Communication-efficient distributed eigenspace estimation with arbitrary node failures

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Abstract

We develop an eigenspace estimation algorithm for distributed environments with arbitrary node failures, where a subset of computing nodes can return structurally valid but otherwise arbitrarily chosen responses. Notably, this setting encompasses several important scenarios that arise in distributed computing and data-collection environments such as silent/soft errors, outliers or corrupted data at certain nodes, and adversarial responses. Our estimator builds upon and matches the performance of a recently proposed non-robust estimator up to an additive $\tilde{O}(\sigma\sqrt{\alpha})$ error, where σ^2 is the variance of the existing estimator and α is the fraction of corrupted nodes.

1 Problem overview and background

Modern machine learning has seen the proliferation of heterogeneous distributed environments for training and deploying data science pipelines. As communication between machines is often the most time-consuming operation in distributed systems, the design of *communication-efficient algorithms* is of paramount importance for scaling to massive datasets [41]. However, the move to distributed environments also adds several additional layers of complexity in the design of algorithms. For example, in the distributed setting we would like our algorithms to be *robust* and providing meaningful answers even in settings where some nodes contain outlier data [4], silently fail during the computation [30, 35], or are compromised and returning malicious results designed to corrupt the central solution.

This work focuses on distributed *eigenspace estimation* in the context of robustness to node-level corruptions. Formally, we assume a computing environment with nodes numbered $i = 1, \ldots, m$, where every node i observes a local version A_i of an unknown symmetric matrix $A \in \mathbb{R}^{d\times d}$; the goal is to approximate the subspace spanned by the $r \ll d$ principal eigenvectors of A. **Distributed PCA** is a standard example in this framework: every machine draws n i.i.d. samples from an unknown distribution $\mathcal{P}$ with covariance matrix A and forms a local empirical covariance matrix A_i. Recently proposed communication-efficient algorithms have every node i transmit V_i, the $d \times r$ matrix of principal eigenvectors of A_i, to a central server, which then aggregates all the local solutions via a carefully-crafted aggregation procedure [10, 20].

We devise and analyze an algorithm that is robust to a wide range of corruptions that can occur to a subset of the computational nodes. In particular, we assume that some fraction α of the computational nodes can respond with completely arbitrary, but structurally valid, responses (i.e., they return arbitrary matrices V_i with orthonormal columns). This model encompasses three common forms of node-level corruption that cannot be easily detected by the central machine in isolation:

Silent/soft errors. While computational errors may be rare on single machines, as distributed workloads span large numbers of nodes the probability that some of them fail becomes significant.

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Though catastrophic failures may be detectable, allowing the central server to simply ignore the output of specific nodes, the more nefarious issue is that of so-called silent (or soft) errors [18, 21, 30]. More specifically, a silent error is one where a node returns an erroneous but structurally valid response to the central machine query. Because the response is structurally valid and the central machine may not have access to the per-node data it is not possible to "validate" the response of each node and, instead, the central estimator must be adapted to be robust to such errors.

Outliers or corrupted data. In certain settings the data collection may be distributed in addition to the computation. If some of the nodes are drawing samples from an invalid or corrupted data source they may introduce gross outliers to the set of responses $\{V_i \mid i \in [m]\}$. Similarly, in the distributed PCA example, while most machines draw a sufficient number of samples, a minority of them may have only a small amount of data available such that the principal eigenspaces of the local empirical covariance matrices are too far from the ground truth, and thus violate standard modelling assumptions in distributed learning. Again, robustness to such outlier responses must be a feature of the estimator since they cannot be detected by individual nodes (as they do not have information about the global problem).

Adversarial responses. In some settings, a subset of nodes may be compromised by an adversary who wishes to influence the central solution by crafting and returning malicious V_i. In fact, the adversarial nodes may be collaborating when constructing their responses. Since the central node does not get to see all the data it cannot validate responses or directly detect adversaries. Therefore, the estimator itself must be adapted to be robust to collections of responses designed to push the solution in specific directions.

The main contribution of our paper is a *communication-efficient algorithm that is robust to node corruptions* (as outlined above) for the distributed eigenspace estimation problem. We note that our corruption model is similar to so-called Byzantine failures [28] in distributed systems.

1.1 Related work

Distributed eigenspace estimation. The problem of distributed eigenspace estimation has been well-studied in the absence of malicious noise. One of the challenges in the distributed setting is aggregating local solutions in the presence of symmetry: for example, if v is an eigenvector of A, both $\pm v$ are valid solutions to our problem. Various works deal with such symmetries in different ways; in the algorithms of [5, 20], the central node averages the *spectral projectors* of the local eigenspaces, and performs an eigendecomposition of the resulting average to approximate the principal eigenspace. This approach is similar to the algorithms of [3, 11, 29], although the latter works focus on distributed *low-rank approximations* and do not address the issue of approximating the principal eigenspace directly. Another standard approach is for the central server to aggregate local solutions after an alignment step designed to remove the orthogonal ambiguity [10, 19, 23] (see also [7] for the non-distributed setting). Indeed, our work builds on the two-stage algorithm presented and analyzed in [10] for the non-robust setting. Finally, we briefly mention a recent line of work [12, 23] that adapts the *shift-and-invert preconditioning* framework [22] to the distributed setting; however, the latter approach leads to algorithms that require multiple rounds of communication.

Robust PCA. The literature contains a number of different formulations for robust principal component analysis. The seminal work of Candés et al. [9] formulated robust PCA as the task of separating an observed matrix $Y \in \mathbb{R}^{d \times d}$ into a low-rank and a sparse component – a slightly different problem from that considered in this paper. Xu et al. [39] considered the problem of approximating a low-dimensional distribution from a set of n i.i.d. samples, a constant fraction of which have been individually corrupted by gross outliers. Follow-up works focused on high-dimensional sparse estimation and applications to sparse robust PCA [2, 16]. However, communication-efficient robust PCA appears to be overlooked. A notable exception is the sketch-based algorithm of [34], although the assumptions therein depart significantly from our setting. Another relevant line of work is that of Byzantine-robust distributed learning (typically focusing on distributed gradient descent); see, e.g., [1, 6, 8, 13, 15, 26, 32, 40] as well as the survey [25]. In these works, an iterative algorithm is distributed across machines that send individual updates to a central server, which combines them using a robust aggregation procedure (e.g., the geometric median [32]). Such works are more general in scope, but typically lead to estimators that require multiple rounds of communication. For minimization of strongly convex quadratics, [40] proposes a one-shot algorithm employing a robust

aggregation scheme. However, this special case departs from our setting as their underlying problem is convex and solved under the assumption that solutions are unique.

1.2 Notation

We let $\mathbb{S}^{d-1}$ denote the unit sphere in d dimensions. We write $\|A\|_{\mathrm{F}} := \sqrt{\langle A, A\rangle}$ and $\|A\|_2 := \sup_{x\in\mathbb{S}^{d-1}} \|Ax\|_2$ for the Frobenius and spectral norms of a matrix $A \in \mathbb{R}^{n\times d}$. We write $\mathbb{O}_{n,r}$ for the set of $n \times r$ matrices with orthonormal columns and $\mathbb{O}_r \equiv \mathbb{O}_{r,r}$. Given $U, V \in \mathbb{O}_{n,r}$ we write

$$\operatorname{dist}(U, V) := \left\|(I - UU^{\mathsf{T}})V\right\|_2 = \left\|(I - VV^{\mathsf{T}})U\right\|_2 \tag{1}$$

for their $\ell_2 \to \ell_2$ subspace distance and $\mathcal{B}_{\mathrm{dist}}(U; r)$ for the scaled unit ball centered at U:

$$\mathcal{B}(U; r) := \{V \in \mathbb{O}_{d,r} \mid \operatorname{dist}(U, V) \le r\}.$$

Finally, we use the notation $A \lesssim B$ to indicate that $A \le cB$ for a dimension-independent constant $c > 0$ and $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$ simultaneously.

2 Robust distributed eigenspace estimation

We now formally introduce the problem setting. In particular, we assume there exists an *unknown* symmetric matrix A with spectral decomposition

$$A = V\Lambda V^{\mathsf{T}} + V_\perp \Lambda_\perp V_\perp^{\mathsf{T}}, \quad V \in \mathbb{O}_{n,r},\ \Lambda = \operatorname{diag}(\{\lambda_i(A)\}_{i=1}^r),\ \Lambda_\perp = \operatorname{diag}(\{\lambda_i(A)\}_{i=r+1}^d), \tag{2}$$

assuming a nonincreasing ordering on the eigenvalues:

$$\lambda_1(A) \ge \cdots \ge \lambda_r(A) > \lambda_{r+1}(A) \ge \cdots \ge \lambda_d(A).$$

Our goal is to approximate the principal r-dimensional eigenspace $\mathcal{V} := \operatorname{span}(V)$ of A given m machines, each of which observes a local version A_i of A, communicating with a central coordinator. We assume that m is even for simplicity. When queried for a response, machine i responds either with an eigenvector matrix spanning the principal eigenspace of the local matrix A_i, or with an arbitrary $d \times r$ matrix with orthonormal columns. The latter case corresponds to so-called *compromised* machines. In contrast, prior work [10, 20] assumes that every machine responds truthfully.

Assumption 1 (Corruption model)**.** There exists a constant $\alpha \in (0, 1/2)$ and an index set $\mathcal{I}_{\mathsf{bad}} \subset [m]$ with $|\mathcal{I}_{\mathsf{bad}}|/m \le \alpha$ such that the following holds: all nodes $i \notin \mathcal{I}_{\mathsf{bad}}$ observe a symmetric matrix $A_i \in \mathbb{R}^{d\times d}$. Moreover, when queried for a response, every node i returns

$$\widehat{V}_i = \begin{cases} V_i, & i \in [m] \setminus \mathcal{I}_{\mathsf{bad}}, \\ Q_i, & i \in \mathcal{I}_{\mathsf{bad}}, \end{cases} \tag{3}$$

where the columns of $V_i \in \mathbb{O}_{d,r}$ span the principal r-dimensional eigenspace of A_i and $Q_i \in \mathbb{O}_{d,r}$ is an arbitrary $d \times r$ matrix with orthonormal columns.

For notational convenience, we also define the set of "good" responses:

$$\mathcal{I}_{\mathsf{good}} := [m] \setminus \mathcal{I}_{\mathsf{bad}}, \quad \text{with} \quad |\mathcal{I}_{\mathsf{good}}| \ge (1-\alpha)m. \tag{4}$$

Furthermore, we require the principal eigenspace of A to be sufficiently separated from its complement and that the local errors $E_i := A_i - A$ are not too large.

Assumption 2. There is a constant $\delta > 0$ such that the following hold:

1. **(Gap)** The matrix A has a nontrivial eigengap:

$$\delta_r(A) := \lambda_r(A) - \lambda_{r+1}(A) \ge \delta. \tag{5}$$

2. **(Approximation)** For all $i \in \mathcal{I}_{\mathsf{good}}$, the local observations satisfy:

$$\|A_i - A\|_2 \le \frac{\delta_r(A)}{8}. \tag{6}$$

The problem admits a natural proxy of difficulty in the form of the *normalized inverse eigengap* κ:

$$(\textbf{Normalized inverse eigengap}) \qquad \kappa := \frac{\|A\|_2}{\delta_r(A)} = \frac{\lambda_1(A)}{\lambda_r(A) - \lambda_{r+1}(A)}. \tag{7}$$

Our algorithm for the robust distributed eigenspace estimation problem is outlined in Algorithm 1, which is essentially a robust version of the Procrustes fixing algorithm from [10]. The latter (non-robust) algorithm operates as follows: first, every machine i computes its local eigenvector matrix V_i and broadcasts it to the central server. Because invariant subspaces do not admit unique representations, naively averaging these estimates can fail to reduce the approximation error further. Instead, the algorithm of [10] first picks one of the local solutions (say V_1) as a reference and "aligns" every other solution with it by solving a so-called *Procrustes problem*:

$$Z_i := \operatorname*{argmin}_{U \in \mathbb{O}_r} \|V_i U - V_1\|_{\mathrm{F}}, \quad i = 2, \ldots, m. \tag{8}$$

After the alignment step (8), the solution of which is available in closed form via the SVD [24], the central coordinator computes and returns the empirical average $(1/m)\sum_{i=1}^m V_i Z_i$.

