On the Convergence of mSGD and AdaGrad for Stochastic Optimization

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Abstract

As one of the most fundamental stochastic optimization algorithms, stochastic gradient descent (SGD) has been intensively developed and extensively applied in machine learning in the past decade. There have been some modified SGD-type algorithms, which outperform the SGD in many competitions and applications in terms of convergence rate and accuracy, such as momentum-based SGD (mSGD) and adaptive gradient algorithm (AdaGrad). Despite these empirical successes, the theoretical properties of these algorithms have not been well established due to technical difficulties. With this motivation, we focus on convergence analysis of mSGD and AdaGrad for any smooth (possibly non-convex) loss functions in stochastic optimization. First, we prove that the estimate sequence of mSGD is asymptotically convergent to a connected set of stationary points with probability one, which is more general than existing works on subsequence convergence or convergence of time averages. Moreover, we prove that the loss function of mSGD decays at a certain rate faster than that of SGD. In addition, we prove the estimate sequence of AdaGrad is asymptotically convergent to a connected set of stationary points with probability one. Also, this result extends the results from the literature on subsequence convergence and the convergence of time averages. Despite the generality of the above convergence results, we have relaxed some assumptions of gradient noises, convexity of loss functions, as well as boundedness of estimate sequences.

1 Introduction

In recent years, the rapid development of machine learning has stimulated a lot of applications of optimization algorithms to employing tremendous data in practical scenarios. Many optimization algorithms of machine learning are based on gradient descent (GD). A typical GD algorithm, minimizing a loss function $g(\boldsymbol{\theta}) \in \mathbb{R}$ via seeking an N-dimensional real-valued parameter $\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta} \in \mathbb{R}^N} g(\boldsymbol{\theta})$, writes as follows

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n - \varepsilon_n \nabla_{\boldsymbol{\theta}_n} g(\boldsymbol{\theta}_n), \tag{1}$$

where $\boldsymbol{\theta}_n$ is the estimate of $\boldsymbol{\theta}^*$ at step n, ε_n is a positive step size (learning rate) to be designed, and $\nabla_{\boldsymbol{\theta}_n} g(\boldsymbol{\theta}_n)$ stands for the gradient of $g(\boldsymbol{\theta}_n)$ at step n. When certain conditions are satisfied, $\boldsymbol{\theta}_n$ in equation 1 will converge to the optimal solution. However, since the computation of $\nabla_{\boldsymbol{\theta}_n} g(\boldsymbol{\theta}_n)$ relies on all data at each step, it is inefficient to apply GD-based algorithms (e.g., equation 1) when the data amount is very large. Therefore, more and more attention has been given on how to accelerate the GD-based algorithms. One of the attempts is stochastic gradient descent (SGD) which originated from Robbins & Monro (1951). Instead of calculating $\nabla_{\boldsymbol{\theta}_n} g(\boldsymbol{\theta}_n)$ over all data, SGD algorithms use a relatively small proportion of data to estimate the gradient, i.e., $\nabla_{\boldsymbol{\theta}_n} g(\boldsymbol{\theta}_n, \boldsymbol{\xi}_n)$ where $\boldsymbol{\xi}_n$ is a random vector introduced to choose a subset of data for each update. Then the SGD modified from GD equation 1 is as follows

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n - \varepsilon_n \nabla_{\boldsymbol{\theta}_n} g(\boldsymbol{\theta}_n, \boldsymbol{\xi}_n). \tag{2}$$

Besides reducing gradient computation, $g(\boldsymbol{\theta}_n, \boldsymbol{\xi}_n)$ can be used as an estimate of the gradient for the scenarios where the accurate gradient is unavailable due to external noises.

In recent years, SGD has shown its prominent advantages in dealing with high dimensional optimization problems such as regularized empirical risk minimization and training deep neural networks (see, e.g. Graves et al. (2013); Nguyen et al. (2018); Hinton & Salakhutdinov (2006); Krizhevsky et al. (2012) and references therein). In Nguyen et al. (2018); Bottou (2012), the convergence and the convergence rate of SGD have been analyzed.

Despite outstanding successes of SGD, it usually has a relatively slow convergence rate induced by random gradients and it provides fluctuating estimates in the learning process. In order to improve the theoretical and empirical performance of SGD, there have been a number of investigations, including three typical algorithms for accelerating the convergence rate: (1) The momentum-based stochastic gradient descend (mSGD), which reduces the update variance by averaging the past gradients (Polyak, 1964); (2) The adaptive gradient algorithm (AdaGrad), which replaces the step size of SGD with adaptive step size (Duchi et al., 2011); (3) Adaptive momentum gradient algorithm (Adam), which integrates mSGD and AdaGrad (Kingma & Ba, 2015). However, the convergence results of these algorithms have not been well established, especially for non-convex loss functions. In this paper, we focus on the investigation of mSGD and AdaGrad. We believe that the convergence analysis results in this paper are helpful for the future research on the convergence or generalization of Adam and other stochastic optimization algorithms.

The technique of mSGD is originally developed by Polyak (Polyak, 1964) for the acceleration of convergence rate of gradient-based methods. A typical expression of mSGD is as follows (Zaręba et al., 2015; Sutskever et al., 2013)

$$v_n = \alpha v_{n-1} + \varepsilon_n \nabla_{\theta_n} g(\theta_n, \xi_n), \quad \theta_{n+1} = \theta_n - v_n, \tag{3}$$

where $\alpha \in [0,1)$ and $\varepsilon_n > 0$ are relatively momentum coefficient and step size (learning rate), respectively. Another type of mSGD with the name of stochastic heavy ball (SHB) has also been studied (Gitman et al., 2019; Gupal & Bazhenov, 1972),

$$v_n = \beta_n v_{n-1} + (1-\beta_n) \nabla_{\theta_n} g(\theta_n, \xi_n), \quad \theta_{n+1} = \theta_n - \gamma_n v_n, \tag{4}$$

where $\beta_n \in (0,1)$ and γ_n are relatively momentum coefficient and step size (learning rate). In recent years, mSGD has been widely employed in the applications of deep learning such as image classification (Krizhevsky et al., 2012), fault diagnosis (Tang et al., 2018), statistical image reconstruction (Kim et al., 2014), etc. Moreover, a number of variants on momentum are emerging, see, e.g., synthesized Nesterov variants (SNV) (Williams & Lovett, 2016), robust momentum (Cyrus et al., 2018), and PID-control based methods (An et al., 2018). The importance of momentum in deep learning has been illustrated in Sutskever et al. (2013) through experiments. However, the theoretical analysis for convergence and convergence rate of mSGD needs further investigation, especially for non-convex loss functions which are common in deep learning. Most existing results are established with guaranteed subsequence convergence or convergence of time averages[1] (see Yang et al. (2016); Gitman et al. (2019); Polyak (1977); Kaniovskii (1983) and references therein). Nevertheless, there is a still distance to asymptotic convergence, which is usually more general and useful in practical applications requiring stable parameter estimates for each realization.

Convergence analysis for SGD unveils that the convergence rate heavily depends on the choice of step size ε_n (Robbins & Monro, 1951; Nguyen et al., 2018), which may consume a huge amount of efforts on fine tune. In order to deal with this problem, the authors in Duchi et al. (2011) proposed AdaGrad with adaptive step size based on the square summation of previous gradients. The form of AdaGrad is as follows (Ward et al., 2019; Duchi et al., 2011)

$$S_n = S_{n-1} + \left\| \nabla_{\theta_n} g(\theta_n, \xi_n) \right\|^2, \quad \theta_{n+1} = \theta_n - \frac{\alpha_0}{\sqrt{S_n}} \nabla_{\theta_n} g(\theta_n, \xi_n), \tag{5}$$

where $\alpha_0 > 0$ is a constant. In recent years, AdaGrad has shown its effectiveness in the field of sparse optimization (Duchi et al., 2013), tensor factorization (Lacroix et al., 2018), and deep learning (Heaton & Jeff, 2018). Some algorithm variants like RMSProp (Tieleman & Hinton, 2012) and SAdaGrad (Chen et al., 2018) are also studied. However, there are few results on the convergence of AdaGrad. Most of these results only prove subsequence

[1]The convergence definitions are given in Section 3.

convergence or convergence of time averages (see, e.g. Zou et al. (2019); Chen et al. (2019); Défossez et al. (2020); Ward et al. (2019)). Although Li & Orabona (2019) studied the asymptotic convergence of a modified AdaGrad algorithm, the result is not applicable to AdaGrad in equation 5. Thus, the asymptotic convergence of AdaGrad in equation 5 is still open.

In this theoretical paper, we aim to establish the convergence of mSGD and AdaGrad under mild conditions. The main contributions of this paper are three-fold:

- We prove that the estimate sequence of mSGD is asymptotically convergent to a connected set of stationary points for possibly non-convex loss function almost surely (i.e., with probability one), which is more general than existing works on subsequence convergence.
- We quantify the convergence rate of mSGD for the loss functions. Through this convergence rate we can get a theoretical explanation of why mSGD can be seen as an acceleration of SGD. Moreover, we provide the convergence rate of mean-square gradients and connect it to the convergence of time averages.
- We prove the estimate sequence of AdaGrad is asymptotically convergent to a connected set of stationary points almost surely for possible non-convex loss functions. The convergence result for the AdaGrad extends the subsequence convergence in the literature.

The remainder of the paper is organized as follows. In Section 2, we introduce the related work considering the convergence of mSGD and AdaGrad. The main results of the paper are given in Section 3, where we study the convergence and convergence rate of mSGD as well as the convergence of AdaGrad. Section 4 concludes the whole paper. Sections 5 and 6 are Code of Ethics and Reproducibility, respectively. The proofs are given in Appendix.

2 Related work

Convergence of mSGD: For the normalized mSGD (SHB), Polyak (Polyak, 1977; 1964) and Kaniovski (Kaniovskii, 1983) studied its convergence (subsequence convergence and convergence of time averages) properties for convex loss functions. Igor Gitman (Gitman et al., 2019) provided some convergence results of mSGD (SHB) for non-convex loss functions, but there is a considerable distance to the asymptotic convergence. Moreover, there is a requirement for uniform boundedness of a noise term in Gitman et al. (2019), i.e., $\mathbb{E}(\|\nabla_{\theta_n} g(\theta_n, \xi_n) - \nabla_{\theta_n} g(\theta_n\|) \leq \delta$, which confines the application scope of mSGD (SHB). In addition, the designs of momentum coefficients in Polyak (1977; 1964); Kaniovskii (1983); Gitman et al. (2019) are not consistent with some practical applications (Smith et al., 2018; Sutskever et al., 2013). Therefore, the asymptotic convergence of mSGD for convex and non-convex loss functions needs further investigation.

Convergence rate of mSGD: Despite outstanding empirical successes of mSGD, there are few results on convergence rate of mSGD. In these results, Mai & Johansson (2020) studied a class of convex loss functions, and obtained a convergence rate of time averages without reflecting the role of momentum parameter. Gitman et al. (2019) and Nicolas Loizou & Richtárik (2020) respectively investigated the asymptotic convergence rate of mSGD (SHB) by restricting to quadratic loss functions. Liu et al. (2020) studied the properties of SHB, where the relation between the loss function of SHB and the step size in every step was studied under the setting that α_n and β_n are constants. The convergence rate of time averages was also studied. However, since the momentum parameters are not consistent with some applications (Smith et al., 2018; Sutskever et al., 2013) and the standard mSGD in equation 3 is not covered, further studies are needed.

Convergence of AdaGrad: In the original work for AdaGrad (Duchi et al., 2011), it was proved that AdaGrad can converge faster in the time averages sense if gradients are sparse and the loss function is convex. Similar results were established by Chen et al. (2019), and Ward et al. (2019). Zou et al. (2019) and Défossez et al. (2020) established convergence results in the subsequence sense. Asymptotic convergence was obtained in Li & Orabona

(2019) for non-convex functions, but the form of the algorithm is no longer standard, as discussed in Duchi et al. (2011). Although such a change alleviates the difficulty in the proof of asymptotic convergence, it cannot be applied to the study of the AdaGrad in equation 5. Moreover, they required that a noise term is of point-wise boundedness (i.e., $\left\|\nabla_{\theta_n} g(\theta_n, \xi_n) - \nabla_{\theta_n} g(\theta_n)\right\| \leq \delta$, where δ is a positive constant), which however is relatively restrictive.

3 Main results

In this section, we provide the main results of this paper, including the analysis of convergence and convergence rate of mSGD in equation 2 and the analysis of convergence of AdaGrad in equation 5. In the following, $\mathbb{R}^N$ denotes the N-dimensional Euclidean space and $\|\cdot\|$ stands for the 2-norm, i.e., the Euclidean norm. To proceed, we need some definitions, consisting of asymptotic convergence, subsequence convergence, mean-square convergence, and convergence of time averages.

Definition 1 (Asymptotic convergence) A sequence $\{x_n\}$ is asymptotically convergent to a set $\mathbb{K}$, if $\lim_{n\to+\infty}\left(\inf_{x\in\mathbb{K}}\left\|x_n - x\right\|\right) = 0$.

Definition 2 (Subsequence convergence) A sequence $\{x_n\}$ converges in subsequence to a set $\mathbb{K}$, if there exists at least one subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $\lim_{n\to+\infty}\left(\inf_{x\in\mathbb{K}}\left\|x_{k_n} - x\right\|\right) = 0$.

Definition 3 (Mean-square convergence) A stochastic sequence $\{x_n\}$ converges in mean square to a fixed vector x, if $\lim_{n\to+\infty}\mathbb{E}(\|x_n - x\|^2) = 0$.

Definition 4 (Convergence of time averages) A stochastic sequence $\{x_n\}$ converges in time averages to a fixed vector x, if $\lim_{T\to+\infty}\frac{1}{T}\sum_{n=1}^{T}\mathbb{E}(\|x_n - x\|^2) = 0$.

It is obvious that asymptotic convergence implies subsequence convergence, and that mean-square convergence ensures convergence of time averages, but not vice versa.

3.1 Convergence of mSGD

In this subsection, with the help of some stochastic approximation techniques (Chen, 2006), we aim to prove that θ_n in equation 3 is asymptotically convergent to a connected component J^* of the set $J := \{\theta | \|\nabla_\theta g(\theta)\| = 0\}$ almost surely (a.s.) under proper conditions. When this connected component degenerates to a stationary point θ^*, it holds that $\theta_n \to \theta^*$, a.s..

In contrast to the existing works of subsequence convergence (cf. Zou et al. (2019); Chen et al. (2019); Défossez et al. (2020); Ward et al. (2019)), we aim to prove that θ_n of mSGD in equation 3 is able to achieve asymptotic convergence. Since mSGD in equation 3 is a stochastic algorithm, we aim to establish its a.s. asymptotic convergence. To proceed, we need some reasonable assumptions with respect to noise sequence $\{\xi_n\}$ and loss function $g(\theta)$.

Assumption 1 Noise sequence $\{\xi_n\}$ are mutually independent and independent of θ_1 and v_0, such that $g(x) = \mathbb{E}_{\xi_n}\left(g(x, \xi_n)\right)$ for any $x \in \mathbb{R}^N$.

Assumption 2 (Loss function assumption) Loss function $g(\theta)$ satisfies the following conditions:

1) $g(\theta)$ is a non-negative and continuously differentiable function.

2) The set of stationary points of $\|\nabla_\theta g(\theta)\|$ is not an empty set, that is

$$J := \{\theta | \|\nabla_\theta g(\theta)\| = 0\} \neq \emptyset.$$

3) $\nabla_\theta g(\theta)$ satisfies the Lipschitz condition, i.e., there is a scalar $c > 0$, such that for any $x, y \in \mathbb{R}^N$

$$\left\|\nabla_x g(x) - \nabla_y g(y)\right\| \leq c\|x - y\|.$$

4) There is a scalar $M > 0$ such that for any $\theta \in \mathbb{R}^N$ and positive integer n,

$$\mathbb{E}_{\xi_n}\left(\left\|\nabla_\theta g(\theta) - \nabla_\theta g(\theta, \xi_n)\right\|^2\right) \leq M\big(1 + g(\theta)\big). \tag{6}$$

Assumption 1 and conditions 1)–3) of Assumption 2 are common in the literature (Gitman et al., 2019). Assumption 2 does not pose any requirement on the convexity of $g(\theta)$. In other words, we allow any convex or non-convex loss functions $g(\theta)$ as long as they satisfy this assumption. Condition 4) corresponds to the condition in Shalev-Shwartz et al. (2011); Nemirovski et al. (2009); Hazan & Kale (2014); Gitman et al. (2019); Yang et al. (2016); Polyak (1977); Kaniovskii (1983) where the following inequality is assumed to hold for any $\theta \in \mathbb{R}^N$ and positive integer n,

$$\mathbb{E}_{\xi_n}\left(\left\|\nabla_\theta g(\theta) - \nabla_\theta g(\theta, \xi_n)\right\|^2\right) \leq K, \ \forall \theta \in \mathbb{R}^N, \tag{7}$$

where K is a positive scalar. Note that equation 6 reduces to equation 7 if $g(\theta)$ is uniformly upper bounded over the space $\mathbb{R}^N$. Since equation 6 does not need this uniform boundedness, it substantially extends equation 7 such that mSGD is also applicable to the scenarios with unbounded loss functions.

Different from deterministic GD-type algorithms with a constant step size, in order to ensure the convergence of mSGD, we need a decreasing step size for counteracting the randomness induced by noise $\{\xi_n\}$ (Gitman et al., 2019; Robbins & Monro, 1951). Specifically, we make the following assumption on step size ε_n together with a fixed momentum coefficient α.

Assumption 3 Momentum coefficient $\alpha \in [0, 1)$ and the sequence of step size ε_n is positive, monotonically decreasing to zero, such that $\sum_{n=1}^{+\infty} \varepsilon_n = +\infty$ and $\sum_{n=1}^{+\infty} \varepsilon_n^2 < +\infty$.

The setting of ε_n in Assumption 3 is consistent with stochastic approximation for root seeking of functions (Chen, 2006) as well as some stochastic optimization algorithms (Gupal & Bazhenov, 1972; Polyak, 1977; Kaniovskii, 1983; Gitman et al., 2019). An explicit example of step size ε_n satisfying Assumption 3 is $\varepsilon_n = 1/n$. Although SHB in equation 4 shares a similar expression as mSGD, the provided conditions of step sizes in Gupal & Bazhenov (1972); Polyak (1977); Kaniovskii (1983); Gitman et al. (2019) are not applicable to the general cases of mSGD with static α (Smith et al., 2018; Sutskever et al., 2013).

In the analysis of asymptotic convergence of mSGD, the following lemma plays an important role.

Lemma 1 Consider the mSGD in equation 3. If Assumptions 1–3 hold, then for $\forall \theta_1 \in \mathbb{R}^N$ and $v_0 \in \mathbb{R}^N$, there is a scalar $T(\theta_1, v_0)$, such that $\mathbb{E}\left(g(\theta_n)\right) < T(\theta_1, v_0)$ for any $n \geq 1$.

