Stochastic Gradient Descent-Ascent: Unified Theory and New Efficient Methods

Aleksandr Beznosikov
Moscow Institute of Physics and Technology, Russian Federation
Eduard Gorbunov
Mohamed bin Zayed University of Artificial Intelligence, United Arab Emirates
Hugo Berard
Mila, Université de Montréal, Canada
Nicolas Loizou
Johns Hopkins University, USA

Abstract

Stochastic Gradient Descent-Ascent (SGDA) is one of the most prominent algorithms for solving min-max optimization and variational inequalities problems (VIP) appearing in various machine learning tasks. The success of the method led to several advanced extensions of the classical SGDA, including variants with arbitrary sampling, variance reduction, coordinate randomization, and distributed variants with compression, which were extensively studied in the literature, especially during the last few years. In this paper, we propose a unified convergence analysis that covers a large variety of stochastic gradient descent-ascent methods, which so far have required different intuitions, have different applications and have been developed separately in various communities. A key to our unified framework is a parametric assumption on the stochastic estimates. Via our general theoretical framework, we either recover the sharpest known rates for the known special cases or tighten them. Moreover, to illustrate the flexibility of our approach we develop several new variants of SGDA such as a new variance-reduced method (L-SVRGDA), new distributed methods with compression (QSGDA, DIANA-SGDA, VR-DIANA-SGDA), and a new method with coordinate randomization (SEGA-SGDA). Although variants of the new methods are known for solving minimization problems, they were never considered or analyzed for solving min-max problems and VIPs. We also demonstrate the most important properties of the new methods through extensive numerical experiments.

1. Introduction

Min-max optimization and, more generally, variational inequality problems (VIPs) appear in a wide range of research areas including but not limited to statistics [5], online learning [15], game theory [58], and machine learning [23]. Motivated by applications in these areas, in this paper, we focus on solving the following regularized VIP: Find $x^* \in \mathbb{R}^d$ such that

$$\langle F(x^*), x - x^* \rangle + R(x) - R(x^*) \geq 0 \quad \forall x \in \mathbb{R}^d, \tag{1}$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is some operator and $R : \mathbb{R}^d \to \mathbb{R}$ is a regularization term (a proper lower semicontinuous convex function), which is assumed to have a simple structure. This problem is quite general and covers a wide range of possible problem formulations. For example, when operator $F(x)$

© A. Beznosikov, E. Gorbunov, H. Berard & N. Loizou.

is the gradient of a convex function f, then problem (1) is equivalent to the composite minimization problem [7], i.e., minimization of $f(x) + R(x)$. Problem (1) is also a more abstract formulation of the min-max problem

$$\min_{x_1 \in Q_1} \max_{x_2 \in Q_2} f(x_1, x_2), \tag{2}$$

with convex-concave continuously differentiable f. In that case, first-order optimality conditions imply that (2) is equivalent to (1) with $x = (x_1^\top, x_2^\top)^\top$, $F(x) = (\nabla_{x_1} f(x_1, x_2)^\top, -\nabla_{x_2} f(x_1, x_2)^\top)^\top$, and $R(x) = \delta_{Q_1}(x_1) + \delta_{Q_2}(x_2)$, where $\delta_Q(\cdot)$ is an indicator function of the set Q [1]. In addition to formulate the constraints, regularization R allows us to enforce some properties to the solution x^*, e.g., sparsity [7, 13].

More precisely, we are interested in the situations when operator F is accessible through the calls of unbiased stochastic oracle. This is natural when F has an expectation form $F(x) = \mathbb{E}_{\xi \sim \mathcal{D}}[F_\xi(x)]$ or a finite-sum form $F(x) = \frac{1}{n}\sum_{i=1}^n F_i(x)$. In the context of machine learning, $\mathcal{D}$ corresponds to some unknown distribution on the data, n corresponds to the number of samples, and F_ξ, F_i denote vector fields corresponding to the samples ξ, and i, respectively [22, 52].

One of the most popular methods for solving (1) is Stochastic Gradient Descent-Ascent[1] (SGDA) [20, 59]. However, besides its rich history, SGDA only recently was analyzed without using strong assumptions on the noise [52] such as uniformly bounded variance. In the last few years, several powerful algorithmic techniques like variance reduction [62, 79] and coordinate-wise randomization [69], were also combined with SGDA resulting in better algorithms. However these methods were analyzed under different assumptions, using different analysis approaches, and required different intuitions. Moreover, to the best of our knowledge, fruitful directions such as communication compression for distributed versions of SGDA or linearly converging variants of coordinate-wise methods for regularized VIPs were never considered in the literature before.

All of these facts motivate the importance and necessity of a novel general analysis of SGDA unifying several special cases and providing the ability to design and analyze new SGDA-like methods filling existing gaps in the theoretical understanding of the method.

In this work, we develop such unified analysis.

1.1. Technical Preliminaries

Throughout the paper, we assume that (1) has at least one solution and operator F is *μ-quasi-strongly monotone* and *ℓ-star-cocoercive*: there exist constants $\mu \geq 0$ and $\ell > 0$ such that for all $x \in \mathbb{R}^d$

$$\langle F(x) - F(x^*), x - x^* \rangle \geq \mu \|x - x^*\|^2, \tag{3}$$

$$\|F(x) - F(x^*)\|^2 \leq \ell \langle F(x) - F(x^*), x - x^* \rangle, \tag{4}$$

where $x^* = \text{proj}_{X^*}(x) := \arg\min_{y \in X^*} \|y - x\|$ is the projection of x on the solution set X^* of (1). If $\mu = 0$, inequality (3) is known as variational stability condition [37], which is weaker than standard monotonicity: $\langle F(x) - F(y), x - y \rangle \geq 0$ for all $x, y \in \mathbb{R}^d$. It is worth mentioning that there exist examples of non-monotone operators satisfying (3) with $\mu > 0$ [52]. Condition (4) is a relaxation of standard cocoercivity $\|F(x) - F(y)\|^2 \leq \ell \langle F(x) - F(y), x - y \rangle$. At this point let us highlight that it is possible for an operator F to satisfy (4) and not be Lipschitz continuous [52].

1. This name is usually used in the min-max setup. Although we consider a more general problem formulation, we keep the name SGDA to highlight the connection with min-max problems.

This emphasizes the wider applicability of the ℓ-*star-cocoercivity* compared to ℓ-cocoercivity. We emphasize that in our convergence analysis we do not assume ℓ-cocoercivity nor L-Lipschitzness of F.

We consider SGDA for solving (1) in its general form:

$$x^{k+1} = \text{prox}_{\gamma_k R}(x^k - \gamma_k g^k), \tag{5}$$

where g^k is an unbiased estimator of $F(x^k)$, $\gamma_k > 0$ is a stepsize at iteration k, and $\text{prox}_{\gamma R}(x) := \arg\min_{y\in\mathbb{R}^d} \left\{R(y) + \|y-x\|^2/2\gamma\right\}$ is a proximal operator defined for any $\gamma > 0$ and $x \in \mathbb{R}^d$. While g^k gives an information about operator F at step k, proximal operator is needed to take into account regularization term R. We assume that function R is such that $\text{prox}_{\gamma R}(x)$ can be easily computed for all $x \in \mathbb{R}^d$. This is a standard assumption satisfied for many practically interesting regularizers [7]. By default we assume that $\gamma_k \equiv \gamma > 0$ for all $k \geq 0$.

1.2. Our Contributions

- **Unified analysis of SGDA.** We propose a general assumption on the stochastic estimates and the problem (1) (Assumption 1) and show that several variants of SGDA (5) satisfy this assumption. In particular, through our approach we cover SGDA with arbitrary sampling [52], variance reduction, coordinate randomization, and compressed communications. Under Assumption 1 we derive general convergence results for quasi-strongly monotone (Theorem 1) and monotone problems (Theorem 3).
- **Extensions of known methods and analysis.** As a by-product of the generality of our theoretical framework, we derive new results for the proximal extensions of several known methods such as *proximal* SGDA-AS [52] and *proximal* SGDA with coordinate randomization [69]. Moreover, we close some gaps on the convergence of known methods, e.g., we derive the first convergence guarantees in the monotone case for SGDA-AS [52] and SAGA-SGDA [62] and we obtain the first result on the convergence of SAGA-SGDA for (averaged star-)cocoercive operators.
- **Sharp rates for known special cases.** For the known methods fitting our framework our general theorems either recover the best rates known for these methods (SGDA-AS) or tighten them (SGDA-SAGA, Coordinate SGDA).
- **New methods.** The flexibility of our approach allows us to develop and analyze several new variants of SGDA. Guided by algorithmic advances for solving minimization problems we propose a new variance-reduced method (L-SVRGDA), new distributed methods with compression (QSGDA, DIANA-SGDA, VR-DIANA-SGDA), and a new method with coordinate randomization (SEGA-SGDA). We show that the proposed new methods fit our theoretical framework and, using our general theorems, we obtain tight convergence guarantees for them. Although the analogs of these methods are known for solving minimization problems [3, 33–35, 43, 56], they were never considered for solving min-max and variational inequality problems. Therefore, by proposing and analyzing these new methods we close several gaps in the literature on SGDA. For example, VR-DIANA-SGDA is the first SGDA-type *linearly converging* distributed stochastic method with compression and SEGA-SGDA is the first *linearly converging* coordinate method for solving *regularized* VIPs.
- **Numerical evaluation.** In numerical experiments, we illustrate the most important properties of the new methods. The numerical results corroborate our theoretical findings.

Throughout the paper, we provide necessary comparison with closely related work. Additional works relevant to our paper are discussed in Appendix A.

2. Unified Analysis of SGDA

Key assumption. We start by introducing the following parametric assumption, which is a central part of our approach.

Assumption 1 *We assume that for all $k \geq 0$ the estimator g^k from (5) is unbiased: $\mathbb{E}_k\left[g^k\right] = F(x^k)$, where $\mathbb{E}_k[\cdot]$ denotes the expectation w.r.t. the randomness at iteration k. Next, we assume that there exist non-negative constants $A, B, C, D_1, D_1 \geq 0$, $\rho \in (0, 1]$ and a sequence of (possibly random) non-negative variables $\{\sigma_k\}_{k\geq 0}$ such that for all $k \geq 0$*

$$\mathbb{E}_k\left[\|g^k - g^{*,k}\|^2\right] \leq 2A\langle F(x^k) - g^{*,k}, x^k - x^{*,k}\rangle + B\sigma_k^2 + D_1, \tag{6}$$

$$\mathbb{E}_k\left[\sigma_{k+1}^2\right] \leq 2C\langle F(x^k) - g^{*,k}, x^k - x^{*,k}\rangle + (1-\rho)\sigma_k^2 + D_2, \tag{7}$$

where $x^{*,k} = proj_{X^*}(x^k)$ and $g^{*,k} = F(x^{*,k})$.

While unbiasedness of g^k is a standard assumption, inequalitites (6)-(7) are new and require clarifications. For simplicity, assume that $\sigma_k^2 \equiv 0$, $F(x^*) = 0$ for all $x^* \in X^*$, and focus on (6). In this case, (6) gives an upper bound for the second moment of the stochastic estimate g^k. For example, such a bound follows from expected cocoercivity assumption [52], where A denotes some expected/averaged (star-)cocoercivity constant and D_1 stands for the variance at the solution (see also Section 3). When F is not necessary zero on X^*, the shift $g^{*,k}$ helps to take this fact into account. Finally, the sequence $\{\sigma_k^2\}_{k\geq 0}$ is typically needed to capture the variance reduction process, parameter B is typically some numerical constant, C is another constant related to (star-)cocoercivity, and D_2 is the remaining noise that is not handled by variance reduction process. As we show in the next sections, inequalitites (6)-(7) hold for various SGDA-type methods.

We point out that Assumption 1 is inspired by similar assumptions appeared in Gorbunov et al. [24, 26]. However, the difference between our assumption and the ones appeared in these papers is significant: Gorbunov et al. [24] focuses only on solving minimization problems and as a result, their assumption includes a much simpler quantity (function suboptimality), instead of the $\langle F(x^k) - g^{*,k}, x^k - x^{*,k}\rangle$, in the right-hand sides of (6)-(7). The assumption proposed in Gorbunov et al. [26], is designed specifically for analyzing vanilla Stochastic EG, it does not have $\{\sigma_k^2\}_{k\geq 0}$ sequence (not able to capture variants of Stochastic EG with variance reduction, quantization, nor coordinate-wise randomization) and works only for (1) with $R(x) \equiv 0$. For more detailed comparison of our approach and this line of work, see Appendix A.

Quasi-strongly monotone case. Under Assumption 1 and quasi-strong monotonicity of F, we derive the following general result.

Theorem 1 *Let F be μ-quasi-strongly monotone ($\mu > 0$) and let Ass. 1 hold. Assume that $0 < \gamma \leq \min\left\{1/\mu, 1/2(A+CM)\right\}$ for some[2] $M > B/\rho$. Then the iterates of SGDA (5), satisfy:*

$$\mathbb{E}[V_k] \leq \left(1 - \min\left\{\gamma\mu, \rho - \frac{B}{M}\right\}\right)^k V_0 + \frac{\gamma^2(D_1 + MD_2)}{\min\left\{\gamma\mu, \rho - B/M\right\}}. \tag{8}$$

where the Lyapunov function V_k is defined by $V_k = \|x^k - x^{*,k}\|^2 + M\gamma^2\sigma_k^2$ for all $k \geq 0$.

2. When $B = 0$, we suppose $M = 0$ and $B/M := 0$ in all following expressions.

The above theorem states that SGDA (5) converges linearly to the neighborhood of the solution. The size of the neighborhood is proportional to the noises D_1 and D_2. When $D_1 = D_2 = 0$, i.e., the method is variance reduced, it converges linearly to the exact solution in expectation. However, in general, to achieve any predefined accuracy, one needs to reduce the size of the neighborhood somehow. One possible way to that is use a proper stepsize schedule. We formalize this discussion in the following result.

Corollary 2 *Let the assumptions of Theorem 1 hold. Consider two possible cases.*

Case 1. *Let $D_1 = D_2 = 0$. Then, for any $K \geq 0$, $M = {}^{2B}/_{\rho}$, and $\gamma = \min\left\{{}^{1}/_{\mu}, {}^{1}/_{2(A+{}^{2BC}/_{\rho})}\right\}$, the iterates of* SGDA*, given by (5), satisfy: $\mathbb{E}[V_K] \leq V_0 \exp\left(-\min\left\{\frac{\mu}{2(A+{}^{2BC}/_{\rho})}, \frac{\rho}{2}\right\} K\right)$.*

Case 2. *Let $D_1 + MD_2 > 0$. For any $K \geq 0$ and $M = {}^{2B}/_{\rho}$ one can choose $\{\gamma_k\}_{k\geq 0}$ as follows:*

$$\gamma_k = \frac{1}{h} \text{ if } K \leq \frac{h}{\mu} \text{ or } \left(K > \frac{h}{\mu} \text{ and } k < k_0\right), \quad \text{and} \quad \gamma_k = \frac{2}{\mu(\kappa + k - k_0)} \text{ if } K > \frac{h}{\mu} \text{ and } k \geq k_0,$$

where $h = \max\left\{2(A + {}^{2BC}/_{\rho}), {}^{2\mu}/_{\rho}\right\}$, $\kappa = {}^{2h}/_{\mu}$ and $k_0 = \lceil {}^{K}/_{2} \rceil$. For this choice of γ_k, the iterates of SGDA*, given by (5), satisfy:*

$$\mathbb{E}[V_K] \leq \frac{32hV_0}{\mu} \exp\left(-\frac{\mu}{h}K\right) + \frac{36(D_1 + {}^{2BD_2}/_{\rho})}{\mu^2 K}.$$

Monotone case. When $\mu = 0$, we additionally assume that F is monotone. Similar to minimization, in the case of $\mu = 0$, the squared distance to the solution is not a valid measure of convergence. To introduce an appropriate convergence measure, we make the following assumption.

Assumption 2 *There exists a compact convex set $\mathcal{C}$ (with the diameter $\Omega_{\mathcal{C}} := \max_{x,y\in\mathcal{C}} \|x - y\|$) such that $X^* \subset \mathcal{C}$.*

In this settings, we focus on the following quantity called a restricted gap-function [60] defined for any $z \in \mathbb{R}^d$ and any $\mathcal{C} \subset \mathbb{R}^d$ satisfying Assumption 2:

$$\text{Gap}_{\mathcal{C}}(z) := \max_{u\in\mathcal{C}} \left[\langle F(u), z - u\rangle + R(z) - R(u)\right]. \tag{9}$$

Assumption 2 and function $\text{Gap}_{\mathcal{C}}(z)$ are standard for the convergence analysis of methods for solving (1) with monotone F [1, 60]. Additional discussion is left to Appendix D.2.

Under these assumptions, Assumption 1, and star-cocoercivity we derive the following general result.

Theorem 3 *Let F be monotone, ℓ-star-cocoercive and let Assumptions 1, 2 hold. Assume that $0 < \gamma \leq {}^{1}/_{2(A+{}^{BC}/_{\rho})}$. Then for all $K \geq 0$ the iterates of* SGDA*, given by (5), satisfy*

$$\begin{aligned}
\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}} \|x^0 - u\|^2\right]}{2\gamma K} + 9\gamma \max_{x^*\in X^*} \|F(x^*)\|^2 \\
& + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + (4A + \ell + {}^{8BC}/_{\rho}) \cdot \frac{\|x^0 - x^{*,0}\|^2}{K} \\
& + \left(4 + (4A + \ell + {}^{8BC}/_{\rho})\,\gamma\right) \frac{\gamma B\sigma_0^2}{\rho K} \\
& + \gamma(2 + \gamma\,(4A + \ell + {}^{8BC}/_{\rho}))(D_1 + {}^{2BD_2}/_{\rho}).
\end{aligned} \tag{10}$$

The above result establishes $\mathcal{O}(1/K)$ rate of convergence to the accuracy proportional to the stepsize γ multiplied by the noise term $D_1 + 2BD_2/\rho$ and $\max_{x^*\in X^*}\|F(x^*)\|^2$. We notice that if $R \equiv 0$ in (1), then $F(x^*) = 0$, meaning that in this case, the second term from (10) equals zero. Otherwise, even in the deterministic case one needs to use small stepsizes to ensure the convergence to any predefined accuracy (see Corollary 17 in Appendix D.2). The term proportional to $\max_{x^*\in X^*}\|F(x^*)\|^2$ can be removed under the assumption that F is cocoercive (see Appendix D.3).

3. SGDA with Arbitrary Sampling

We start our consideration of special cases with a standard SGDA (5) with $g^k = F_{\xi^k}(x^k), \xi^k \sim \mathcal{D}$ under so-called *expected cocoercivity* assumption from Loizou et al. [52], which we properly adjust to the setting of regularized VIPs.

Assumption 3 (Expected Cocoercivity) *We assume that stochastic operator $F_\xi(x), \xi \sim \mathcal{D}$ is such that for all $x \in \mathbb{R}^d$, $\mathbb{E}_{\mathcal{D}}\left[\|F_\xi(x) - F_\xi(x^*)\|^2\right] \leq \ell_{\mathcal{D}}\langle F(x) - F(x^*), x - x^*\rangle$, where $x^* = proj_{X^*}(x)$.*

When $R(x) \equiv 0$, this assumption recovers the original one from Loizou et al. [52]. We also emphasize that for operator F Assumption 3 implies only star-cocoercivity.

Following Loizou et al. [52], we mainly focus on finite-sum case and its stochastic reformulation: we consider a random *sampling* vector $\xi = (\xi_1, \ldots, \xi_n)^\top \in \mathbb{R}^n$ having a distribution $\mathcal{D}$ such that $\mathbb{E}_{\mathcal{D}}[\xi_i] = 1$ for all $i \in [n]$. Using this we can rewrite $F(x) = \frac{1}{n}\sum_{i=1}^n F_i(x)$ as

$$F(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}_{\mathcal{D}}[\xi_i F_i(x)] = \mathbb{E}_{\mathcal{D}}\left[F_\xi(x)\right], \tag{11}$$

where $F_\xi(x) = \frac{1}{n}\sum_{i=1}^n \xi_i F_i(x)$. Such a reformulation allows to handle a wide range of samplings: the only assumption on $\mathcal{D}$ is $\mathbb{E}_{\mathcal{D}}[\xi_i] = 1$ for all $i \in [n]$. Therefore, this setup is often referred to as *arbitrary sampling* [29, 30, 32, 49, 50, 64, 65, 67]. We elaborate on several special cases in Appendix E.5.

In this setting, SGDA with Arbitrary Sampling (SGDA-AS)[3] fits our framework.

Proposition 4 *Let Assumption 3 hold. Then,* SGDA-AS *satisfies Assumption 1 with $A = \ell_{\mathcal{D}}$, $D_1 = 2\sigma_*^2 := 2\max_{x^*\in X^*}\mathbb{E}_{\mathcal{D}}\left[\|F_\xi(x^*) - F(x^*)\|^2\right]$, $B = 0$, $\sigma_k^2 \equiv 0$, $C = 0$, $\rho = 1$, $D_2 = 0$.*

Plugging these parameters to Theorem 1 we recover the result[4] from Loizou et al. [52] when $R(x) \equiv 0$ and generalize it to the case of $R(x) \not\equiv 0$ without sacrificing the rate. Applying Corollary 2, we establish the rate of convergence to the exact solution.

Corollary 5 *Let F be μ-quasi-strongly monotone and Assumption 3 hold. Then for all $K > 0$ there exists a choice of γ (see (48)) for which the iterates of* SGDA-AS*, satisfy:*

$$\mathbb{E}[\|x^K - x^{*,K}\|^2] = \mathcal{O}\left(\frac{\ell_{\mathcal{D}}\Omega_0^2}{\mu}\exp\left(-\frac{\mu}{\ell_{\mathcal{D}}}K\right) + \frac{\sigma_*^2}{\mu^2 K}\right),$$

where $\Omega_0^2 = \|x^0 - x^{*,0}\|^2$.

3. For the pseudo-code of SGDA-AS see Algorithm 1 in Appendix E.
4. In the main part of the paper, we focus on μ-quasi strongly monotone case with $\mu > 0$. For simplicity, we provide here the rates of convergence to the exact solution. Further details, including the rates in monotone case, are left to the Appendix.

For the different stepsize schedule, Loizou et al. [52] derive the convergence rate $\mathcal{O}(1/K + 1/K^2)$ which is inferior to our rate, especially when σ_*^2 is small. In addition, Loizou et al. [52] consider explicitly only uniform minibatch sampling without replacement as a special case of arbitrary sampling. In Appendix E.5, we discuss another prominent sampling strategy called importance sampling. In Section 6, we provide numerical experiments verifying our theoretical findings and showing the benefits of importance sampling over uniform sampling for SGDA.

4. SGDA with Variance Reduction

In this section, we focus on variance reduced variants of SGDA for solving finite-sum problems $F(x) = \frac{1}{n}\sum_{i=1}^n F_i(x)$. We start with the Loopless Stochastic Variance Reduced Gradient Descent-Ascent (L-SVRGDA), which is a generalization of the L-SVRG algorithm proposed in Hofmann et al. [34], Kovalev et al. [43]. L-SVRGDA (see Alg. 2) follows the update rule (5) with

$$g^k = F_{j_k}(x^k) - F_{j_k}(w^k) + F(w^k), \quad w^{k+1} = \begin{cases} x^k, & \text{with prob. } p, \\ w^k, & \text{with prob. } 1-p, \end{cases} \tag{12}$$

where in k^{th} iteration j_k is sampled uniformly at random from $[n]$. Here full operator F is computed once w^k is updated, which happens with probability p. Typically, p is chosen as $p \sim 1/n$ ensuring that the expected cost of 1 iteration equals $\mathcal{O}(1)$ oracle calls, i.e., computations of $F_i(x)$ for some $i \in [n]$.

We introduce the following assumption about operators F_i.

Assumption 4 (Averaged Star-Cocoercivity) *We assume that there exists a constant $\widehat{\ell} > 0$ such that for all $x \in \mathbb{R}^d$*

$$\frac{1}{n}\sum_{i=1}^n \|F_i(x) - F_i(x^*)\|^2 \leq \widehat{\ell}\langle F(x) - F(x^*), x - x^*\rangle, \tag{13}$$

where $x^* = proj_{X^*}(x)$.

For example, if F_i is ℓ_i-cocoercive for $i \in [n]$, then (13) holds with $\widehat{\ell} \leq \max_{i\in[n]} \ell_i$. Next, if F_i is L_i-Lipschitz for all $i \in [n]$ and F is μ-quasi strongly monotone, then (13) is satisfied for $\widehat{\ell} \in [\overline{L}, \overline{L}^2/\mu]$, where $\overline{L}^2 = \frac{1}{n}\sum_{i=1}^n L_i^2$.

Moreover, for the analysis of variance reduced variants of SGDA we also use uniqueness of the solution.

Assumption 5 (Unique Solution) *We assume that the solution set X^* of problem (1) is a singleton: $X^* = \{x^*\}$.*

These assumptions are sufficient to derive validity of Assumption 1 for L-SVRGDA estimator.

Proposition 6 *Let Assumptions 4 and 5 hold. Then,* L-SVRGDA *satisfies Assumption 1 with $A = \widehat{\ell}$, $B = 2$, $\sigma_k^2 = \frac{1}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^*)\|^2$, $C = p\widehat{\ell}/2$, $\rho = p$, $D_1 = D_2 = 0$.*

Plugging these parameters in our general results on the convergence of SGDA-type algorithms we derive the convergence results for L-SVRGDA, see Table 1 and Appendix F.1 for the details.

Table 1: Summary of the complexity results for variance reduced methods for solving (1). By complexity we mean the number of oracle calls required for the method to find x such that $\mathbb{E}[\|x-x^*\|^2] \leq \varepsilon$. Dependencies on numerical and logarithmic factors are hidden. By default, operator F is assumed to be **μ-strongly monotone** and, as the result, the solution is unique. Our results rely on **μ-quasi strong monotonicity** of F (3), but we also assume uniqueness of the solution. Methods supporting $R(x) \not\equiv 0$ are highlighted with *. Our results are highlighted in green. Notation: $\overline{\ell}$, $\overline{L}$ = averaged cocoercivity/Lipschitz constants depending on the sampling strategy, e.g., for uniform sampling $\overline{\ell}^2 = \frac{1}{n}\sum_{i=1}^n \ell_i^2$, $\overline{L}^2 = \frac{1}{n}\sum_{i=1}^n L_i^2$ and for importance sampling $\overline{\ell} = \frac{1}{n}\sum_{i=1}^n \ell_i$, $\overline{L} = \frac{1}{n}\sum_{i=1}^n L_i$; $\widehat{\ell}$ = averaged star-cocoercivity constant from Assumption 4.

Method	Citation	Assumptions	Complexity
SVRE [(1)]	[16]	F_i is ℓ_i-cocoer.	$n + \frac{\overline{\ell}}{\mu}$
EG-VR * [(1)]	[1]	F_i is L_i-Lip.	$n + \sqrt{n}\frac{\overline{L}}{\mu}$
SVRGDA *	[62]	F_i is L_i-Lip.	$n + \frac{\overline{L}^2}{\mu^2}$
SAGA-SGDA *	[62]	F_i is L_i-Lip.	$n + \frac{\overline{L}^2}{\mu^2}$
VR-AGDA	[79]	F_i is $L_{\max}$-Lip.[(2)]	$\min\left\{n + \frac{L_{\max}^9}{\mu^9}, n^{2/3}\frac{L_{\max}^3}{\mu^3}\right\}$
L-SVRGDA *	**This paper**	As. 4	$n + \frac{\widehat{\ell}}{\mu}$
SAGA-SGDA *	**This paper**	As. 4	$n + \frac{\widehat{\ell}}{\mu}$

[(1)] The method is based on Extragradient update rule.
[(2)] Yang et al. [79] consider saddle point problems satisfying so-called two-sided PL condition, which is weaker than strong-convexity-strong-concavity of the objective function.

Moreover, in Appendix F.2, we show that SAGA-SGDA [62] fits our framework and using our general analysis we tighten the convergence rates for this method.

We compare our convergence guarantees with known results in Table 1. We note that by neglecting importance sampling scenario, in the worst case, our convergence results match the best-known results for SGDA-type methods, i.e., ones derived in Palaniappan and Bach [62]. Indeed, this follows from $\widehat{\ell} \in [\overline{L}, \overline{L}^2/\mu]$. Next, when the difference between $\overline{\ell}$ and $\widehat{\ell}$ is not significant, our complexity results match the one derived in Chavdarova et al. [16] for SVRE, which is EG-type method. Although in general, $\overline{\ell}$ might be smaller than $\widehat{\ell}$, our analysis does not require cocoercivity of each F_i and it works for $R(x) \not\equiv 0$. Finally, Alacaoglu and Malitsky [1] derive a better rate (when $n = \mathcal{O}(\overline{L}^2/\mu^2)$), but their method is based on EG. Therefore, our results match the best-known ones in the literature on SGDA-type methods.

5. Distributed SGDA with Compression

In this section, we consider the distributed version of (1), i.e., we assume that $F(x) = \frac{1}{n}\sum_{i=1}^n F_i(x)$, where $\{F_i\}_{i=1}^n$ are distributed across n devices connected with parameter-server in a centralized fashion. Each device i has an access to the computation of the unbiased estimate of F_i at the given point. Typically, in these settings, the communication is a bottleneck, especially when n and d are huge. This means that in the naive distributed implementations of SGDA, communication rounds take much more time than local computations on the clients. Various approaches are used to circumvent this issue.

One of them is based on the usage of compressed communications. We focus on the unbiased compression operators.

Definition 7 *Operator $\mathcal{Q}:\mathbb{R}^d\to\mathbb{R}^d$ (possibly randomized) is called unbiased compressor/quantization if there exists a constant $\omega\geq 1$ such that for all $x\in\mathbb{R}^d$*

$$\mathbb{E}[\mathcal{Q}(x)]=x,\quad \mathbb{E}[\|\mathcal{Q}(x)-x\|^2]\leq\omega\|x\|^2. \tag{14}$$

In this paper, we consider compressed communications in the direction from clients to the server. The simplest method with compression – QSGDA (Alg. 4) – can be described as SGDA (5) with $g^k=\frac{1}{n}\sum_{i=1}^n\mathcal{Q}(g_i^k)$. Here g_i^k are stochastic estimators satisfying the following assumption[5].