2.1 Technical challenges and main result

To robustify the algorithm described above against node failures, we need the following ingredients:

Reference estimation. In the presence of corruptions one must guard against the possibility of choosing an outlier as a reference solution (which would render the alignment step (8) useless). The first step of our algorithm robustly determines a reference guaranteed to have nontrivial alignment with the ground truth (Algorithm 2).

Solution aggregation. With the robust reference at hand, the next step of the algorithm aligns other local solutions with it. However, since some of the solutions are outliers, we use a robust mean estimation algorithm in the last step of Algorithm 1 to compute the empirical average only over inliers (and possible "benign" outliers) with high probability.

Combining the above ingredients requires working around additional technical challenges. The first challenge is showing that averaging over inliers that have been aligned with $\widehat{V}_{\mathsf{ref}}$ is equivalent (up to small error) to averaging over inliers that have been aligned with the ground truth V. To do so, we leverage a path-independence result from [37] to view the aligned estimates as the eigenvectors arising from a carefully chosen sequence of perturbations to the unknown matrix A (Lemma 1).

After aligning local estimates, we approximate the empirical mean over inliers with a spectral filtering algorithm. Existing analyses of that algorithm focus on vector-valued inputs and lead to error bounds in the Euclidean norm (or other vector norms, see [36]). Invoking these error bounds in a black-box fashion for matrix-valued inputs (by flattening every eigenvector matrix to a (dr)-dimensional vector) has multiple drawbacks. On one hand, it leads to unnatural scaling as the final result depends on the empirical covariance over flattened inputs. On the other hand, the resulting error bounds are with respect to the Frobenius norm, while the spectral norm is a standard error measure in eigenspace estimation. To avoid this, we modify the iterative filtering algorithm so that inputs are not flattened and obtain error bounds with respect to the spectral norm (3).

We analyze each ingredient of Algorithm 1 separately, in Sections 2.2 to 2.4; all proofs appear in the supplementary material. Our analysis is almost completely deterministic: indeed, the only source of randomness is the filtering algorithm used in the final stage (Algorithm 5).

Algorithm 1 Robust distributed eigenspace estimation

Input: responses $\{\widehat{V}_i\}_{i=1,\ldots,m}$, corruption fraction α, failure prob. p, error parameter ω.

$\widehat{V}_{\mathsf{ref}} := \texttt{RobustReferenceEstimator}\left(\widehat{V}_1, \ldots, \widehat{V}_m\right)$. ▷ Algorithm 2; Section 2.2

$\{\widetilde{V}_i\}_{i=1,\ldots,m} := \texttt{ProcrustesFixing}\left(\{\widehat{V}_1, \ldots, \widehat{V}_m\}, \widehat{V}_{\mathsf{ref}}\right)$ ▷ Algorithm 3; Section 2.3

$\bar{V} := \texttt{AdaptiveFilter}(\{\widetilde{V}_1, \ldots, \widetilde{V}_m\}, 6, \omega, p, \alpha)$ ▷ Algorithm 5; Section 2.4

return $\bar{V}$

Our main theorem on the performance of Algorithm 1 follows below. The result is first stated for inputs from a generic eigenspace estimation problem. In Section 3, we apply it in a black-box fashion to specialize its results for distributed PCA.

Theorem 1. *Let Assumptions 1 and 2 hold and suppose that the corruption level α satisfies*

$$\varphi := \alpha + \frac{6\log(1/p)}{m} < \frac{1}{12}. \tag{9}$$

For a fixed failure probability p and error parameter ω, Algorithm 1 returns $\bar{V} \in \mathbb{R}^{d\times r}$ satisfying:

$$\operatorname{dist}(\bar{V}, V) \lesssim \underbrace{\frac{1}{\delta}\left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i\in\mathcal{I}_{\mathsf{good}}} A_i - A \right\|_2}_{E_{\mathsf{oracle}}} + \underbrace{\frac{\kappa^2}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i\in\mathcal{I}_{\mathsf{good}}} \left(\frac{\|A_i - A\|_2}{\delta} \right)^2}_{E_{\mathsf{high}}} + \underbrace{\sqrt{\varphi \max(\omega, \sigma^2)}}_{E_{\mathsf{robust}}}. \tag{10}$$

with probability at least $1 - 2\log(6/\omega)\cdot p$. Moreover, the variance σ^2 satisfies

$$\sigma^2 \le \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i\in\mathcal{I}_{\mathsf{good}}} V_i V_i^{\mathsf{T}} - VV^{\mathsf{T}} \right\|_2 + 2\cdot \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i\in\mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V \right\|_2. \tag{11}$$

Remark 1 (Error bound interpretation)**.** *The error bound (10) decomposes naturally into three terms that can be controlled individually in applications. The first term, E_{oracle}, corresponds to an oracle estimator that approximates V via the principal eigenspace of $1/|\mathcal{I}_{\mathsf{good}}| \sum_{i\in\mathcal{I}_{\mathsf{good}}} A_i$. The second term, E_{high}, represents high-order errors that occur as a result of the alignment step in Algorithm 3 that are dominated by E_{oracle} in "typical" situations. Finally, the term E_{robust} quantifies the effect of outliers, vanishing as the fraction of corrupted nodes $\alpha \downarrow 0$. We comment on the scaling of E_{robust} relative to the error of the non-robust algorithm in the context of distributed PCA in Section 3.*

2.2 The robust reference estimator

This section focuses on the analysis of Algorithm 2, which yields the robust reference estimator $\widehat{V}_{\mathsf{ref}}$ used to remove the orthogonal ambiguity from local solutions. We note that the construction of the estimator dates back to the seminal work of Nemirovski and Yudin [31].

Algorithm 2 `RobustReferenceEstimator`$(Y_1, \ldots, Y_m)$

for $i = 1, \ldots, m$ **do**
  $\varepsilon_i := \min\{r \ge 0 \mid |\mathcal{B}_{\mathsf{dist}}(Y_i; r) \cap \{Y_i\}_{i=1}^m| > \frac{m}{2}\}$
return $Y_{i_\star}$, where $i_\star := \operatorname{argmin}_{i\in[m]} \varepsilon_i$

Remark 2. *The quantities ε_i in Algorithm 2 can be found in time $O(m^2 d r^2)$ by first computing $r_j := \operatorname{dist}(Y_i, Y_j)$ for all $j \ne i$ and setting $\varepsilon_i :=$ `median`$(\{r_j\}_{j\ne i})$.*

Note that even though $\widehat{V}_{\mathsf{ref}}$ could be chosen among some of the compromised samples, its construction ensures that it inherits the accuracy of the majority of the responses.

Proposition 1 (Robust reference estimator)**.** *Given a sample $\{Y_1, \ldots, Y_m\}$ where $Y_i \in \mathbb{O}_{d,r}$ and $|\{i \in [m] \mid \operatorname{dist}(Y_i, V) \le \varepsilon\}| > m/2$ for a fixed $\varepsilon > 0$, Algorithm 2 outputs $Y_{i_\star}$ satisfying*

$$\operatorname{dist}(Y_{i_\star}, V) \le 3\varepsilon. \tag{12}$$

2.3 The `ProcrustesFixing` algorithm

In this section, we formally introduce the Procrustes-fixing procedure and show that it properly aligns all the non-compromised responses given the reference solution described in Section 2.2. The procedure is described in Algorithm 3; it accepts a set of $d \times r$ matrices with orthonormal columns as well as a reference matrix Y_{ref} of the same shape.

The work [10] provides an error bound for the `ProcrustesFixing` algorithm under idealized conditions; namely, that the reference solution is equal to the ground truth V.

Algorithm 3 $\texttt{ProcrustesFixing}(\{Y_1, \ldots, Y_m\}, Y_{\mathsf{ref}})$

for $i = 1, \ldots, m$ **do**
  $\widetilde{Y}_i := Y_i Z_i, \quad \text{where} \quad Z_i := \operatorname{argmin}_{Z \in \mathbb{O}_r} \|Y_i Z - Y_{\mathsf{ref}}\|_{\mathrm{F}}$ ▷ Procrustes alignment
return $\{\widetilde{Y}_i \mid i \in [m]\}$

Theorem 2 (Theorem 2 in [10])**.** *Let Assumption 2 hold and let*

$$\tilde{V} := \frac{1}{|S|} \sum_{i \in S} \widetilde{V}_i, \quad \textit{where} \quad \{\widetilde{V}_i\}_{i \in S} = \texttt{ProcrustesFixing}(\{V_i\}_{i \in S}, V), \; S \subset \mathcal{I}_{\mathsf{good}}.$$

Then the following bound holds:

$$\left\|\tilde{V} - V\right\|_2 \lesssim \frac{1}{\delta^2} \frac{1}{|S|} \sum_{i \in S} \|A_i - A\|_2^2 + \frac{1}{\delta} \left\| \frac{1}{|S|} \sum_{i \in S} A_i - A \right\|_2. \tag{13}$$

While the setting of Theorem 2 is idealized, when the reference chosen by Algorithm 2 is sufficiently close to V one would expect that the aligned estimates are not far from their ideal version. The next Lemma shows that aligning the local solutions with $\widehat{V}_{\mathsf{ref}}$ is equivalent to aligning with the ground truth V, up to higher-order errors.

Lemma 1. *Let $V_i \in \mathbb{O}_{d,r}$ span the principal r-dimensional eigenspace of the matrix A_i and let $V \in \mathbb{O}_{d,r}$ span the principal r-dimensional invariant subspace of A. Suppose that there is a $V_{\mathsf{ref}} \in \mathbb{O}_{d,r}$ satisfying $\operatorname{dist}(V_{\mathsf{ref}}, V) = \varepsilon < \delta_r(A)/8$, and define the sets of aligned estimates*

$$V_i^{\mathsf{ideal}} := V_i \cdot \operatorname*{argmin}_{Z \in \mathbb{O}_r} \|V_i Z - V\|_{\mathrm{F}}, \quad V_i^{\mathsf{corr}} := V_i \cdot \operatorname*{argmin}_{Z \in \mathbb{O}_r} \|V_i Z - V_{\mathsf{ref}}\|_{\mathrm{F}}.$$

Then for any $i \in \mathcal{I}_{\mathsf{good}}$ the following holds:

$$\left\|V_i^{\mathsf{ideal}} - V_i^{\mathsf{corr}}\right\|_2 \lesssim \frac{1}{\delta^2} \max\left\{\|A_i - A\|_2^2, \|A\|_2^2 \varepsilon^2\right\}. \tag{14}$$

Putting everything together, we arrive at a *deterministic* characterization of the error attained by the empirical average over any subset of responses that come from non-compromised nodes and have been aligned with the robust reference estimator. Note that this characterization does not immediately translate to an algorithm, since the set of compromised nodes is not known a-priori.