Lemma 1 actually guarantees stability of mSGD. Let $\mathscr{F}_n = \sigma(\theta_1, v_0, \{\xi_i\}_{i=1}^n)$ be the minimal σ-algebra generated by $\theta_1, v_0, \{\xi_i\}_{i=1}^n$. As a result, θ_n is adapted to $\mathscr{F}_{n-1}$. Then for any $n \in \mathbb{N}_+$, it holds that

$$\begin{aligned}
&\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2\right) = \mathbb{E}\left(\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right) \\
=&\mathbb{E}\left(\mathbb{E}_{\xi_n}\left(\left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2\right)\right) \leq M\Big(1 + \mathbb{E}\left(g(\theta_n)\right)\Big) \leq M(1 + T(\theta_1, v_0)),
\end{aligned} \tag{8}$$

where the second equality follows from the independence between ξ_n and $\{\theta_1, v_0, \{\xi_i\}_{i=1}^{n-1}\}$ in Assumption 1, and the last two inequalities hold due to 4) in Assumption 2 and Lemma 1, respectively. Intuitively, the result in equation 8 means that the fluctuation induced by random noise $\{\xi_i\}_{i=1}^{n-1}$ is well restrained. Note that the derived result in equation 8 is totally different from equation 7 required in the literature (Shalev-Shwartz et al., 2011; Nemirovski

et al., 2009; Hazan & Kale, 2014; Gitman et al., 2019; Yang et al., 2016; Polyak, 1977; Kaniovskii, 1983), since equation 7 needs a uniformly upper bound K over the whole space (i.e., $\theta \in \mathbb{R}^N$) which is difficult to satisfy when loss function $g(\theta)$ is quadratic or cubic with respect to θ over unbounded parameter space. In contrast, regardless of the order of $g(\theta)$ with respect to θ, equation 8 ensures boundedness of $\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2\right)$ for learning any fixed true parameter θ^*. This reflects a favorable learning process against random noise ξ_n for dealing with general loss functions $g(\theta)$. This result paves the way to the following theorem on asymptotic convergence of mSGD.

Theorem 1 Consider the mSGD in equation 3. If Assumptions 1–3 hold, then for $\forall \theta_1 \in \mathbb{R}^N$ and $\forall v_0 \in \mathbb{R}^N$, there exists a connected set $J^* \subseteq J$ such that the estimate θ_n is convergent to the set J^* almost surely, i.e.,

$$\lim_{n\to\infty} d(\theta_n, J^*) = 0, \qquad \text{a.s.}$$

where $d(x, J^*) = \inf_y\{\|x-y\|, y \in J^*\}$ denotes the distance between point x and set J^*.

In Theorem 1, we prove that the estimates of mSGD asymptotically converge to a connected set of stationary points almost surely. When this connected set degenerates to a stationary point θ^*, it holds that $\theta_n \to \theta^*$, a.s.. The result enables engineers for the design of proper momentum coefficients and step sizes (like $\alpha = 0.9$, $\varepsilon_n = 0.1/n$) in the related applications of mSGD with mathematic guarantee.

3.2 Convergence rate of mSGD

In this subsection, we analyze the convergence rate of mSGD. Before that, given positive real sequences $\{a_n\}$ and $\{b_n\}$, we let $a_n = O(b_n)$ if there is a constant $c > 0$, such that $a_n/b_n < c$ for any $n \geq 1$. In order to quantitative analysis. We need some extra assumptions.

Assumption 4 (Loss function assumption) Loss function $g(\theta)$ satisfies the following conditions:

1) $g(\theta)$ is a non-negative and continuously differentiable function. The set of its stationary points $J = \{\theta | \|\nabla_\theta g(\theta)\| = 0\}$ is a bounded set which has only finite connected components $J_1, ..., J_n$. In addition, there is $\varepsilon_1 > 0$, such that for any $i \in \{1, 2, \ldots, n\}$ and $0 < d(\theta, J_i) < \varepsilon_1$, it holds that $\left|g(\theta) - g_i\right| \neq 0$, where $g_i = \{g(\theta) | \theta \in J_i\}$ is a constant.

2) For any $i \in \{1, 2, \ldots, n\}$, it holds that

$$\liminf_{d(\theta, J_i)\to 0} \frac{\|\nabla_\theta g(\theta)\|^2}{g(\theta) - g_i} \geq s > 0.$$

Assumption 4 does not require any convexity but some slightly stronger conditions for the loss function around equilibrium points.

Theorem 2 Consider the mSGD in equation 3. If Assumptions 1-4 hold, for $\forall v_0 \in \mathbb{R}^N$ and $\forall \theta_1 \in \mathbb{R}^N$, it holds that

$$\mathbb{E}\left(\|\nabla_{\theta_n} g(\theta_n)\|^2\right) = O\left(e^{-\sum_{i=1}^n \frac{s\varepsilon_i}{p(1-\alpha)^2}}\right), \tag{9}$$

where $p = \exp\left\{\sum_{k=1}^\infty M\varepsilon_k^2\right\}$ and M is defined in condition 4) of Assumption 2.

In Theorem 2, let $\alpha = 0$, then we can obtain the convergence rate of SGD,

$$\mathbb{E}\left(\|\nabla_{\theta_n} g(\theta_n)\|^2\right) = O\left(e^{-\sum_{i=1}^n \frac{s}{p}\varepsilon_i}\right).$$

We can observe that the convergence rate of mSGD is quicker than that of SGD in this sense. Moreover, Theorem 2 provides a stronger characterization of convergence rate than some existing works considering the convergence rate of time averages $\frac{1}{T}\sum_{i=1}^T \mathbb{E}(\|\nabla_{\theta_n} g(\theta_n)\|^2) =$

$O(T^l)$ ($l=-1$ in Liu et al. (2020) and $l=-1/2$ in Mai & Johansson (2020)). In the following, we will elaborate on this point. From equation 9, there exists a scalar $t_0>0$ such that $\forall n>0$

$$\frac{\mathbb{E}\left(\|\nabla_{\theta_n} g(\theta_n)\|^2\right)}{\exp\left\{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}\right\}}<t_0,$$

implying

$$\frac{\frac{1}{T}\sum_{n=1}^{T}\mathbb{E}\left(\|\nabla_{\theta_n} g(\theta_n)\|^2\right)}{\frac{1}{T}\sum_{n=1}^{T}\exp\left\{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}\right\}}<t_0.$$

So it holds that

$$\frac{1}{T}\sum_{n=1}^{T}\mathbb{E}\left(\|\nabla_{\theta_n} g(\theta_n)\|^2\right)=O\left(\frac{1}{T}\sum_{n=1}^{T}e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right).$$

Let $\varepsilon_n=\frac{1}{n}$, then we know that $\sum_{n=1}^{T}\varepsilon_n=O(\ln T)$. Hence,

$$\frac{1}{T}\sum_{n=1}^{T}e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}=O\left(\frac{1}{T}\sum_{n=1}^{T}e^{-\frac{s\ln(n)}{p(1-\alpha)^2}}\right)=O\left(\frac{1}{T}\sum_{n=1}^{T}n^{-\frac{s}{p(1-\alpha)^2}}\right).$$

If $\frac{s}{p(1-\alpha)^2}<1$, $\sum_{n=1}^{T}n^{-\frac{s}{p(1-\alpha)^2}}=O\left(T^{-\frac{s}{p(1-\alpha)^2}+1}\right)$, so we have

$$\frac{1}{T}\sum_{n=1}^{T}e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}=\frac{1}{T}O\left(T^{-\frac{s}{p(1-\alpha)^2}+1}\right)=O\left(T^{-\frac{s}{p(1-\alpha)^2}}\right).$$

If $\frac{s}{p(1-\alpha)^2}=1$, it holds that $\sum_{n=1}^{T}n^{-\frac{s}{p(1-\alpha)^2}}=O(\ln T)$, and hence

$$\frac{1}{T}\sum_{n=1}^{T}e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}=\frac{1}{T}O(\ln T)=O\left(\frac{\ln T}{T}\right).$$

If $\frac{s}{p(1-\alpha)^2}>1$, then $\sum_{n=1}^{T}n^{-\frac{s}{p(1-\alpha)^2}}:=C_\alpha(T)\to C_\alpha$ as $T\to+\infty$, where C_α is a positive constant, so

$$\frac{1}{T}\sum_{n=1}^{T}e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}=\frac{C_\alpha(T)}{T}=O\left(\frac{C_\alpha}{T}\right).$$

Note that the coefficient C_α depends on s, p, and α, where s is the constant given in Assumption 4 only depending on $g(\theta)$, and p is given in Theorem 2 and relies on M in 4) of Assumption 2 and step size ε_n. The above observation indicates that setting α close to one makes mSGD achieve convergence rate of time averages of order $O(1/T)$. As remarked previously, larger α implies smaller coefficient C_α and thus quicker convergence rate. Interestingly, since $C_\alpha(T)$ is monotonically increasing, the convergence rate has a relatively small coefficient when T is small. This could illustrate why mSGD can achieve better performance in the early phase of iteration.

3.3 Convergence of AdaGrad

In this subsection, we aim to establish the convergence of AdaGrad. Compared to the study of mSGD, the design of adaptive step size increases the technical difficulties. To proceed, the required conditions on loss function $g(\theta)$ are summarized as follows.

Assumption 5 Loss function $g(\theta)$ in equation 5 satisfies the following conditions:

1) $g(\theta)$ is a non-negative and continuously differentiable function. The set of its stationary points $J=\{\theta|\|\nabla_\theta g(\theta)\|=0\}$ is a bounded set which has only finite connected components $J_1,\ldots,J_n$. In addition, there is $\varepsilon_1>0$, such that for any i and $0<d(\theta,J_i)<\varepsilon_1$, it holds that $\left|g(\theta)-g_i\right|\neq 0$, where $g_i=\{g(\theta)|\theta\in J_i\}$ is a constant.

2) The gradient $\nabla_\theta g(\theta)$ satisfies the Lipschitz condition, i.e., for any $x,y\in\mathbb{R}^N$,

$$\left\|\nabla_x g(x)-\nabla_y g(y)\right\|\leq c\|x-y\|.$$

3) There are two constants $M' > 0$ and $a > 0$ such that for any $\theta \in \mathbb{R}^N$ and $n \in \mathbb{N}_+$,

$$E_{\xi_n}\left(\left\|\nabla_\theta g(\theta,\xi_n)\right\|^2\right) \leq M'\left\|\nabla_\theta g(\theta)\right\|^2 + a. \tag{10}$$

Condition 2) is the same as in 2) of Assumption 2, and condition 1) is relatively weak. Assumption 5 does not need any convexity of loss function, and can be satisfied by general loss functions in machine learning. Similar to condition 4) of Assumption 2, condition 3) is a condition to restrain the noise influence. Equation 10 is milder than $\|\nabla_\theta g(\theta,\xi_n) - \nabla_\theta g(\theta)\| \leq S$ *a.s.* (Li & Orabona, 2019; Défossez et al., 2020), which is relatively restrictive in dealing with unbounded noises, e.g., $\nabla_\theta g(\theta,\xi_n) = \nabla_\theta g(\theta) + \xi_n$, where ξ_n is independent identically distributed and Gaussian.

Compared to SGD and mSGD, there are more challenges in analyzing the convergence of AdaGrad. The challenges mainly come from two aspects: (1) since AdaGrad does not have a decreasing step size, the noise influence on AdaGrad can not be restrained as mSGD which is with the help of decreasing step size satisfying Assumption 3; (2) adaptive step size of AdaGrad in equation 5 (i.e., $\alpha_0/\sqrt{\sum_{i=1}^n \|\nabla_{\theta_i} g(\theta_i,\xi_i)\|^2}$) is a random variable conditionally dependent of ξ_n given $\sigma\{\theta_1,\ \xi_1,...,\ \xi_{n-1}\}$. Then when we deal with terms in the proof like

$$\frac{\alpha_0}{\sqrt{S_n}}\nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n) \quad \text{or} \quad \frac{\alpha_0^2}{S_n}\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2, \tag{11}$$

we can not make the conditional expectation to transform equation 11 to

$$\frac{\alpha_0}{\sqrt{S_n}}\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \quad \text{or} \quad \frac{\alpha_0^2}{S_n} E_{\xi_n}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2\right).$$

In the literature, Li & Orabona (2019) changed the step size to $\alpha_0/\sqrt{\sum_{i=1}^{n-1} \|\nabla_{\theta_i} g(\theta_i,\xi_i)\|^2}$ to make it conditionally independent of ξ_n given $\sigma\{\theta_1,\ \xi_1,...,\ \xi_{n-1}\}$, which however is no longer the standard AdaGrad (Duchi et al., 2011; Chen et al., 2019; Ward et al., 2019; Défossez et al., 2020).

Before we provide the main theorem of this subsection, a useful lemma is worth discussing.

Lemma 2 Let $\{\alpha_k\}$, $\{\beta_k\}$ being non-negative random variable sequences, such that the following conditions hold almost surely

- $\sum_{k=1}^{+\infty} \alpha_k = \infty$;
- $\sum_{k=1}^{+\infty} \alpha_k \beta_k < +\infty$;

Then there exists a subsequence $\{\beta_{n_k}\}$ of $\{\beta_k\}$, such that $\beta_{n_k} \xrightarrow{k\to\infty} 0$ almost surely.

Proof 1 We aim to prove $\liminf_{k\to+\infty} \beta_k = 0$ by contradiction. Suppose $\liminf_{k\to+\infty} \beta_k = u > 0$, then $\exists k_0$, such that $\forall k > k_0$, $\beta_k > u/2$. It follows that

$$\sum_{k=k_0+1}^{+\infty} \alpha_k < \frac{2}{u}\sum_{k=k_0+1}^{+\infty} \alpha_k \beta_k < +\infty. \tag{12}$$

Obviously, there is a contradiction between equation 12 and the condition $\sum_{k=1}^{+\infty} \alpha_k = +\infty$. Thus, $\liminf_{k\to+\infty} \beta_k = 0$. So we get that there exists a subsequence $\{\beta_{n_k}\}$ of $\{\beta_k\}$ holds $\beta_{n_k} \xrightarrow{k\to\infty} 0$ almost surely.

This lemma, inspired by Proposition 2 in Ya. I. Alber (Alber et al., 1998), is quite useful in the convergence analysis of AdaGrad. Then we are ready to provide the convergence result of AdaGrad in the following theorem.

Theorem 3 Consider the AdaGrad in equation 5. If Assumptions 1 and 5 hold, then for $\forall \theta_1 \in \mathbb{R}^N$ and $S_0 = 0$, there exists a connected component of the set $J^* \subseteq J$, such that the estimate θ_n is convergent to the set J^* almost surely, i.e.,

$$\lim_{n\to\infty} d(\theta_n, J^*) = 0. \qquad \text{a.s.}$$

In Theorem 3, we prove the standard AdaGrad is convergent almost surely, in contrast to the convergence of time averages in Chen et al. (2019); Ward et al. (2019) which focus on the metric $\frac{1}{T}\sum_{i=1}^{T}\mathbb{E}(\|\nabla_{\theta_n} g(\theta_n)\|^2)$, or the subsequence convergence in Zou et al. (2019); Défossez et al. (2020), focusing on $\min_{n=0,1,2,\ldots,T}\mathbb{E}(\|\nabla_{\theta_n} g(\theta_n)\|)$.

4 Conclusion

In this paper, we studied the convergence of two algorithms extensively used in machine learning applications, namely, momentum-based stochastic gradient descent (mSGD) and adaptive step stochastic gradient descent (AdaGrad). By considering general loss functions (either convex or non-convex), we first establish the almost sure asymptotic convergence of mSGD. Moreover, we find the convergence rate of the mSGD and reveal that the mSGD indeed has a faster convergence rate than the SGD. Furthermore, we prove AdaGrad is convergent almost surely under mild conditions. Subsequence convergence and convergence of time averages in the literature are substantially extended in this work to asymptotic convergence.

5 Code of Ethics

This is a theoretical paper focusing on the investigation for the convergence of mSGD and AdaGrad optimization algorithms. The results in this paper do not have issues in fairness, inappropriate potential applications impact, privacy and security issues, legal compliance, research Integrity Issues, or responsible research practice (e.g., IRB, documentation, research ethics). This paper does no involve human subjects, practices to data set releases, potentially harmful insights, methodologies and applications, or pontential conflicts of interest and sponsorship.

6 Reproducibility

This is a theoretical paper focusing on the investigation for the convergence of mSGD and AdaGrad optimization algorithms. The developed results are provided in the main paper, i.e., Theorems 1-3. In Appendix, the proofs of these theorems are provided together with some useful lemmas as well as their proofs. An outline of these proofs is given in Section A in Appendix.

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A Appendix Outline

Section B aims to verify the convergence results of mSGD. Several auxiliary lemmas are first provided, followed by their proofs in Sections B.1-B.8, where the proof of Lemma 1 in the main paper is given in Section B.4. Theorems 1-2 are proved in Sections B.9 and B.10, respectively. Section C aims to verify the convergence results of AdaGrad. Some auxiliary lemmas are given at the beginning, followed by their proofs in Sections C.1-C.5. Theorem 3 is proved in Section C.6.

B Convergence and convergence rate of mSGD

The following lemmas are used for the proofs of Theorems 1-2. Hereafter, $\mathbb{N}_+$ denotes the set of all positive integers. A function f is in a (differentiability) class C^k if its l–th derivatives exist and are continuous, where $l \leq k$.

Lemma 3 Suppose $f(x) \in C^1 \quad (x \in \mathbb{R}^N)$ with gradient satisfying the following Lipschitz condition

$$\|\nabla f(x) - \nabla f(y)\| \leq c\|x-y\|,$$

then for any $x, y \in \mathbb{R}^N$, it holds that

$$f(y) + \nabla f(y)^T(x-y) - \frac{c}{2}\|x-y\|^2 \leq f(x) \leq f(y) + \nabla f(y)^T(x-y) + \frac{c}{2}\|x-y\|^2.$$

Lemma 4 Suppose $f(x) \in C^1 \quad (x \in \mathbb{R}^N)$ with gradient satisfying the following Lipschitz condition

$$\left\|\nabla f(x) - \nabla f(y)\right\| \leq c\|x-y\|,$$

and the set $S = \{x|\nabla f(x) = 0\}$ only has finite connected components $S_1, S_2, ..., S_m$. Furthermore, assume there exists $\varepsilon_1 > 0$, such that for any $i = 1, 2, ..., m$ and $x \in \{x|0 < d(x, S_i) < \varepsilon_1\}$, it holds that $\left|f(x) - f_i\right| \neq 0$, where $f_i = f(x)$ for $x \in S_i$. Then for any $i = 1, 2, ..., m$, if there is $\varepsilon_0 > 0$ satisfying $d(x, S_i) < \varepsilon_0$, it follows that

$$\left\|\nabla f(x)\right\|^2 \leq 2c\left|f(x) - f_i\right|.$$

Lemma 5 (Wang et al., 2019) Suppose that $\{X_n\} \in \mathbb{R}^N$ is an $\mathscr{L}_2$ martingale difference sequence, and $(X_n, \mathscr{F}_n)$ is an adaptive process. Then it holds that $\sum_{k=0}^{\infty} X_k < +\infty$ *a.s.*, if

$$\sum_{n=1}^{\infty} \mathbb{E}(\|X_n\|^2) < +\infty, \quad \text{or} \quad \sum_{n=1}^{\infty} \mathbb{E}\left(\|X_n\|^2 \middle| \mathscr{F}_{n-1}\right) < +\infty. \quad a.s.$$

Lemma 6 Suppose that $\{X_n\} \in \mathbb{R}^N$ is a non-negative sequence of random variables, then it holds that $\sum_{n=0}^{\infty} X_n < +\infty$ *a.s.*, if $\sum_{n=0}^{\infty} \mathbb{E}\left(X_n\right) < +\infty$.

Lemma 7 Suppose $\{v_n\}$ is a sequence generated by mSGD in equation 3. Under Assumptions 1–3, it holds that $\sum_{n=0}^{+\infty} \mathbb{E}\left(\|v_n\|^2\right) < c(v_0, \theta_1)$, and $\sum_{n=0}^{+\infty} \|v_n\|^2 < +\infty$ *a.s.*, where $c(v_0, \theta_1)$ is a constant only related to v_0 and θ_1.

