theta^{(l)}\|_1\sum_{l\in[K]}\|\theta^{(l)}\|_1(\sqrt{P_{kl}}-\sqrt{P_{k^*l}})^2}{\|\theta\|_1^2 2\sqrt{r_0}}\\ &\geq \frac{1}{24K^2C_1^4C_2^2}\frac{\frac{1}{KC_2}\|\theta\|_1\sum_{l\in[K]}\|\theta^{(l)}\|_1(\sqrt{P_{kl}}-\sqrt{P_{k^*l}})^2}{\|\theta\|_1^2 2\sqrt{r_0}}\\ &= \frac{1}{48\sqrt{r_0}K^3C_1^4C_2^3}\frac{\sum_{l\in[K]}\|\theta^{(l)}\|_1(\sqrt{P_{kl}}-\sqrt{P_{k^*l}})^2}{\|\theta\|_1} \qquad (124)\end{aligned}$$

Substituting (124) into (113), we obtain

$$\mathbb{P}(\hat{y} \neq k^* | B_0, \pi^* = e_{k^*}) \leq \bar{C} \sum_{k=1}^{K} \exp\Big(-\frac{C_0}{c_0^2} \min\{1, \frac{1}{C_3}\} \tilde{\beta}_k^2 \theta^* \|\theta\|_1\Big)$$
$$\leq \bar{C} \sum_{k=1}^{K} \exp\Big(-C_8 \theta^* \sum_{l=1}^{K} \|\theta^{(l)}\|_1 (\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2\Big) \tag{125}$$

where $C_8 = \frac{C_0}{c_0^2} \min\{1, \frac{1}{C_3}\} \frac{1}{48\sqrt{r_0} K^3 C_1^4 C_2^3} = \min\{1, \frac{1}{C_3}\} \frac{C_0}{48\sqrt{r_0} K^3 c_0^2 C_1^4 C_2^3}$.

As a result,

$$\frac{1}{\bar{C}K^2} \mathrm{Risk}(\hat{y}|B_0) = \frac{1}{\bar{C}K^2} \sum_{k*=1}^{K} \mathbb{P}(\hat{y} \neq k^* | B_0, \pi^* = e_{k^*})$$
$$\leq \frac{1}{\bar{C}K^2} \sum_{k*=1}^{K} \bar{C} \sum_{k=1}^{K} \exp\Big(-C_8 \theta^* \sum_{l=1}^{K} \|\theta^{(l)}\|_1 (\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2\Big)$$
$$= \frac{1}{K^2} \sum_{k*=1}^{K} \sum_{k=1}^{K} \exp\Big(-C_8 \theta^* \sum_{l=1}^{K} \|\theta^{(l)}\|_1 (\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2\Big)$$
$$\leq \max_{k \neq k^* \in [K]} \exp\Big(-C_8 \theta^* \sum_{l=1}^{K} \|\theta^{(l)}\|_1 (\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2\Big)$$
$$= \exp\Big(-C_8 \theta^* \min_{k \neq k^* \in [K]} \sum_{l=1}^{K} \|\theta^{(l)}\|_1 (\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2\Big) \tag{126}$$

Comparing (112) and (126), we have

$$\frac{-\log(\frac{1}{2\bar{C}K^2} \mathrm{Risk}(\hat{y}|B_0))}{-\log(\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\})} \geq \frac{\log(2) + C_8 \mathcal{I}}{\log(2) + 2\sqrt{2}\mathcal{I}} \tag{127}$$

where $\mathcal{I} = \theta^* \min_{k \neq k^* \in [K]} \sum_{l=1}^{K} \|\theta^{(l)}\|_1 (\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2$ denotes the efficient information in the data.