Proposition 2 (Error of clean samples)**.** *Let $\widehat{V}_{\mathsf{ref}}$ be the output of Algorithm 2 given inputs $\widehat{V}_1, \ldots, \widehat{V}_m$. For any index set $S \subset \mathcal{I}_{\mathsf{good}}$ and $i \in S$, define*

$$V_i^{\mathsf{corr}} := V_i \cdot \operatorname*{argmin}_{Z \in \mathbb{O}_r} \left\|V_i Z - \widehat{V}_{\mathsf{ref}}\right\|_{\mathrm{F}}; \quad V_i^{\mathsf{ideal}} := V_i \cdot \operatorname*{argmin}_{Z \in \mathbb{O}_r} \|V_i Z - V\|_{\mathrm{F}}.$$

Suppose that $\operatorname{dist}(V, \widehat{V}_{\mathsf{ref}}) = \varepsilon < \delta_r(A)/8$. Then the following bound holds:

$$\left\| \frac{1}{|S|} \sum_{i \in S} V_i^{\mathsf{corr}} - V \right\|_2 \lesssim \frac{1}{\delta^2 |S|} \sum_{i \in S} \max\left(\|A_i - A\|_2^2, \|A\|_2^2 \varepsilon^2\right) + \frac{1}{\delta} \left\| \frac{1}{|S|} \sum_{i \in S} A_i - A \right\|_2. \tag{15}$$

Proof. From the triangle inequality, Lemma 1 and Theorem 2 it follows that

$$\begin{aligned}
\left\| \frac{1}{|S|} \sum_{i \in S} V_i^{\mathsf{corr}} - V \right\|_2 &= \left\| \frac{1}{|S|} \sum_{i \in S} V_i^{\mathsf{corr}} - V_i^{\mathsf{ideal}} + V_i^{\mathsf{ideal}} - V \right\|_2 \\
&\leq \left\| \frac{1}{|S|} \sum_{i \in S} V_i^{\mathsf{corr}} - V_i^{\mathsf{ideal}} \right\|_2 + \left\| \frac{1}{|S|} \sum_{i \in S} V_i^{\mathsf{ideal}} - V \right\|_2 \\
&\lesssim \frac{1}{\delta^2 |S|} \sum_{i \in S} \max\left(\|A_i - A\|_2^2, \|A\|_2^2 \varepsilon^2\right) + \frac{1}{\delta} \left\| \frac{1}{|S|} \sum_{i \in S} A_i - A \right\|_2.
\end{aligned}$$

□

2.4 Analysis of robust mean estimation

We now analyze the last phase of the algorithm, which approximates V using the robust mean of the aligned samples. The mean estimation procedure used is the randomized iterative filtering method shown in Algorithm 4, which is an appropriate modification of the spectral filtering algorithm from the robust statistics literature for matrix-valued inputs. Its analysis is available in the Appendix, while Theorem 3 below summarizes its guarantees.

Algorithm 4 `Filter`$(S := \{X_i\}_{i=1,\dots,m}, \lambda_{\mathsf{ub}})$

Compute empirical mean and covariance:

$$\theta_S := \frac{1}{|S|}\sum_{i\in S} X_i, \quad \Sigma_S := \frac{1}{|S|}\sum_{i\in S}(X_i-\theta_S)(X_i-\theta_S)^\mathsf{T}.$$

Compute leading eigenpair (λ, v) of Σ_S.
if $\lambda < 18\lambda_{\mathsf{ub}}$ **then**
  return θ_S
else
  Compute outlier scores $\tau_i := v^\mathsf{T}(X_i-\theta_S)(X_i-\theta_S)^\mathsf{T}v$ for $i \in S$.
  Sample Z from S following $\mathbb{P}(Z = X_i) = \frac{\tau_i}{\sum_{j\in S}\tau_j}$.
  return `Filter`$(S \setminus \{Z\}, \lambda_{\mathsf{ub}})$

Theorem 3. *Suppose $G_0 \subset [m]$ and that the corruption level α and failure probability p satisfy*

$$\alpha + \frac{6\log(1/p)}{m} \le \frac{1}{12} \quad \text{and} \quad |G_0| \ge (1-\alpha)m. \tag{16}$$

Moreover, let $\Sigma_{G_0} := \frac{1}{|G_0|}\sum_{i\in G_0}(X_i-\theta_{G_0})(X_i-\theta_{G_0})^\mathsf{T}$, where $\theta_{G_0} := \frac{1}{|G_0|}\sum_{i\in G_0} X_i$, and fix $\lambda_{\mathsf{ub}} \ge \|\Sigma_{G_0}\|_2$. Then Algorithm 4 invoked with $\{X_i\}_{i=1}^m$ and λ_{ub} returns an estimate $\theta_{\lambda_{\mathsf{ub}}}$ satisfying

$$\mathbb{P}\left(\left\|\theta_{\lambda_{\mathsf{ub}}} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \ge 18\sqrt{5\lambda_{\mathsf{ub}}}\left(\alpha + \frac{4\log(1/p)}{m}\right)^{1/2}\right) \le p. \tag{17}$$

Remark 3. *The upper bound $\alpha < 1/12$ appears to be a proof artifact and could likely be improved by optimizing choices of constants in the proof of Theorem 3. Numerical evidence in Section 3.1 suggests that the breakdown point of Algorithm 4 is closer to the natural limit of $1/2$.*

The error in (17) scales with the upper bound λ_{ub}, which may be far from the "optimal" $\|\Sigma_{G_0}\|_2$. We describe an adaptive version of Algorithm 4 that achieves this at a logarithmic additional cost. Indeed, suppose an upper bound on α is available and the unknown parameter $\|\Sigma_{G_0}\|_2$ lies in an interval $[\lambda_{\mathsf{lb}}, \lambda_{\mathsf{ub}}]$. We construct a search grid $\mathcal{G}$ as follows:

$$\mathcal{G} := \left\{2^j \mid j \in \{j_{\mathsf{lo}}, j_{\mathsf{hi}}\}\right\}, \quad j_{\mathsf{lo}} := \lfloor \log_2(\lambda_{\mathsf{lb}})\rfloor, \quad j_{\mathsf{hi}} := \lceil \log_2(\lambda_{\mathsf{ub}})\rceil. \tag{18}$$

We are now in good shape to describe our estimator. To simplify notation, we define the error proxy

$$f(\lambda; p, \alpha) := 18\sqrt{5\lambda}\left(\alpha + \frac{4\log(1/p)}{m}\right)^{1/2}. \tag{19}$$

Our estimator, $\theta_{\hat\lambda}$, is implemented in Alg. 5 and defined as:

$$\theta_{\hat\lambda}, \text{ where } \hat\lambda := \min\left\{\lambda \in \mathcal{G} \mid \|\theta_\lambda - \theta_{\lambda'}\|_2 \le f(\lambda; p, \alpha) + f(\lambda'; p, \alpha),\ \forall \lambda' \in \mathcal{G} \cap [\lambda, \infty)\right\}. \tag{20}$$

Algorithm 5 `AdaptiveFilter`$(S = \{X_i\}_{i=1,\dots,m}, \lambda_{\mathsf{ub}}, \lambda_{\mathsf{lb}}, p, \alpha)$

Set up search grid: $j_{\mathsf{lo}} := \lfloor \log_2 \lambda_{\mathsf{lb}}\rfloor$, $j_{\mathsf{hi}} := \lceil \log_2 \lambda_{\mathsf{ub}}\rceil$.
for $j = j_{\mathsf{hi}}, \dots, j_{\mathsf{lo}}$ **do**
  $\theta_{2^j} =$ `Filter`$(S, 2^j)$ ▷ Algorithm 4
  if $\exists k > j$ such that $\|\theta_{2^j} - \theta_{2^k}\| > f(2^j; p, \alpha) + f(2^k; p, \alpha)$ **then** ▷ f defined in (19)
    return $\theta_{2^{j+1}}$
return $\theta_{2^{j_{\mathsf{lo}}}}$

If $\|\Sigma_{G_0}\|_2 \in [\lambda_{\mathsf{lb}}, \lambda_{\mathsf{ub}}]$, the estimator attains the optimal error up to a constant while the success probability degrades only logarithmically, as shown by the following Proposition.

Proposition 3. *If G_0, p, and α satify (16) and $\|\Sigma_{G_0}\|_2 \le \lambda_{\mathsf{ub}}$, the estimator $\theta_{\hat{\lambda}}$ from (20) satisfies*

$$\left\|\theta_{\hat{\lambda}} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \le 171\sqrt{\max\left\{\|\Sigma_{G_0}\|_2, \lambda_{\mathsf{lb}}\right\}}\left(\alpha + \frac{4\log(1/p)}{m}\right)^{1/2}$$

with probability at least $1 - 2\log_2\left(\lambda_{\mathsf{ub}}/\lambda_{\mathsf{lb}}\right)p$.

2.5 Proof sketch of main theorem

We briefly sketch the proof of Theorem 1 here. We decompose

$$\operatorname{dist}(\bar{V}, V) \lesssim \|\bar{V} - V\|_2 \le \underbrace{\left\|\bar{V} - \frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}}\widetilde{V}_i\right\|_2}_{\Delta_1} + \underbrace{\left\|\frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}}\widetilde{V}_i - V\right\|_2}_{\Delta_2}$$

The error Δ_1 can be directly controlled by applying Proposition 3 with $G_0 \equiv \mathcal{I}_{\mathsf{good}}$ combined with the fact that the spectral norm of the matrix $\Sigma_{\mathcal{I}_{\mathsf{good}}}$ admits the upper bound in (21); we refer the reader to the supplement for a complete derivation.

$$\|\Sigma_{\mathcal{I}_{\mathsf{good}}}\|_2 \le \left\|\frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}} V_iV_i^{\mathsf{T}} - VV^{\mathsf{T}}\right\|_2 + 2\left\|\frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}}\widetilde{V}_i - V\right\|_2 \tag{21}$$

Finally, we control the error Δ_2 by invoking Proposition 2 with $S = \mathcal{I}_{\mathsf{good}}$, since

$$\widetilde{V}_i = V_i \cdot \operatorname*{argmin}_{Z\in\mathbb{O}_r}\left\|V_iZ - \widehat{V}_{\mathsf{ref}}\right\|_{\mathrm{F}}, \quad \text{for all } i \in \mathcal{I}_{\mathsf{good}}.$$

Combining the resulting upper bounds yields the error in Theorem 1. □

Remark 4. *Both of the terms in (21) are typically small and can be directly controlled in concrete applications such as distributed PCA. Moreover, while the bound in (21) is not directly computable, it is immediate that $\|\Sigma_{\mathcal{I}_{\mathsf{good}}}\|_2 \lesssim 1$ and thus we may initialize Algorithm 5 with $\lambda_{\mathsf{ub}} = O(1)$ in the absence of a finer upper bound.*

3 Robust distributed PCA

In this section, we specialize the results of Section 2 to robust distributed PCA for subgaussian distributions. We first formalize the sampling model for the problem.

Assumption 3 (Subgaussian data)**.** Every machine $i \in \mathcal{I}_{\mathsf{good}}$ draws $\{X_j^{(i)}\}_{j=1}^n \overset{\text{iid}}{\sim} \mathcal{P}$, where $\mathcal{P}$ is a zero-mean, subgaussian distribution with covariance matrix $A := \mathbb{E}_{X\sim\mathcal{P}}\left[XX^{\mathsf{T}}\right]$, and forms $A_i := \frac{1}{n}\sum_{j=1}^n X_j^{(i)}(X_j^{(i)})^{\mathsf{T}}$.

Our main theorem follows directly from Theorem 1 and control of $\|\Sigma_{\mathcal{I}_{\mathsf{good}}}\|_2$ under Assumption 3.

Theorem 4. *Let Assumptions 1 to 3 hold and suppose that $n \gtrsim \kappa^2 \cdot (r_\star + \log(mn/p))$ and α, m and p satisfy (9). Then Algorithm 1 initialized with $\omega = \sqrt{1/mn}$ returns a $\bar{V}$ satisfying*

$$\operatorname{dist}(\bar{V}, V) \lesssim \sqrt{\varrho\left(\alpha + \frac{\log(1/p)}{m}\right)} + \kappa\sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}} + \kappa^4 \cdot \frac{r\left(r_\star + \log(mn\log(mn)/p)\right)}{n}, \tag{22}$$

with probability at least $1 - 2/n - p$. Here, $r_\star := \operatorname{Tr}(A)/\|A\|_2$ and ϱ is given by

$$\varrho := \kappa\sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}} + \max\left\{\kappa^2\sqrt{r}, \kappa^4\right\} \cdot \frac{r_\star + \log(mn\log(mn)/p)}{n}.$$

When $\kappa \asymp 1$, high-order terms in Theorem 4 can be discarded and we arrive at the following:

Corollary 1. *Assume that the conditions of Theorem 4 hold and $\kappa \asymp 1$. Then:*

$$\operatorname{dist}(\bar{V}, V) \lesssim \sqrt{\alpha + \frac{\log(\log(mn)/p)}{m}} \cdot \left(\frac{r(r_\star + \log(n))}{(1-\alpha)mn}\right)^{1/4} + \sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}}$$

with probability at least $1 - 2/n - p$.