Lemma 8 Suppose $\{\theta_n\}$ is a sequence generated by mSGD in equation 3. Under Assumptions 1–3, it holds that: for $n \geq 1$,

$$\sum_{t=1}^{n} \varepsilon_t \mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t)\|^2\right) < B(v_0, \theta_1) < +\infty, \qquad \sum_{t=1}^{n} \varepsilon_t \|\nabla_{\theta_t} g(\theta_t)\|^2 < +\infty.$$

where $B(v_0, \theta_1) > 0$ is a constant only related to v_0 and θ_1.

Lemma 9 Suppose $\{\theta_n\}$ is a sequence generated by mSGD in equation 3 such that $\{g(\theta_n)\}$ converges a.s.. If Assumptions 1–3 hold, and

$$g(\theta_{n+1}) \leq \zeta_n - b\sum_{t=1}^{n} \varepsilon_t \left\|\nabla g(\theta_t)\right\|^2 \ a.s., \tag{13}$$

where ζ_n is a random variable such that $\lim_{n\to+\infty} \zeta_n = \zeta < \infty$ *a.s.*, and b is a positive constant, then there exists a connected component J^* of $J := \{\theta|\nabla_\theta g(\theta) = 0\}$, such that

$$d(\theta_n, J^*) \xrightarrow{a.s.} 0.$$

B.1 Proof of Lemma 3

For any $x, y \in \mathbb{R}^N$, we construct a unary function

$$h(\theta) = f\big(y + \theta(x-y)\big) \quad \big(\theta \in [0,1]\big).$$

It is easy to see $h \in C^1$. Then $\forall \theta \in [0,1]$, it holds that

$$\begin{aligned} f(x) = h(1) = h(0) + \int_0^1 h'(t)dt &= f(y) + \int_0^1 \nabla f\big(y + t(x-y)\big)^T (x-y)dt \\ &= f(y) + \nabla f(y)^T (x-y) + \int_0^1 \Big(\nabla f\big(y+t(x-y)\big) - \nabla f(y)\Big)^T (x-y)dt. \end{aligned}$$

Then we get

$$\begin{aligned} &\big|f(x) - f(y) - \nabla f(y)^T (x-y)\big| = \left| \int_0^1 \Big(\nabla f\big(y+t(x-y)\big) - \nabla f(y)\Big)^T (x-y)dt \right| \\ &\leq \int_0^1 \Big\|\nabla f\big(y+t(x-y)\big) - \nabla f(y)\Big\| \|x-y\| dt. \end{aligned} \tag{14}$$

Since ∇f satisfies the Lipschitz condition, it holds that

$$\|\nabla f(x+t(x-y)) - \nabla f(y)\| \leq c\|t(x-y)\| \leq ct\|x-y\|. \tag{15}$$

Substituting equation 15 into equation 14 yields

$$\big|f(x) - f(y) - \nabla f(y)^T (x-y)\big| \leq c\|x-y\|^2 \int_0^1 t dt = \frac{c}{2}\|x-y\|^2,$$

which leads to

$$f(y) - \nabla f(y)^T (x-y) - \frac{c}{2}\|x-y\|^2 \leq f(x) \leq f(y) - \nabla f(y)^T (x-y) + \frac{c}{2}\|x-y\|^2. \tag{16}$$

B.2 Proof of Lemma 4

First we construct a closed and bounded set S which satisfies $S \supset \bigcup_{i=1}^m S_i$. Since $\|\nabla f(x)\|$ ($x \in S$) is a continuous function on a closed set, $\|\nabla f(x)\|$ ($x \in S$) is a uniformly continuous function. Then $\exists\ \varepsilon_2 > 0$, $\forall\ S_i$, if $d(x, S_i) < \varepsilon_2$, there is $\|\nabla f(x)\| < c\varepsilon_1/4$. We assign $\varepsilon_0 = \min\{\varepsilon_1/4, \varepsilon_2\}$.

Let $S_i' = \{x | d(x, S_i) < \varepsilon_1\}$, $S_i'' = \{x | d(x, S_i) < \varepsilon_0\}$. Since $d(x, S_i)$ is a continuous function, it holds that $\forall x_0 \in S_i''$. $\exists x_1 \in S_i$, such that $\overline{x_0 x_1} \subset S_i''$ ($\overline{x_0 x_1}$ is a straight line connecting x_1 and x_0). Let $x_2 = \arg\min_{x \in L_i} d(x, x_0)$, where $L_i = \{x | x \in \overline{x_0 x_1}, x \in S_i\}$. Then we construct a unary function

$$g(t) = \begin{cases} \left| f\left(x_2 + t\frac{x_0 - x_2}{\|x_0 - x_2\|}\right) - f_i \right|, & \text{if } t > 0 \\ \left| f\left(x_2 - t\frac{x_0 - x_2}{\|x_0 - x_2\|}\right) - f_i \right|, & \text{if } t < 0. \end{cases}$$

Denote

$$t_0 = \inf_{t>0} \left\{ x_2 + t\frac{x_0 - x_2}{\|x_0 - x_2\|} \notin S_i' \right\}.$$

It follows that $g(t) \geq 0$ ($t \in (-t_0, 0) \cup (0, t_0)$), $g(0) = 0$ and $g(t) \in C^1$. Due to $g(t) \in C^1$, g' satisfies the *Lipschitz condition* with the *Lipschitz constant* c. Then through the *Lipschitz condition* of f', we get that

$$g(t') - g(t'') = \int_{t''}^{t'} \big(g'(x) - g'(t'')\big)dx + \int_{t''}^{t'} g'(t'')dx \geq -\int_{t''}^{t'} c|t'' - x|dx + \int_{t''}^{x} g'(t'')dx.$$

Then we take a special value of t' and t'', that is $t' = \|x_0 - x_2\|$, $t'' = t' - g'(t')/c$. Also, it holds that

$$g(t'') = \left| f\left(x_2 + \left(t' - \frac{g'(t')}{c}\right)\frac{x_2 - x_0}{\|x_2 - x_0\|}\right) - f_i \right| = \left| f\left(x_0 - \frac{g'(t')}{c}\frac{x_2 - x_0}{\|x_2 - x_0\|}\right) - f_i \right|.$$

On the other hand, there is

$$\left\| x_0 - \frac{g'(t')}{c}\frac{x_2 - x_0}{\|x_2 - x_0\|} - x_2 \right\| \leq \|x_0 - x_2\| + \left\| \frac{g'(t')}{c}\frac{x_2 - x_0}{\|x_2 - x_0\|} \right\| = \|x_0 - x_2\| + \frac{\|\nabla f(x_0)\|}{c}. \tag{17}$$

Due to $x_0 \in S_i''$, we know $\|\nabla f(x_0)\|/c \leq \varepsilon_1/4$. Substitute into equation 17, we get that

$$\left\| x_0 - \frac{g'(t')}{c}\frac{x_2 - x_0}{\|x_2 - x_0\|} - x_2 \right\| \leq \|x_0 - x_2\| + \frac{\varepsilon_1}{4} \leq \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4} < \varepsilon_1.$$

Then it holds that

$$x_0 - \frac{g'(t')}{c}\frac{x_2 - x_0}{\|x_2 - x_0\|} \in S_i'.$$

So $t'' \in (-t_0, t_0)$, which leads to $g(t'') > 0$. As a result,

$$g(t') \geq g(t') - g(t'') \geq -\int_{t''}^{t'} c|t'' - x|dx + \int_{t''}^{x} g'(t'')dx = \frac{1}{2c}(g'(t'))^2 = \frac{1}{2c}\|\nabla f(x_0)\|^2.$$

Due to the arbitrariness of x_0, we draw the conclusion that $\exists \varepsilon_0 = \min\{\varepsilon_1/4, \varepsilon_2\}$, $\forall S_i$ $(i = 1, 2, ..., m)$, if $d(x, S_i) < \varepsilon_0$, there is

$$\left\|\nabla f(x)\right\|^2 \leq 2c\left|f(x) - f_i\right|.$$

B.3 Proof of Lemma 6

For $\forall \delta > 0$, we have

$$\begin{aligned} &P\left(\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}\left(\sum_{t=n}^{m} X_t \geq \delta\right)\right) \\ &= \lim_{n\to+\infty} P\left(\bigcup_{m=n}^{+\infty}\left(\sum_{t=n}^{m} X_t \geq \delta\right)\right) = \lim_{n\to+\infty} P\left(\sum_{t=n}^{+\infty} X_t \geq \delta\right) \leq \lim_{n\to+\infty}\frac{1}{\delta}\mathbb{E}\left(\sum_{t=n}^{+\infty} X_t\right), \end{aligned}$$

where the last inequality is due to Markov inequality. Since $\sum_{t=1}^{\infty}\mathbb{E}\left(X_t\right) < \infty$, it follows that

$$P\left(\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}\left(\sum_{t=n}^{m} X_t \geq \delta\right)\right) \leq \lim_{n\to+\infty}\frac{1}{\delta}\mathbb{E}\left(\sum_{t=n}^{+\infty} X_t\right) = 0. \tag{18}$$

By Cauchy's convergence test, we have $\sum_{t=1}^{\infty} X_t < \infty, \quad a.s.$

B.4 Proof of Lemma 1

Recall the mSGD algorithm in equation 3

$$v_n = \alpha v_{n-1} + \varepsilon_n \nabla_{\theta_n} g(\theta_n, \xi_n) \tag{19}$$

$$\theta_{n+1} = \theta_n - v_n. \tag{20}$$

Equation equation 19 is equivalent to

$$v_n = \alpha v_{n-1} + \varepsilon_n \nabla_{\theta_n} g(\theta_n) + \varepsilon_n\big(\nabla_{\theta_n} g(\theta_n, \xi_n) - \nabla_{\theta_n} g(\theta_n)\big).$$

Under Assumption 2 2), it follows from Lemma 3 that

$$-\nabla_{\theta_t} g(\theta_t)^{\mathsf{T}} v_t - \frac{c}{2}\|v_t\|^2 \leq g(\theta_{t+1}) - g(\theta_t) \leq -\nabla_{\theta_t} g(\theta_t)^{\mathsf{T}} v_t + \frac{c}{2}\|v_t\|^2. \tag{21}$$

In this subsection, we just use the right side of equation 21. The left side will be used in the next subsection. Consider $\nabla_{\theta_t} g(\theta_t)^{\mathsf{T}} v_t$ in the following

$$\begin{aligned} \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}} v_t &= (\nabla_{\theta_t} g(\theta_t))^{\mathsf{T}}(\alpha v_{t-1} + \varepsilon_t \nabla_{\theta_n} g(\theta_t, \xi_t)) \\ &= \alpha(\nabla_{\theta_{t-1}} g(\theta_{t-1}) + \nabla_{\theta_t} g(\theta_t) - \nabla_{\theta_{t-1}} g(\theta_{t-1}))^{\mathsf{T}} v_{t-1} + \varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}} \nabla_{\theta_t} g(\theta_t, \xi_t) \\ &= \alpha \nabla_{\theta_{t-1}} g(\theta_{t-1})^{\mathsf{T}} v_{t-1} + \alpha(\nabla_{\theta_t} g(\theta_t) - \nabla_{\theta_{t-1}} g(\theta_{t-1}))^{T} v_{t-1} + \varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}} \nabla_{\theta_t} g(\theta_t, \xi_t) \end{aligned}$$

Recursively applying the above equation yields

$$\begin{aligned}\nabla_{\theta_t} g(\theta_t)^\top v_t = \alpha^{t-1}\nabla_{\theta_1} g(\theta_1)^\top v_1 + \sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1}))^T v_{i-1} \\ + \sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i \nabla_{\theta_i} g(\theta_i)^\top \nabla_{\theta_i} g(\theta_i,\xi_i).\end{aligned} \tag{22}$$

By substituting the above equation into equation 21 and noting $-(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1}))^\top v_{i-1} \le \left\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1})\right\| \|v_{i-1}\| \le c\|v_{t-1}\|^2$, we obtain

$$\begin{aligned}& g(\theta_{t+1}) - g(\theta_t) \\ \le & -\alpha^{t-1}\nabla_{\theta_1} g(\theta_1)^\top v_1 - \sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i \nabla_{\theta_i} g(\theta_i)^\top \nabla_{\theta_i} g(\theta_i,\xi_i) + \frac{c}{2}\|v_t\|^2 \\ & - \sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_t} g(\theta_t) - \nabla_{\theta_{t-1}} g(\theta_{t-1}))^T v_{t-1} \\ < & -\alpha^{t-1}\nabla_{\theta_1} g(\theta_1)^\top v_1 - \sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i \nabla_{\theta_i} g(\theta_i)^\top \nabla_{\theta_i} g(\theta_i,\xi_i) + c\sum_{i=1}^{t}\alpha^{t-i}\|v_i\|^2.\end{aligned} \tag{23}$$

Denote $Z(t) := \prod_{k=t}^{+\infty}(1+M_0\varepsilon_k^2)$, where

$$M_0 = \frac{cM}{\alpha^{1-\delta}(1-\alpha^\delta)(1-\alpha)^2},$$

where $\delta > 0$ is a constant and M is introduced in Assumption 2 4). Here we define M_0, $Z(t)$ and δ to facilitate the proof equation 34. From Assumption 3, it holds that $\sum_{t=1}^{+\infty}\varepsilon_t^2 < +\infty$. Thus, $\sum_{t=1}^{+\infty} M_0\varepsilon_t^2 < +\infty$. From a general inequality $\ln(1+x) \le x$ for $x > -1$, we get

$$Z(t) \le \prod_{k=1}^{+\infty}(1+M_0\varepsilon_k^2) = \exp\left\{\sum_{k=1}^{\infty}\ln(1+M_0\varepsilon_k^2)\right\} \le \exp\left\{\sum_{k=1}^{\infty} M_0\varepsilon_k^2\right\} < +\infty,$$

which means that $Z(t)$ is uniformly upper bounded. Then multiplying $Z(t+1)$ on both sides of equation 23 and taking the mathematical expectation yield

$$\begin{aligned}Z(t+1)\,\mathbb{E}\left(g(\theta_{t+1}) - g(\theta_t)\right) \le & -Z(t+1)\alpha^{t-1}\,\mathbb{E}\left(\nabla_{\theta_1} g(\theta_1)^\top v_1\right) + c\sum_{i=1}^{t}\alpha^{t-i}Z(i+1)\,\mathbb{E}\left(\|v_i\|^2\right) \\ & - Z(t+1)\sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i\,\mathbb{E}\left\|\nabla_{\theta_i} g(\theta_i)\right\|^2 \\ < & -Z(t+1)\alpha^{t-1}\,\mathbb{E}\left(\nabla_{\theta_1} g(\theta_1)^\top v_1\right) + c\sum_{i=1}^{t}\alpha^{t-i}Z(i+1)\,\mathbb{E}\left(\|v_i\|^2\right),\end{aligned} \tag{24}$$

where the first and second inequalities are respectively due to $Z(i+1) < Z(i)$ and $Z(t+1)\sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i\,\mathbb{E}\left\|\nabla_{\theta_i} g(\theta_i)\right\|^2 > 0$.

Next, we aim to analyze $c\sum_{i=1}^{t}\alpha^{t-i}Z(i+1)\,\mathbb{E}\left(\|v_i\|^2\right)$ in equation 24. It is proved in Appendix B.5 that

$$\begin{aligned}Z(i+1)\,\mathbb{E}\|v_i\|^2 \le & \alpha^{(1+\delta)i}Z(1)\,\mathbb{E}\|v_0\|^2 + \frac{1}{\alpha^{1-\delta}}\sum_{k=1}^{i}\alpha^{(1+\delta)(i-k)}\varepsilon_k^2 Z(k+1)\,\mathbb{E}\left(\|\nabla_{\theta_k} g(\theta_k) - \nabla_{\theta_k} g(\theta_k,\xi_k)\|^2\right) \\ & - \frac{2}{\alpha^{1-\delta}}\sum_{k=1}^{i}\alpha^{(1+\delta)(i-k)}Z(k+1)\Big(\mathbb{E}\left(\varepsilon_k g(\theta_{k+1})\right) - \mathbb{E}\left(\varepsilon_{k-1} g(\theta_k)\right)\Big).\end{aligned} \tag{25}$$

Taking a weighted sum of equation 25 yields

$$\begin{aligned}
&\sum_{i=1}^{t} \alpha^{t-i} Z(i+1)\, \mathbb{E}\, \|v_i\|^2 \\
\leq &\sum_{i=1}^{t} \alpha^{t-i} \alpha^{(1+\delta)i} Z(1)\, \mathbb{E}\left(\|v_0\|^2\right) \\
&+ \sum_{i=1}^{t} \alpha^{t-i} \left(\frac{1}{\alpha^{1-\delta}} \sum_{k=1}^{i} \alpha^{(1+\delta)(i-k)} \varepsilon_k^2 Z(k+1)\, \mathbb{E}\left(\|\nabla_{\theta_k} g(\theta_k) - \nabla_{\theta_k} g(\theta_k, \xi_k)\|^2\right) \right) \\
&- \sum_{i=1}^{t} \alpha^{t-i} \left(\frac{2}{\alpha^{1-\delta}} \sum_{k=1}^{i} \alpha^{(1+\delta)(i-k)} Z(k+1) \left(\mathbb{E}\left(\varepsilon_k g(\theta_{k+1})\right) - \mathbb{E}\left(\varepsilon_{k-1} g(\theta_k)\right) \right) \right) := A + B + C.
\end{aligned} \tag{26}$$

We derive that

$$A = \left(\sum_{i=1}^{t} \alpha^{\delta i} \right) \alpha^t Z(1)\, \mathbb{E}\left(\|v_0\|^2\right) \leq \frac{\alpha^{t-1}\alpha^{\delta}}{1-\alpha^{\delta}} Z(1)\, \mathbb{E}\left(\|v_0\|^2\right), \tag{27}$$

$$\begin{aligned}
B &= \frac{1}{\alpha^{1-\delta}} \sum_{i=1}^{t} \sum_{k=1}^{i} \alpha^{t-k+\delta(i-k)} \left(\varepsilon_k^2 Z(k+1)\, \mathbb{E}\left(\|\nabla_{\theta_k} g(\theta_k) - \nabla_{\theta_k} g(\theta_k, \xi_k)\|^2\right) \right) \\
&\leq \frac{1}{\alpha^{1-\delta}(1-\alpha^{\delta})} \sum_{k=1}^{t} \alpha^{t-k} Z(k+1) \varepsilon_k^2\, \mathbb{E}\left(\|\nabla_{\theta_k} g(\theta_k) - \nabla_{\theta_k} g(\theta_k, \xi_k)\|^2\right),
\end{aligned} \tag{28}$$

$$\begin{aligned}
C &= -\frac{2}{\alpha^{1-\delta}} \sum_{k=1}^{t} \sum_{i=k}^{t} \alpha^{t-k+\delta(i-k)} Z(k+1) \left(\mathbb{E}\left(\varepsilon_k g(\theta_{k+1})\right) - \mathbb{E}\left(\varepsilon_{k-1} g(\theta_k)\right) \right) \\
&= -\frac{2}{\alpha^{1-\delta}} \sum_{k=1}^{t} \left(\sum_{i=0}^{t-k} \alpha^{\delta i} \right) \alpha^{t-k} Z(k+1) \left(\mathbb{E}\left(\varepsilon_k g(\theta_{k+1})\right) - \mathbb{E}\left(\varepsilon_{k-1} g(\theta_k)\right) \right).
\end{aligned} \tag{29}$$