Assumption 6 (Bounded variance) *All stochastic realizations g_i^k are unbiased and have bounded variance, i.e., for all $i\in[n]$ and $k\geq 0$ the following holds:*

$$\mathbb{E}[g_i^k]=F_i(x^k),\quad \mathbb{E}[\|g_i^k-F_i(x^k)\|^2]\leq\sigma_i^2. \tag{15}$$

Despite its simplicity, QSGDA was never considered in the literature on solving min-max problems and VIPs. It turns out that under such assumptions QSGDA satisfies our Assumption 1.

Proposition 8 *Let F be ℓ-star-cocoercive and Assumptions 4, 6 hold. Then,* QSGDA *satisfies Assumption 1 with $A=\frac{3\ell}{2}+\frac{9\omega\widehat{\ell}}{2n}$, $B=0$, $\sigma_k^2\equiv 0$, $D_1=\frac{3(1+3\omega)\sigma^2+9\omega\zeta_*^2}{n}$, $C=0$, $\rho=1$, $D_2=0$, where $\sigma^2=\frac{1}{n}\sum_{i=1}^n\sigma_i^2,\zeta_*^2:=\frac{1}{n}\max_{x*\in X^*}\sum_{i=1}^n\|F_i(x^*)\|^2$.*

As for the other special cases, we derive the convergence results for QSGDA using our general theorems (see Table 2 and Appendix G.1 for the details). The proposed method is simple, but have a significant drawback: even in the deterministic case ($\sigma=0$), QSGDA does not converge linearly unless $\zeta_*^2=0$. However, when the data on clients is arbitrary heterogeneous the dissimilarity measure ζ_*^2 is strictly positive and can be large (even when $R(x)\equiv 0$).

To resolve this issue, we propose a more advanced scheme based on DIANA update [35, 56] – DIANA-SGDA (Alg. 5). In a nutshell, DIANA-SGDA is SGDA (5) with g^k defined as follows:

$$\Delta_i^k=g_i^k-h_i^k,\quad h_i^{k+1}=h_i^k+\alpha\mathcal{Q}(\Delta_i^k), \tag{16}$$

$$g^k=h^k+\frac{1}{n}\sum_{i=1}^n\mathcal{Q}(\Delta_i^k),\quad h^{k+1}=\frac{1}{n}\sum_{i=1}^n h_i^{k+1}=h^k+\alpha\frac{1}{n}\sum_{i=1}^n\mathcal{Q}(\Delta_i^k), \tag{17}$$

where the first two lines correspond to the local computations on the clients and the last two lines – to the server-side computations. Taking into account the update rule for h^{k+1}, one can notice that DIANA-SGDA requires workers to send only vectors $\mathcal{Q}(\Delta_i^k)$ to the server at step k, i.e., the method uses only compressed workers-server communications.

As we show next, DIANA-SGDA fits our framework.

Proposition 9 *Let Assumptions 4, 5, 6 hold. Suppose that $\alpha\leq 1/(1+\omega)$. Then,* DIANA-SGDA *with quantization (14) satisfies Assumption 1 with $\sigma_k^2=\frac{1}{n}\sum_{i=1}^n\|h_i^k-F_i(x^*)\|^2$ and $A=\left(\frac{1}{2}+\frac{\omega}{n}\right)\widehat{\ell}$, $B=\frac{2\omega}{n}$, $D_1=\frac{(1+\omega)\sigma^2}{n}$, $C=\frac{\alpha\widehat{\ell}}{2}$, $\rho=\alpha$, $D_2=\alpha\sigma^2$, , where $\sigma^2=\frac{1}{n}\sum_{i=1}^n\sigma_i^2$.*

5. We use this assumption for illustrating the flexibility of the framework. It is possible to consider Arbitrary Sampling setup as well.

DIANA-SGDA can be considered as a variance-reduced method, since it reduces the term proportional to $\omega\zeta_*^2$ that the bound for QSGDA contains (see Table 2 and Appendix G.2 for the details). As the result, when $\sigma = 0$, i.e., workers compute $F_i(x)$ at each step, DIANA-SGDA enjoys linear convergence to the exact solution.

Next, when local operators F_i have a finite-sum form $F_i(x) = \frac{1}{m}\sum_{j=1}^m F_{ij}(x)$, one can combine L-SVRGDA and DIANA-SGDA as follows: consider the scheme from (16)-(17) with

$$g_i^k = F_{ij_k}(x^k) - F_{ij_k}(w^k) + F(w_i^k), \quad w_i^{k+1} = \begin{cases} x^k, & \text{with prob. } p, \\ w_i^k, & \text{with prob. } 1-p, \end{cases} \tag{18}$$

where j_k is sampled uniformly at random from $[n]$. We call the resulting method VR-DIANA-SGDA (Alg. 6) and we note that its analog for solving minimization problems (VR-DIANA) was proposed and analyzed in Horváth et al. [35].

To cast VR-DIANA-SGDA as special case of our general framework, we need to make the following assumption.

Assumption 7 *We assume that there exists a constant $\widetilde{\ell} > 0$ such that for all $x \in \mathbb{R}^d$*

$$\frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m \|F_{ij}(x) - F_{ij}(x^*)\|^2 \leq \widetilde{\ell}\langle F(x) - F(x^*), x - x^*\rangle, \tag{19}$$

where $x^* = proj_{X^*}(x)$.

Using Assumption 7 and previously introduced conditions, we get the following result.

Proposition 10 *Let F be ℓ-star-cocoercive and Assumptions 4, 5, 7 hold. Suppose that $\alpha \leq \min\left\{\frac{p}{3}, \frac{1}{1+\omega}\right\}$. Then,* VR-DIANA-SGDA *satisfies Assumption 1 with $A = \frac{\ell}{2} + \frac{\widetilde{\ell}}{n} + \frac{\omega(\widehat{\ell}+\widetilde{\ell})}{n}$, $B = \frac{2(\omega+1)}{n}$, $\sigma_k^2 = \frac{1}{n}\sum_{i=1}^n \|h_i^k - F_i(x^*)\|^2 + \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m \|F_{ij}(w_i^k) - F_{ij}(x^*)\|^2$, $C = \frac{p\widetilde{l}}{2} + \alpha(\widetilde{\ell} + \widehat{\ell})$, $\rho = \alpha$, $D_1 = D_2 = 0$.*

Since $D_1 = D_2 = 0$, our general results imply linear convergence of VR-DIANA-SGDA when $\mu > 0$ (see the details in Appendix G.3). That is, VR-DIANA-SGDA is the *first linearly converging distributed* SGDA*-type method with compression*. We compare it with MASHA1 [12] in Table 2. Firstly, let us note that MASHA1 is a method based on EG, and its convergence guarantees depend on the Lipschitz constants. In addition, we note that the complexity of MASHA1 could be better than the one of VR-DIANA-SGDA when cocoercivity constants are large compared to Lipschitz ones. However, our compleixty bound has better dependency on quantization parameter ω, number of clients n, and the size of the local dataset m. These parameters can be large meaning that the improvement is noticeable.

6. Numerical Experiments

To illustrate our theoretical results, we conduct several numerical experiments on quadratic games, which are defined through the affine operator: $F(x) = \frac{1}{n}\sum_{i=1}^n \mathbf{A}_i x + b_i$, where each matrix $\mathbf{A}_i \in \mathbb{R}^{d\times d}$ is non-symmetric with all eigenvalues having strictly positive real part. Enforcing all the eigenvalues to have strictly positive real part ensures that the operator is strongly monotone and cocoercive. We consider two different settings: (i) problem without constraints, and (ii) problem

Table 2: Summary of the complexity results for distributed methods with unbiased compression for solving distributed (1) with $F = \frac{1}{n}\sum_{i=1}^n F_i(x)$. By complexity we mean the number of communication rounds required for the method to find x such that $\mathbb{E}[\|x - x^*\|^2] \leq \varepsilon$. Dependencies on numerical and logarithmic factors are hidden. $\mathbb{E}$ stands for the setup, when $F_i(x) = \mathbb{E}_{\xi_i}[F_{\xi_i}(x)]$; Σ denotes the case, when $F_i(x) = \frac{1}{m}\sum_{j=1}^m F_{ij}(x)$. Our results rely on **$\mu$-quasi strong monotonicity** of F (3), but we also assume uniqueness of the solution. Methods supporting $R(x) \not\equiv 0$ are highlighted with *. Our results are highlighted in green. Notation: $\sigma^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$ – averaged upper bound for the variance (see Ass. 6 for the definition of σ_i^2); ω = quantization parameter (see Def. 7); $\zeta_*^2 = \frac{1}{n}\max_{x*\in X^*}\sum_{i=1}^n \|F_i(x^*)\|^2$; $L_{\max} = \max_{i\in[n]} L_i$; $\widetilde{\ell}$ = averaged star-cocoercivity constant from Ass. 7.

Setup	Method	Citation	Assumptions	Complexity
$\mathbb{E}$	QSGDA *	**This paper**	As. 4, 6	$\frac{\ell}{\mu} + \frac{\omega\widehat{\ell}}{n\mu} + \frac{(1+\omega)\sigma^2+\omega\zeta_*^2}{n\mu^2\varepsilon}$
	DIANA-SGDA *	**This paper**	As. 4, 6	$\omega + \frac{\ell}{\mu} + \frac{\omega\widehat{\ell}}{n\mu} + \frac{(1+\omega)\sigma^2}{n\mu^2\varepsilon}$
Σ	MASHA1 *[(1)]	[12]	F_i is L_i-Avg. Lip.[(2)]	$m + \omega + \frac{L_{\max}\sqrt{(m+\omega)(1+\frac{\omega}{n})}}{\mu}$
	VR-DIANA-SGDA *	**This paper**	As. 4, 7	$m + \omega + \frac{\ell}{\mu} + \frac{(1+\omega)(\widehat{\ell}+\widetilde{\ell})}{n\mu} + \frac{(1+\omega)\max\{m,\omega\}\widetilde{\ell}}{nm\mu}$

[(1)] The method is based on Extragradient update rule.
[(2)] This means that for all $x, y \in \mathbb{R}^d$ and $i \in [n]$ the following inequality holds: $\frac{1}{m}\sum_{j=1}^m \|F_{ij}(x) - F_{ij}(y)\|^2 \leq L_i^2\|x - y\|^2$.

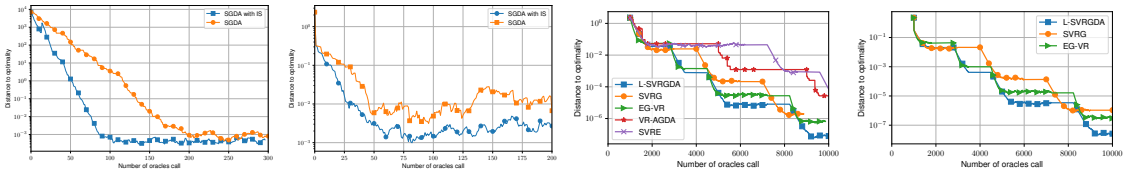

Figure 1: The first two plots correspond to the comparison of Uniform Sampling (US) vs Importance Sampling (IS): the first plot shows the result for the problem without constraints, the second one – with constraints. As expected by theory IS converges faster and to a smaller neighborhood than US. The last two plots provide a comparison of variance reduced methods: the third plot shows the result for the problem without constraints, the fourth one – with constraints. Note that L-SVRGDA is very competitive, and outperforms all the other methods.

that has ℓ_1 regularization and constraints forcing the solution to lie in the ℓ_∞-ball of radius r. In all experiments, we use a constant step-size for all methods which was selected manually using a grid-search and picking the best performing step-size for each methods. For further details about the experiments and additional experiments for distributed methods see Appendix B.

Uniform sampling (US) vs Important sampling (IS). We note that Loizou et al. [52] which studies SGDA-AS does not consider IS explicitly. Although we show the theoretical benefits of IS in comparison to US in Appendix E.5, here we provide a numerical comparison to illustrate the superiority of IS (on both constrained and unconstrained quadratic games). We choose the matrices $\mathbf{A}_i$ such that $\ell_{\max} = \max_i \ell_i \gg \bar{\ell}$. In this case, our theory predicts that IS should perform better than US. We provide the results in Fig. 1. We observe that indeed SGDA with IS converges faster and to a smaller neighborhood than SGDA with US. This observation perfectly corroborates our theory.

Comparison of variance reduced methods. In this experiment, we test the performance of our proposed L-SVRGDA (Alg. 2) and compare it to other variance reduced methods on quadratic games, see Fig. 1. In particular, we compare it to SVRG [62], SVRE [16], EG-VR [1] and VR-AGDA [79].

In the constrained setting, we only compare L-SVRGDA to SVRG and EG-VR, since they are the only methods from this list that handle constrained settings. For loopless variants we choose $p = \frac{1}{n}$ and for the non-loopless variants we pick the number of inner-loop iteration to be n. We observe that all methods converge linearly and that L-SVRGDA is competitive with the other considered variance-reduced methods, converging slightly faster than all of them.

REFERENCES

[1] Ahmet Alacaoglu and Yura Malitsky. Stochastic variance reduction for variational inequality methods. *arXiv preprint arXiv:2102.08352*, 2021.

[2] Ahmet Alacaoglu, Yura Malitsky, and Volkan Cevher. Forward-reflected-backward method with variance reduction. *Computational optimization and applications*, 80(2):321–346, 2021.

[3] Dan Alistarh, Demjan Grubic, Jerry Li, Ryota Tomioka, and Milan Vojnovic. Qsgd: Communication-efficient sgd via gradient quantization and encoding. *Advances in Neural Information Processing Systems*, 30:1709–1720, 2017.

[4] Waïss Azizian, Franck Iutzeler, Jérôme Malick, and Panayotis Mertikopoulos. The last-iterate convergence rate of optimistic mirror descent in stochastic variational inequalities. In *Conference on Learning Theory*, pages 326–358. PMLR, 2021.

[5] Francis Bach. The "η-trick" or the effectiveness of reweighted least-squares, 2019.

[6] Heinz H Bauschke, Patrick L Combettes, et al. *Convex analysis and monotone operator theory in Hilbert spaces*, volume 408. Springer, 2011.

[7] Amir Beck. *First-order methods in optimization*. Society for Industrial and Applied Mathematics (SIAM), 2017.

[8] Aleksandr Beznosikov, Samuel Horváth, Peter Richtárik, and Mher Safaryan. On biased compression for distributed learning. *arXiv preprint arXiv:2002.12410*, 2020.

[9] Aleksandr Beznosikov, Abdurakhmon Sadiev, and Alexander Gasnikov. Gradient-free methods with inexact oracle for convex-concave stochastic saddle-point problem. In *International Conference on Mathematical Optimization Theory and Operations Research*, pages 105–119. Springer, 2020.

[10] Aleksandr Beznosikov, Valentin Samokhin, and Alexander Gasnikov. Distributed saddle-point problems: Lower bounds, optimal algorithms and federated gans. *arXiv preprint arXiv:2010.13112*, 2020.

[11] Aleksandr Beznosikov, Vasilii Novitskii, and Alexander Gasnikov. One-point gradient-free methods for smooth and non-smooth saddle-point problems. In *International Conference on Mathematical Optimization Theory and Operations Research*, pages 144–158. Springer, 2021.

[12] Aleksandr Beznosikov, Peter Richtárik, Michael Diskin, Max Ryabinin, and Alexander Gasnikov. Distributed methods with compressed communication for solving variational inequalities, with theoretical guarantees. *arXiv preprint arXiv:2110.03313*, 2021.

[13] Emmanuel J Candes, Michael B Wakin, and Stephen P Boyd. Enhancing sparsity by reweighted ℓ_1 minimization. *Journal of Fourier analysis and applications*, 14(5):877–905, 2008.

[14] Yair Carmon, Yujia Jin, Aaron Sidford, and Kevin Tian. Variance reduction for matrix games. *Advances in Neural Information Processing Systems*, 32, 2019.

[15] Nicolo Cesa-Bianchi and Gábor Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

[16] Tatjana Chavdarova, Gauthier Gidel, François Fleuret, and Simon Lacoste-Julien. Reducing noise in GAN training with variance reduced extragradient. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019.

[17] Constantinos Daskalakis, Stratis Skoulakis, and Manolis Zampetakis. The complexity of constrained min-max optimization. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1466–1478, 2021.

[18] Damek Davis and Wotao Yin. A three-operator splitting scheme and its optimization applications. *Set-valued and variational analysis*, 25(4):829–858, 2017.

[19] Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. *Advances in neural information processing systems*, 27, 2014.

[20] Vladimir Fedorovich Dem'yanov and Aleksandr Borisovich Pevnyi. Numerical methods for finding saddle points. *USSR Computational Mathematics and Mathematical Physics*, 12(5): 11–52, 1972.

[21] Jelena Diakonikolas, Constantinos Daskalakis, and Michael Jordan. Efficient methods for structured nonconvex-nonconcave min-max optimization. In *International Conference on Artificial Intelligence and Statistics*, pages 2746–2754. PMLR, 2021.

[22] Gauthier Gidel, Hugo Berard, Gaëtan Vignoud, Pascal Vincent, and Simon Lacoste-Julien. A variational inequality perspective on generative adversarial networks. In *International Conference on Learning Representations (ICLR)*, 2019.

[23] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In Z. Ghahramani, M. Welling, C. Cortes, N. Lawrence, and K. Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 27. Curran Associates, Inc., 2014.

[24] Eduard Gorbunov, Filip Hanzely, and Peter Richtarik. A Unified Theory of SGD: Variance Reduction, Sampling, Quantization and Coordinate Descent. In Silvia Chiappa and Roberto Calandra, editors, *Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics*, volume 108 of *Proceedings of Machine Learning Research*, pages 680–690. PMLR, 26–28 Aug 2020.

[25] Eduard Gorbunov, Dmitry Kovalev, Dmitry Makarenko, and Peter Richtarik. Linearly converging error compensated sgd. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 20889–20900. Curran Associates, Inc., 2020. URL https://proceedings.neurips.cc/paper/2020/file/ef9280fbc5317f17d480e4d4f61b3751-Paper.pdf.

[26] Eduard Gorbunov, Hugo Berard, Gauthier Gidel, and Nicolas Loizou. Stochastic extragradient: General analysis and improved rates. *arXiv preprint arXiv:2111.08611*, 2021.

[27] Eduard Gorbunov, Konstantin P. Burlachenko, Zhize Li, and Peter Richtarik. MARINA: Faster non-convex distributed learning with compression. In Marina Meila and Tong Zhang, editors, *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pages 3788–3798. PMLR, 18–24 Jul 2021. URL https://proceedings.mlr.press/v139/gorbunov21a.html.

[28] Eduard Gorbunov, Nicolas Loizou, and Gauthier Gidel. Extragradient method: $\mathrm{O}(1/K)$ last-iterate convergence for monotone variational inequalities and connections with cocoercivity. *arXiv preprint arXiv:2110.04261*, 2021.

[29] Robert Gower, Othmane Sebbouh, and Nicolas Loizou. Sgd for structured nonconvex functions: Learning rates, minibatching and interpolation. In *International Conference on Artificial Intelligence and Statistics*, pages 1315–1323. PMLR, 2021.

[30] Robert Mansel Gower, Nicolas Loizou, Xun Qian, Alibek Sailanbayev, Egor Shulgin, and Peter Richtárik. SGD: General Analysis and Improved Rates. In *Proceedings of the 36th International Conference on Machine Learning*, volume 97 of *Proceedings of Machine Learning Research*, pages 5200–5209, 2019.

[31] Yuze Han, Guangzeng Xie, and Zhihua Zhang. Lower complexity bounds of finite-sum optimization problems: The results and construction. *arXiv preprint arXiv:2103.08280*, 2021.

[32] Filip Hanzely and Peter Richtárik. Accelerated coordinate descent with arbitrary sampling and best rates for minibatches. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 304–312. PMLR, 2019.

[33] Filip Hanzely, Konstantin Mishchenko, and Peter Richtárik. SEGA: Variance reduction via gradient sketching. *Advances in Neural Information Processing Systems*, 31, 2018.

[34] Thomas Hofmann, Aurelien Lucchi, Simon Lacoste-Julien, and Brian McWilliams. Variance reduced stochastic gradient descent with neighbors. *Advances in Neural Information Processing Systems*, 28, 2015.

[35] Samuel Horváth, Dmitry Kovalev, Konstantin Mishchenko, Sebastian Stich, and Peter Richtárik. Stochastic distributed learning with gradient quantization and variance reduction. *arXiv preprint arXiv:1904.05115*, 2019.

[36] Yu-Guan Hsieh, Franck Iutzeler, Jérôme Malick, and Panayotis Mertikopoulos. On the convergence of single-call stochastic extra-gradient methods. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d'Alché-Buc, E. Fox, and R. Garnett, editors, *Advances in Neural Information Processing Systems*, volume 32. Curran Associates, Inc., 2019.

[37] Yu-Guan Hsieh, Franck Iutzeler, Jérôme Malick, and Panayotis Mertikopoulos. Explore aggressively, update conservatively: Stochastic extragradient methods with variable stepsize scaling. *Advances in Neural Information Processing Systems*, 33, 2020.

[38] Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. *Advances in Neural Information Processing Systems*, 26, 2013.

[39] Anatoli Juditsky, Arkadi Nemirovski, and Claire Tauvel. Solving variational inequalities with stochastic mirror-prox algorithm. *Stochastic Systems*, 1(1):17–58, 2011.

[40] Sai Praneeth Karimireddy, Quentin Rebjock, Sebastian Stich, and Martin Jaggi. Error feedback fixes signsgd and other gradient compression schemes. In *International Conference on Machine Learning*, pages 3252–3261. PMLR, 2019.

[41] Ahmed Khaled, Othmane Sebbouh, Nicolas Loizou, Robert M. Gower, and Peter Richtárik. Unified analysis of stochastic gradient methods for composite convex and smooth optimization. *arXiv preprint arXiv:2006.11573*, 2020.

[42] Galina M Korpelevich. The extragradient method for finding saddle points and other problems. *Matecon*, 12:747–756, 1976.

[43] Dmitry Kovalev, Samuel Horváth, and Peter Richtárik. Don't jump through hoops and remove those loops: SVRG and Katyusha are better without the outer loop. In *Algorithmic Learning Theory*, 2020.

[44] Chris Junchi Li, Yaodong Yu, Nicolas Loizou, Gauthier Gidel, Yi Ma, Nicolas Le Roux, and Michael I Jordan. On the convergence of stochastic extragradient for bilinear games with restarted iteration averaging. *arXiv preprint arXiv:2107.00464*, 2021.

[45] Zhize Li, Dmitry Kovalev, Xun Qian, and Peter Richtarik. Acceleration for compressed gradient descent in distributed and federated optimization. In *International Conference on Machine Learning*, pages 5895–5904. PMLR, 2020.

[46] Hongzhou Lin, Julien Mairal, and Zaid Harchaoui. Catalyst acceleration for first-order convex optimization: from theory to practice. *Journal of Machine Learning Research*, 18(1):7854–7907, 2018.

[47] Tianyi Lin, Zhengyuan Zhou, Panayotis Mertikopoulos, and Michael Jordan. Finite-time last-iterate convergence for multi-agent learning in games. In *International Conference on Machine Learning*, pages 6161–6171. PMLR, 2020.

[48] Sijia Liu, Songtao Lu, Xiangyi Chen, Yao Feng, Kaidi Xu, Abdullah Al-Dujaili, Mingyi Hong, and Una-May O'Reilly. Min-max optimization without gradients: Convergence and applications to black-box evasion and poisoning attacks. In *International Conference on Machine Learning*, pages 6282–6293. PMLR, 2020.

[49] Nicolas Loizou and Peter Richtárik. Convergence analysis of inexact randomized iterative methods. *SIAM Journal on Scientific Computing*, 42(6):A3979–A4016, 2020.

[50] Nicolas Loizou and Peter Richtárik. Momentum and stochastic momentum for stochastic gradient, newton, proximal point and subspace descent methods. *Computational Optimization and Applications*, 77(3):653–710, 2020.

[51] Nicolas Loizou, Hugo Berard, Alexia Jolicoeur-Martineau, Pascal Vincent, Simon Lacoste-Julien, and Ioannis Mitliagkas. Stochastic hamiltonian gradient methods for smooth games. In *International Conference on Machine Learning*, pages 6370–6381. PMLR, 2020.

[52] Nicolas Loizou, Hugo Berard, Gauthier Gidel, Ioannis Mitliagkas, and Simon Lacoste-Julien. Stochastic gradient descent-ascent and consensus optimization for smooth games: Convergence analysis under expected co-coercivity. *Advances in Neural Information Processing Systems*, 34, 2021.

[53] Luo Luo, Guangzeng Xie, Tong Zhang, and Zhihua Zhang. Near optimal stochastic algorithms for finite-sum unbalanced convex-concave minimax optimization. *arXiv preprint arXiv:2106.01761*, 2021.

[54] Yura Malitsky and Matthew K Tam. A forward-backward splitting method for monotone inclusions without cocoercivity. *SIAM Journal on Optimization*, 30(2):1451–1472, 2020.

[55] Panayotis Mertikopoulos and Zhengyuan Zhou. Learning in games with continuous action sets and unknown payoff functions. *Mathematical Programming*, 173(1):465–507, 2019.

[56] Konstantin Mishchenko, Eduard Gorbunov, Martin Takáč, and Peter Richtárik. Distributed learning with compressed gradient differences. *arXiv preprint arXiv:1901.09269*, 2019.

[57] Konstantin Mishchenko, Dmitry Kovalev, Egor Shulgin, Peter Richtarik, and Yura Malitsky. Revisiting stochastic extragradient. In Silvia Chiappa and Roberto Calandra, editors, *Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics*, volume 108 of *Proceedings of Machine Learning Research*, pages 4573–4582. PMLR, 26–28 Aug 2020.

[58] Oskar Morgenstern and John Von Neumann. *Theory of games and economic behavior*. Princeton university press, 1953.

[59] Arkadi Nemirovski, Anatoli Juditsky, Guanghui Lan, and Alexander Shapiro. Robust stochastic approximation approach to stochastic programming. *SIAM Journal on Optimization*, 19(4): 1574–1609, 2009.

[60] Yurii Nesterov. Dual extrapolation and its applications to solving variational inequalities and related problems. *Mathematical Programming*, 109(2):319–344, 2007.

[61] Yurii Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical programming*, 120(1):221–259, 2009.

[62] Balamurugan Palaniappan and Francis Bach. Stochastic variance reduction methods for saddle-point problems. In *Advances in Neural Information Processing Systems*, pages 1416–1424, 2016.

[63] Leonid Denisovich Popov. A modification of the arrow-hurwicz method for search of saddle points. *Mathematical notes of the Academy of Sciences of the USSR*, 28(5):845–848, 1980.

[64] Xun Qian, Zheng Qu, and Peter Richtárik. Saga with arbitrary sampling. In *International Conference on Machine Learning*, pages 5190–5199. PMLR, 2019.

[65] Xun Qian, Zheng Qu, and Peter Richtárik. L-svrg and l-katyusha with arbitrary sampling. *Journal of Machine Learning Research*, 22(112):1–47, 2021.

[66] Xun Qian, Peter Richtárik, and Tong Zhang. Error compensated distributed sgd can be accelerated. *Advances in Neural Information Processing Systems*, 34, 2021.

[67] Peter Richtárik and Martin Takác. Stochastic reformulations of linear systems: algorithms and convergence theory. *SIAM Journal on Matrix Analysis and Applications*, 41(2):487–524, 2020.

[68] Peter Richtárik, Igor Sokolov, and Ilyas Fatkhullin. EF21: A new, simpler, theoretically better, and practically faster error feedback. In *Advances in Neural Information Processing Systems*, 2021.

[69] Abdurakhmon Sadiev, Aleksandr Beznosikov, Pavel Dvurechensky, and Alexander Gasnikov. Zeroth-order algorithms for smooth saddle-point problems. In *International Conference on Mathematical Optimization Theory and Operations Research*, pages 71–85. Springer, 2021.

[70] Frank Seide, Hao Fu, Jasha Droppo, Gang Li, and Dong Yu. 1-bit stochastic gradient descent and its application to data-parallel distributed training of speech dnns. In *Fifteenth Annual Conference of the International Speech Communication Association*, 2014.

[71] Chaobing Song, Zhengyuan Zhou, Yichao Zhou, Yong Jiang, and Yi Ma. Optimistic dual extrapolation for coherent non-monotone variational inequalities. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 14303–14314. Curran Associates, Inc., 2020.

[72] Sebastian U Stich. Unified optimal analysis of the (stochastic) gradient method. *arXiv preprint arXiv:1907.04232*, 2019.

[73] Sebastian U Stich, Jean-Baptiste Cordonnier, and Martin Jaggi. Sparsified sgd with memory. In *Proceedings of the 32nd International Conference on Neural Information Processing Systems*, pages 4452–4463, 2018.

[74] Vladislav Tominin, Yaroslav Tominin, Ekaterina Borodich, Dmitry Kovalev, Alexander Gasnikov, and Pavel Dvurechensky. On accelerated methods for saddle-point problems with composite structure. *arXiv preprint arXiv:2103.09344*, 2021.

[75] Sharan Vaswani, Francis Bach, and Mark Schmidt. Fast and faster convergence of sgd for over-parameterized models and an accelerated perceptron. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 1195–1204. PMLR, 2019.

[76] Bang Công Vũ. A splitting algorithm for dual monotone inclusions involving cocoercive operators. *Advances in Computational Mathematics*, 38(3):667–681, 2013.

[77] Zhongruo Wang, Krishnakumar Balasubramanian, Shiqian Ma, and Meisam Razaviyayn. Zeroth-order algorithms for nonconvex minimax problems with improved complexities. *arXiv preprint arXiv:2001.07819*, 2020.

[78] Wei Wen, Cong Xu, Feng Yan, Chunpeng Wu, Yandan Wang, Yiran Chen, and Hai Li. Terngrad: ternary gradients to reduce communication in distributed deep learning. In *Proceedings of the 31st International Conference on Neural Information Processing Systems*, pages 1508–1518, 2017.

[79] Junchi Yang, Negar Kiyavash, and Niao He. Global convergence and variance reduction for a class of nonconvex-nonconcave minimax problems. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 1153–1165. Curran Associates, Inc., 2020.

[80] TaeHo Yoon and Ernest K Ryu. Accelerated algorithms for smooth convex-concave minimax problems with $\mathrm{O}(1/k^2)$ rate on squared gradient norm. In *International Conference on Machine Learning*, pages 12098–12109. PMLR, 2021.