We briefly compare the error of Algorithm 1 to that of its non-robust counterpart from [10] when $\kappa \asymp 1$. The latter algorithm returns an estimate $\bar{V}^{\mathsf{nonrobust}}$ satisfying $\operatorname{dist}(\bar{V}^{\mathsf{nonrobust}}, V) = \tilde{\mathcal{O}}\left(\sqrt{r_\star/mn}\right)$. Ignoring the $\sqrt{r}$ factors, which are likely an artifact of our proof, our algorithm also introduces an additive error of the order $\tilde{\mathcal{O}}(\sqrt{\alpha/1-\alpha} \cdot (r_\star r/mn)^{1/4})$. Note that for a constant absolute number of corruptions $\alpha \propto 1/m$, this additive factor scales as

$$\sqrt{\frac{\alpha}{1-\alpha}} \left(\frac{r_\star r}{mn}\right)^{1/4} \lesssim \left(\frac{r_\star r}{m^3 n}\right)^{1/4}.$$

If m and n are comparable, this is similar to the error of the non-robust algorithm up to an $(r/r_\star)^{1/4}$ factor. Therefore, the performance of Algorithm 1 degrades gracefully as a function of the corruption level under not too restrictive assumptions on the ratio m/n.

3.1 Numerical study

We provide a brief numerical illustration of the performance of Algorithm 1 on data sampled from an unknown Gaussian distribution $\mathcal{D} := \mathcal{N}(0, V\Lambda V^\mathsf{T} + V_\perp \Lambda_\perp V_\perp^\mathsf{T})$, where $[V \quad V_\perp] \in \mathbb{O}_d$ is a random $d \times d$ orthogonal matrix and $\Lambda, \Lambda_\perp$ are generated according to the following model:

$$\Lambda = I_r, \quad (\Lambda_\perp)_{jj} = (1-\delta)\eta^j, \quad j = 1, \ldots, d-r, \quad \text{where} \quad \eta = 1 - \frac{1-\delta}{r_\star - r} \in (0,1). \tag{23}$$

We simulate an adversary by replacing the first $\lfloor \alpha m \rfloor$ responses by the same $V_{\mathsf{adv}} \in \mathbb{O}_{d,r}$, chosen to be near-orthogonal to V. We fix the gap $\delta = 0.25$ throughout. We compare Alg. 1 (labelled `Robust` in our plots) against two baselines: the algorithm from [10] (labelled `Naive`), which corresponds to Alg. 3 using the first response – which is always corrupted in our experiment – as the reference followed by naive averaging; and a version of Alg. 1 without the robust mean estimation step (labelled `Procrustes`). Our implementation always removes the sample with the largest outlier score in each step of Alg. 4 and uses a simplified error proxy $f(\lambda; \alpha) := \sqrt{\lambda\alpha}$ instead of (19) in Alg. 5.

Our experiment is illustrated in Figure 1. Clearly, the baseline methods break down in the presence of corruption, yielding solutions nearly orthogonal to V as α approaches $1/2$. In contrast, the error of Alg. 1 degrades gracefully with α. We note that our algorithm yields a nontrivial solution even when almost half of the measurements are corrupted ($\alpha = 45\%$), in line with intuition suggesting that $\alpha_\star = 1/2$ is a natural breakdown point for outlier-robust algorithms.

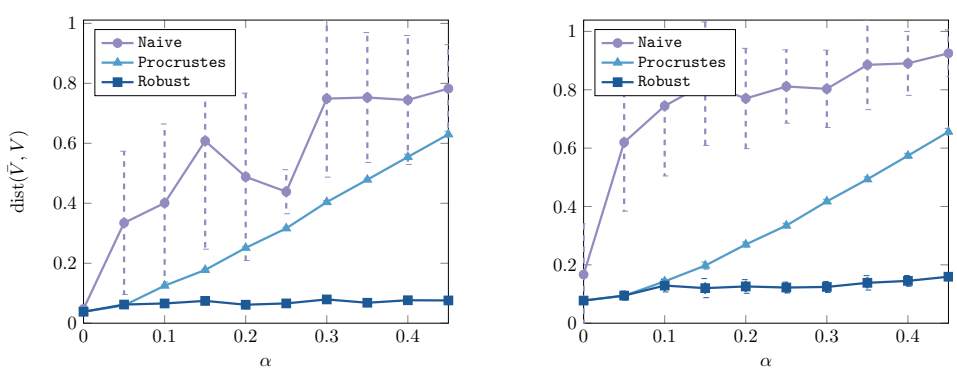

Figure 1: Robust distributed PCA with $m = 150$, $n = 50r$, $r_\star = 2r$, and $\kappa = 5$ under Model (23). We report the mean subspace distance $\pm$ one standard deviation over 10 independent runs for subspace dimension $r = 5$ (**left**) and $r = 10$ (**right**).

In addition, we study the effect of the number of per-machine samples n and total number of machines m in distributed PCA. We fix $r = 5$, $r_\star = 2r$, $\kappa = 5$ and $\delta = 0.25$ and experiment with

$$(m,n) = \left\{(32, 2^i) \mid i = 5, \ldots, 10\right\}, \quad (m,n) = \left\{(2^i, 128) \mid i = 4, \ldots, 9\right\}, \quad \alpha \in \{0.25, 0.45\}.$$

The results can be found in Fig. 2 and are consistent with our theory: for fixed m, a small number of samples per machine n leads to inaccurate local solutions and inaccurate final estimates for all variants, while increasing n leads to a decrease in error for all configurations. On the other hand, the robust algorithm produces good estimates for all attempted values of m after fixing a sufficiently large sample size n, and increasing m only decreases the error for the robust variant. Finally, we observe that the error of the non-robust variants plateaus since the corruption fraction α is bounded away from zero.

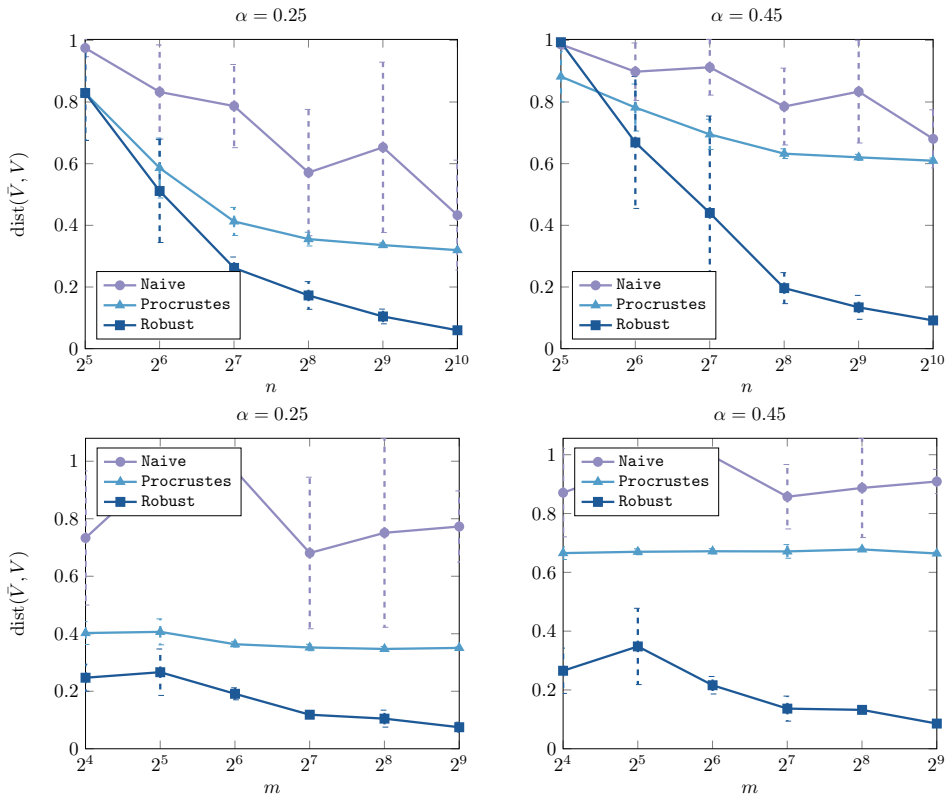

Figure 2: Top: Effect of sample size n for fixed number of machines $m = 32$. **Bottom**: Effect of number of machines m for fixed sample size $n = 128$.

4 Discussion

We presented a communication-efficient algorithm for distributed eigenspace estimation that is robust to $\alpha = 1/12$ fraction of nodes returning structurally valid but otherwise potentially adversarial responses. Numerical evidence suggests its breakdown point is closer to the (optimal) $\alpha_* = 1/2$, which might be achievable by an improved analysis of the filtering procedure in Alg 4. Our adaptive version of the filtering procedure in Alg. 5 trades off knowledge of (an upper bound on) the corruption level α with the need for a tight bound on $\|\Sigma_{\mathcal{I}_{\text{good}}}\|_2$. Alternatively, one can design a version of Algorithm 5 that is adaptive to the corruption level α using a similar construction that evaluates the error proxy $f(\lambda; p, \alpha)$ for different values of α and fixed $\lambda \approx \|\Sigma_{\mathcal{I}_{\text{good}}}\|_2$ instead.

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1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes]
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
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   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
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   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
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Appendix

A Auxiliary results

In this section, we present a few supporting results. The first result is a path independence lemma for perturbations of eigenvectors. It first appeared in [37]; the eigengap condition in the statement of the Lemma is justified in [10, Lemma 5].

Lemma 2 (Path independence)**.** *Let $A \in \mathbb{R}^{d\times d}$ be a fixed* symmetric *matrix and let $\hat{A} := A + E$, where E is a symmetric perturbation. Suppose that we can write*

$$\hat{A} - A = E_0 + E_1 = F_0 + F_1,$$

where E_0, E_1, F_0, F_1 are symmetric matrices, and define the intermediate *matrices*

$$\hat{A}_1 := A + E_0,\ \hat{A}_2 = \hat{A}_1 + E_1, \quad \tilde{A}_1 := A + F_0,\ \tilde{A}_2 = \tilde{A}_1 + F_1.$$

Fix any $V \in O(d,r)$ whose columns span the principal r-dimensional invariant subspace of A and construct the leading eigenvector matrices $\hat{V}_1$, $\hat{V}_2 \in O(d,r)$ of $\hat{A}_1$ and $\hat{A}_2$ such that

$$\min_{U\in\mathbb{O}_r} \left\|\hat{V}_1 U - V\right\|_F = \left\|\hat{V}_1 - V\right\|_F, \quad \min_{U\in\mathbb{O}_r} \left\|\hat{V}_2 U - \hat{V}_1\right\|_F = \left\|\hat{V}_2 - \hat{V}_1\right\|_F.$$

Further, let $\tilde{V}_1$ and $\tilde{V}_2$ be the leading eigenvector matrices of $\tilde{A}_1$ and $\tilde{A}_2$, constructed in a similar fashion. Then, $\hat{V}_2$ and $\tilde{V}_2$ both span principal invariant subspaces of $\hat{A} = A + E$. Moreover, they satisfy

$$\hat{V}_2 = \tilde{V}_2 + T, \quad \|T\|_2 \lesssim \frac{\varepsilon^2}{\delta^2}, \quad \varepsilon := \max\left\{\|E_0\|_2, \|E_1\|_2, \|F_0\|_2, \|F_1\|_2\right\},$$

as long as A satisfies $\delta_r(A) \geq 4\varepsilon$.

Lemma 3. *Suppose that $U \in \mathbb{O}_{d,r}$ satisfies $\operatorname{dist}(U,V) \leq \varepsilon < 1/2$, where V is the principal eigenvector matrix of a symmetric matrix A with eigengap $\delta_r(A) := \lambda_r(A) - \lambda_{r+1}(A) > 0$. Then there exists a symmetric matrix B such that the following hold:*

1. *$\|A - B\|_2 \leq 8\|A\|_2 \varepsilon$ and $\delta_r(B) = \delta_r(A)$.*
2. *U is the principal eigenvector matrix of B.*

Proof. We prove Item 1 first. To that end, we can write $A = A_1 + A_2$, where $A_1 := V\Sigma_1 V^\mathsf{T}$ and $A_2 := V_\perp \Sigma_2 V_\perp^\mathsf{T}$. We consider the following matrix B:

$$B = U\Sigma_1 U^\mathsf{T} + U_\perp \Sigma_2 U_\perp^\mathsf{T}, \tag{24}$$

where $U_\perp^\mathsf{T} U = 0$ and $U_\perp \in \mathbb{O}_{d,d-r}$. From (24) and the condition $\Sigma_1 \succ \Sigma_2$, it follows that U is a principal eigenvector matrix for B. Moreover, the gap condition on A immediately translates to the claimed gap condition for B.