Substituting equation 27–equation 29 into equation 24 yields

$$\begin{aligned}
&Z(t+1)\, \mathbb{E}\left(\left(g(\theta_{t+1}) - g(\theta_t) \right) \right) \\
\leq &- Z(t+1) \alpha^{t-1}\, \mathbb{E}\left(\nabla_{\theta_1} g(\theta_1)^{\mathsf{T}} v_1 \right) + \frac{\alpha^{t-1} c \alpha^{\delta}}{1-\alpha^{\delta}} Z(1)\, \mathbb{E}\left(\|v_0\|^2\right) \\
&+ \frac{c}{\alpha^{1-\delta}(1-\alpha^{\delta})} \sum_{i=1}^{t} \alpha^{t-i} Z(i+1) \varepsilon_i^2\, \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\|^2\right) \\
&- \frac{2c}{\alpha^{1-\delta}} \sum_{i=1}^{t} \left(\sum_{k=0}^{t-i} \alpha^{\delta k} \right) \alpha^{t-i} Z(i+1) \left(\mathbb{E}\left(\varepsilon_i g(\theta_{i+1})\right) - \mathbb{E}\left(\varepsilon_{i-1} g(\theta_i)\right) \right).
\end{aligned} \tag{30}$$

Construct a sequence $\{V_n\}$ as follows

$$V_n = \sum_{t=1}^{n} \left(\frac{1}{2-\alpha} \right)^{n-t} Z(t+1)\, \mathbb{E}\left(\left(g(\theta_{t+1}) - g(\theta_t) \right) \right). \tag{31}$$

By substituting equation 30 into equation 31 following the way of equation 27–equation 29, we have

$$\begin{aligned}
V_n \leq &\frac{\alpha^n (2-\alpha)}{1-\alpha} Z(1) \left| \mathbb{E}\left(\nabla_{\theta_1} g(\theta_1)^{\mathsf{T}} v_1 \right) \right| + \frac{c \alpha^n \alpha^{\delta} (2-\alpha)}{(1-\alpha)(1-\alpha^{\delta})} Z(1)\, \mathbb{E}\left(\|v_0\|^2\right) \\
&+ \frac{c}{\alpha^{1-\delta}(1-\alpha^{\delta})(1-\alpha)^2} \sum_{t=1}^{n} \left(\frac{1}{2-\alpha} \right)^{n-i} Z(t+1) \varepsilon_t^2\, \mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t) - \nabla_{\theta_t} g(\theta_t, \xi_t)\|^2\right) \\
&- \frac{2c}{\alpha^{1-\delta}(1-\alpha^{\delta})} \sum_{t=1}^{n} f(n-t) \left(\frac{1}{2-\alpha} \right)^{n-t} Z(t+1) \left(\mathbb{E}\left(\varepsilon_t g(\theta_{t+1})\right) - \mathbb{E}\left(\varepsilon_{t-1} g(\theta_t)\right) \right),
\end{aligned}$$

where $f(n-t)$ is defined as follows

$$f(n-t) = \sum_{k=1}^{n-t} \big(\alpha(2-\alpha)\big)^k - \alpha^\delta \sum_{k=1}^{n-t} \big(\alpha^{1+\delta}(2-\alpha)\big)^k.$$

Move the last term to the left-hand side of the above inequality, then we have

$$\begin{aligned}
&\sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)} g(\theta_{t+1}) - e_t^{(n-1)} g(\theta_t)\right)\Big) \\
\leq &\frac{\alpha^n(2-\alpha)}{1-\alpha} Z(1) \Big|\mathbb{E}\left(\nabla_{\theta_1} g(\theta_1)^\top v_1\right)\Big| + \frac{c\alpha^n\alpha^\delta(2-\alpha)}{(1-\alpha)(1-\alpha^\delta)} Z(1)\,\mathbb{E}\left(\|v_0\|^2\right) \\
&+ \frac{c}{\alpha^{1-\delta}(1-\alpha^\delta)(1-\alpha)^2} \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\varepsilon_t^2\,\mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t) - \nabla_{\theta_t} g(\theta_t,\xi_t)\|^2\right),
\end{aligned} \tag{32}$$

where

$$e_{t+1}^{(n)} = \left(1 + \frac{2c\varepsilon_t f(n-t)}{\alpha^{1-\delta}(1-\alpha^\delta)}\right). \tag{33}$$

Because of $\alpha < 1$, it holds that $f(n-t) > 0$ and $e_{t+1}^{(n)} > 1$. It follows from Assumption 3 that

$$\begin{aligned}
&\frac{c}{\alpha^{1-\delta}(1-\alpha^\delta)(1-\alpha)^2} \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\varepsilon_t^2\,\mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t) - \nabla_{\theta_t} g(\theta_t,\xi_t)\|^2\right) \\
\leq &\sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1) M_0 \varepsilon_t^2 e_t^{(n-1)} \big(1 + \mathbb{E}(g(\theta_t))\big),
\end{aligned} \tag{34}$$

where

$$M_0 = \frac{cM}{\alpha^{1-\delta}(1-\alpha^\delta)(1-\alpha)^2}.$$

Calculate $f(n-t)$, then we obtain

$$f(n-t) = \sum_{k=1}^{n-t} \big(\alpha(2-\alpha)\big)^k - \alpha^\delta \sum_{k=1}^{n-t} \big(\alpha^{1+\delta}(2-\alpha)\big)^k > 0.$$

It holds that

$$\begin{aligned}
&\sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)} g(\theta_{t+1}) - e_t^{(n-1)} g(\theta_t)\right)\Big) \\
&- \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1) M_0 \varepsilon_t^2 e_t^{(n-1)} \big(1 + \mathbb{E}(g(\theta_t))\big) \\
&= \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)} \big(1 + g(\theta_{t+1})\big)\right) \\
&- \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)(1 + M_0\varepsilon_t^2)\,\mathbb{E}\left(e_t^{(n-1)} \big(1 + g(\theta_t)\big)\right) \\
&- \frac{c}{\alpha^{1-\delta}(1-\alpha^\delta)} \sum_{t=1}^{n} f(n-t) \left(\frac{1}{2-\alpha}\right)^{n-t} (\varepsilon_t - \varepsilon_{t-1}) \\
&> \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)} \big(1 + g(\theta_{t+1})\big)\right) \\
&- \sum_{t=1}^{n} \left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)(1 + M_0\varepsilon_t^2)\,\mathbb{E}\left(e_t^{(n-1)} \big(1 + g(\theta_t)\big)\right).
\end{aligned} \tag{35}$$

Substituting equation 34 and equation 35 into equation 32 yields

$$\begin{aligned}
&\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)}\big(1+g(\theta_{t+1})\big)\right)\\
&-\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)(1+M_0\varepsilon_t^2)\,\mathbb{E}\left(e_{t}^{(n-1)}\big(1+g(\theta_{t})\big)\right)\\
\leq&\frac{\alpha^n(2-\alpha)}{1-\alpha}Z(1)\left|\mathbb{E}\left(\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1\right)\right|+\frac{c\alpha^n\alpha^{\delta}(2-\alpha)}{2(1-\alpha)(1-\alpha^{\delta})}Z(1)\,\mathbb{E}\left(\|v_0\|^2\right).
\end{aligned} \tag{36}$$

According to the definition of $Z(t)$, we have

$$Z(t)=\prod_{k=t}^{+\infty}(1+M_0\varepsilon_k^2)=(1+M_0\varepsilon_t^2)\prod_{k=t+1}^{+\infty}(1+M_0\varepsilon_k^2)=(1+M_0\varepsilon_t^2)Z(t+1).$$

Let

$$F_n:=\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)}\big(1+g(\theta_{t+1})\big)\right),$$

and it follows from equation 36 that

$$\begin{aligned}
F_n-F_{n-1}\leq&Z(1)\left(\frac{1}{2-\alpha}\right)^{n-1}\mathbb{E}\left(g(\theta_1)\right)+\frac{\alpha^n(2-\alpha)}{1-\alpha}Z(1)\left|\mathbb{E}\left(\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1\right)\right|\\
&+\frac{c\alpha^n\alpha^{\delta}(2-\alpha)}{2(1-\alpha)(1-\alpha^{\delta})}Z(1)\,\mathbb{E}\left(\|v_0\|^2\right).
\end{aligned} \tag{37}$$

Denote $p=\exp\left\{\sum_{k=1}^{\infty}M_0\varepsilon_k^2\right\}$. By taking the summation of equation 37, we obtain

$$\begin{aligned}
F_n\leq&Z(1)\,\mathbb{E}\left(e_1\big(1+g(\theta_1)\big)\right)\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{t-1}+\frac{p(2-\alpha)}{1-\alpha}\left|\mathbb{E}\left(\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1\right)\right|\sum_{t=1}^{n}\alpha^t\\
&+\frac{c\alpha^{\delta}(2-\alpha)}{2(1-\alpha)(1-\alpha^{\delta})}Z(1)\,\mathbb{E}\left(\|v_0\|^2\right)\sum_{t=1}^{n}\alpha^t\\
<&\frac{p(2-\alpha)}{1-\alpha}\,\mathbb{E}\left(e_1\big(1+g(\theta_1)\big)\right)+\frac{p\alpha(2-\alpha)}{(1-\alpha)^2}\left|\mathbb{E}\left(\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1\right)\right|\\
&+\frac{c\alpha^{1+\delta}(2-\alpha)}{2(1-\alpha)^2(1-\alpha^{\delta})}Z(1)\,\mathbb{E}\left(\|v_0\|^2\right).
\end{aligned}$$

Define

$$\begin{aligned}
T(\theta_1,v_0)=&1+\frac{p(2-\alpha)}{1-\alpha}\,\mathbb{E}\left(e_1\big(1+g(\theta_1)\big)\right)+\frac{p\alpha(2-\alpha)}{(1-\alpha)^2}\left|\mathbb{E}\left(\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1\right)\right|\\
&+\frac{c\alpha^{1+\delta}(2-\alpha)}{2(1-\alpha)^2(1-\alpha^{\delta})}Z(1)\,\mathbb{E}\left(\|v_0\|^2\right).
\end{aligned}$$

It follows from the relationship between $g(\theta_{n+1})$ and F_n that

$$\begin{aligned}
\mathbb{E}\left(1+g(\theta_{n+1})\right)&<p\left(\frac{1}{2-\alpha}\right)^{n-n}Z(n+1)\,\mathbb{E}\left(e_{n+1}^{(n)}\big(1+g(\theta_{n+1})\big)\right)\\
&<p\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{n-t}Z(t+1)\,\mathbb{E}\left(e_{t+1}^{(n)}\big(1+g(\theta_{t+1})\big)\right)\\
&<pF_n<1+T(\theta_1,v_0),
\end{aligned}$$

which leads to $\mathbb{E}\left(g(\theta_n)\right)<T(\theta_1,v_0)$.

B.5 Proof of equation 25

We consider

$$\begin{aligned}
&\varepsilon_i g(\theta_{i+1}) - \varepsilon_{i-1} g(\theta_i) \\
&= \varepsilon_i \big(g(\theta_{i+1} - g(\theta_i)\big) + (\varepsilon_i - \varepsilon_{i-1}) g(\theta_i) \\
&\leq \varepsilon_i \big(g(\theta_{i+1}) - g(\theta_i)\big) \\
&\leq -\varepsilon_i \nabla_{\theta_i} g(\theta_i)^\mathsf{T} v_i + \frac{c}{2}\varepsilon_i \|v_i\|^2 \\
&= (-\varepsilon_i \nabla_{\theta_i} g(\theta_i, \xi_i)^\mathsf{T} v_i - (\varepsilon_i \nabla_{\theta_i} g(\theta_i) - \varepsilon_i \nabla_{\theta_i} g(\theta_i, \xi_i))^\mathsf{T} v_i + \frac{c}{2}\varepsilon_i \|v_i\|^2 \\
&= \alpha v_i^\mathsf{T} v_{i-1} - \|v_i\|^2 + \varepsilon_i^2 \left\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\right\|^2 \\
&\quad - \big(\varepsilon_i \alpha v_{i-1} + \varepsilon_i^2 \nabla_{\theta_i} g(\theta_i)\big)^\mathsf{T} \big(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i))\big) + \frac{c}{2}\varepsilon_i \|v_i\|^2,
\end{aligned} \tag{38}$$

where the first inequality is due to $\varepsilon_i \leq \varepsilon_{i-1}$ in Assumption 3, and the last equality is from equation 20. Since ξ_i and θ_i are independent, taking the mathematical expectation of equation 38 and noting that

$$\mathbb{E}\left(\big(\varepsilon_i \alpha v_{i-1} + \varepsilon_i^2 \nabla_{\theta_i} g(\theta_i)\big)^\mathsf{T} \big(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i))\big)\right) = 0,$$

yield

$$\begin{aligned}
&\mathbb{E}\left(\varepsilon_i g(\theta_{i+1})\right) - \mathbb{E}\left(\varepsilon_{i-1} g(\theta_i)\right) \\
\leq &\alpha \mathbb{E}(v_i^\mathsf{T} v_{i-1}) - \mathbb{E}\left(\|v_i\|^2\right) + \varepsilon_i^2 \mathbb{E}\left(\left\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\right\|^2\right) + \frac{c}{2}\varepsilon_i \mathbb{E}\left(\|v_i\|^2\right).
\end{aligned} \tag{39}$$

Moreover, it holds that

$$\begin{aligned}
&\mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\|^2\right) \\
= &\mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i, \xi_i)\|^2\right) - \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i)\|^2\right) \\
= &\frac{1}{\varepsilon_i^2} \mathbb{E}\left(\|v_i - \alpha v_{i-1}\|^2\right) - \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i)\|^2\right) \\
= &\frac{1}{\varepsilon_i^2} \left(\mathbb{E}\left(\|v_i\|^2\right) + \alpha^2 \mathbb{E}\left(\|v_{i-1}\|^2\right) - 2\alpha \mathbb{E}(v_i^\mathsf{T} v_{i-1})\right) - \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i)\|^2\right).
\end{aligned} \tag{40}$$

Combining equation 39 and equation 40, we get

$$\begin{aligned}
&\mathbb{E}\left(\varepsilon_i g(\theta_{i+1})\right) - \mathbb{E}\left(\varepsilon_{i-1} g(\theta_i)\right) \\
\leq &-\frac{1}{2}\left(\mathbb{E}\left(\|v_i\|^2\right) - \alpha^2 \mathbb{E}\left(\|v_{i-1}\|^2\right)\right) - \frac{\varepsilon_i^2}{2} \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i)\|^2\right) + \frac{c}{2}\varepsilon_i \mathbb{E}\left(\|v_i\|^2\right) \\
&+ \frac{\varepsilon_i^2}{2} \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\|^2\right).
\end{aligned} \tag{41}$$

Since $\varepsilon_i \to 0$, given any $\delta > 0$, there is an integer $i_0 \geq 0$, such that for $i \geq i_0$, $1 - c\varepsilon_i > \alpha^{1-\delta} (\delta > 0)$. Since i_0 is finite, without loss of generality, we assume $i_0 = 0$ for convenience, i.e., $1 - c\varepsilon_i > \alpha^{1-\delta} (\delta > 0) (i \geq 1)$. Thus, we have

$$\begin{aligned}
&\mathbb{E}\left(\varepsilon_i g(\theta_{i+1})\right) - \mathbb{E}\left(\varepsilon_{i-1} g(\theta_i)\right) \\
\leq &-\frac{\alpha^{1-\delta}}{2}\left(\mathbb{E}\left(\|v_i\|^2\right) - \alpha^{1+\delta} \mathbb{E}\left(\|v_{i-1}\|^2\right)\right) + \frac{\varepsilon_i^2}{2} \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\|^2\right).
\end{aligned} \tag{42}$$

Multiplying both sides of equation 42 by $Z(i+1)$, and noticing that $Z(i) > Z(i+1)$, we have

$$\begin{aligned}
&\left(Z(i+1) \mathbb{E}\left(\|v_i\|^2\right) - \alpha^{1+\delta} Z(i) \mathbb{E}\left(\|v_{i-1}\|^2\right)\right) \\
\leq &-\frac{2}{\alpha^{1-\delta}} Z(i+1)\left(\mathbb{E}\left(\varepsilon_i g(\theta_{i+1})\right) - \mathbb{E}\left(\varepsilon_{i-1} g(\theta_i)\right)\right) + \frac{\varepsilon_i^2}{\alpha^{1-\delta}} Z(i+1) \mathbb{E}\left(\|\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_i} g(\theta_i, \xi_i)\|^2\right).
\end{aligned}$$

Then equation 25 is obtained by recursively applying the above inequality.

B.6 Proof of Lemma 7

From equation 41, we have

$$\begin{aligned}&\mathbb{E}\left(\varepsilon_n g(\theta_{n+1})\right)-\mathbb{E}\left(\varepsilon_0 g(\theta_1)\right)\\<&-\frac{1}{2}\sum_{t=1}^{n}\mathbb{E}\left(\|v_t\|^2\right)+\frac{\alpha^2}{2}\sum_{t=1}^{n}\mathbb{E}\left(\|v_{t-1}\|^2\right)-\sum_{t=1}^{n}\frac{\varepsilon_t^2}{2}\mathbb{E}\left(\|\nabla_{\theta_t}g(\theta_t)\|^2\right)+\frac{c}{2}\sum_{t=1}^{n}\varepsilon_t\mathbb{E}\left(\|v_t\|^2\right)\\&+\sum_{t=1}^{n}\frac{\varepsilon_t^2}{2}\mathbb{E}\left(\|\nabla_{\theta_t}g(\theta_t)-\nabla_{\theta_t}g(\theta_t,\xi_t)\|^2\right).\end{aligned}\tag{43}$$

It follows from Assumption 2 5) and Lemma 1 that

$$\mathbb{E}\left(\|\nabla_{\theta_t}g(\theta_t)-\nabla_{\theta_t}g(\theta_t,\xi_t)\|^2\right)<M(1+T(\theta_1,v_0))<+\infty.$$

Because of $\sum_{t=1}^{n}\varepsilon_t^2<+\infty$, there is a scalar $\bar{M}>0$ such that for $\forall n$

$$\sum_{t=1}^{n}\frac{\varepsilon_t^2}{2}\mathbb{E}\left(\|\nabla_{\theta_t}g(\theta_t)-\nabla_{\theta_t}g(\theta_t,\xi_t)\|^2\right)<\bar{M}<+\infty.$$

Then it follows from equation 43 that

$$\begin{aligned}&\frac{1}{2}\sum_{t=1}^{n}(1-\alpha^2-c\varepsilon_t)\mathbb{E}\left(\|v_t\|^2\right)\\\leq&\bar{M}+\varepsilon_0 g(\theta_1)-\varepsilon_n\mathbb{E}\left(g(\theta_{n+1})\right)+\frac{\alpha^2}{2}\mathbb{E}\left(\|v_0\|^2-\|v_n\|^2\right)\\&-\sum_{t=1}^{n}\frac{\varepsilon_t^2}{2}\mathbb{E}\left(\|\nabla_{\theta_t}g(\theta_t)\|^2\right)\\<&\bar{M}+\varepsilon_0 g(\theta_1)+\alpha^2\mathbb{E}\left(\|v_0\|^2\right)/2<K,\end{aligned}$$

where K is a positive scalar. Since $\varepsilon_n\to 0$ when n is large enough, it holds that $\frac{1}{5}(1-\alpha^2)<1-\alpha^2-c\varepsilon_n$. Without loss of generality, assume $\frac{1}{5}(1-\alpha^2)<1-\alpha^2-c\varepsilon_n^2$ for $n\geq 0$, so $\sum_{t=1}^{n}\mathbb{E}\left(\|v_t\|^2\right)<\frac{10K}{1-\alpha^2}<+\infty$. By Lemma 6, we obtain $\sum_{t=1}^{n}\|v_t\|^2<+\infty$.