Notice that since $\mathcal{I} \geq 0$, when $C_8 \geq 2\sqrt{2}$, $\frac{\log(2)+C_8\mathcal{I}}{\log(2)+2\sqrt{2}\mathcal{I}} \geq 1$; when $C_8 \leq 2\sqrt{2}$, $\frac{\log(2)+C_8\mathcal{I}}{\log(2)+2\sqrt{2}\mathcal{I}} \geq \frac{\log(2)\frac{C_8}{2\sqrt{2}}+C_8\mathcal{I}}{\log(2)+2\sqrt{2}\mathcal{I}} = \frac{C_8}{2\sqrt{2}}$. Therefore,

$$\frac{-\log(\frac{1}{2\bar{C}K^2} \mathrm{Risk}(\hat{y}|B_0))}{-\log(\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\})} \geq \frac{\log(2) + C_8 \mathcal{I}}{\log(2) + 2\sqrt{2}\mathcal{I}} \geq \min\{1, \frac{C_8}{2\sqrt{2}}\} \tag{128}$$

Take $\tilde{c}_2 = \min\{1, \frac{C_8}{2\sqrt{2}}, \frac{1}{2\bar{C}K^2}\}$. Then $\tilde{c}_2$ only depends on $K, C_1, C_2, C_3, c_3, r_0$ (recall that both C_0 and c_0 only depend on $K, C_1, C_2, C_3, c_3, r_0$, and $\bar{C} = 2$), and

$$\frac{-\log(\tilde{c}_2 \mathrm{Risk}(\hat{y}|B_0))}{-\log(\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\})} \geq \frac{-\log(\frac{1}{2\bar{C}K^2} \mathrm{Risk}(\hat{y}|B_0))}{-\log(\inf_{\tilde{y}}\{\mathrm{Risk}(\tilde{y})\})} \geq \min\{1, \frac{C_8}{2\sqrt{2}}\} \geq \tilde{c}_2 \tag{129}$$

This concludes our proof. □

I PROOF OF THEOREM 4, 5

I.1 PROOF OF THEOREM 4

Theorem 4. *Consider the DCBM model where (9)-(10) hold. We apply SCORE+ to obtain $\hat{\Pi}_{\mathcal{U}\setminus\{i\}}$ and plug it into the above algorithm. As $n \to \infty$, suppose for some constant $q_0 > 0$, $\min_{i \in \mathcal{U}} \theta_i \geq q_0 \max_{i \in \mathcal{U}} \theta_i$, $\beta_n \|\theta_{\mathcal{U}}\| \geq q_0 \sqrt{\log(n)}$, $\beta_n^2 \|\theta\|_1 \min_{i \in \mathcal{U}} \theta_i \to \infty$, and $\beta_n^2 \|\theta\|_1 \min_k \{\|\theta_{\mathcal{L}}^{(k)}\|_1\} \to \infty$. Then, $\frac{1}{|\mathcal{U}|} \sum_{i \in \mathcal{U}} \mathbb{P}(\hat{y}_i \neq k_i) \to 0$, so the in-sample classification algorithm in section 4 is consistent.*

Proof. Let $i^* = arg\max_{i\in\mathcal{U}} \mathbb{P}(\hat{y}_i \neq k_i)$. Then

$$\frac{1}{|\mathcal{U}|}\sum_{i\in\mathcal{U}} \mathbb{P}(\hat{y}_i \neq k_i) \leq \mathbb{P}(\hat{y}_{i^*} \neq k_{i^*})$$

Notice that the assumptions of theorem 4 directly imply the assumptions of Corollary 2 when taking i^* as the new node. Hence, regard i^* as the new node and leveraging on Corollary 2, we have

$$\mathbb{P}(\hat{y}_{i^*} \neq k_{i^*}) \to 0$$

Therefore,

$$\frac{1}{|\mathcal{U}|}\sum_{i\in\mathcal{U}} \mathbb{P}(\hat{y}_i \neq k_i) \to 0$$

In other words, the in-sample classification algorithm in section 4 is consistent. □