It remains to bound the distance between A and B. We write

$$\|A - B\|_2 \leq \left\|U\Sigma_1 U^\mathsf{T} - A_1\right\|_2 + \left\|U_\perp \Sigma_2 U_\perp^\mathsf{T} - A_2\right\|_2.$$

To upper bound the first term on the right-hand side above, we use the spectral projectors $P_U := UU^\mathsf{T}$ and $P_{U_\perp} = I - P_U$ to decompose it into

$$\begin{aligned}
\left\|U\Sigma_1 U^\mathsf{T} - A_1\right\|_2 &\leq \left\|U\Sigma_1 U^\mathsf{T} - P_U A_1\right\|_2 + \|P_{U_\perp} A_1\|_2 \\
&\leq \left\|U\Sigma_1 U^\mathsf{T} - P_U A_1 P_U\right\|_2 + \|P_U A_1 P_{U_\perp}\|_2 + \|P_{U_\perp} A_1\|_2 \\
&\leq \left\|\Sigma_1 - U^\mathsf{T} V \Sigma_1 V^\mathsf{T} U\right\|_2 + 2\|\Sigma_1\|_2 \|P_{U_\perp} V\|_2 \\
&\leq \left\|\Sigma_1 (I - V^\mathsf{T} U)\right\|_2 + \left\|(I - V^\mathsf{T} U)\Sigma_1 V^\mathsf{T} U\right\|_2 + 2\|\Sigma_1\|_2 \|P_{U_\perp} V\|_2 \\
&\leq 2\|\Sigma_1\|_2 \varepsilon + 4\|\Sigma_1\|_2 \varepsilon^2,
\end{aligned}$$

where the last inequality follows from the inequality $\|P_{U_\perp} V\|_2 = \operatorname{dist}(U,V) = \varepsilon$ and Lemma 4. A similar argument shows that $\|U_\perp \Sigma_2 U_\perp^\mathsf{T} - A_2\|_2 \leq 2\|\Sigma_2\|_2 \varepsilon + 4\|\Sigma_2\|_2 \varepsilon^2$. Taking into account the bound $\varepsilon < 1/2$ completes the proof. □

Lemma 4 (Modified $\sin\theta$ distance)**.** *Let $U, V \in O(d,r)$ satisfy $\operatorname{dist}(U,V) = \alpha < 1$. Then the following holds:*

$$\left\| I - U^{\mathsf{T}} V \right\|_2 \leq 2\alpha^2.$$

Proof. Let $P\Sigma Q^{\mathsf{T}}$ be the singular value decomposition of $U^{\mathsf{T}}V$. Recall that [14, Eq. (2.5)]:

$$\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r); \quad \sigma_i = \cos(\theta_i),$$

where $\theta_i \in [0, \pi/2]$ and $\|\sin\Theta\|_2 = \alpha$, following [14, Lemma 2.5]. From our assumptions, it follows that

$$\begin{aligned}
\left\| I - U^{\mathsf{T}} V \right\|_2 &= \left\| P(I - \Sigma) Q^{\mathsf{T}} \right\|_2 \\
&= \left\| I - \Sigma \right\|_2 \\
&= \max_{i \in [r]} \{1 - \cos(\theta_i)\} \\
&= \max_{i \in [r]} \{2 \sin^2(\theta_i/2)\} \\
&\leq 2 \max_{i \in [r]} \sin^2(\theta_i) \\
&= 2 \left\| \sin\Theta \right\|_2^2,
\end{aligned}$$

with the last inequality following from $0 \leq \sin(\theta/2) \leq \sin(\theta)$ for any $\theta \in [0, \pi/2]$. □

B Additional proofs

This section includes proofs that were omitted from the main text.

B.1 Proof of Proposition 1

Proof. Define $\mathcal{C} := \{i \in [m] \mid \operatorname{dist}(Y_i, V) \leq \varepsilon\}$ with $|\mathcal{C}| > \frac{m}{2}$. Now, we consider any pair (i,j) with $i, j \in \mathcal{C}$. By the triangle inequality,

$$\operatorname{dist}(Y_i, Y_j) \leq \operatorname{dist}(Y_i, V) + \operatorname{dist}(Y_j, V) \leq 2\varepsilon, \quad \text{for all } i, j \in \mathcal{C}. \tag{25}$$

Now, fix $i_\star$ to be any index for which $\operatorname{dist}(Y_i, Y_j) \leq 2\varepsilon$ for at least $m/2$ other indices $j \neq i_\star$ (such an index always exists because $|\mathcal{C}| \geq \frac{m}{2} + 1$). For any such $i_\star$, there must be another index j satisfying $\operatorname{dist}(Y_j, V) \leq \varepsilon$ and $\operatorname{dist}(Y_j, Y_{i_\star}) \leq 2\varepsilon$. Therefore,

$$\operatorname{dist}(Y_{i_\star}, V) \leq \operatorname{dist}(Y_{i_\star}, Y_j) + \operatorname{dist}(Y_j, V) \leq 3\varepsilon.$$

□

B.2 Proof of Lemma 1

Proof. Recall that V is an eigenvector matrix of A that satisfies

$$\min_{Z \in \mathbb{O}_r} \|V - V_{\mathsf{ref}} Z\|_{\mathrm{F}} = \|V - V_{\mathsf{ref}}\|_{\mathrm{F}}.$$

From Lemma 3, it follows that the columns of V_{ref} span the principal eigenspace of a matrix B with nontrivial eigengap that satisfies $\|A - B\|_2 \lesssim \|A\|_2 \varepsilon$. We now relate V_i^{ideal} to V_i^{corr} using the aforementioned path independence result.

To that end, note that V_i^{ideal} is the leading eigenvector matrix of

$$A_i := A + (A_i - A) + 0,$$

that has been maximally aligned with V (in the sense of Frobenius distance). On the other hand, the Procrustes estimates V_i^{corr} are given by the leading eigenvector matrices of

$$A_i := A + (B - A) + (A_i - B),$$

since V_{ref} is the leading eigenvector of B nearest to V and V_i^{corr} is formed as the leading eigenvector of A_i nearest to V_{ref}. Applying Lemma 2 with $E_0 := A_i - A$, $E_1 = \mathbf{0}$, $F_0 := B - A$ and $F_1 := A_i - B$, we obtain

$$V_i^{\mathsf{corr}} = V_i^{\mathsf{ideal}} + \mathcal{O}\left(\frac{1}{\delta^2}\max\left\{\|A_i - A\|_2^2, \|B - A\|_2^2, \|A_i - B\|_2^2\right\}\right).$$

Finally, we note the following upper bound

$$\|A_i - B\|_2^2 \lesssim \|A_i - A\|_2^2 + \|A - B\|_2^2 \lesssim \max\left(\|A_i - A\|_2^2, \|A\|_2^2\,\varepsilon^2\right),$$

which concludes the proof. □

B.3 Proof of Proposition 3

Proof. Let j_{crit} be the smallest index for which $2^j \geq \max\{\lambda_{\mathsf{lb}}, \|\Sigma_{G_0}\|_2\}$. For a fixed corruption fraction α and failure probability p, define the events

$$\mathcal{E}_j := \left\{\left\|\theta_{2^j} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \leq f(2^j; p, \alpha)\right\}, \quad \mathcal{E} := \bigcap_{j=j_{\mathsf{crit}}}^{j_{\mathsf{hi}}} \mathcal{E}_j.$$

From Theorem 3 in the main text and a union bound, it follows that

$$\begin{aligned}
\mathbb{P}(\mathcal{E}) &\geq 1 - \sum_{j\in\{j_{\mathsf{crit}},\ldots,j_{\mathsf{hi}}\}} \mathbb{P}\left(\left\|\theta_{2^j} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \geq f(2^j; p, \alpha)\right) \\
&\geq 1 - (j_{\mathsf{hi}} - j_{\mathsf{crit}}) \cdot p \\
&\geq 1 - 2\log_2\left(\frac{\lambda_{\mathsf{ub}}}{\lambda_{\mathsf{lb}}}\right) p.
\end{aligned}$$

Let us write $\theta_* := \frac{1}{|G_0|}\sum_{i\in G_0} X_i$. Conditioned on the event $\mathcal{E}$, for any $j, j' \geq j_{\mathsf{crit}}$ we have

$$\|\theta_{2^j} - \theta_{2^{j'}}\|_2 \leq \|\theta_{2^j} - \theta_*\|_2 + \|\theta_{2^{j'}} - \theta_*\|_2 \leq f(2^j; p, \alpha) + f(2^{j'}; p, \alpha).$$

Consequently, it follows that $2^{j_{\mathsf{crit}}}$ satisfies the condition of the estimator, and therefore

$$\hat{\lambda} \leq 2^{j_{\mathsf{crit}}} \leq 2\max\{\lambda_{\mathsf{lb}}, \|\Sigma_{G_0}\|_2\}.$$

Finally, the desired claim follows since

$$\begin{aligned}
\|\theta_{\hat{\lambda}} - \theta_*\|_2 &\leq \|\theta_{\hat{\lambda}} - \theta_{2^{j_{\mathsf{crit}}}}\|_2 + \|\theta_{2^{j_{\mathsf{crit}}}} - \theta_*\|_2 \\
&\leq f(\hat{\lambda}; p, \alpha) + 2f(2^{j_{\mathsf{crit}}}; p, \alpha) \\
&\leq 3f(2^{j_{\mathsf{crit}}}; p, \alpha) \\
&\leq 171\sqrt{\|\Sigma_{G_0}\|_2}\left(\alpha + \frac{4\log(1/p)}{m}\right)^{1/2}.
\end{aligned}$$

□

The next Lemma provides an upper bound on the operator norm of the empirical covariance $\Sigma_{\mathcal{I}_{\mathsf{good}}}$.

Lemma 5. *Suppose that $\widehat{V}_{\mathsf{ref}}$ satisfies $\delta_r(A) \geq 8\,\mathrm{dist}(\widehat{V}_{\mathsf{ref}}, V)$. Then we have*

$$\|\Sigma_{\mathcal{I}_{\mathsf{good}}}\|_2 \leq \left\|\frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}} V_i V_i^{\mathsf{T}} - VV^{\mathsf{T}}\right\|_2 + 2\left\|\frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V\right\|_2. \tag{26}$$

Proof. Let μ denote the empirical mean over $\mathcal{I}_{\mathsf{good}}$. We have

$$\mu = \frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}} \widetilde{V}_i,$$

$$
\begin{aligned}
\Sigma_{\mathcal{I}_{\mathsf{good}}} &= \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} (\widetilde{V}_i - \mu)(\widetilde{V}_i - \mu)^\mathsf{T} \\
&= \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i \widetilde{V}_i^\mathsf{T} - \mu\mu^\mathsf{T} \\
&= \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i \widetilde{V}_i^\mathsf{T} - VV^\mathsf{T} + VV^\mathsf{T} - \mu\mu^\mathsf{T} \\
&= \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i \widetilde{V}_i^\mathsf{T} - VV^\mathsf{T} + (V - \mu)(V + \mu)^\mathsf{T},
\end{aligned}
$$

where $V \in \mathbb{O}_{d,r}$ spans the principal eigenspace of A and satisfies

$$
\min_{Z \in \mathbb{O}_r} \left\| VZ - \widehat{V}_{\mathsf{ref}} \right\|_{\mathrm{F}} = \left\| V - \widehat{V}_{\mathsf{ref}} \right\|_{\mathrm{F}}.
$$

We now bound the spectral norm of $\Sigma_{\mathcal{I}_{\mathsf{good}}}$. Indeed, we have

$$
\begin{aligned}
\left\| \Sigma_{\mathcal{I}_{\mathsf{good}}} \right\|_2 &\leq \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i \widetilde{V}_i^\mathsf{T} - VV^\mathsf{T} \right\|_2 + \left\| V + \mu \right\|_2 \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V \right\|_2 \\
&\leq \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} V_i V_i^\mathsf{T} - VV^\mathsf{T} \right\|_2 + 2 \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V \right\|_2,
\end{aligned}
$$

using the fact that $\widetilde{V}_i \widetilde{V}_i^\mathsf{T} = V_i V_i^\mathsf{T}$ for all $i \in \mathcal{I}_{\mathsf{good}}$. □

