Flows and Diffusions on the Neural Manifold

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Abstract

Diffusion and flow-based generative models have achieved remarkable success in domains such as image synthesis, video generation, and natural language modeling. In this work, we extend these advances to *weight space learning* by leveraging recent techniques to incorporate structural priors derived from optimization dynamics. Central to our approach is modeling the trajectory induced by gradient descent as a trajectory inference problem. We unify several trajectory inference techniques towards matching a gradient flow, providing a theoretical framework for treating optimization paths as inductive bias. We further explore architectural and algorithmic choices, including reward fine-tuning by adjoint matching, the use of autoencoders for latent weight representation, conditioning on task-specific context data, and adopting informative source distributions such as Kaiming uniform. Experiments demonstrate that our method matches or surpasses baselines in generating in-distribution weights, improves initialization for downstream training, and supports fine-tuning to enhance performance. Finally, we illustrate a practical application in safety-critical systems: detecting harmful covariate shifts, where our method outperforms the closest comparable baseline.

1 Introduction

Flow matching (FM) (Albergo and Vanden-Eijnden, 2023; Lipman et al., 2023; Liu et al., 2023) is a prominent fixture in generative modeling tasks from imaging (Lipman et al., 2023; Tong et al., 2024; Esser et al., 2024; Liu et al., 2024a) to language (Gat et al., 2024; Shaul et al., 2024; Campbell et al., 2024). However, its application to neural network weights has not been explored. By leveraging the principled, yet versatile training of FM, we aim to generate task-specific weights on novel tasks.

The natural question is: why generate task-specific weights instead of relying on conventional training methods? One compelling reason is efficiency. If we can train a meta-model to produce classifiers conditioned only on the evaluation dataset, then generating weights reduces to a single inference pass of our flow or diffusion model. This motivation parallels recent work in zero- and few-shot learning (Zhang et al., 2024; Soro et al., 2025), where generalization to new tasks is achieved with minimal or no training. Further on efficiency, conditionally generated weights can also serve as a strong, head-start initialization for downstream fine-tuning, which we later evaluate on corrupted datasets. This approach is especially practical when training a large number of smaller networks, such as in applications involving implicit neural representations (Essakine et al., 2025). Finally, we argue that learning to generate neural network weights opens a new perspective: it allows us to reinterpret diverse problems as questions on weight space. We illustrate this view in Section 4 through an application to detecting harmful covariate shift.

In this paper, we introduce flow matching as a new class of methods for generating neural network weights, designed to incorporate structural priors such as training trajectories and source distributions. Under this framework, we may cast our goal as one of *trajectory inference* (Lavenant et al., 2024), reconstructing the continuous-time dynamics $t \mapsto p_t^*$ given easy sampling from the marginal distributions $(p_{t_k}^*)_{k=0}^K$. In practice, temporal observations are sparse, necessitating methods that can sensibly interpolate between observed timepoints, often leveraging biases in data. Indeed, we further ground our approach in the *manifold hypothesis* (Bengio et al., 2013), which posits that natural data lies on a low-dimensional submanifold of the ambient

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space, representing a bias that may be incorporated as a prior. Drawing on the Lottery Ticket Hypothesis (Frankle and Carbin, 2019; Zhang et al., 2021; Liu et al., 2024b) as well as the body of work on pruning (Cheng et al., 2024), we extend this intuition to weight space: neural networks themselves tend to lie on a low-dimensional structure, which we refer to as the *neural manifold* (N$\mathcal{M}$). We will make use of these observations to motivate the various experiments conducted in Section 4.

In this work, we make preliminary steps toward understanding flows and diffusions on the neural manifold. Our contributions include: **1)** we unify and prove characterizations of various methods to approximate a gradient descent trajectory, enabling more accurate modeling of our priors; **2)** we incorporate theoretical considerations to design flow- and diffusion-based approaches for generating weights that match or exceed conventionally trained models on in-distribution tasks, provide better initializations for downstream training, and allows for conditioning on context data to retrieve pre-trained weights from a distribution pre-trained on various datasets; **3)** we incorporate a fine-tuning mechanism, grounded in adjoint matching (Domingo-Enrich et al., 2025), to enhance performance, and **4)** we show how this can be used to detect harmful covariate shifts that outperforms the closest comparable baseline, supporting our motivation to reinterpret problems as questions on weight space.

2 Preliminaries

Conditional flow models. Chen et al. (2019) first introduced continuous normalizing flows as an effective data generation process through modeling dynamics. Simulation-free methods improve on this concept by simplifying the training objective (Albergo and Vanden-Eijnden, 2023; Lipman et al., 2023; Liu et al., 2023). Following the formulation of Lipman et al. (2023), given random variables $\bar{\mathbf{x}}_0 \sim p_0$ and $\bar{\mathbf{x}}_1 \sim p_1$ a data distribution, define a reference flow $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_t)_{t\in[0,1]}$ where $\bar{\mathbf{x}}_t = \beta_t \bar{\mathbf{x}}_0 + \alpha_t \bar{\mathbf{x}}_1$ with the constraint that $\alpha_0 = \beta_1 = 0$ and $\alpha_1 = \beta_0 = 1$. The aim of flow modeling is to learn a sequence $\mathbf{x} = (\mathbf{x}_t)_{t\in[0,1]}$ which has the same marginal distribution as $\bar{\mathbf{x}}$. To make this a feasible task, we describe this process as an ODE: $d\mathbf{x}_t = v(\mathbf{x}_t, t)dt$ where $\mathbf{x}_0 \sim \mathcal{N}(0, \boldsymbol{I})$. Training proceeds by first parameterizing $v(\mathbf{x}_t, t)$ by a neural network θ and matching the reference flow velocity, i.e. $u(\mathbf{x}_t, t) := \frac{d}{dt}\bar{\mathbf{x}}_t$. This would, however, be an unfeasible training objective, therefore, we condition on samples from the distribution $\mathbf{x}_1 \sim p_1$ and train

$$\mathcal{L}_{\text{cfm}}(\theta) = \mathbb{E}_{\text{t}\sim U[0,1], \mathbf{x}_1 \sim p_1, \mathbf{x}_t \sim p_t(\cdot|\mathbf{x}_1)} ||v_\theta(\mathbf{x}_t, t) - u(\mathbf{x}_t, t \mid \mathbf{x}_1)||. \tag{1}$$

Lipman et al. (2023) proved that this loss produces the same gradients as the marginal loss, thus optimizing it will result in convergence to the reference $u(\mathbf{x}_t, t)$. Moreover, we can always marginalize an independent conditioning variable $\boldsymbol{y}$ on v_θ, u – this will serve as our context conditioning vector.

Modeling probability paths. The straight line paths advocated in the conventional flow matching framework (Lipman et al., 2023) can be contextualized in the broader framework of matching interpolants $(\mathbf{x}_t)_{t\in[0,1]}$ to minimize a given energy function. In the conventional case, the kinetic energy $\mathcal{E}(\mathbf{x}_t, \dot{\mathbf{x}}_t) = \mathbb{E}_{\text{t}\sim U[0,1]} ||\dot{\mathbf{x}}_t||^2$ is minimized (Shaul et al., 2023), however, more general energies have been considered (Neklyudov et al., 2023; 2024; Kapusniak et al., 2024; Liu et al., 2024c) to broaden the class of learnable paths. Of interest to us is the concept of finding interpolants that depend on prior knowledge as provided through samples. For instance, Metric Flow Matching (Kapusniak et al., 2024) learns parametric interpolants that minimize

$$\mathbb{E}_{(\mathbf{x}_0, \mathbf{x}_1)\sim\pi} \mathcal{E}_g(\mathbf{x}_t) = \mathbb{E}_{\text{t}\sim U[0,1], (\mathbf{x}_0, \mathbf{x}_1)\sim\pi} ||\dot{\mathbf{x}}_t||^2_{g(\mathbf{x}_t)}, \tag{2}$$

where π is some distribution on the product, typically $p_0 \otimes p_1$, and g is a data-dependent Riemannian metric on the ambient space $\mathbb{R}^d$. Alternatively, Rohbeck et al. (2025) proposed a more stable option using cubic splines for multi-marginal flow matching. Most related to our setting is the task of modeling distributions evolved via gradient flow: $\frac{d}{dt}p_t = -\nabla \cdot (p_t \nabla s_t)$. For instance, TrajectoryNet (Tong et al., 2020) is an older work that uses continuous normalizing flows to model cellular trajectories under a spatial potential. More recently, JKOnet and JKOnet* (Bunne et al., 2022; Terpin et al., 2024) exploited the connection between diffusion processes and energy-minimizing probability paths (in Wasserstein space) to model a Wasserstein gradient flow by learning the drift potential $V_\theta(\cdot, t)$ in a diffusion process.

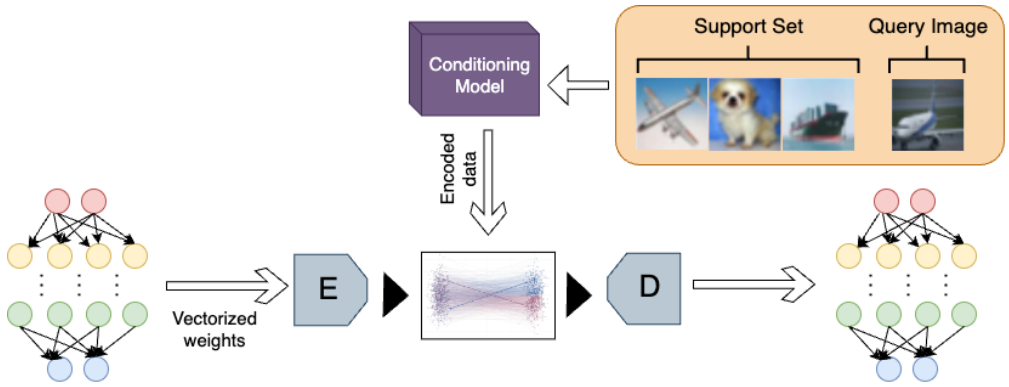

Figure 1: A schematic of the training process of N$\mathcal{M}$-CFM w/ VAE for conditional generation. Given a set of pre-trained target weights and a support set of images, we apply the conditioned flow model to pushforward a sample of the latent prior towards encoded target weights. The decoder is used during inference where we start from a sample of the latent prior and pushforward towards the target distribution with a trained vector field $v_\theta(\cdot, t; \boldsymbol{y})$ where $\boldsymbol{y}$ is the support set embedding.

Harmful covariate shift detection. Covariate shifts refers to changes in the test data distribution $p_{\text{test}}(x)$ as compared to the training distribution $p_{\text{train}}(x)$ while the relation between inputs and outputs remain fixed, i.e. $p_{\text{test}}(y|x) = p_{\text{train}}(y|x)$. Importantly, we do not require labels to determine this shift, thus it is practical to do so in a standard deployment setting. Prior work in this domain include deep kernel MMD (Liu et al., 2020), H-divergences (Zhao et al., 2022), and Detectron (Ginsberg et al., 2023). As Detectron requires minimal tuning and is most performant in low-data regimes ($N < 100$ samples), we emphasize the use of this approach. In particular, Detectron (Ginsberg et al., 2023) builds off of *selective classification*—building classifiers that accept/reject test data depending on closeness to the training distribution—and *PQ-learning* (Goldwasser et al., 2020) that extends the conventional theory of PAC learning to arbitrary test distributions by employing selective classification. The main idea considers the generalization set $\mathcal{R}$ of a classifier f_θ and samples $\mathcal{Q}$ from an unknown distribution. The strategy is to fine-tune constrained disagreement classifiers (CDCs) to agree with f_θ on $\mathcal{R}$ but disagree on $\mathcal{Q}$. If $\mathcal{Q} \subset \mathcal{R}$, then it will be difficult to disagree on $\mathcal{Q}$, but if the CDCs behave inconsistently on $\mathcal{Q}$, that suggests a covariate shift. Notably, this method is sample-efficient, agnostic to classifier architecture, and may be used in tests of statistical significance.

3 Methods

In this section, we build towards our approach. Proofs are provided in App. A.

3.1 Flow matching on neural network parameters

For the purpose of our analysis, let us restrict our view to neural networks that are optimized by gradient descent (GD) algorithms to minimize a loss $\mathcal{L}(\theta_k) := d(\mathcal{M}_\theta(\boldsymbol{X}) - \boldsymbol{Y})$, where $\mathcal{M}_\theta$ is a neural network parameterized by trainable weights $\theta \in \mathbb{R}^p$, $\boldsymbol{X} \in \mathbb{R}^{N \times D}$ are inputs, $\boldsymbol{Y} \in \mathbb{R}^{N \times c}$ are labels, and d is some differentiable distance function, such as cross-entropy. To minimize via GD, parameter updates are done by $\theta_{k+1} = \theta_k - \alpha \nabla \mathcal{L}(\theta_k)$ given some learning rate $\alpha > 0$. Taking the learning rate to zero, we can view parameter evolution as a *gradient flow*. For simplicity, we assume that the updates are deterministic (contrary to stochastic gradient descent which randomly selects training batches), and defer to App. D an approach incorporating stochasticity via Schrödinger bridges. For now, the sole source of randomness is the initialization $\theta_0 \sim p_0$. Within this framework, we can write down a continuity equation.

Theorem 1 (Informal)**.** *Let $\theta_0 \sim p_0$ be the initialized network parameters, and $\Omega \subset \mathbb{R}^p$ a compact subset. Supposing that $\mathcal{L} : \Omega \to \mathbb{R}$ is C^1 in the network parameters and the updates $\theta_t \in \Omega$, we have $p_t = \text{Law}(\theta_t)$*

with

$$\partial_t p_t - \nabla \cdot (p_t \nabla \mathcal{L}) = 0. \tag{3}$$

Proof. The idea is to view GD as an iterated minimization scheme on $\mathcal{P}(\Omega)$ with functional $\mathcal{F}(p_t) = \int_\Omega \mathcal{L}(x)\, dp_t(x)$. Alternatively, this may be viewed as the necessary first-order optimality conditions of a JKO scheme (Lanzetti et al., 2024; Terpin et al., 2024). We provide the full proof in App. A. □

Lipman et al. (2023) formulated flow matching as learning a vector field $v_t : \mathbb{R}^p \to \mathbb{R}^p$ to generate a probability path p_t according to $\partial_t p_t + \nabla \cdot (p_t v_t) = 0$. Theorem 1 suggests that the ideal vector field is precisely $v_t(\theta) = -\nabla \mathcal{L}(\theta)$. Moreover, we have a few natural choices for p_0; see App. B for a brief remark on the effect on p_t.

3.2 Approximating Eqn. 3

The problem of learning the continuous dynamics of a system governed by a continuity equation has been studied in many forms in existing literature. In this section, we discuss some of these methods and unify them towards approximating Eqn. 3, but is also of independent interest.

3.2.1 Matching gradients

Frameworks which match gradients have a learnable function $\Psi_\theta(x, t)$ which is trained to match a regression target. Two representatives of this approach are the works on *action matching* (Neklyudov et al., 2023; 2024) and the JKOnet family (Bunne et al., 2022; Terpin et al., 2024).

Action matching. The action matching setup presumes an initial distribution q_0, a velocity field $v : [0, 1] \times \Omega \to \mathbb{R}^p$, and a continuity equation which describes the dynamics: $\frac{d}{dt} q_t = -\nabla \cdot (q_t v_t)$. Neklyudov et al. (2023, Theorem 2.1) showed that, under mild conditions on q_t, a unique function $s_t^*(x)$ termed the *action* may be defined such that $v_t(x) = \nabla s_t^*(x)$ and the continuity equation $\frac{d}{dt} q_t = -\nabla \cdot (q_t \nabla s_t^*)$ is satisfied. One can readily see the connection with Eqn. 3: $s_t^* \equiv \mathcal{L}$ up to a constant. Therefore, the *action gap* is

$$AG(s, s^*) = \frac{1}{2} \int_0^1 \mathbb{E}_{q_t(x)} ||\nabla s_t(x) - \nabla s_t^*(x)||^2 \, dt = \frac{1}{2} \int_0^1 \mathbb{E}_{q_t(x)} ||\nabla s_t(x) - \nabla \mathcal{L}(x)||^2 \, dt. \tag{4}$$

This optimization is clearly intractable because of the required access to $\nabla \mathcal{L}$, therefore the authors computed a more tractable variational objective for optimization (Neklyudov et al., 2023, Theorem 2.2.). To close this exposition, we recall a bound on the 2-Wasserstein distance which will be a recurring theme in this section.

Proposition 1 (Prop. A.1 of Neklyudov et al. (2023))**.** *Suppose the curl-free vector field ∇s_t is continuously differentiable in (t, x), and uniformly Lipschitz in x throughout $[0, 1] \times \mathbb{R}^p$ with Lipschitz constant K. Let $\hat{q}_t$ denote the density path induced by ∇s_t. Then,*

$$W_2^2(\hat{q}_\tau, q_\tau) \leq e^{(1+2K)\tau} \int_0^\tau ||\nabla s_t(x) - \nabla \mathcal{L}(x)||^2 \, dt. \tag{5}$$

JKOnet. The JKOnet family considers the problem of modeling the Fokker-Planck equation

$$\partial_t p_t(x) = \nabla \cdot (p_t(x) \nabla V(x)) + \beta \nabla^2 p_t(x), \tag{6}$$

given some potential V. The seminal work Jordan et al. (1998) related such equations to a variational objective in Wasserstein space, namely

$$\mu_{t+1} = \operatorname*{arg\,min}_{\mu \in \mathcal{P}(\mathbb{R}^p)} J(\mu) + \frac{1}{2\tau} W_2^2(\mu, \mu_t) \quad \text{where} \quad J(\mu) := \int_{\mathbb{R}^p} V(x) p_t(x) \, dx + \beta \int_{\mathbb{R}^p} p_t(x) \log(p_t(x)) \, dx, \tag{7}$$

and step size $\tau > 0$. Focusing on the most recent presentation, termed JKOnet* (Terpin et al., 2024), consider the Euclidean analog of Eqn. 7 and its first-order optimality condition: $\nabla J(x_{t+1}) + (x_{t+1} - x_t)/\tau = 0$, where

$x_t \sim \mu_t$. If we let $\beta = 0$, then by Terpin et al. (2024, Prop. 3.1), we have the minimization objective

$$\int_{\mathbb{R}^p \times \mathbb{R}^p} \left|\left| \nabla V(x_{t+1}) + \frac{1}{\tau}(x_{t+1} - x_t) \right|\right|^2 d\pi_t(x_t, x_{t+1}) = 0, \tag{8}$$

where π_t is the optimal coupling between μ_t and μ_{t+1}. It can be readily seen that when $\beta = 0$, Eqn. 6 matches the form of Eqn. 3, and the requisite setup for action matching; in particular, the bound Eqn. 5 applies. Moreover, the objective Eqn. 8 is an intuitive description of matching the best linear approximation of the gradient $\nabla\mathcal{L}$. Motivated by its simplicity, we opt towards JKOnet* as the representative approach.

3.2.2 Multi-marginal flow matching

The multi-marginal flow matching (MMFM) objective is similar to that of CFM in Eqn. 3. The difference lies in the definition of the regression target. Suppose we wish to generate marginal densities $p^*_{t_0}, p^*_{t_1}, \ldots, p^*_{t_K}$. Instead of sampling $x \sim p_1$, we sample $z = (x_0, \ldots, x_K)$ independently from each of the marginal densities. To align with the CFM objective, the reference path $p_t(x|z)$ must be a piecewise affine Gaussian path, and its mean a linear interpolation between the $K + 1$ samples. Formally,

$$\mu_t(z) = \sum_{k=0}^{K-1} \left(x_k + \frac{t - t_k}{t_{k+1} - t_k}(x_{k+1} - x_k) \right) \cdot \mathbf{1}_{[t_k, t_{k+1})}(t) \tag{9}$$

and the regression target ought to be

$$u_t(x \mid z) = \sum_{k=0}^{K-1} \frac{x_{k+1} - x_k}{t_{k+1} - t_k} \cdot \mathbf{1}_{[t_k, t_{k+1})}(t). \tag{10}$$

With the usual marginalization of u_t, i.e. $u_t(x)p_t(x) = \mathbb{E}_{q(z)}[u_t(x \mid z)p_t(x \mid z)]$, we can argue by checking the continuity equation Tong et al. (2024, Theorem 3.1) that p_t is generated by u_t.