[81] Deming Yuan, Qian Ma, and Zhen Wang. Dual averaging method for solving multi-agent saddle-point problems with quantized information. *Transactions of the Institute of Measurement and Control*, 36(1):38–46, 2014.

[82] Dao Li Zhu and Patrice Marcotte. Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities. *SIAM Journal on Optimization*, 6(3):714–726, 1996.

Supplementary Material
Stochastic Gradient Descent-Ascent: Unified Theory and New Efficient Methods

Contents

1 Introduction **1**
- 1.1 Technical Preliminaries . . . 2
- 1.2 Our Contributions . . . 3

2 Unified Analysis of SGDA **4**

3 SGDA with Arbitrary Sampling **6**

4 SGDA with Variance Reduction **7**

5 Distributed SGDA with Compression **8**

6 Numerical Experiments **10**

A Further Related Work **21**

B Missing Details on Numerical Experiments **24**
- B.1 Setup . . . 24
- B.2 Additional remarks about Fig. 1 . . . 24
- B.3 Numerical Experiments with Distributed Methods . . . 25

C Auxiliary Results and Technical Lemmas **27**

D Proof of The Main Results **29**
- D.1 Quasi-Strongly Monotone Case . . . 29
- D.2 Monotone Case . . . 32
- D.3 Cocoercive Case . . . 43

E SGDA with Arbitrary Sampling: Missing Proofs and Details **48**
- E.1 Proof of Proposition 4 . . . 48
- E.2 Analysis of SGDA-AS in the Quasi-Strongly Monotone Case . . . 48
- E.3 Analysis of SGDA-AS in the Monotone Case . . . 49
- E.4 Analysis of SGDA-AS in the Cocoercive Case . . . 50
- E.5 Missing Details on Arbitrary Sampling . . . 50

F SGDA with Variance Reduction: Missing Proofs and Details **54**
- F.1 L-SVRGDA . . . 54
  - F.1.1 Proof of Proposition 6 . . . 54
  - F.1.2 Analysis of L-SVRGDA in the Quasi-Strongly Monotone Case . . . 55
  - F.1.3 Analysis of L-SVRGDA in the Monotone Case . . . 56
  - F.1.4 Analysis of L-SVRGDA in the Cocoercive Case . . . 56
- F.2 SAGA-SGDA . . . 57
  - F.2.1 SAGA-SGDA Fits Assumption 1 . . . 57
  - F.2.2 Analysis of SAGA-SGDA in the Quasi-Strongly Monotone Case . . . 59

F.2.3 Analysis of SAGA-SGDA in the Monotone Case . . . 59
F.2.4 Analysis of SAGA-SGDA in the Cocoercive Case . . . 60
F.3 Discussion of the Results in the Monotone and Cocoercive Cases . . . 60

G Distributed SGDA with Compression: Missing Proofs and Details **61**
G.1 QSGDA . . . 61
G.1.1 Proof of Proposition 8 . . . 61
G.1.2 Analysis of QSGDA in the Quasi-Strongly Monotone Case . . . 63
G.1.3 Analysis of QSGDA in the Monotone Case . . . 64
G.1.4 Analysis of QSGDA in the Cocoercive Case . . . 64
G.2 DIANA-SGDA . . . 65
G.2.1 Proof of Proposition 9 . . . 65
G.2.2 Analysis of DIANA-SGDA in the Quasi-Strongly Monotone Case . . . 66
G.2.3 Analysis of DIANA-SGDA in the Monotone Case . . . 66
G.2.4 Analysis of DIANA-SGDA in the Cocoercive Case . . . 67
G.3 VR-DIANA-SGDA . . . 68
G.3.1 Proof of Proposition 10 . . . 68
G.3.2 Analysis of VR-DIANA-SGDA in the Quasi-Strongly Monotone Case . . . 73
G.3.3 Analysis of VR-DIANA-SGDA in the Monotone Case . . . 74
G.3.4 Analysis of VR-DIANA-SGDA in the Cocoercive Case . . . 75
G.4 Discussion of the Results in the Monotone and Cocoercive Cases . . . 76

H Coordinate SGDA **77**
H.1 CSGDA . . . 77
H.1.1 CSGDA Fits Assumption 1 . . . 77
H.1.2 Analysis of CSGDA in the Quasi-Strongly Monotone Case . . . 78
H.1.3 Analysis of CSGDA in the Monotone Case . . . 78
H.1.4 Analysis of CSGDA in the Cocoercive Case . . . 79
H.2 SEGA-SGDA . . . 79
H.2.1 SEGA-SGDA Fits Assumption 1 . . . 79
H.2.2 Analysis of SEGA-SGDA in the Quasi-Strongly Monotone Case . . . 80
H.2.3 Analysis of SEGA-SGDA in the Monotone Case . . . 80
H.2.4 Analysis of SEGA-SGDA in the Cocoercive Case . . . 81
H.3 Comparison with Related Work . . . 82

Appendix A. Further Related Work

The references necessary to motivate our work and connect it to the most relevant literature are included in the appropriate sections of the main body of the paper. Here we present a broader view of the literature, including some more references to papers of the area that are not directly related with our work.

Variants of the key assumption in prior work & Detailed comparison to our results. Here we would like to provide more details on the comparison with the closely related works [24, 26, 52].

As we mention in the main part of the paper, Gorbunov et al. [24] focus on solving the much simpler minimization problems using SGD. In particular, their Assumption 4.1 requires a function suboptimality (or Bregman divergence) for the upper bound, a concept that cannot be used in VI problems (there are no functions). Thus, the difference of the two notions does not solely lie on the norm bound, but begins at the deeper, conceptual level. In addition, we focus also on monotone VIs (non-quasi-strongly monotone), while Gorbunov et al. [24] consider only the class of quasi-strongly convex minimization problems.

Next, Gorbunov et al. [26] provide convergence guarantees for vanilla SEG under the arbitrary sampling paradigm. Their analysis is not able to capture SEG with variance reduction, quantization, and coordinate-wise randomization. In contrast, our approach covers variants of SGDA with variance reduction, quantization and coordinate-wise randomization. We are able to capture these more advanced variants by using sequence $\{\sigma_k^2\}_{k\geq 0}$ (see (7)) in our key assumption, and this is a major difference between our approach and the approach of Gorbunov et al. [26]. In addition, our analysis works for the case $R(x) \not\equiv 0$. Although the generalization of the analysis to the case of non-zero R might be trivial in the quasi-strongly monotone case, for the monotone case this is definitely not straightforward. Finally, for the monotone case, we do not require large batch-sizes to achieve any predefined accuracy, while analysis of SEG in [26] does (see Appendix B in their work).

Finally, we highlight again that Loizou et al. [52] focus only on uniform minibatch SGDA for solving quasi-strongly monotone problems. This is only a special case of our approach (see Section 3). We note that even in this scenario, through our analysis we were able to provide faster convergence by considering SGDA with importance sampling (see Appendix E.5 and Fig. 1).

Stochastic methods for solving VIPs. Although this paper is devoted to SGDA-type methods, we briefly mention here the works studying other popular stochastic methods for solving VIPs based on different algorithmic schemes such as Extragradient (EG) method [42] and Optimistic Gradient (OG) method [63]. The first analysis of Stochastic EG for solving (quasi-strongly) monotone VIPs was proposed in Juditsky et al. [39] and then was extended and generalized in various ways [10, 26, 37, 44, 57]. Stochastic OG was studied in Azizian et al. [4], Gidel et al. [22], Hsieh et al. [36]. In addition, lightweight second-order methods like stochastic Hamiltonian methods and stochastic consensus optimization were studied in [51], and [52], respectively.

Analysis of SGDA. SGDA is usually analyzed under uniformly bounded variance assumption. That is, $\mathbb{E}[\|g^k - F(x^k)\|^2 \mid x^k] \leq \sigma^2$ is typically assumed to get convergence guarantees [55, 59, 79]. This assumption rarely holds, especially for unconstrained VIPs: it is easy to construct an example of (1) with F being a finite sum of linear operators such that the variance is unbounded. Lin et al. [47] provide a convergence analysis of SGDA under a relative random noise assumption allowing to handle some special cases not covered by uniformly bounded variance assumption. However, relative noise is also a quite strong assumption and usually requires a special type of noise appearing

in coordinate methods[6] or in the training of overparameterized models [75]. In their recent work, Loizou et al. [52] proposed a new weak condition called expected cocoercivity. This assumption fits our theoretical framework (see Section 3) and does not imply strong conditions on the variance of the stochastic estimator but it is stronger than star-cocoercivity of operator F.

Variance reduction for VIPs. The first variance-reduced variants of SGDA (SVRGDA and SAGA-SGDA – analogs of SVRG [38] and SAGA [19]) for solving (1) with strongly monotone operator F having a finite-sum form with Lipschitz summands were proposed in Palaniappan and Bach [62]. For two-sided PL min-max problems without regularization Yang et al. [79] proposed a variance-reduced version of SGDA with alternating updates. Since the considered class of problems includes non-strongly-convex-non-strongly-concave min-max problems, the rates from Yang et al. [79] are inferior to Palaniappan and Bach [62]. There are also several works studying variance-reduced methods based on different methods rather than SGDA. Chavdarova et al. [16] proposed a combination of SVRG and Extragradient (EG) [42] called SVRE and analyzed the method for strongly monotone VIPs without regularization and with cocoercive summands F_i. The cocoercivity assumption was relaxed to averaged Lipschitzness in Alacaoglu and Malitsky [1], where the authors proposed another variance-reduced version of EG (EG-VR) based on Loopless variant of SVRG [34, 43]. Loizou et al. [51] studied stochastic Hamiltonian gradient descent (SHGD), and propose the first stochastic variance reduced Hamiltonian method, named L-SVRHG, for solving stochastic bilinear games and and stochastic games satisfying a "sufficiently bilinear" condition. Moreover, Loizou et al. [51] provided the first set of global non-asymptotic last-iterate convergence guarantees for a stochastic game over a non-compact domain, in the absence of strong monotonicity assumptions.

We should highlight that the rates from Alacaoglu and Malitsky [1] match the lower bounds from Han et al. [31]. Under additional assumptions similar results were achieved in Carmon et al. [14]. Alacaoglu et al. [2] developed variance-reduced method (FoRB-VR) based on Forward-Reflected-Backward algorithm [54], but the derived rates are inferior to those from Alacaoglu and Malitsky [1].

Using Catalyst acceleration framework of Lin et al. [46], Palaniappan and Bach [62], Tominin et al. [74] achieve (neglecting extra logarithmic factors) similar rates as in Alacaoglu and Malitsky [1] and Luo et al. [53] derive even tighter rates for min-max problems. However, as all Catalyst-based approaches, these methods require solving an auxiliary problem at each iteration, which reduces their practical efficiency.

Communication compression for VIPs. While distributed methods with compression were extensively studied for solving minimization problems both for unbiased compression operators [3, 27, 35, 41, 45, 56, 78] and biased compression operators [8, 25, 40, 66, 68, 70, 73], much less is known for min-max problems and VIPs. To the best of our knowledge, the first work on distributed methods with compression for min-max problems is Yuan et al. [81], where the authors proposed a distributed version of Dual Averaging [61] with rounding and showed a convergence to the neighborhood of the solution that cannot be reduced via standard tricks like increasing the batchsize or decreasing the stepsize. More recently, Beznosikov et al. [12] proposed new distributed variants of EG with unbiased/biased compression for solving (1) with (strongly) monotone and Lipschitz operator F. Beznosikov et al. [12] obtained the first linear convergence guarantees on distributed VIPs with compressed communication.

6. For example, see inequality (66) from Appendix H in the case when there is no regularization term, i.e., when $R(x) \equiv 0$ and, as a result, $F(x^*) = 0$ for all $x^* \in X^*$.

On quasi-strong monotonicity and star-cocoercivity. In this work we focus on quasi-strongly monotone VI problems, a class of structured non-monotone operators for which we are able to provide tight convergence guarantees and avoid the standard issues (cycling and divergence of the methods) appearing in the more general non-monotone regime.

Since in general non-monotone problems, finding approximate first-order locally optimal solutions is intractable [17, 21], it is reasonable to consider class of problems that satisfy special structural assumptions on the objective function for which these intractability barriers can be bypassed. Examples of problems belong in this category are the ones of our work which satisfy (3) or, for example, the two-sided PL condition [79] or the error-bound condition [37]. It is worth highlighting that quasi-strong monotone problems were considered in Gorbunov et al. [26], Loizou et al. [52], Mertikopoulos and Zhou [55], Song et al. [71] as well.

Cocoercivity is a classical assumption in the literature on VIPs [82] and operator splittings [18, 76]. It can be interpreted as an intermediate notion between monotonicity and strong monotonicity. In general, it is stronger than monotonicity and Lipschitzness of the operator, e.g., simple bilinear games are non-cocoercive. From Cauchy-Swartz's inequality, one can show that a ℓ-co-coercive operator is ℓ-Lipschitz. In single-objective minization, one can prove the converse statement by using convex duality. Thus, a gradient of a function is L–co-coercive if and only if the function is convex and L-smooth (i.e. L-Lipschitz gradients) [6]. However, in general, a L-Lipchitz operator is *not* L–co-coercive. Star-cocoercivity is a new notion recently introduced in [52] and is weaker than classical cocoercivity and can be achieved via a proper transformation of quasi-monotone Lipschitz operator [28]. Moreover, any μ-quasi strongly monotone L-Lipschitz operator F is ℓ-star-cocoercive with $\ell \in [L, L^2/\mu]$ and there exist examples of operators that are quasi-strongly monotone and star-cocoercive but neither monotone nor Lipschitz [52].

Coordinate and zeroth-order methods for solving min-max problems and VIPs. Coordinate methods for solving VIPs are rarely considered in the literature. The most relevant results are given in the literature on zeroth-order methods for solving min-max problems. Although some of them can be easily extended to the coordinate versions of methods for solving VIPs, these methods are usually considered and analyzed for min-max problems. The closest work to our paper is Sadiev et al. [69]: they propose and analyze several zeroth-order variants of SGDA and Stochastic EG with two-point feedback oracle for solving strongly-convex-strongly-concave and convex-concave smooth min-max problems with bounded domain. Moreover, Sadiev et al. [69] consider firmly smooth convex-concave min-max problems which is an analog of cocoercivity for min-max problems. There are also papers focusing on different problems like non-sonvex-strongly-concave smooth min-max problems [48, 77], non-smooth strongly-convex-strongly-concave and convex-concave min-max problems [9] and on different methods like ones that use one-point feedback oracle [11]. These works are less relevant to our paper than Sadiev et al. [69]. Moreover, the results derived in these papers are inferior to the ones from Sadiev et al. [69].

Appendix B. Missing Details on Numerical Experiments

The code for the experiments is available here: https://anonymous.4open.science/r/sgda-8572/README.md

B.1. Setup

We consider the special case of (1) with F and R defined as follows:

$$F(x) = \frac{1}{n}\sum_{i=1}^{n} F_i(x), \quad F_i(x) = \mathbf{A}_i x + b_i, \tag{20}$$

$$R(x) = \lambda\|x\|_1 + \delta_{B_r(0)}(x) = \lambda\|x\|_1 + \begin{cases} 0, & \text{if } \|x\|_\infty \le r, \\ +\infty, & \text{if } \|x\|_\infty > r, \end{cases} \tag{21}$$

where each matrix $\mathbf{A}_i \in \mathbb{R}^{d\times d}$ is non-symmetric with all eigenvalues with strictly positive real part, $b_i \in \mathbb{R}^d$, $r > 0$ is the radius of ℓ_∞-ball, and $\lambda \ge 0$ is regularization parameter. One can show (see Example 6.22 from Beck [7]) that for the given $R(x)$ prox operator has an explicit formula:

$$\text{prox}_{\gamma R}(x) = \text{sign}\,(x)\min\left\{\max\left\{|x| - \gamma\lambda, 0\right\}, r\right\}, \tag{22}$$

where $\text{sign}(\cdot)$ and $|\cdot|$ are component-wise operators. The considered problem generalizes the following quadratic game:

$$\min_{\|x_1\|_\infty \le r}\max_{\|x_2\|_\infty \le r} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2}x_1^\top \mathbf{A}_{1,i}x_1 + x_1^\top \mathbf{A}_{2,i}x_2 - \frac{1}{2}x_2^\top \mathbf{A}_{3,i}x_2 + b_{1,i}^\top x_1 - b_{2,i}^\top x_2 + \lambda\|x_1\|_1 - \lambda\|x_2\|_1$$

with $\mu_i\mathbf{I} \preccurlyeq \mathbf{A}_{1,i} \preccurlyeq L_i\mathbf{I}$ and $\mu_i\mathbf{I} \preccurlyeq \mathbf{A}_{3,i} \preccurlyeq L_i\mathbf{I}$. Indeed, the above problem is a special case of (1)+(21) with

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{A}_i = \begin{pmatrix} \mathbf{A}_{1,i} & \mathbf{A}_{2,i} \\ -\mathbf{A}_{2,i} & \mathbf{A}_{3,i} \end{pmatrix}, \quad b_i = \begin{pmatrix} b_{1,i} \\ b_{2,i} \end{pmatrix},$$
$$R(x) = \lambda\|x_1\|_1 + \lambda\|x_2\|_1 + \delta_{B_r(0)}(x_1) + \delta_{B_r(0)}(x_2).$$

In our experiments, to generate the non-symmetric matrices $\mathbf{A}_i \in \mathbb{R}^{d\times d}$ defined in (21), we first sample real random matrices $\mathbf{B}_i$ where the elements of the matrices are sampled from a normal distribution. We then compute the eigendecomposition of the matrices $\mathbf{B}_i = \mathbf{Q}_i\mathbf{D}_i\mathbf{Q}_i^{-1}$, where the $\mathbf{D}_i$ are diagonal matrices with complex numbers on the diagonal. Next, we construct the matrices $\mathbf{A}_i = \Re(\mathbf{Q}_i\mathbf{D}_i^+\mathbf{Q}_i^{-1})$ where $\Re(\mathbf{M})_{i,j} = \Re(\mathbf{M}_{i,j})$ and $\mathbf{D}_i^+$ is obtained by transforming all the elements of $\mathbf{D}_i$ to have positive real part. This process ensures that the eigenvalues of $\mathbf{A}_i$ all have positive real part, and thus that $F(x)$ is strongly monotone and cocoercive. The $b_i \in \mathbb{R}^d$ are sampled from a normal distribution with variance $100/d$. For all the experiments we choose $n = 1000$ and $d = 100$. For the distributed experiments we simulate $m = 10$ nodes on a single machine with 2 CPUs.

B.2. Additional remarks about Fig. 1

In all figures of the paper, we plot the distance to optimality as a function of the number of oracle calls. When using variance reduced methods we sometimes have to compute the full-batch gradient, and thus have to make n oracle calls. This is why we observe "steps" for variance reduced methods in Fig. 1, we observe a "step" every-time the full batch gradient is computed.

B.3. Numerical Experiments with Distributed Methods

In our last experiment, we consider a distributed version of the quadratic game, in which we assume that $F(x) = \frac{1}{n}\sum_{i=1}^{n} F_i(x)$ with each $\{F_i\}_{i=1}^{n}$ having similar form to (20). The information about operator F_i is stored on node i only. We compare the distributed methods proposed in the paper: QSGDA, DIANA-SGDA, and VR-DIANA-SGDA. For the quantization we use the RandK sparsification [8] with $K = 5$. We show our findings in Fig. 2 and 3, where the performance is measured both in terms of number of oracle calls and the number of bits communicated from workers to the server. In both figures, we can clearly see the advantage of using quantization in terms of reducing the communication cost compared to the baseline SGDA. We also observe that VR-DIANA-SGDA achieves linear convergence to the solution. However, DIANA-SGDA performs similarly to QSGDA since the noise σ^2 is larger than the dissimilarity constant ζ_*^2.

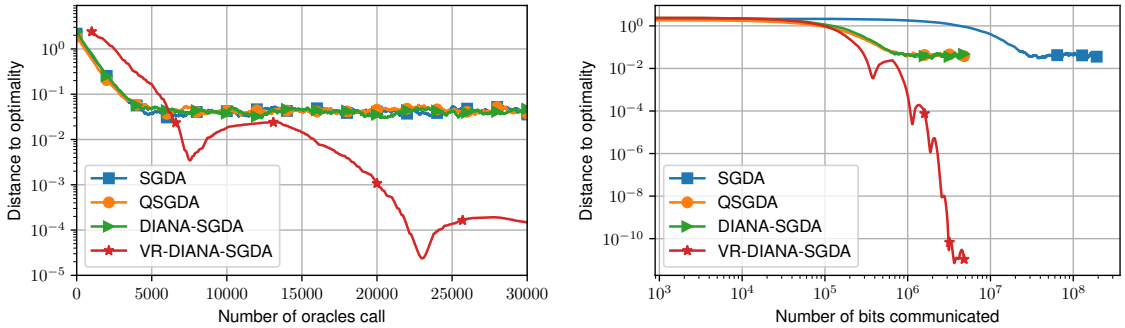

Figure 2: Comparison of algorithms in distributed setting **Left:** Number of oracle calls. **Right:** Number of bits communicated.

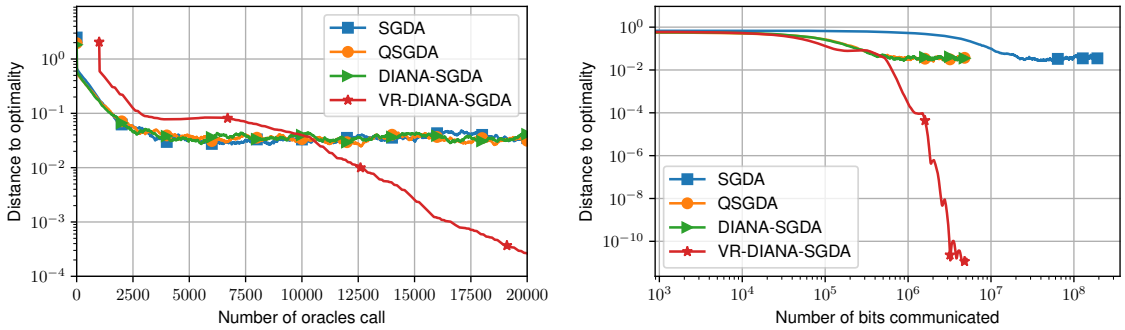

Figure 3: Results on distributed quadratic games with constraints. **Letf:** Number of oracle calls. **Right:** Number of bits communicated between nodes.

To illustrate further the difference between DIANA-SGDA and QSGDA, we conduct an additional experiment with full-batched methods ($\sigma = 0$), see Fig. 4. We consider the full-batch version of QSGDA and DIANA-SGDA. This enables us to separate the noise coming from the quantization from the noise coming from the stochasticity. We observe that when using full-batch DIANA-SGDA converges linearly to the solution while QSGDA only converges to a neighborhood of the solution.

An interesting observation is that although the convergence is linear, the distance to optimality is not monotonically decreasing, this does not contradicts the theory.

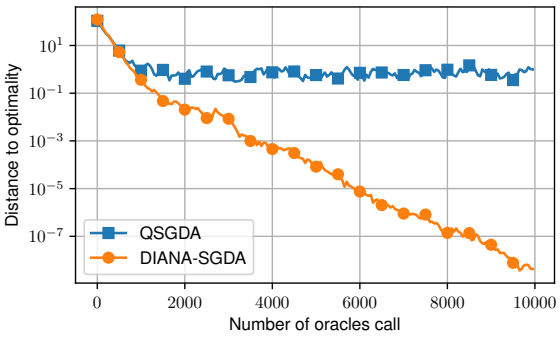

Figure 4: QSGDA vs DIANA-SGDA: DIANA-SGDA converges linearly to the solution while QSGDA only converges to a neighborhood of the solution.

Appendix C. Auxiliary Results and Technical Lemmas

Useful inequalities. In our proofs, we often apply the following inequalities that hold for any $a, b \in \mathbb{R}^d$ and $\alpha > 0$:

$$\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2, \tag{23}$$

$$\langle a, b\rangle \leq \frac{1}{2\alpha}\|a\|^2 + \frac{\alpha}{2}\|b\|^2. \tag{24}$$

Useful lemmas. The following lemma from Stich [72] allows us to derive the rates of convergence to the exact solution.

Lemma 11 (Simplified version of Lemma 3 from [72]) *Let the non-negative sequence $\{r_k\}_{k\geq 0}$ satisfy the relation*

$$r_{k+1} \leq (1 - a\gamma_k)r_k + c\gamma_k^2$$

for all $k \geq 0$, parameters $a > 0$, $c \geq 0$, and any non-negative sequence $\{\gamma_k\}_{k\geq 0}$ such that $\gamma_k \leq 1/h$ for some $h \geq a$, $h > 0$. Then, for any $K \geq 0$ one can choose $\{\gamma_k\}_{k\geq 0}$ as follows:

$$\begin{aligned} \text{if } K \leq \frac{h}{a}, &\qquad \gamma_k = \frac{1}{h}, \\ \text{if } K > \frac{h}{a} \text{ and } k < k_0, &\qquad \gamma_k = \frac{1}{h}, \\ \text{if } K > \frac{h}{a} \text{ and } k \geq k_0, &\qquad \gamma_k = \frac{2}{a(\kappa + k - k_0)}, \end{aligned}$$

where $\kappa = 2h/a$ and $k_0 = \lceil K/2 \rceil$. For this choice of γ_k the following inequality holds:

$$r_K \leq \frac{32hr_0}{a}\exp\left(-\frac{aK}{2h}\right) + \frac{36c}{a^2K}.$$

In the analysis of monotone case, we rely on the classical result from proximal operators theory.

Lemma 12 (Theorem 6.39 (iii) from Beck [7]) *Let R be a proper lower semicontinuous convex function and $x^+ = prox_{\gamma R}(x)$. Then for all $z \in \mathbb{R}^d$ the following inequality holds:*

$$\langle x^+ - x, z - x^+\rangle \geq \gamma\left(R(x^+) - R(z)\right).$$

Finally, we rely on the following technical lemma for handling the sums arising in the proofs for the monotone case.

Lemma 13 *Let $K > 0$ be a positive integer and $\eta_1, \eta_2, \ldots, \eta_K$ be random vectors such that $\mathbb{E}_k[\eta_k] := \mathbb{E}[\eta_k \mid \eta_1, \ldots, \eta_{k-1}] = 0$ for $k = 2, \ldots, K$. Then*

$$\mathbb{E}\left[\left\|\sum_{k=1}^K \eta_k\right\|^2\right] = \sum_{k=1}^K \mathbb{E}[\|\eta_k\|^2]. \tag{25}$$

Proof We start with the following derivation:

$$
\begin{aligned}
\mathbb{E}\left[\left\|\sum_{k=1}^{K}\eta_k\right\|^2\right] &= \mathbb{E}[\|\eta_K\|^2] + 2\mathbb{E}\left[\left\langle \eta_K, \sum_{k=1}^{K-1}\eta_k\right\rangle\right] + \mathbb{E}\left[\left\|\sum_{k=1}^{K-1}\eta_k\right\|^2\right] \\
&= \mathbb{E}[\|\eta_K\|^2] + 2\mathbb{E}\left[\mathbb{E}_K\left[\left\langle \eta_K, \sum_{k=1}^{K-1}\eta_k\right\rangle\right]\right] + \mathbb{E}\left[\left\|\sum_{k=1}^{K-1}\eta_k\right\|^2\right] \\
&= \mathbb{E}[\|\eta_K\|^2] + 2\mathbb{E}\left[\left\langle \mathbb{E}_K[\eta_K], \sum_{k=1}^{K-1}\eta_k\right\rangle\right] + \mathbb{E}\left[\left\|\sum_{k=1}^{K-1}\eta_k\right\|^2\right] \\
&= \mathbb{E}[\|\eta_K\|^2] + \mathbb{E}\left[\left\|\sum_{k=1}^{K-1}\eta_k\right\|^2\right].
\end{aligned}
$$

Applying similar steps to $\mathbb{E}\left[\left\|\sum_{k=1}^{K-1}\eta_k\right\|^2\right], \mathbb{E}\left[\left\|\sum_{k=1}^{K-2}\eta_k\right\|^2\right], \ldots, \mathbb{E}\left[\left\|\sum_{k=1}^{2}\eta_k\right\|^2\right]$, we get the result. ■

Appendix D. Proof of The Main Results

In this section, we provide complete proofs of our main results.

D.1. Quasi-Strongly Monotone Case

We start with the case when F satisfies (3) with $\mu > 0$. For readers convenience, we restate the theorems below.