B.4 Proof of Theorem 3

In this section, we modify the proof of [33, Theorem 4] to derive guarantees for robust mean estimation with matrix-valued inputs. We recall some notation used therein: given the set of "good" samples G_0 and the initial sample $S_0 = \{1, \ldots, m\}$, we denote

$$
\begin{aligned}
S_k &= \{\text{points remaining after } k \text{ recursive calls to } \texttt{Filter}\}, \\
G_k &= S_k \cap G_0, \\
B_k &= S_k \setminus G_0, \\
\alpha &= \frac{m - |G_0|}{m}.
\end{aligned} \tag{27}
$$

Moreover, given any set $S \subset [m]$, we write

$$
\Sigma_S := \frac{1}{|S|} \sum_{i \in S} (X_i - \mu_S)(X_i - \mu_S)^\mathsf{T}, \quad \text{where} \quad \mu_S := \frac{1}{|S|} \sum_{i \in S} X_i. \tag{28}
$$

In our proofs, we frequently employ the total variation distance d_{TV}. For discrete distributions P_1, P_2 on a common sample space Ω, d_{TV} is given by

$$
d_{\mathrm{TV}}(P_1, P_2) = \frac{1}{2} \left\| P_1 - P_2 \right\|_1 = \frac{1}{2} \sum_{x \in \Omega} \left| P_1(x) - P_2(x) \right|. \tag{29}
$$

Finally, we define the events $\mathcal{E}_k$, where $k \in \mathbb{N}$, as below:

$$
\mathcal{E}_k := \left\{ \sum_{i \in G_k} \tau_i \geq \frac{1}{\gamma} \sum_{j \in S_k} \tau_j \right\}, \quad k = 0, 1, \ldots \tag{30}
$$

Our proof essentially traces the proof of [33, Theorem 4] but for the case of matrix-valued inputs to the `Filter` algorithm. The first result has already been shown in [33], as its proof is independent of the shape of the inputs.

Lemma 6 (See [33, Lemma 6]). *Let $T := \inf \{k \in \mathbb{N} \mid \mathcal{E}_k \text{ is true}\}$. Then we have:*

$$
\mathbb{P}\left(T \geq 3(m - |G_0|) + 18 \log(1/p) \right) \leq p. \tag{31}
$$

The remainder of the proof is devoted to showing that, as soon as some $\mathcal{E}_k$ is true, `Filter` will terminate with a good estimate. Throughout, we condition on the event

$$\mathcal{E} := \{T \leq T_p\}, \quad \text{where} \quad T_p := 3(m - |G_0|) + 18\log(1/p), \tag{32}$$

which holds with probability at least $1 - p$.

Theorem 5. *Suppose that α, p and N satisfy*

$$3\alpha + \frac{18\log(1/p)}{m} \leq \frac{1}{4}. \tag{33}$$

Then the following hold simultaneously with probability at least $1 - p$:

1. $\texttt{Filter}(S_0, \|\Sigma_{G_0}\|_2)$ *terminates after at most T_p iterations;*
2. *The output of* $\texttt{Filter}(S_0, \|\Sigma_{G_0}\|_2)$, $\theta_{\|\Sigma_{G_0}\|_2}$, *satisfies*

$$\left\|\theta_{\|\Sigma_{G_0}\|_2} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \leq 18\sqrt{5\,\|\Sigma_{G_0}\|_2}\left(\alpha + \frac{4\log(1/p)}{m}\right)^{1/2}. \tag{34}$$

Remark 5. *While we prove the Theorem for the case $\lambda_{\mathtt{ub}} = \|\Sigma_{G_0}\|_2$, a straightforward modification of the proof shows that when $\lambda_{\mathtt{ub}} \geq \|\Sigma_{G_0}\|_2$, we have*

$$\left\|\theta_{\lambda_{\mathtt{ub}}} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \leq 18\sqrt{5\lambda_{\mathtt{ub}}}\left(\alpha + \frac{4\log(1/p)}{m}\right)^{1/2}.$$

Proof of Theorem 5. We condition on the event $\mathcal{E}$ from (32), which holds with probability at least $1 - p$. This implies that there is some index $k \leq T_p$ such that

$$\sum_{i\in G_k} \tau_i \geq \frac{1}{\gamma}\sum_{j\in S_k} \tau_j.$$

From Lemma 11, we obtain that the empirical covariance satisfies $\|\Sigma_{S_k}\|_2 \leq 18\,\|\Sigma_{G_0}\|_2$, and thus the algorithm terminates after at most k steps. We have the following cases:

1. The termination condition was first triggered at the kth step. In that case, Lemma 11 directly implies the desired inequality.
2. The algorithm terminated at some index $\ell < k$. Then it follows from Lemma 12 that

$$\eta := d_{\mathrm{TV}}(\mathrm{Unif}(S_\ell), \mathrm{Unif}(G_0)) \leq 5\alpha + \frac{20\log(1/p)}{N}. \tag{35}$$

At the same time, Lemma 7 implies that

$$\left\|\theta_{\|\Sigma_{G_0}\|_2} - \frac{1}{|G_0|}\sum_{i\in G_0} X_i\right\|_2 \leq \frac{\sqrt{\eta}}{1-\sqrt{\eta}}\cdot\left(\|\Sigma_{S_\ell}\|_2^{1/2} + \|\Sigma_{G_0}\|_2^{1/2}\right). \tag{36}$$

From the termination condition, we obtain that

$$\|\Sigma_{S_\ell}\|_2 \leq 18\,\|\Sigma_{G_0}\|_2. \tag{37}$$

Combining Eqs. (35) to (37) yields the desired bound.

□

The next few Lemmas are supporting statements used in the proof of Theorem 5.

Lemma 7. *Let $S = \{X_1, \ldots, X_m\}$ where $X_i \in \mathbb{R}^{d\times r}$ and suppose that P_1, P_2 are discrete distributions supported over $[m]$ with $d_{\mathrm{TV}}(P_1, P_2) = \eta$. Then the following holds:*

$$\|\mathbb{E}_{P_1}[X_i] - \mathbb{E}_{P_2}[X_i]\|_2 \leq \frac{\sqrt{\eta}}{1-\sqrt{\eta}}\cdot\left(\|\Sigma_{P_1}\|_2^{1/2} + \|\Sigma_{P_2}\|_2^{1/2}\right), \tag{38}$$

where the matrices Σ_{P_i} are defined as:

$$\Sigma_{P_i} = \mathbb{E}_{X\sim P_i}\left[(X - \mathbb{E}_{P_i}[X])(X - \mathbb{E}_{P_i}[X])^{\mathsf{T}}\right].$$

Proof. Following the proof of [27, Lemma 2.1], we consider a coupling between P_1 and P_2 such that $\mathbb{P}(X = X') \geq 1 - \eta$. Denoting $\|X\|_{L^2} := \sqrt{\mathbb{E}[X^2]}$, we have

$$\begin{aligned}
\|\mathbb{E}_{P_1}[X] - \mathbb{E}_{P_2}[X']\|_2 &= \sup_{u,v\in\mathcal{B}} \langle u, (\mathbb{E}_{P_1}[X] - \mathbb{E}_{P_2}[X'])v\rangle \\
&= \sup_{u,v\in\mathcal{B}} \mathbb{E}\left[\langle u, (X - X')v\rangle \mathbf{1}\{X \neq X'\}\right] \\
&\leq \mathbb{E}\left[\mathbf{1}\{X \neq X'\}^2\right]^{1/2} \cdot \sup_{u,v\in\mathcal{B}} \mathbb{E}\left[\langle u, (X - X')v\rangle^2\right]^{1/2} \\
&\leq \sqrt{\eta} \cdot \sup_{u,v\in\mathcal{B}} \|\langle u, (X - X')v\rangle\|_{L^2}. \qquad (39)
\end{aligned}$$

Let $\mu_1 := \mathbb{E}_{P_1}[X]$ and $\mu_2 = \mathbb{E}_{P_2}[X]$. Since $\|\cdot\|_{L^2}$ is a norm, the triangle inequality implies that

$$\begin{aligned}
\sup_{u,v\in\mathcal{B}} \|\langle u, (X - X')v\rangle\|_{L^2} &= \sup_{u,v\in\mathcal{B}} \|\langle u, (X - \mu_1 + \mu_1 - \mu_2 + \mu_2 - X')v\rangle\|_{L^2} \\
&\leq \sup_{u,v\in\mathcal{B}} \|\langle u, (X - \mu_1)v\rangle\|_{L^2} + \sup_{u,v\in\mathcal{B}} \|\langle u, (X' - \mu_2)v\rangle\|_{L^2} \qquad (40) \\
&\quad + \sup_{u,v\in\mathcal{B}} \|\langle u, (\mu_1 - \mu_2)v\rangle\|_{L^2}.
\end{aligned}$$

We now upper bound the remaining terms. For the first one, we have

$$\begin{aligned}
\sup_{u,v\in\mathcal{B}} \|\langle u, (X - \mu_1)v\rangle\|_{L^2} &= \sup_{u,v\in\mathcal{B}} \mathbb{E}\left[\langle u, (X - \mu_1)v\rangle^2\right]^{1/2} \\
&= \sup_{u,v\in\mathcal{B}} \mathbb{E}[\mathrm{Tr}(u^\mathsf{T}(X - \mu_1)\underbrace{vv^\mathsf{T}}_{\preceq I_d}(X - \mu_1)^\mathsf{T}u)]^{1/2} \\
&\leq \sup_{u\in\mathcal{B}} \mathbb{E}\left[\mathrm{Tr}\left(u^\mathsf{T}(X - \mu_1)(X - \mu_1)^\mathsf{T}u\right)\right]^{1/2} \\
&= \left(\sup_{u\in\mathcal{B}} \left\langle u, \mathbb{E}\left[(X - \mu_1)(X - \mu_1)^\mathsf{T}\right]u\right\rangle\right)^{1/2} \\
&= \|\Sigma_{P_1}\|_2^{1/2}, \qquad (41)
\end{aligned}$$

where the penultimate equality uses linearity of the trace operator and the last equality is the definition of the spectral norm for symmetric positive semidefinite matrices. Similar arguments also yield

$$\sup_{u,v\in\mathcal{B}} \|\langle u, (X' - \mu_2)v\rangle\|_{L^2} \leq \|\Sigma_{P_2}\|_2^{1/2}, \qquad (42)$$

$$\sup_{u,v\in\mathcal{B}} \|\langle u, (\mu_1 - \mu_2)v\rangle\|_{L^2} \leq \|\mathbb{E}_{P_1}[X] - \mathbb{E}_{P_2}[X']\|_2. \qquad (43)$$

Plugging Eqs. (40) to (43) back into Eq. (39) and rearranging yields the expected result:

$$\|\mathbb{E}_{P_1}[X] - \mathbb{E}_{P_2}[X']\|_2 \leq \frac{\sqrt{\eta}}{1 - \sqrt{\eta}} \left(\|\Sigma_{P_1}\|_2^{1/2} + \|\Sigma_{P_2}\|_2^{1/2}\right).$$

□

Lemma 8. *Let $G \subset S \subset [m]$. Moreover, let μ_S and μ_G be their respective empirical means, and let v be the leading eigenvector of Σ_S so that the outlier scores satisfy*

$$\tau_i = \left\langle v, (X_i - \mu_S)(X_i - \mu_S)^\mathsf{T}v\right\rangle, \quad \forall i \in S.$$

Moreover, define $\eta := 1 - {}^{|G|}/_{|S|}$ and fix a $\gamma \in (0, {}^1/_\eta)$. Then, we have the implication

$$\|\Sigma_S\|_2 \geq (1 - \eta)^2 \left(\frac{\gamma}{1 - \gamma\eta}\right) \|\Sigma_G\|_2 \implies \sum_{j\in G} \tau_j \leq \frac{1}{\gamma} \sum_{i\in S} \tau_i. \qquad (44)$$