In addition, it is natural to think that if the timepoints $(t_0, \ldots, t_K)$ were dense enough, then its limit is the true probability path $t \mapsto p^*_t$. Rohbeck et al. (2025, Proposition 2) proved that the MMFM objective is equivalent to solving K CFM objectives. By analogy, we consider a MMFM setup equivalent to K OT-CFM objectives, and we show that the reference path p_t approaches p^*_t in the sense of the action gap (cf. Prop. 1).

Theorem 2. *Suppose the true marginals evolve according to $\frac{d}{dt}p^*_t = -\nabla \cdot (p^*_t \nabla s^*_t)$. Define $q(z)$ such that marginalizing q with respect to all variables except x_k, x_{k+1} yields the coupling $p_{t_k} \otimes (T^{k+1}_k)_{\#} p_{t_k}$, where T^{k+1}_k is the transport map from p_{t_k} to $p_{t_{k+1}}$. Then,*

$$\lim_{|t_k - t_{k+1}| \to 0} \int_0^1 \mathbb{E}_{p_t(x)} ||u_t(x) - \nabla s^*_t(x)||_2^2 \, dt = 0. \tag{11}$$

3.2.3 Learned proxy matching

In this section, given its importance as the reference flow we will match during FM training, we discuss how close of a *proxy curve* (generalization of μ_t in Eqn. 9) we can obtain within a family of paths that minimizes an energy functional. This will include methods such as Metric Flow Matching (Kapusniak et al., 2024), but also shares similarities with GSBM (Liu et al., 2024c). To motivate this, note that one could "sample" from p_t, by saving neural network parameters over the course of training on different initial samples $\theta_0 \sim p_0$. This approach can be used to build a dataset of intermediate weights $\mathcal{D} = \bigcup_{t \in [0,1]} \mathcal{D}_t$ of weights saved over the course of training. Following Kapusniak et al. (2024), if we define a data-dependent metric $g : \mathbb{R}^p \to \mathcal{S}_{++}(p)$, which is a smooth map parameterized by the dataset $\mathcal{D}$, we may compute a smooth energy-minimizing curve $\gamma_{\mathbf{x}_0, \mathbf{x}_1} := \arg\min_{\gamma_0 = \mathbf{x}_0, \gamma_1 = \mathbf{x}_1} \mathcal{E}_g(\gamma_t)$ between fixed points $(\mathbf{x}_0, \mathbf{x}_1) \sim \pi$ that can be shown to stay close to the data (Kapusniak et al., 2024, Proposition 1). Further, we may perform this minimization by training geodesic interpolants $\mathbf{x}_{t,\eta} \approx \gamma(t; \eta)$, see App. E for details. Following, we develop a framework using energy

functionals, which are data-dependent in practice, motivated by the connection between the metric and potential approach (Kapusniak et al., 2024, App. C.1): $||\dot{x}_t||_{g(x_t)} = ||\dot{x}_t||_2^2 + V(x_t; x_0, x_1)$, where V is a potential function depending on the boundary conditions. In this setting, we seek to characterize choices of energy V to minimize the W_2 distance between the proxy probability path, which evolves by v_θ, and the reference p_t in Eqn. 3. We start by writing down a continuity equation for the proxy path (cf. Theorem 1).

Theorem 3 (Proxy reference path)**.** *Suppose the Lagrangian $L(x_t, \dot{x}_t, t) = ||\dot{x}_t||_2^2 + V_t(x_t, \dot{x}_t)$ is Tonelli and strongly convex in velocity. The Lagrangian optimal transport map T exists between p_0 and p_1. Moreover, there exists a locally Lipschitz, locally bounded vector field w s.t.*

$$\partial_t \hat{p}_t + \nabla \cdot (\hat{p}_t w_t) = 0 \tag{12}$$

satisfies $\hat{p}_t = \mathrm{Law}(\gamma_t)$ where γ is a random, smooth Lagrangian-minimizing curve and (γ_0, γ_1) is an optimal coupling of p_0, p_1.

The following discussion will focus on quantifying closeness of the reference p and $\hat{p}$. Our objective is to characterize functionals that would induce good proxy trajectories which remain close to Eqn. 3. Hence, we ought to assume some regularity for V, in particular, we want the Lagrangian to be Tonelli. In Prop. 3 (App. A), this definition of the learned path is used to find an expression for the W_2 distance that accounts for the closeness of w_t to the loss gradient $-\nabla\mathcal{L}$.

Following the action matching discussion, we define an *action gap* for the data-dependent energy that will be used for a more intuitive and practical W_2-bound; see App. D for an analogue with entropy-regularization.

$$AG(p, \hat{p}) := \frac{1}{2} \int_0^1 \mathbb{E}_\gamma ||\nabla\mathcal{L}(\gamma_t) + \dot{\gamma}_t||^2 \, dt, \tag{13}$$

where the expectation is taken over all Lagrangian-minimizing curves s.t. (γ_0, γ_1) is an optimal coupling under the Lagrangian in Theorem 3.

We note that this definition is unconventional as it uses curves obtained from $\hat{p}$ in Theorem 3. However, we believe this formulation to better match the interpretation of finding a proxy trajectory that reflects evolution via GD minimization. Intuitively, our Lagrangian-minimizing curves ought to have derivatives close to the gradient descent direction along its length, and the objective is to vary the potential V to minimize $AG(p, \hat{p})$. Next, we adapt a result from Neklyudov et al. (2023); Albergo and Vanden-Eijnden (2023) to bound the Wasserstein distance in terms of the action gap (cf. Prop. 1)

Proposition 2. *Suppose that $\nabla\mathcal{L}$ is uniformly Lipschitz in x with Lipschitz constant K. Then,*

$$W_2^2(p_t, \hat{p}_t) \leq e^{(1+2K)t} \int_0^t \mathbb{E}_\gamma [\nabla\mathcal{L}(\gamma_s) + \dot{\gamma}_s] \, ds. \tag{14}$$

As the loss is arbitrary, it is not guaranteed that the action gap vanishes. For one, the smoothness assumption on γ means that eccentric losses cannot be fit exactly. However, as detailed in App. E, we can, in practice, weaken the smoothness assumption to match a learned minimizing curve or the simpler cubic splines (motivated by Rohbeck et al. (2025)). We also remark that optimizing this functional V is challenging in practice, requiring learned interpolants as discussed in App. E, or modeling a drift potential directly as in JKOnet (Bunne et al., 2022; Terpin et al., 2024). To conclude, we make use of a natural smoothness assumption on the gradient descent path to prove a bound on AG.

Theorem 4 (Informal)**.** *Suppose for each $(x_0, x_1) \sim \pi$ there exists a smooth connecting curve $\tilde{\gamma}$ whose* ***gradient difference*** *$||\nabla\mathcal{L}(\gamma_t) + \dot{\gamma}_t||^2$ is small for all $t \in [0, 1]$. Moreover, if $V(\gamma_t, \dot{\gamma}_t) \geq 0$ does not approach zero when the gradient difference is large, and is sufficiently small when the gradient difference is itself small, then the Lagrangian minimizing curve has a small gradient difference for all $t \in [0, 1]$.*

3.3 Practical synthesis of neural network parameters

We describe the components of our approach below and leave more details to Appendix F, G. Concerning notation, we use the N$\mathcal{M}$- prefix to denote our methods, e.g. N$\mathcal{M}$-CFM to denote conditional flow matching.

Weight encoder. Due to the often intractable size of weight space, it is sometimes necessary for modeling to take place in latent space. We justify this design by appealing to work on the Lottery Ticket Hypothesis (Frankle and Carbin, 2019; Zhang et al., 2021; Liu et al., 2024b) as well as the body of work on pruning (Cheng et al., 2024), which suggests that, like natural data, neural networks live on a low-dimensional manifold within its ambient space. There are a variety of encoders to choose from, such as the typical variational autoencoder (VAE) (Kingma and Welling, 2022), and more specialized encoders for neural network parameters (Kofinas et al., 2024; Putterman et al., 2024). However, to simplify matters as we are mostly focused on the generative aspect, we stick to the VAE as used in Soro et al. (2025).

Generative meta-model. The backbone of our meta-learning framework is a conditional FM model following Tong et al. (2024), alongside the multi-marginal variant (MMFM) that matches piecewise-linear interpolants; see App. E for a detailed discussion of interpolants. We also experiment with using the JKOnet* method in a few tests; see details in App. F. Preliminary experiments were also done with learned interpolants such as Metric Flow Matching, but we found them to be unstable, likely due to the sparsity of data and the large ambient space favoring simpler interpolants. Exploiting the flexibility of FM to use a non-Gaussian prior, we use the Kaiming uniform or normal initializations (He et al., 2015a), as the source p_0. The data distribution p_1 of base model weights is experiment-dependent, however, broadly they are obtained by conventional pre-training methods or from a model zoo (Schürholt et al., 2022).

Reward fine-tuning. FM models lend themselves to the recently proposed reward fine-tuning method, based on the adjoint ODE (Domingo-Enrich et al., 2025), which casts stochastic optimal control as a regression problem. The goal of this method is to modify the base generative distribution $\hat{p}_1^{\text{base}}$ to generate the reward-tilted distribution $p_1^*(x) \propto p_1^{\text{base}}(x)\exp(r(x))$ via the *Adjoint Matching* algorithm. Naturally, in our setting, we suppose p_1^{base} is obtained from meta-training and governs classifiers that predict on $\mathcal{D}_1$, but we wish to modify the meta-model to generate base models that predict on $\mathcal{D}_2$. Therefore, we set the reward $r(X_1) := -\mathcal{L}_2(X_1)$ where $\mathcal{L}_2$ is a loss on $\mathcal{D}_2$ such as cross-entropy and proceed with adjoint matching.

Table 1: Best validation accuracy of unconditional N$\mathcal{M}$ CNN3 generation for various datasets. *orig* denotes base models trained conventionally by SGD, *p-diff* those generated with p-diff (Wang et al., 2024), and || denotes equality with the above row. MMFM(k) and JKO(k) indicates the number of marginal distributions considered in addition to p_0 and p_1.

	CIFAR100			CIFAR10			MNIST			STL10		
	orig.	N$\mathcal{M}$	p-diff.	orig.	N$\mathcal{M}$	p-diff.	orig.	N$\mathcal{M}$	p-diff	orig.	N$\mathcal{M}$	p-diff
N$\mathcal{M}$-CFM w/ VAE	25.62	26.01	25.99	63.38	64.32	63.37	98.93	98.91	98.93	53.88	53.50	53.86
N$\mathcal{M}$-CFM	||	25.31	||	||	62.52	||	||	98.52	||	||	53.49	||
N$\mathcal{M}$-MMFM(2)	||	21.16	||	||	63.35	||	||	98.53	||	||	53.20	||
N$\mathcal{M}$-MMFM(3)	||	24.53	||	||	63.34	||	||	98.62	||	||	53.53	||
N$\mathcal{M}$-MMFM(4)	||	22.89	||	||	61.33	||	||	98.39	||	||	53.08	||
N$\mathcal{M}$-JKO(3)	||	24.84	||	||	62.60	||	||	98.59	||	||	53.44	||
N$\mathcal{M}$-JKO(4)	||	24.97	||	||	63.35	||	||	98.57	||	||	53.21	||

4 Experiments

First, we confirm various properties that are to be expected of weight generation models. Next, we explore reward fine-tuning courtesy of adjoint matching, finishing with an application to detect harmful covariate shifts. Further details are provided in App. G.

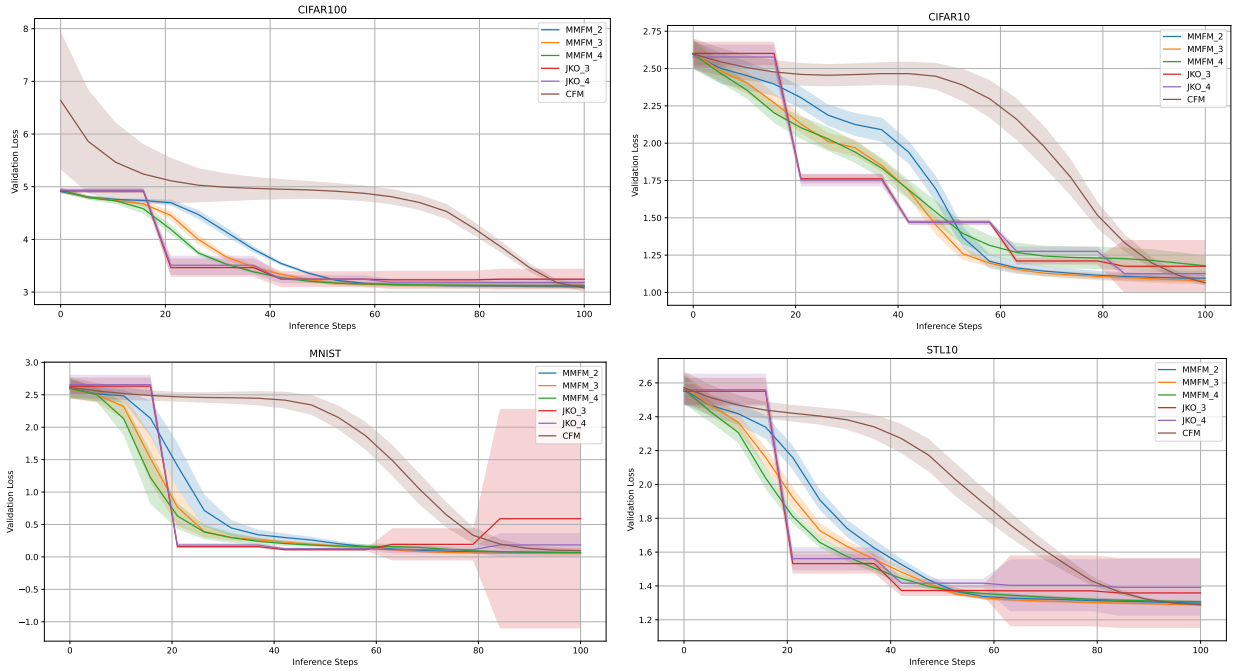

Figure 2: Base model validation loss over the course of inference for various N$\mathcal{M}$ methods. The plots were computed on 20 out of 100 intermediate timepoints for MMFM and CFM, but restricted by design to k timepoints for JKO(k).

4.1 Basic properties of N$\mathcal{M}$ methods

Unconditional generation. We first evaluate the basic modeling capabilities of the flow meta-model. The target distribution p_1 is obtained by training a variety of base models on common datasets: CIFAR10, CIFAR100, STL10, and MNIST, and saving weight checkpoints across 100 epochs of training. For large models, we may choose to generate only a subset of the weights. In our case, we generate the batch norm parameters for ResNet-18 (He et al., 2015b), ViT-base (Dosovitskiy et al., 2021) and ConvNext-tiny (Liu et al., 2022), and the full medium-CNN (Schürholt et al., 2022), which we denote CNN3 corresponding to the number of convolutional layers. The objective is to train a separate meta-model for each dataset and validate its base model reconstruction on classifying its corresponding test set. Focusing on CNN3, Table 1 shows that we are able to match base models trained conventionally and with p-diff (Wang et al., 2024), with extra results in Table 7. In addition, Figure 2 visualizes how validation loss changes over inference timesteps for the various N$\mathcal{M}$ methods. We observe that methods incorporating trajectory information (MMFM and JKO) observe a faster decrease in validation loss over inference steps. This suggests that gradient descent converges rather quickly to the final validation loss and so learning the intermediate marginal distributions leads to faster inference-time convergence. Finally, due to computational constraints, we restrict further experiments to the best observed MMFM and JKO settings: in our case, MMFM(3) and JKO(4); see App. G for a discussion of computation times.

Model retrieval. Following (Soro et al., 2025), we perform model retrieval to test whether the meta-model can distinguish weights of the base model given conditioning samples from the dataset the base model was trained on. The base model is the same CNN3 model and we obtain weight checkpoints from the trained on MNIST, Fashion-MNIST (FMNIST), CIFAR10, and STL10 across 50 epochs of conventional training. Unlike in the previous test, we will train just a single meta-model conditioned on context samples from their respective training sets passed through a CLIP (Radford et al., 2021) encoder. During validation, we pass in a random support sample from one of the four datasets and generate the *full* CNN3. In Table 2, we see that most of our mean top-5 validation accuracy matches that of the base models. There is a noticeable decline

in N$\mathcal{M}$-MMFM accuracy; this is due to the higher complexity of the target vector field as opposed to the straight N$\mathcal{M}$-CFM targets. With a larger meta-model, we were able to get higher accuracies, but the results here serve to inform the reader of this downside.

Downstream initialization. Next, we repeat the model retrieval test, but instead we obtain weights across 10 epochs of conventional training. We opted for the encoded runs if possible for efficiency. The generated weights are used as initialization before fine-tuning another 20 epochs. As shown in Table 3, our initialization enjoys faster convergence, even for corrupted datasets, highlighting the generalization capability of our meta-model.

4.2 Reward fine-tuning

Adjoint matching. Continuing from the exposition in Section 3, we note that our noise injection is different from the noise schedule assumed in the original derivations (Domingo-Enrich et al., 2025). As such, we modified the computations slightly as given in App. F.4. However, since the conventional flow matching algorithm injects a very small noise in the Gaussian path, i.e. $p_t(x_t|x_0, x_1) = \mathcal{N}(\mu_t, \sigma_t^2 I)$, where practically $\sigma_t = 10^{-3}$, we found that divisions by σ_t in the adjoint matching algorithm will explode quantities in the loss. To resolve this issue, we derived a deterministic version of the adjoint matching algorithm, which we found to work much better with more sensible norms. Further, we derived a multi-marginal variant of the adjoint matching algorithm, however, we focus our resources only on the CFM case. All computations may be found in App. F.4.

Table 2: Mean validation accuracy of top-5 N$\mathcal{M}$ model retrievals

	CIFAR10	STL10	MNIST	FMNIST
Original	63.38	53.88	98.93	89.77
N$\mathcal{M}$-CFM	62.89	53.47	98.69	90.24
N$\mathcal{M}$-CFM w/ VAE	62.79	53.41	98.54	90.59
N$\mathcal{M}$-MMFM	53.45	49.62	92.86	76.55
N$\mathcal{M}$-MMFM w/ VAE	63.87	52.86	98.36	89.73
N$\mathcal{M}$-JKO[1]	62.04	51.16	97.44	87.75

Table 3: Mean validation accuracy over top-5 fine-tuned generated weights post-retrieval. The $\widetilde{\text{tilde}}$ indicates corrupted datasets, on which the model was not trained.