B.7 Proof of Lemma 8

Through Taylor expansion, we derive that

$$\begin{aligned}&g(\theta_{t+1})-g(\theta_t)\\=&\nabla_{\theta_{\zeta_t}}g(\theta_{\zeta_t})^T(\theta_{t+1}-\theta_t)=-\nabla_{\theta_t}g(\theta_t)^T v_t+\left(\nabla_{\theta_{\zeta_t}}g(\theta_{\zeta_t})-\nabla_{\theta_t}g(\theta_t)\right)^T(\theta_{t+1}-\theta_t),\end{aligned}\tag{44}$$

where θ_{ζ_t} means a point between θ_t and θ_{t+1}. Substituting equation 22 into equation 44 yields

$$\begin{aligned}g(\theta_{t+1})-g(\theta_t)=&-\Big(\alpha^{t-1}\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1+\sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i}g(\theta_i)-\nabla_{\theta_{i-1}}g(\theta_{i-1}))^T v_{i-1}\\&+\sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i\nabla_{\theta_i}g(\theta_i)^{\mathsf{T}}\nabla_{\theta_i}g(\theta_i,\xi_i)\Big)+\left(\nabla_{\theta_{\zeta_t}}g(\theta_{\zeta_t})-\nabla_{\theta_t}g(\theta_t)\right)^T(\theta_{t+1}-\theta_t).\end{aligned}$$

It follows that

$$\begin{aligned}g(\theta_{n+1})=&g(\theta_1)+\sum_{t=1}^{n}\left(g(\theta_{t+1})-g(\theta_t)\right)\\=&g(\theta_1)-\frac{1-\alpha^n}{1-\alpha}\nabla_{\theta_1}g(\theta_1)^{\mathsf{T}}v_1+\frac{1-\alpha^n}{1-\alpha}\varepsilon_1\nabla_{\theta_1}g(\theta_1)\nabla_{\theta_1}g(\theta_1,\xi_1)\\&-\sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t\nabla_{\theta_t}g(\theta_t)\nabla_{\theta_t}g(\theta_t,\xi_t)+\sum_{t=1}^{n}\left(\nabla_{\theta_{\zeta_t}}g(\theta_{\zeta_t})-\nabla_{\theta_t}g(\theta_t)\right)^T(\theta_{t+1}-\theta_t)\\&-\sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i}g(\theta_i)-\nabla_{\theta_{i-1}}g(\theta_{i-1}))^T v_{i-1}.\end{aligned}\tag{45}$$

Take the mathematical expectation of equation 45, and notice Assumption 2 3), then we have

$$\begin{aligned}\mathbb{E}\left(g(\theta_{n+1})\right) \leq & \mathbb{E}\left(g(\theta_1)\right) + \frac{\alpha}{1-\alpha}\left|\nabla_{\theta_1} g(\theta_1)^{\mathsf{T}} v_1\right| + \frac{1}{1-\alpha}\varepsilon_1 \mathbb{E}\left(\|\nabla_{\theta_1} g(\theta_1)\|^2\right) \\ & - \sum_{t=1}^{n} \varepsilon_t \mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t)\|^2\right) + c\sum_{t=1}^{n} \mathbb{E}\left(\|v_t\|^2\right) + \frac{2c}{1-\alpha}\sum_{t=1}^{n} \mathbb{E}\left(\|v_t\|^2\right).\end{aligned}$$

From Lemma 7, it follows that for some positive constant Q,

$$c\sum_{t=1}^{n} \mathbb{E}\left(\|v_t\|^2\right) + \frac{2c}{1-\alpha}\sum_{t=1}^{n} \mathbb{E}\left(\|v_t\|^2\right) < Q.$$

Hence,

$$\begin{aligned}\mathbb{E}\left(g(\theta_{n+1})\right) < & Q + \mathbb{E}\left(g(\theta_1)\right) + \frac{\alpha}{1-\alpha}\left|\nabla_{\theta_1} g(\theta_1)^{\mathsf{T}} v_1\right| + \frac{1}{1-\alpha}\varepsilon_1 \mathbb{E}\left(\|\nabla_{\theta_1} g(\theta_1)\|^2\right) \\ & - \sum_{t=1}^{n} \varepsilon_t \mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t)\|^2\right).\end{aligned}$$

As a result,

$$\begin{aligned}\sum_{t=1}^{n} \varepsilon_t \mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t)\|^2\right) < & Q + \mathbb{E}\left(g(\theta_1)\right) + \frac{\alpha}{1-\alpha}\left|\nabla_{\theta_1} g(\theta_1)^{\mathsf{T}} v_1\right| + \frac{1}{1-\alpha}\varepsilon_1 \mathbb{E}\left(\|\nabla_{\theta_1} g(\theta_1)\|^2\right) \\ & - \mathbb{E}\left(g(\theta_{n+1})\right) \\ < & +\infty.\end{aligned}$$

From Lemma 6, we have $\sum_{t=1}^{n} \varepsilon_t \|\nabla_{\theta_t} g(\theta_t)\|^2 < +\infty$ *a.s.*.

B.8 Proof of Lemma 9

We divide the proof into three steps.

The first step is to prove

$$\liminf_{n\to+\infty} \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 = 0 \ a.s..$$

Suppose the above conclusion does not hold, i.e.,

$$\liminf_{n\to+\infty} \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 > s^2 > 0 \ a.s.,$$

where s is a random variable depending on sample paths. Then it holds that

$$P\left(\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}\left(\left\|\nabla_{\theta_m} g(\theta_m)\right\|^2 > \frac{1}{4}s^2\right)\right) = 1.$$

From equation 13 and $\lim_{n\to+\infty}\zeta_n = \zeta < \infty$ *a.s.*, if $\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 > \frac{1}{4}s^2$, it holds that

$$g(\theta_{n+1}) \leq \zeta_n - b\sum_{i=n}^{+\infty}\varepsilon_i = -\infty.$$

It follows that

$$P\big(\lim_{n\to+\infty} g(\theta_{n+1}) = -\infty\big) \geq P\left(\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}\left(\left\|\nabla_{\theta_m} g(\theta_m)\right\|^2 > \frac{1}{4}s^2\right)\right) = 1.$$

As a result, $\lim_{n\to+\infty} g(\theta_n) = -\infty$, meaning a contradiction. Hence,

$$\liminf_{n\to+\infty} \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 = 0 \ a.s. \tag{46}$$

Step 2 is to prove that the set $\{\theta_n\}$ has an accumulation point contained in J with probability one. From equation 46, we have $\forall \varepsilon > 0$

$$P\left(\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}\left(\left\|\nabla_{\theta_m} g(\theta_m)\right\|^2 > \varepsilon\right)\right) = 0.$$

Since $\|\nabla_\theta g(\theta)\|^2$ is continuous, $\forall \delta > 0$, it holds that

$$P\left(\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}\left(\inf_{\theta\in J}\left\|\theta_m - \theta\right\|^2 > \delta\right)\right) = 0.$$

In addition, under the given conditions, J is a closed set. It holds that

$$P\left(\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}\left(\min_{\theta\in J}\left\|\theta_m - \theta\right\|^2 > \delta\right)\right) = 0.$$

For convenience, let $\theta_n^* := \arg\min_{\theta\in J}\|\theta_n - \theta\|^2$, then

$$P\left(\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}\left(\left(\left\|\theta_m - \theta_m^*\right\| \le \sqrt{\delta}\right)\right)\right) = 1. \tag{47}$$

It follows that

$$P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcap_{i=1}^{+\infty}\left(\left\|\theta_{m_i} - \theta_{m_i}^*\right\| \le \sqrt{\delta}\right)\right) = 1, \tag{48}$$

where $\mathscr{S}_{\mathbb{N}}$ is the set of all infinite subsequences of $\mathbb{N}$. Since $\{\theta_{m_i}^*\} \subset J$, $\{\theta_{m_i}^*\}$ is bounded. From the accumulative point principle, the event

$$\bigcap_{n=1}^{+\infty}\bigcup_{i,j\ge n}\left(\|\theta_{m_i}^* - \theta_{m_j}^*\| \le \sqrt{\delta}\right),$$

is a deterministic event, i.e., being true for every sample path. Thus

$$\bigcup_{\{m_{k_i}\}\in\mathscr{S}_{\mathbb{N}}^{(i)}}\bigcap_{i,j\ge 1}\left(\left\|\theta_{m_{k_i}} - \theta_{m_{k_j}}\right\| \le \sqrt{\delta}\right)$$

is a deterministic event as well, where $\mathscr{S}_{\mathbb{N}}^{(i)}$ is the set of all infinite subsequences of $\{m_i\}$. Due to the arbitrariness of $\{m_i\}$, we get that

$$\bigcap_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcup_{\{m_{k_i}\}\in\mathscr{S}_{\mathbb{N}}^{(i)}}\bigcap_{i,j\ge 1}\left(\left\|\theta_{m_{k_i}} - \theta_{m_{k_j}}\right\| \le \sqrt{\delta}\right) \tag{49}$$

is a deterministic event. Therefore,

$$P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_i}-\theta^*_{m_j}\right\|<\sqrt{\delta}\right)\bigcap\left(\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\bigcap\left(\left\|\theta_{m_j}-\theta^*_{m_j}\right\|<\sqrt{\delta}\right)\right) \tag{50}$$

$$\begin{aligned}
=&P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcup_{\{m_{k_i}\}\in\mathscr{S}^{(i)}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_{k_i}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\bigcap\left(\left\|\theta_{m_{k_i}}-\theta^*_{m_{k_i}}\right\|<\sqrt{\delta}\right)\right.\\
&\left.\bigcap\left(\left\|\theta_{m_{k_j}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\right)\\
\geq&P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcup_{\{m_{k_i}\}\in\mathscr{S}^{(i)}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_{k_i}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\bigcap\left(\bigcap_{i=1}^{+\infty}\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\right)\\
=&P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\left(\bigcap_{i=1}^{+\infty}\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\bigcap\bigcup_{\{m_{k_i}\}\in\mathscr{S}^{()}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_{k_i}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\right)\\
\geq&P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\left(\bigcap_{i=1}^{+\infty}\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\right.\\
&\left.\bigcap\left(\bigcap_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcup_{\{m_{k_i}\}\in\mathscr{S}^{(i)}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_{k_i}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\right)\right)\\
=&P\left(\left(\bigcap_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcup_{\{m_{k_i}\}\in\mathscr{S}^{(i)}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_{k_i}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\right)\right.\\
&\left.\bigcap\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\left(\bigcap_{i=1}^{+\infty}\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\right)\right)
\end{aligned}$$

Combine (48) and (49), then we get that

$$\begin{aligned}
&P\left(\left(\bigcap_{\{m_{k_i}\}\in\mathscr{S}_{\mathbb{N}}}\bigcup_{\{m_i\}\in\mathscr{S}^{(i)}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_{k_i}}-\theta^*_{m_{k_j}}\right\|<\sqrt{\delta}\right)\right)\right.\\
&\left.\bigcap\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\left(\bigcap_{i=1}^{+\infty}\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\right)\right)=1.
\end{aligned}$$

So

$$\begin{aligned}
&P\left(\bigcup_{\{m_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcap_{i,j\geq 1}\left(\left\|\theta^*_{m_i}-\theta^*_{m_j}\right\|<\sqrt{\delta}\right)\bigcap\left(\left\|\theta_{m_i}-\theta^*_{m_i}\right\|<\sqrt{\delta}\right)\right.\\
&\left.\bigcap\left(\left\|\theta_{m_j}-\theta^*_{m_j}\right\|<\sqrt{\delta}\right)\right)=1.
\end{aligned}$$

Thus it holds that

$$
\begin{aligned}
&P\left(\bigcap_{t=1}^{+\infty}\bigcup_{i,j>t}\left(\left(\|\theta_i^*-\theta_j^*\|\le\sqrt{\delta}\right)\bigcap\left(\|\theta_i-\theta_j\|\le 3\sqrt{\delta}\right)\right)\right.\\
\ge&P\left(\bigcap_{t=1}^{+\infty}\bigcup_{i,j>t}\left(\left(\|\theta_i^*-\theta_j^*\|\le\sqrt{\delta}\right)\bigcap\left(\|\theta_i-\theta_i^*\|\le\sqrt{\delta}\right)\bigcap\left(\|\theta_j-\theta_j^*\|\le\sqrt{\delta}\right)\right)\right)\\
=&P\left(\bigcup_{\{m_{k_i}\}\in\mathscr{S}_{\mathbb{N}}}\bigcap_{i,j\ge1}\left(\|\theta_{m_i}^*-\theta_{m_j}^*\|<\sqrt{\delta}\right)\bigcap\left(\|\theta_{m_i}-\theta_{m_i}^*\|<\sqrt{\delta}\right)\right.\\
&\left.\bigcap\left(\|\theta_{m_j}-\theta_{m_j}^*\|<\sqrt{\delta}\right)\right)\\
=&1.
\end{aligned}
$$

This means we can find two "same" convergent subsequences $\{\theta_{k_n}\}$ and $\{\theta_{k_n}^*\}$ with probability one. Because J is a closed set and $\{\theta_{k_n}^*\}\subset J$, we know that there exists $\theta''\in J$ such that

$$\lim_{n\to+\infty}\theta_{k_n}=\theta''\,a.s. \tag{51}$$

This indicates that we can find an accumlative point of $\{\theta_n\}$ in J with probability one. Denote the connected component containing θ'' by J^*.

Step 3 is to prove that there is a connected component J^* of J such that $\theta_n\to J^*$ a.s. From equation 51, we have

$$P\left(\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}\left(\|\theta_m-\theta''\|<\delta'\right)\right)=1.$$

Since $g(\theta)$ is continuous, we have

$$P\left(\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}\left(\left|g(\theta_m)-g(\theta'')\right|<\delta''\right)\right)=1.$$

Since $\left\{g(\theta_n)\right\}$ converges a.s., let M be the limit, i.e., $\lim_{n\to+\infty}g(\theta_n)=M\ a.s.$ Hence $g(\theta'')=M$, and $g(\theta)=M$ for all $\theta\in J^*$. Define $A=\{\theta|g(\theta)=M\}$. From $g(\theta_n)\to M$ a.s. and the continuity of $g(\theta)$, it holds that $\theta_n\to A$ a.s. Obviously, $J^*\subseteq A$. If $J^*=A$, then the conclusion follows. Now assume $J^*\subsetneqq A$. From the definition of J^*, it follows that there are two disjoint open sets V and H such that $A\subset H\bigcup V$, $J^*\subset V$, $A/J^*\subset H$, $H'\bigcap V'=\emptyset$ (H' and V' are closures of H and V). So we just need to prove that $\theta_n\to V$ a.s. A contradiction argument is to be used. Suppose that $\{\theta_n\}$ has accumulation points in V and H simultaneously with a probability larger than zero. That is,

$$P\left(\left(\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}(\theta_m\in H)\right)\bigcap\left(\bigcap_{s=1}^{+\infty}\bigcup_{t=s}^{+\infty}(\theta_t\in V)\right)\bigcap\left(\bigcup_{r=1}^{+\infty}\bigcap_{v=r}^{+\infty}(\theta_v\in H\cup V)\right)\right)>0. \tag{52}$$

Expanding equation 52, we get

$$P\left(\bigcup_{r=1}^{+\infty}\left(\bigcap_{(m,n)\in\mathbb{N}\times\mathbb{N}}\bigcup_{i\ge m,j\ge n}\left((\theta_i\in V)\bigcap(\theta_j\in H)\right)\right)\bigcap\left(\bigcap_{v=r}^{+\infty}(\theta_v\in H\cup V)\right)\right)>0.$$

Note that the event

$$\bigcap_{(m,n)\in\mathbb{N}\times\mathbb{N}}\bigcup_{i\ge m,j\ge n}\left((\theta_i\in V)\bigcap(\theta_j\in H)\right)$$

can be written as

$$\bigcup_{\{\alpha_i\}\in\mathscr{S}_{\mathbb{N}},\{\beta_i\}\in\mathscr{S}_{\mathbb{N}}}\bigcap_{i=1}^{+\infty}\left((\theta_{\alpha_i}\in V)\bigcap(\theta_{\beta_i}\in H)\right),$$

where $\{\alpha_i\} \cap \{\beta_i\} = \emptyset$. The event

$$\left(\bigcap_{(m,n)\in\mathbb{N}\times\mathbb{N}} \bigcup_{i\geq m, j\geq n} \left((\theta_i \in V) \bigcap (\theta_j \in H) \right) \right) \bigcap \left(\bigcap_{v=r}^{+\infty} (\theta_v \in H \cup V) \right) \tag{53}$$

means $\forall t \geq r$, θ_t must belong to one of H and V. So $\{\alpha_i\} \cup \{\beta_i\}$ must include the set $\{r, r+1, r+2, ...\}$. Denote $\{\alpha_i\}$ and $\{\beta_i\}$ as $\{\alpha_i^{(r)}\}$ and $\{\beta_i^{(r)}\}$, and then equation 53 can be written as

$$\bigcup_{\{\alpha_i^{(r)}\}\in\mathscr{S}_{\mathbb{N}}, \{\beta_i^{(r)}\}\in\mathscr{S}_{\mathbb{N}}} \bigcap_{i=1}^{+\infty} \left((\theta_{\alpha_i^{(r)}} \in V) \bigcap (\theta_{\beta_i^{(r)}} \in H) \right).$$

So we get

$$P\left(\bigcup_{r=1}^{+\infty} \bigcup_{\{\alpha_i^{(r)}\}, \{\beta_i^{(r)}\}} \bigcap_{i=1}^{+\infty} \left((\theta_{\alpha_i^{(r)}} \in V) \bigcap (\theta_{\beta_i^{(r)}} \in H) \right) \right) > 0. \tag{54}$$

It follows from $V' \bigcap H' = \emptyset$ that

$$\inf_{\theta^{(1)}\in H, \theta^{(2)}\in V} \left\| \theta^{(1)} - \theta^{(2)} \right\| = s > 0, \tag{55}$$

where s is a constant. By Lemma 7, it holds that $\forall s > 0$

$$P\left(\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \|\theta_m - \theta_{m+1}\| \geq s \right) = 0. \tag{56}$$

Now we consider $\{\alpha_i^{(r)} + 1\}$. In fact, $\left\{\alpha_i^{(r)} + 1\right\}$ and $\left\{\beta_i^{(r)}\right\}$ have a finite number of identical elements with probability one. If not, then

$$P\left(\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \left((\theta_{\alpha_m^{(r)}} \in V) \bigcap (\theta_{\alpha_m^{(r)}+1} \in H) \right) \right) > 0.$$

By equation 55, it follows that

$$\begin{aligned}
&P\left(\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \left\| \theta_m - \theta_{m+1} \right\| \geq s \right) \\
\geq &P\left(\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \left\| \theta_{\alpha_m^{(r)}} - \theta_{\alpha_m^{(r)}+1} \right\| \geq s \right) \\
\geq &P\left(\bigcap_{n=1}^{+\infty} \bigcup_{m=n}^{+\infty} \left(\left(\theta_{\alpha_m^{(r)}} \in V \right) \bigcap \left(\theta_{\alpha_m^{(r)}+1} \in H \right) \right) \right) > 0,
\end{aligned}$$

which however contradicts with equation 56. Hence $\left\{\alpha_i^{(r)} + 1\right\}$ and $\left\{\beta_i^{(r)}\right\}$ have a finite number of identical elements with probability one. From $\left\{\alpha_i^{(r)}\right\} \bigcup \left\{\beta_i^{(r)}\right\} \supset \{r, r+1, r+2, ...\}$, we know that

$$P\left(\bigcup_{n=1}^{+\infty} \bigcap_{m=n}^{+\infty} (\alpha_m^{(r)} + 1) \in \left\{\alpha_i^{(r)}\right\} \right) = 1. \tag{57}$$