Theorem 14 (Theorem 1) *Let F be μ-quasi-strongly monotone with $\mu > 0$ and Assumption 1 hold. Assume that*

$$0 < \gamma \leq \min\left\{\frac{1}{\mu}, \frac{1}{2(A+CM)}\right\} \tag{26}$$

for some $M > {}^{B}/_{\rho}$. Then for the Lyapunov function $V_k = \|x^k - x^{*,k}\|^2 + M\gamma^2\sigma_k^2$, and for all $k \geq 0$ we have

$$\mathbb{E}[V_k] \quad \leq \quad \left(1 - \min\left\{\gamma\mu, \rho - \frac{B}{M}\right\}\right)^k \mathbb{E}[V_0] + \frac{\gamma^2(D_1 + MD_2)}{\min\left\{\gamma\mu, \rho - {}^{B}/_{M}\right\}}. \tag{27}$$

Proof First of all, we recall a well-known fact about proximal operators: for any solution x^* of (1) we have

$$x^* = \text{prox}_{\gamma R}(x^* - \gamma F(x^*)). \tag{28}$$

Using this and non-expansiveness of proximal operator, we derive

$$\begin{aligned}
\|x^{k+1} - x^{*,k+1}\|^2 &\leq \|x^{k+1} - x^{*,k}\|^2 \\
&= \left\|\text{prox}_{\gamma R}(x^k - \gamma g^k) - \text{prox}_{\gamma R}(x^{*,k} - \gamma F(x^{*,k}))\right\|^2 \\
&\leq \left\|x^k - \gamma g^k - x^{*,k} - \gamma F(x^{*,k})\right\|^2 \\
&= \|x^k - x^{*,k}\|^2 - 2\gamma\left\langle x^k - x^{*,k}, g^k - F(x^{*,k})\right\rangle + \gamma^2\|g^k - F(x^{*,k})\|^2.
\end{aligned}$$

Next, we take an expectation $\mathbb{E}_k[\cdot]$ w.r.t. the randomness at iteration k and get

$$\begin{aligned}
\mathbb{E}_k\left[\|x^{k+1} - x^{*,k+1}\|^2\right] &= \|x^k - x^{*,k}\|^2 - 2\gamma\left\langle x^k - x^{*,k}, F(x^k) - F(x^{*,k})\right\rangle \\
&\quad + \gamma^2\mathbb{E}_k\left[\left\|g^k - F(x^{*,k})\right\|^2\right] \\
&\overset{(6)}{\leq} \|x^k - x^{*,k}\|^2 - 2\gamma\left\langle x^k - x^*, F(x^k) - F(x^{*,k})\right\rangle \\
&\quad + \gamma^2\left(2A\left\langle x^k - x^{*,k}, F(x^k) - F(x^{*,k})\right\rangle + B\sigma_k^2 + D_1\right).
\end{aligned}$$

Summing up this inequality with (7) multiplied by $M\gamma^2$, we obtain

$$\begin{aligned}
&\mathbb{E}_k\left[\|x^{k+1}-x^{*,k+1}\|^2\right]+M\gamma^2\mathbb{E}_k[\sigma_{k+1}^2]\\
&\qquad\leq \|x^k-x^{*,k}\|^2-2\gamma\left\langle x^k-x^{*,k},F(x^k)-F(x^{*,k})\right\rangle\\
&\qquad\quad+\gamma^2\left(2A\left\langle x^k-x^{*,k},F(x^k)-F(x^{*,k})\right\rangle+B\sigma_k^2+D_1\right)\\
&\qquad\quad+M\gamma^2\left(2C\left\langle x^k-x^{*,k},F(x^k)-F(x^{*,k})\right\rangle+(1-\rho)\sigma_k^2+D_2\right)\\
&\qquad=\|x^k-x^{*,k}\|^2+M\gamma^2\left(1-\rho+\frac{B}{M}\right)\sigma_k^2+\gamma^2(D_1+MD_2)\\
&\qquad\quad-2\gamma\left(1-\gamma(A+CM)\right)\left\langle x^k-x^{*,k},F(x^k)-F(x^{*,k})\right\rangle. \qquad (29)
\end{aligned}$$

Since $\gamma\leq\frac{1}{2(A+CM)}$ the factor $-2\gamma\left(1-\gamma(A+CM)\right)$ is non-positive. Therefore, applying strong quasi-monotonicity of F, we derive

$$\begin{aligned}
\mathbb{E}_k\left[\|x^{k+1}-x^{*,k+1}\|^2+M\gamma^2\sigma_{k+1}^2\right] &\leq \left(1-2\gamma\mu\left(1-\gamma(A+CM)\right)\right)\|x^k-x^{*,k}\|^2\\
&\quad+M\gamma^2\left(1-\rho+\frac{B}{M}\right)\sigma_k^2+\gamma^2(D_1+MD_2).
\end{aligned}$$

Using $\gamma\leq\frac{1}{2(A+CM)}$ and the definition $V_k=\|x^k-x^{*,k}\|^2+M\gamma^2\sigma_k^2$, we get

$$\begin{aligned}
\mathbb{E}_k\left[V_{k+1}\right] &\leq \left(1-\gamma\mu\right)\|x^k-x^{*,k}\|^2+M\gamma^2\left(1-\rho+\frac{B}{M}\right)\sigma_k^2+\gamma^2(D_1+MD_2)\\
&\leq \left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)V_k+\gamma^2(D_1+MD_2).
\end{aligned}$$

Next, we take the full expectation from the above inequality and establish the following recurrence:

$$\mathbb{E}\left[V_{k+1}\right] \leq \left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)\mathbb{E}[V_k]+\gamma^2(D_1+MD_2). \qquad (30)$$

Unrolling the recurrence, we derive

$$\begin{aligned}
\mathbb{E}\left[V_k\right] &\leq \left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)^k\mathbb{E}[V_0]+\gamma^2(D_1+MD_2)\sum_{t=0}^{k-1}\left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)^t\\
&\leq \left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)^k\mathbb{E}[V_0]+\gamma^2(D_1+MD_2)\sum_{t=0}^{\infty}\left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)^t\\
&= \left(1-\min\left\{\gamma\mu,\rho-\frac{B}{M}\right\}\right)^k\mathbb{E}[V_0]+\frac{\gamma^2(D_1+MD_2)}{\min\left\{\gamma\mu,\rho-{}^{B}/_{M}\right\}},
\end{aligned}$$

which finishes the proof. ■

Using this and Lemma 11, we derive the following result about the convergence to the exact solution.

Corollary 15 (Corollary 2) *Let the assumptions of Theorem 1 hold. Consider two possible cases.*

1. *Let $D_1 = D_2 = 0$. Then, for any $K \geq 0$, $M = {}^{2B}/_{\rho}$, and*

$$\gamma = \min\left\{\frac{1}{\mu}, \frac{1}{2(A + {}^{2BC}/_{\rho})}\right\} \tag{31}$$

we have

$$\mathbb{E}[V_K] \leq \mathbb{E}[V_0] \exp\left(-\min\left\{\frac{\mu}{2(A + {}^{2BC}/_{\rho})}, \frac{\rho}{2}\right\} K\right). \tag{32}$$

2. *Let $D_1 + MD_2 > 0$. Then, for any $K \geq 0$ and $M = {}^{2B}/_{\rho}$ one can choose $\{\gamma_k\}_{k\geq 0}$ as follows:*

$$\begin{aligned} \text{if } K \leq \frac{h}{\mu}, \qquad & \gamma_k = \frac{1}{h}, \\ \text{if } K > \frac{h}{\mu} \text{ and } k < k_0, \qquad & \gamma_k = \frac{1}{h}, \\ \text{if } K > \frac{h}{\mu} \text{ and } k \geq k_0, \qquad & \gamma_k = \frac{2}{\mu(\kappa + k - k_0)}, \end{aligned} \tag{33}$$

where $h = \max\{2(A + {}^{2BC}/_{\rho}), {}^{2\mu}/_{\rho}\}$, $\kappa = {}^{2h}/_{\mu}$ and $k_0 = \lceil {}^{K}/_{2} \rceil$. For this choice of γ_k the following inequality holds:

$$\begin{aligned} \mathbb{E}[V_K] \quad \leq \quad & 32 \max\left\{\frac{2(A + {}^{2BC}/_{\rho})}{\mu}, \frac{2}{\rho}\right\} \mathbb{E}[V_0] \exp\left(-\min\left\{\frac{\mu}{2(A + {}^{2BC}/_{\rho})}, \frac{\rho}{4}\right\} K\right) \\ & + \frac{36(D_1 + {}^{2BD_2}/_{\rho})}{\mu^2 K}. \end{aligned} \tag{34}$$

Proof The first part of the corollary follows from Theorem 1 due to

$$\left(1 - \min\left\{\gamma\mu, \rho - \frac{B}{M}\right\}\right)^K = \left(1 - \min\left\{\gamma\mu, \frac{\rho}{2}\right\}\right)^K \leq \exp\left(-\min\left\{\gamma\mu, \frac{\rho}{2}\right\} K\right).$$

Plugging (31) in the above inequality, we derive (32). Next, we consider the case when $D_1 + MD_2 > 0$. First, we notice that (30) holds for non-constant stepsizes γ_k such that

$$0 < \gamma_k \leq \min\left\{\frac{1}{\mu}, \frac{1}{2(A + CM)}\right\}.$$

Therefore, for any $k \geq 0$ we have

$$\begin{aligned} \mathbb{E}\left[V_{k+1}\right] \quad & \leq \quad \left(1 - \min\left\{\gamma_k\mu, \rho - \frac{B}{M}\right\}\right) \mathbb{E}[V_k] + \gamma_k^2 (D_1 + MD_2) \\ & \overset{M = {}^{2B}/_{\rho}}{=} \quad \left(1 - \min\left\{\gamma_k\mu, {}^{\rho}/_{2}\right\}\right) \mathbb{E}[V_k] + \gamma_k^2 (D_1 + {}^{2BD_2}/_{\rho}). \end{aligned}$$

Secondly, we assume that for all $k \geq 0$

$$0 < \gamma_k \leq \min\left\{\frac{\rho}{2\mu}, \frac{1}{2(A + CM)}\right\}.$$

Applying this to the recurrence for $\mathbb{E}[V_k]$, we obtain

$$\mathbb{E}\left[V_{k+1}\right] \quad \leq \quad (1-\gamma_k\mu)\,\mathbb{E}[V_k] + \gamma_k^2(D_1 + {}^{2BD_2}/_{\rho}).$$

It remains to apply Lemma 11 with $r_k = \mathbb{E}[V_k]$, $a = \mu$, $c = D_1 + {}^{2BD_2}/_{\rho}$, and $h = \max\left\{2(A + {}^{2BC}/_{\rho}), {}^{2\mu}/_{\rho}\right\}$ to the above recurrence. ■

D.2. Monotone Case

Next, we consider the case when $\mu = 0$. Before deriving the proof, we provide additional discussion of the setup.

We emphasize that the maximum in (9) is taken over the compact set $\mathcal{C}$ containing the solution set X^*. Therefore, the quantity $\text{Gap}_{\mathcal{C}}(z)$ is a valid measure of convergence [60]. We point out that the iterates x^k do not have to lie in $\mathcal{C}$. Our analysis works for the problems with unbounded and bounded domains (see Alacaoglu and Malitsky [1], Nesterov [60] for similar setups).

Another popular convergence measure for the case when $R(x) \equiv 0$ in (1) is $\|F(x^k)\|^2$. Although the squared norm of the operator is a weaker guarantee, it is easier to compute in practice and better suited for non-monotone problems [80]. Nevertheless, $\|F(x^k)\|^2$ is not a valid measure of convergence for (1) with $R(x) \not\equiv 0$. Therefore, we focus on $\text{Gap}_{\mathcal{C}}(z)$ in the monotone case.[7]

Theorem 16 (Theorem 3) *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2 hold. Assume that*

$$0 < \gamma \leq \frac{1}{2(A + {}^{BC}/_{\rho})}. \tag{35}$$

Then for the function $\text{Gap}_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ we have

$$\begin{aligned}
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} \\
& + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + (4A + \ell + {}^{8BC}/_{\rho}) \cdot \frac{\|x^0 - x^{*,0}\|^2}{K} \\
& + \left(4 + (4A + \ell + {}^{8BC}/_{\rho})\,\gamma\right)\frac{\gamma B\sigma_0^2}{\rho K} \\
& + \gamma(2 + \gamma\,(4A + \ell + {}^{8BC}/_{\rho}))(D_1 + {}^{2BD_2}/_{\rho}) \\
& + 9\gamma \max_{x^*\in X^*}\|F(x^*)\|^2.
\end{aligned} \tag{36}$$

Proof First, we apply the classical result about proximal operators (Lemma 12) with $x^+ = x^{k+1}$, $x = x^k - \gamma g^k$, and $z = u$ for arbitrary point $u \in \mathbb{R}^d$:

$$\langle x^{k+1} - x^k + \gamma g^k, u - x^{k+1}\rangle \quad \geq \quad \gamma\left(R(x^{k+1}) - R(u)\right).$$

Multiplying by the factor of 2 and making small rearrangement, we get

$$2\gamma\langle g^k, u - x^k\rangle + 2\langle x^{k+1} - x^k, u - x^k\rangle + 2\langle x^{k+1} - x^k + \gamma g^k, x^k - x^{k+1}\rangle \quad \geq \quad 2\gamma\left(R(x^{k+1}) - R(u)\right)$$

7. When $R(x) \equiv 0$, our analysis can be modified to get the guarantees on the squared norm of the operator.

implying

$$\begin{aligned}2\gamma\left(\langle F(x^k), x^k - u\rangle + R(x^{k+1}) - R(u)\right) &\leq 2\langle x^{k+1} - x^k, u - x^k\rangle + 2\gamma\langle F(x^k) - g^k, x^k - u\rangle \\ &\quad +2\langle x^{k+1} - x^k, x^k - x^{k+1}\rangle + 2\gamma\langle g^k, x^k - x^{k+1}\rangle.\end{aligned}$$

Next, we use a squared norm decomposition $\|a+b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b\rangle$, and obtain

$$\begin{aligned}2\gamma\left(\langle F(x^k), x^k - u\rangle + R(x^{k+1}) - R(u)\right) &\leq \|x^{k+1} - x^k\|^2 + \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ &\quad +2\gamma\langle F(x^k) - g^k, x^k - u\rangle \\ &\quad -2\|x^{k+1} - x^k\|^2 + 2\gamma\langle g^k, x^k - x^{k+1}\rangle. \qquad (37)\end{aligned}$$

Then, due to $2\langle a, b\rangle \leq \|a\|^2 + \|b\|^2$ we have

$$\begin{aligned}2\gamma\left(\langle F(x^k), x^k - u\rangle + R(x^{k+1}) - R(u)\right) &\leq \|x^{k+1} - x^k\|^2 + \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ &\quad +2\gamma\langle F(x^k) - g^k, x^k - u\rangle \\ &\quad -2\|x^{k+1} - x^k\|^2 + \gamma^2\|g^k\|^2 + \|x^k - x^{k+1}\|^2 \\ &= \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ &\quad +2\gamma\langle F(x^k) - g^k, x^k - u\rangle + \gamma^2\|g^k\|^2.\end{aligned}$$

Monotonicity of F implies $\langle F(u), x^k - u\rangle \leq \langle F(x^k), x^k - u\rangle$, allowing us to continue our derivation as follows:

$$\begin{aligned}2\gamma\left(\langle F(u), x^k - u\rangle + R(x^{k+1}) - R(u)\right) &\leq \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ &\quad +2\gamma\langle F(x^k) - g^k, x^k - u\rangle + \gamma^2\|g^k\|^2 \\ &= \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ &\quad +2\gamma\langle F(x^k) - g^k, x^k - u\rangle \\ &\quad +\gamma^2\|g^k - g^{*,k} + g^{*,k}\|^2 \\ &\overset{(23)}{\leq} \|x^k - u\|^2 - \|x^{k+1} - u\|^2 \\ &\quad +2\gamma\langle F(x^k) - g^k, x^k - u\rangle \\ &\quad +2\gamma^2\|g^k - g^{*,k}\|^2 + 2\gamma^2\|g^{*,k}\|^2.\end{aligned}$$

Summing up the above inequality for $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned}
2\gamma \sum_{k=0}^{K-1} \Big(\langle F(u), x^k - u \rangle + R(x^{k+1}) - R(u) \Big) &\leq \sum_{k=0}^{K-1} \|x^k - u\|^2 - \sum_{k=0}^{K-1} \|x^{k+1} - u\|^2 \\
&\quad + 2\gamma^2 \sum_{k=0}^{K-1} \|g^{*,k}\|^2 \\
&\quad + 2\gamma \sum_{k=0}^{K-1} \langle F(x^k) - g^k, x^k - u \rangle \\
&\quad + 2\gamma^2 \sum_{k=0}^{K-1} \|g^k - g^{*,k}\|^2 \\
&= \|x^0 - u\|^2 - \|x^K - u\|^2 + 2\gamma^2 \sum_{k=0}^{K-1} \|g^{*,k}\|^2 \\
&\quad + 2\gamma \sum_{k=0}^{K-1} \langle F(x^k) - g^k, x^k - u \rangle \\
&\quad + 2\gamma^2 \sum_{k=0}^{K-1} \|g^k - g^{*,k}\|^2.
\end{aligned}$$

Next, we divide both sides by $2\gamma K$

$$\begin{aligned}
\frac{1}{K} \sum_{k=0}^{K-1} \Big(\langle F(u), x^k - u \rangle + R(x^{k+1}) - R(u) \Big) &\leq \frac{\|x^0 - u\|^2 - \|x^K - u\|^2}{2\gamma K} + \frac{\gamma}{K} \sum_{k=0}^{K-1} \|g^{*,k}\|^2 \\
&\quad + \frac{1}{K} \sum_{k=0}^{K-1} \langle F(x^k) - g^k, x^k - u \rangle \\
&\quad + \frac{\gamma}{K} \sum_{k=0}^{K-1} \|g^k - g^{*,k}\|^2
\end{aligned}$$

and, after small rearrangement, we obtain

$$\begin{aligned}
\frac{1}{K} \sum_{k=0}^{K-1} \Big(\langle F(u), x^{k+1} - u \rangle + R(x^{k+1}) - R(u) \Big) &\leq \frac{\|x^0 - u\|^2 - \|x^K - u\|^2}{2\gamma K} + \frac{\langle F(u), x^K - x^0 \rangle}{K} \\
&\quad + \frac{\gamma}{K} \sum_{k=0}^{K-1} \|g^{*,k}\|^2 \\
&\quad + \frac{1}{K} \sum_{k=0}^{K-1} \langle F(x^k) - g^k, x^k - u \rangle \\
&\quad + \frac{\gamma}{K} \sum_{k=0}^{K-1} \|g^k - g^{*,k}\|^2.
\end{aligned}$$

Applying Jensen's inequality for convex function R, we get $R\left(\frac{1}{K}\sum_{k=0}^{K-1} x^{k+1}\right) \leq \frac{1}{K}\sum_{k=0}^{K-1} R(x^{k+1})$. Plugging this in the previous inequality, we derive for u^* being a projection of u on X^*

$$
\begin{aligned}
&\left\langle F(u), \left(\frac{1}{K}\sum_{k=0}^{K-1} x^{k+1}\right) - u\right\rangle + R\left(\frac{1}{K}\sum_{k=0}^{K-1} x^{k+1}\right) - R(u) \\
&\qquad \leq \frac{\|x^0 - u\|^2 - \|x^K - u\|^2}{2\gamma K} + \frac{\langle F(u), x^K - x^0\rangle}{K} + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^{*,k}\|^2 \\
&\qquad\quad + \frac{1}{K}\sum_{k=0}^{K-1}\langle F(x^k) - g^k, x^k - u\rangle + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^k - g^{*,k}\|^2 \\
&\qquad \overset{(24)}{\leq} \frac{\|x^0 - u\|^2 - \|x^K - u\|^2}{2\gamma K} + \frac{\|x^K - x^0\|^2}{4\gamma K} + \frac{4\gamma}{K}\|F(u) - F(u^*) + F(u^*)\|^2 \\
&\qquad\quad + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^{*,k}\|^2 + \frac{1}{K}\sum_{k=0}^{K-1}\langle F(x^k) - g^k, x^k - u\rangle + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^k - g^{*,k}\|^2 \\
&\qquad \overset{(23)}{\leq} \frac{\|x^0 - u\|^2 - \|x^K - u\|^2}{2\gamma K} + \frac{\|x^0 - u\|^2 + \|x^K - u\|^2}{2\gamma K} + \frac{8\gamma}{K}\|F(u) - F(u^*)\|^2 \\
&\qquad\quad + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^{*,k}\|^2 + 8\gamma\|F(u^*)\|^2 + \frac{1}{K}\sum_{k=0}^{K-1}\langle F(x^k) - g^k, x^k - u\rangle \\
&\qquad\quad + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^k - g^{*,k}\|^2 \\
&\qquad \overset{(4)}{\leq} \frac{\|x^0 - u\|^2}{\gamma K} + \frac{8\gamma\ell^2\|u - u^*\|^2}{K} + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\qquad\quad + \frac{1}{K}\sum_{k=0}^{K-1}\langle F(x^k) - g^k, x^k - u\rangle + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^k - g^{*,k}\|^2.
\end{aligned}
$$

Next, we take maximum from the both sides in $u \in \mathcal{C}$, which gives $\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)$ in the left-hand side by definition (9), and take the expectation of the result:

$$
\begin{aligned}
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] &\leq \frac{\mathbb{E}\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{\gamma K} + \frac{8\gamma\ell^2\mathbb{E}\left[\max_{u\in\mathcal{C}}\|u-u^*\|^2\right]}{K} \\
&\quad +9\gamma \max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\quad +\frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle\right] + \frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k-g^{*,k}\|^2\right] \\
&\leq \frac{\mathbb{E}\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + 9\gamma \max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\quad +\frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle\right] \\
&\quad +\frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k-g^{*,k}\|^2\right]. \qquad (38)
\end{aligned}
$$

In the last step, we also use that $X^* \subset \mathcal{C}$ and $\Omega_{\mathcal{C}} := \max_{x,y\in\mathcal{C}}\|x-y\|$ (Assumption 2).

It remains to upper bound the terms from the last two lines of (38). We start with the first one. Since

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k\rangle\right] &= \mathbb{E}\left[\sum_{k=0}^{K-1}\langle \mathbb{E}[F(x^k)-g^k \mid x^k], x^k\rangle\right] = 0, \\
\mathbb{E}\left[\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^0\rangle\right] &= \sum_{k=0}^{K-1}\left\langle \mathbb{E}[F(x^k)-g^k], x^0\right\rangle = 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle\right] &= \frac{1}{K}\mathbb{E}\left[\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k\rangle\right] \\
&\quad +\frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, -u\rangle\right] \\
&= \frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, -u\rangle\right] \\
&= \frac{1}{K}\mathbb{E}\left[\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^0\rangle\right] \\
&\quad +\frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, -u\rangle\right] \\
&= \mathbb{E}\left[\max_{u\in\mathcal{C}}\left\langle \frac{1}{K}\sum_{k=0}^{K-1}(F(x^k)-g^k), x^0-u\right\rangle\right] \\
&\overset{(24)}{\leq} \mathbb{E}\left[\max_{u\in\mathcal{C}}\left\{\frac{\gamma K}{2}\left\|\frac{1}{K}\sum_{k=0}^{K-1}(F(x^k)-g^k)\right\|^2+\frac{1}{2\gamma K}\|x^0-u\|^2\right\}\right] \\
&= \frac{\gamma}{2K}\mathbb{E}\left[\left\|\sum_{k=0}^{K-1}(F(x^k)-g^k)\right\|^2\right]+\frac{1}{2\gamma K}\max_{u\in\mathcal{C}}\|x^0-u\|^2.
\end{aligned}
$$

We notice that $\mathbb{E}[F(x^k)-g^k \mid F(x^0)-g^0,\ldots,F(x^{k-1})-g^{k-1}]=0$ for all $k\geq 1$, i.e., conditions of Lemma 13 are satisfied. Therefore, applying Lemma 13, we get

$$
\begin{aligned}
\frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle\right] &\leq \frac{\gamma}{2K}\sum_{k=0}^{K-1}\mathbb{E}[\|F(x^k)-g^k\|^2] \\
&\quad +\frac{1}{2\gamma K}\max_{u\in\mathcal{C}}\|x^0-u\|^2. \qquad (39)
\end{aligned}
$$

Combining (38) and (39), we derive

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] &\leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K}+\frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K}+9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\quad +\frac{\gamma}{2K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k-F(x^k)\|^2\right] \\
&\quad +\frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k-g^{*,k}\|^2\right] \qquad (40)\\
&\overset{(23)}{\leq} \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K}+\frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K}+9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\quad +\frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|F(x^k)-g^{*,k}\|^2\right]+\frac{2\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k-g^{*,k}\|^2\right].
\end{aligned}$$

Using ℓ-star-cocoercivity of F together with the first part of Assumption 1, we continue our derivation as follows:

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] &\leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K}+\frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K}+2\gamma D_1+9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\quad +\frac{\gamma(4A+\ell)}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\langle F(x^k)-g^{*,k},x^k-x^{*,k}\rangle\right]+\frac{2\gamma B}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2\right] \\
&= \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K}+\frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K}+2\gamma D_1+9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\
&\quad +\frac{\gamma(4A+\ell)}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\langle F(x^k)-g^{*,k},x^k-x^{*,k}\rangle\right] \\
&\quad +\frac{2\gamma B}{K}\left(1+\frac{1}{\rho}\right)\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2\right]-\frac{2\gamma B}{\rho K}\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2\right].
\end{aligned}$$

Next, we use the second part of Assumption 1 and get

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + 2\gamma D_1 + 9\gamma \max_{x^*\in X^*}\|F(x^*)\|^2 \\
& + \frac{\gamma(4A+\ell)}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\langle F(x^k)-g^{*,k}, x^k - x^{*,k}\rangle\right] \\
& + \frac{2\gamma B}{K}\left(1+\frac{1}{\rho}\right)\sum_{k=1}^{K-1}\mathbb{E}\left[2C\langle F(x^{k-1})-g^{*,k-1}, x^{k-1}-x^{*,k-1}\rangle\right] \\
& + \frac{2\gamma B}{K}\left(1+\frac{1}{\rho}\right)\sum_{k=1}^{K-1}\mathbb{E}\left[(1-\rho)\sigma_{k-1}^2 + D_2\right] \\
& + \frac{2\gamma B}{K}\left(1+\frac{1}{\rho}\right)\sigma_0^2 - \frac{2\gamma B}{\rho K}\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2\right] \\
\leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{2\gamma B(1+{}^1\!/\!_\rho)}{K}\sigma_0^2 \\
& + 2\gamma\left(D_1 + B(1+{}^1\!/\!_\rho)D_2\right) \\
& + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 + \frac{\gamma(4A+\ell)}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\langle F(x^k)-g^{*,k}, x^k-x^{*,k}\rangle\right] \\
& + \frac{2\gamma B}{K}\left(1+\frac{1}{\rho}\right)\sum_{k=0}^{K-2}\mathbb{E}\left[2C\langle F(x^k)-g^{*,k}, x^k-x^{*,k}\rangle + (1-\rho)\sigma_k^2\right] \\
& - \frac{2\gamma B}{\rho K}\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2\right] \\
\leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{2\gamma B(1+{}^1\!/\!_\rho)}{K}\sigma_0^2 \\
& + 2\gamma\left(D_1 + B(1+{}^1\!/\!_\rho)D_2\right) + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\
& + \left(4A+\ell+4BC(1+{}^1\!/\!_\rho)\right)\frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\langle F(x^k)-g^{*,k}, x^k-x^{*,k}\rangle\right] \\
& + \frac{2\gamma B}{K}(1-\rho)\left(1+\frac{1}{\rho}\right)\sum_{k=0}^{K-2}\mathbb{E}\left[\sigma_k^2\right] - \frac{2\gamma B}{\rho K}\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2\right].
\end{aligned}
$$

Since $(1-\rho)(1+{}^1/_\rho) = -\rho + {}^1/_\rho \leq {}^1/_\rho$, the last row is non-positive and we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{2\gamma B(1+{}^1/_\rho)}{K}\sigma_0^2 \\ & +2\gamma\left(D_1 + B(1+{}^1/_\rho)D_2\right) + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\ & +\frac{\gamma\left(4A+\ell+4BC(1+{}^1/_\rho)\right)}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\langle F(x^k)-g^{*,k}, x^k-x^{*,k}\rangle\right].\end{aligned} \tag{41}$$

Note that inequality (29) from the proof of Theorem 1 is derived using Assumption 1 only. With $M = {}^B/_\rho$ it gives

$$\begin{aligned}\mathbb{E}\left[\|x^{k+1}-x^{*,k+1}\|^2\right] + \frac{\gamma^2 B}{\rho}\mathbb{E}[\sigma_{k+1}^2] \quad \leq \quad & \mathbb{E}\left[\|x^k-x^{*,k}\|^2\right] + \frac{\gamma^2 B}{\rho}\mathbb{E}\left[\sigma_k^2\right] + \gamma^2(D_1 + {}^{BD_2}/_\rho) \\ & -2\gamma\left(1-\gamma(A+{}^{BC}/_\rho)\right)\mathbb{E}\left[\left\langle x^k-x^{*,k}, F(x^k)-g^{*,k}\right\rangle\right].\end{aligned}$$

Since $\gamma \leq {}^1/_{2(A+BC/\rho)}$ we obtain

$$\begin{aligned}\gamma\mathbb{E}\left[\left\langle x^k-x^{*,k}, F(x^k)-g^{*,k}\right\rangle\right] \quad \leq \quad & \mathbb{E}\left[\|x^k-x^{*,k}\|^2\right] + \frac{\gamma^2 B}{\rho}\mathbb{E}\left[\sigma_k^2\right] - \mathbb{E}\left[\|x^{k+1}-x^{*,k+1}\|^2\right] \\ & -\frac{\gamma^2 B}{\rho}\mathbb{E}[\sigma_{k+1}^2] + \gamma^2(D_1 + {}^{BD_2}/_\rho).\end{aligned}$$

Plugging this inequality in (41), we derive

$$\begin{aligned}\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{2\gamma B(1+{}^1/_\rho)}{K}\sigma_0^2 \\ & +2\gamma\left(D_1 + B(1+{}^1/_\rho)D_2\right) + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2 \\ & +\left(4A+\ell+4BC(1+{}^1/_\rho)\right)\cdot\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|x^k-x^{*,k}\|^2\right] \\ & -\left(4A+\ell+4BC(1+{}^1/_\rho)\right)\cdot\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|x^{k+1}-x^{*,k+1}\|^2\right] \\ & +\left(4A+\ell+4BC(1+{}^1/_\rho)\right)\cdot\frac{\gamma^2 B}{\rho K}\sum_{k=0}^{K-1}\mathbb{E}\left[\sigma_k^2-\sigma_{k+1}^2\right] \\ & +\gamma^2\left(4A+\ell+4BC(1+{}^1/_\rho)\right)\cdot(D_1 + {}^{BD_2}/_\rho) \\ \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \left(4A+\ell+{}^{8BC}/_\rho\right)\cdot\frac{\|x^0-x^{*,0}\|^2}{K} \\ & +\left(4+\left(4A+\ell+{}^{8BC}/_\rho\right)\gamma\right)\frac{\gamma B\sigma_0^2}{\rho K} \\ & +\gamma\left(\left(2+\gamma\left(4A+\ell+{}^{8BC}/_\rho\right)\right)(D_1 + {}^{2BD_2}/_\rho) + 9\max_{x^*\in X^*}\|F(x^*)\|^2\right),\end{aligned} \tag{42}$$

where in the last inequality we use $1 + {}^1/_\rho \leq {}^2/_\rho$. ■

Corollary 17 *Let the assumptions of Theorem 3 hold. Then, for all K one can choose γ as*

$$\gamma = \min\left\{\frac{1}{4A+\ell+{}^{8BC}/_\rho}, \frac{\Omega_{0,\mathcal{C}}\sqrt{\rho}}{\widehat{\sigma}_0\sqrt{B}}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{K(D_1+{}^{2BD_2}/_\rho)}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\}, \tag{43}$$

where $\Omega_0 := \|x^0 - x^{*,0}\|^2$ and $\Omega_{0,\mathcal{C}}$, $\widehat{\sigma}_0$, and G_* are some upper bounds for $\max_{u\in\mathcal{C}}\|x^0-u\|$, σ_0, and $\max_{x^*\in X^*}\|F(x^*)\|$ respectively. This choice of γ implies $\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^K x^k\right)\right]$ equals

$$\mathcal{O}\left(\frac{(A+\ell+{}^{BC}/_\rho)(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)+\ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}\widehat{\sigma}_0\sqrt{B}}{\sqrt{\rho}K} + \frac{\Omega_{0,\mathcal{C}}(\sqrt{D_1+{}^{BD_2}/_\rho}+G_*)}{\sqrt{K}}\right).$$