Epoch	Method	$\widetilde{\text{CIFAR10}}$	$\widetilde{\text{STL10}}$	$\widetilde{\text{MNIST}}$	$\widetilde{\text{FMNIST}}$	CIFAR10	STL10	MNIST	FMNIST
0	RandomInit	$\sim 10\%$	$\sim 10\%$	$\sim 10\%$	$\sim 10\%$	$\sim 10\%$	$\sim 10\%$	$\sim 10\%$	$\sim 10\%$
	N$\mathcal{M}$-CFM w/ VAE	39.66 ± 0.08	36.85 ± 0.36	90.93 ± 0.35	72.44 ± 0.27	44.63 ± 0.05	40.06 ± 0.02	95.57 ± 0.05	83.68 ± 0.24
	N$\mathcal{M}$-MMFM w/ VAE	39.08 ± 0.22	21.98 ± 0.89	90.55 ± 0.23	74.08 ± 0.08	41.89 ± 0.37	36.41 ± 1.54	90.02 ± 1.09	83.38 ± 0.29
	N$\mathcal{M}$-JKO	40.53 ± 0.27	18.83 ± 0.30	90.80 ± 0.20	74.45 ± 0.15	45.04 ± 0.19	41.76 ± 0.21	95.42 ± 0.22	83.90 ± 0.04
1	RandomInit	34.05 ± 1.13	16.64 ± 1.61	83.21 ± 0.57	66.81 ± 0.74	36.27 ± 2.05	22.13 ± 2.11	96.52 ± 0.20	77.21 ± 0.26
	N$\mathcal{M}$-CFM w/ VAE	44.68 ± 0.16	38.18 ± 0.40	94.02 ± 0.42	76.43 ± 0.22	48.28 ± 0.14	41.66 ± 0.19	97.38 ± 0.10	85.27 ± 0.30
	N$\mathcal{M}$-MMFM w/ VAE	43.96 ± 0.37	29.44 ± 1.25	94.71 ± 0.35	79.85 ± 0.47	47.98 ± 0.32	39.93 ± 1.45	97.68 ± 0.13	85.50 ± 0.12
	N$\mathcal{M}$-JKO	45.04 ± 0.31	22.26 ± 0.08	94.94 ± 0.17	80.63 ± 0.20	49.61 ± 0.29	42.41 ± 0.31	97.39 ± 0.02	85.48 ± 0.13
5	RandomInit	46.55 ± 0.80	25.08 ± 1.13	92.53 ± 0.28	79.08 ± 0.93	47.74 ± 1.33	31.55 ± 2.00	98.24 ± 0.03	84.87 ± 0.40
	N$\mathcal{M}$-CFM w/ VAE	47.41 ± 0.13	39.98 ± 0.29	95.22 ± 0.09	79.88 ± 0.61	51.69 ± 0.14	42.62 ± 0.08	98.04 ± 0.05	86.76 ± 0.15
	N$\mathcal{M}$-MMFM w/ VAE	47.06 ± 0.45	35.24 ± 0.72	95.92 ± 0.19	82.08 ± 0.11	51.70 ± 0.21	41.35 ± 0.65	98.51 ± 0.02	86.64 ± 0.14
	N$\mathcal{M}$-JKO	48.14 ± 0.08	26.00 ± 0.28	95.92 ± 0.11	82.87 ± 0.19	53.01 ± 0.11	43.38 ± 0.29	98.13 ± 0.05	86.75 ± 0.04
20	RandomInit	50.28 ± 0.43	33.63 ± 0.99	95.81 ± 0.18	82.36 ± 0.42	51.35 ± 1.21	44.16 ± 1.28	98.51 ± 0.05	88.25 ± 0.69
	N$\mathcal{M}$-CFM w/ VAE	52.25 ± 0.18	41.18 ± 0.28	96.41 ± 0.08	83.42 ± 0.23	55.66 ± 0.23	44.38 ± 0.16	98.25 ± 0.05	88.02 ± 0.17
	N$\mathcal{M}$-MMFM w/ VAE	52.57 ± 0.73	39.80 ± 0.48	97.01 ± 0.21	84.38 ± 0.08	55.85 ± 0.76	44.10 ± 0.26	98.85 ± 0.03	88.29 ± 0.03
	N$\mathcal{M}$-JKO	52.59 ± 0.02	34.53 ± 0.35	96.82 ± 0.10	84.89 ± 0.20	56.65 ± 0.27	45.06 ± 0.21	98.43 ± 0.03	88.10 ± 0.14
30	RandomInit	52.99 ± 0.55	37.79 ± 0.55	96.55 ± 0.22	84.16 ± 0.66	56.05 ± 1.21	45.80 ± 1.19	98.55 ± 0.05	88.55 ± 0.66

Support of classifier weights. Notably, the method of reward fine-tuning *cannot* be applied for arbitrary meta-model fine-tuning. Indeed, this follows from the fact that supp $p_1^* =$ supp p_1^{base}. Empirically, we found that fine-tuning a N$\mathcal{M}$-CFM model trained to generate CIFAR10 classifiers into one that generates STL10 classifiers fails at achieving reasonable accuracies, often just slightly better than a Kaiming uniform initialization. This suggests that the support of classifier weights, trained for different datasets, are mostly disjoint. We stress that this property of the support is a function of both the downstream data *and* the model architecture. Indeed, due to the small size of the CNN3, the parameters that predict on different datasets e.g.

[1] JKOnet expects the trajectory to evolve via gradient flow. This is not guaranteed in the latent space, hence we only show the un-encoded variant.

CIFAR10 and STL10, will differ considerably, but this may not be the case for larger neural networks which possess a larger generalization set. To summarize our hypothesis: *the support set of CNN3 weights trained for different datasets are narrow and mostly disjoint, thus, small changes in the training data will noticeably affect the support w.r.t. validation accuracy.*

We found experimental evidence to support this hypothesis, and also to suggest that reward fine-tuning goes a long way towards improving validation accuracy on out-of-distribution data. Indeed, we find accuracies to be bounded above, often far below the validation accuracy obtained from SGD fine-tuning for the most corrupted data. We defer results and discussion to App. G.4, specifically Tables 10 and 11. Given this finding, we use it to approach the problem of harmful covariate shifts.

4.3 Detecting harmful covariate shifts

Training CDCs. Continuing the exposition in Section 2, we specify the training regiment of CDCs. Let $g(\cdot)$ represent one CDC and $f(\cdot)$ the base classifier; further, let $\mathbf{P} = \{(x_i, y_i)\}_{i=1}^n$ be samples from the generalization set $\mathcal{R}$ and $\mathbf{Q} = \{\tilde{x}_i\}_{i=1}^m$ from the unknown distribution. Our objective is for g to maintain performance on $\mathbf{P}$, but to disagree with f on $\mathbf{Q}$. Naturally, we use the cross-entropy loss ℓ_{ce} on $\mathbf{P}$, but on $\mathbf{Q}$, Ginsberg et al. (2023) introduces the *disagreement-cross-entropy*:

$$\ell_{dce}(\hat{y}, f(x_i)) = \frac{1}{1-N} \sum_{c=1}^{N} \mathbf{1}_{f(x_i) \neq c} \log(\hat{y}_c),$$

where N denotes the total number of classes. We combine these objectives by minimizing the CDC loss:

$$\ell_{cdc}(\mathbf{P}, \mathbf{Q}) := \frac{1}{|\mathbf{P} \cup \mathbf{Q}|} \left(\sum_{(x_i, y_i) \in \mathbf{P}} \ell_{ce}(g(x_i), y_i) + \lambda \sum_{\tilde{x}_i \in \mathbf{Q}} \ell_{dce}(g(\tilde{x}_i), f(\tilde{x}_i)) \right) \tag{15}$$

To test for shift, Detectron (Ginsberg et al., 2023) compares g trained with $\mathbf{Q}$ sampled from the unknown distribution ($g_{\mathbf{Q}}$) against $\mathbf{Q} = \mathbf{P}^*$ ($g_{\mathbf{P}}$), i.e. samples from the generalization set. In particular, the disagreement rate or the class entropy for each case is obtained and hypothesis tested. In both cases, the disagreement and entropy are higher if $\mathbf{Q}$ represents a significant shift.

Table 4: True positive rate at the 5% significance level (TPR@5) and area under receiver operating characteristic curve (AUROC) for detection of harmful covariate shift on CIFAR10.1 and Camelyon17. We test on both the disagreement rate (DAR) and the entropy, setting $\lambda = \kappa/(|\mathbf{Q}| + 1)$. See App. G for details on choosing κ. The best result for each column and our method are **bolded**.

TPR@5	CIFAR10			Camelyon		
$\lvert\mathbf{Q}\rvert$	10	20	50	10	20	50
Detectron (DAR), $\kappa = 1$	0	0	0	0	$.10 \pm .10$	0
Detectron (DAR), κ vary	0	0	$.10 \pm .10$	$.10 \pm .10$	$.20 \pm .13$	$.50 \pm .17$
Meta-detectron (DAR), κ vary	$.53 \pm .13$	$.47 \pm .13$	$.53 \pm .13$	$.73 \pm .12$	$\mathbf{.40 \pm .13}$	$\mathbf{.68 \pm .10}$
Detectron (Entropy), $\kappa = 1$	$\mathbf{.60 \pm .17}$	$.40 \pm .16$	$.50 \pm .17$	0	0	0
Detectron (Entropy), κ vary	$.60 \pm .16$	$.10 \pm .10$	$.10 \pm .10$	0	0	0
Meta-detectron (Entropy), κ vary	$.47 \pm .13$	$\mathbf{.93 \pm .07}$	**1.00**	**1.00**	0	$.24 \pm .09$
AUROC	CIFAR10			Camelyon		
$\lvert\mathbf{Q}\rvert$	10	20	50	10	20	50
Detectron (DAR), $\kappa = 1$	0.515	0.595	0.485	0.59	0.595	0.795
Detectron (DAR), κ vary	0.480	0.495	0.665	0.665	0.750	0.875
Meta-detectron (DAR), κ vary	**0.876**	0.838	0.900	0.867	0.760	**0.930**
Detectron (Entropy), $\kappa = 1$	0.740	0.695	0.850	0.345	0.610	0.720
Detectron (Entropy), κ vary	0.775	0.740	0.785	0.490	0.445	0.660
Meta-detectron (Entropy), κ vary	0.809	**0.987**	**1.000**	**1.000**	**0.836**	0.755

Table 5: In-distribution validation accuracy before and after reward fine-tuning.

	CIFAR10		
$\lvert\mathbf{Q}\rvert$	10	20	50
Meta-detectron ($\mathbf{P}^*$), κ vary	$60.78 \pm 0.21 \to 61.09 \pm 0.17$	$60.97 \pm 0.16 \to 60.49 \pm 0.25$	$60.78 \pm 0.09 \to 60.38 \pm 0.24$
Meta-detectron ($\mathbf{Q}$), κ vary	$60.97 \pm 0.15 \to 60.76 \pm 0.26$	$61.61 \pm 0.30 \to 61.11 \pm 0.13$	$61.14 \pm 0.15 \to 60.59 \pm 0.26$
	Camelyon		
$\lvert\mathbf{Q}\rvert$	10	20	50
Meta-detectron ($\mathbf{P}^*$), κ vary	$92.10 \pm 0.35 \to 92.12 \pm 0.46$	$92.09 \pm 0.54 \to 91.95 \pm 0.48$	$92.31 \pm 0.21 \to 90.43 \pm 0.82$
Meta-detectron ($\mathbf{Q}$), κ vary	$92.36 \pm 0.34 \to 91.89 \pm 0.66$	$90.61 \pm 1.45 \to 91.38 \pm 0.23$	$92.66 \pm 0.24 \to 91.01 \pm 0.36$

Motivation. The problem of detecting covariate shift is not just about the data—a lot of modern neural networks are robust to such changes. The essence of the problem is whether or not the classifier weights required to predict on $\mathbf{Q}$ differs from the current classifier. Therefore, the preceding analysis regarding the support set is pertinent: we require a method of analysis that is sensitive to detecting changes in the classifier weights required to predict on a new set.

Meta-detectron. Our approach, termed *meta-detectron*, builds on reward fine-tuning by adjoint matching. We start by meta-training a N$\mathcal{M}$-CFM meta-model to classify on each of the datasets (this is identical to the unconditional generation setup). Next, we reward fine-tune, maintaining the procedure of sampling from the meta-model at each iteration to compute the reward, but now the reward function is $r(X_1) = -\ell_{cdc}(\mathbf{P}, \mathbf{Q}; X_1)$, where X_1 serves the role of g, and the original meta-model generates the base classifier f in Eqn. 15. As the method requires training $g_{\mathbf{P}}$ and $g_{\mathbf{Q}}$, we likewise fine-tune two different meta-models depending on the disagreement set, and compare the disagreement rate and entropy of the generated $g_{\mathbf{P}}$ and $g_{\mathbf{Q}}$. We motivate this approach by noting the importance of the support set of the original learned target distribution. In particular, it's more likely for the support to contain classifiers that disagree on an out-of-distribution set $\mathbf{Q}$ than it is to contain those disagreeing on $\mathbf{P}^*$. That being said, we expect the disagreement rate to be more conservative overall due to the tightness of the support.

Discussion. We evaluate on CNN3 with experiments following Ginsberg et al. (2023) on CIFAR10.1 (Recht et al., 2019), where shift comes from the dataset pipeline, and Camelyon17 (Veeling et al., 2018), which consists of histopathological slides from multiple hospitals. Table 4 shows the *True Positive Rate at 5% Significance Level (TPR@5)* and *AUROC* aggregated over 10 randomly chosen seeds for sampling $\mathbf{P}^*$ and $\mathbf{Q}$ of varying sample sizes. In addition, we ablated over the weight λ; see App. G.5 for details and further results. Compared to the original tests (Ginsberg et al., 2023, Table 1) on ResNet-18, we observe that covariate shift is highly architecture dependent. This is expected as CNN3 underfits CIFAR10 ($\sim 63\%$ validation accuracy). Our approach accounts for this as the base classifiers are generated directly by the fine-tuned meta-models. We also observe–though not shown–lower disagreement rates overall, which pays off in the TPR@5 as the $\mathbf{P}^*$ disagreement rates are close to zero in all cases, and confirms the conservative nature of our method. Importantly, we also observe in Table 5 that the validation accuracy on $\mathbf{P}$ is mostly unchanged. Regarding meta-training behavior, Figure 3 shows that the AUROC increases sharply early in the reward fine-tuning phase, requiring only about 50 batch iterations to reach its peak. This coincides with a marked decrease in ℓ_{cdc}. However, we also note some instability in the AUROC throughout training, particularly in the Camelyon experiments, where fluctuations are more pronounced; see App. G.5 for more figures.

5 Related work

Flow matching for trajectory inference. The flow matching framework (Albergo and Vanden-Eijnden, 2023; Lipman et al., 2023; Liu et al., 2023) gives way to a few methods of controlling the trajectory of the inference path, from the simple multi-marginal approach (Rohbeck et al., 2025), to approaches with more sophisticated interpolants (Neklyudov et al., 2023; 2024; Kapusniak et al., 2024; Pooladian et al., 2024; Rohbeck et al., 2025). A traditional application of trajectory inference is single cell RNA-sequencing (Tong

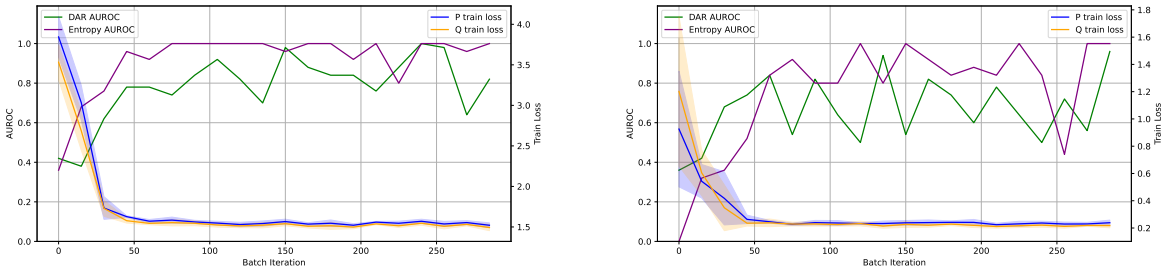

Figure 3: Plots illustrating how AUROC and ℓ_{cdc} evolves over reward fine-tuning iterations for CIFAR10 and Camelyon17 when $|\mathbf{Q}| = 20$. See App. G.5 for more figures

et al., 2020; Neklyudov et al., 2024; Kapusniak et al., 2024), however, a similar problem arises in weight generation. For a broad mathematical overview, see (Lavenant et al., 2024).

Weight generation. Recent approaches to generating neural network parameters centers around learning a distribution over pre-trained weights and generalizing this ability with the help of conditioning information. Most similar to our approach in this regard are the various diffusion-based approaches (Soro et al., 2025; Zhang et al., 2024; Wang et al., 2024) which have been used to generate neural network weights with a focus on in-context learning tasks such as zero- and few-shot learning. However, flexibility is limited by its restriction to Gaussian processes and a sluggish inference speed. More broadly, we may categorize this form of learning as meta-learning (Fifty et al., 2024; Hu et al., 2022; Zhmoginov et al., 2022), which aims to learn concepts from a few demonstrations. It is therefore natural that the literature has two evaluation settings: in-distribution tasks and out-of-distribution (OOD) tasks. With enough training and capacity, it's clear *meta-models* (i.e. models trained on multiple data distributions) should excel at in-distribution tasks. However, generalization to novel tasks often presents a challenge to meta-learning and weight generation frameworks (Wang et al., 2024; Schürholt et al., 2024; Soro et al., 2025), including non-diffusion-based approaches Knyazev et al. (2021; 2023). See Appendix C for further related works.

6 Conclusion, Limitations, and Future Work

In this work, we have provided a preliminary investigation of the latest dynamical generative models for weight generation with applications to covariate shift detection. Due to the large size of modern neural network architectures, limited resources constrain our study to architectures with $< 10^6$ parameters. To address other concerns such as training dataset diversity and the lack of experiments incorporating stochastic weight evolution, future research directions include: **1)** exploration of architectures specializing in neural network weights to expedite meta-training, **2)** incorporating Schrödinger bridge matching approaches to address stochastic weight evolution. Moreover, the methods here open up a plethora of other applications. For instance, we may experiment with **1)** more traditional meta-learning tests such as zero- and few-shot learning; **2)** model merging by superposition of the inference ODE/SDE (Skreta et al., 2025); or **3)** network constrained problems such as generating binary or Lipschitz neural networks.

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Appendix

This appendix consist of details left out in the main text. First, we complete the proofs of the results in the text. Next, we perform a more comprehensive review of the related literature. Following, we fill in a few technical results that were promised in the main text, before closing with architecture and training settings.

A Proofs

Theorem 1:
Let $\theta_0 \sim p_0$ be the initialized network parameters residing on $\Omega \subset \mathbb{R}^p$ a compact set. Supposing that $\mathcal{L} : \Omega \to \mathbb{R}$ is C^1 and the gradient descent updates θ_t reside in Ω. Define $p_t = \mathrm{Law}(\theta_t)$ and further assume $p_t > 0$ a.e., then it satisfies the continuity equation Eqn. 3.

Proof of Theorem 1. We follow the reasoning of Santambrogio (2015, Ch. 8). First, we provide context as for the discretized formulation of a gradient flow. Let $F : \mathbb{R}^d \to \bar{\mathbb{R}}$ be lower semi-continuous and is bounded below as $F(x) \geq C_1 - C_2|x|^2$ for some $C_1, C_2 \geq 0$. Consider the formal problem

$$\begin{cases} x'(t) = -\nabla F(x) \\ x(0) = x_0 \end{cases}.$$

This is understood as a Cauchy problem which happens to be a gradient flow. If we fix a small time step $\tau > 0$, this problem has a discretization

$$x_{k+1}^\tau \in \arg\min_{x \in \mathbb{R}^d} F(x) + \frac{|x - x_k^\tau|^2}{2\tau}$$

We now generalize this discretization scheme and show that its limit is a solution to the gradient flow.

Define $J(p) := \int_\Omega \mathcal{L}\, dp$. We now have the scheme

$$p_{k+1}^\tau \in \arg\min_{p \in \mathcal{P}(\Omega)} J(p) + \frac{W_2^2(p, p_k^\tau)}{2\tau}.$$

Claim: The minimization above produces a minimizer $p = p_{k+1}^\tau$. If we modify

$$\tilde{J}(p) = \begin{cases} J(p) & p \ll \mathrm{Leb}^p \\ +\infty & \text{otherwise} \end{cases}$$

then this is a unique minimizer.

Proof. As Ω is compact, it's a well-known result that $\mathcal{P}(\Omega)$ is compact in the weak topology (Parthasarathy, 2005). Moreover, Santambrogio (2015, Prop. 7.1) gives lower semi-continuity of J, which is enough for existence. Moreover, from Santambrogio (2015, Prop. 7.19), we have: if $p \ll \mathrm{Leb}^p$, then $W_2^2(\cdot, p)$ is strictly convex. Since $J(p)$ is linear in p, we have that the minimization objective is strictly convex with the modification to $\tilde{J}$, thus the minimizer is unique. □

Note: due to the nice properties of having $p \ll \mathrm{Leb}^p$, and considering that we do not lose much generality, we will use $\tilde{J}$ for the rest of the proof.

Claim: The first variation $\frac{\delta J}{\delta p}(p) = \mathcal{L}$ for all $p \in \mathcal{P}(\Omega)$.