Hence $\left\{\beta_i^{(r)}\right\}$ is a finite sequence with probability one, otherwise it contradicts with equation 54. Thus, the assumption does not hold, and $\theta_n \to V$ a.s. Furthermore, $\theta_n \to J^*$. Therefore, there is a connected component J^* of J such that

$$\lim_{n\to\infty} d(\theta_n, J^*) = 0.$$

B.9 Proof of Theorem 1

First of all we aim to prove that $g(\theta_{n+1})$ is convergent almost surely. Divide equation 45 into four parts as follows.

$$\begin{aligned} g(\theta_{n+1}) = & \underbrace{g(\theta_1) - \frac{1-\alpha^n}{1-\alpha}\nabla_{\theta_1} g(\theta_1)^{\mathsf{T}} v_1 + \frac{1-\alpha^n}{1-\alpha}\varepsilon_1 \nabla_{\theta_1} g(\theta_1)\nabla_{\theta_1} g(\theta_1,\xi_1)}_{(A)} \\ & \underbrace{-\sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}}\nabla_{\theta_t} g(\theta_t,\xi_t)}_{(B)} + \underbrace{\sum_{t=1}^{n}\left(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\right)^T(\theta_{t+1}-\theta_t)}_{(C)} \\ & \underbrace{-\sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1}))^T v_{i-1}}_{(D)}. \end{aligned} \tag{58}$$

Due to $\alpha < 1$, α^n is tending to zero, which ensures the convergence of part (A). For (C), we consider the absolute value of $\sum_{t=n}^{m}$. It follows from Assumption 2 3) that

$$\begin{aligned} & \left|\sum_{t=n}^{m}\left(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\right)^T(\theta_{t+1}-\theta_t)\right| \le \sum_{t=n}^{m}\left|\left(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\right)^T(\theta_{t+1}-\theta_t)\right| \\ & \le \sum_{t=n}^{m}\left\|\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\right\|\|v_t\| \le c\sum_{t=n}^{m}\|v_t\|^2. \end{aligned}$$

Through Lemma 7, we get $\sum_{t=n}^{m}\|v_t\|^2 \to 0 \;\; a.s.$, leading to

$$\left|\sum_{t=n}^{m}\left(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\right)^T(\theta_{t+1}-\theta_t)\right| \to 0 \;\; a.s..$$

Through *Cauchy's test for convergence*, we know that (C) is convergent almost surely. By using the same function, it holds that (D) is convergent almost surely. For (B), we have

$$\begin{aligned} (B) = & \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}}\nabla_{\theta_t} g(\theta_t,\xi_t) \\ = & \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t\|\nabla_{\theta_t} g(\theta_t)\|^2 + \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}}\left(\nabla_{\theta_t} g(\theta_t,\xi_t) - \nabla_{\theta_t} g(\theta_t)\right). \end{aligned} \tag{59}$$

From Lemma 7, it follows that (C) is convergent almost surely. For (B), it holds that

$$\begin{aligned} (B) = & \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}}\nabla_{\theta_t} g(\theta_t,\xi_t) \\ = & \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t\|\nabla_{\theta_t} g(\theta_t)\|^2 + \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}}\left(\nabla_{\theta_t} g(\theta_t,\xi_t) - \nabla_{\theta_t} g(\theta_t)\right). \end{aligned} \tag{60}$$

By Lemma 8, we know

$$\sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t\|\nabla_{\theta_t} g(\theta_t)\|^2 < +\infty \;\; a.s..$$

From Lemmas 5 and 8 it follows that

$$\sum_{t=1}^{n-1}\frac{1-\alpha^{n-t}}{1-\alpha}\varepsilon_{t+1} \nabla_{\theta_t} g(\theta_t)^{\mathsf{T}}\left(\nabla_{\theta_t} g(\theta_t,\xi_t) - \nabla_{\theta_t} g(\theta_t)\right)$$

is convergent a.s. Thus (B) is convergent a.s., and $g(\theta_{n+1})$ is convergent a.s.. Substituting equation 60 into equation 58 leads to

$$g(\theta_{n+1}) \le \zeta_n' - \sum_{t=1}^{n}\varepsilon_t\left\|\nabla_{\theta_t} g(\theta_t)\right\|^2,$$

where $\{\zeta_n'\}$ is defined as follows

$$\begin{aligned}\zeta_n' =& g(\theta_1) - \frac{1-\alpha^n}{1-\alpha}\nabla_{\theta_1} g(\theta_1)^\top v_1 + \frac{1-\alpha^n}{1-\alpha}\varepsilon_1 \nabla_{\theta_1} g(\theta_1)\nabla_{\theta_1} g(\theta_1,\xi_1) \\ &- \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\varepsilon_t \nabla_{\theta_t} g(\theta_t)^\top\big(\nabla_{\theta_t} g(\theta_t,\xi_t) - \nabla_{\theta_t} g(\theta_t)\big) + \sum_{t=1}^{n}\Big(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\Big)^T(\theta_{t+1}-\theta_t) \\ &- \sum_{t=1}^{n}\frac{1-\alpha^{n-t+1}}{1-\alpha}\sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1}))^T v_{i-1}.\end{aligned}$$

we know $\{\zeta_n'\}$ is convergent a.s. It follows from Lemma 9 that there exists a connected component J^* of J such that $\lim_{n\to\infty} d(\theta_n, J^*) = 0$.

B.10 Proof of Theorem 2

First of all we can get that

$$\begin{aligned} g(\theta_{t+1}) - g(\theta_t) = -\Big(&\alpha^{t-1}\nabla_{\theta_1} g(\theta_1)^\top v_1 + \sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1}))^T v_{i-1} \\ &+ \sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i \nabla_{\theta_i} g(\theta_i)^\top \nabla_{\theta_i} g(\theta_i,\xi_i)\Big) + \Big(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\Big)^T(\theta_{t+1}-\theta_t). \end{aligned} \tag{61}$$

From Theorem 1, it follows that $g(\theta_n)$ is convergent a.s., and it is orbitally convergent to g_i $(i = 1,2,...,N)$ a.s.. Then it holds that

$$\lim_{n\to+\infty} g(\theta_n) = \sum_{i=1}^{N} I_i g_i \quad a.s.,$$

where

$$I_i = \begin{cases} 1 & \lim_{n\to+\infty} g(\theta_n) = g_i \\ 0 & \lim_{n\to+\infty} g(\theta_n) \neq g_i \end{cases}$$

For convenient, we let $g^* = \sum_{i=1}^{\infty} I_i g_i$. Then we make some transformation on equation 61

$$\begin{aligned} \big(g(\theta_{t+1}) - g^*\big) - \big(g(\theta_t) - g^*\big) = -\Big(&\alpha^{t-1}\nabla_{\theta_1} g(\theta_1)^\top v_1 + \sum_{i=1}^{t-1}\alpha^{t-i}(\nabla_{\theta_i} g(\theta_i) - \nabla_{\theta_{i-1}} g(\theta_{i-1}))^T v_{i-1} \\ &+ \sum_{i=2}^{t}\alpha^{t-i}\varepsilon_i \nabla_{\theta_i} g(\theta_i)^\top \nabla_{\theta_i} g(\theta_i,\xi_i)\Big) + \Big(\nabla_{\theta_{\zeta_t}} g(\theta_{\zeta_t}) - \nabla_{\theta_t} g(\theta_t)\Big)^T(\theta_{t+1}-\theta_t). \end{aligned} \tag{62}$$

Then we make some transformations, take absolute values, take the mathematical expectation, and use same techniques in Theorem 1. It follows that $F_n' - F_{n-1}' \le P_{n-1} - Q_{n-1}$, where P_{n-1}, Q_{n-1}, and F_n' are defined as follows

$$\begin{aligned} P_{n-1} =& \frac{1}{(1-\alpha)^2}\left(\frac{1}{2-\alpha}\right)^{n-1}\varepsilon_1 \mathbb{E}\left(\|\nabla_{\theta_1} g(\theta_1)\|^2\right) + Z(1)L\left(\frac{1}{2-\alpha}\right)^{n-1}\mathbb{E}\left(g(\theta_1)\right) \\ &- \frac{\alpha^n(2-\alpha)}{1-\alpha}Z(n+1)\mathbb{E}\left(\nabla_{\theta_1} g(\theta_1)^\top v_1\right) + \frac{c\alpha^n\alpha^\delta(2-\alpha)}{2(1-\alpha)(1-\alpha^\delta)}Z(1)\mathbb{E}\left(\|v_0\|^2\right), \\ F_n' =& \sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{n-t} Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1}) - g^*\right|\right), \\ Q_{n-1} =& \frac{1}{(1-\alpha)^2}\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{n-t}\varepsilon_t \mathbb{E}\left(\|\nabla_{\theta_t} g(\theta_t)\|^2\right), \end{aligned} \tag{63}$$

where $g^* = \inf_{\theta\in\mathbb{R}^N} g(\theta)$. Then we have

$$F_n' \le \left(1 - \frac{Q_{n-1}}{F_{n-1}'}\right)F_{n-1}' + P_{n-1}. \tag{64}$$

Derive $Q_n/\varepsilon_{n+1}F_n'$ as follows

$$\begin{aligned}\frac{Q_n}{\varepsilon_{n+1}F_n'} &= \frac{1}{(1-\alpha)^2}\frac{\sum_{t=1}^{n+1}\left(\frac{1}{2-\alpha}\right)^{n+1-t}\varepsilon_{t+1}\mathbb{E}\left(\|\nabla_{\theta_t}g(\theta_t)\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1})-g^*\right|\right)}\\ &= \frac{1}{(1-\alpha)^2}\frac{\left(\frac{1}{2-\alpha}\right)^{n}\varepsilon_2\mathbb{E}\left(\|\nabla_{\theta_1}g(\theta_1)\|^2\right)+\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1})-g^*\right|\right)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\liminf_{n\to+\infty}\frac{Q_n}{\varepsilon_{n+1}F_n'}\\ &= \frac{1}{(1-\alpha)^2}\liminf_{n\to+\infty}\frac{\left(\frac{1}{2-\alpha}\right)^{n}\varepsilon_2\mathbb{E}\left(\|\nabla_{\theta_1}g(\theta_1)\|^2\right)+\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1})-g^*\right|\right)}\\ &\geq \frac{1}{(1-\alpha)^2}\liminf_{n\to+\infty}\frac{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1})-g^*\right|\right)}.\end{aligned}\tag{65}$$

From equation 33, we have

$$\begin{aligned}e_{t+1}^{(n)} &= 1+\frac{2c\varepsilon_t f(n-t)}{\alpha^{1-\delta}(1-\alpha^{\delta})}\\ &= 1+\frac{2c\varepsilon_t}{\alpha^{1-\delta}(1-\alpha^{\delta})}\left(\sum_{k=1}^{n-t}\left(\alpha(2-\alpha)\right)^k-\alpha^{\delta}\sum_{k=1}^{n-t}\left(\alpha^{1+\delta}(2-\alpha)\right)^k\right)\\ &\leq 1+C_\alpha\varepsilon_t,\end{aligned}\tag{66}$$

where $C_\alpha=\frac{2c(2-\alpha)}{\alpha^{-\delta}(1-\alpha^{\delta})(1-\alpha(2-\alpha))}$. Substituting equation 66 into equation 65 yields

$$\begin{aligned}&\liminf_{n\to+\infty}\frac{Q_n}{\varepsilon_{n+1}F_n'}\\ &\geq \frac{1}{(1-\alpha)^2}\liminf_{n\to+\infty}\frac{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1})-g^*\right|\right)}.\end{aligned}\tag{67}$$

Then we proceed with the proof under two different conditions, namely, $\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)=+\infty$ and $\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)<+\infty$.

First, if $\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)=+\infty$ (the proof for this condition is up to equation 77). It follows from Assumption 4 2) and the uniform convergence and *O'stolz theorem* that

$$\begin{aligned}&\liminf_{n\to+\infty}\frac{Q_n}{\varepsilon_{n+1}F_n'}\\ &\geq \frac{1}{(1-\alpha)^2}\liminf_{n\to+\infty}\frac{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)\mathbb{E}\left(\left(1+C_\alpha\varepsilon_t\right)\left|g(\theta_{t+1})-g^*\right|\right)}\\ &\geq \frac{1}{(1-\alpha)^2}\liminf_{n\to+\infty}\frac{\mathbb{E}\left(\|\nabla_{\theta_{n+1}}g(\theta_{n+1})\|^2\right)}{Z(n+1)\mathbb{E}\left(\left(1+C_\alpha\varepsilon_t\right)\left|g(\theta_{t+1})-g^*\right|\right)}\\ &\geq \frac{1}{(1-\alpha)^2}\liminf_{n\to+\infty}\frac{s}{Z(n+1)\left(1+C_\alpha\varepsilon_n\right)}=\frac{s}{p(1-\alpha)^2},\end{aligned}\tag{68}$$

where $p = \exp\left\{\sum_{k=1}^{\infty} M\varepsilon_k^2\right\}$. By using O'stolz theorem on $Q_n/F_n^{'}$, it follows that

$$\begin{aligned}
\lim_{n\to+\infty}\frac{Q_n}{F_n'} &= \frac{1}{(1-\alpha)^2}\lim_{n\to+\infty}\frac{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\varepsilon_{t+1}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)\mathbb{E}\left(e_{t+1}^{(n)}\left|g(\theta_{t+1})-g^*)\right|\right)} \\
&\leq \frac{1}{(1-\alpha)^2}\lim_{n\to+\infty}\frac{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\varepsilon_{t+1}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)}{\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}Z(t+1)\mathbb{E}\left|g(\theta_{t+1})-g^*\right|} \\
&= \frac{1}{(1-\alpha)^2}\lim_{n\to+\infty}\frac{\varepsilon_n\mathbb{E}\left(\|\nabla_{\theta_n}g(\theta_n)\|^2\right)}{Z(n+1)\mathbb{E}\left|g(\theta_{n+1})-g^*\right|} \\
&\leq \frac{1}{(1-\alpha)^2}\lim_{n\to+\infty}\frac{t\varepsilon_n\mathbb{E}\left(g(\theta_{n+1})-g^*\right)}{Z(n+1)\mathbb{E}\left|g(\theta_{n+1})-g^*\right|} = 0.
\end{aligned}$$

So we conclude that $\exists n_0 \in \mathbb{N}_+$, such that $\forall n \geq n_0$,

$$\frac{Q_n}{F_n'} < 1 - \frac{3-\alpha}{2(2-\alpha)}. \tag{69}$$

From equation 64, it follows that

$$F_n' \leq \prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)F_{n_0}' + \sum_{t=n_0+1}^{n}\prod_{i=t}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)P_{t-1}. \tag{70}$$

Using the inequality $\ln(1+x) \leq x$ and equation 68, we get

$$\begin{aligned}
&\prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right) = \exp\left(\sum_{i=n_0}^{n-1}\ln\left(1-\frac{Q_i}{F_i'}\right)\right) \leq \exp\left(-\sum_{i=n_0}^{n-1}\frac{Q_i}{F_i'}\right) \leq ke^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}} \\
&= O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right),
\end{aligned} \tag{71}$$

where k is a constant. It follows that

$$\frac{\sum_{t=n_0+1}^{n-1}\prod_{i=t}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)P_{t-1}}{\prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)F_1'} = \sum_{t=n_0+1}^{n}\frac{P_{t-1}}{\prod_{i=n_0}^{t-1}\left(1-\frac{Q_i}{F_i'}\right)}. \tag{72}$$

From equation 63, we have

$$\begin{aligned}
P_{t-1} =& \frac{1}{(1-\alpha)^2}\left(\frac{1}{2-\alpha}\right)^{t-1}\varepsilon_1\mathbb{E}\left(\|\nabla_{\theta_1}g(\theta_1)\|^2\right) + Z(1)L\left(\frac{1}{2-\alpha}\right)^{t-1}\mathbb{E}\left(g(\theta_1)\right) \\
&+ \frac{c\alpha^t\alpha^\delta(2-\alpha)}{2(1-\alpha)(1-\alpha^\delta)}Z(1)\mathbb{E}\left(\|v_0\|^2\right) = \bar{p}\left(\frac{1}{2-\alpha}\right)^{t-1} + \bar{q}\alpha^{t-1},
\end{aligned} \tag{73}$$

where $\bar{p} = \frac{1}{(1-\alpha)^2}\varepsilon_1\mathbb{E}\left(\|\nabla_{\theta_1}g(\theta_1)\|^2\right) + Z(1)L\mathbb{E}\left(g(\theta_1)\right)$ and $\bar{q} = \frac{c\alpha^{\delta+1}(2-\alpha)}{2(1-\alpha)(1-\alpha^\delta)}Z(1)\mathbb{E}\left(\|v_0\|^2\right)$. Substituting equation 73 into equation 72 and noting equation 69 yield

$$\begin{aligned}
&\frac{\sum_{t=n_0+1}^{n-1}\prod_{i=t}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)P_{t-1}}{\prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)F_1'} \\
&= \sum_{t=n_0+1}^{n}\frac{P_{t-1}}{\prod_{i=n_0}^{t-1}\left(1-\frac{Q_i}{F_i'}\right)} < \sum_{t=n_0+1}^{n}\frac{\bar{p}\left(\frac{1}{2-\alpha}\right)^{t-1}}{\prod_{i=n_0}^{t-1}\left(1-\frac{Q_i}{F_i'}\right)} + \sum_{t=n_0+1}^{n}\frac{\bar{q}\alpha^{t-1}}{\prod_{i=n_0}^{t-1}\left(1-\frac{Q_i}{F_i'}\right)} \\
&\leq \sum_{t=n_0+1}^{n}\frac{\bar{p}\left(\frac{1}{2-\alpha}\right)^{t-1}}{\left(\frac{3-\alpha}{2(2-\alpha)}\right)^{t-n_0}} + \sum_{t=n_0+1}^{n}\frac{\bar{q}\alpha^{t-1}}{\left(\frac{3-\alpha}{2(2-\alpha)}\right)^{t-n_0}} \\
&< \left(\frac{3-\alpha}{2(2-\alpha)}\right)^{n_0-1}(\bar{p}+\bar{q})\sum_{t=n_0+1}^{n}\left(\frac{2}{3-\alpha}\right)^{t-1} < c_0,
\end{aligned} \tag{74}$$

where c_0 is a positive constant. It follows that

$$\sum_{t=n_0+1}^{n-1}\prod_{i=t}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)P_{t-1} < c_0\prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)F_1' = O\left(\prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)F_1'\right) = O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right). \tag{75}$$

Combine equation 70, equation 71 and equation 75, then we get

$$F_n' \leq \prod_{i=n_0}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)F_1' + \sum_{t=n_0+1}^{n-1}\prod_{i=t}^{n-1}\left(1-\frac{Q_i}{F_i'}\right)P_{t-1} = O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right). \tag{76}$$

In addition, we have

$$\mathbb{E}\left(g(\theta_{t+1})-g^*\right) < F_n' = O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right). \tag{77}$$

Second, if $\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right) < +\infty$, it holds that

$$\lim_{n\to+\infty}\left(\frac{1}{2-\alpha}\right)^{-n}\mathbb{E}\left(\|\nabla_{\theta_{n+1}}g(\theta_{n+1})\|^2 = 0,\right.$$

that is

$$\mathbb{E}\left(\|\nabla_{\theta_{n+1}}g(\theta_{n+1})\|^2\right) = O\left(\left(\frac{1}{2-\alpha}\right)^n\right).$$

Under Assumption 4 2), we have

$$\limsup_{n\to+\infty}\frac{\mathbb{E}\left(g(\theta_{n+1})-g^*\right)}{\left(\frac{1}{2-\alpha}\right)^n} \leq \limsup_{n\to+\infty}\frac{\mathbb{E}\left(\left\|\nabla_{\theta_{n+1}}g(\theta_{n+1})\right\|^2\right)}{s\left(\frac{1}{2-\alpha}\right)^n} \leq \frac{1}{s}\frac{\mathbb{E}\left(\left\|\nabla_{\theta n+1}g(\theta_{n+1})\right\|^2\right)}{\left(\frac{1}{2-\alpha}\right)^n} = 0.$$

It follows that

$$\mathbb{E}\left|g(\theta_{n+1})-g^*\right| = O\left(\left(\frac{1}{2-\alpha}\right)^n\right).$$

And then we compare $\left(\frac{1}{2-\alpha}\right)^n$ with $e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}$. There is

$$\frac{\left(\frac{1}{2-\alpha}\right)^n}{\exp\left(-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}\right)} = \frac{\exp\left(-n\ln(2-\alpha)\right)}{\exp\left(-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}\right)} = \exp\left(\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2} - n\ln(2-\alpha)\right).$$

It follows from Assumption 3 that $\varepsilon_n \to 0$, indicating $\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2} - n\ln(2-\alpha) < 0$ when n is sufficiently large. Thus, we have

$$\limsup_{n\to+\infty}\frac{\left(\frac{1}{2-\alpha}\right)^n}{\exp\left(-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}\right)} = \limsup_{n\to+\infty}\exp\left(\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2} - n\ln(2-\alpha)\right) < 1.$$

Then it holds that

$$\mathbb{E}\left|g(\theta_{n+1})-g^*\right| = O\left(\left(\frac{1}{2-\alpha}\right)^n\right) = O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right). \tag{78}$$

Combining equation 78 and equation 77 leads to

$$\mathbb{E}\left|g(\theta_{n+1})-g^*\right| = O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right)$$

no matter $\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right) = +\infty$ or $\sum_{t=1}^{n}\left(\frac{1}{2-\alpha}\right)^{-t}\mathbb{E}\left(\|\nabla_{\theta_{t+1}}g(\theta_{t+1})\|^2\right)$ $< +\infty$. It follows from Lemma 4 that

$$\mathbb{E}\left(\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2\right) = O\left(e^{-\sum_{i=1}^{n}\frac{s\varepsilon_i}{p(1-\alpha)^2}}\right)$$

C Convergence of AdaGrad

The following lemmas are used for the proof of Theorem 3.