Proof First of all, the choice of γ from (43) implies (35) since

$$\frac{1}{4A+\ell+{}^{8BC}/_\rho} \leq \frac{1}{2\left(A+{}^{BC}/_\rho\right)}.$$

Using (10), the definitions of $\Omega_{0,\mathcal{C}}$, $\widehat{\sigma}_0$, G_*, and $\gamma \leq {}^1/_{(4A+\ell+{}^{8BC}/_\rho)}$, we get

$$\begin{aligned}
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^K x^k\right)\right] &\leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + (4A+\ell+{}^{8BC}/_\rho)\cdot\frac{\|x^0-x^{*,0}\|^2}{K} \\
&\quad + \left(4+(4A+\ell+{}^{8BC}/_\rho)\gamma\right)\frac{\gamma B\sigma_0^2}{\rho K} \\
&\quad + \gamma\left((2+\gamma\left(4A+\ell+{}^{8BC}/_\rho\right))(D_1+{}^{2BD_2}/_\rho) + 9\max_{x^*\in X^*}\|F(x^*)\|^2\right) \\
&\leq \frac{3\Omega_{0,\mathcal{C}}^2}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{(4A+\ell+{}^{8BC}/_\rho)\Omega_0^2}{K} \\
&\quad + \left(4+(4A+\ell+{}^{8BC}/_\rho)\gamma\right)\frac{\gamma B\widehat{\sigma}_0^2}{\rho K} \\
&\quad + \gamma\left((2+\gamma\left(4A+\ell+{}^{8BC}/_\rho\right))(D_1+{}^{2BD_2}/_\rho) + 9G_*^2\right) \\
&\leq \frac{3\Omega_{0,\mathcal{C}}^2}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{(4A+\ell+{}^{8BC}/_\rho)\Omega_0^2}{K} + \frac{5\gamma B\widehat{\sigma}_0^2}{\rho K} \\
&\quad + 3\gamma\left(D_1 + \frac{2BD_2}{\rho} + 3G_*^2\right).
\end{aligned}$$

Finally, we apply (43):

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] &\leq \frac{3\Omega_{0,\mathcal{C}}^2}{2\min\left\{\frac{1}{4A+\ell+{}^{8BC}/_{\rho}}, \frac{\Omega_{0,\mathcal{C}}\sqrt{\rho}}{\widehat{\sigma}_0\sqrt{B}}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{K(D_1+{}^{2BD_2}/_{\rho})}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\}K} + \frac{1}{\ell}\cdot\frac{8\ell^2\Omega_{\mathcal{C}}^2}{K} \\
&\quad + \frac{(4A+\ell+{}^{8BC}/_{\rho})\Omega_0^2}{K} + \frac{\Omega_{0,\mathcal{C}}\sqrt{\rho}}{\widehat{\sigma}_0\sqrt{B}}\cdot\frac{\gamma B\widehat{\sigma}_0^2}{\rho K} \\
&\quad + \frac{\Omega_{0,\mathcal{C}}}{\sqrt{K(D_1+{}^{2BD_2}/_{\rho})}}\cdot 3\left(D_1+\frac{2BD_2}{\rho}\right) + \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\cdot 9G_*^2 \\
&= \mathcal{O}\Bigg(\frac{(A+\ell+{}^{BC}/_{\rho})(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)+\ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}\widehat{\sigma}_0\sqrt{B}}{\sqrt{\rho}K} \\
&\qquad\qquad + \frac{\Omega_{0,\mathcal{C}}(\sqrt{D_1+{}^{BD_2}/_{\rho}}+G_*)}{\sqrt{K}}\Bigg).
\end{aligned}
$$

■

D.3. Cocoercive Case

The upper bound from Theorem 3 contains the term proportional to $\max_{x^*\in X^*}\|F(x^*)\|^2$, which is non-zero in general. Therefore, even when there is no noise the method with constant stepsize converges only to some error proportional to $\max_{x^*\in X^*}\|F(x^*)\|^2$. To resolve this issue we assume ℓ-cocoercivity of F, i.e., we assume that

$$\|F(x)-F(y)\|^2 \leq \ell\langle F(x)-F(y), x-y\rangle \quad \forall x,y\in\mathbb{R}^d.$$

Theorem 18 *Let F be ℓ-cocoercive and Assumptions 1, 2 hold. Assume that*

$$0<\gamma\leq \min\left\{\frac{1}{\ell}, \frac{1}{2(A+{}^{BC}/_{\rho})}\right\}. \tag{44}$$

Then for the function $Gap_{\mathcal{C}}(z)$ from (9) and for all $K\geq 0$ we have

$$\begin{aligned}\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] \leq\ & \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + (6A+3\ell+{}^{12BC}/_{\rho})\cdot\frac{\|x^0-x^{*,0}\|^2}{K}\\ &+\left(6+(6A+3\ell+{}^{12BC}/_{\rho})\,\gamma\right)\frac{\gamma B\sigma_0^2}{\rho K}\\ &+\gamma(3+\gamma\,(6A+3\ell+{}^{12BC}/_{\rho}))(D_1+{}^{2BD_2}/_{\rho}).\end{aligned} \tag{45}$$

Proof We start the proof from (37).

$$\begin{aligned}2\gamma\left(\langle F(x^k), x^k-u\rangle + R(x^{k+1})-R(u)\right) \leq\ & \|x^{k+1}-x^k\|^2+\|x^k-u\|^2-\|x^{k+1}-u\|^2\\ &+2\gamma\langle F(x^k)-g^k, x^k-u\rangle\\ &-2\|x^{k+1}-x^k\|^2+2\gamma\langle g^k, x^k-x^{k+1}\rangle\\ =\ & \|x^k-u\|^2-\|x^{k+1}-u\|^2\\ &+2\gamma\langle F(x^k)-g^k, x^k-u\rangle\\ &-\|x^{k+1}-x^k\|^2+2\gamma\langle F(u), x^k-x^{k+1}\rangle\\ &+2\gamma\langle g^k-F(u), x^k-x^{k+1}\rangle.\end{aligned}$$

Then, due to $2\langle a,b\rangle\leq\|a\|^2+\|b\|^2$ we have

$$\begin{aligned}2\gamma\left(\langle F(x^k), x^k-u\rangle + R(x^{k+1})-R(u)\right) \leq\ & \|x^k-u\|^2-\|x^{k+1}-u\|^2\\ &+2\gamma\langle F(x^k)-g^k, x^k-u\rangle\\ &-\|x^{k+1}-x^k\|^2+2\gamma\langle F(u), x^k-x^{k+1}\rangle\\ &+\gamma^2\|g^k-F(u)\|^2+\|x^k-x^{k+1}\|^2\\ =\ & \|x^k-u\|^2-\|x^{k+1}-u\|^2\\ &+2\gamma\langle F(x^k)-g^k, x^k-u\rangle\\ &+2\gamma\langle F(u), x^k-x^{k+1}\rangle+\gamma^2\|g^k-F(u)\|^2.\end{aligned}$$

Next, we add $2\gamma\left(\langle F(u), x^{k+1}-u\rangle - \langle F(x^k), x^k-u\rangle\right)$ to both sides of the previous inequality.

$$\begin{aligned}
2\gamma\left(\langle F(u), x^{k+1}-u\rangle + R(x^{k+1}) - R(u)\right) &\leq \|x^k-u\|^2 - \|x^{k+1}-u\|^2 \\
&\quad +2\gamma\langle F(u)-g^k, x^k-u\rangle + \gamma^2\|g^k-F(u)\|^2 \\
&\leq \|x^k-u\|^2 - \|x^{k+1}-u\|^2 \\
&\quad -2\gamma\langle F(x^k)-F(u), x^k-u\rangle \\
&\quad -2\gamma\langle g^k-F(x^k), x^k-u\rangle \\
&\quad +2\gamma^2\|g^k-F(x^k)\|^2 + 2\gamma^2\|F(x^k)-F(u)\|^2.
\end{aligned}$$

Using that F is ℓ-co-cocoercive, we get

$$\begin{aligned}
2\gamma\left(\langle F(u), x^{k+1}-u\rangle + R(x^{k+1}) - R(u)\right) &\leq \|x^k-u\|^2 - \|x^{k+1}-u\|^2 \\
&\quad -\frac{2\gamma}{\ell}\|F(x^k)-F(u)\|^2 \\
&\quad -2\gamma\langle g^k-F(x^k), x^k-u\rangle \\
&\quad +2\gamma^2\|g^k-F(x^k)\|^2 + 2\gamma^2\|F(x^k)-F(u)\|^2 \\
&= \|x^k-u\|^2 - \|x^{k+1}-u\|^2 \\
&\quad -\frac{2\gamma}{\ell}\left(1-\gamma\ell\right)\|F(x^k)-F(u)\|^2 \\
&\quad -2\gamma\langle g^k-F(x^k), x^k-u\rangle \\
&\quad +2\gamma^2\|g^k-F(x^k)\|^2.
\end{aligned}$$

With $\gamma \leq \frac{1}{\ell}$, we have

$$\begin{aligned}
2\gamma\left(\langle F(u), x^{k+1}-u\rangle + R(x^{k+1}) - R(u)\right) &\leq \|x^k-u\|^2 - \|x^{k+1}-u\|^2 \\
&\quad -2\gamma\langle g^k-F(x^k), x^k-u\rangle \\
&\quad +2\gamma^2\|g^k-F(x^k)\|^2.
\end{aligned}$$

Summing up the above inequality for $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned}
2\gamma\sum_{k=0}^{K-1}\left(\langle F(u), x^{k+1}-u\rangle + R(x^{k+1}) - R(u)\right) &\leq \sum_{k=0}^{K-1}\|x^k-u\|^2 - \sum_{k=0}^{K-1}\|x^{k+1}-u\|^2 \\
&\quad +2\gamma^2\sum_{k=0}^{K-1}\|g^k-F(x^k)\|^2 \\
&\quad +2\gamma\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle.
\end{aligned}$$

Next, we divide both sides by $2\gamma K$

$$\begin{aligned}\frac{1}{K}\sum_{k=0}^{K-1}\Big(\langle F(u), x^{k+1}-u\rangle + R(x^{k+1}) - R(u)\Big) \quad &\leq \quad \frac{\|x^0-u\|^2 - \|x^K-u\|^2}{2\gamma K} \\ &\quad + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^k - F(x^k)\|^2 \\ &\quad + \frac{1}{K}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle.\end{aligned}$$

Applying Jensen's inequality for convex function R, we get $R\left(\frac{1}{K}\sum_{k=0}^{K-1}x^{k+1}\right) \leq \frac{1}{K}\sum_{k=0}^{K-1}R(x^{k+1})$.

$$\begin{aligned}\left\langle F(u), \left(\frac{1}{K}\sum_{k=0}^{K-1}x^{k+1}\right) - u\right\rangle + R\left(\frac{1}{K}\sum_{k=0}^{K-1}x^{k+1}\right) - R(u) \\ \leq \frac{\|x^0-u\|^2 - \|x^K-u\|^2}{2\gamma K} \\ + \frac{\gamma}{K}\sum_{k=0}^{K-1}\|g^k - F(x^k)\|^2 \\ + \frac{1}{K}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle.\end{aligned}$$

Next, we take maximum from the both sides in $u \in \mathcal{C}$, which gives $\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)$ in the left-hand side by definition (9), and take the expectation of the result:

$$\begin{aligned}\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] \quad &\leq \quad \frac{\mathbb{E}\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{\gamma K} + \frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k - F(x^k)\|^2\right] \\ &\quad + \frac{1}{K}\mathbb{E}\left[\max_{u\in\mathcal{C}}\sum_{k=0}^{K-1}\langle F(x^k)-g^k, x^k-u\rangle\right].\end{aligned}$$

Using the estimate (39), we get

$$\begin{aligned}\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] \quad &\leq \quad \frac{\max_{u\in\mathcal{C}}\|x^0-u\|^2}{\gamma K} + \frac{\gamma}{K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k - F(x^k)\|^2\right] \\ &\quad + \frac{\gamma}{2K}\sum_{k=0}^{K-1}\mathbb{E}[\|F(x^k)-g^k\|^2] + \frac{1}{2\gamma K}\max_{u\in\mathcal{C}}\|x^0-u\|^2 \\ &\leq \quad \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + \frac{3\gamma}{2K}\sum_{k=0}^{K-1}\mathbb{E}\left[\|g^k - F(x^k)\|^2\right].\end{aligned}$$

It remains to estimate $\frac{1}{K}\sum\limits_{k=0}^{K-1}\mathbb{E}\left[\|g^k - F(x^k)\|^2\right]$. This was done in the previous proof (see from (40) to (42)). Then, we finally have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] &\leq \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + (6A+3\ell+{}^{12BC}/_{\rho})\cdot\frac{\|x^0-x^{*,0}\|^2}{K}\\ &\quad + (6+(6A+3\ell+{}^{12BC}/_{\rho})\gamma)\frac{\gamma B\sigma_0^2}{\rho K}\\ &\quad +\gamma(3+\gamma(6A+3\ell+{}^{12BC}/_{\rho}))(D_1+{}^{2BD_2}/_{\rho}).\end{aligned}$$

■

Corollary 19 *Let the assumptions of Theorem 18 hold. Then, for all K one can choose γ as*

$$\gamma = \min\left\{\frac{1}{6A+3\ell+{}^{12BC}/_{\rho}}, \frac{\Omega_{0,\mathcal{C}}\sqrt{\rho}}{\widehat{\sigma}_0\sqrt{B}}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{K(D_1+{}^{2BD_2}/_{\rho})}}\right\}, \tag{46}$$

where $\Omega_0 := \|x^0 - x^{*,0}\|^2$ and $\Omega_{0,\mathcal{C}}$, and $\widehat{\sigma}_0$ are some upper bounds for $\max_{u\in\mathcal{C}}\|x^0-u\|$, and σ_0 respectively. This choice of γ implies $\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right]$ equals

$$\mathcal{O}\left(\frac{(A+\ell+{}^{BC}/_{\rho})(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)}{K} + \frac{\Omega_{0,\mathcal{C}}\widehat{\sigma}_0\sqrt{B}}{\sqrt{\rho}K} + \frac{\Omega_{0,\mathcal{C}}\sqrt{D_1+{}^{BD_2}/_{\rho}}}{\sqrt{K}}\right).$$

Proof First of all, the choice of γ from (46) implies (35) since

$$\frac{1}{6A+3\ell+{}^{12BC}/_{\rho}} \leq \frac{1}{2\left(A+{}^{BC}/_{\rho}\right)}.$$

Using (45), the definitions of $\Omega_{0,\mathcal{C}}$, $\widehat{\sigma}_0$, and $\gamma \leq {}^{1}/_{(6A+3\ell+{}^{12BC}/_{\rho})}$, we get

$$\begin{aligned}\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] &\leq \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + (6A+3\ell+{}^{12BC}/_{\rho})\cdot\frac{\|x^0-x^{*,0}\|^2}{K}\\ &\quad + (6+(6A+3\ell+{}^{12BC}/_{\rho})\gamma)\frac{\gamma B\sigma_0^2}{\rho K}\\ &\quad +\gamma(3+\gamma(6A+3\ell+{}^{12BC}/_{\rho}))(D_1+{}^{2BD_2}/_{\rho})\\ &\leq \frac{3\Omega_{0,\mathcal{C}}^2}{2\gamma K} + \frac{(6A+3\ell+{}^{12BC}/_{\rho})\Omega_0^2}{K}\\ &\quad + (6+(6A+3\ell+{}^{12BC}/_{\rho})\gamma)\frac{\gamma B\widehat{\sigma}_0^2}{\rho K}\\ &\quad +\gamma(3+\gamma(6A+3\ell+{}^{12BC}/_{\rho}))(D_1+{}^{2BD_2}/_{\rho})\\ &\leq \frac{3\Omega_{0,\mathcal{C}}^2}{2\gamma K} + \frac{(6A+3\ell+{}^{12BC}/_{\rho})\Omega_0^2}{K} + \frac{7\gamma B\widehat{\sigma}_0^2}{\rho K} + 4\gamma\left(D_1+\frac{2BD_2}{\rho}\right).\end{aligned}$$

Finally, we apply (43):

$$
\begin{aligned}
\mathbb{E}\left[\mathrm{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] &\leq \frac{3\Omega_{0,\mathcal{C}}^2}{2\min\left\{\frac{1}{6A+3\ell+{}^{12BC}/_{\rho}}, \frac{\Omega_{0,\mathcal{C}}\sqrt{\rho}}{\widehat{\sigma}_0\sqrt{B}}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{K(D_1+{}^{2BD_2}/_{\rho})}}\right\}K} \\
&\quad + \frac{(6A+3\ell+{}^{12BC}/_{\rho})\Omega_0^2}{K} + \frac{\Omega_{0,\mathcal{C}}\sqrt{\rho}}{\widehat{\sigma}_0\sqrt{B}}\cdot\frac{\gamma B\widehat{\sigma}_0^2}{\rho K} \\
&\quad + \frac{\Omega_{0,\mathcal{C}}}{\sqrt{K(D_1+{}^{2BD_2}/_{\rho})}}\cdot 4\left(D_1+\frac{2BD_2}{\rho}\right) \\
&= \mathcal{O}\left(\frac{(A+\ell+{}^{BC}/_{\rho})(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)}{K} + \frac{\Omega_{0,\mathcal{C}}\widehat{\sigma}_0\sqrt{B}}{\sqrt{\rho}K}\right. \\
&\qquad \left. + \frac{\Omega_{0,\mathcal{C}}\sqrt{D_1+{}^{BD_2}/_{\rho}}}{\sqrt{K}}\right).
\end{aligned}
$$

■

Appendix E. SGDA with Arbitrary Sampling: Missing Proofs and Details

Algorithm 1 SGDA-AS: Stochastic Gradient Descent-Ascent with Arvitrary Sampling

Input: starting point $x^0 \in \mathbb{R}^d$, distribution $\mathcal{D}$, stepsize $\gamma > 0$, number of steps K
for $k = 0$ **to** $K - 1$ **do**
    Sample $\xi^k \sim \mathcal{D}$ independently from previous iterations and compute $g^k = F_{\xi^k}(x^k)$
    $x^{k+1} = \text{prox}_{\gamma R}(x^k - \gamma g^k)$

end for

E.1. Proof of Proposition 4

Proposition 20 (Proposition 4) *Let Assumption 3 hold. Then,* SGDA *satisfies Assumption 1 with*

$$A = \ell_{\mathcal{D}}, \quad B = 0, \quad \sigma_k^2 \equiv 0, \quad D_1 = 2\sigma_*^2 := 2 \max_{x^* \in X^*} \mathbb{E}_{\mathcal{D}} \left[\|F_\xi(x^*) - F(x^*)\|^2 \right],$$
$$C = 0, \quad \rho = 1, \quad D_2 = 0.$$

Proof To prove the result, it is sufficient to derive an upper bound for $\mathbb{E}_k \left[\|g^k - F(x^{*,k})\|^2 \right]$:

$$\begin{aligned}
\mathbb{E}_k \left[\|g^k - F(x^{*,k})\|^2 \right] &= \mathbb{E}_{\mathcal{D}} \left[\|F_{\xi^k}(x^k) - F(x^{*,k})\|^2 \right] \\
&\leq 2\mathbb{E}_{\mathcal{D}} \left[\|F_{\xi^k}(x^k) - F_{\xi^k}(x^{*,k})\|^2 \right] + 2\mathbb{E}_{\mathcal{D}} \left[\|F_{\xi^k}(x^{*,k}) - F(x^{*,k})\|^2 \right] \\
&\overset{Ass.(3)}{\leq} 2\ell_{\mathcal{D}} \langle F(x^k) - F(x^{*,k}), x^k - x^{*,k} \rangle + 2\sigma_*^2,
\end{aligned}$$

where $\sigma_*^2 := \max_{x^* \in X^*} \mathbb{E}_{\mathcal{D}} \left[\|F_\xi(x^*) - F(x^*)\|^2 \right]$. The above inequality implies that Assumption 1 holds with

$$A = \ell_{\mathcal{D}}, \quad B = 0, \quad \sigma_k^2 \equiv 0, \quad D_1 = 2\sigma_*^2 := 2 \max_{x^* \in X^*} \mathbb{E}_{\mathcal{D}} \left[\|F_\xi(x^*) - F(x^*)\|^2 \right],$$
$$C = 0, \quad \rho = 1, \quad D_2 = 0.$$

■

E.2. Analysis of SGDA-AS in the Quasi-Strongly Monotone Case

Plugging the parameters from the above proposition in Theorem 1 and Corollary 2 we get the following results.

Theorem 21 *Let F be μ-quasi strongly monotone, Assumption 3 hold, and $0 < \gamma \leq 1/2\ell_{\mathcal{D}}$. Then, for all $k \geq 0$ the iterates produced by* SGDA-AS *satisfy*

$$\mathbb{E} \left[\|x^k - x^{*,k}\|^2 \right] \leq (1 - \gamma\mu)^k \|x^0 - x^{0,*}\|^2 + \frac{2\gamma\sigma_*^2}{\mu}. \tag{47}$$

Corollary 22 (Corollary 5) *Let the assumptions of Theorem 21 hold. Then, for any $K \geq 0$ one can choose $\{\gamma_k\}_{k\geq 0}$ as follows:*

$$\begin{aligned} \text{if } K \leq \frac{2\ell_{\mathcal{D}}}{\mu}, \quad & \gamma_k = \frac{1}{2\ell_{\mathcal{D}}}, \\ \text{if } K > \frac{2\ell_{\mathcal{D}}}{\mu} \text{ and } k < k_0, \quad & \gamma_k = \frac{1}{2\ell_{\mathcal{D}}}, \\ \text{if } K > \frac{2\ell_{\mathcal{D}}}{\mu} \text{ and } k \geq k_0, \quad & \gamma_k = \frac{2}{4\ell_{\mathcal{D}} + \mu(k-k_0)}, \end{aligned} \tag{48}$$

where $k_0 = \lceil K/2 \rceil$. For this choice of γ_k the following inequality holds for SGDA-AS*:*

$$\mathbb{E}[\|x^K - x^{*,K}\|^2] \leq \frac{64\ell_{\mathcal{D}}}{\mu}\|x^0 - x^{*,0}\|^2 \exp\left(-\frac{\mu}{2\ell_{\mathcal{D}}}K\right) + \frac{72\sigma_*^2}{\mu^2 K}.$$

E.3. Analysis of SGDA-AS in the Monotone Case

In the monotone case, using Theorem 3, we establish the new result for SGDA-AS.

Theorem 23 *Let F be monotone ℓ-star-cocoercive and Assumptions 1, 2, 3 hold. Assume that $\gamma \leq 1/2\ell_{\mathcal{D}}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* SGDA-AS *satisfy*

$$\begin{aligned} \mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\max_{u\in\mathcal{C}}\|x^0 - u\|^2}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{(4\ell_{\mathcal{D}} + \ell)\|x^0 - x^{*,0}\|^2}{K} \\ & + 2\gamma(2 + \gamma(4\ell_{\mathcal{D}} + \ell))\sigma_*^2 + 9\gamma \max_{x^*\in X^*}\|F(x^*)\|^2. \end{aligned}$$

Next, we apply Corollary 17 and get the following rate of convergence to the exact solution.

Corollary 24 *Let the assumptions of Theorem 23 hold. Then $\forall K > 0$ and*

$$\gamma = \min\left\{\frac{1}{4\ell_{\mathcal{D}} + \ell}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{2K}\sigma_*}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\} \tag{49}$$

the iterates produced by SGDA-AS *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{(\ell_{\mathcal{D}} + \ell)(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2) + \ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}(\sigma_* + G_*)}{\sqrt{K}}\right).$$

As we already mentioned before, the above result is new for SGDA-AS: the only known work on SGDA-AS [52] focuses on the μ-quasi-strongly monotone case only with $\mu > 0$. Moreover, neglecting the dependence on problem/noise parameters, the derived convergence rate $\mathcal{O}\left(1/K + 1/\sqrt{K}\right)$ is standard for the analysis of stochastic methods for solving monotone VIPs [39].

E.4. Analysis of SGDA-AS in the Cocoercive Case

In the cocoercive case, using Theorem 18, we establish the new result for SGDA-AS.

Theorem 25 *Let F be ℓ-cocoercive and Assumptions 1, 2, 3 hold. Assume that $\gamma \leq 1/2\ell_{\mathcal{D}}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* SGDA-AS *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + \frac{(6\ell_{\mathcal{D}}+3\ell)\|x^0-x^{*,0}\|^2}{K} + 2\gamma(3+\gamma(6\ell_{\mathcal{D}}+3\ell))\sigma_*^2.$$

Next, we apply Corollary 19 and get the following rate of convergence to the exact solution.

Corollary 26 *Let the assumptions of Theorem 25 hold. Then $\forall K > 0$ and*

$$\gamma = \min\left\{\frac{1}{6\ell_{\mathcal{D}}+3\ell}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{2K}\sigma_*}\right\} \tag{50}$$

the iterates produced by SGDA-AS *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{(\ell_{\mathcal{D}}+\ell)(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)}{K} + \frac{\Omega_{0,\mathcal{C}}\sigma_*}{\sqrt{K}}\right).$$

E.5. Missing Details on Arbitrary Sampling

In the main part of the paper, we discuss the Arbitrary Sampling paradigm and, in particular, using our general theoretical framework, we obtain convergence guarantees for SGDA under Expected Cocoercivity assumption (Assumption 3). In this section, we give the particular examples of arbitrary sampling fitting this setup. In all the examples below, we focus on a special case of stochastic reformulation from (11) and assume that for all $i \in [n]$ operator F_i is (ℓ_i, X^*)-cocoercive, i.e., for all $i \in [n]$ and $x \in \mathbb{R}^d$ we have

$$\|F_i(x) - F_i(x^*)\|^2 \leq \ell_i\langle F_i(x) - F_i(x^*), x - x^*\rangle, \tag{51}$$

where x^* is the projection of x on X^*. Note that (51) holds whenever F_i are cocoercive.

Uniform Sampling. We start with the classical uniform sampling: let $\mathbb{P}\{\xi = ne_i\} = 1/n$ for all $i \in [n]$, where $e_i \in \mathbb{R}^n$ is the i-th coordinate vector from the standard basis in $\mathbb{R}^n$. Then, $\mathbb{E}[\xi_i] = 1$ for all $i \in [n]$ and Assumption 3 holds with $\ell_{\mathcal{D}} = \max_{i\in[n]} \ell_i$:

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}\left[\|F_\xi(x) - F_\xi(x^*)\|^2\right] &= \frac{1}{n}\sum_{i=1}^{n}\|F_i(x) - F_i(x^*)\|^2 \\
&\overset{(51)}{\leq} \frac{1}{n}\sum\left(\ell_i\langle F_i(x) - F_i(x^*), x - x^*\rangle\right) \\
&\leq \max_{i\in[n]} \ell_i\langle F(x) - F(x^*), x - x^*\rangle
\end{aligned}$$

In this case, Corollaries 22 and 24 imply the following rate for SGDA in μ-quasi strongly monotone, monotone and cocoercive cases respectively:

$$
\begin{aligned}
\mathbb{E}[\|x^K - x^{*,K}\|^2] &\leq \frac{64\max_{i\in[n]}\ell_i}{\mu}\|x^0 - x^{*,0}\|^2 \exp\left(-\frac{\mu}{2\max_{i\in[n]}\ell_i}K\right) + \frac{72\sigma^2_{*,\text{US}}}{\mu^2 K},\\
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^K x^k\right)\right] &= \mathcal{O}\left(\frac{(\max_{i\in[n]}\ell_i + \ell)(\Omega^2_{0,\mathcal{C}} + \Omega^2_0) + \ell\Omega^2_{\mathcal{C}}}{K} + \frac{\Omega_{0,\mathcal{C}}(\sigma_{*,\text{US}} + G_*)}{\sqrt{K}}\right),\\
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^K x^k\right)\right] &= \mathcal{O}\left(\frac{(\max_{i\in[n]}\ell_i + \ell)(\Omega^2_{0,\mathcal{C}} + \Omega^2_0)}{K} + \frac{\Omega_{0,\mathcal{C}}\sigma_{*,\text{US}}}{\sqrt{K}}\right),
\end{aligned}
\tag{52}
$$

where $\sigma^2_{*,\text{US}} := \max_{x^*\in X^*}\frac{1}{n}\sum_{i=1}^n \|F_i(x^*) - F(x^*)\|^2$.