Proof. We note that since Ω is compact, $\mathcal{L}$ is presumed continuous, and p a finite measure, we have that $J(p) < \infty$ for all $p \in \mathcal{P}(\Omega)$. This is sufficient to show that p is always regular for J. Moreover, the first variation satisfies:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} J(p + \epsilon\chi) = \int \frac{\delta J}{\delta p}(p)\, d\chi.$$

By linearity: $J(p + \epsilon\chi) = J(p) + \epsilon \int \mathcal{L}\, d\chi$, and therefore $\frac{\delta J}{\delta p}(p) = \mathcal{L}$. □

It was remarked Santambrogio (2015, Remark 7.13) that the first variation of the transport cost $\mathcal{T}_c$ for a continuous cost c is exactly the Kantorovich potential φ *only if it is unique*. In our case, uniqueness stems from Santambrogio (2015, Prop. 7.18). Given our preparation, we are now ready to state the main Lemma.

Lemma 1. *Let T_{k+1}^τ be the optimal transport map from p_{k+1}^τ to p_k^τ (note the reverse), then we have the velocity*

$$v_{k+1}^\tau := \frac{\iota - T_{k+1}^\tau}{\tau} = -\nabla\mathcal{L} \quad a.e. \tag{16}$$

Proof. Since the first variation is linear in the functional argument, we have that the first variation

$$\frac{\delta(J + W_2^2(\cdot, p_k^\tau)/2\tau)}{\delta p}(p) = \mathcal{L} + \varphi/\tau.$$

By Santambrogio (2015, Prop. 7.20), we have $\mathcal{L} + \varphi/\tau = C$ for some constant C a.e. (precisely, on all $supp\, p_{k+1}^\tau$, which is assume > 0 a.e.). Differentiating, we have

$$\nabla\varphi = \iota - T_{k+1}^\tau = -\tau\nabla\mathcal{L} \quad \text{a.e.}$$

□

Before proceeding further, we provide a simple bound on the 2-Wasserstein distance of consecutive iterates: by optimality

$$J(p_{k+1}^\tau) + \frac{W_2^2(p_{k+1}^\tau, p_k^\tau)}{2\tau} \leq J(p_k^\tau),$$

therefore

$$\sum_k \frac{W_2^2(p_{k+1}^\tau, p_k^\tau)}{2\tau} \leq \sum_k 2(J(p_k^\tau) - J(p_{k+1}^\tau)) \leq 2J(p_0) =: C.$$

where we telescoped the sum and used $\inf J \geq 0$ as $\mathcal{L} \geq 0$.

For us to take a limit, we need to fine p_t^τ for values in $(k\tau, (k+1)\tau)$. Following Santambrogio (2015, Ch. 8), take

$$p_t^\tau = \left(\frac{k\tau - t}{\tau} v_k^\tau + \iota\right)_\# p_k^\tau \quad \text{for } t \in ((k-1)\tau, k\tau).$$

Moreover, v_t^τ ought to be defined so that it advects p^τ over time, and following the intuition of interpolating between discrete values, we require

$$||v_t^\tau||_{L^2(p_t^\tau)} = |(p^\tau)'|(t) = \lim_{t' \to 0} \frac{W_2(p_{t+t'}^\tau, p_t^\tau)}{|t'|} = \frac{W_2(p_{k-1}^\tau, p_k^\tau)}{\tau}.$$

In fact,

$$v_t^\tau = \frac{\iota - T_k^\tau}{\tau} \circ ((k\tau - t)v_k^\tau + \iota)^{-1}$$

works. Define the momentum $E^\tau = p^\tau v^\tau$.

Now we bound the maximal E^τ:

$$|E^\tau|([0,1] \times \Omega) = \int_0^1 \int_\Omega |v_t^\tau| \, dp_t^\tau \, dt = \int_0^1 ||v_t^\tau||_{L^1(p_t^\tau)} \, dt \leq \int_0^1 ||v_t^\tau||_{L^2(p_t^\tau)} \, dt.$$

Using the Cauchy-Schwarz inequality

$$\int_0^1 ||v_t^\tau||_{L^2(p_t^\tau)} \, dt \leq \int_0^1 ||v_t^\tau||^2_{L^2(p_t^\tau)} \, dt = \sum_k \tau \left(\frac{W_2(p_{k-1}^\tau, p_k^\tau)}{\tau}\right)^2 \leq C^2. \tag{17}$$

Moreover, we see for $0 < s < t < 1$,

$$W_2(p_t^\tau, p_s^\tau) \leq \int_s^t |(p^\tau)'|(r) \, dr \leq (t-s)^{1/2} \left(\int_s^t |(p^\tau)'|(r)^2 \, dr\right)^{1/2}$$

But since $|(p^\tau)'|(r)^2 = ||v_r^\tau||^2_{L^2(p_t^\tau)}$, we have

$$W_2(p_t^\tau, p_s^\tau) \leq C^2(t-s)^{1/2}. \tag{18}$$

This provides a uniform Hölder bound on $t \mapsto p_t^\tau$, which implies uniform boundedness and equicontinuity. Therefore, up to subsequences, both E^τ and p^τ converges to a limit as $\tau \to 0$: by Arzela-Ascoli, we have

$p^\tau \to p$ uniformly on W_2. Moreover, by boundedness of $|E^\tau|$, we have weak compactness on the space of measures, and therefore, there exists a subsequence s.t. $E^\tau \to E$ weakly.

Now, checking the distributional test, e.g. Santambrogio (2015, Definition 4.1), we have that $\partial_t p_t^\tau + \nabla \cdot E^\tau = 0$. Moreover, the distributional test passes to the limit as v_t^τ is continuous, $\int\int ||v_t^\tau|^2 \, dp_t^\tau dt \leq C^2$, and $p^\tau \to p$ uniformly, and $E^\tau \to E$ weakly. In other words, we have $\partial_t p_t + \nabla \cdot E = 0$. We are left to compute E.

Claim: $E = -p\nabla\mathcal{L}$.

The trick here is to instead consider simpler curves than the interpolation we defined above. In particular, consider $\tilde{p}_t^\tau = p_{k+1}^\tau$ for $t \in (k\tau, (k+1)\tau)$ and $\tilde{v}_t^\tau = v_{k+1}^\tau$. Likewise, define $\tilde{E}^\tau = \tilde{p}^\tau \tilde{v}^\tau$. Since

$$\frac{W_2(p_{k+1}^\tau, p_k^\tau)}{\tau} = \frac{1}{\tau}\left(\int |\iota - T_{k+1}^\tau|^2 \, dp_{k+1}^\tau\right)^{1/2} = ||\tilde{v}_{k+1}^\tau||_{L^2(p_{k+1}^\tau)},$$

we have $||\tilde{v}_{k+1}^\tau||_{L^2(p_t^\tau)} = ||v_{k+1}^\tau||_{L^2(p_t^\tau)}$. This implies that $\tilde{E}^\tau$ has the same bound as in Eqn. 17. Moreover, by using Eqn. 18, we see $W_2(p_t^\tau, \tilde{p}_t^\tau) \leq C^2\sqrt{\tau}$. Hence, we have the same convergence: $\tilde{p}^\tau \to p$ uniformly. One needs to be more careful with the momentum. Let $\tilde{E}^\tau \to \tilde{E}$ weakly, we look to prove $\int f \cdot d\tilde{E} = \int f \cdot dE$ for all $f : [0,1] \times \Omega \to \mathbb{R}^p$ Lipschitz. We compute:

$$\left|\int f \cdot d\tilde{E}^\tau - \int f \cdot dE^\tau\right| \leq \int_0^1 \int_\Omega |f \circ ((k\tau - t)v_{\kappa(t)}^\tau + \iota) - f||v_{\kappa(t)}^\tau| \, dp^\tau \, dt \leq Lip(f)\tau \int_0^1 \int_\Omega |\tilde{v}_t^\tau|^2 \, dp^\tau \, dt,$$

where $\kappa(t)$ returns the smallest multiple $k\tau \geq t$, and the RHS is bounded by $Lip(f)C^2\tau$. Thus, as $\tau \to 0$, we see that $E = \tilde{E}$.

But now, $\tilde{E}^\tau = \tilde{v}^\tau \tilde{p}^\tau = -\tilde{p}^\tau \nabla\mathcal{L}$ by Lemma 1. Therefore, for any $f \in C_c^1((0,1) \times \Omega, \mathbb{R}^p)$, we test:

$$\int f \cdot d\tilde{E}^\tau = -\int f \cdot \nabla\mathcal{L} p^\tau.$$

As $\mathcal{L} \in C^1$, we know the integrand is continuous, so we can just pass the limit as $\tau \to 0$, meaning that $\tilde{E} = E = -p\nabla\mathcal{L}$ invoking the fact that $\tilde{p}^\tau \to p$ uniformly. □

Theorem 2:
Suppose the true marginals evolve according to $\frac{d}{dt}p_t^* = -\nabla \cdot (p_t^* \nabla s_t^*)$. Define $q(z)$ such that marginalizing q with respect to all variables except x_k, x_{k+1} yields the coupling $p_{t_k} \otimes (T_k^{k+1})_\# p_{t_k}$, where T_k^{k+1} is the transport map from p_{t_k} to $p_{t_{k+1}}$. Then,

$$\lim_{|t_k - t_{k+1}| \to 0} \int_0^1 \mathbb{E}_{p_t(x)} ||u_t(x) - \nabla s_t^*(x)||_2^2 \, dt = 0.$$

Proof of Theorem 2. By Ambrosio et al. (2006, Prop. 8.4.6), we have that

$$\nabla s_t^* = \lim_{h \to 0} \frac{1}{h}(T^*(p_t^*, p_{t+h}^*) - \mathrm{id}), \tag{19}$$

where $T^*(p_t^*, p_{t+h}^*)$ is the unique transport map between densities p_t^* and p_{t+h}^*. Fixing a small $h > 0$, define

$$d_t(x) := \sum_{m=0}^{M-1} \mathbf{1}_{[hm, h(m+1))}(t) \frac{T_m^*(x) - x}{h}, \qquad \text{where } T_m^* = T^*(p_{hm}^*, p_{h(m+1)}^*).$$

The idea of our next steps is to instead consider the normed difference between u_t and d_t, and later conclude by the triangle inequality.

Recall that

$$u_t(x) = \mathbb{E}_{q(z)} \frac{u_t(x|z)p_t(x|z)}{p_t(x)} = \sum_{k=0}^{K-1} \mathbf{1}_{[t_k, t_{k+1})}(t) \int u_t(x|x_k, T_k^{k+1}(x_k)) \frac{p_t(x|x_k, T_k^{k+1}(x_k))}{p_t(x)} \, dp_{t_k}(x_k), \tag{20}$$

where we used the assumed disintegration of $q(z)$. We compute

$$\mathbb{E}_{p_t(x)}||u_t(x) - d_t(x)||_2^2 = \int ||u_t(x) - d_t(x)||_2^2 \, dp_t(x) \tag{21}$$

$$= \int \left|\left| \int \left[\frac{T_k^{k+1}(x_k) - x_k}{t_{k+1} - t_k} - \frac{T_m^*(x) - x}{h} \right] \frac{p_t(x|x_k, T_k^{k+1}(x_k))}{p_t(x)} \, dp_{t_k}(x_k) \right|\right|_2^2 dp_t(x) \tag{22}$$

$$\leq \int \int \left|\left| \frac{T_k^{k+1}(x_k) - x_k}{t_{k+1} - t_k} - \frac{T_m^*(x) - x}{h} \right|\right|_2^2 \frac{p_t(x|x_k, T_k^{k+1}(x_k))}{p_t(x)} \, dp_{t_k}(x_k) \, dp_t(x). \tag{23}$$

assuming that $t \in [t_k, t_{k+1})$ and using the fact

$$d_t(x) = \int d_t(x) \frac{p_t(x|x_k, T_k^{k+1}(x_k))}{p_t(x)} \, dp_t(x_k).$$

Claim: The reference p_t is an absolutely continuous curve in W_2-space.

Proof. Let $0 \leq s < t \leq 1$ be given. It is known that affine Gaussian paths are absolutely continuous, in particular this means that there exists $g_k \in L^1([0,1])$ such that $W_2(p_{t_k}, p_{t_{k+1}}) \leq \int_{t_k}^{t_{k+1}} g_k(\tau) \, d\tau$. If $[s,t] \subset [t_k, t_{k+1}]$, then we conclude. Otherwise, consider $W_2(p_s, p_t) \leq W_2(p_s, p_{t_k}) + W_2(p_{t_k}, p_t)$. □

The continuity claim allows us to take the limit: $W_2^2(p_{t_k}, p_t) \to 0$ as $|t_k - t| \to 0$. Moreover, by Santambrogio (2015, Theorem 5.10), we have $p_{t_k} \rightharpoonup p_t$, i.e. narrow convergence.

Claim: Suppose $p_{t_k} \rightharpoonup p_{hm}^*$ and $p_{t_{k+1}} \rightharpoonup p_{h(m+1)}^*$. We have $||T^*(p_{t_k}, p_{t_{k+1}}) - T^*(p_{hm}, p_{h(m+1)})||_{L^2(p_t)} \to 0$.

Proof. Let $t < \min(t_k, h(m+1))$. We first prove this claim for $T^*(p_t, p_{t_{k+1}}) \to T^*(p_t, p_{h(m+1)})$. Villani (2008, Cor. 5.23) gives convergence in measure (in this case, p_t is the measure). Thus, the argument for $L^2(p_t)$ convergence follows from a typical analysis argument. To simplify notation, let $T \equiv T_t^{h(m+1)}$ and $(T_n)_{n \geq 1}$ the approaching sequence.

Suppose not, i.e. there exists a subsequence (T_{n_i}) such that $||T_{n_i} - T||_{L^2(p_t)} \geq \epsilon$ for some $\epsilon > 0$. By convergence in p_t-measure, there exists a further subsequence $(T_{n_{i_j}})$ that converges to T pointwise p_t-a.e. Since $T_n, T \in L^2(p_t)$, by the Dominated Convergence Theorem $||T_{n_{i_j}} - T||_{L^2(p_t)} \to 0$, and hence $\int |T_{n_{i_j}}(x)| \, dp_t(x) \to \int |T(x)| \, dp_t(x)$. At this point, we have proven that for every subsequence (taken implicitly) of integral $\int |T_n(x)| \, dp_t(x)$, there exists a further subsequence that converges to the limit. It is a well-known result in analysis that this shows convergence of the original sequence of integrals, i.e. $||T_n - T||_{L^2(p_t)} \to 0$.

We now apply the previous result three times to the following and conclude:

$$\begin{aligned} ||T^*(p_{t_k}, p_{t_{k+1}}) - T^*(p_{hm}, p_{h(m+1)})||_{L^2(p_t)} \leq \; & ||T^*(p_{t_k}, p_{t_{k+1}}) - T^*(p_{t_k}, p_t)||_{L^2(p_t)} \\ & + ||T^*(p_{t_k}, p_t) - T^*(p_{hm}, p_t)||_{L^2(p_t)} \\ & + ||T^*(p_{hm}, p_t) - T^*(p_{hm}, p_{h(m+1)})||_{L^2(p_t)} \end{aligned}$$

□

To finish, we remind the reader that the densities $p_{t_k} = p_{t_k}^*$, therefore, the supposition of this claim is fulfilled by the observation of narrow convergence: $p_{t_k}^* \rightharpoonup p_{hm}^*$ and $p_{t_{k+1}}^* \rightharpoonup p_{h(m+1)}^*$. Therefore, combining the claim with Eqn. 23, we can make $\mathbb{E}_{p_t(x)}||u_t(x) - d_t(x)||_2^2 < \delta/2$ for some small $\delta > 0$. Then, if we had chosen a small enough h, we would have $\mathbb{E}_{p_t(x)}||\nabla s_t^*(x) - d_t(x)||_2^2 < \delta/2$. Combining these bounds, we conclude.

□

Theorem 3:
Suppose the Lagrangian $L(x_t, \dot{x}_t, t) = ||\dot{x}_t||_2^2 + V_t(x_t, \dot{x}_t)$ is Tonelli and strongly convex in velocity. The Lagrangian optimal transport map T exists between p_0 and p_1. Moreover, there exists a locally Lipschitz, locally bounded vector field w s.t.

$$\partial_t \hat{p}_t + \nabla \cdot (\hat{p}_t w_t) = 0$$

satisfies $\hat{p}_t = \text{Law}(\gamma_t)$ where γ is a random, smooth Lagrangian-minimizing curve and (γ_0, γ_1) is an optimal coupling of p_0, p_1.

Proof of Theorem 3. First, let us recall the definition of a *Tonelli* Lagrangian. Following (Schachter, 2017), it satisfies:

1. L does not depend on time.
2. L is C^2.
3. L is strictly convex in velocity.
4. There exists a constant c_0 and a function $\theta : \mathbb{R}^p \to \mathbb{R}$ with superlinear growth, i.e. $\lim_{|v| \to \infty} \theta(v)/|v| = \infty$, with $\theta \geq 0$ s.t. $L(x, v) \geq c_0 + \theta(v)$.

Then, as noted in Schachter (2017, Ch. 3.3), the Lagrangian optimal transport problem has a solution, specifically a map $T : \mathbb{R}^p \to \mathbb{R}^p$ which pushforwards p_0 to p_1. For our purposes, we should also note that there exists a unique optimal trajectory $\sigma : [0, 1] \times \mathbb{R}^p \to \mathbb{R}^p$ s.t.

$$\sigma = \text{arginf}_{\sigma : [0,1] \times \mathbb{R}^p \to \mathbb{R}^p} \left\{ \int_0^1 \int_{\mathbb{R}^p} L(\sigma(t, x), \dot{\sigma}(t, x))\, dp_0(x)\, dt : \sigma(1, \cdot)_\# p_0 = p_1 \right\}.$$

Using Schachter (2017, Prop. 3.4.4), we have a velocity field $w : [0, 1] \times \mathbb{R}^p \to \mathbb{R}^p$ satisfying $\dot{\sigma}(t, x) = w_t(\sigma(t, x))$ which is locally Lipschitz and locally bounded. Then, by Schachter (2017, Prop. 3.4.3), the path defined by $\hat{p}_t = (\sigma_t)_\# p_0$ and the velocity field w_t satisfies the continuity equation

$$\partial_t \hat{p}_t + \nabla \cdot (\hat{p}_t w_t) = 0$$

in the sense of distributions. All that's left is to show that $\hat{p}_t = \text{Law}(\gamma_t)$ where γ is drawn from the Lagrangian-minimizing curves. However, this follows from Villani (2008, Thm. 7.21) as $(\hat{p}_t)_{t \in [0,1]}$ minimizes

$$\mathbb{A}(p) = \inf_\sigma \int_0^1 \int_{\mathbb{R}^p} L(\sigma(t, x), \dot{\sigma}(t, x))\, dp_0(x)\, dt,$$

applying the equivalence between (iii) and (i) of Thm. 7.21. □

Proposition 2: Suppose that $\nabla \mathcal{L}$ is uniformly Lipschitz in x with Lipschitz constant K. Then,

$$W_2^2(p_t, \hat{p}_t) \leq e^{(1+2K)t} \int_0^t \mathbb{E}_\gamma [\nabla \mathcal{L}(\gamma_s) + \dot{\gamma}_s]\, ds.$$

Proof of Prop. 2. First, note from Villani (2008, Thm. 7.21) that as $p_t, \hat{p}_t$ are continuous paths, there exists dynamical optimal couplings of (p_0, p_1), as defined in Villani (2008, Def. 7.20), which we shall denote γ_*, γ respectively. In particular, we use that $p_t = \text{Law}(\gamma_*(t))$ and $\hat{p}_t = \text{Law}(\gamma(t))$, and that both $(\gamma(0), \gamma(1))$ and $(\gamma_*(0), \gamma_*(1))$ are distributed according to π the optimal coupling between (p_0, p_1). Understanding this, we may define

$$Q_t := \mathbb{E}_{(\gamma(0), \gamma(1)) \sim \pi} ||\gamma_*(t) - \gamma(t)||^2 = \mathbb{E}_{(\gamma_*(0), \gamma_*(1)) \sim \pi} ||\gamma_*(t) - \gamma(t)||^2 \geq W_2^2(p_t, \hat{p}_t). \tag{24}$$

Furthermore, as the reference path $(p_t)_{t\in[0,1]}$ follows the continuity equation Eqn. 3, $\dot{\gamma}(t) = -\nabla\mathcal{L}(\gamma_*(t))$. Thus, considering the time-derivative, we have

$$\begin{aligned}\frac{\partial Q_t}{\partial t} &= 2\int \langle \gamma(t)-\gamma_*(t), -\nabla\mathcal{L}(\gamma_*(t)) - \dot{\gamma}(t)\rangle \, d\pi(\gamma_0,\gamma_1) \\ &= 2\int \langle \gamma(t)-\gamma_*(t), -\nabla\mathcal{L}(\gamma_*(t)) + \nabla\mathcal{L}(\gamma(t))\rangle \, d\pi(\gamma_0,\gamma_1) \\ &\qquad + 2\int \langle \gamma(t)-\gamma_*(t), -\nabla\mathcal{L}(\gamma(t)) - \dot{\gamma}(t)\rangle \, d\pi(\gamma_0,\gamma_1)\end{aligned}$$

The first term may be bounded by Lipschitzness of $\nabla\mathcal{L}$:

$$2\langle \gamma(t)-\gamma_*(t), -\nabla\mathcal{L}(\gamma_*(t)) + \nabla\mathcal{L}(\gamma(t))\rangle \le 2K||\gamma(t)-\gamma_*(t)||^2.$$

The second term may be bounded by:

$$\begin{aligned}&||\gamma(t)-\gamma_*(t)||^2 - 2\langle \gamma(t)-\gamma_*(t), -\nabla\mathcal{L}(\gamma(t)) - \dot{\gamma}(t)\rangle + ||\nabla\mathcal{L}(\gamma(t)) + \dot{\gamma}(t)||^2 \ge 0, \\ &2\langle \gamma(t)-\gamma_*(t), -\nabla\mathcal{L}(\gamma(t)) - \dot{\gamma}(t)\rangle \le ||\gamma(t)-\gamma_*(t)||^2 + ||\nabla\mathcal{L}(\gamma(t)) + \dot{\gamma}(t)||^2\end{aligned}$$

In summary,

$$\frac{\partial Q_t}{\partial t} \le (1+2K)Q_t + \int ||\nabla\mathcal{L}(\gamma(t)) + \dot{\gamma}(t)||^2 \, d\pi(\gamma_0,\gamma_1),$$

then by Gronwall's inequality:

$$Q_t \le \exp(t(1+2K)) \int_0^t \int ||\nabla\mathcal{L}(\gamma(t)) + \dot{\gamma}(t)||^2 \, d\pi(\gamma_0,\gamma_1),$$

using the fact that $Q_0 = 0$, and now we conclude by the fact that $W_2^2(p_t,\hat{p}_t) \le Q_t$. □

Proposition 3 (Adapted from Cor. 5.25 Santambrogio (2015))**.** *Suppose that $(p,\hat{p})$ resides in a compact domain $\Omega \subset \mathbb{R}^p$ and suppose that $p_t, \hat{p}_t$ are absolutely continuous w.r.t. Lebesgue measure for every t. Further, if we assume $p, \hat{p}$ are absolutely continuous curves in $W_2(\Omega)$, then*

$$W_2^2(p_t,\hat{p}_t) = 2\int_0^1 \int_\Omega (x - T_t(x)) \cdot (\nabla\mathcal{L}(x) + w_t(T_t(x))) \, dt \, dp_t(x), \tag{25}$$

where T_t is the optimal transport map between p_t and $\hat{p}_t$ for the cost $|x-y|^2/2$.