Lemma 10 Suppose $\{\theta_n\}$ is a sequence generated by AdaGrad in equation 5, and Assumptions 1 and 5 hold. If $\left\|\nabla_{\theta_n} g(\theta)\right\|^2 > a$ where a is given in Assumption 5 5), then for any $n \in \mathbb{N}_+, \theta_1 \in \mathbb{R}^N$, and $\varepsilon \in (0, \frac{1}{2})$, it holds that

$$\begin{aligned}\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \leq & \frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\ & - \frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\ & + 4M^2\alpha_0^3 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n^{1+\varepsilon}} + X_n^{(\varepsilon)} + Y_n^{(\varepsilon)},\end{aligned}$$

where

$$\begin{aligned}X_n^{(\varepsilon)} &= \frac{\alpha_0}{2}\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right) \\ Y_n^{(\varepsilon)} &= \frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right).\end{aligned}$$

Lemma 11 Suppose $\{\theta_n\}$ is a sequence generated by AdaGrad in equation 5, and Assumptions 1 and 5 hold. If $\left\|\nabla_{\theta_n} g(\theta)\right\|^2 \leq a$ where a is given in Assumption 5 5), then for any $n \in \mathbb{N}_+, \theta_1 \in \mathbb{R}^N$, and $\varepsilon \in (0, \frac{1}{2})$, it holds that

$$\begin{aligned}\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \leq & -\frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0^3 c^2 (M+1)^2}{2}\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}} \\ & + \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2 + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\ & + \frac{\alpha_0(M+1)}{2}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) + \frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\ & + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n^{1+\varepsilon}} + A_n^{(\varepsilon)} + B_n^{(\varepsilon)},\end{aligned}$$

where

$$\begin{aligned}A_n^{(\varepsilon)} &= \frac{\alpha_0}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n, \xi_n)\right) \\ B_n^{(\varepsilon)} &= \frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{2a(M+1)S_{n-1}^{\frac{1}{2}+\varepsilon}}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right).\end{aligned}$$

Lemma 12 Suppose $\{\theta_n\}$ is a sequence generated by AdaGrad in equation 5, and Assumptions 1 and 5 hold. Then $\forall n \in \mathbb{N}_+$, $\forall \theta_1 \in \mathbb{R}^N$ $\forall \varepsilon \in (0, \frac{1}{2})$, it holds that

$$\sum_{k=3}^{n} \frac{\left\|\nabla_{\theta_k} g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}} < +\infty \quad a.s..$$

Lemma 13 Suppose $\{\theta_n\}$ is a sequence generated by AdaGrad in equation 5, and Assumptions 1 and 5 hold. Then $\forall n \in \mathbb{N}_+$ $\forall \theta_1 \in \mathbb{R}^N$ $\forall \varepsilon \in (0, \frac{1}{2})$, $\exists \zeta < +\infty$, it holds that

$$\frac{g(\theta_{n+1}) - g^*}{S_{n+1}^{\varepsilon}} \leq \zeta < +\infty \quad a.s..$$

Lemma 14 Suppose $\{\theta_n\}$ is a sequence generated by AdaGrad in equation 5 subject to $S_n = \sum_{k=1}^{n} \|\nabla_{\theta_k} g(\theta_k, \xi_k)\|^2 = +\infty$ $a.s.$. Under Assumptions 1 and 5, $\forall n \in \mathbb{N}_+$ $\forall \theta_1 \in \mathbb{R}^N$, $\forall \varepsilon_0 \in (0, \frac{3}{8})$, it holds that

$$\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon_0}} \to 0 \quad a.s..$$

Lemma 15 Suppose that $\{X_n\} \in \mathbb{R}^N$ is a vector sequence and $f(x) \in C^1$ is a monotonically non-increasing non-negative function with $\int_a^{+\infty} f(x)dx < \infty$ $(\forall a > 0)$. Then $\forall N \in \mathbb{N}_+$, it holds that

$$\sum_{n=1}^{N} \|X_n\|^2 f\left(\sum_{k=1}^{n} \|X_k\|^2\right) < \int_{\|X_1\|^2}^{\sum_{k=1}^{N} \|X_k\|^2} f(x)dx < \sum_{n=1}^{N} \|X_n\|^2 f\left(\sum_{k=1}^{n-1} \|X_k\|^2\right).$$

C.1 Proof of Lemma 10

First of all, it follows from Lemma 3 that

$$\begin{aligned} g(\theta_{n+1}) - g(\theta_n) &\leq \nabla_{\theta_n} g(\theta_n)^T (\theta_{n+1} - \theta_n) + \frac{c\alpha_0^2}{2} \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n} \\ &= -\frac{\alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n, \xi_n)}{\sqrt{S_n}} + \frac{c\alpha_0^2}{2} \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n}, \end{aligned} \tag{79}$$

where

$$S_n = \sum_{k=1}^{n} \left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2.$$

Note that

$$\begin{aligned} &\left\| \frac{1}{\sqrt{M+1}} \nabla_{\theta_n} g(\theta_n, \xi_n) - \sqrt{M+1} \nabla_{\theta_n} g(\theta_n) \right\|^2 \\ &= \frac{1}{M+1} \left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 + (M+1) \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - 2\nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n, \xi_n), \end{aligned} \tag{80}$$

where $M = M' + 2$ and M' is defined in Assumption 5 3). Substitute equation 80 into equation 79, then we get that

$$\begin{aligned} &g(\theta_{n+1}) - g(\theta_n) \\ &\leq -\frac{\alpha_0}{2} \left(\frac{1}{M+1} \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{\sqrt{S_n}} + (M+1) \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}} \right) \\ &+ \frac{\alpha_0}{2} \frac{1}{\sqrt{S_n}} \left\| \frac{1}{\sqrt{M+1}} \nabla_{\theta_n} g(\theta_n, \xi_n) - \sqrt{M+1} \nabla_{\theta_n} g(\theta_n) \right\|^2 + \frac{c\alpha_0^2}{2} \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n}. \end{aligned} \tag{81}$$

Due to $S_n \geq S_{n-1}$, it follows that

$$\begin{aligned} &\frac{\alpha_0}{2} \frac{1}{\sqrt{S_n}} \left\| \frac{1}{\sqrt{M+1}} \nabla_{\theta_n} g(\theta_n, \xi_n) - \sqrt{M+1} \nabla_{\theta_n} g(\theta_n) \right\|^2 \\ &\leq \frac{\alpha_0}{2} \frac{1}{\sqrt{S_{n-1}}} \left\| \frac{1}{\sqrt{M+1}} \nabla_{\theta_n} g(\theta_n, \xi_n) - \sqrt{M+1} \nabla_{\theta_n} g(\theta_n) \right\|^2. \end{aligned} \tag{82}$$

Substitute equation 82 into equation 81, then we have

$$
\begin{aligned}
& g(\theta_{n+1}) - g(\theta_n) \\
& \leq -\frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{\sqrt{S_n}} + (M+1)\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}}\right) \\
& + \frac{\alpha_0}{2}\frac{1}{\sqrt{S_{n-1}}}\left\|\frac{1}{\sqrt{M+1}}\nabla_{\theta_n} g(\theta_n,\xi_n) - \sqrt{M+1}\nabla_{\theta_n} g(\theta_n)\right\|^2 + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n}.
\end{aligned} \tag{83}
$$

Notice that

$$
\begin{aligned}
& \frac{\alpha_0}{2}\frac{1}{\sqrt{S_{n-1}}}\left\|\frac{1}{\sqrt{M+1}}\nabla_{\theta_n} g(\theta_n,\xi_n) - \sqrt{M+1}\nabla_{\theta_n} g(\theta_n)\right\|^2 \\
& = \frac{\alpha_0}{2}\frac{1}{\sqrt{S_{n-1}}}\left(\frac{1}{M+1}\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 + (M+1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - 2\nabla_{\theta_n} g(\theta_n,\xi_n)^T\nabla_{\theta_n} g(\theta_n)\right) \\
& = \frac{\alpha_0}{2}\frac{1}{\sqrt{S_{n-1}}}\left(\frac{1}{M+1}\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 + (M+1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - 2\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2\right) \\
& + \frac{\alpha_0}{\sqrt{S_{n-1}}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n,\xi_n)\right).
\end{aligned} \tag{84}
$$

Substitute equation 84 into equation 83, and divide both sides of the inequality by S_n^{ε} ($\varepsilon < \frac{1}{2}$), then we get

$$
\begin{aligned}
& \frac{g(\theta_{n+1})}{S_n^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
& \leq -\frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}} + (M+1)\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
& + \frac{\alpha_0}{2}\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\left(\frac{1}{M+1}\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 + (M+1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - 2\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2\right) \\
& + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} + \frac{\alpha_0}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n,\xi_n)\right).
\end{aligned}
$$

Notice that $\frac{g(\theta_{n+1})}{S_n^{\varepsilon}} > \frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}$, then we obtain

$$
\begin{aligned}
& \frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
& \leq -\frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}} + (M+1)\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
& + \frac{\alpha_0}{2}\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\left(\frac{1}{M+1}\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 + (M+1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - 2\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2\right) \\
& + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} + \frac{\alpha_0}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n,\xi_n)\right).
\end{aligned} \tag{85}
$$

Rearrange the above inequality, then it holds that

$$
\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
&\leq -\frac{\alpha_0}{2}(M+1)\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}} \\
&+ \frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + (M-1)\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} + X_n^{(\varepsilon)} \\
&= \frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&+ \frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{(M-1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{(M+1)\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&+ \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} + X_n^{(\varepsilon)}.
\end{aligned}
\tag{86}
$$

$X_n^{(\varepsilon)}$ is defined as follow

$$
X_n^{(\varepsilon)} = \frac{\alpha_0}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n,\xi_n)\right).
$$

Due to $\|\nabla_{\theta_n} g(\theta_n)\|^2 > a$, we have

$$
\begin{aligned}
&\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right) \leq M\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + a \\
&< (M+1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2,
\end{aligned}
\tag{87}
$$

Moreover, using the Taylor formula, we obtain

$$
\begin{aligned}
&\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 = \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}) + \left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right)\right\|^2 \\
&= \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 + 2\nabla_{\theta_{n-1}} g(\theta_{n-1})^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right) \\
&+ \left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 \leq \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 \\
&+ 2\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|\left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right\| + \left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2.
\end{aligned}
$$

Under Assumption 5 3), we get that

$$
\begin{aligned}
&\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 \\
&+ 2\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|\left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right\| + \left\|\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 \\
&\leq \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 + \frac{2\alpha_0 c}{\sqrt{S_{n-1}}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\| \\
&+ c^2\alpha_0^2\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}}.
\end{aligned}
\tag{88}
$$

From inequality $2a^T b \leq \lambda\|a\|^2 + \frac{1}{\lambda}\|b\|^2 \quad (\lambda > 0)$, it follows that

$$
\begin{aligned}
&(M-1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \\
&\leq (M+1)\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 - \frac{M-1}{4M-3}\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + 4M^2\alpha_0^2 c^2\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}}.
\end{aligned}
\tag{89}
$$

By substituting equation 87 into equation 89, we have

$$
\begin{aligned}
&(M-1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + \frac{1}{M+1}\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right) \\
&\leq (M+1)\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2 - \frac{M-1}{4M-3}\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + 4M^2\alpha_0^2 c^2\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}},
\end{aligned}
\tag{90}
$$

Divide both sides of equation 90 by $S_{n-1}^{\frac{1}{2}+\varepsilon}$, and notice $\frac{M-1}{4M-3} > \frac{1}{5}$ from $M > 2$, then it holds that

$$\begin{aligned}
&\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} \\
&\leq (M+1)\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{5}\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + 4M^2\alpha_0^2 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}} \\
&+ \frac{2}{\alpha_0} Y_n^{(\varepsilon)},
\end{aligned} \tag{91}$$

where

$$Y_n^{(\varepsilon)} = \frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right).$$

Making some simple transformations on equation 91 leads to

$$\begin{aligned}
&\frac{\alpha_0}{2}\left(\frac{1}{M+1}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{(M-1)\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{(M+1)\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&\leq -\frac{\alpha_0}{10}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + 4M^2\alpha_0^3 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}} + Y_n^{(\varepsilon)}.
\end{aligned} \tag{92}$$

Substitute equation 92 into equation 86, then we get

$$\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
&\leq \frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&- \frac{\alpha_0}{10}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + 4M^2\alpha_0^3 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} \\
&+ X_n^{(\varepsilon)} + Y_n^{(\varepsilon)}.
\end{aligned} \tag{93}$$

It follows that

$$\begin{aligned}
&-\frac{\alpha_0}{10}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} = -\frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} \\
&= -\frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&- \frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} \\
&\leq -\frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right).
\end{aligned} \tag{94}$$

Substituting equation 94 into equation 93 yields

$$\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
&\leq \frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&- \frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&+ 4M^2\alpha_0^3 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} + X_n^{(\varepsilon)} + Y_n^{(\varepsilon)}.
\end{aligned} \tag{95}$$

C.2 Proof of Lemma 11

First of all, dividing both sides of equation 79 by S_n^{ε} yields

$$\frac{g(\theta_{n+1})}{S_n^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \leq -\frac{\alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)}{S_n^{\frac{1}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}. \tag{96}$$

Due to $S_{n+1} \geq S_n$, it holds that

$$\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \leq -\frac{\alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)}{S_n^{\frac{1}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}. \tag{97}$$

Then we make some transformations to obtain that

$$\begin{aligned}
&-\frac{\alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)}{S_n^{\frac{1}{2}+\varepsilon}} \\
&= -\frac{\alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&\leq -\frac{\alpha_0 \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \alpha_0\left(\frac{(M+1)a}{2} + \frac{1}{2(M+1)a}\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2\right) \\
&\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&\leq -\frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + \frac{(M+1)\alpha_0 a}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&+ \left(\frac{\alpha_0}{2a(M+1)}\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right)\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}} + A_n^{(\varepsilon)} + B_n^{(\varepsilon)}.
\end{aligned} \tag{98}$$

where

$$\begin{aligned}
A_n^{(\varepsilon)} &= \frac{\alpha_0}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 - \nabla_{\theta_n} g(\theta_n)^T \nabla_{\theta_n} g(\theta_n,\xi_n)\right) \\
B_n^{(\varepsilon)} &= \frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{2a(M+1)S_{n-1}^{\frac{1}{2}+\varepsilon}}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right).
\end{aligned} \tag{99}$$

Due to $\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq a$, we get

$$\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right) \leq M\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + a \leq (M+1)a. \tag{100}$$

Substitute it into equation 98, then we get

$$\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_n)}{S_n^{\varepsilon}}\leq-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}+\frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right)\\
&+\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{2S_{n-1}^{\frac{1}{2}+\varepsilon}}+\frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}+A_n^{(\varepsilon)}+B_n^{(\varepsilon)}\\
&=-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{2S_{n-1}^{\frac{1}{2}+\varepsilon}}+\frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right)\\
&+\frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}+A_n^{(\varepsilon)}+B_n^{(\varepsilon)}.
\end{aligned}\tag{101}$$

We make some transformations on $-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{2S_{n-1}^{\frac{1}{2}+\varepsilon}}$ to obtain that

$$\begin{aligned}
&-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{2S_{n-1}^{\frac{1}{2}+\varepsilon}}\leq-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}}\\
&=-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}+\frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right).
\end{aligned}\tag{102}$$

Then we use inequality $2a^Tb\leq\lambda\|a\|^2+\frac{1}{\lambda}\|b\|^2\quad(\lambda>0)$ on equation 88 to get

$$\begin{aligned}
&\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2-\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2\leq\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{10(M+1)}\\
&+\frac{10\alpha_0^2c^2(M+1)}{S_{n-1}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2+\frac{\alpha_0^2c^2}{S_{n-1}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2.
\end{aligned}\tag{103}$$

Divide both sides of equation 103 by $S_{n-1}^{\frac{1}{2}+\varepsilon}$ and notice $S_{n-2}\leq S_{n-1}\leq S_n$, then we have

$$\begin{aligned}
&\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\leq\frac{1}{M+1}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{10S_{n-2}^{\frac{1}{2}+\varepsilon}}\\
&+\frac{10\alpha_0^2c^2(M+1)}{S_{n-1}^{1+\varepsilon}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2+\frac{\alpha_0^2c^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2.
\end{aligned}\tag{104}$$

Then we calculate $\frac{\alpha_0}{2}(M+1)equation\ 104+equation\ 102$

$$\begin{aligned}
&-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{2S_{n-1}^{\frac{1}{2}+\varepsilon}}+\frac{\alpha_0(M+1)}{2}\left(\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right)\\
&\leq-\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}}+5\alpha_0^3c^2(M+1)^2\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}}\\
&+\frac{(M+1)\alpha_0^3c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2+\frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right).
\end{aligned}$$

Move $\frac{\alpha_0(M+1)}{2}\left(\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right)$ to the right-hand side of the above inequality, then we have

$$
\begin{aligned}
&-\frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{2S_{n-1}^{\frac{1}{2}+\varepsilon}} \leq -\frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}} + 5\alpha_0^3 c^2 (M+1)^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}} \\
&+\frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2 + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&+\frac{\alpha_0(M+1)}{2}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right).
\end{aligned} \tag{105}
$$

Substitute equation 105 into equation 101, then we have

$$
\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
&\leq -\frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}} + 5\alpha_0^3 c^2 (M+1)^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}} \\
&+\frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2 + \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&+\frac{\alpha_0(M+1)}{2}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) + \frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&+\frac{c\alpha_0^2}{2}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} + A_n^{(\varepsilon)} + B_n^{(\varepsilon)}.
\end{aligned} \tag{106}
$$

C.3 Proof of Lemma 12

First of all, it holds that

$$
\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}} \\
&= I\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq a\right)\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}}\right) + I\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 > a\right)\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_n)}{S_n^{\varepsilon}}\right).
\end{aligned}
$$

$I\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq a\right) \in \mathscr{F}_{n-1}$ is the indicator function such that

$$
I\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq a\right) = \begin{cases} 1 & \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq a \\ 0 & \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 > a \end{cases}
$$

For convenient, we abbreviate $I\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \leq a\right)$ as $I_n^{\leq a}$ and $I\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 > a\right)$ as $I_n^{>a}$ in the following.