I.2 Proof of Theorem 5

Theorem 5. *Suppose the conditions of Corollary 1 hold, where b_0 is properly small , and suppose that $\hat{\Pi}_{\mathcal{U}\setminus\{i\}}$ is b_0-correct for all $i \in \mathcal{U}$. Furthermore, we assume for sufficiently large constant C_3, $\max_{i\in\mathcal{U}} \theta_i \leq \frac{1}{C_3}$, $\max_{i\in\mathcal{U}} \theta_i \leq \min_{k\in[K]} C_3\|\theta_{\mathcal{L}}^{(k)}\|_1$, $\log(|\mathcal{U}|) \leq C_3\beta_n^2\|\theta\|_1 \min_{i\in\mathcal{U}} \theta_i$, and for a constant $r_0 > 0$, $\min_{k\neq\ell}\{P_{k\ell}\} \geq r_0$. Then, there is a constant $\tilde{c}_{21} = \tilde{c}_{21}(K, C_1, C_2, C_3, c_3, r_0) > 0$ such that $[-\log(\tilde{c}_{21}\mathrm{Risk}_{ins}(\hat{y}))]/[-\log(\inf_{\tilde{y}}\{\mathrm{Risk}_{ins}(\tilde{y})\})] \geq \tilde{c}_{21}$, so the in-sample classification algorithm in section 4 is efficient.*

Proof. For $i \in \mathcal{U}$, define individual risk $\mathrm{Risk}(\tilde{y}_i) = \sum_{k^*\in[K]} \mathbb{P}(\tilde{y}_i \neq k^*|\pi_i = e_{k^*})$. Then the in-sample risk $\mathrm{Risk}_{ins}(\tilde{y}) = \frac{1}{|\mathcal{U}|}\sum_{i\in\mathcal{U}} \mathrm{Risk}(\hat{y}_i)$.

The minimizer of $\mathrm{Risk}_{ins}(\tilde{y})$ may not exist, so we define $\tilde{y}^{(0)}$ to be an approximate minimizer such that $\mathrm{Risk}_{ins}(\tilde{y}^{(0)}) \leq 2\inf_{\tilde{y}}\{\mathrm{Risk}_{ins}(\tilde{y})\}$. By the definition of infimum, such $\tilde{y}^{(0)}$ always exists as long as $\inf_{\tilde{y}}\{\mathrm{Risk}_{ins}(\tilde{y})\} > 0$. Notice that for any $i \in \mathcal{U}$, $\inf_{\tilde{y}}\{\mathrm{Risk}_{ins}(\tilde{y})\} \geq \inf_{\tilde{y}_i} \frac{1}{|\mathcal{U}|}\{\mathrm{Risk}(\tilde{y}_i)\}$. Regarding node i as the new node and leveraging on (112), we know that $\inf_{\tilde{y}_i}\{\mathrm{Risk}(\tilde{y}_i)\} > 0$. Hence, $\inf_{\tilde{y}}\{\mathrm{Risk}_{ins}(\tilde{y})\} \geq \inf_{\tilde{y}_i} \frac{1}{|\mathcal{U}|}\{\mathrm{Risk}(\tilde{y}_i)\} > 0$ (note that we are not taking n or $|\mathcal{U}| \to \infty$ here), and $\tilde{y}^{(0)}$ is well-defined.

Let $i^* = arg\max_{i\in\mathcal{U}} \mathrm{Risk}(\hat{y}_i)$, and let k^* be the true label of i^*. Regard $[n]\backslash\{i^*\}$ as the existing nodes in the network and i^* as the new node. By (112) and (126), we have

$$-\log(\frac{1}{2\bar{C}K^2}\mathrm{Risk}(\hat{y}_{i^*})) \geq \log(2) + C_8\mathcal{I}_{i^*} \tag{130}$$

$$-\log(\{\mathrm{Risk}(\tilde{y}_{i^*}^{(0)})\}) \leq -\inf_{\tilde{y}_{i^*}} \log(\{\mathrm{Risk}(\tilde{y}_{i^*})\}) \leq \log(2) + 2\sqrt{2}\mathcal{I}_{i^*} \tag{131}$$

where $\mathcal{I}_{i^*} = \theta_{i^*} \min_{k\neq k^*\in[K]}(\sum_{l=1}^K \|\theta^{(l)}\|_1(\sqrt{P_{kl}} - \sqrt{P_{k^*l}})^2 - \theta_{i^*}(1 - \sqrt{P_{kk^*}})^2)$ denotes the efficient information in the data for classifying node i^*.