Proof of Prop. 3. The assumption aligns with Santambrogio (2015, Cor. 5.25), which gives:

$$\frac{d}{dt}W_2^2(p_t,\hat{p}_t) = 2\int_\Omega (x - T_t(x)) \cdot (\nabla\mathcal{L}(x) - w_t(T_t(x))) \, dp_t(x).$$

Integrating both sides and noting that $p_0 = \hat{p}_0$ yields the desired result.

□

Theorem 4:
Define $g(\gamma,t) := ||\nabla\mathcal{L}(\gamma_t) + \dot{\gamma}_t||^2$. Suppose for each $(x_0,x_1) \sim \pi$ there exists a smooth connecting curve $\tilde{\gamma}$ s.t. $\sup_{0\le t\le 1} g(\tilde{\gamma},t) \le \delta$ for some $\delta > 0$ and length $\int_0^1 ||\dot{\gamma}_t||^2 \le \Gamma$. If there exists $\eta > \delta > 0$ s.t.

1. $V(\gamma_t,\dot{\gamma}_t) \ge 2\Gamma g(\gamma,t)$ for $g(\gamma,t) \ge \eta$, and
2. $V(\gamma_t,\dot{\gamma}_t) \le \min\{\Gamma/\delta, \Gamma\} g(\gamma,t)$ if $g(\gamma,t) \le \delta$,

then $\sup_{0\le t\le 1} g(\gamma_*,t) \le 2\eta$, where γ_* is the Lagrangian-minimizing curve.

Proof of Theorem 4. Suppose not, so that there exists $t_0 \in [0,1]$ s.t. $g(\gamma_*, t_0) = ||\dot{\gamma}_*(t_0) + \nabla\mathcal{L}(\gamma_*(t_0))||^2 > 2\eta$. By continuity of γ_* and $\nabla\mathcal{L}$ w.r.t. time, we have $0 \leq t' \leq t_0$ s.t. for any $t \in [t', t_0]$, $g(\gamma_*, t) \geq \eta$ and t' is chosen s.t. $g(\gamma_*, t') = \eta$. By our assumption on η, it's natural that we look at the action under two different cases. First, if $g(\gamma_*, t) \geq \eta$, we have:

$$\begin{aligned} A(\gamma_*) &\geq \int_{t'}^{t_0} ||\dot{\gamma}_*(t)||_2^2 + V(\gamma_*(t), \dot{\gamma}_*(t))\, dt \\ &\geq \int_{t'}^{t_0} ||\dot{\gamma}_*(t)||_2^2 + 2\Gamma||\dot{\gamma}_*(t) + \nabla\mathcal{L}(\gamma_*(t))||_2^2\, dt \\ &> ||\gamma_*(t') - \gamma_*(t_0)||^2 + 2\Gamma\eta \geq 2\Gamma\eta. \end{aligned}$$

Otherwise, we have by assumption that there exists a smooth connecting curve γ s.t. $d_\nabla(\gamma) \leq \delta < \eta$, hence

$$\begin{aligned} A(\gamma) &= \int_0^1 ||\dot{\gamma}(t)||_2^2 + V(\gamma(t), \dot{\gamma}(t))\, dt \\ &\leq \Gamma + \int_0^1 ||\dot{\gamma}_*(t) + \nabla\mathcal{L}(\gamma_*(t))||_2^2\, dt \\ &\leq \Gamma + \delta \cdot \Gamma/\delta = 2\Gamma. \end{aligned}$$

By minimality of the Lagrangian-minimizing curve γ_*, we have a contradiction. Hence, for all times, $g(\gamma_*, t) \leq 2\eta$, implying the result. □

B Remark on weight initialization

By adapting a well-known result (Ambrosio et al., 2006, Prop. 9.3.2), we can quantify how the choice of weight initialization affects the distribution of converged training weights.

Proposition 4. *Further assume that $\mathcal{L}$ is λ-convex. Given two different weight initializations $p_0^{(0)}$ and $p_0^{(1)}$ on $\mathcal{P}(\Omega)$ that evolves according to Eqn. 3, we have $W_2(p_t^{(0)}, p_t^{(1)}) \leq e^{-\lambda t} W_2(p_0^{(0)}, p_0^{(1)})$.*

Proof of Prop. 4. If $\mathcal{L}$ is λ-convex, then by Ambrosio et al. (2006, Prop. 9.3.2), we have that $L^\dagger(p) := \int_\Omega \mathcal{L}(x)\, dp(x)$ is λ-geodesically convex. Now, we see that Ambrosio et al. (2006, Thm. 11.1.4) applies, and we get the desired inequality. □

The primary hurdle to applying Prop. 4 is that $\mathcal{L}$ is rarely convex in the network parameters. For instance, optimization of a multi-layer perceptron (MLP) is highly non-convex due to non-linear activations and the product between hidden and outer layer weights. Interestingly, given a loss minimization problem on a MLP, we have a corresponding convex optimization problem (Pilanci and Ergen, 2020), i.e. the loss objective is convex in the network parameters and the two problems have identical optimal values. Therefore, with a modified loss function $\mathcal{L}'$, if p_1' is distributed over its minimizers, e.g. gradient descent is used to solve the convex minimization problem until convergence, we can apply Prop. 4 with $\lambda = 0$ and $\mathcal{L}'(\theta_1) = \mathcal{L}(\theta_1)$ is minimal for $\theta_1 \sim p_1'$.

C Related works

Conditional flow matching. The CFM objective, where a conditional vector field is regressed to learn probability paths from a source to target distribution, was first introduced in Lipman et al. (2023). The CFM objective attempts to minimize the expected squared loss of a target conditional vector field (which is conditioned on training data and generates a desired probability path) and an unconditional neural network. The authors showed that optimizing the CFM objective is equivalent to optimizing the unconditional FM objective. Moreover, the further work (Tong et al., 2024) highlighted that certain choices of parameters for the probability paths led to the optimal conditional flow being equivalent to the optimal transport path

between the initial and target data distributions, thus resulting in shorter inference times. However, the original formulations of flow matching assumed that the initial distributions were Gaussian. Pooladian et al. (2023) extended the theory to arbitrary source distributions using minibatch sampling and proved a bound on the variance of the gradient of the objective. Tong et al. (2024) showed that using the 2-Wasserstein optimal transport map as the joint probability distribution of the initial and target data along with straight conditional probability paths results in a marginal vector field that solves the dynamical optimal transport problem between the initial and target distributions.

Neural network parameter generation. Due to the flexibility of neural network as function approximators, it is natural to think that they could be applied to neural network weights. Denil et al. (2014) paved the way for this exploration as their work provided evidence of the redundancy of most network parameterizations, hence showing that paramter generation is a feasible objective. Later, Ha et al. (2017) introduced Hypernetworks which use embeddings of weights of neural network layers to generate new weights and apply their approach to dynamic weight generation of RNNs and LSTMs. A significant portion of our paper's unconditional parameter generation section builds upon the ideas from Wang et al. (2024) and the concurrent work of Soro et al. (2025) where the authors employ a latent diffusion model to generate new parameters for trained image classification networks.

Meta-learning context. Although neural networks are adept at tasks on which they were trained, a common struggle of networks is generalization to unseen tasks. In contrast, humans can often learn new tasks when given only a few examples. A pioneering modern work in this field is MAML (Finn et al., 2017), which learns good initialization parameters for the meta-learner such that it can easily be fine-tuned to new tasks. Their approach utilizes two nested training loops. The inner loop computes separate parameters adapted to each of the training tasks. The outer loop computes the loss using each of these parameters on their respective tasks and updates the model's parameters through gradient descent. However, MAML often had unstable training runs, and so successive works gradually refined the method (Antoniou et al., 2019; Rajeswaran et al., 2019; Zhao et al., 2020; Przewiezlikowski et al., 2022). The aforementioned works typically focus on classification tasks, however, this paradigm allows for great versatility. For instance, Beck et al. (2023) used hypernetworks to generate the parameters of a policy model and (Lee et al., 2023) exploited the in-context learning ability of transformers to general reinforcement learning tasks.

Weight generation for few-shot learning. Following up on the work of meta-learning context, few-shot learning is a natural application of such meta-learning algorithms. An early example is Ravi and Larochelle (2017) who designed a meta-learner based on the computations in an LSTM cell. At each training example in the support set, their meta learner uses the losses and the gradients of the losses of the base learner (in addition to other information from previous training examples) to produce base learner parameters for the next training example. The loss of the base learner on the test examples in the support set is backpropagated through the meta learner's parameters. Moreover, we may leverage the advancements in generative modeling for weight generation. Lee et al. (2023) used transformers for in-context reinforcement learning, but we also see the works of Zhmoginov et al. (2022); Hu et al. (2022); Kirsch et al. (2024); Fifty et al. (2024) use transformers and foundation models. More similar to our method is the body of work on using diffusion models for weight generation (Du et al., 2023; Zhang et al., 2024; Wang et al., 2024; Soro et al., 2025). These methods vary in their approach, some leveraging a relationship between the gradient descent algorithm and the denoising step in diffusion models to design their meta-learning algorithm. Others rely on the modeling capabilities of conditioned latent diffusion models to learn the target distribution of weights. Most evaluations conducted were in-distribution tasks, i.e. tasks sampled from the same data distribution as the training tasks, hence, there is room to explore ways of adapting this approach for out-of-distribution tasks.

D Stochastic formulation of weight evolution

D.1 Setup

The present formulation of gradient descent as gradient flow ignores the crucial role of noise within typical neural network optimization schemes. Stochastic differential equations (SDEs) provides a way to model SGD

as a continuous-time process while taking into account the role of noise. Following Li et al. (2021), we write:

$$dX_t = -\nabla\mathcal{L}(X_t)dt + (\alpha\Sigma(X_t))^{1/2}dW_t \tag{26}$$

where W_t is the Wiener process, $\alpha > 0$ is the learning rate, and $\Sigma(X) = \mathbb{E}[(\nabla\mathcal{L}_\xi(X) - \nabla\mathcal{L}(X))(\nabla\mathcal{L}_\xi(X) - \nabla\mathcal{L}(X))^\top]$; here, ξ is a random variable denoting the random batch of training data in the context of SGD. We may then write the Fokker-Planck-Kolmogorov (FPK) equation

$$\partial_t p_t - \nabla\cdot(p_t\nabla\mathcal{L}) = \frac{\alpha}{2}\sum_{ij}\frac{\partial^2}{\partial x_i\partial x_j}([\Sigma^{1/2}\Sigma^{\top/2}]_{ij}p_t),$$

and the corresponding Schrödinger bridge (SB) problem

$$\mathbb{P}^* := \underset{\mathbb{P}_0=p_0,\mathbb{P}_1=p_1}{\arg\min}\ D_{KL}(\mathbb{P}|\mathbb{Q}) \tag{27}$$

where $\mathbb{Q} = \mathrm{Law}(X)$ as governed by Eqn. 26.

D.2 Solution with known SB matching methods

We run into the issue of well-posedness if the noise covariance is not known a priori, and also if it depends on the state X_t. For now, we are aware of the work by Berlinghieri et al. (2025), which only assumes standard SDE regularity conditions to optimize Eqn. 27. In particular, given a density over time $h(\cdot)$ and samples from a random time t_i, we may construct a state distribution at snapshots $\hat{f}_{t_i}(\cdot)$. For our approximation, we instead use an empirical time density $\hat{h}(\cdot)$ approximated from sampled timesteps, and a candidate SDE model parameterized by θ, $f_{\theta,t}$. Training proceeds by using the maximum mean discrepancy (MMD) to quantify the discrepancy between $\hat{h}\circ f_{\theta,t}$ and $\hat{h}\circ\hat{f}_t$.

To simplify the analysis and align with most SB approximation methods, take $\epsilon_t = \alpha\mathbb{E}_{X_t}[\Sigma(X_t)]$ to form the approximation

$$dY_t = -\nabla\mathcal{L}(Y_t)dt + \sqrt{\epsilon_t}dW_t, \tag{28}$$

and obtain the FPK

$$\partial_t p_t - \nabla\cdot(p_t\nabla\mathcal{L}) = \frac{\epsilon_t}{2}\Delta p_t. \tag{29}$$

In this form, multi-marginal methods (Lavenant et al., 2024; Chen et al., 2023; Shen et al., 2025) may be employed with intermediate weight samples as reference data.

Variational interpolants. The recent work by Shen et al. (2025) sheds some light directly onto the problem of modeling gradient flows. This method allows us to specify a family of possible proxies e.g. those induced by the SDE

$$dX_t = \nabla\Psi_\alpha(X_t)dt + \gamma_\beta dW_t, \tag{30}$$

where α, β are learnable parameters. This method is explored in the context of multi-marginal Schrödinger bridges, but it is trivial to modify it for our purposes:

1. For $i = 0, \ldots, K-1$, consider data anchors $\{\theta^j_{t_i}\}_{j\in[N_i]}$ and $\{\theta^j_{t_{i+1}}\}_{j\in[N_{i+1}]}$.
   (a) Simulate forward to t_{i+1} from $\{\theta^j_{t_i}\}_{j\in[N_i]}$ using Eqn. 30.
   (b) Simulate backward to t_i from $\{\theta^j_{t_{i+1}}\}_{j\in[N_{i+1}]}$ using Eqn. 30.
   (c) Use simulated samples to estimate the drift between t_i and t_{i+1}.
2. Concatenate the estimated drifts and use Stage 2 of Alg. 1 in Shen et al. (2025) to fit α according to the estimated drifts.

The diffusion parameter β can be estimated in an outer loop of the above algorithm, as suggested by Shen et al. (2025), allowing us to match ϵ_t. By following this procedure for flow model training, we can vary our interpolant within a natural family of path distributions, using data anchors to better inform training.

Generalized Schrödinger Bridge Matching. Due to the close relation with our analysis in Sec. 3.2, we further discuss GSBM (Liu et al., 2024c) which employs entropic action matching as the inner loop objective. In particular, given the reference process Eqn. 29, invoke Neklyudov et al. (2023, Prop. 3.1) to obtain a unique entropic action satisfying the FPK equation Eqn. 29. However, in this case, we have a two level optimization: **(1)** perform action matching to obtain the drift u_t^θ for a fixed reference path $(\hat{p}_t)_{t\in[0,1]}$, and **(2)** optimize the marginals $\hat{p}_t$ given the coupling $p_{0,1}^\theta$ evolved according to the learned drift u_t^θ in

$$dX_t = u_t^\theta(X_t)dt + \sqrt{\epsilon_t}dW_t. \tag{31}$$

Most relevant to us is the second stage which involves optimizing the marginal distributions. Given $x_0 \sim p_0$ and x_1 from Eqn. 31, we also obtain the intermediate states $\{X_{t_k}\}$ for $0 < t_1 < \cdots < t_K < 1$. To parameterize the marginals p_t, we assume a Gaussian path $p_t(X_t|x_0, x_1) = \mathcal{N}(\mu_t, \sigma_t^2\boldsymbol{I})$, hence deferring our optimization to μ_t and σ_t. Liu et al. (2024c) uses 1-D splines with the control points $\{X_{t_k}\}$ to obtain μ_t and a uniform sampling of σ_t (with boundary conditions $\sigma_0 = \sigma_1 = 0$). Using this parameterization, we can compute the minimization objective

$$\mathcal{J} = \int_0^1 \mathbb{E}_{p_t(X_t|x_0,x_1)}\left[\frac{1}{2}||u_t^\theta(X_t)||_2^2 + V_t(X_t)\right] dt \tag{32}$$

to optimize the control points X_{t_k} and σ_{t_k}, given some choice of $V_t(\cdot)$.

Returning to the marginals p_t, we wish to relate its evolution with Eqn. 28. Applying, for example, Du et al. (2024, Prop. 3), we can write down the FPK equation

$$\partial_t \hat{p}_t(x) = -\nabla\cdot(\hat{p}_t(x)u_t(x)) + \frac{\epsilon_t}{2}\Delta\hat{p}_t \quad \text{where} \quad u_t(x) = \dot{\mu}_t + \frac{1}{2}\left(\frac{\dot{\sigma}_t}{\sigma_t} - \frac{\epsilon_t}{\sigma_t}\right)(x - \mu_t). \tag{33}$$

Therefore, our analogy to Eqn. 13 would be to choose our energy functional V to minimize the gap

$$\int_0^1 \mathbb{E}_{x\sim\hat{p}_t}||\nabla\mathcal{L}(x) + u_t(x)||_2^2\, dt.$$

E Interpolating paths

There are many variants of interpolating paths that have been used as reference in the flow matching literature. Typically, the conditional probability path is of the form

$$p_t(x|z) = \mathcal{N}(x; \mu_t(z), \sigma_t^2(z)\boldsymbol{I}) \tag{34}$$

with the consideration that the boundary conditions are satisfied, i.e. $\int p_0(x|z)\, dq(z) = p_0(x)$ and $\int p_1(x|z)\, dq(z) = p_1(x)$. The simplest case by Tong et al. (2024) considers a linear interpolant $\mu_t(x_0, x_1) = tx_1 + (1-t)x_0$ with a constant, small variance and $q(x_0, x_1) = p_0(x_0)p_1(x_1)$.