Importance Sampling. Next, we consider a non-uniform sampling strategy – importance sampling: let $\mathbb{P}\left\{\xi = e_i {}^{n\overline{\ell}}/_{\ell_i}\right\} = {}^{\ell_i}/_{n\overline{\ell}}$ for all $i \in [n]$, where $\overline{\ell} = \frac{1}{n}\sum_{i=1}^n \ell_i$. Then, $\mathbb{E}[\xi_i] = 1$ for all $i \in [n]$ and Assumption 3 holds with $\ell_{\mathcal{D}} = \overline{\ell}$:

$$
\begin{aligned}
\mathbb{E}_{\mathcal{D}}\left[\|F_\xi(x) - F_\xi(x^*)\|^2\right] &= \sum_{i=1}^n \frac{\ell_i}{n\overline{\ell}}\left\|\frac{\overline{\ell}}{\ell_i}\left(F_i(x) - F_i(x^*)\right)\right\|^2\\
&= \sum_{i=1}^n \frac{\overline{\ell}}{n\ell_i}\|F_i(x) - F_i(x^*)\|^2\\
&\overset{(51)}{\leq} \frac{\overline{\ell}}{n}\sum\langle F_i(x) - F_i(x^*), x - x^*\rangle\\
&\leq \overline{\ell}\langle F(x) - F(x^*), x - x^*\rangle
\end{aligned}
$$

In this case, Corollaries 22 and 24 imply the following rate for SGDA in μ-quasi strongly monotone, monotone and cocoercive cases respectively:

$$
\begin{aligned}
\mathbb{E}[\|x^K - x^{*,K}\|^2] &\leq \frac{64\overline{\ell}}{\mu}\|x^0 - x^{*,0}\|^2 \exp\left(-\frac{\mu}{2\overline{\ell}}K\right) + \frac{72\sigma^2_{*,\text{IS}}}{\mu^2 K},\\
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^K x^k\right)\right] &= \mathcal{O}\left(\frac{(\overline{\ell} + \ell)(\Omega^2_{0,\mathcal{C}} + \Omega^2_0) + \ell\Omega^2_{\mathcal{C}}}{K} + \frac{\Omega_{0,\mathcal{C}}(\sigma_{*,\text{IS}} + G_*)}{\sqrt{K}}\right),\\
\mathbb{E}\left[\text{Gap}_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^K x^k\right)\right] &= \mathcal{O}\left(\frac{(\overline{\ell} + \ell)(\Omega^2_{0,\mathcal{C}} + \Omega^2_0)}{K} + \frac{\Omega_{0,\mathcal{C}}\sigma_{*,\text{IS}}}{\sqrt{K}}\right)
\end{aligned}
\tag{53}
$$

where $\sigma^2_{*,\text{IS}} := \max_{x^*\in X^*}\frac{1}{n}\sum_{i=1}^n \frac{\ell_i}{\overline{\ell}}\left\|\frac{\overline{\ell}}{\ell_i}F_i(x^*) - F(x^*)\right\|^2$. We emphasize that $\overline{\ell} \leq \max_{i\in[n]}\ell_i$ and, in fact, $\overline{\ell}$ might be much smaller than $\max_{i\in[n]}\ell_i$. Therefore, compared to SGDA with uniform sampling, SGDA with importance sampling has better exponentially decaying term in the quasi-strongly monotone case and converges faster to the neighborhood, if executed with constant stepsize. Moreover, $\sigma^2_{*,\text{IS}} \leq \sigma^2_{*,\text{US}}$, when $\max_{x^*\in X^*}\|F_i(x^*)\| \sim \ell_i$. In this case, SGDA with importance sampling has better $\mathcal{O}({}^1/_K)$ term than SGDA with uniform sampling as well.

Minibatch Sampling With Replacement. Let $\xi = \frac{1}{b}\sum_{i=1}^{b}\xi^i$, where ξ^i are i.i.d. samples from some distribution $\mathcal{D}$ satisfying (11) and Assumption 3. Then, the distribution of ξ satisfies (11) and Assumption 3 as well with the same constant $\ell_{\mathcal{D}}$. Therefore, minibatched versions of uniform sampling and importance sampling fit the framework as well with $\ell_{\mathcal{D}} = \max_{i\in[n]}\ell_i, \sigma_*^2 = \frac{\sigma_{*,\text{US}}^2}{b}$ and $\ell_{\mathcal{D}} = \bar{\ell}, \sigma_*^2 = \frac{\sigma_{*,\text{IS}}^2}{b}$.

Minibatch Sampling Without Replacement. For given batchsize $b \in [n]$ we consider the following sampling strategy: for each subset $S \subseteq [n]$ such that $|S| = b$ we have $\mathbb{P}\left\{\xi = \frac{n}{b}\sum_{i\in S} e_i\right\} = \frac{b!(n-b)!}{n!}$, i.e., S is chosen uniformly at random from all b-element subsets of $[n]$. In the special case, when $R(x) \equiv 0$, Loizou et al. [52] show that this sampling strategy satisfies (11) and Assumption 3 with

$$\ell_{\mathcal{D}} = \frac{n(b-1)}{b(n-1)}\ell + \frac{n-b}{b(n-1)}\max_{i\in[n]}\ell_i, \quad \sigma_*^2 = \frac{n-b}{b(n-1)}\sigma_{*,\text{US}}^2. \tag{54}$$

Clearly, both parameters are smaller than corresponding parameters for minibatched version of uniform sampling with replacement, which indicates the theoretical benefits of sampling without replacement. Plugging the parameters from (54) in Corollaries 22 and 24, we get the rate of convergence for this sampling strategy. Moreover, in the quasi-strongly monotone case, to guarantee $\mathbb{E}[\|x^K - x^{*,K}\|^2] \leq \varepsilon$ for some $\varepsilon > 0$, the method requires

$$\begin{aligned} Kb &= \mathcal{O}\left(\max\left\{\left(b\frac{\ell}{\mu} + \frac{(n-b)}{n}\frac{\max_{i\in[n]}\ell_i}{\mu}\right)\log\frac{\ell_{\mathcal{D}}\|x^0 - x^{*,0}\|^2}{\mu\varepsilon}, \frac{(n-b)\sigma_{*,\text{US}}^2}{n\mu^2\varepsilon}\right\}\right) \\ &= \widetilde{\mathcal{O}}\left(\max\left\{\frac{b\left(\ell - \frac{1}{n}\max_{i\in[n]}\ell_i\right) + \max_{i\in[n]}\ell_i}{\mu}, \frac{(n-b)\sigma_{*,\text{US}}^2}{n\mu^2\varepsilon}\right\}\right) \quad \text{oracle calls,} \end{aligned} \tag{55}$$

where $\widetilde{\mathcal{O}}(\cdot)$ hides numerical and logarithmic factors. One can notice that the first term in the maximum linearly increases in b (since ℓ cannot be smaller than $\frac{1}{n}\max_{i\in[n]}\ell_i$), while the second term linearly decreases in b. The first term in the maximum is lower bounded by $\frac{(n-b)}{n}\frac{\max_{i\in[n]}\ell_i}{\mu}$. Therefore, if $\max_{i\in[n]}\ell_i \geq \frac{\sigma_{*,\text{US}}^2}{\mu\varepsilon}$, the the first term in the maximum is always larger than the second one, meaning that the optimal batchsize, i.e., the batchsize that minimizes oracle complexity (55) neglecting the logarithmic terms, equals $b_* = 1$. Next, if $\max_{i\in[n]}\ell_i < \frac{\sigma_{*,\text{US}}^2}{\mu\varepsilon}$, then there exists a positive value of b such that the first term in the maximum equals the second term. This value equals

$$\frac{n\left(\sigma_{*,\text{US}}^2 - \mu\varepsilon\max_{i\in[n]}\ell_i\right)}{\sigma_*^2 + \mu\varepsilon\left(n\ell - \max_{i\in[n]}\ell_i\right)}.$$

One can easily verify that it is always smaller than n, but it can be non integer and it can be smaller than 1 as well. Therefore, the optimal batchsize is

$$b_* = \begin{cases} 1, & \text{if } \max_{i\in[n]}\ell_i \geq \frac{\sigma_{*,\text{US}}^2}{\mu\varepsilon}, \\ \max\left\{1, \left\lfloor\frac{n\left(\sigma_{*,\text{US}}^2 - \mu\varepsilon\max_{i\in[n]}\ell_i\right)}{\sigma_*^2 + \mu\varepsilon\left(n\ell - \max_{i\in[n]}\ell_i\right)}\right\rfloor\right\}, & \text{otherwise.} \end{cases}$$

We notice that Loizou et al. [52] derive the following formula for the optimal batchsize (ignoring numerical constants):

$$\widetilde{b}_* = \begin{cases} 1, & \text{if } \max_{i\in[n]} \ell_i - \ell \geq \frac{\sigma^2_{*,\text{US}}}{\mu\varepsilon}, \\ \max\left\{1, \left\lfloor \frac{n\left(\sigma^2_{*,\text{US}} - \mu\varepsilon(\max_{i\in[n]} \ell_i - \ell)\right)}{\sigma^2_* + \mu\varepsilon\left(n\ell - \max_{i\in[n]} \ell_i\right)} \right\rfloor \right\}, & \text{otherwise.} \end{cases}$$

However, in terms of $\widetilde{\mathcal{O}}(\cdot)$ both formulas give the same complexity result.

Appendix F. SGDA with Variance Reduction: Missing Proofs and Details

In this section, we provide missing proofs and details for Section 4.

F.1. L-SVRGDA

Algorithm 2 L-SVRGDA: Loopless Stochastic Variance Reduced Gradient Descent-Ascent

Input: starting point $x^0 \in \mathbb{R}^d$, probability $p \in (0,1]$, stepsize $\gamma > 0$, number of steps K
Set $w^0 = x^0$ and compute $F(w^0)$
for $k = 0$ **to** $K-1$ **do**
  Draw a fresh sample j_k from the uniform distribution on $[n]$ and compute $g^k = F_{j_k}(x^k) - F_{j_k}(w^k) + F(w^k)$
  $w^{k+1} = \begin{cases} x^k, & \text{with probability } p, \\ w^k, & \text{with probability } 1-p, \end{cases}$
  $x^{k+1} = \text{prox}_{\gamma R}(x^k - \gamma g^k)$
end for

F.1.1. PROOF OF PROPOSITION 6

Lemma 27 *Let Assumption 4 hold. Then for all $k \geq 0$* L-SVRGDA *satisfies*

$$\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] \leq 2\widehat{\ell}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + 2\sigma_k^2, \tag{56}$$

where $\sigma_k^2 := \frac{1}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^{*,k})\|^2$.

Proof Since $g^k = F_{j_k}(x^k) - F_{j_k}(w^k) + F(w^k)$, we have

$$\begin{aligned}
\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &= \mathbb{E}_k\left[\|F_{j_k}(x^k) - F_{j_k}(w^k) + F(w^k) - F(x^{*,k})\|^2\right] \\
&= \frac{1}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(w^k) + F(w^k) - F(x^{*,k})\|^2 \\
&\leq \frac{2}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^{*,k})\|^2 \\
&\quad + \frac{2}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^{*,k}) - (F(w^k) - F(x^{*,k}))\|^2 \\
&\leq \frac{2}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^{*,k})\|^2 + \frac{2}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^{*,k})\|^2 \\
&\overset{(13)}{\leq} 2\widehat{\ell}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + 2\sigma_k^2.
\end{aligned}$$

■

Lemma 28 *Let Assumptions 4 and 5 hold. Then for all $k \geq 0$* L-SVRGDA *satisfies*

$$\mathbb{E}_k\left[\sigma_{k+1}^2\right] \leq p\widehat{\ell}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + (1-p)\sigma_k^2, \tag{57}$$

where $\sigma_k^2 := \frac{1}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^{*,k})\|^2$.

Proof Using the definitions of σ_{k+1}^2 and w^{k+1} (see (12)), we derive

$$\begin{aligned}
\mathbb{E}_k\left[\sigma_{k+1}^2\right] &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}_k\left[\|F_i(w^{k+1}) - F_i(x^{*,k+1})\|^2\right] \\
&\overset{\text{As. 5}}{=} \frac{1}{n}\sum_{i=1}^n \mathbb{E}_k\left[\|F_i(w^{k+1}) - F_i(x^{*,k})\|^2\right] \\
&= \frac{p}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^{*,k})\|^2 + \frac{1-p}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^{*,k})\|^2 \\
&\overset{(13)}{\leq} p\widehat{\ell}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + (1-p)\sigma_k^2.
\end{aligned}$$

■

The above two lemmas imply that Assumption 1 is satisfied with certain parameters.

Proposition 29 (Proposition 6) *Let Assumptions 4 and 5 hold. Then,* L-SVRGDA *satisfies Assumption 1 with*

$$A = \widehat{\ell}, \quad B = 2, \quad \sigma_k^2 = \frac{1}{n}\sum_{i=1}^n \|F_i(w^k) - F_i(x^*)\|^2, \quad C = \frac{p\widehat{\ell}}{2}, \quad \rho = p, \quad D_1 = D_2 = 0.$$

F.1.2. ANALYSIS OF L-SVRGDA IN THE QUASI-STRONGLY MONOTONE CASE

Plugging the parameters from the above proposition in Theorem 1 and Corollary 2 with $M = \frac{4}{p}$ we get the following results.

Theorem 30 *Let F be μ-quasi strongly monotone, Assumptions 4, 5 hold, and $0 < \gamma \leq 1/6\widehat{\ell}$. Then for all $k \geq 0$ the iterates produced by* L-SVRGDA *satisfy*

$$\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq \left(1 - \min\left\{\gamma\mu, p/2\right\}\right)^k V_0, \tag{58}$$

where $V_0 = \|x^0 - x^*\|^2 + 4\gamma^2\sigma_0^2/p$.

Corollary 31 *Let the assumptions of Theorem 30 hold. Then, for $p = n$, $\gamma = 1/6\widehat{\ell}$ and any $K \geq 0$ we have*

$$\mathbb{E}[\|x^k - x^*\|^2] \leq V_0 \exp\left(-\min\left\{\frac{\mu}{6\widehat{\ell}}, \frac{1}{2n}\right\} K\right).$$

F.1.3. ANALYSIS OF L-SVRGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of L-SVRGDA in the monotone case.

Theorem 32 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2, 4, 5 hold. Assume that $\gamma \leq 1/6\widehat{\ell}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates of* L-SVRGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{\left(12\widehat{\ell}+\ell\right)\|x^0-x^{*,0}\|^2}{K} + \left(4+\left(12\widehat{\ell}+\ell\right)\gamma\right)\frac{2\gamma\sigma_0^2}{pK} + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2.$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 33 *Let the assumptions of Theorem 32 hold and $p = 1/n$. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{12\widehat{\ell}+\ell}, \frac{1}{\sqrt{2n\widehat{\ell}\ell}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\}. \tag{59}$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{(\widehat{\ell}+\ell)(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)+\sqrt{n\widehat{\ell}\ell}\Omega_{0,\mathcal{C}}^2+\ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}G_*}{\sqrt{K}}\right).$$

Proof First of all, (13), (4), and Cauchy-Schwarz inequality imply

$$\begin{aligned}
\sigma_0^2 &= \frac{1}{n}\sum_{i=1}^{n}\|F_i(x^0)-F_i(x^*)\|^2 \\
&\overset{(13)}{\leq} \widehat{\ell}\langle F(x^0)-F(x^*), x^0-x^*\rangle \\
&\leq \widehat{\ell}\|F(x^0)-F(x^*)\|\cdot\|x^0-x^*\| \\
&\leq \widehat{\ell}\ell\|x^0-x^*\|^2 \leq \widehat{\ell}\ell\max_{u\in\mathcal{C}}\|x^0-u\|^2 \leq \widehat{\ell}\ell\Omega_{0,\mathcal{C}}^2.
\end{aligned}$$

Next, applying Corollary 17 with $\widehat{\sigma}_0 := \sqrt{\widehat{\ell}\ell}\Omega_{0,\mathcal{C}}$, we get the result. ■

F.1.4. ANALYSIS OF L-SVRGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of L-SVRGDA in the cocoercive case.

Theorem 34 *Let F be ℓ-cocoercive and Assumptions 1, 2, 4, 5 hold. Assume that $\gamma \leq 1/6\widehat{\ell}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates of* L-SVRGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + \frac{\left(18\widehat{\ell}+3\ell\right)\|x^0-x^{*,0}\|^2}{K} + \left(6+\left(18\widehat{\ell}+3\ell\right)\gamma\right)\frac{2\gamma\sigma_0^2}{pK}.$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 35 *Let the assumptions of Theorem 34 hold and $p = 1/n$. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{18\widehat{\ell} + 3\ell}, \frac{1}{\sqrt{2n\widehat{\ell}\ell}}\right\}. \tag{60}$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{(\widehat{\ell} + \ell)(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2) + \sqrt{n\widehat{\ell}\ell}\Omega_{0,\mathcal{C}}^2}{K}\right).$$

F.2. SAGA-SGDA

In this section, we show that SAGA-SGDA [62] fits our theoretical framework and derive new results for this method under averaged star-cocoercivity.

Algorithm 3 SAGA-SGDA [62]

1: **Input:** starting point $x^0 \in \mathbb{R}^d$, stepsize $\gamma > 0$, number of steps K
2: Set $w_i^0 = x^0$ and compute $F_i(w_i^0)$ for all $i \in [n]$
3: **for** $k = 0$ **to** $K - 1$ **do**
4: Draw a fresh sample j_k from the uniform distribution on $[n]$ and compute $g^k = F_{j_k}(x^k) - F_{j_k}(w_{j_k}^k) + \frac{1}{n}\sum_{i=1}^n F_i(w_i^k)$
5: Set $w_{j_k}^{k+1} = x^k$ and $w_i^{k+1} = w_i^k$ for $i \neq j_k$
6: $x^{k+1} = \text{prox}_{\gamma R}(x^k - \gamma g^k)$
7: **end for**

F.2.1. SAGA-SGDA FITS ASSUMPTION 1

Lemma 36 *Let Assumption 4 hold. Then for all $k \geq 0$* SAGA-SGDA *satisfies*

$$\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] \leq 2\widehat{\ell}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + 2\sigma_k^2, \tag{61}$$

where $\sigma_k^2 := \frac{1}{n}\sum_{i=1}^n \|F_i(w_i^k) - F_i(x^{*,k})\|^2$.

Proof For brevity, we introduce a new notation: $S^k = \frac{1}{n}\sum_{i=1}^n F_i(w_i^k)$. Since $g^k = F_{j_k}(x^k) - F_{j_k}(w_{j_k}^k) + S^k$, we have

$$\begin{aligned}
\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &= \mathbb{E}_k\left[\|F_{j_k}(x^k) - F_{j_k}(w_{j_k}^k) + S^k - F(x^{*,k})\|^2\right] \\
&= \frac{1}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(w_i^k) + S^k - F(x^{*,k})\|^2 \\
&\leq \frac{2}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^{*,k})\|^2 \\
&\quad + \frac{2}{n}\sum_{i=1}^n \|F_i(w_i^k) - F_i(x^{*,k}) - (S^k - F(x^{*,k}))\|^2 \\
&\leq \frac{2}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^{*,k})\|^2 + \frac{2}{n}\sum_{i=1}^n \|F_i(w_i^k) - F_i(x^{*,k})\|^2 \\
&\overset{(13)}{\leq} 2\widehat{\ell}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + 2\sigma_k^2.
\end{aligned}$$

■

Lemma 37 *Let Assumptions 4 and 5 hold. Then for all $k \geq 0$* SAGA-SGDA *satisfies*

$$\mathbb{E}_k\left[\sigma_{k+1}^2\right] \leq \frac{\widehat{\ell}}{n}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + (1 - 1/n)\sigma_k^2, \tag{62}$$

where $\sigma_k^2 := \frac{1}{n}\sum_{i=1}^n \|F_i(w_i^k) - F_i(x^{*,k})\|^2$.

Proof Using the definitions of σ_{k+1}^2 and w_i^{k+1}, we derive

$$\begin{aligned}
\mathbb{E}_k\left[\sigma_{k+1}^2\right] &= \frac{1}{n}\sum_{i=1}^n \mathbb{E}_k\left[\|F_i(w_i^{k+1}) - F_i(x^{*,k+1})\|^2\right] \\
&\overset{\text{As. 5}}{=} \frac{1}{n}\sum_{i=1}^n \mathbb{E}_k\left[\|F_i(w_i^{k+1}) - F_i(x^{*,k})\|^2\right] \\
&= \frac{1}{n^2}\sum_{i=1}^n \|F_i(x^k) - F_i(x^{*,k})\|^2 + \frac{1 - 1/n}{n}\sum_{i=1}^n \|F_i(w_i^k) - F_i(x^{*,k})\|^2 \\
&\overset{(13)}{\leq} \frac{\widehat{\ell}}{n}\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle + (1 - 1/n)\sigma_k^2.
\end{aligned}$$

■

The above two lemmas imply that Assumption 1 is satisfied with certain parameters.

Proposition 38 *Let Assumptions 4 and 5 hold. Then,* SAGA-SGDA *satisfies Assumption 1 with*

$$A = \widehat{\ell}, \quad B = 2, \quad \sigma_k^2 = \frac{1}{n}\sum_{i=1}^n \|F_i(w_i^k) - F_i(x^*)\|^2, \quad C = \frac{\widehat{\ell}}{2n}, \quad \rho = \frac{1}{n}, \quad D_1 = D_2 = 0.$$

F.2.2. ANALYSIS OF SAGA-SGDA IN THE QUASI-STRONGLY MONOTONE CASE

Applying Theorem 1 and Corollary 2 with $M = 4n$, we get the following results.

Theorem 39 *Let F be μ-quasi strongly monotone, Assumptions 4, 5 hold, and $0 < \gamma \leq 1/6\widehat{\ell}$. Then for all $k \geq 0$ the iterates produced by* SAGA-SGDA *satisfy*

$$\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq \left(1 - \min\left\{\gamma\mu, 1/2n\right\}\right)^k V_0, \tag{63}$$

where $V_0 = \|x^0 - x^*\|^2 + 4n\gamma^2\sigma_0^2$.

Corollary 40 *Let the assumptions of Theorem 39 hold. Then, for $\gamma = 1/6\widehat{\ell}$ and any $K \geq 0$ we have*

$$\mathbb{E}[\|x^K - x^*\|^2] \leq V_0 \exp\left(-\min\left\{\frac{\mu}{6\widehat{\ell}}, \frac{1}{2n}\right\} K\right).$$

F.2.3. ANALYSIS OF SAGA-SGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of SAGA-SGDA in the monotone case.

Theorem 41 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2, 4, 5 hold. Assume that $\gamma \leq 1/6\widehat{\ell}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* SAGA-SGDA *satisfy*

$$\begin{aligned}\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\max_{u\in\mathcal{C}}\|x^0 - u\|^2}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{\left(12\widehat{\ell} + \ell\right)\|x^0 - x^{*,0}\|^2}{K} \\ & + \left(4 + \left(12\widehat{\ell} + \ell\right)\gamma\right)\frac{2\gamma\sigma_0^2}{pK} + 9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2.\end{aligned}$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 42 *Let the assumptions of Theorem 41 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{12\widehat{\ell} + \ell}, \frac{1}{\sqrt{2n\widehat{\ell}\ell}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\}, \tag{64}$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{(\widehat{\ell} + \ell)(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2) + \sqrt{n\widehat{\ell}\ell}\Omega_{0,\mathcal{C}}^2 + \ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}G_*}{\sqrt{K}}\right).$$

Proof Since σ_0 for SAGA-SGDA and L-SVRGDA are the same, the proof of this corollary is identical to the one for Corollary 33. ■

F.2.4. ANALYSIS OF SAGA-SGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of SAGA-SGDA in the cocoercive case.

Theorem 43 *Let F be ℓ-cocoercive and Assumptions 1, 2, 4, 5 hold. Assume that $\gamma \leq 1/6\widehat{\ell}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* SAGA-SGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\max_{u\in\mathcal{C}}\|x^0-u\|^2}{2\gamma K} + \frac{\left(18\widehat{\ell}+3\ell\right)\|x^0-x^{*,0}\|^2}{K} + \left(6+\left(18\widehat{\ell}+3\ell\right)\gamma\right)\frac{2\gamma\sigma_0^2}{pK}.$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 44 *Let the assumptions of Theorem 43 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{18\widehat{\ell}+3\ell}, \frac{1}{\sqrt{2n\widehat{\ell}\ell}}\right\}, \tag{65}$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{(\widehat{\ell}+\ell)(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)+\sqrt{n\widehat{\ell}\ell}\Omega_{0,\mathcal{C}}^2}{K}\right).$$

F.3. Discussion of the Results in the Monotone and Cocoercive Cases

Among the papers mentioned in the related work on variance-reduced methods (see Section A), only Alacaoglu and Malitsky [1], Alacaoglu et al. [2], Carmon et al. [14], Luo et al. [53], Tominin et al. [74] consider monotone (convex-concave) and Lipschitz (smooth) VIPs (min-max problems) without assuming strong monotonicity (strong-convexity-strong-concavity) of the problem. In this case, Alacaoglu and Malitsky [1] derive $\mathcal{O}\left(n+\frac{\sqrt{n}L}{K}\right)$ convergence rate (neglecting the dependence on the quantities like $\Omega_{0,\mathcal{C}}^2 = \max_{u\in\mathcal{C}}\|x^0-u\|^2$), which is optimal for the considered setting [31]. Under additional assumptions a similar rate is derived in Carmon et al. [14]. Luo et al. [53], Tominin et al. [74] also achieve this rate but using Catalyst. Finally, Alacaoglu et al. [2] derive $\mathcal{O}\left(n+\frac{nL}{K}\right)$, which is worse than the one from Alacaoglu and Malitsky [1]. Our results for monotone and star-cocoercive regularized VIPs give $\mathcal{O}\left(\frac{\sqrt{n\ell\widehat{\ell}}+\widehat{\ell}}{K}+\frac{G_*}{\sqrt{K}}\right)$ rate, which is typically worse than $\mathcal{O}\left(n+\frac{\sqrt{n}L}{K}\right)$ rate from Alacaoglu and Malitsky [1] due to the relation between cocoercivity constants and Lipschitz constants (even when $R(x)\equiv 0$, i.e., $G_*=0$). However, in general, it is possible that star-cocoercivity holds, while Lipschitzness does not [52]. As for cocoercive case, we obtain $\mathcal{O}\left(\frac{\sqrt{n\ell\widehat{\ell}}+\widehat{\ell}}{K}\right)$, which matches the rate from Alacaoglu and Malitsky [1] up to the difference between cocoercivity and Lipschitz constants. Moreover, we emphasize here that Alacaoglu and Malitsky [1] and other works do not consider SGDA as the basis for their methods. To the best of our knowledge, our results are the first ones for variance-reduced SGDA-type methods derived in the monotone case without assuming (quasi-)strong monotonicity.

Appendix G. Distributed SGDA with Compression: Missing Proofs and Details

In this section, we provide missing proofs and details for Section 5.

G.1. QSGDA

In this section (and in the one about DIANA-SGDA), we assume that each F_i has an expectation form: $F_i(x) = \mathbb{E}_{\xi_i \sim \mathcal{D}_i}[F_{\xi_i}(x)]$.

Algorithm 4 QSGDA: Quantized Stochastic Gradient Descent-Ascent

Input: starting point $x^0 \in \mathbb{R}^d$, stepsize $\gamma > 0$, number of steps K
for $k = 0$ **to** $K - 1$ **do**
  Broadcast x^k to all workers
  for $i = 1, \ldots, n$ in parallel **do**
    Compute g_i^k and send $\mathcal{Q}(g_i^k)$ to the server
  end for
  $g^k = \frac{1}{n} \sum_{i=1}^n \mathcal{Q}(g_i^k)$
  $x^{k+1} = \text{prox}_{\gamma R}\left(x^k - \gamma g^k\right)$
end for

G.1.1. PROOF OF PROPOSITION 8

Proposition 45 (Proposition 8) *Let F be ℓ-star-cocoercive and Assumptions 4, 6 hold. Then,* QSGDA *with quantization (14) satisfies Assumption 1 with*

$$A = \left(\frac{3\ell}{2} + \frac{9\omega\widehat{\ell}}{2n}\right), \quad D_1 = \frac{3(1+3\omega)\sigma^2 + 9\omega\zeta_*^2}{n}, \quad \sigma_k^2 = 0, \quad B = 0,$$
$$C = 0, \quad \rho = 1, \quad D_2 = 0,$$

where $\sigma^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$ and $\zeta_*^2 = \frac{1}{n}\max_{x^* \in X^*}\left[\sum_{i=1}^n \|F_i(x^*)\|^2\right]$.

Proof Since $g^k = \frac{1}{n}\sum_{i=1}^{n} \mathcal{Q}\left(g_i^k\right)$, $\mathcal{Q}\left(g_1^k\right), \ldots, \mathcal{Q}\left(g_n^k\right)$ are independent for fixed $g_1^k, \ldots, g_n^k$, and $g_1^k, \ldots, g_n^k$ are independent for fixed x^k, we have

$$
\begin{aligned}
\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &= \mathbb{E}_k\left[\left\|\frac{1}{n}\sum_{i=1}^{n}\mathcal{Q}\left(g_i^k\right) - F(x^{*,k})\right\|^2\right] \\
&= \mathbb{E}_k\left[\left\|\frac{1}{n}\sum_{i=1}^{n}\left[\mathcal{Q}\left(g_i^k\right) - g_i^k + g_i^k - F_i(x^k)\right] + F(x^k) - F(x^{*,k})\right\|^2\right] \\
&\leq 3\mathbb{E}_k\left[\left\|\frac{1}{n}\sum_{i=1}^{n}[\mathcal{Q}\left(g_i^k\right) - g_i^k]\right\|^2\right] + 3\mathbb{E}_k\left[\left\|\frac{1}{n}\sum_{i=1}^{n}[g_i^k - F_i(x^k)]\right\|^2\right] \\
&\quad +3\left\|F(x^k) - F(x^{*,k})\right\|^2 \\
&= \frac{3}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|\mathcal{Q}\left(g_i^k\right) - g_i^k\right\|^2\right] + \frac{3}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|g_i^k - F_i(x^k)\right\|^2\right] \\
&\quad +3\left\|F(x^k) - F(x^{*,k})\right\|^2.
\end{aligned}
$$

Next, we use Assumption 6, $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$, and the definition of quantization (14) and get

$$
\begin{aligned}
\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &\leq \frac{3\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|g_i^k\right\|^2\right] + \frac{3\sigma^2}{n} + 3\left\|F(x^k) - F(x^{*,k})\right\|^2 \\
&\leq \frac{3\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|g_i^k - F_i(x^k) + F_i(x^k) - F_i(x^{*,k}) + F_i(x^{*,k})\right\|^2\right] \\
&\quad + \frac{3\sigma^2}{n} + 3\left\|F(x^k) - F(x^{*,k})\right\|^2 \\
&\leq \frac{9\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|g_i^k - F_i(x^k)\right\|^2\right] + \frac{9\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|F_i(x^k) - F_i(x^{*,k})\right\|^2\right] \\
&\quad + \frac{9\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|F_i(x^{*,k})\right\|^2\right] + \frac{3\sigma^2}{n} + 3\left\|F(x^k) - F(x^{*,k})\right\|^2 \\
&\overset{(15)}{\leq} \frac{9\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|F_i(x^k) - F_i(x^{*,k})\right\|^2\right] + 3\left\|F(x^k) - F(x^{*,k})\right\|^2 \\
&\quad + \frac{9\omega}{n^2}\sum_{i=1}^{n}\mathbb{E}_k\left[\left\|F_i(x^{*,k})\right\|^2\right] + \frac{3(1+3\omega)\sigma^2}{n}.
\end{aligned}
$$

Star-cocoercivity of F and Assumption 4 give

$$\begin{aligned}\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &\leq \left(3\ell + \frac{9\omega}{n}\widehat{\ell}\right)\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle \\ &\quad + \frac{9\omega}{n^2}\sum_{i=1}^n \mathbb{E}_k\left[\left\|F_i(x^{*,k})\right\|^2\right] + \frac{3(1+3\omega)\sigma^2}{n} \\ &\leq \left(3\ell + \frac{9\omega}{n}\widehat{\ell}\right)\langle F(x^k) - F(x^{*,k}), x^k - x^{*,k}\rangle \\ &\quad + \frac{9\omega}{n^2}\max_{x*\in X^*}\left[\sum_{i=1}^n \|F_i(x^*)\|^2\right] + \frac{3(1+3\omega)\sigma^2}{n}.\end{aligned}$$

■

G.1.2. ANALYSIS OF QSGDA IN THE QUASI-STRONGLY MONOTONE CASE

Applying Theorem 1 and Corollary 2, we get the following results.