As a result, we have

$$\begin{aligned}
\frac{-\log(\frac{1}{4\bar{C}K^2}\text{Risk}_{ins}(\hat{y}))}{-\log(\inf_{\tilde{y}}\{\text{Risk}_{ins}(\tilde{y})\})} &\geq \frac{-\log(\frac{1}{4\bar{C}K^2}\text{Risk}_{ins}(\hat{y}))}{-\log(\frac{1}{2}\text{Risk}_{ins}(\tilde{y}^{(0)}))} \\
&= \frac{-\log(\frac{1}{4\bar{C}K^2}\frac{1}{|\mathcal{U}|}\sum_{i\in\mathcal{U}}\text{Risk}(\hat{y}_i))}{-\log(\frac{1}{2|\mathcal{U}|}\sum_{i\in\mathcal{U}}\text{Risk}(\tilde{y})_i^{(0)})} \\
&\geq \frac{-\log(\frac{1}{4\bar{C}K^2}\text{Risk}(\hat{y}_{i^*}))}{-\log(\frac{1}{|2\mathcal{U}|}\text{Risk}(\tilde{y})_{i^*}^{(0)})} \\
(By\ (130),(131)) &\geq \frac{\log(4)+C_8\mathcal{I}_{i^*}}{\log(4)+2\sqrt{2}\mathcal{I}_{i^*}+\log(|\mathcal{U}|)} \qquad (132)
\end{aligned}$$

Notice that

$$\begin{aligned}
&\mathcal{I}_{i^*} \\
&= \theta_{i^*}\min_{k\neq k^*\in[K]}(\sum_{l=1}^{K}\|\theta^{(l)}\|_1(\sqrt{P_{kl}}-\sqrt{P_{k^*l}})^2-\theta_{i^*}(1-\sqrt{P_{kk^*}})^2) \\
&\geq \theta_{i^*}\min_{k\neq k^*\in[K]}(\|\theta^{(k)}\|_1(\sqrt{P_{kk}}-\sqrt{P_{k^*k}})^2+\|\theta^{(k^*)}\|_1(\sqrt{P_{kk^*}}-\sqrt{P_{k^*k^*}})^2-\theta_{i^*}(1-\sqrt{P_{kk^*}})^2) \\
&\text{(By identification condition that } P_{kk}=P_{k^*k^*}=1) \\
&= \theta_{i^*}\min_{k\neq k^*\in[K]}((\|\theta^{(k)}\|_1+\|\theta^{(k^*)}\|_1-\theta_{i^*})(1-\sqrt{P_{k^*k}})^2) \\
&\text{(The true label of node } i^* \text{ is } k^*) \\
&\geq \theta_{i^*}\min_{k\neq k^*\in[K]}(\|\theta^{(k)}\|_1(1-\sqrt{P_{k^*k}})^2) \qquad (133)
\end{aligned}$$

By assumption 9, for any $k\in[K]$,

$$\begin{aligned}
(1-\sqrt{P_{k^*k}})^2 &= \frac{(1-P_{k^*k})^2}{(1+\sqrt{P_{k^*k}})^2} \\
&= \frac{(2-2P_{k^*k})^2}{4(1+\sqrt{P_{k^*k}})^2} \\
&\text{(By identification condition that } P_{kk}=P_{k^*k^*}=1) \\
&= \frac{(P_{kk}+Pk^*k^*-2P_{k^*k})^2}{4(1+\sqrt{P_{k^*k}})^2} \\
&= \frac{((e_k-e_{k^*})'P(e_k-e_{k^*}))^2}{4(1+\sqrt{P_{k^*k}})^2} \\
&\geq \frac{(|\lambda_{min}(P)|\|e_k-e_{k^*}\|^2)^2}{4(1+\sqrt{C_1})^2} \\
&\geq \frac{\beta_n^2}{(1+\sqrt{C_1})^2} \qquad (134)
\end{aligned}$$

By assumption 10, for any $k\in[K]$

$$\begin{aligned}
\|\theta^{(k)}\|_1 &\geq \min_{k\neq k^*\in[K]}\|\theta^{(k)}\|_1 \\
&\geq \frac{1}{C_2}\max_{k\neq k^*\in[K]}\|\theta^{(k)}\|_1 \\
&\geq \frac{1}{KC_2}\sum_{k\neq k^*\in[K]}\|\theta^{(k)}\|_1 \\
&= \frac{1}{KC_2}\|\theta\|_1 \qquad (135)
\end{aligned}$$