Since we are most concerned with the inducing vector field (see Eqn. 1), we would like a simple vector field that induce the desired conditional probability path. The Gaussian path Eqn. 34 is known (Rohbeck et al., 2025) to have flow

$$\phi_t(x|z) = \mu_t(z) + \sigma_t(z)\left(\frac{x - \mu_0(z)}{\sigma_0(z)}\right)$$

which, in fact, has a unique inducing vector field (Lipman et al., 2023)

$$u_t(x|z) = \frac{\sigma_t'(z)}{\sigma_t(z)}(x_t - \mu_t(z)) + \mu_t'(z). \tag{35}$$

The above suggests that if we have a desired interpolating path μ_t, the vector field to match is known from Eqn. 35. Following, we discuss a few learned variants such as JKOnet (Bunne et al., 2022; Terpin et al., 2024), gWOT (Lavenant et al., 2024), and a varying proxy by Shen et al. (2025). We close by discussing the simple interpolants we use in our experiments.

JKOnet. Deceptively, we first discuss a learned method which does not produce interpolants per se, but instead an energy functional J_ξ used in a JKO scheme. Looking at the proof of Theorem 1 in App. A, we see the relation of Eqn. 3 to a JKO flow. Indeed, J_ξ attempts to approximate $\int_\Omega \mathcal{L}\, dp$ from samples of the weight population by a neural network (Bunne et al., 2022; Terpin et al., 2024). As this is a learned method, we incur some computational cost, but also we require a dense sampling of intermediate weights to construct the necessary population samples. In return, we could even utilize this network as the meta-learner directly, or use it as a reference for flow model training.

Lifted curves. In spline interpolation, it is best if data points correspond to different timesteps to better capture the trajectory over time. If instead we have a sampling of population over time, it is more natural to consider matching marginal path distributions. Lavenant et al. (2024) (gWOT) provides a framework for exactly this. When we have data from N_i samples at various time points t_i, $\{\theta^j_{t_i}\}_{i\in[K]}$, we may form the empirical distribution

$$\hat{\rho}_{t_i} = \frac{1}{N_i}\sum_{j=1}^{N_i} \delta_{\theta^j_{t_i}}.$$

To produce smoother interpolants, we also introduce a regularizer and Gaussian convolution with a kernel of width h to obtain $\hat{\rho}^h_{t_i}$. Specifically, we minimize (by gradient descent) the convex functional

$$F_{K,\lambda,h}(\mathbf{R}) := \sigma^2 D_{KL}(\mathbf{R}|\mathbf{W}^\sigma) + \frac{1}{\lambda}\sum_{i=1}^{K} |t_{i+1} - t_i| D_{KL}(\hat{\rho}^h_{t_i}|\mathbf{R}_{t_i})$$

over a law on paths $\mathbf{R} \in \mathcal{P}(\Omega)$.

Data-dependent geodesics. Kapusniak et al. (2024) made use of a data-dependent metric to compute a geodesic. In practice, the interpolant is obtained through training a neural network to minimize

$$L_{\text{mfm}}(\theta) = \mathbb{E}_{\text{t}\sim U[0,1],(\mathbf{x}_0,\mathbf{x}_1)\sim\pi} ||v_\theta(\mathbf{x}_{t,\eta^*}, t) - \dot{\mathbf{x}}_{t,\eta^*}||^2_{g(\mathbf{x}_{t,\eta^*})}. \tag{36}$$

Comparing with Eqn. 35, note that we are using a small, constant variance, so indeed we only match the interpolant derivative. Moreover, as g is a data-dependent metric, its optimization towards a suitable proxy path reduces to learning a parameterization of g w.r.t. the intermediate weights $\mathcal{D}$. Here, we note the MFM framework fits without issue into the proxy path framework of Sec. 3.2 primarily because geodesics are presumed smooth in time. Intuitively, due to randomness in the training process, some intermediate weights $\theta \in \mathcal{D}$ ought to be weighed less than others so that the induced geodesic γ better matches the true loss gradient.

Cubic splines. Following the recent work by Rohbeck et al. (2025), we can fit a cubic spline to conditional data points and use this curve as the reference interpolant. Cubic splines are obtained by optimizing the variational objective

$$\mu_t(x_0, \ldots, x_k) = \arg\min_{\gamma\in\mathcal{H}^2([t_0,t_K])} \int_{t_0}^{t_K} ||\ddot{\gamma}(t)||_2^2\, dt \tag{37}$$

where $x_k = \gamma(t_k)$ and $\mathcal{H}^2([t_0, t_K])$ denotes the class of functions that has absolutely continuous first derivative and weak second derivative on the interval $[t_0, t_K]$. The conditioning data may be sampled, say from our dataset $\mathcal{D}$. Moreover, Rohbeck et al. (2025) considered class-conditional trajectories. In our setting, this could mean weight trajectories from different training datasets. Thus, we sample intermediate weights $(x^c_0, \ldots, x^c_K)$, where c denotes the training set, and we may use techniques such as classifier-free guidance on v_θ to improve training. Note that as the interpolant is entirely contingent on the intermediate distributions, and specifically the intermediate samples, interpolant optimization cannot be done within this approach. Instead, we rely on a faithful sampling $\mathcal{D}$ from the reference Eqn. 3 to provide a good proxy path.

Piecewise linear interpolants. We conclude by discussing the most straightforward method of incorporating conditional data. In particular, if we have points $z = (x_0, \ldots, x_K)$ at times $(t_0, \ldots, t_K)$, we have the

conditional vector field

$$u_t(x|z) = \sum_{k=0}^{K-1} \frac{x_{k+1} - x_k}{t_{k+1} - t_k} \mathbf{1}_{[t_k, t_{k+1})}(t). \tag{38}$$

Within our conceptual framework, we may write

$$\frac{1}{2}\int_0^1 \mathbb{E}_\gamma ||\nabla \mathcal{L}(\gamma_t) + \dot{\gamma}_t||^2 \, dt = \frac{1}{2}\int_0^1 \sum_{k=0}^{K-1} \mathbf{1}_{[t_k, t_{k+1})}(t) \mathbb{E}_{(x_k, x_{k+1}) \sim p_{t_k} \otimes p_{t_{k+1}}} ||\nabla \mathcal{L}(\gamma_t) + \frac{x_{k+1} - x_k}{t_{k+1} - t_k}||^2$$

as the action gap. As the interpolant is entirely contingent on the intermediate distributions, and specifically the intermediate samples, interpolant optimization cannot be done within this approach. However, this approach has desirable limiting properties w.r.t. the sampling of $\mathcal{D}$. In particular, recalling Eqn. 3, if we let $\sigma : [0, 1] \times \mathbb{R}^p \to \mathbb{R}^p$ be the flow of the drift in Eqn. 3, the expression

$$\mathbb{E}_{(x_k, x_{k+1}) \sim p_{t_k} \otimes (\sigma_{t_{k+1} - t_k})_\# p_{tk}} ||\nabla \mathcal{L}(\gamma_t) + \frac{x_{k+1} - x_k}{t_{k+1} - t_k}||^2$$

goes to zero as $t_{k+1} - t_k \to 0$. In other words, if the intermediate samples are drawn from the same training trajectory, and are sampled with sufficient time-density, the gap can be made arbitrarily small.

F Implementation details

Here, we expound on the implementation of our approach. See Figure 1 for a schematic of the training and inference process.

F.1 Pre-trained model acquisition

Datasets and architectures. We conduct experiments on a wide range of datasets, including CIFAR-10/100 (Krizhevsky and Hinton, 2009), STL-10 (Coates et al., 2011), Fashion-MNIST (Xiao et al., 2017), CIFAR10.1 (Recht et al., 2019), and Camelyon17 (Veeling et al., 2018). To evaluate our meta-model's ability to generate new subsets of network parameters, we conduct experiments on ResNet-18 (He et al., 2015b), ViT-Base (Dosovitskiy et al., 2021), ConvNeXt-Tiny (Liu et al., 2022), the latter two are sourced from timm Wightman (2019). As we shall detail below, small CNN architectures from a model zoo (Schürholt et al., 2022) are also used for full-model generations.

Model pre-training and checkpointing. For better control over the target distribution p_1, in experiments involving ResNet-18, ViT-Base, and ConvNeXt-Tiny, we pre-train these base models from scratch on their respective datasets. We follow Wang et al. (2024) and train the base models until their accuracy stabilizes (in practice, we train all base models for 100 epochs). Depending on the experiment, we save checkpoints differently. If only the converged weights are needed, we save 200 weights at every iteration past 100 epochs. Otherwise, if we require intermediate weights, then we specify the number of saving epochs and the number of weights to save in such an epoch. For instance, we may have 100 save epochs and 100 saves per epoch, meaning that we save at *every* training epoch and save weights in the first 100 iterations of each epoch.

Model zoo. The model zoo used for meta-training in the model retrieval setting, was sourced from (Schürholt et al., 2022). As the base model, we employed their CNN-medium architecture, which consists of three convolutional layers and contains 10,000-11,000 parameters, depending on the number of input channels. We pre-trained these models as described above.

F.2 Variational autoencoder

The variational autoencoder follows the implementation of Soro et al. (2025), which employs a UNet architecture for auto-encoding. In particular, given a set of model weights $\{\mathcal{M}_i\}_{i=1}^N$, we first flatten the weights to obtain vectors $\boldsymbol{w}_i \in \mathbb{R}^{d_i}$. For the sake of uniformity, we always zero-pad vectors to $d = \max_i d_i$. Alternatively, we allow for layer-wise vectorization: set a chunk size ℓ which corresponds to the weight

dimension of a network layer. Then, zero-pad $\boldsymbol{w}_i$ to be a multiple of ℓ, say $\tilde{d}$. This allows us to partition into k equal length vectors $\boldsymbol{w}_{i,k} \in \mathbb{R}^{\tilde{d}/k}$. Typically, larger models benefit from layer-wise vectorization.

Subsequently, we train a VAE to obtain an embedding of such vectors by optimizing the objective:

$$L_{\text{VAE}}(\theta, \phi) := -\mathbb{E}_{\boldsymbol{z} \sim q(\boldsymbol{z}|\boldsymbol{w})}[\log p_\theta(\boldsymbol{w}|\boldsymbol{z}) + \beta D_{KL}(q_\phi(\boldsymbol{z}|\boldsymbol{w})||p(\boldsymbol{z}))] \tag{39}$$

where $\boldsymbol{w}$ is the vectorized weights, $\boldsymbol{z}$ is the embedding we are learning, and p_θ, q_ϕ are the reconstruction and posterior distributions respectively. Moreover, we fix the prior $p(\boldsymbol{z})$ to be a $(0, 1)$-Gaussian and the weight is set to be $\beta = 10^{-5}$. For layer-wise vectorization, we simply change the input dimensions to match the chunk size. Upon decoding, we concatenate the chunks to re-form the weight vector.

As for training, the VAE was trained with the objective in equation 39. Moreover, following p-diff (Wang et al., 2024), we add Gaussian noise to the input and latent vector, i.e. given noise factors σ_{in} and σ_{lat} with encoder f_ϕ and decoder f_θ, we have

$$\boldsymbol{z} = f_\phi(\boldsymbol{w} + \xi_{in}),\ \hat{\boldsymbol{w}} = f_\theta(\boldsymbol{z} + \xi_{lat}) \quad \text{where} \quad \xi_{in} \sim \mathcal{N}(0, \sigma_{in}^2 \boldsymbol{I}),\ \xi_{lat} \sim \mathcal{N}(0, \sigma_{lat}^2 \boldsymbol{I}).$$

A new VAE is trained at every instantiation of the N$\mathcal{M}$-CFM model as architectures often differ in their input dimension for different experiments. However, they are trained with different objectives: the VAE is trained to minimize reconstruction loss. In all experiments, we fix $\sigma_{in} = 0.001$ and $\sigma_{lat} = 0.01$.

Algorithm 1 Sampling Trajectories from Weight Tensor

Require: Number of save epochs N_{epochs}, savepoints per epoch S, classifier size D, tensor of classifier weights $X \in \mathbb{R}^{N_{epochs} \times S \times D}$, number of time samples $K \leq N_{epochs}$.

Ensure: Sampled tensor $W \in \mathbb{R}^{K \times S \times D}$

1: Flatten X to shape $[N_{epochs} \cdot S, D]$ ▷ Assumes first dim. sorted by training iteration.
2: Sample $K \cdot S$ indices I uniformly from $[0, N_{epochs} S)$
3: Extract $X[I]$ and reshape to $W \in \mathbb{R}^{K \times S \times D}$
4: **if** add_noise **then**
5: $\quad W \leftarrow W + \epsilon$ ▷ $\epsilon \sim \mathcal{N}(0, 10^{-3})$
6: **end if**
7: **return** W

F.3 Generative meta-model

Multi-marginal flow matching. Multi-marginal flow matching (MMFM) proceeds in the same regression paradigm as flow matching models, with the difference being the regression target is now Eqn. 38 (piece-wise linear interpolation). In practice, $z = (x_0, \ldots, x_K)$, where we have $K = 3, 4$, or 5, as evaluated in Table 1, and the elements of z are sampled from $p_{t_0} \otimes \cdots \otimes p_{t_K}$ constituted by samples obtained from base model pre-training App. F.1. Typically, we save a lot more checkpoints than needed, and so we need to subsample them by Algorithm 1 to create the training dataset. For better training, we may also choose to sample weight initializations, i.e. $x_0 \in z$, for each batch. This can be done easily by using `torch.nn.init` to reset parameters of a module to the desired initialization, e.g. Kaiming uniform. In fact, sampling the weight initialization is all that is required for the validation step.

JKOnet*. The dataloading aspect of JKOnet training is exactly the same as MMFM, so we focus our attention to the training regiment. Following Terpin et al. (2024), we may formally write our loss as

$$\sum_{k=0}^{K-1} \int_{\mathbb{R}^D \times \mathbb{R}^D} ||\nabla V_\theta(x_t, t) + (x_{t+1} - x_t)/\Delta t||^2 \, d\gamma(x_t, x_{t+1}). \tag{40}$$

Recalling the gradient descent formula, the time argument is not necessary for this loss, but we found it to improve empirical performance. Moreover, the $\gamma(\cdot, \cdot\cdot)$ distribution is traditionally an optimal coupling of consecutive marginals $p_{t_k}, p_{t_{k+1}}$, however, we have a more natural choice: x_t and x_{t+1} ought to be

Table 6: Model architectures and hyperparameters. The JKOnet model uses the same UNet as N$\mathcal{M}$-CFM and N$\mathcal{M}$-MMFM, but with the up-sampling section replaced by pooling and linear layers.

Weight Encoder (UNet)	
Architecture	VAE
Latent Space Size	$4 \times 4 \times 4$
Upsampling/Downsampling Layers	5
Channel Multiplication (per Downsampling Layer)	(1, 1, 2, 2, 2)
ResNet Blocks (per Layer)	2
KL-Divergence Weight	1e-5
N$\mathcal{M}$-CFM and N$\mathcal{M}$-MMFM Model (UNet)	
Input Size w/ VAE	$4 \times 4 \times 4$
Input Size w/o VAE	variable
Upsampling/Downsampling Layers	4
Channel Multiplication (per Downsampling Layer)	(2, 2, 2, 2)
ResNet Blocks (per Layer)	2

checkpoints saved consecutively during pre-training. Indeed, we found this to be the superior choice in terms of performance. Moreover, an important note is in order: since JKOnet* expects x_t to evolve by a gradient flow, this imposes a condition on how the checkpoints are obtained. We found that meta-training **only works if pre-training is done using the SGD optimizer**. Modern optimizers such as Adam notably fails for JKOnet*. We also connect this point to the footnote in Table 2: just because the weights evolve by a gradient flow does *not* mean that the encoded weights behave the same way. Indeed, model retrieval in latent space fails when we use JKOnet*. This suggests an avenue for further research: construct an autoencoder that preserves gradient flow in latent space, where preferably the latent gradient flow can be deduced/estimated from the original gradient steps.

Architecture. The neural network used for flow matching is the UNet from D2NWG (Soro et al., 2025). The specific hyperparameters used for the CFM model varies between experiments, so we leave this discussion to the next section. For experiments such as model retrieval where we require a conditioning vector, this is implemented by concatenating a context feature vector (e.g. images are passed into a CLIP encoder (Radford et al., 2021), and optionally an attention module if we have a set of context images) to the last dimension (same axis as the neural network parameters); of course, we ignore these extra features after the forward pass.

F.4 Reward fine-tuning

Adjoint matching with $\sigma_t = 10^{-3}$. We start in Domingo-Enrich et al. (2025, App. C.5). As we are using N$\mathcal{M}$-CFM, let $Y_0 \sim p_0$ and $Y_1 \sim p_1$, then we may write

$$\begin{aligned} v(x,t) &= \mathbb{E}[\alpha_t Y_1 + \beta_t Y_0 \mid x = \alpha_t Y_1 + \beta_t Y_0] \\ &= \mathbb{E}[\frac{\dot{\alpha}_t(x - \beta_t Y_0)}{\alpha_t} + \dot{\beta}_t Y_0 \mid x = \alpha_t Y_1 + \beta_t Y_0] \\ &= \frac{\dot{\alpha}_t}{\alpha_t} x + (\dot{\beta}_t - \frac{\dot{\alpha}_t}{\alpha_t}\beta_t)\mathbb{E}[Y_0 \mid x = \alpha_t Y_1 + \beta_t Y_0], \end{aligned} \tag{41}$$

using the fact that $Y_1 = (x - \beta_t Y_0)/\alpha_t$. On a practical note, we typically have $\alpha_t = t$, $\beta_t = 1 - t$. A central piece of the theory consists of relating this vector field v with the score function $s(x,t)$. We note that once this relationship is established with σ_t, then the rest of the adjoint matching derivation follows. The crucial step is in writing the score function; following from Domingo-Enrich et al. (2025, Eqn. 92),

$$s(x,t) = \frac{\mathbb{E}[p_{t|1}(x|Y_1)\nabla \log p_{t|1}(x|Y_1)]}{p_t(x)}, \quad \text{where} \quad p_{t|1}(x|Y_1) = \frac{\exp(-||x - \alpha_t Y_1||^2/2\beta_t^2)}{(2\pi\beta_t^2)^{D/2}}. \tag{42}$$

It suffices to change $p_{t|1}$:

$$p_{t|1}(x|Y_1) = \frac{\exp(-||x - \alpha_t Y_1||^2/(2\sigma_t)^2)}{(2\pi\sigma_t^2)^{D/2}} \implies \nabla \log p_{t|1}(x|Y_1) = -\frac{x - \alpha_t Y_1}{\sigma_t^2} \tag{43}$$

and combine with Eqn. 42 to obtain $s(x,t) = -\beta_t \mathbb{E}[Y_0 \mid x = \beta_t Y_0 + \alpha_t Y_1]/\sigma_t^2$. Further combining with Eqn. 41, we have the correspondence

$$v(x,t) = \frac{\dot{\alpha}_t}{\alpha_t}x + \left(\frac{\dot{\alpha}_t}{\alpha_t}\beta_t - \dot{\beta}_t\right)\frac{\sigma_t^2}{\beta_t}s(x,t) \iff s(x,t) = \frac{\beta_t\left(v(x,t) - \frac{\dot{\alpha}_t}{\alpha_t}x\right)}{\sigma_t^2\left(\frac{\dot{\alpha}_t}{\alpha_t}\beta_t - \dot{\beta}_t\right)}. \tag{44}$$

Concluding, we may thus write the SDE that mirrors Domingo-Enrich et al. (2025, Eqn. 6):

$$\begin{aligned} dX_t &= b(X_t,t)\,dt + \epsilon(t)\,dB_t \\ &= \left[\frac{\dot{\alpha}_t}{\alpha_t}x + s(x,t)\left(\left(\frac{\dot{\alpha}_t}{\alpha_t}\beta_t - \dot{\beta}_t\right)\frac{\sigma_t^2}{\beta_t} + \epsilon(t)^2/2\right)\right]dt + \epsilon(t)\,dB_t. \end{aligned} \tag{45}$$

Algorithm 2 Deterministic adjoint matching for fine-tuning flow models

Require: Pre-trained FM velocity field v^{base}, step size h, number of fine-tuning iterations N, trajectory batch size M, dataset batch size m, initialized $v^{ft} = v^{base}$, cross-entropy loss $\mathcal{L}$.
Ensure: Reward fine-tuned FM velocity field v^{ft}.