Proof. Recall that the (normalized) sum of outlier scores over the set G is given by

$$\begin{aligned}
\frac{1}{|G|}\left\langle v, \sum_{i\in G}(X_i - \mu_S)(X_i - \mu_S)^\mathsf{T}v\right\rangle &= \frac{1}{|G|}\left\langle v, \sum_{i\in G}(X_i - \mu_G)(X_i - \mu_G)^\mathsf{T}v\right\rangle \\
&\quad + \left\langle v, (\mu_S - \mu_G)(\mu_S - \mu_G)^\mathsf{T}v\right\rangle
\end{aligned}$$

$$= \langle v, \Sigma_G v \rangle + \left\langle v, (\mu_S - \mu_G)(\mu_S - \mu_G)^\mathsf{T} v \right\rangle . \tag{45}$$

We now simplify the second term. Indeed, we have

$$\begin{aligned} \mu_S - \mu_G &= \frac{1}{|S|} \sum_{i \in G} X_i + \frac{1}{|S|} \sum_{i \in S \setminus G} X_i - \frac{1}{|G|} \sum_{i \in G} X_i \\ &= \left(1 - \frac{|G|}{|S|}\right) (\mu_{S \setminus G} - \mu_G) \end{aligned} \tag{46}$$

For brevity, denote $\eta := \frac{|S \setminus G|}{|S|}$. Plugging (46) back into (45), we obtain

$$\frac{1}{|G|} \sum_{j \in G} \tau_j = \langle v, \Sigma_G v \rangle + \eta^2 \left\langle v, (\mu_{S \setminus G} - \mu_G)(\mu_{S \setminus G} - \mu_G)^\mathsf{T} v \right\rangle \tag{47}$$

We now bound the second term in (47). From [17, Lemma 2.4], it follows that

$$\langle v, \Sigma_S v \rangle = (1 - \eta) \langle v, \Sigma_G v \rangle + \eta \langle v, \Sigma_{S \setminus G} v \rangle + \eta(1 - \eta) \left\langle v, (\mu_{S \setminus G} - \mu_G)(\mu_{S \setminus G} - \mu_G)^\mathsf{T} v \right\rangle$$

Rearranging and multiplying by $\eta/(1 - \eta)$ gives

$$\begin{aligned} \eta^2 \left\langle v, (\mu_{S \setminus G} - \mu_G)(\mu_{S \setminus G} - \mu_G)^\mathsf{T} v \right\rangle &= \frac{\eta}{1 - \eta} \langle v, \Sigma_S v \rangle - \eta \langle v, \Sigma_G v \rangle - \frac{\eta^2}{(1 - \eta)} \langle v, \Sigma_{S \setminus G} v \rangle \\ &\leq \frac{\eta}{1 - \eta} \langle v, \Sigma_S v \rangle - \eta \langle v, \Sigma_G v \rangle . \end{aligned}$$

Plugging back into Eq. (47) and using the fact that $|G| = |S| (1 - \eta)$, we obtain

$$\begin{aligned} \sum_{j \in G} \tau_j &\leq |G| (1 - \eta) \langle v, \Sigma_G v \rangle + \frac{|G| \eta}{1 - \eta} \langle v, \Sigma_S v \rangle \\ &\leq |G| (1 - \eta) \|\Sigma_G\|_2 + (|S| - |G|) |S| \|\Sigma_S\|_2 \end{aligned} \tag{48}$$

Finally, replacing $|G| = |S| (1 - \eta)$ in (48) and rearranging, we obtain

$$\|\Sigma_G\|_2 \leq (\gamma^{-1} - \eta) \frac{\|\Sigma_S\|_2}{(1 - \eta)^2} \implies \sum_{j \in G} \tau_j \leq \frac{1}{\gamma} \sum_{i \in S} \tau_i.$$

□

Lemma 9. *Suppose that (33) is true. Then the following holds for any $k \leq T_p$:*

$$\frac{|S_k \setminus G_k|}{|S_k|} \leq \frac{4\alpha}{3}.$$

Proof. Recall that $B_k = S_k \setminus G_k$ and notice that

$$\frac{|B_k|}{|S_k|} = \frac{|B_k|}{|S_0|} \frac{|S_0|}{|S_k|} \leq \frac{|B_0|}{|S_0|} \frac{|S_0|}{|S_0| - T_p} = \alpha \cdot \frac{1}{1 - (3\alpha + \frac{18 \log(1/p)}{m})} \leq \frac{4\alpha}{3},$$

where the first inequality follows from the fact that $|B_k| \leq |B_0|$. □

Lemma 10. *For any integer k, the sets G_k and G_0 satisfy*

$$\|\Sigma_{G_k}\|_2 \leq \frac{|G_0|}{|G_k|} \|\Sigma_{G_0}\|_2 \tag{49}$$

Proof. We expand the definition of Σ_{G_0} and rewrite:

$$\Sigma_{G_0} = \frac{1}{|G_0|} \sum_{i \in G_0} (X_i - \mu_{G_0})(X_i - \mu_{G_0})^\mathsf{T}$$

$$= \underbrace{\frac{1}{|G_0|}\sum_{i\in G_k}(X_i-\mu_{G_0})(X_i-\mu_{G_0})^\mathsf{T}}_{T_1} + \underbrace{\frac{1}{|G_0|}\sum_{i\in G_0\setminus G_k}(X_i-\mu_{G_0})(X_i-\mu_{G_0})^\mathsf{T}}_{T_2}$$

We now rewrite the first term in the above sum using

$$\begin{aligned}
T_1 &= \frac{1}{|G_0|}\sum_{i\in G_k}(X_i-\mu_{G_k}+\mu_{G_k}-\mu_{G_0})(X_i-\mu_{G_k}+\mu_{G_k}-\mu_{G_0})^\mathsf{T} \\
&= \frac{|G_k|}{|G_0|}\Sigma_{G_k} + \frac{|G_k|}{|G_0|}\left(\frac{1}{|G_k|}\sum_{i\in G_k}(X_i-\mu_{G_k})\right)(\mu_{G_k}-\mu_{G_0})^\mathsf{T} \\
&\quad + \frac{|G_k|}{|G_0|}(\mu_{G_k}-\mu_{G_0})\left(\frac{1}{|G_k|}\sum_{i\in G_k}X_i-\mu_{G_k}\right)^\mathsf{T} + \frac{|G_k|}{|G_0|}(\mu_{G_k}-\mu_{G_0})(\mu_{G_k}-\mu_{G_0})^\mathsf{T} \\
&= \frac{|G_k|}{|G_0|}\left(\Sigma_{G_k}+(\mu_{G_k}-\mu_{G_0})(\mu_{G_k}-\mu_{G_0})^\mathsf{T}\right)
\end{aligned}$$

Letting $v\in\mathbb{S}^{d-1}$ and using the fact that T_2 is positive semidefinite, we arrive at

$$\langle v,\Sigma_{G_0}v\rangle = \frac{|G_k|}{|G_0|}\left(\langle v,\Sigma_{G_k}v\rangle + \left\|(\mu_{G_k}-\mu_{G_0})^\mathsf{T}v\right\|^2\right) + \langle v,T_2v\rangle \geq \frac{|G_k|}{|G_0|}\langle v,\Sigma_{G_k}v\rangle \tag{50}$$

Finally, taking suprema over both sides yields the desired inequality. □

Lemma 11. *Suppose that* (33) *is true and that the following inequality holds for some index $k\leq T_p$:*

$$\sum_{i\in G_k}\tau_i \geq \frac{1}{\gamma}\sum_{j\in S_k}\tau_j. \tag{51}$$

Then the empirical means satisfy

$$\left\|\mathbb{E}_{\mathrm{Unif}(G_0)}[X]-\mathbb{E}_{\mathrm{Unif}(S_k)}[X]\right\|_2 \leq 18\left(5\alpha+\frac{20\log(1/p)}{m}\right)^{1/2}\|\Sigma_{G_0}\|_2^{1/2}.$$

Proof. Let $P_1 := \mathrm{Unif}(G_0)$ and $P_2 := \mathrm{Unif}(S_k)$. From Lemma 7, it follows that

$$\left\|\mathbb{E}_{P_1}[X]-\mathbb{E}_{P_2}[X]\right\|_2 \leq \frac{\sqrt{d_{\mathrm{TV}}(P_1,P_2)}}{1-\sqrt{d_{\mathrm{TV}}(P_1,P_2)}}\cdot\left(\|\Sigma_{G_0}\|_2^{1/2}+\|\Sigma_{G_k}\|_2^{1/2}\right). \tag{52}$$

Since (51) is the reverse of (44), we obtain

$$\begin{aligned}
\|\Sigma_{S_k}\|_2 &\leq (1-\eta)^2\frac{\gamma}{1-\gamma\eta}\|\Sigma_{G_k}\|_2 \\
&\leq \frac{3}{1-6\alpha}\|\Sigma_{G_k}\|_2 \\
&\leq 6\cdot\frac{|G_0|}{|G_k|}\|\Sigma_{G_0}\|_2,
\end{aligned}$$

where the first inequality follows from the contrapositive of Lemma 8, the second inequality from $\gamma=3$ and Lemma 9, and the last inequality follows by our assumption on α. Now, let $K\leq T_p$ be the number of samples in G_0 that were removed by the algorithm. We have

$$\frac{|G_0|}{|G_k|} = \frac{m-|B_0|}{m-|B_0|-K} \leq \frac{m-|B_0|}{m-|B_0|-T_p} \leq \frac{m-|B_0|}{m-18\log(1/p)-4|B_0|} = \frac{1-\alpha}{1-4\alpha-\frac{18\log(1/p)}{m}}$$

From (33), we additionally have that

$$1-(4\alpha+\frac{18\log(1/p)}{m}) \geq 1-\frac{4}{3}\left(3\alpha+\frac{18\log(1/p)}{m}\right) \geq \frac{1}{3} \implies \|\Sigma_{S_k}\|_2 \leq 18\|\Sigma_{G_0}\|_2.$$

Substituting the above into (52) and using Lemma 12 yields the desired bound:

$$\begin{aligned}\|\mathbb{E}_{P_1}[X] - \mathbb{E}_{P_2}[X]\|_2 &\leq \frac{\left(5\alpha + \frac{20\log(1/p)}{m}\right)^{1/2}}{1 - \left(5\alpha + \frac{20\log(1/p)}{m}\right)^{1/2}} \left(\|\Sigma_{G_0}\|_2^{1/2} + \sqrt{18}\,\|\Sigma_{G_0}\|_2^{1/2}\right) \\ &\leq 18\left(5\alpha + \frac{20\log(1/p)}{m}\right)^{1/2} \|\Sigma_{G_0}\|_2^{1/2}.\end{aligned}$$

□

Lemma 12. *Suppose $k \leq T_p$ and (33) holds. Then we have that*

$$d_{\mathrm{TV}}(\mathrm{Unif}(S_k), \mathrm{Unif}(G_0)) \leq 5\alpha + \frac{20\log(1/p)}{m}. \tag{53}$$

Proof. We let $P_1 := \mathrm{Unif}(S_k)$, $P_2 := \mathrm{Unif}(G_0)$ and $P_3 := \mathrm{Unif}(G_k)$, and write $K \leq k \leq T_p$ for the number of samples originally in G_0 that were removed by the `Filter` algorithm by the kth step. From the triangle inequality, it follows that

$$\begin{aligned}d_{\mathrm{TV}}(P_1, P_2) &\leq d_{\mathrm{TV}}(P_1, P_3) + d_{\mathrm{TV}}(P_2, P_3) \\ &= \frac{|S_k| - |G_k|}{|S_k|} + \frac{|G_0| - |G_k|}{|G_0|} \\ &= \frac{m - k - (m - |B_0| - K)}{m - k} + \frac{K}{m - |B_0|} \\ &= \frac{|B_0| + (K - k)}{m - k} + \frac{K}{m - |B_0|} \\ &\leq \frac{|B_0|}{m - k} + \frac{T_p}{m - |B_0|} \\ &\leq \frac{|B_0|}{m - T_p} + \frac{T_p}{m - |B_0|},\end{aligned}$$

where the second line follows from Lemma 13 and the last two inequalities follow from $K \leq m$ and $m \leq T_p$. Finally, using Lemma 6 and Eq. (33), we obtain

$$\begin{aligned}\frac{|B_0|}{m - T_p} + \frac{T_p}{m - |B_0|} &= \frac{\alpha}{1 - \frac{T_p}{m}} + \frac{\frac{T_p}{m}}{1 - \alpha} \\ &\leq \frac{\alpha}{1 - \frac{18\log(1/p)}{m} - 3\alpha} + \frac{\frac{18\log(1/p)}{m} + 3\alpha}{1 - \alpha} \\ &\leq \frac{4\alpha}{3} + \frac{\frac{18\log(1/p)}{m} + 3\alpha}{1 - \alpha} \\ &\leq \frac{4\alpha}{3} + \frac{\frac{18\log(1/p)}{m} + 3\alpha}{1 - \frac{1}{12}} \\ &\leq 5\alpha + \frac{20\log(1/p)}{m}.\end{aligned}$$