Theorem 46 *Let F be μ-quasi strongly monotone, ℓ-star-cocoercive, Assumptions 4, 6 hold, and*

$$0 < \gamma \leq \frac{1}{3\ell + \frac{9\omega\widehat{\ell}}{n}}.$$

Then, for all $k \geq 0$ the iterates produced by QSGDA *satisfy*

$$\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq (1-\gamma\mu)^k\|x^0 - x^*\|^2 + \gamma\frac{3(1+3\omega)\sigma^2 + 9\omega\zeta_*^2}{n\mu}.$$

Corollary 47 *Let the assumptions of Theorem 46 hold. Then, for any $K \geq 0$ one can choose $\{\gamma_k\}_{k\geq 0}$ as follows:*

$$\begin{aligned}&\text{if } K \leq \frac{1}{\mu}\cdot\left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right), && \gamma_k = \left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right)^{-1}, \\ &\text{if } K > \frac{1}{\mu}\cdot\left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right) \text{ and } k < k_0, && \gamma_k = \left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right)^{-1}, \\ &\text{if } K > \frac{1}{\mu}\cdot\left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right) \text{ and } k \geq k_0, && \gamma_k = \frac{2}{(6\ell + {}^{18\omega\widehat{\ell}}/_{n} + \mu(k-k_0))},\end{aligned}$$

where $k_0 = \lceil {}^{K}/_{2}\rceil$. For this choice of γ_k the following inequality holds:

$$\begin{aligned}\mathbb{E}[\|x^K - x^{*,K}\|^2] &\leq \frac{32(3\ell + {}^{9\omega\widehat{\ell}}/_{n})}{\mu}\|x^0 - x^{*,0}\|^2\exp\left(-\frac{\mu}{(3\ell + {}^{9\omega\widehat{\ell}}/_{n})}K\right) \\ &\quad + \frac{36}{\mu^2 K}\cdot\frac{3(1+3\omega)\sigma^2 + 9\omega\zeta_*^2}{n}.\end{aligned}$$

G.1.3. ANALYSIS OF QSGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of QSGDA in the monotone case.

Theorem 48 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2, 4, 6 hold. Assume that $\gamma \leq \left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right)^{-1}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* QSGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \left(7\ell + \frac{18\omega\widehat{\ell}}{n}\right)\cdot\frac{\|x^0-x^{*,0}\|^2}{K}$$
$$+\gamma\left(2+\gamma\left(7\ell+\frac{18\omega\widehat{\ell}}{n}\right)\right)\cdot\frac{3(1+3\omega)\sigma^2+9\omega\zeta_*^2}{n}$$
$$+9\gamma\max_{x^*\in X^*}\left[\|F(x^*)\|^2\right]$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 49 *Let the assumptions of Theorem 48 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{7\ell+\frac{18\omega\widehat{\ell}}{n}}, \frac{\Omega_{0,\mathcal{C}}\sqrt{n}}{\sqrt{3K(1+3\omega)\sigma^2+9K\omega\zeta_*^2}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\}.$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{\left(\ell+\omega\widehat{\ell}/n\right)\left(\Omega_{0,\mathcal{C}}^2+\Omega_0^2\right)+\ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}(\sigma\sqrt{1+\omega}+G_*\sqrt{n}+\zeta_*\sqrt{\omega})}{\sqrt{nK}}\right).$$

G.1.4. ANALYSIS OF QSGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of QSGDA in the cocoercive case.

Theorem 50 *Let F be ℓ-cocoercive and Assumptions 1, 2, 4, 6 hold. Assume that $\gamma \leq \left(3\ell + \frac{9\omega\widehat{\ell}}{n}\right)^{-1}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* QSGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \left(10\ell + \frac{27\omega\widehat{\ell}}{n}\right)\cdot\frac{\|x^0-x^{*,0}\|^2}{K}$$
$$+\gamma\left(3+\gamma\left(10\ell+\frac{27\omega\widehat{\ell}}{n}\right)\right)\cdot\frac{3(1+3\omega)\sigma^2+9\omega\zeta_*^2}{n}.$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 51 *Let the assumptions of Theorem 50 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{10\ell+\frac{27\omega\widehat{\ell}}{n}}, \frac{\Omega_{0,\mathcal{C}}\sqrt{n}}{\sqrt{3K(1+3\omega)\sigma^2+9K\omega\zeta_*^2}}\right\}.$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{\left(\ell+\omega\widehat{\ell}/n\right)\left(\Omega_{0,\mathcal{C}}^2+\Omega_0^2\right)}{K} + \frac{\Omega_{0,\mathcal{C}}(\sigma\sqrt{1+\omega}+\zeta_*\sqrt{\omega})}{\sqrt{nK}}\right).$$

G.2. DIANA-SGDA

Algorithm 5 DIANA-SGDA: DIANA Stochastic Gradient Descent-Ascent [35, 56]

Input: starting points $x^0, h_1^0, \ldots, h_n^0 \in \mathbb{R}^d$, $h^0 = \frac{1}{n}\sum_{i=1}^n h_i^0$, stepsizes $\gamma, \alpha > 0$, number of steps K
for $k = 0$ **to** $K-1$ **do**
  Broadcast x^k to all workers
  for $i = 1, \ldots, n$ in parallel **do**
    Compute g_i^k and $\Delta_i^k = g_i^k - h_i^k$
    Send $\mathcal{Q}(\Delta_i^k)$ to the server
    $h_i^{k+1} = h_i^k + \alpha\mathcal{Q}(\Delta_i^k)$
  end for
  $g^k = h^k + \frac{1}{n}\sum\limits_{i=1}^n \mathcal{Q}(\Delta_i^k) = \frac{1}{n}\sum\limits_{i=1}^n (h_i^k + \mathcal{Q}(\Delta_i^k))$
  $x^{k+1} = \mathrm{prox}_{\gamma R}\left(x^k - \gamma g^k\right)$
  $h^{k+1} = h^k + \alpha\frac{1}{n}\sum\limits_{i=1}^n \mathcal{Q}(\Delta_i^k) = \frac{1}{n}\sum\limits_{i=1}^n h_i^k$
end for

G.2.1. PROOF OF PROPOSITION 9

The following result follows from Lemmas 1 and 2 from [35]. It holds in our settings as well, since it does not rely on the exact form of $F_i(x^k)$.

Lemma 52 (Lemmas 1 and 2 from [35]) *Let Assumptions 5, 6 hold. Suppose that $\alpha \leq 1/(1+\omega)$. Then, for all $k \geq 0$* DIANA-SGDA *satisfies*

$$\begin{aligned}
\mathbb{E}_k\left[g^k\right] &= F(x^k),\\
\mathbb{E}_k\left[\|g^k - F(x^*)\|^2\right] &\leq \left(1 + \frac{2\omega}{n}\right)\frac{1}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^*)\|^2 + \frac{2\omega\sigma_k^2}{n} + \frac{(1+\omega)\sigma^2}{n},\\
\mathbb{E}_k\left[\sigma_{k+1}^2\right] &\leq (1-\alpha)\sigma_k^2 + \frac{\alpha}{n}\sum_{i=1}^n \|F_i(x^k) - F_i(x^*)\|^2 + \alpha\sigma^2,
\end{aligned}$$

where $\sigma_k^2 = \frac{1}{n}\sum\limits_{i=1}^n \|h_i^k - F_i(x^*)\|^2$ and $\sigma^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$.

The lemma above implies that Assumption 1 is satisfied with certain parameters.

Proposition 53 (Proposition 9) *Let Assumptions 4, 5, 6 hold. Suppose that $\alpha \leq \frac{1}{1+\omega}$. Then,* DIANA-SGDA *with quantization (14) satisfies Assumption 1 with $\sigma_k^2 = \frac{1}{n}\sum_{i=1}^n \|h_i^k - F_i(x^*)\|^2$ and*

$$A = \left(\frac{1}{2} + \frac{\omega}{n}\right)\widehat{\ell}, \quad B = \frac{2\omega}{n}, \quad D_1 = \frac{(1+\omega)\sigma^2}{n}, \quad C = \frac{\alpha\widehat{\ell}}{2}, \quad \rho = \alpha, \quad D_2 = \alpha\sigma^2.$$

Proof To get the result, one needs to apply Assumption 4 to estimate $\frac{1}{n}\sum_{i=1}^{n}\|F_i(x^k)-F_i(x^*)\|^2$ from Lemma 52. ■

G.2.2. ANALYSIS OF DIANA-SGDA IN THE QUASI-STRONGLY MONOTONE CASE

Applying Theorem 1 and Corollary 2 with $M=\frac{4\omega}{\alpha n}$, we get the following results.

Theorem 54 *Let F be μ-quasi strongly monotone, Assumptions 4, 5, 6 hold, $\alpha \leq 1/(1+\omega)$, and*

$$0<\gamma\leq\frac{1}{\left(1+\frac{6\omega}{n}\right)\widehat{\ell}}.$$

Then, for all $k\geq 0$ the iterates produced by DIANA-SGDA *satisfy*

$$\mathbb{E}\left[\|x^k-x^*\|^2\right]\leq\left(1-\min\left\{\gamma\mu,\frac{\alpha}{2}\right\}\right)^k\mathbb{E}[V_0]+\frac{\gamma^2\sigma^2(1+5\omega)}{n\cdot\min\left\{\gamma\mu,\alpha/2\right\}},$$

where $V_0=\|x^0-x^*\|^2+4\omega\gamma^2\sigma_0^2/\alpha n$.

Corollary 55 *Let the assumptions of Theorem 9 hold. Then, for any $K\geq 0$ one can choose $\alpha=1/(1+\omega)$ and $\{\gamma_k\}_{k\geq 0}$ as follows:*

$$\begin{aligned}
&\text{if } K\leq\frac{h}{\mu}, && \gamma_k=\frac{1}{h},\\
&\text{if } K>\frac{h}{\mu} \text{ and } k<k_0, && \gamma_k=\frac{1}{h},\\
&\text{if } K>\frac{h}{\mu} \text{ and } k\geq k_0, && \gamma_k=\frac{2}{2h+\mu(k-k_0)},
\end{aligned}$$

where $h=\max\left\{\left(1+\frac{6\omega}{n}\right)\widehat{\ell},2\mu(1+\omega)\right\}$, $k_0=\lceil K/2\rceil$. For this choice of γ_k the following inequality holds:

$$\begin{aligned}
\mathbb{E}[\|x^K-x^{*,K}\|^2] \quad\leq\quad & 32\max\left\{\frac{\left(1+\frac{6\omega}{n}\right)\widehat{\ell}}{\mu},2(1+\omega)\right\}V_0\exp\left(-\min\left\{\frac{\mu}{\widehat{\ell}(1+\frac{6\omega}{n})},\frac{1}{1+\omega}\right\}K\right)\\
&+\frac{36(1+5\omega)\sigma^2}{\mu^2nK}.
\end{aligned}$$

G.2.3. ANALYSIS OF DIANA-SGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of DIANA-SGDA in the monotone case.

Theorem 56 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2, 4, 5, 6 hold. Assume that*

$$0<\gamma\leq\frac{1}{\left(1+\frac{4\omega}{n}\right)\widehat{\ell}}.$$

Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by DIANA-SGDA *satisfy*

$$\begin{aligned}\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] &\leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \left(2\widehat{\ell} + \frac{12\omega\widehat{\ell}}{n} + \ell\right)\frac{\|x^0-x^{*,0}\|^2}{K} \\ &\quad + \left(4+\gamma\left(2\widehat{\ell}+\frac{12\omega\widehat{\ell}}{n}+\ell\right)\right)\frac{\gamma B\sigma_0^2}{\rho K} \\ &\quad +\gamma\left(\left(2+\gamma\left(2\widehat{\ell}+\frac{12\omega\widehat{\ell}}{n}+\ell\right)\right)\left(\frac{(1+5\omega)\sigma^2}{n}\right)\right) \\ &\quad +9\gamma\max_{x^*\in X^*}\|F(x^*)\|^2.\end{aligned}$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 57 *Let the assumptions of Theorem 56 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\left(\ell + 2\widehat{\ell} + \frac{12\omega\widehat{\ell}}{n}\right)^{-1}, \frac{\sqrt{\alpha n}}{\sqrt{2\omega\widehat{\ell}\ell}}, \frac{\Omega_{0,\mathcal{C}}}{\sigma\sqrt{K^{(1+3\omega)}/n}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\},$$

This choice of γ implies that $\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right]$ equals

$$\mathcal{O}\left(\frac{(\ell+\widehat{\ell}+\omega\widehat{\ell}/n)(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)+\ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}^2\sqrt{\widehat{\ell}\ell}\sqrt{\omega}}{\sqrt{\alpha n}K} + \frac{\Omega_{0,\mathcal{C}}(\sqrt{(1+\omega)\sigma^2/n}+G_*)}{\sqrt{K}}\right).$$

Proof The proof follows from the next upper bound $\widehat{\sigma}_0^2$ for σ_0^2 with initialization $h_i^0 = F_i(x^0)$

$$\begin{aligned}\sigma_0^2 &= \frac{1}{n}\sum_{i=1}^{n}\|F_i(x^0)-F_i(x^*)\|^2 \\ &\leq \widehat{\ell}\langle F(x^0)-F(x^*), x^0-x^*\rangle \\ &\leq \widehat{\ell}\|F(x^0)-F(x^*)\|\cdot\|x^0-x^*\| \\ &\leq \widehat{\ell}\ell\|x^0-x^*\|^2 \leq \widehat{\ell}\ell\max_{u\in\mathcal{C}}\|x^0-u\|^2 \leq \widehat{\ell}\ell\Omega_{0,\mathcal{C}}^2.\end{aligned}$$

Next, applying Corollary 17 with $\widehat{\sigma}_0 := \sqrt{\widehat{\ell}\ell}\Omega_{0,\mathcal{C}}$, we get the result. ■

G.2.4. ANALYSIS OF DIANA-SGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of DIANA-SGDA in the cocoercive case.

Theorem 58 *Let F be ℓ-cocoercive and Assumptions 1, 2, 4, 5, 6 hold. Assume that*

$$0 < \gamma \leq \frac{1}{\left(1+\frac{4\omega}{n}\right)\widehat{\ell}}.$$

Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by DIANA-SGDA *satisfy*

$$\begin{aligned}\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] &\leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \left(3\widehat{\ell} + \frac{18\omega\widehat{\ell}}{n} + 3\ell\right)\frac{\|x^0-x^{*,0}\|^2}{K} \\ &\quad + \left(6 + \gamma\left(4\widehat{\ell} + \frac{18\omega\widehat{\ell}}{n} + 3\ell\right)\right)\frac{\gamma B\sigma_0^2}{\rho K} \\ &\quad + \gamma\left(\left(3 + \gamma\left(3\widehat{\ell} + \frac{18\omega\widehat{\ell}}{n} + 3\ell\right)\right)\left(\frac{(1+5\omega)\sigma^2}{n}\right)\right).\end{aligned}$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 59 *Let the assumptions of Theorem 58 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\left(3\ell + 3\widehat{\ell} + \frac{18\omega\widehat{\ell}}{n}\right)^{-1}, \frac{\sqrt{\alpha n}}{\sqrt{2\omega\widehat{\ell}\ell}}, \frac{\Omega_{0,\mathcal{C}}}{\sigma\sqrt{K^{(1+3\omega)}/n}}\right\},$$

This choice of γ implies that $\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right]$ equals

$$\mathcal{O}\left(\frac{(\ell + \widehat{\ell} + \omega\widehat{\ell}/n)(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2)}{K} + \frac{\Omega_{0,\mathcal{C}}^2\sqrt{\widehat{\ell}\ell}\sqrt{\omega}}{\sqrt{\alpha n}K} + \frac{\Omega_{0,\mathcal{C}}\sqrt{(1+\omega)\sigma^2/n}}{\sqrt{K}}\right).$$

G.3. VR-DIANA-SGDA

In this section, we assume that each F_i has a finite-sum form: $F_i(x) = \frac{1}{m}\sum_{j=1}^{m} F_{ij}(x)$.

G.3.1. PROOF OF PROPOSITION 10

Lemma 60 (Modification of Lemmas 3 and 7 from [35]) *Let F be ℓ-star-cocoercive and Assumptions 4, 5, 7 hold. Then for all $k \geq 0$* VR-DIANA-SGDA *satisfies*

$$\begin{aligned}\mathbb{E}_k\left[g^k\right] &= F(x^k), \\ \mathbb{E}_k\left[\|g^k - F(x^*)\|\right] &\leq \left(\ell + \frac{2\widetilde{\ell}}{n} + \frac{2\omega(\widehat{\ell}+\widetilde{\ell})}{n}\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{2(\omega+1)}{n}\sigma_k^2,\end{aligned}$$

where $\sigma_k^2 = \frac{H^k}{n} + \frac{D^k}{nm}$ with $H^k = \sum_{i=1}^{n}\left\|h_i^k - F_i(x^*)\right\|^2$ and $D^k = \sum_{i=1}^{n}\sum_{j=1}^{m}\left\|F_{ij}(w_i^k) - F_{ij}(x^*)\right\|^2$.

Proof First of all, we derive unbiasedness:

$$\mathbb{E}\left[g^k\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\mathcal{Q}(g_i^k - h_i^k) + h_i^k\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[g_i^k - h_i^k + h_i^k\right] = \frac{1}{n}\sum_{i=1}^{n}F_i(x^k) = F(x^k).$$

Algorithm 6 VR-DIANA-SGDA: VR-DIANA Stochastic Gradient Descent-Ascent [35]

Input: starting points $x^0, h_1^0, \ldots, h_n^0 \in \mathbb{R}^d$, $h^0 = \frac{1}{n}\sum\limits_{i=1}^n h_i^0$, probability $p \in (0,1]$ stepsizes $\gamma, \alpha > 0$, number of steps K,
for $k = 0$ **to** $K-1$ **do**
  Broadcast x^k to all workers
  for $i = 1, \ldots, n$ in parallel **do**
    Draw a fresh sample j_i^k from the uniform distribution on $[m]$ and compute $g_i^k = F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k) + F_i(w_i^k)$
    $w_i^{k+1} = \begin{cases} x^k, & \text{with probability } p, \\ w_i^k, & \text{with probability } 1-p, \end{cases}$
    $\Delta_i^k = g_i^k - h_i^k$
    Send $\mathcal{Q}(\Delta_i^k)$ to the server
    $h_i^{k+1} = h_i^k + \alpha \mathcal{Q}(\Delta_i^k)$
  end for
  $g^k = h^k + \frac{1}{n}\sum\limits_{i=1}^n \mathcal{Q}(\Delta_i^k) = \frac{1}{n}\sum\limits_{i=1}^n (h_i^k + \mathcal{Q}(\Delta_i^k))$
  $x^{k+1} = \text{prox}_{\gamma R}\left(x^k - \gamma g^k\right)$
  $h^{k+1} = h^k + \alpha \frac{1}{n}\sum\limits_{i=1}^n \mathcal{Q}(\Delta_i^k) = \frac{1}{n}\sum\limits_{i=1}^n h_i^k$
end for

By definition of the variance we get

$$\mathbb{E}_{\mathcal{Q}}\left[\left\|g^k - F(x^*)\right\|^2\right] = \underbrace{\left\|\mathbb{E}_{\mathcal{Q}}\left[g^k\right] - F(x^*)\right\|^2}_{T_1} + \underbrace{\mathbb{E}_{Q}\left[\left\|g^k - \mathbb{E}_{Q}\left[g^k\right]\right\|^2\right]}_{T_2}.$$

Next, we derive the upper bounds for terms T_1 and T_2 separately. For T_2 we use unbiasedness of quantization and independence of workers:

$$\begin{aligned} T_2 &= \mathbb{E}_{Q}\left[\left\|\frac{1}{n}\sum_{i=1}^n \mathcal{Q}(g_i^k - h_i^k) - (g_i^k - h_i^k)\right\|^2\right] \\ &= \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}_{\mathcal{Q}}\left[\left\|\mathcal{Q}(g_i^k - h_i^k) - (g_i^k - h_i^k)\right\|^2\right] \overset{(14)}{\leq} \frac{\omega}{n^2}\sum_{i=1}^n \left\|g_i^k - h_i^k\right\|^2. \end{aligned}$$

Taking $\mathbb{E}_k[\cdot]$ from the both sides of the above inequality, we derive

$$
\begin{aligned}
\mathbb{E}_k\left[T_2\right] &\leq \frac{\omega}{n^2}\sum_{i=1}^n \mathbb{E}_k\left[\left\|g_i^k - h_i^k\right\|^2\right] = \frac{\omega}{n^2}\sum_{i=1}^n \left(\left\|\mathbb{E}_k\left[g_i^k - h_i^k\right]\right\|^2 + \mathbb{E}_k\left[\left\|g_i^k - h_i^k - \mathbb{E}_k\left[g_i^k - h_i^k\right]\right\|^2\right]\right)\\
&= \frac{\omega}{n^2}\sum_{i=1}^n \left(\left\|F_i(x^k) - h_i^k\right\|^2 + \mathbb{E}_k\left[\left\|g_i^k - F_i(x^k)\right\|^2\right]\right)\\
&= \frac{\omega}{n^2}\sum_{i=1}^n \left(\left\|F_i(x^k) - h_i^k\right\|^2 + \mathbb{E}_k\left[\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k) - \mathbb{E}_k\left[F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k)\right]\right\|^2\right]\right)\\
&\leq \frac{\omega}{n^2}\sum_{i=1}^n \left(\left\|F_i(x^k) - h_i^k\right\|^2 + \mathbb{E}_k\left[\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k)\right\|^2\right]\right)\\
&\leq \frac{2\omega}{n^2}\sum_{i=1}^n \left(\left\|h_i^k - F_i(x^\star)\right\|^2 + \left\|F_i(x^k) - F_i(x^\star)\right\|^2\right)\\
&\quad + \frac{2\omega}{n^2}\sum_{i=1}^n \left(\mathbb{E}_k\left[\left\|F_{ij_i^k}(w_i^k) - F_{ij_i^k}(x^\star)\right\|^2\right] + \mathbb{E}_k\left[\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(x^\star)\right\|^2\right]\right).
\end{aligned}
$$

Since j_i^k is sampled uniformly at random from $[m]$, we have

$$
\begin{aligned}
\mathbb{E}_k\left[T_2\right] &\leq \frac{2\omega}{n^2}\sum_{i=1}^n \left(\left\|h_i^k - F_i(x^\star)\right\|^2 + \left\|F_i(x^k) - F_i(x^\star)\right\|^2\right)\\
&\quad + \frac{2\omega}{mn^2}\sum_{i=1}^n\sum_{j=1}^m \left(\mathbb{E}_k\left[\left\|F_{ij}(w_i^k) - F_{ij}(x^\star)\right\|^2\right] + \mathbb{E}_k\left[\left\|F_{ij}(x^k) - F_{ij}(x^\star)\right\|^2\right]\right)\\
&\overset{(13),(19)}{\leq} \frac{2\omega}{n^2}H^k + \frac{2\omega}{mn^2}D^k + \frac{2\omega(\widehat{\ell} + \widetilde{\ell})}{n}\langle F(x^k) - F(x^*), x^k - x^*\rangle.
\end{aligned}
$$

In last line, we also use the definitions of H^k, D^k. For T_1 we use definition of g^k:

$$
T_1 = \left\|\frac{1}{n}\sum_{i=1}^n \mathbb{E}_{\mathcal{Q}}\left[\mathcal{Q}(g_i^k - h_i^k) + h_i^k\right] - F(x^*)\right\|^2 = \left\|\frac{1}{n}\sum_{i=1}^n g_i^k - F(x^*).\right\|^2
$$

Next, we estimate $\mathbb{E}_k[T_1]$ similarly to $\mathbb{E}_k[T_2]$:

$$
\begin{aligned}
\mathbb{E}_k\left[T_1\right] &= \mathbb{E}_k\left[\left\|\frac{1}{n}\sum_{i=1}^n g_i^k - F(x^*)\right\|^2\right] = \left\|\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[g_i^k\right] - F(x^*)\right\|^2 + \mathbb{E}_k\left[\left\|\frac{1}{n}\sum_{i=1}^n \left(g_i^k - \mathbb{E}\left[g_i^k\right]\right)\right\|_2^2\right] \\
&= \left\|F(x^k) - F(x^*)\right\|^2 + \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}_k\left[\left\|g_i^k - F_i(x^k)\right\|^2\right] \\
&\overset{(4)}{\leq} \ell\langle F(x^k) - F(x^*), x^k - x^*\rangle \\
&\quad + \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}\left[\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k) - \mathbb{E}_k\left[F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k)\right]\right\|^2\right] \\
&\leq \ell\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{1}{n^2}\sum_{i=1}^n \mathbb{E}_k\left[\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k)\right\|^2\right] \\
&= \ell\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{1}{mn^2}\sum_{i=1}^n\sum_{j=1}^m \left\|F_{ij}(x^k) - F_{ij}(w_i^k)\right\|^2 \\
&\leq \ell\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{2}{mn^2}\sum_{i=1}^n\sum_{j=1}^m \left(\left\|F_{ij}(w_i^k) - F_{ij}(x^\star)\right\|^2 + \left\|F_{ij}(x^k) - F_{ij}(x^\star)\right\|^2\right) \\
&\overset{(19)}{\leq} \left(\ell + \frac{2\widetilde{\ell}}{n}\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{2}{mn^2}D^k.
\end{aligned}
$$

Finally, summing $\mathbb{E}\left[T_1\right]$ and $\mathbb{E}\left[T_2\right]$ we get

$$
\begin{aligned}
\mathbb{E}\left[\left\|g^k - F(x^*)\right\|^2\right] &= \mathbb{E}\left[T_1 + T_2\right] \\
&\leq \left(\ell + \frac{2\widetilde{\ell}}{n}\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{2}{mn^2}D^k \\
&\quad + \frac{2\omega}{n^2}H^k + \frac{2\omega}{mn^2}D^k + \frac{2\omega(\widehat{\ell} + \widetilde{\ell})}{n}\langle F(x^k) - F(x^*), x^k - x^*\rangle \\
&\leq \left(\ell + \frac{2\widetilde{\ell}}{n} + \frac{2\omega(\widehat{\ell} + \widetilde{\ell})}{n}\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle + \frac{2\omega}{n^2}H^k + \frac{2(\omega+1)}{mn^2}D^k,
\end{aligned}
$$

which concludes the proof since $\sigma_k^2 = \frac{H^k}{n} + \frac{D^k}{nm}$. ■

Lemma 61 (Modification of Lemmas 5 and 6 from [35]) *Let F be ℓ-star-cocoercive and Assumptions 4, 5, 7 hold. Suppose that $\alpha \leq \min\left\{\frac{p}{3}; \frac{1}{1+\omega}\right\}$. Then for all $k \geq 0$* VR-DIANA-SGDA *satisfies*

$$
\mathbb{E}_k\left[\sigma_{k+1}^2\right] \quad \leq \quad (1-\alpha)\sigma_k^2 + \left(p\widetilde{\ell} + 2\alpha(\widetilde{\ell} + \widehat{\ell})\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle,
$$

where $\sigma_k^2 = \frac{H^k}{n} + \frac{D^k}{nm}$ with $H^k = \sum_{i=1}^n \left\|h_i^k - F_i(x^*)\right\|^2$ and $D^k = \sum_{i=1}^n\sum_{j=1}^m \left\|F_{ij}(w_i^k) - F_{ij}(x^*)\right\|^2$.