for $n \in [0, \dots, N-1]$ **do**
  Sample M trajectories $\boldsymbol{X} = (X_t)_{0\le t\le 1}$ with an Euler solver with step size h and $X_0 \sim p_0$.
  $\{(x_i, y_i)\}_{i=1}^m \sim \mathcal{D}$ ▷ Sample from classifier dataset
  For each of the M trajectories, evaluate classifier on predicted weights
    $\ell(X_1) = \sum_{i=1}^m \mathcal{L}(\text{NNET}_{X_1}(x_i), y_i)$. ▷ $\ell(X_1)$ is a vector of size M
  $\tilde{a}_{t-h} = \tilde{a}_t + h\tilde{a}_t^\top \nabla_{X_t} v^{base}(X_t,t), \quad \tilde{a}_1 = \nabla_{X_1}\ell(X_1)$ ▷ Backward solve the lean adjoint ODE
  Detach from computation graphs: $X_t = \texttt{stopgrad}(X_t)$ and $\tilde{a}_t = \texttt{stopgrad}(\tilde{a}_t)$.
  $\mathcal{L}_{AM}(\theta) = \sum_{t\in\{0,\dots,1-h\}} ||v_\theta^{ft}(X_t,t) - (v^{base}(X_t,t) - \tilde{a}_t)||^2$.
  Compute $\nabla_\theta \mathcal{L}_{AM}$ and optimize as usual.
end for

Deterministic in practice. The key quantity in adjoint matching is the memory-less schedule given as $\epsilon(t) = \sqrt{2\eta_t}$, where reading from Eqn. 45, $\eta_t = \left(\frac{\dot{\alpha}_t}{\alpha_t}\beta_t - \dot{\beta}_t\right)\frac{\sigma_t^2}{\beta_t}$. Plugging in $\alpha_t = t$, $\beta_t = 1-t$, we have

$$\epsilon(t) = \sqrt{\frac{2\sigma_t^2}{1-t}((1-t)/t+1)} = \sqrt{\frac{2\sigma_t^2}{t(1-t)}} = \sigma_t\sqrt{\frac{2}{t(1-t)}}.$$

Following the suggestions in Domingo-Enrich et al. (2025, App. H), we add a small value h to both terms in the denominator. In practice, this means that $\sqrt{\frac{2}{t(1-t)}} \le 10$, whereas $\sigma_t = 10^{-3}$. Looking at the adjoint matching algorithm, this clearly presents a problem of scale. We found in practice that a deterministic variant, i.e. $\epsilon(t) = 0$ works well in our evaluations. To formulate this modified algorithm, note that our primary objective is to optimize over the control u in $dX_t = (b(X_t,t) + \epsilon(t)u(X_t,t))\,dt + \epsilon(t)\,dB_t$. The original formulation includes the correction term to the drift, and defines $v^{ft}(x,t)$ as

$$b(x,t) + \epsilon(t)u(x,t) = 2v^{ft}(x,t) - \frac{\dot{\alpha}_t}{\alpha_t}x.$$

We simplify by instead defining $b(x,t) = v^{base}(x,t)$ and $v^{ft}(x,t) = b(x,t) + u(x,t)$. Consequently, this simplifies the adjoint ODE and the regression target, leading to Algorithm 2.

Multi-marginal adjoint matching. Since this paper touches upon multi-marginal flow matching in its experiments, we think it natural to extend the adjoint matching computations to the multi-marginal setting. In this section, we provide the derivations and leave experimentation to future work. Suppose we are given

$z = (Y_0, \dots, Y_K)$ at times $(t_0, \dots, t_K) \subset [0,1]$, we first need to find an expression for a sampled point x analagous to $x = \beta_t Y_0 + \alpha_t Y_1$. Motivated by Eqn. 38, define

$$x = \sum_{k=0}^{K-1} [(1 - s_t^{(k)})Y_k + s_t^{(k)} Y_{k+1}] \mathbf{1}_{[t_k, t_{k+1})}(t), \quad \text{where } s_t^{(k)} = \frac{t - t_k}{t_{k+1} - t_k}. \tag{46}$$

Differentiating w.r.t. t, we indeed get $u_t(x|z)$ in Eqn. 38; moreover, to simplify notatoin, set $r_t^{(k)} = 1 - s_t^{(k)}$. The key insight is that only one of the terms in Eqn. 46 is non-zero for any $t \in [0,1]$, therefore

$$\begin{aligned} v(x,t) &= \sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \mathbb{E}[\dot{r}_t^{(k)} Y_k + \dot{s}_t^{(k)} Y_{k+1} \mid x] \\ &= \sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \mathbb{E}\left[\dot{r}_t^{(k)} Y_k + \dot{s}_t^{(k)} \frac{x - r_t^{(k)} Y_k}{s_t^{(k)}} \mid x\right] \\ &= \sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \left[\frac{\dot{s}_t^{(k)}}{s_t^{(k)}} x + \left(r_t^{(k)} - \frac{\dot{s}_t^{(k)}}{s_t^{(k)}} r_t^{(k)}\right) \mathbb{E}[Y_k \mid x]\right]. \end{aligned} \tag{47}$$

Like before, the key step is to relate the velocity v and the score function. Note that the backward conditional probabilty is now

$$p_{t|1}(x|z) = \sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) p_{t|t+1}(x|Y_{k+1}), \tag{48}$$

where again we use σ_t:

$$p_{t|t+1}(x|Y_{k+1}) = \frac{\exp(-||x - s_t^{(k)} Y_{k+1}||^2 / 2\sigma_t^2)}{(2\pi\sigma_t^2)^{D/2}} \implies \nabla \log p_{t|t+1}(x|Y_{k+1}) = -\frac{x - s_t^{(k)} Y_{k+1}}{\sigma_t^2}. \tag{49}$$

Since $\mathbb{E}, \nabla$ are linear, we may move around the sum as we please. Therefore, analagous to the CFM case:

$$s(x,t) = -\sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \mathbb{E}\left[\frac{x - s_t^{(k)} Y_{k+1}}{\sigma_t^2} \middle| x\right] = -\sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \frac{r_t^{(k)}}{\sigma_t^2} \mathbb{E}[Y_k \mid x]. \tag{50}$$

At this point, it is a matter of using the same trick as in Eqn. 47: there must be at most one $k \in [0, \dots, K-1]$ such that $\mathbb{E}[Y_k \mid x] = -\sigma_t^2 s(x,t)/r_t^{(k)}$. Therefore,

$$\begin{aligned} v(x,t) &= \sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \left[\frac{\dot{s}_t^{(k)}}{s_t^{(k)}} x + \frac{\sigma_t^2}{r_t^{(k)}} \left(\frac{\dot{s}_t^{(k)}}{s_t^{(k)}} r_t^{(k)} - r_t^{(k)}\right) s(x,t)\right] \\ \iff s(x,t) &= \sum_k \mathbf{1}_{[t_k, t_{k+1})}(t) \left[\frac{r_t^{(k)}}{\sigma_t^2 \left(\frac{\dot{s}_t^{(k)}}{s_t^{(k)}} r_t^{(k)} - r_t^{(k)}\right)} \left(v(x,t) - \frac{\dot{s}_t^{(k)}}{s_t^{(k)}} x\right)\right] \end{aligned} \tag{51}$$

Now recall that

$$\frac{dX_t}{dt} = v(X_t, t) \iff dX_t = \left(v(X_t, t) + \frac{\epsilon(t)^2}{2} s(x,t)\right) dt + \epsilon(t)\, dB_t$$

as given in Domingo-Enrich et al. (2025, Eqns. 3, 4). Let $b(X_t, t)$ denote the drift term in the SDE above. Plugging in v and s from Eqn. 51, we find

$$\eta_t = \sum_{k=0}^{K-1} \mathbf{1}_{[t_k, t_{k+1})}(t) \frac{\sigma_t^2 \left(\frac{\dot{s}_t^{(k)}}{s_t^{(k)}} r_t^{(k)} - r_t^{(k)}\right)}{r_t^{(k)}}. \tag{52}$$

Therefore, the memory-less noise schedule $\epsilon(t) = \sqrt{2\eta_t}$ yields

$$b(X_t, t) = \sum_{k=0}^{K-1} \mathbf{1}_{[t_k, t_{k+1})}(t) \left[2v(X_t, t) - \frac{\dot{s}_t^{(k)}}{s_t^{(k)}} X_t\right]$$

and by analogy we define the v^{ft} via a control function u:

$$b(x,t) + \epsilon(t)u(x,t) = \sum_{k=0}^{K-1} \mathbf{1}_{[t_k,t_{k+1})}(t) \left[2v^{ft}(x,t) - \frac{\dot{s}_t^{(k)}}{s_t^{(k)}} x \right]. \tag{53}$$

Table 7: Best validation accuracy of unconditional N$\mathcal{M}$ generation for various datasets. *orig* denotes base models trained conventionally by SGD, *p-diff* those generated with p-diff (Wang et al., 2024), and || denotes equality with the above row. The base models that precede CNN3 are all partially generated, specifically we focus on generating just the batch norm parameters.

	CIFAR100			CIFAR10			MNIST			STL10		
Base Models	orig.	N$\mathcal{M}$	p-diff.	orig.	N$\mathcal{M}$	p-diff.	orig.	N$\mathcal{M}$	p-diff	orig.	N$\mathcal{M}$	p-diff
ResNet-18: N$\mathcal{M}$-CFM w/ VAE	71.45	71.42	71.40	94.54	94.36	94.36	99.68	99.65	99.65	62.00	62.00	62.24
ViT-base: N$\mathcal{M}$-CFM w/ VAE	85.95	85.86	85.85	98.20	98.11	98.12	99.41	99.38	99.36	96.15	95.77	95.80
ConvNext-tiny: N$\mathcal{M}$-CFM w/ VAE	85.06	85.12	85.17	98.03	97.89	97.90	99.42	99.41	99.40	95.95	95.63	95.63
CNN3: N$\mathcal{M}$-CFM w/ VAE	26.62	27.01	26.99	63.38	64.32	63.37	98.93	98.91	98.93	53.88	53.50	53.86
CNN3: N$\mathcal{M}$-CFM	||	25.31	||	||	62.52	||	||	98.52	||	||	53.49	||
CNN3: N$\mathcal{M}$-MMFM(2)	||	21.16	||	||	63.35	||	||	98.53	||	||	53.20	||
CNN3: N$\mathcal{M}$-MMFM(3)	||	24.53	||	||	63.34	||	||	98.62	||	||	53.53	||
CNN3: N$\mathcal{M}$-MMFM(4)	||	22.89	||	||	61.33	||	||	98.39	||	||	53.08	||
CNN3: N$\mathcal{M}$-JKO(3)	||	24.84	||	||	62.60	||	||	98.59	||	||	53.44	||
CNN3: N$\mathcal{M}$-JKO(4)	||	24.97	||	||	63.35	||	||	98.57	||	||	53.21	||

G Experimental details

In this section, we provide further experimental details such as hyperparameters and computation times, alongside some extra results.

G.1 Unconditional generation

Unconditional generation involves two stages: first is the training of base models. We choose a Resnet18, ViT-B, ConvNext-tiny, and CNN3 for our base models and provide the training parameters in Table 8. Next, we train the generative meta-model; Table 9 lists the training settings. We found that in most cases, the autoencoder and N$\mathcal{M}$-CFM converge after 1000 epochs. One should note that when using Kaiming uniform for the source p_0, it is also necessary to train the autoencoder on this distribution. On a NVIDIA A40 card, we estimate training time to be around 2 hours when the VAE is used. Un-encoded training generally takes double that time. Inference requires less than 1 minute to complete for both encoded and un-encoded runs, and it takes a few seconds to go through the validation set to compute the classification accuracy. Notably, the JKO runs were about 2× **as fast** since the model output is a scalar. This means we can compress inputs through the downsampling layers of the UNet, leading to a smaller parameter count. Extra results are presented in Table 7.

G.2 Model retrieval

For each dataset, we first sample 30 images as representatives, and then pre-compute its CLIP (Radford et al., 2021) embeddings. We then have a choice of how to aggregate the 30 embeddings. In our case, we use a light multi-head attention module and a linear layer to compress the context into a vector. Generally, we maintain the feature dimension—the version of CLIP we use returns a 768-dimensional vector. To condition the flow/diffusion model with this context, we concatenate to the channel dimension if we are using a VAE, or to the last dimension otherwise. One should note that the number of parameters differ between datasets; for instance, the three-channel datasets have slightly more parameters than their one-channel counterparts. Hence, we apply a simple zero-padding to standardize the input dimension. On a NVIDIA A40 card, we estimate a training time of 2 hours for the VAE and 3 hours for the CFM to achieve our level of accuracy (w/ VAE) and about 5 hours un-encoded. Inference times remain the same as in *unconditional generation*.

Table 8: Task training settings. Note that only CNN3 was evaluated for MMFM and JKOnet experiments, so it's the only model type with non-zero *Num. of save epochs* and *Savepoints per epoch*.

Model Type	ResNet18	ViT & ConvNext	CNN
Optimizer	SGD	AdamW	SGD
Initial training LR	0.1	1×10^{-4}	0.1
Training scheduler	MultiStepLR	CosineAnnealingLR	MultiStepLR
Layer params. saved	Last 2 BN layers	Last 2 BN layers	All layers
Num. of save epochs	0	0	100
Savepoints per epoch	0	0	100
Num. final weights saved	200	200	200
Saved parameter count	2048	3072	[10565, 12743]
Training epochs	100	100	100
Batch Size	64	128	128

Table 9: Training settings for modules in the unconditional generation experiment. The number of JKOnet inference steps depends on the number of intermediate marginal distributions we are modeling.

Model Type	Autoencoder	CFM w/ AE	MMFM	JKOnet
Optimizer	AdamW	AdamW	AdamW	AdamW
(LR, weight decay)	(1e-4, 2e-6)	(1e-4, 2e-6)	(3e-4, 2e-6)	(5e-3, 2e-6)
Num. inference steps	n/a	100	100	variable
Weight initialization	n/a	Kaiming uniform	Kaiming uniform	Kaiming uniform
Epochs	500	1000	1000	1000
Batch size	64	64	64	64

G.3 Downstream initialization

In this evaluation, we obtain weights from model retrieval (albeit trained with weights before convergence), and fine-tune on their corresponding dataset, i.e. if we retrieve an MNIST classifier, we fine-tune on MNIST and $\widetilde{\text{MNIST}}$. Fine-tuning is done conventionally with the task training settings in Table 8. The corruption of datasets is done by applying the following transformations: random horizontal flip, random rotation (max 15 deg), color jittering, Gaussian blur (kernel size of 3, $\sigma \in [0.1, 2.0]$).

G.4 Reward fine-tuning

Training settings. For the corresponding experiments, we used an AdamW optimizer with learning rate 2×10^{-5} and weight decay 5×10^{-4}. We use a trajectory batch size of 8 (denoted M in Algorithm 2) and a dataset batch size $m = 128/512$ depending on the dataset. We also clip gradients at norm 1.0 and set the number of fine-tuning iterations to 150. We use a cosine annealing scheduler with $\eta_{\min} = 10^{-6}$. The step size was set to $h = 0.025$, meaning our trajectory consists of 40 timesteps. As suggested in Domingo-Enrich et al. (2025, App. H), we only evaluated gradients at 20 out of 40 timesteps: 10 of the last timesteps and a uniform sample of 10 from the first 30 timesteps.

Training tricks. We also introduce a few tricks. First, Algorithm 2 suggests that we must take a fine-tuning step every time we sample a dataset batch. We instead opt to average the starting lean adjoint $\tilde{a}_1$ over 3 batch samples; we found this to result in more stable training losses. In fact, this is all the iterations done per epoch, so if $N = 150$, we only have 450 batch iterations total. Moreover, as $\tilde{a}_1$ is the gradient of a classifier loss, we experimented with treating it like a stochastic gradient descent step, which means including a learning rate, momentum, and weight decay parameter. To clarify, this involves saving the $\tilde{a}_1$ values from previous training iterations. It does not seem to help much, other than the learning rate. We set the reward learning rate to 1.5 and momentum to 0.01.

Padding regularization. Another trick specific to N$\mathcal{M}$-CFM-All is to use padding regularization. As mentioned in the model retrieval section, zero-padding is applied to standardize the input dimension given the variability between dataset classifiers. This trick indexes the padded elements in the input tensor and adds its ℓ_2-norm to the loss, thus coercing it towards zero. Notably, this is an implicit method of conditioning on

the dataset. For example, if we are regularizing more elements of the input, that suggests we are classifying on a one-channel dataset since these classifiers require more padding.

Weight augmentation. In our experiments, we also tried augmenting the network weight data acquired from pre-training in an attempt to expand supp $\hat{p}_1$. This is done by simple Gaussian noise ($\sigma = 5 \times 10^{-3}$), dropout ($p = 0.02$), and mix-up. Recall that mix-up involves sampling a data pair (w_1, w_2) and an interpolation parameter $\alpha \sim \text{Uniform}[0, 1]$, and returning an interpolation $(1 - \alpha)w_1 + \alpha w_2$.

Corruption levels. As part of our experiments to get a sense of the width of classifier supports, we applied increasing corruption to the base datasets. The transformations are as follows:

1. Level 1: random horizontal flip, random rotation (max 15 deg).
2. Level 2: Level 1 + color jittering, and Gaussian blur (kernel size of 3 and $\sigma \in [0.1, 2.0]$).
3. Level 3: Level 2 + random erasing with $p = 0.5$, scale in [0.2, 0.5], and ratio in [0.3, 3.3].

Table 10: Mean validation accuracy over five generated classifiers after reward fine-tuning on increasingly corrupted datasets. The -All suffix indicates that the CFM was trained on classifiers of CIFAR10, STL10, MNIST, and FMNIST, whereas +A indicates weight augmentation, and +P indicates regularization on padded values. The arrow '→' indicates the accuracy before (left) and after (right) reward fine-tuning.

	CIFAR10		
Corruption Level	0	1	2
SGD fine-tuning	63.38 → 63.38	59.93 → 60.91	24.18 → 49.90
N$\mathcal{M}$-CFM	$62.53 \pm 0.02 \to 63.33 \pm 0.08$	$58.65 \pm 0.22 \to 60.34 \pm 0.76$	$24.84 \pm 0.93 \to 34.15 \pm 0.74$
N$\mathcal{M}$-CFM-All	$28.92 \pm 11.19 \to 61.90 \pm 0.22$	$23.57 \pm 16.43 \to 57.65 \pm 0.91$	$22.72 \pm 1.56 \to 34.59 \pm 1.88$
N$\mathcal{M}$-CFM-All+A	$45.07 \pm 15.64 \to 56.24 \pm 1.65$	$37.53 \pm 15.16 \to 55.99 \pm 1.17$	$19.89 \pm 4.11 \to 32.44 \pm 1.73$
N$\mathcal{M}$-CFM-All+P	$50.18 \pm 18.49 \to 60.29 \pm 0.62$	$51.36 \pm 14.51 \to 56.64 \pm 1.14$	$21.84 \pm 3.69 \to 33.97 \pm 1.23$
N$\mathcal{M}$-CFM-All+A+P	$39.25 \pm 17.68 \to 58.85 \pm 1.33$	$41.59 \pm 15.05 \to 55.91 \pm 1.79$	$19.21 \pm 3.30 \to 33.79 \pm 2.00$
	MNIST		
Corruption Level	0	1	2
SGD fine-tuning	98.93 → 98.93	96.58 → 97.78	18.8 → 97.55
N$\mathcal{M}$-CFM	$98.52 \pm 0.01 \to 98.79 \pm 0.04$	$95.87 \pm 0.01 \to 97.01 \pm 2.27$	$15.68 \pm 0.17 \to 91.21 \pm 3.05$
N$\mathcal{M}$-CFM-All	$60.77 \pm 27.58 \to 97.56 \pm 1.20$	$35.74 \pm 27.14 \to 59.32 \pm 27.88$	$26.68 \pm 22.11 \to 65.40 \pm 34.17$
N$\mathcal{M}$-CFM-All+A	$37.67 \pm 37.07 \to 95.84 \pm 0.53$	$49.42 \pm 37.93 \to 92.50 \pm 8.99$	$11.14 \pm 4.65 \to 35.32 \pm 28.25$
N$\mathcal{M}$-CFM-All+P	$30.28 \pm 33.54 \to 95.72 \pm 1.03$	$49.24 \pm 38.96 \to 94.65 \pm 0.41$	$28.54 \pm 24.07 \to 78.87 \pm 21.20$
N$\mathcal{M}$-CFM-All+A+P	$27.57 \pm 34.27 \to 95.88 \pm 0.35$	$58.32 \pm 38.92 \to 94.59 \pm 0.30$	$12.68 \pm 6.16 \to 88.64 \pm 3.92$

Support of classifier weights. To get a sense of how the weight distributions change as the dataset changes, see Table 10. In this experiment, we reward fine-tuned the N$\mathcal{M}$-CFM meta-model on increasingly corrupted versions of the base training dataset. The affect of the corruption is noticeable on the support as reward fine-tuning, which is constrained within the support set, fails to reach the accuracy of conventional

Table 11: Mean validation accuracy over five generated classifiers after reward fine-tuning on four datasets. The -All suffix indicates that the CFM was trained on classifiers of CIFAR10, STL10, MNIST, and FMNIST, whereas +A indicates weight augmentation, and +P indicates regularization on padded values. The arrow '→' indicates the accuracy before (left) and after (right) reward fine-tuning.