From the assumption, $\log(|\mathcal{U}|) \leq C_3\beta_n^2\|\theta\|_1 \min_{i\in\mathcal{U}} \theta_i$. Plugging (134) (135) into (133), we obtain

$$\begin{aligned}\mathcal{I}_{i^*} &\geq \theta_{i^*} \min_{k\neq k^*\in[K]} (\frac{1}{KC_2(1+\sqrt{C_1})^2}\beta_n^2\|\theta\|_1) \\ &\geq \frac{1}{KC_2(1+\sqrt{C_1})^2}\beta_n^2\|\theta\|_1 \min_{i\in\mathcal{U}} \theta_i \\ &\geq \frac{1}{KC_2C_3(1+\sqrt{C_1})^2} log(|\mathcal{U}|) \end{aligned} \tag{136}$$

In other words,

$$log(|\mathcal{U}|) \leq KC_2C_3(1+\sqrt{C_1})^2\mathcal{I}_{i^*} \tag{137}$$

Substituting (137) into (132), we have

$$\frac{-\log(\frac{1}{4\bar{C}K^2}\text{Risk}_{ins}(\hat{y}))}{-\log(\inf_{\tilde{y}}\{\text{Risk}_{ins}(\tilde{y})\})} \geq \frac{\log(4)+C_8\mathcal{I}_{i^*}}{\log(4)+(2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2)\mathcal{I}_{i^*}} \tag{138}$$

Similar to the proof of Theorem 3, notice that since $\mathcal{I}_{i^*} \geq 0$, when $C_8 \geq 2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2$,

$$\frac{\log(4)+C_8\mathcal{I}_{i^*}}{\log(4)+(2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2)\mathcal{I}_{i^*}} \geq 1 \tag{139}$$

; when $C_8 \leq 2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2$,

$$\begin{aligned}\frac{\log(4)+C_8\mathcal{I}_{i^*}}{\log(4)+(2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2)\mathcal{I}_{i^*}} &\geq \frac{\log(4)\frac{C_8}{2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2}+C_8\mathcal{I}_{i^*}}{\log(4)+(2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2)\mathcal{I}_{i^*}} \\ &= \frac{C_8}{2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2} \end{aligned} \tag{140}$$

Therefore,

$$\begin{aligned}\frac{-\log(\frac{1}{4\bar{C}K^2}\text{Risk}_{ins}(\hat{y}))}{-\log(\inf_{\tilde{y}}\{\text{Risk}_{ins}(\tilde{y})\})} &\geq \frac{\log(4)+C_8\mathcal{I}_{i^*}}{\log(4)+(2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2)\mathcal{I}_{i^*}} \\ &\geq \min\{1, \frac{C_8}{2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2}\} \end{aligned} \tag{141}$$

Take $\tilde{c}_{21} = \min\{1, \frac{C_8}{2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2}, \frac{1}{4\bar{C}K^2}\}$. Then $\tilde{c}_{21}$ only depends on $K, C_1, C_2, C_3, c_3, r_0$ (recall that both C_0 and c_0 only depend on $K, C_1, C_2, C_3, c_3, r_0$, and $\bar{C} = 2$), and

$$\begin{aligned}\frac{-\log(\tilde{c}_{21}\text{Risk}_{ins}(\hat{y}))}{-\log(\inf_{\tilde{y}}\{\text{Risk}_{ins}(\tilde{y})\})} &\geq \frac{-\log(\frac{1}{4\bar{C}K^2}\text{Risk}_{ins}(\hat{y}))}{-\log(\inf_{\tilde{y}}\{\text{Risk}_{ins}(\tilde{y})\})} \\ &\geq \min\{1, \frac{C_8}{2\sqrt{2}+KC_2C_3(1+\sqrt{C_1})^2}\} \\ &\geq \tilde{c}_{21} \end{aligned} \tag{142}$$

This concludes our proof. □