Proof We start with considering H^{k+1}:

$$\begin{aligned}
\mathbb{E}_k\left[H^{k+1}\right] &= \mathbb{E}_k\left[\sum_{i=1}^n \left\|h_i^{k+1} - F_i(x^\star)\right\|^2\right] \\
&= \sum_{i=1}^n \left\|h_i^k - F_i(x^\star)\right\|^2 + \sum_{i=1}^n \mathbb{E}_k\left[2\langle \alpha\mathcal{Q}(g_i^k - h_i^k), h_i^k - F_i(x^\star)\rangle + \alpha^2\left\|\mathcal{Q}(g_i^k - h_i^k)\right\|^2\right] \\
&\overset{(14)}{\leq} H^k + \sum_{i=1}^n \mathbb{E}_k\left[2\alpha\langle g_i^k - h_i^k, h_i^k - F_i(x^\star)\rangle + \alpha^2(\omega+1)\left\|g_i^k - h_i^k\right\|^2\right].
\end{aligned}$$

Since $\alpha \leq 1/(\omega+1)$, we have

$$\begin{aligned}
\mathbb{E}_k\left[H^{k+1}\right] &\leq H^k + \mathbb{E}_k\left[\sum_{i=1}^n \alpha\langle g_i^k - h_i^k, g_i^k + h_i^k - 2F_i(x^\star)\rangle\right] \\
&= H^k + \mathbb{E}_k\left[\sum_{i=1}^n \alpha\langle g_i^k - F_i(x^\star) + F_i(x^\star) - h_i^k, g_i^k - F_i(x^\star) + h_i^k - F_i(x^\star)\rangle\right] \\
&= H^k + \mathbb{E}_k\left[\sum_{i=1}^n \alpha\left(\left\|g_i^k - F_i(x^\star)\right\|^2 - \left\|h_i^k - F_i(x^\star)\right\|^2\right)\right] \\
&= H^k(1-\alpha) + \mathbb{E}_k\left[\sum_{i=1}^n \alpha\left(\left\|g_i^k - F_i(x^\star)\right\|^2\right)\right] \\
&\leq H^k(1-\alpha) + \sum_{i=1}^n\left(2\alpha\mathbb{E}_k\left[\left\|g_i^k - F_i(x^k)\right\|^2\right] + 2\alpha\left\|F_i(x^k) - F_i(x^\star)\right\|^2\right) \\
&= H^k(1-\alpha) + \sum_{i=1}^n \mathbb{E}_k\left[2\alpha\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k) - \mathbb{E}_k\left[F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k)\right]\right\|^2\right] \\
&\quad + 2\alpha\sum_{i=1}^n\left\|F_i(x^k) - F_i(x^\star)\right\|^2 \\
&\leq H^k(1-\alpha) + \sum_{i=1}^n\left(\mathbb{E}_k\left[2\alpha\left\|F_{ij_i^k}(x^k) - F_{ij_i^k}(w_i^k)\right\|^2\right] + 2\alpha\left\|F_i(x^k) - F_i(x^\star)\right\|^2\right) \\
&\leq H^k(1-\alpha) + \frac{2\alpha}{m}\sum_{i=1}^n\sum_{j=1}^m\left(\left\|F_{ij}(x^k) - F_{ij}(x^\star)\right\|^2 + \left\|F_{ij}(w_i^k) - F_{ij}(x^\star)\right\|^2\right) \\
&\quad + 2\alpha\sum_{i=1}^n\left\|F_i(x^k) - F_i(x^\star)\right\|_2^2 \\
&\overset{(13),(19)}{\leq} H^k(1-\alpha) + \frac{2\alpha}{m}\sum_{i=1}^n\sum_{j=1}^m\left\|F_{ij}(w_{ij}^k) - F_{ij}(x^\star)\right\|_2^2 \\
&\quad + 2\alpha n(\widetilde{\ell} + \widehat{\ell})\langle F(x^k) - F(x^*), x^k - x^*\rangle \\
&= H^k(1-\alpha) + \frac{2\alpha}{m}D^k + 2\alpha n(\widetilde{\ell} + \widehat{\ell})\langle F(x^k) - F(x^*), x^k - x^*\rangle.
\end{aligned}$$

Next, we consider D^{k+1}

$$\begin{aligned}
\mathbb{E}_k\left[D^{k+1}\right] &= \sum_{i=1}^n \sum_{j=1}^m \mathbb{E}_k\left[\left\|F_{ij}(w_i^{k+1}) - F_{ij}(x^\star)\right\|^2\right] \\
&= \sum_{i=1}^n \sum_{j=1}^m \left[(1-p)\left\|F_{ij}(w_{ij}^k) - F_{ij}(x^\star)\right\|_2^2 + p\left\|F_{ij}(x^k) - F_{ij}(x^\star)\right\|_2^2\right] \\
&\overset{(19)}{\leq} D^k(1-p) + nmp\widetilde{\ell}\langle F(x^k) - F(x^*), x^k - x^*\rangle.
\end{aligned}$$

It remains put the upper bounds on D^{k+1}, H^{k+1} together and use the definition of σ_{k+1}^2:

$$\begin{aligned}
\mathbb{E}_k\left[\sigma_{k+1}^2\right] &= \frac{\mathbb{E}_k\left[H^{k+1}\right]}{n} + \frac{\mathbb{E}_k\left[D^{k+1}\right]}{nm} \\
&\leq (1-\alpha)\frac{H^k}{n} + (1+2\alpha-p)\frac{D^k}{nm} + \left(p\widetilde{\ell} + 2\alpha(\widetilde{\ell}+\widehat{\ell})\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle
\end{aligned}$$

With $\alpha \leq \frac{p}{3}$ we get $-p \leq -3\alpha$, implying

$$\begin{aligned}
\mathbb{E}_k\left[\sigma_{k+1}^2\right] &\leq (1-\alpha)\frac{H^k}{n} + (1-\alpha)\frac{D^k}{nm} + \left(p\widetilde{\ell} + 2\alpha(\widetilde{\ell}+\widehat{\ell})\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle \\
&= (1-\alpha)\sigma_k^2 + \left(p\widetilde{\ell} + 2\alpha(\widetilde{\ell}+\widehat{\ell})\right)\langle F(x^k) - F(x^*), x^k - x^*\rangle.
\end{aligned}$$

■

The above two lemmas imply that Assumption 1 is satisfied with certain parameters.

Proposition 62 (Proposition 10) *Let F be ℓ-star-cocoercive and Assumptions 4, 5, 7 hold. Suppose that $\alpha \leq \min\left\{\frac{p}{3}; \frac{1}{1+\omega}\right\}$. Then,* VR-DIANA-SGDA *satisfies Assumption 1 with*

$$A = \left(\frac{\ell}{2} + \frac{\widetilde{\ell}}{n} + \frac{\omega(\widehat{\ell}+\widetilde{\ell})}{n}\right), \quad B = \frac{2(\omega+1)}{n},$$

$$\sigma_k^2 = \frac{1}{n}\sum_{i=1}^n \left\|h_i^k - F_i(x^*)\right\|^2 + \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m \left\|F_{ij}(w_i^k) - F_{ij}(x^*)\right\|^2,$$

$$C = \left(\frac{p\widetilde{l}}{2} + \alpha(\widetilde{\ell}+\widehat{\ell})\right), \quad \rho = \alpha \leq \min\left\{\frac{p}{3}; \frac{1}{1+\omega}\right\}, \quad D_1 = D_2 = 0.$$

G.3.2. ANALYSIS OF VR-DIANA-SGDA IN THE QUASI-STRONGLY MONOTONE CASE

Applying Theorem 1 and Corollary 2 with $M = \frac{4(\omega+1)}{n\alpha}$, we get the following results.

Theorem 63 *Let F be μ-quasi strongly monotone, ℓ-star-cocoercive and Assumptions 4, 5, 7 hold. Suppose that $\alpha \leq \min\left\{\frac{p}{3}; \frac{1}{1+\omega}\right\}$ and*

$$0 < \gamma \leq \left(\ell + \frac{10(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{4(\omega+1)p\widetilde{l}}{\alpha n}\right)^{-1}.$$

Then for all $k \geq 0$ the iterates of VR-DIANA-SGDA *satisfy*

$$\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq (1 - \min\{\gamma\mu, 1/\alpha n\})^k V_0,$$

where $V_0 = \|x^0 - x^*\|^2 + \frac{4(\omega+1)\gamma^2}{n\alpha}\sigma_0^2$.

Corollary 64 *Let the assumptions of Theorem 63 hold. Then, for* $p = \frac{1}{m}$, $\alpha = \min\left\{\frac{1}{3m}, \frac{1}{1+\omega}\right\}$,

$$\gamma = \left(\ell + \frac{10(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{4(\omega+1)\max\{3m, 1+\omega\}\widetilde{\ell}}{nm}\right)^{-1}$$

and any $K \geq 0$ *we have*

$$\mathbb{E}[\|x^k - x^*\|^2] \leq V_0 \exp\left(-\min\left\{\frac{\mu}{\ell + \frac{10(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{4(\omega+1)\max\{3m,1+\omega\}\widetilde{\ell}}{nm}}, \frac{1}{6m}, \frac{1}{2(1+\omega)}\right\} K\right).$$

G.3.3. ANALYSIS OF VR-DIANA-SGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of VR-DIANA-SGDA in the monotone case.

Theorem 65 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2, 4, 5, 7 hold. Assume that*

$$0 < \gamma \leq \left(\ell + \frac{6(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{2(\omega+1)p\widetilde{l}}{\alpha n}\right)^{-1}$$

and $\alpha = \min\left\{\frac{p}{3}, \frac{1}{1+\omega}\right\}$. *Then for* $Gap_{\mathcal{C}}(z)$ *from (9) and for all* $K \geq 0$ *the iterates produced by* VR-DIANA-SGDA *satisfy*

$$\begin{aligned}\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}}\|x^0 - u\|^2\right]}{2\gamma K} \\ & + \left(3\ell + \frac{12(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{8(\omega+1)p\widetilde{l}}{\alpha n}\right)\cdot\frac{\|x^0 - x^{*,0}\|^2}{K} \\ & + \left(4 + \gamma\left(3\ell + \frac{12(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{8(\omega+1)p\widetilde{l}}{\alpha n}\right)\right)\frac{\gamma B\sigma_0^2}{\rho K}.\end{aligned}$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 66 *Let the assumptions of Theorem 65 hold. Then* $\forall K > 0$ *one can choose* $p = \frac{1}{m}$, $\alpha = \min\left\{\frac{1}{3m}, \frac{1}{1+\omega}\right\}$ *and* γ *as*

$$\begin{aligned}\gamma \quad = \quad \min\Bigg\{ & \frac{1}{3\ell + \frac{12(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{8(\omega+1)\max\{3m,1+\omega\}\widetilde{\ell}}{mn}}, \\ & \frac{\Omega_{0,\mathcal{C}}\sqrt{n}}{\Omega_{0,\mathcal{C}}\sqrt{2\max\{3m, 1+\omega\}(\omega+1)(\widetilde{\ell}+\widehat{\ell})\ell}}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\Bigg\}.\end{aligned}$$

This choice of α and γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] = \mathcal{O}\Bigg(\frac{\left(\ell + {}^{(\omega+1)(\widehat{\ell}+\widetilde{\ell})}/_{n} + {}^{(\omega+1)\max\{m,\omega\}\widetilde{\ell}}/_{mn}\right)(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2) + \ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}}^2\sqrt{\max\{m,\omega\}(\omega+1)(\widetilde{\ell}+\widehat{\ell})\ell}}{\sqrt{n}K} + \frac{\Omega_{0,\mathcal{C}}G_*}{\sqrt{K}}\Bigg).$$

Proof The proof follows from the next upper bound $\widehat{\sigma}_0^2$ for σ_0^2 with initialization $h_i^0 = F_i(x^0)$ and $w_i = x_0$

$$\begin{aligned}
\sigma_0^2 &= \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}\|F_{ij}(x^0) - F_{ij}(x^*)\|^2 + \frac{1}{n}\sum_{i=1}^{n}\|F_i(x^0) - F_i(x^*)\|^2 \\
&\leq (\widetilde{\ell}+\widehat{\ell})\langle F(x^0) - F(x^*), x^0 - x^*\rangle \\
&\leq (\widetilde{\ell}+\widehat{\ell})\|F(x^0) - F(x^*)\| \cdot \|x^0 - x^*\| \\
&\leq (\widetilde{\ell}+\widehat{\ell})\ell\|x^0 - x^*\|^2 \leq (\widetilde{\ell}+\widehat{\ell})\ell\max_{u\in\mathcal{C}}\|x^0 - u\|^2 \leq (\widetilde{\ell}+\widehat{\ell})\ell\Omega_{0,\mathcal{C}}^2.
\end{aligned}$$

Next, applying Corollary 17 with $\widehat{\sigma}_0 := \sqrt{(\widetilde{\ell}+\widehat{\ell})\ell}\Omega_{0,\mathcal{C}}$, we get the result. ■

G.3.4. ANALYSIS OF VR-DIANA-SGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of VR-DIANA-SGDA in the cocoercive case.

Theorem 67 *Let F be ℓ-cocoercive and Assumptions 1, 2, 4, 5, 7 hold. Assume that*

$$0 < \gamma \leq \left(\ell + \frac{6(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{2(\omega+1)p\widetilde{l}}{\alpha n}\right)^{-1}$$

and $\alpha = \min\left\{\frac{p}{3}, \frac{1}{1+\omega}\right\}$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by VR-DIANA-SGDA *satisfy*

$$\begin{aligned}
\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K}x^k\right)\right] &\leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0 - u\|^2\right]}{2\gamma K} \\
&\quad + \left(6\ell + \frac{18(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{12(\omega+1)p\widetilde{l}}{\alpha n}\right)\cdot\frac{\|x^0 - x^{*,0}\|^2}{K} \\
&\quad + \left(6 + \gamma\left(6\ell + \frac{18(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{12(\omega+1)p\widetilde{l}}{\alpha n}\right)\right)\frac{\gamma B\sigma_0^2}{\rho K}.
\end{aligned}$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 68 *Let the assumptions of Theorem 67 hold. Then* $\forall K > 0$ *one can choose* $p = \frac{1}{m}$, $\alpha = \min\left\{\frac{1}{3m}, \frac{1}{1+\omega}\right\}$ *and* γ *as*

$$\gamma = \min\left\{\frac{1}{6\ell + \frac{18(\omega+1)(\widehat{\ell}+\widetilde{\ell})}{n} + \frac{12(\omega+1)\max\{3m,1+\omega\}\widetilde{\ell}}{mn}}, \frac{\Omega_{0,\mathcal{C}}\sqrt{n}}{\Omega_{0,\mathcal{C}}\sqrt{2\max\{3m,1+\omega\}(\omega+1)(\widetilde{\ell}+\widehat{\ell})\ell}}\right\}.$$

This choice of α *and* γ *implies*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{\left(\ell + {}^{(\omega+1)(\widehat{\ell}+\widetilde{\ell})}/_{n} + {}^{(\omega+1)\max\{m,\omega\}\widetilde{\ell}}/_{mn}\right)\left(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2\right)}{K} + \frac{\Omega_{0,\mathcal{C}}^2\sqrt{\max\{m,\omega\}(\omega+1)(\widetilde{\ell}+\widehat{\ell})\ell}}{\sqrt{n}K}\right).$$

G.4. Discussion of the Results in the Monotone and Cocoercive Cases

Beznosikov et al. [12] also consider monotone case and derive the following rate for MASHA1 (neglecting the dependence on Lipschitz parameters and the quantities like $\Omega_{0,\mathcal{C}}^2 = \max_{u\in\mathcal{C}}\|x^0 - u\|^2$): $\mathcal{O}\left(\sqrt{(m+\omega)(1+{}^{\omega}/_{n})}\frac{1}{K}\right)$. In general, due to the term proportional to ${}^{1}/_{\sqrt{K}}$ and due to the relation between (star-)cocoercivity constants and Lipschitz constants our rate $\mathcal{O}\left(\frac{(1+\omega)}{nK} + \frac{(1+\omega)\max\{m,\omega\}}{mnK} + \frac{\sqrt{\max\{m,\omega\}(1+\omega)}}{\sqrt{n}K} + \frac{G_*}{\sqrt{K}}\right)$ our rate is worse than the one from Beznosikov et al. [12] (even when $R(x) \equiv 0$, i.e., $G_* = 0$). However, when the difference between cocoercivity and Lipschitz constants is not significant, and m, n or ω are sufficiently large, our result in the cocoercive case (Corollary 68) might be better. Moreover, we emphasize here that Beznosikov et al. [12] do not consider SGDA as the basis for their methods. To the best of our knowledge, our results are the first ones for distributed SGDA-type methods with compression derived in the monotone case without assuming (quasi-)strong monotonicity.

Appendix H. Coordinate SGDA

In this section, we focus on the coordinate versions of SGDA. To denote i-th component of the vector x we use $[x]_i$. Vectors $e_1, \ldots, e_d \in \mathbb{R}^d$ form a standard basis in $\mathbb{R}^d$.

H.1. CSGDA

Algorithm 7 CSGDA: Coordinate Stochastic Gradient Descent-Ascent

Input: starting point $x^0 \in \mathbb{R}^d$, stepsize $\gamma > 0$, number of steps K
for $k = 0$ **to** $K - 1$ **do**
  Sample uniformly at random $j \in [d]$
  $g^k = de_j[F(x^k)]_j$
  $x^{k+1} = \text{prox}_{\gamma R}\left(x^k - \gamma g^k\right)$
end for

H.1.1. CSGDA FITS ASSUMPTION 1

Proposition 69 *Let F be ℓ-star-cocoercive. Then,* CSGDA *satisfies Assumption 1 with*

$$A = d\ell, \quad D_1 = 2d \max_{x*\in X^*}\left[\|F(x^*)\|^2\right], \quad \sigma_k^2 = 0, \quad B = 0, \quad C = 0, \quad \rho = 1, \quad D_2 = 0.$$

Proof First of all, for all $a \in \mathbb{R}^d$ and for random index j uniformly distributed on $[d]$ we have $\mathbb{E}_j[\|e_j[a]_j\|^2] = \frac{1}{d}\sum_{i=1}^d [a]_j^2 = \frac{1}{d}\|a\|^2$. Using this and $g^k = de_j[F(x^k)]_j$, we derive

$$\begin{aligned}
\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &= \mathbb{E}_k\left[\|de_j[F(x^k) - F(x^{*,k})]_j + de_j[F(x^{*,k})]_j - F(x^{*,k})\|^2\right] \\
&\leq 2\mathbb{E}_k\left[\|de_j[F(x^k) - F(x^{*,k})]_j\|^2\right] + 2\mathbb{E}_k\left[\|de_j[F(x^{*,k})]_j - F(x^{*,k})\|^2\right] \\
&= 2d\|F(x^k) - F(x^{*,k})\|^2 + 2\mathbb{E}_k\left[\|de_j[F(x^{*,k})]_j - \mathbb{E}_k[de_j[F(x^{*,k})]_j]\|^2\right] \\
&\leq 2d\|F(x^k) - F(x^{*,k})\|^2 + 2\mathbb{E}_k\left[\|de_j[F(x^{*,k})]_j\|^2\right] \\
&= 2d\|F(x^k) - F(x^{*,k})\|^2 + 2d\|F(x^{*,k})\|^2. \qquad (66)
\end{aligned}$$

Finally, the star-cocoercivity of F implies

$$\begin{aligned}
\mathbb{E}_k\left[\|g^k - F(x^{*,k})\|^2\right] &\leq 2d\ell\langle F(x^k) - F(x^{*,k}), x^k - x^*\rangle + 2d\|F(x^{*,k})\|^2 \\
&\leq 2d\ell\langle F(x^k) - F(x^{*,k}), x^k - x^*\rangle + 2d \max_{x*\in X^*}\left[\|F(x^*)\|^2\right].
\end{aligned}$$

■

H.1.2. ANALYSIS OF CSGDA IN THE QUASI-STRONGLY MONOTONE CASE

Applying Theorem 1 and Corollary 2, we get the following results.

Theorem 70 *Let F be μ-quasi strongly monotone and ℓ-star-cocoercive, $0 < \gamma \leq 1/2d\ell$. Then for all $k \geq 0$*

$$\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq (1-\gamma\mu)^k \|x^0 - x^{*,0}\|^2 + \frac{2\gamma d}{\mu} \cdot \max_{x^* \in X^*} \left[\|F(x^*)\|^2\right].$$

Corollary 71 *Let the assumptions of Theorem 70 hold. Then, for any $K \geq 0$ one can choose $\{\gamma_k\}_{k\geq 0}$ as follows:*

$$\begin{aligned} \text{if } K \leq \frac{2d\ell}{\mu}, &\qquad \gamma_k = \frac{1}{2d\ell}, \\ \text{if } K > \frac{2d\ell}{\mu} \text{ and } k < k_0, &\qquad \gamma_k = \frac{1}{2d\ell}, \\ \text{if } K > \frac{2d\ell}{\mu} \text{ and } k \geq k_0, &\qquad \gamma_k = \frac{2}{\mu(4d\ell + \mu(k-k_0))}, \end{aligned}$$

where $k_0 = \lceil K/2 \rceil$. For this choice of γ_k the following inequality holds:

$$\mathbb{E}[V_K] \leq \frac{64d\ell}{\mu}\|x^0 - x^{*,0}\|^2 \exp\left(-\frac{\mu K}{2d\ell}\right) + \frac{72d}{\mu^2 K} \cdot \max_{x^* \in X^*} \left[\|F(x^*)\|^2\right].$$

H.1.3. ANALYSIS OF CSGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of CSGDA in the monotone case.

Theorem 72 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2 hold. Assume that $\gamma \leq 1/2d\ell$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by CSGDA satisfy*

$$\begin{aligned} \mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \quad \leq \quad & \frac{3\left[\max_{u\in\mathcal{C}} \|x^0 - u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + \frac{5d\ell\|x^0 - x^{*,0}\|^2}{K} \\ & + 20\gamma d \cdot \max_{x^* \in X^*} \left[\|F(x^*)\|^2\right]. \end{aligned}$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 73 *Let the assumptions of Theorem 72 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{5d\ell}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{2dK}}\right\}.$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{d\ell(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2) + \ell\Omega_{\mathcal{C}}^2}{K} + \frac{\Omega_{0,\mathcal{C}} G_*}{\sqrt{K}}\right).$$

H.1.4. ANALYSIS OF CSGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of CSGDA in the cocoercive case.

Theorem 74 *Let F be ℓ-cocoercive and Assumptions 1, 2 hold. Assume that $\gamma \leq 1/2d\ell$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* CSGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0-u\|^2\right]}{2\gamma K} + \frac{9d\ell\|x^0-x^{*,0}\|^2}{K} + 16\gamma d \cdot \max_{x*\in X^*}\left[\|F(x^*)\|^2\right].$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 75 *Let the assumptions of Theorem 74 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{9d\ell}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{2dK}}\right\}.$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{d\ell(\Omega_{0,\mathcal{C}}^2+\Omega_0^2)}{K} + \frac{\Omega_{0,\mathcal{C}}G_*}{\sqrt{K}}\right).$$

H.2. SEGA-SGDA

In this section, we consider a modification of SEGA [33] – the linearly converging coordinate method for composite optimization problems working even for non-separable regularizers.

Algorithm 8 SEGA-SGDA: SEGA Stochastic Gradient Descent-Ascent [33]

1: **Input:** starting point $x^0 \in \mathbb{R}^d$, stepsize $\gamma > 0$, number of steps K
2: Set $h^0 = 0$
3: **for** $k = 0$ **to** $K-1$ **do**
4: Sample uniformly at random $j \in [d]$
5: $h^{k+1} = h^k + e_j([F(x^k)]_j - h_j^k)$
6: $g^k = de_j([F(x^k)]_j - h_j^k) + h^k$
7: $x^{k+1} = \text{prox}_{\gamma R}\left(x^k - \gamma g^k\right)$
8: **end for**

H.2.1. SEGA-SGDA FITS ASSUMPTION 1

The following result from Hanzely et al. [33] does not rely on the fact that $F(x)$ is the gradient of some function. Therefore, it holds in our settings as well.

Lemma 76 (Lemmas A.3 and A.4 from [33]) *Let Assumption 5 hold. Then for all $k \geq 0$* SEGA-SGDA *satisfies*

$$\begin{aligned}\mathbb{E}_k\left[\|g^k - F(x^*)\|^2\right] &\leq 2d\|F(x^k) - F(x^*)\|^2 + 2d\sigma_k^2,\\ \mathbb{E}_k\left[\sigma_{k+1}^2\right] &\leq \left(1 - \frac{1}{d}\right)\sigma_k^2 + \frac{1}{d}\|F(x^k) - F(x^*)\|^2,\end{aligned}$$

where $\sigma_k^2 = \|h^k - F(x^*)\|^2$.

The lemma above implies that Assumption 1 is satisfied with certain parameters.

Proposition 77 *Let F be ℓ-star-cocoercive and Assumption 5 holds. Then,* SEGA-SGDA *satisfies Assumption 1 with $\sigma_k^2 = \|h^k - F(x^*)\|^2$ and*

$$A = d\ell, \quad B = 2d, \quad D_1 = 0, \quad C = \frac{\ell}{2d}, \quad \rho = \frac{1}{d}, \quad D_2 = 0.$$

Proof The result follows from Lemma 76 and star-cocoercivity of F. ■

H.2.2. ANALYSIS OF SEGA-SGDA IN THE QUASI-STRONGLY MONOTONE CASE

Applying Theorem 1 and Corollary 2 with $M = 4d^2$, we get the following results.

Theorem 78 *Let F be μ-quasi strongly monotone, ℓ-star-cocoercive, Assumption 5 holds, and $0 < \gamma \leq \frac{1}{6d\ell}$. Then, for all $k \geq 0$ the iterates produced by* SEGA-SGDA *satisfy*

$$\mathbb{E}\left[\|x^k - x^*\|^2\right] \leq \left(1 - \min\left\{\gamma\mu, \frac{1}{2d}\right\}\right)^k \cdot V_0,$$

where $V_0 = \|x^0 - x^*\|^2 + 4d^2\gamma^2\sigma_0^2$.

Corollary 79 *Let the assumptions of Theorem 78 hold. Then, for $\gamma = \frac{1}{6d\ell}$ and any $K \geq 0$ we have*

$$\mathbb{E}[\|x^k - x^*\|^2] \leq V_0 \exp\left(-\min\left\{\frac{\mu}{6d\ell}, \frac{1}{2d}\right\} K\right).$$

H.2.3. ANALYSIS OF SEGA-SGDA IN THE MONOTONE CASE

Next, using Theorem 3, we establish the convergence of CSGDA in the monotone case.

Theorem 80 *Let F be monotone, ℓ-star-cocoercive and Assumptions 1, 2, 5 hold. Assume that $\gamma \leq 1/6d\ell$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* SEGA-SGDA *satisfy*

$$\begin{aligned}\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq{}& \frac{3\left[\max_{u\in\mathcal{C}}\|x^0 - u\|^2\right]}{2\gamma K} + \frac{8\gamma\ell^2\Omega_{\mathcal{C}}^2}{K} + 13d\ell \cdot \frac{\|x^0 - x^{*,0}\|^2}{K}\\ &+ (4 + 13\gamma d\ell)\frac{2d\gamma\sigma_0^2}{K} + 9\gamma \cdot \max_{x^*\in X^*}\left[\|F(x^*)\|^2\right].\end{aligned}$$

Applying Corollary 17, we get the rate of convergence to the exact solution.

Corollary 81 *Let the assumptions of Theorem 80 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{13d\ell}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{2}G^* d}, \frac{\Omega_{0,\mathcal{C}}}{G_*\sqrt{K}}\right\}.$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{d\ell(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2) + \ell\Omega_{\mathcal{C}}^2}{K} + \frac{d\Omega_{0,\mathcal{C}}G_*}{K} + \frac{\Omega_{0,\mathcal{C}}G_*}{\sqrt{K}}\right).$$

Proof The proof follows from the next upper bound $\widehat{\sigma}_0^2$ for σ_0^2 with initialization $h_0 = 0$

$$\sigma_0^2 = \|h_0 - F(x^*)\|^2 = \|F(x^*)\|^2 \leq G_*^2.$$

■

H.2.4. ANALYSIS OF SEGA-SGDA IN THE COCOERCIVE CASE

Next, using Theorem 18, we establish the convergence of CSGDA in the cocoercive case.

Theorem 82 *Let F be ℓ-cocoercive and Assumptions 1, 2, 5 hold. Assume that $\gamma \leq 1/6d\ell$. Then for $Gap_{\mathcal{C}}(z)$ from (9) and for all $K \geq 0$ the iterates produced by* SEGA-SGDA *satisfy*

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] \leq \frac{3\left[\max_{u\in\mathcal{C}}\|x^0 - u\|^2\right]}{2\gamma K} + 21d\ell \cdot \frac{\|x^0 - x^{*,0}\|^2}{K} + (6 + 21\gamma d\ell)\frac{2d\gamma\sigma_0^2}{K}.$$

Applying Corollary 19, we get the rate of convergence to the exact solution.

Corollary 83 *Let the assumptions of Theorem 82 hold. Then $\forall K > 0$ one can choose γ as*

$$\gamma = \min\left\{\frac{1}{21d\ell}, \frac{\Omega_{0,\mathcal{C}}}{\sqrt{2}G^* d}\right\}.$$

This choice of γ implies

$$\mathbb{E}\left[Gap_{\mathcal{C}}\left(\frac{1}{K}\sum_{k=1}^{K} x^k\right)\right] = \mathcal{O}\left(\frac{d\ell(\Omega_{0,\mathcal{C}}^2 + \Omega_0^2)}{K} + \frac{d\Omega_{0,\mathcal{C}}G_*}{K}\right).$$

Table 3: Summary of the complexity results for zeroth-order methods with two-points feedback oracles for solving (1). By complexity we mean the number of oracle calls required for the method to find x such that $\mathbb{E}[\|x - x^*\|^2] \leq \varepsilon$. By default, operator F is assumed to be **μ-strongly monotone** and, as the result, the solution is unique. Our results rely on **μ-quasi strong monotonicity** of F (3). Methods supporting $R(x) \not\equiv 0$ are highlighted with *. Our results are highlighted in green. Notation: $q =$ the parameter depending on the proximal setup, $q = 2$ in Euclidean case and $q = +\infty$ in the ℓ_1-proximal setup; $G_* = \max_{x* \in X^*} \|F(x^*)\|$, which is zero when $R(x) \equiv 0$.

Method	Citation	Assumptions	Complexity
zoscESVIA [(1)]	[69]	F is L-Lip.[(2)]	$\widetilde{\mathcal{O}}\left(d\frac{L}{\mu}\right)$
zoVIA	[69]	F is L-Lip.[(2)]	$\widetilde{\mathcal{O}}\left(d^{2/q}\frac{L^2}{\mu^2}\right)$
CSGDA *	**This paper**	F is ℓ-cocoer.	$\widetilde{\mathcal{O}}\left(d\frac{\ell}{\mu} + \frac{dG_*^2}{\mu^2\varepsilon}\right)$
SEGA-SGDA *	**This paper**	F is ℓ-cocoer. As. 5	$\widetilde{\mathcal{O}}\left(d + d\frac{\ell}{\mu}\right)$

[(1)] The method is based on Extragradient update rule. Moreover, at each step full operator is approximated.
[(2)] The problem is defined on a bounded set.

H.3. Comparison with Related Work

The summary of rates in the (quasi-) strongly monotone case is provided in Table 3. First of all, our results are the first convergence for solving *regularized* VIPs via coordinate methods. In particular, SEGA-SGDA is the first linearly converging coordinate method for solving regularized VIPs. Next, when $q = 2$ in zoVIA from Sadiev et al. [69], i.e., Euclidean proximal setup is used, our rate for SEGA-SGDA is better than the one derived for zoVIA in Sadiev et al. [69] since $\ell \leq {}^{L^2}/_{\mu}$. Finally, zoscESVIA might have better rate, but it is based on EG and it uses approximation of each component of operator F at each iteration, which makes one iteration of the method costly.

In the monotone and cocoercive cases, our result and the results from Sadiev et al. [69] are comparable modulo the difference between (star-)cocoercivity and Lipschitz constants.