□

Lemma 13. *Consider a pair of discrete sets S, S' such that $S' \subset S$. We have:*

$$d_{\mathrm{TV}}(\mathrm{Unif}(S), \mathrm{Unif}(S')) = \frac{|S| - |S'|}{|S|}. \tag{54}$$

Proof. Using the fact that $d_{\mathrm{TV}}(p, q) = \frac{1}{2}\|p - q\|_1$, we have:

$$d_{\mathrm{TV}}(\mathrm{Unif}(S), \mathrm{Unif}(S')) = \frac{1}{2}\left(\sum_{x \in S \cap S'} \left|\frac{1}{|S|} - \frac{1}{|S'|}\right| + \sum_{x \in S \setminus S'} \frac{1}{|S|}\right)$$

$$= \frac{1}{2}\left(1 - \frac{|S'|}{|S|} + \frac{|S| - |S'|}{|S|}\right)$$
$$= 1 - \frac{|S'|}{|S|}.$$

□

B.5 Proof of Theorem 4

We now present the proof of the main theorem on distributed PCA. We first recall that

$$A_i = \frac{1}{n}\sum_{j=1}^{n} X_j^{(i)}(X_j^{(i)})^\mathsf{T}; \quad i \in \mathcal{I}_{\mathsf{good}},$$

where $X_j^{(i)} \overset{\text{iid}}{\sim} \mathcal{P}$, and that the responses $V_i \in \mathbb{O}_{d,r}$ span the leading r-dimensional eigenspace of A_i. Under this model, the local errors $E_i := A_i - A$ as well as the error of the empirical average over the inliers are bounded with high probability. We will condition on the following events for the remainder of this section:

$$\begin{aligned}
\mathcal{E}_1 &= \left\{\max_{i\in\mathcal{I}_{\mathsf{good}}} \|A_i - A\|_2 \le \min\left\{\frac{\delta}{8}, C_1 \|A\|_2 \sqrt{\frac{r_\star + \log(m/p)}{n}}\right\}\right\}, \\
\mathcal{E}_2 &= \left\{\left\|\frac{1}{|\mathcal{I}_{\mathsf{good}}|}\sum_{i\in\mathcal{I}_{\mathsf{good}}} A_i - A\right\|_2 \le C_2 \|A\|_2 \sqrt{\frac{r_\star + \log(n)}{|\mathcal{I}_{\mathsf{good}}|\, n}}\right\}.
\end{aligned} \tag{55}$$

Lemma 14. *Suppose that $n \gtrsim \kappa^2 \cdot (r_\star + \log(mn/p))$. Then the following hold:*

$$\mathbb{P}(\mathcal{E}_1) \le p, \quad \mathbb{P}(\mathcal{E}_2) \le \frac{2}{n}. \tag{56}$$

Proof. The bound on $\mathbb{P}(\mathcal{E}_2)$ in Eq. (56) follows from an application of [38, Exercise 9.2.5] and the assumed lower bound on n. On the other hand, the same result yields

$$\mathbb{P}\left(\|A_i - A\|_2 \ge C_1 \|A\|_2 \left(\sqrt{\frac{r_\star + \log(m/p)}{n}} + \frac{r_\star + \log(m/p)}{n}\right)\right) \le \frac{p}{m},$$

for any fixed $i \in \mathcal{I}_{\mathsf{good}}$. From the lower bound on n, it follows that

$$\frac{r_\star + \log(m/p)}{n} \le \sqrt{\frac{r_\star + \log(m/p)}{n}} \quad \text{and} \quad C_1 \|A\|_2 \sqrt{\frac{r_\star + \log(m/p)}{n}} < \frac{\delta}{8}.$$

Finally, taking a union bound over $\mathcal{I}_{\mathsf{good}}$ recovers the bound on $\mathbb{P}(\mathcal{E}_1)$. □

An immediate corollary is a bound on the error of `RobustReferenceEstimator`.

Corollary 2. *There is a universal constant C_{ref} such that the output of Alg. 2 satisfies*

$$\operatorname{dist}(\widehat{V}_{\mathsf{ref}}, V) \le C_{\mathsf{ref}}\kappa \cdot \sqrt{\frac{r_\star + \log(m/p)}{n}}.$$

Proof. From the bound $\alpha < \frac{1}{2}$ and the conditioning on $\mathcal{E}_1$, we deduce the existence of an index set S' such that $|S'| > \frac{m}{2}$, and

$$\operatorname{dist}(\widehat{V}_i, V) \le \frac{\|A_i - A\|_2}{\delta - \frac{\delta}{4}} \le \frac{2C_1 \|A\|_2}{\delta}\sqrt{\frac{r_\star + \log(m/p)}{n}}, \quad \text{for all } i \in S',$$

where the first bound on $\operatorname{dist}(\widehat{V}_i, V)$ follows from the Davis-Kahan theorem [14, Theorem 2.7] and the fact that $\|A_i - A\|_2 \le \frac{\delta}{8}$ for any $i \notin \mathcal{I}_{\text{bad}}$. From Proposition 1 in the main text, it follows that

$$\operatorname{dist}(\widehat{V}_{\mathsf{ref}}, V) \le \underbrace{6C_1}_{C_{\mathsf{ref}}} \frac{\|A\|_2}{\delta}\sqrt{\frac{r_\star + \log(m/p)}{n}}.$$

□

The next Proposition instantiates the bounds of Lemma 5 for for the case of distributed PCA.

Proposition 4. *In the setting of Lemma 5, the matrix $\Sigma_{\mathcal{I}_{\mathsf{good}}}$ satisfies*

$$\|\Sigma_{\mathcal{I}_{\mathsf{good}}}\|_2 \lesssim \kappa \sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}} + \kappa^2 \cdot \frac{\sqrt{r}(r_\star + \log(n))}{n} + \kappa^4 \cdot \frac{r_\star + \log(m/p)}{n}. \tag{57}$$

Proof. From Lemma 5, it follows that

$$\|\Sigma_{\mathcal{I}_{\mathsf{good}}}\|_2 \leq \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} V_i V_i^\mathsf{T} - VV^\mathsf{T} \right\|_2 + 2 \cdot \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V \right\|_2$$

From Proposition 2 in the main text and conditioning $\mathcal{E}_1$ and $\mathcal{E}_2$, we have

$$\begin{aligned}
\left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V \right\|_2 &\lesssim \frac{1}{\delta} \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} A_i - A \right\|_2 \\
&\quad + \left(\frac{\|A\|_2}{\delta} \right)^2 \max\left(C_1^2, C_{\mathsf{ref}}^2 \left(\frac{\|A\|_2}{\delta} \right)^2 \right) \cdot \frac{r_\star + \log(m/p)}{n} \\
&\leq C_2 \kappa \sqrt{\frac{r_\star + \log(n)}{(1-\alpha)mn}} + \kappa^2 \max\left(C_1^2, C_{\mathsf{ref}}^2 \kappa^2\right) \cdot \frac{r_\star + \log(m/p)}{n} \\
&\lesssim \kappa \sqrt{\frac{r_\star + \log(n)}{(1-\alpha)mn}} + \kappa^4 \cdot \frac{r_\star + \log(m/p)}{n}.
\end{aligned} \tag{58}$$

On the other hand, using [20, Theorem 2], we have that

$$\left\| \frac{1}{\mathcal{I}_{\mathsf{good}}} \sum_{i \in \mathcal{I}_{\mathsf{good}}} V_i V_i^\mathsf{T} - VV^\mathsf{T} \right\|_2 \lesssim \kappa \sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}} + \kappa^2 \frac{\sqrt{r}(r_\star + \log(n))}{n}.$$

Putting all the bounds together yields (57). □

We now invoke Proposition 3 and recall that ϱ is defined as

$$\varrho := \kappa \sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}} + \kappa^2 \cdot \frac{\sqrt{r}(r_\star + \log(n))}{n} + \kappa^4 \cdot \frac{r_\star + \log(m/p)}{n} \tag{59}$$

From that and Proposition 4, it follows that Alg. 5 from the main text invoked with $\lambda_{\mathsf{lb}} = \omega := \sqrt{1/mn}$ and $\lambda_{\mathsf{ub}} = 6$ outputs an estimate satisfying

$$\left\| \bar{V} - \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i \right\|_2 \lesssim \sqrt{\max\{\varrho, \omega\}} \cdot \left(\alpha + \frac{\log(1/p)}{m} \right)^{1/2} \tag{60}$$

$$= \sqrt{\varrho} \cdot \left(\alpha + \frac{\log(1/p)}{m} \right)^{1/2} \tag{61}$$

with failure probability at most $2\log_2(6/\omega)p$. Finally, from Eqs. (58) and (61) it follows that

$$\begin{aligned}
\|\bar{V} - V\|_2 &\leq \left\| \bar{V} - \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i \right\|_2 + \left\| \frac{1}{|\mathcal{I}_{\mathsf{good}}|} \sum_{i \in \mathcal{I}_{\mathsf{good}}} \widetilde{V}_i - V \right\|_2 \\
&\lesssim \sqrt{\varrho} \left(\alpha + \frac{\log(1/p)}{m} \right)^{1/2} + \kappa \sqrt{\frac{r(r_\star + \log(n))}{(1-\alpha)mn}} + \kappa^4 \frac{\sqrt{r}(r_\star + \log(m/p))}{n}.
\end{aligned}$$

In particular, the success probability is at least (given that ω is set as $\sqrt{1/mn}$):

$$1 - p - \frac{2}{n} - 2\log_2\left(\frac{6}{\omega}\right) p \geq 1 - \frac{2}{n} - 2\log_2(6mn)p.$$

Letting $p := \frac{p'}{2\log_2(6mn)}$ and relabeling $p' \to p$ yields the result.

C Experiment details

The numerical experiments in the main text were coded in `Julia` and run on a machine with Intel(R) Core(TM) i7-7700 CPU, 16GB of RAM and a GNU/Linux environment. The code is attached as part of the supplementary material and consists of a library called `RobustDistributedPCA.jl` and a subfolder `scripts/` that contains the scripts reproducing the experiments in the manuscript. We refer the reader to the included `README.md` file for installation and usage instructions.

Notes on implementation. Our implementations of the `Filter` and `AdaptiveFilter` algorithms deviate from the theory in the following ways:

1. In `Filter`, we remove the point with the largest outlier score by default:

   $$Z := \operatorname*{argmax}_{j \in S} \tau_j.$$

   The randomized selection can be enabled by passing `randomized = true` to the appropriate function. Please see the attached `README.md` file for details.

2. The error proxy $f(\lambda; p, \alpha)$ we use in `AdaptiveFilter` has been simplified to

   $$f(\lambda; p, \alpha) \equiv f(\lambda; \alpha) = \sqrt{\lambda \alpha}.$$

   The reason is twofold: on one hand, we suspect that many of the constants involved in the original definition of f are artifacts of our proof and are much lower in practice; on the other hand, since the default behavior of `Filter` is not randomized, the term owing to failure probability in f can be removed.

3. Finally, we change the stopping condition of `AdaptiveFilter` to $\lambda \leq \lambda_{\mathsf{ub}}$ in order to simplify constants. As the value of λ_{ub} is determined adaptively, this does not affect the output of the algorithm.