	CIFAR10	STL10	MNIST	FMNIST
N$\mathcal{M}$-CFM-All	$28.92 \pm 11.19 \to 61.90 \pm 0.22$	$42.21 \pm 11.68 \to 52.63 \pm 0.14$	$60.77 \pm 27.58 \to 97.56 \pm 1.20$	$43.12 \pm 34.22 \to 88.49 \pm 1.09$
N$\mathcal{M}$-CFM-All+A	$45.07 \pm 15.64 \to 56.24 \pm 1.65$	$27.21 \pm 11.44 \to 50.66 \pm 1.71$	$37.67 \pm 37.07 \to 95.84 \pm 0.53$	$55.49 \pm 33.85 \to 85.65 \pm 2.76$
N$\mathcal{M}$-CFM-All+P	$50.18 \pm 18.49 \to 60.29 \pm 0.62$	$22.05 \pm 6.32 \to 52.11 \pm 0.36$	$30.28 \pm 33.54 \to 95.72 \pm 1.03$	$32.07 \pm 29.62 \to 86.93 \pm 1.97$
N$\mathcal{M}$-CFM-All+A+P	$39.25 \pm 17.68 \to 58.85 \pm 1.33$	$21.27 \pm 5.06 \to 50.49 \pm 0.77$	$27.57 \pm 34.27 \to 95.88 \pm 0.35$	$46.20 \pm 34.08 \to 84.21 \pm 5.30$

fine-tuning. This holds true even for mild corruption schemes, suggesting the ideal classifier support on the corrupted set has little intersection with the original support, indicating narrowness of the set. To see that different classifiers have mostly disjoint supports, we try expanding supp p_1^{base} by training on classifiers for different datasets. To verify this, we trained a new N$\mathcal{M}$-CFM model on classifiers for different datasets *without* any context conditioning. Since the fine-tuned target distribution ought to classify only one dataset, the hope is for fine-tuning to redirect the velocity field towards this one classifier distribution. This intuition turns out to be insufficient, as shown by the N$\mathcal{M}$-CFM-All rows in the table, further supporting our hypothesis. Moreover, this result holds with weight augmentations. The results also suggest that context conditioning is necessary for consistent validation accuracies, as convincingly shown in the MNIST case. Indeed, the padding regularization is an implicit form of context conditioning as the MNIST classifiers—expecting one-channel images—are slightly smaller than the 3-channel dataset classifiers. Further reinforcing our hypothesis, we also provide results of a N$\mathcal{M}$-CFM model trained on all four datasets fine-tuned to generate classifiers for each in Table 11.

Computation times. On a NVIDIA A40 GPU, one full training run takes about 2 and a half hours. During evaluation, we sample generated weights 5 times and validate on the test dataset; this test completes in under 5 minutes.

Table 12: True positive rate at the 5% significance level (TPR@5) and area under receiver operating characteristic curve (AUROC) for detection of harmful covariate shift on CIFAR10.1 and Camelyon17. We test on both the disagreement rate (DAR) and the entropy, setting $\lambda = \kappa/(|\mathbf{Q}| + 1)$. The best result for each column and our method are **bolded**.

TPR@5	CIFAR10			Camelyon		
$\lvert\mathbf{Q}\rvert$	10	20	50	10	20	50
Detectron (DAR), $\kappa = 1$	0	0	0	0	$.10 \pm .10$	0
Detectron (DAR), κ match	0	0	0	$.30 \pm .15$	$.20 \pm .13$	$\mathbf{.50 \pm .17}$
Meta-detectron (DAR)	$.33 \pm .13$	$.47 \pm .13$	$.27 \pm .12$	$\mathbf{.80 \pm .11}$	$\mathbf{.40 \pm .13}$	$.42 \pm .15$
Detectron (Entropy), $\kappa = 1$	$\mathbf{.60 \pm .17}$	$.40 \pm .16$	$.50 \pm .17$	0	0	0
Detectron (Entropy), κ match	$.50 \pm .17$	$.10 \pm .10$	$.20 \pm .13$	$.10 \pm .10$	0	0
Meta-detectron (Entropy)	$.27 \pm .12$	$\mathbf{.93 \pm .07}$	$\mathbf{.93 \pm .07}$	0	0	$.27 \pm .12$
AUROC	CIFAR10			Camelyon		
$\lvert\mathbf{Q}\rvert$	10	20	50	10	20	50
Detectron (DAR), $\kappa = 1$	0.515	0.595	0.485	0.590	0.595	0.795
Detectron (DAR), κ match	0.495	0.485	0.560	0.690	0.795	**0.935**
Meta-detectron (DAR)	**0.849**	0.838	0.938	**0.900**	0.760	0.806
Detectron (Entropy), $\kappa = 1$	0.740	0.695	0.850	0.345	0.610	0.720
Detectron (Entropy), κ match	0.735	0.730	0.820	0.510	0.455	0.600
Meta-detectron (Entropy)	0.716	**0.987**	**0.996**	0.747	**0.836**	0.847

Table 13: In-distribution validation accuracy before and after reward fine-tuning.

	CIFAR10		
$\lvert\mathbf{Q}\rvert$	10	20	50
Meta-detectron ($\mathbf{P}^*$)	$61.81 \pm 0.17 \to 61.01 \pm 0.14$	$60.97 \pm 0.16 \to 60.49 \pm 0.25$	$60.64 \pm 0.19 \to 60.32 \pm 0.30$
Meta-detectron ($\mathbf{Q}$)	$60.84 \pm 0.27 \to 60.85 \pm 0.27$	$61.61 \pm 0.30 \to 61.11 \pm 0.13$	$60.90 \pm 0.15 \to 61.15 \pm 0.28$
	Camelyon		
$\lvert\mathbf{Q}\rvert$	10	20	50
Meta-detectron ($\mathbf{P}^*$)	$92.75 \pm 0.24 \to 91.38 \pm 0.72$	$92.84 \pm 0.21 \to 91.71 \pm 0.59$	$92.63 \pm 0.15 \to 92.39 \pm 0.22$
Meta-detectron ($\mathbf{Q}$)	$92.85 \pm 0.27 \to 92.47 \pm 0.21$	$92.60 \pm 0.19 \to 92.44 \pm 0.34$	$93.03 \pm 0.18 \to 92.70 \pm 0.42$

G.5 Detecting harmful covariate shifts

Training. The only settings that were changed from reward fine-tuning is the learning rate, which is now 1.5×10^{-5} and the number of fine-tuning epochs $N = 100$. The computation time varies between CIFAR10 and Camelyon17. The former completes in 2 hours, whereas the latter requires 3 and a half hours. The difference stems from the higher image resolution of Camelyon17, resulting in more parameters in the classifier.

Choosing λ. Following the exposition in Ginsberg et al. (2023), the choice of λ can be motivated by a counting argument. We suppose that agreeing with the base classifier on a sample of $\mathbf{P}$ incurs a reward of 1 and disagreeing on a sample of $\mathbf{Q}$ incurs a reward of λ. Originally, to encourage agreement of $\mathbf{P}$ as the primary objective, λ is set so that the reward obtained from disagreeing on *all* samples of $\mathbf{Q}$ is less than agreeing on just *one* sample of $\mathbf{P}$, i.e. $\lambda|\mathbf{Q}| < 1$, giving $\lambda = \frac{1}{|\mathbf{Q}|+1}$. However, this argument can be generalized slightly. As reward fine-tuning is a more conservative approach, we want to increase the reward for disagreeing on $\mathbf{Q}$. For instance, we may want the reward obtained from disagreeing on *all* samples of $\mathbf{Q}$ to be about the same as agreeing on $\kappa > 0$ samples of $\mathbf{P}$. This gives $\lambda = \kappa/(|\mathbf{Q}| + 1)$ as the ℓ_{dce} weight.

Choosing κ. We tried $\kappa = 1, 3, 6, 9, |\mathbf{Q}| + 1$ for Camelyon17 and $\kappa = 1, 32, 40, 50, |\mathbf{Q}| + 1$ for CIFAR. We ran two experiments with different weight settings. In our fixed run, seen in Tables 12 and 13, we used $\kappa = 4$ for Camelyon17 and $\kappa = 32$ for CIFAR10. The reason for the lower κ values for Camelyon17 stems from the number of classes: recall that Camelyon17 is requires a binary classifier, whereas there are 10 classes in the CIFAR10 dataset. The dataset batch size also matters: we used a batch size of 128 for Camelyon17 (as the images are larger) and 512 for CIFAR10. In the run where we varied κ over the sample size $|\mathbf{Q}|$, we started at a reference point: $\kappa = 32$ for CIFAR10 at $|\mathbf{Q}| = 20$ and scaled naturally, i.e. $\kappa = 16$ when $|\mathbf{Q}| = 10$ and $\kappa = 80$ when $|\mathbf{Q}| = 50$. Likewise, we used the reference $\kappa = 4$ for Camelyon17 at $|\mathbf{Q}| = 20$ and scaled accordingly.

Shift detection test. We used a standard two-sample test identical to Ginsberg et al. (2023). Given the original $\mathbf{P}$ and the unknown $\mathbf{Q}$, we would like to rule out the null hypothesis $\mathbf{P} = \mathbf{Q}$ at the 5% significance level by comparing two statistics: *entropy* and the *disagreement rate*. The definition of entropy we use measures uncertainty over classes in the logits.

$$\text{Entropy}(x) = \sum_{c=1}^{N} \hat{p}_c \log \hat{p}_c \quad \text{where } \hat{p} = \frac{f(x) + g(x)}{2}, \tag{54}$$

where f is our base classifier and g is the generated classifier. In contrast to Detectron, we do not use CDC ensembles in our method. We use a Kolmogorov-Smirnov test to compute the p-value for covariate shift on the entropy distributions, comparing $g_{\mathbf{P}^*}$ and $g_{\mathbf{Q}}$. Intuitively, when $\mathbf{Q}$ is out-of-distribution, the generated classifier predicts with high entropy on $\mathbf{Q}$ and low on $\mathbf{P}^*$.

Regarding the disagreement rate, the null hypothesis is represented by $\mathbb{E}[\phi_{\mathbf{Q}}] \leq \mathbb{E}[\phi_{\mathbf{P}^*}]$ where ϕ is the disagreement rate and expectation is taken over trial seeds. This comes from the idea that its easier to learn to reject from a distribution that is not in the training set (since the base classifier f will also be unsure). The test result is considered significant at α% when $\phi_{\mathbf{Q}}$ is greater than the $(1 - \alpha)$ quantile of $\phi_{\mathbf{P}^*}$. In practice $\alpha = 5\%$.

Further results. See Tables 12 and 13 for the results when κ is fixed. We also show more plots in Figures 4 and 5.

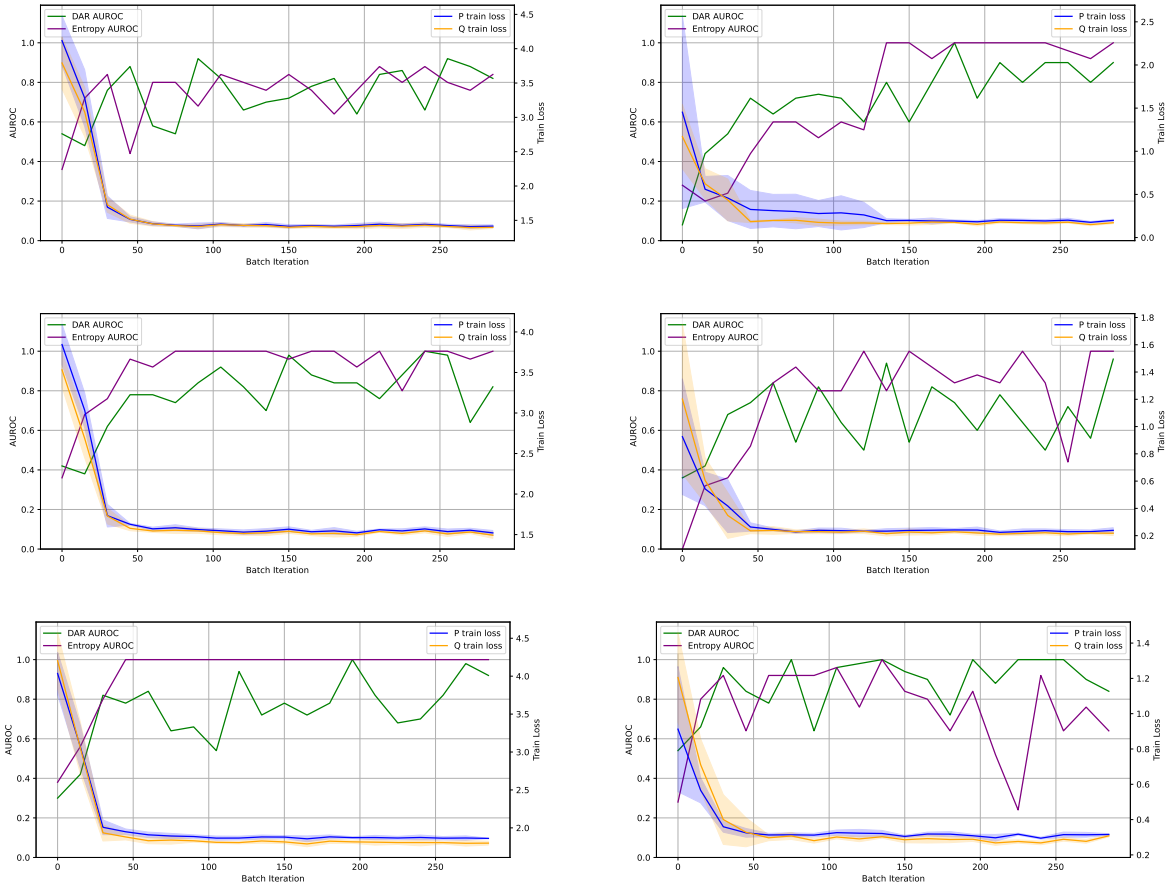

Figure 4: Plots illustrating how AUROC and ℓ_{cdc} evolves over reward fine-tuning iterations for CIFAR10 and Camelyon17 when $|\mathbf{Q}| = 10, 20, 50$ for *varying* κ ordered from top to bottom.

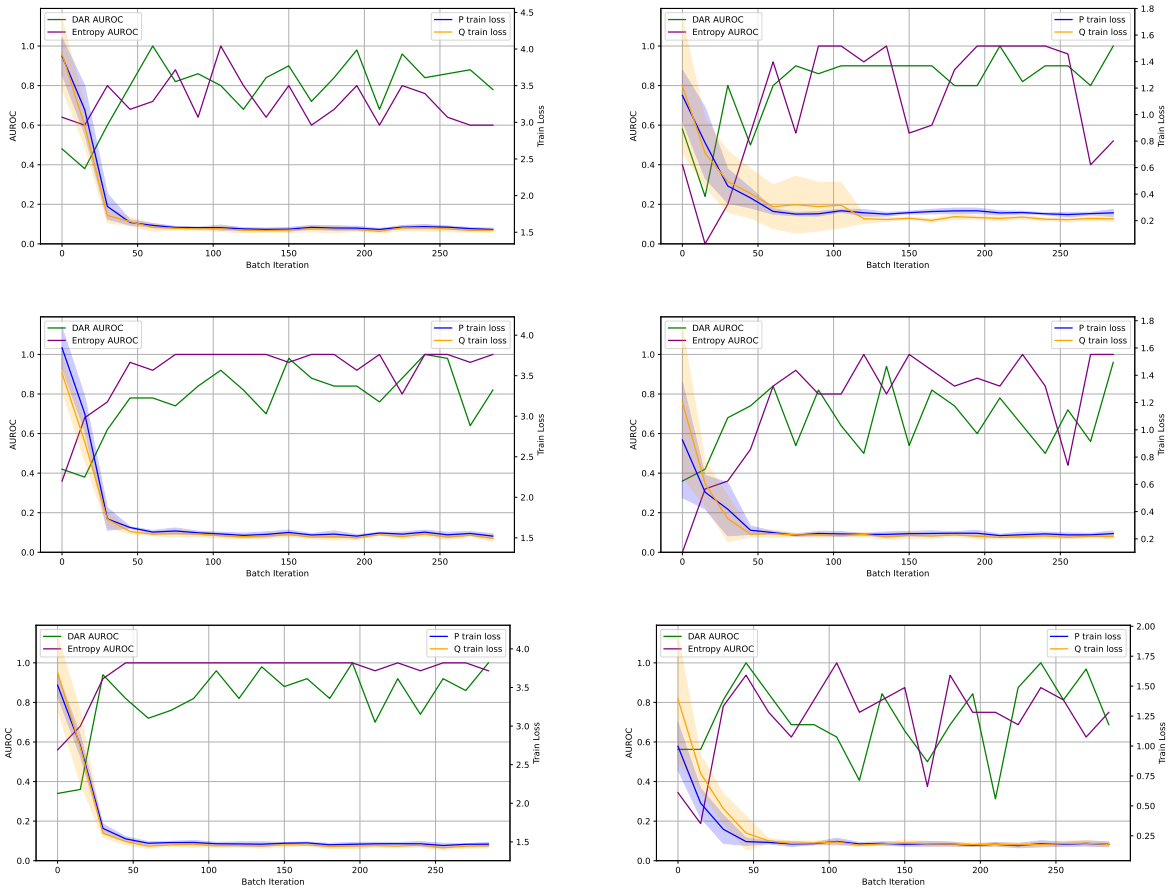

Figure 5: Plots illustrating how AUROC and ℓ_{cdc} evolves over reward fine-tuning iterations for CIFAR10 and Camelyon17 when $|\mathbf{Q}| = 10, 20, 50$ for *fixed* κ ordered from top to bottom.

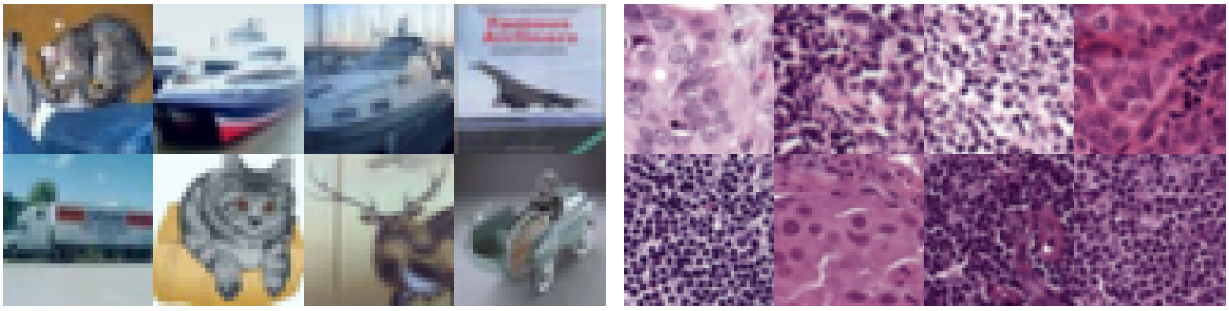

Figure 6: Samples of $\mathbf{P}^*$ (**top**) and $\mathbf{Q}$ (**bottom**) from the CIFAR10 (**left**) and Camelyon17 (**right**) evaluation of Meta-Detectron.