Through Lemma 11, we get that

$$
\begin{aligned}
& I_n^{\leq a}\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_n)}{S_n^{\varepsilon}}\right) \\
& \leq -I_n^{\leq a}\frac{\alpha_0\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{20S_{n-1}^{\frac{1}{2}+\varepsilon}}+5\alpha_0^3c^2(M+1)^2I_n^{\leq a}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}} \\
& +I_n^{\leq a}\frac{(M+1)\alpha_0^3c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2+\frac{\alpha_0}{20}I_n^{\leq a}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
& +\frac{\alpha_0(M+1)}{2}I_n^{\leq a}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right)+\frac{\alpha_0a(M+1)}{2}I_n^{\leq a}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
& +\frac{c\alpha_0^2}{2}I_n^{\leq a}\frac{\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}+I_n^{\leq a}A_n^{(\varepsilon)}+I_n^{\leq a}B_n^{(\varepsilon)}.
\end{aligned}
\tag{107}
$$

Through Lemma 10, we get

$$
\begin{aligned}
& I_n^{>a}\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_n)}{S_n^{\varepsilon}}\right) \\
& \leq -I_n^{>a}\frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}+\frac{\alpha_0(M+1)}{2}`I_n^{>a}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
& +\frac{\alpha_0}{20}I_n^{>a}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
& 4M^2\alpha_0^3c^2I_n^{>a}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1}),\xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}}+\frac{c\alpha_0^2}{2}I_n^{>a}\frac{\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} \\
& +I_n^{>a}X_n^{(\varepsilon)}+I_n^{>a}Y_n^{(\varepsilon)}.
\end{aligned}
\tag{108}
$$

Calculate *equation* 107 + *equation* 108, then it holds that

$$
\begin{aligned}
& I_n^{\leq a}\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_n)}{S_n^{\varepsilon}}\right)+I_n^{>a}\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_n)}{S_n^{\varepsilon}}\right) \\
& \leq \frac{\alpha_0}{20}(I_n^{\leq a}+I_n^{>a})\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
& +\frac{\alpha_0}{2}(M+1)(I_n^{\leq a}+I_n^{>a})\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
& -\frac{\alpha_0}{20}\left(I_n^{\leq a}+I_n^{>a}\right)\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}+4M^2\alpha_0^3c^2I_n^{>a}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1}),\xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}} \\
& +\frac{c\alpha_0^2}{2}I_n^{>a}\frac{\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}+5\alpha_0^3c^2(M+1)^2I_n^{\leq a}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}} \\
& +I_n^{\leq a}\frac{(M+1)\alpha_0^3c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2 \\
& +\frac{\alpha_0a(M+1)}{2}I_n^{\leq a}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right)+\frac{c\alpha_0^2}{2}I_n^{\leq a}\frac{\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} \\
& +I_n^{\leq a}A_n^{(\varepsilon)}+I_n^{\leq a}B_n^{(\varepsilon)}+I_n^{>a}X_n^{(\varepsilon)}+I_n^{>a}Y_n^{(\varepsilon)}.
\end{aligned}
\tag{109}
$$

Notice $I_n^{\leq a} \leq 1$, then we get

$$\frac{\alpha_0 a(M+1)}{2} I_n^{\leq a}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right) \leq \frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right). \tag{110}$$

Substitute equation 110 into equation 109, then we get

$$\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_n)}{S_n^{\varepsilon}} \\
&\leq \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&+\frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}}\right) \\
&-\frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}+4M^2\alpha_0^3 c^2 I_n^{>a}\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}),\xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}+\varepsilon}}+\frac{c\alpha_0^2}{2} I_n^{>a}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}} \\
&+5\alpha_0^3 c^2(M+1)^2 I_n^{\leq a}\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}^{1+\varepsilon}}+I_n^{\leq a}\frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2 \\
&+\frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_n^{\frac{1}{2}+\varepsilon}}\right)+\frac{c\alpha_0^2}{2} I_n^{\leq a}\frac{\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2}{S_n^{1+\varepsilon}}+I_n^{\leq a}A_n^{(\varepsilon)}+I_n^{\leq a}B_n^{(\varepsilon)}+I_n^{>a}X_n^{(\varepsilon)} \\
&+I_n^{>a}Y_n^{(\varepsilon)}.
\end{aligned} \tag{111}$$

We make a summation of equation 111 to get

$$\begin{aligned}
&\sum_{k=3}^{n}\left(\frac{g(\theta_{k+1})}{S_{k+1}^{\varepsilon}}-\frac{g(\theta_k)}{S_k^{\varepsilon}}\right) \\
&\leq \frac{\alpha_0}{20}\sum_{k=3}^{n}\left(\frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1})\right\|^2}{S_{k-2}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_k} g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}\right) \\
&+\frac{\alpha_0}{2}(M+1)\sum_{k=3}^{n}\left(\frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1})\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}-\frac{\left\|\nabla_{\theta_k} g(\theta_k)\right\|^2}{S_k^{\frac{1}{2}+\varepsilon}}\right) \\
&-\frac{\alpha_0}{20}\sum_{k=3}^{n}\frac{\left\|\nabla_{\theta_k} g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}+4M^2\alpha_0^3 c^2\sum_{k=3}^{n} I_k^{>a}\frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1}),\xi_{k-1}\right\|^2}{S_{k-1}^{\frac{3}{2}+\varepsilon}} \\
&+\frac{c\alpha_0^2}{2}\sum_{k=3}^{n} I_k^{>a}\frac{\left\|\nabla_{\theta_k} g(\theta_k,\xi_k)\right\|^2}{S_k^{1+\varepsilon}}+5\alpha_0^3 c^2(M+1)^2\sum_{k=2}^{n} I_k^{\leq a}\frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1},\xi_{k-1})\right\|^2}{S_{k-1}^{1+\varepsilon}} \\
&+\sum_{k=3}^{n} I_k^{\leq a}\frac{(M+1)\alpha_0^3 c^2}{2S_{k-1}^{\frac{3}{2}+\varepsilon}}\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1},\xi_{k-1})\right\|^2 \\
&+\frac{\alpha_0 a(M+1)}{2}\sum_{k=3}^{n}\left(\frac{1}{S_{k-1}^{\frac{1}{2}+\varepsilon}}-\frac{1}{S_k^{\frac{1}{2}+\varepsilon}}\right)+\frac{c\alpha_0^2}{2}\sum_{k=3}^{n} I_k^{\leq a}\frac{\left\|\nabla_{\theta_k} g(\theta_k,\xi_k)\right\|^2}{S_k^{1+\varepsilon}} \\
&+\sum_{k=3}^{n}\left(I_k^{\leq a}A_k^{(\varepsilon)}+I_k^{\leq a}B_k^{(\varepsilon)}+I_k^{>a}X_k^{(\varepsilon)}+I_k^{>a}Y_k^{(\varepsilon)}\right).
\end{aligned} \tag{112}$$

It follows from Lemma 15 that

$$\sum_{k=3}^{n}\frac{\left\|\nabla_{\theta_k} g(\theta_k,\xi_k)\right\|^2}{S_k^{1+\varepsilon}} \leq \int_{S_3}^{+\infty}\frac{1}{x^{1+\varepsilon}}dx=\frac{1}{\varepsilon S_3^{\varepsilon}}, \tag{113}$$

and

$$\sum_{k=3}^{n} \frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1}, \xi_{k-1})\right\|^2}{S_{k-1}^{1+\varepsilon}} \leq \int_{S_2}^{+\infty} \frac{1}{x^{1+\varepsilon}} dx = \frac{1}{\varepsilon S_2^{\varepsilon}}, \tag{114}$$

and

$$\sum_{k=3}^{n} \frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1}, \xi_{k-1})\right\|^2}{S_{k-1}^{\frac{3}{2}+\varepsilon}} \leq \int_{S_2}^{+\infty} \frac{1}{x^{\frac{3}{2}+\varepsilon}} dx = \frac{2}{(1+2\varepsilon) S_2^{\frac{1}{2}+\varepsilon}}. \tag{115}$$

Due to $I_k^{\leq a} \leq 1$ and $I_k^{>a} \leq 1$, we get that

$$\begin{aligned}
&4M^2\alpha_0^3 c^2 \sum_{k=3}^{n} I_k^{>a} \frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1}), \xi_{k-1}\right\|^2}{S_{k-1}^{\frac{3}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2} \sum_{k=3}^{n} I_k^{>a} \frac{\left\|\nabla_{\theta_k} g(\theta_k, \xi_k)\right\|^2}{S_k^{1+\varepsilon}} \\
&+ 5\alpha_0^3 c^2 (M+1)^2 \sum_{k=3}^{n} I_k^{\leq a} \frac{\left\|\nabla_{\theta_{k-1}} g(\theta_{k-1}, \xi_{k-1})\right\|^2}{S_{k-1}^{1+\varepsilon}} \\
&+ \sum_{k=3}^{n} I_k^{\leq a} \frac{(M+1)\alpha_0^3 c^2}{2S_{k-1}^{\frac{3}{2}+\varepsilon}} \left\|\nabla_{\theta_{k-1}} g(\theta_{k-1}, \xi_{k-1})\right\|^2 + \frac{c\alpha_0^2}{2} \sum_{k=3}^{n} I_k^{\leq a} \frac{\left\|\nabla_{\theta_k} g(\theta_k, \xi_k)\right\|^2}{S_k^{1+\varepsilon}} \\
&\leq \frac{8M^2\alpha_0^3 c^2}{(1+2\varepsilon) S_2^{\frac{1}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2\varepsilon S_3^{\varepsilon}} + \frac{5\alpha_0^3 c^2 (M+1)^2}{\varepsilon S_2^{\varepsilon}} + \frac{\alpha_0^3 c^2 (M+1)}{(1+2\varepsilon) S_2^{\frac{1}{2}+\varepsilon}} + \frac{c\alpha_0^2}{2\varepsilon S_3^{\varepsilon}} := K.
\end{aligned} \tag{116}$$

Substituting equation 116 into equation 112 leads to

$$\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_3)}{S_3^{\varepsilon}} \\
&\leq \frac{\alpha_0}{20} \left(\frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{S_1^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} \right) + \frac{\alpha_0}{2}(M+1) \left(\frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{S_2^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}} \right) \\
&- \frac{\alpha_0}{20} \sum_{k=3}^{n} \frac{\left\|\nabla_{\theta_k} g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0 a(M+1)}{2} \left(\frac{1}{S_2^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}} \right) + K \\
&+ \sum_{k=3}^{n} \left(I_k^{\leq a} A_k^{(\varepsilon)} + I_k^{\leq a} B_k^{(\varepsilon)} + I_k^{>a} X_k^{(\varepsilon)} + I_k^{>a} Y_k^{(\varepsilon)} \right).
\end{aligned}$$

It is obvious that

$$\begin{aligned}
&\frac{\alpha_0}{20} \left(\frac{\left\|\nabla_{\theta_0} g(\theta_0)\right\|^2}{S_1^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}} \right) + \frac{\alpha_0}{2}(M+1) \left(\frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{S_2^{\frac{1}{2}+\varepsilon}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_n^{\frac{1}{2}+\varepsilon}} \right) \\
&+ \frac{\alpha_0 a(M+1)}{2} \left(\frac{1}{S_2^{\frac{1}{2}+\varepsilon}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon}} \right) + K \leq \frac{\alpha_0}{20} \frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{S_1^{\frac{1}{2}+\varepsilon}} + \frac{\alpha_0(M+1)}{2} \frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{S_2^{\frac{1}{2}+\varepsilon}} \\
&+ \frac{\alpha_0 a(M+1)}{2S_2^{\frac{1}{2}+\varepsilon}} + K := L.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}} - \frac{g(\theta_3)}{S_3^{\varepsilon}} \\
&\leq -\frac{\alpha_0}{20} \sum_{k=3}^{n} \frac{\left\|\nabla_{\theta_k} g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}} + L + \sum_{k=3}^{n} \left(I_k^{\leq a} A_k^{(\varepsilon)} + I_k^{\leq a} B_k^{(\varepsilon)} + I_k^{>a} X_k^{(\varepsilon)} + I_k^{>a} Y_k^{(\varepsilon)} \right).
\end{aligned} \tag{117}$$

Note that $\{I_k^{\leq a}A_k\}$, $\{I_k^{\leq a}B_k\}$, $\{I_k^{>a}X_k\}$ and $\{I_k^{>a}Y_k\}$ are all martingale difference sequences, thus it follows that $\mathbb{E}\left(\sum_{k=1}^{n}\left(I_k^{\leq a}A_k+I_k^{\leq a}B_k+I_k^{>a}X_k+I_k^{>a}Y_k\right)\right)=0$. Then we calculate mathematical expectation on equation 117

$$\mathbb{E}\left(\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}-\frac{g(\theta_3)}{S_3^{\varepsilon}}\right)\leq-\frac{\alpha_0}{20}\mathbb{E}\left(\sum_{k=3}^{n}\frac{\left\|\nabla_{\theta_k}g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}\right)+L.$$

That is

$$\mathbb{E}\left(\sum_{k=3}^{n}\frac{\left\|\nabla_{\theta_k}g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}\right)<\frac{20}{\alpha_0}\left(\frac{g(\theta_3)}{S_3^{\varepsilon}}+L\right)<+\infty. \tag{118}$$

From Lemma 6, it holds that

$$\sum_{k=3}^{n}\frac{\left\|\nabla_{\theta_k}g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}<+\infty\ \ a.s.. \tag{119}$$

C.4 Proof of Lemma 13

It follows from equation 117 that

$$\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}\leq\frac{g(\theta_3)}{S_3^{\varepsilon}}+L+\sum_{k=3}^{n}\left(I_k^{\leq a}A_k^{(\varepsilon)}+I_k^{\leq a}B_k^{(\varepsilon)}+I_k^{>a}X_k^{(\varepsilon)}+I_k^{>a}Y_k^{(\varepsilon)}\right). \tag{120}$$

From equation 99, we obtain

$$\sum_{k=3}^{n}\mathbb{E}\left(\left\|I_k^{\leq a}A_k^{(\varepsilon)}\right\|^2\right)\leq\sum_{k=3}^{n}\mathbb{E}\left(I_k^{\leq a}\frac{\alpha_0^2}{S_{n-1}^{1+2\varepsilon}}\left(\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2-\nabla_{\theta_n}g(\theta_n)^T\nabla_{\theta_n}g(\theta_n,\xi_n)\right)^2\right).$$

With inequality $(a+b)^2\leq 2(a^2+b^2),\ 2a^Tb\leq a^2+b^2\ \ (a,\ b>0)$ and equation 118, we get

$$\begin{aligned}
&\mathbb{E}\left(I_k^{\leq a}\frac{\alpha_0^2}{S_{n-1}^{1+2\varepsilon}}\left(\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2-\nabla_{\theta_n}g(\theta_n)^T\nabla_{\theta_n}g(\theta_n,\xi_n)\right)^2\right)\\
&\leq 2\mathbb{E}\left(\frac{\alpha_0^2\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{1+2\varepsilon}}\left(I_k^{\leq a}\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2\right)\right)+2\mathbb{E}\left(\frac{\alpha_0^2I_k^{\leq a}}{S_{n-1}^{1+2\varepsilon}}\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2\left\|\nabla_{\theta_n}g(\theta_n,\xi_n)\right\|^2\right)\\
&\leq 2(M+2)\mathbb{E}\left(\frac{\alpha_0^2a\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{1+\varepsilon}}\right)\\
&\leq 2(M+2)\alpha_0^2a\mathbb{E}\left(\frac{\alpha_0^2a\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\right)\\
&<40(M+2)\alpha_0a\left(\frac{g(\theta_1)}{S_1^{\varepsilon}}+L\right).
\end{aligned} \tag{121}$$

It follows from Lemma 5 that $\sum_{k=3}^{n}I_k^{\leq a}A_k^{(\varepsilon)}$ is convergent a.s. Similarly, $\sum_{k=3}^{n}I_k^{\leq a}B_k^{(\varepsilon)}$, $\sum_{k=3}^{n}I_k^{>a}X_k^{(\varepsilon)}$ and $\sum_{k=3}^{n}I_k^{>a}Y_k^{(\varepsilon)}$ are both convergent a.s. It follows that

$$\sum_{k=3}^{n}\left(I_k^{\leq a}A_k^{(\varepsilon)}+I_k^{\leq a}B_k^{(\varepsilon)}+I_k^{>a}X_k^{(\varepsilon)}+I_k^{>a}Y_k^{(\varepsilon)}\right)<\xi'<+\infty\ \ a.s.,$$

$$\frac{g(\theta_{n+1})}{S_{n+1}^{\varepsilon}}\leq\frac{g(\theta_3)}{S_3^{\varepsilon}}+L+\xi'<+\infty\ \ a.s.$$

For convenience, let $\xi=\frac{g(\theta_3)}{S_3^{\varepsilon}}+L+\xi'$. Thus, it holds that

$$\frac{g(\theta_{n+1})-g(\theta^*)}{S_{n+1}^{\varepsilon}}<\xi<+\infty\ \ a.s.. \tag{122}$$

C.5 Proof of Lemma 14

First of all, $\forall\ 0<\varepsilon_0<\frac{3}{8}$, there is $0<\frac{4}{3}\varepsilon_0<\frac{1}{2}$. From Lemma 12, it follows that

$$\sum_{n=4}^{+\infty}\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{4}{3}\varepsilon_0}}<+\infty\quad a.s.. \tag{123}$$

It follows from equation 88 that

$$\begin{aligned}&\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2-\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2\leq\frac{\alpha_0 c}{\sqrt{S_{n-1}}}\left(\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2+\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\right)\\&+\frac{\alpha_0^2c^2}{S_{n-1}}\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2.\end{aligned} \tag{124}$$

Divide both sides of equation 124 by $S_{n-1}^{\varepsilon_0}$ and notice that $S_n>S_{n-1}>S_{n-2}$, then we get

$$\begin{aligned}&\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\varepsilon_0}}-\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\varepsilon_0}}\leq\frac{\alpha_0 c\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon_0}}+\frac{\alpha_0 c\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\varepsilon_0}}\\&+\frac{\alpha_0^2c^2\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-2}^{1+\varepsilon_0}}.\end{aligned} \tag{125}$$

Note that $0<\frac{2}{3}\varepsilon_0<\frac{1}{2}$, then it follows from Assumption 5 4) and Lemma 13 that

$$\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{2}{3}\varepsilon_0}}\leq\frac{t\big(g(\theta_{n-1})-g(\theta^*)\big)}{S_{n-2}^{\frac{2}{3}\varepsilon_0}}<t\xi<+\infty.$$

It follows that

$$\begin{aligned}&\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\varepsilon_0}}-\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\varepsilon_0}}\leq\frac{\alpha_0 ct\xi}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}+\frac{\alpha_0 cMt\xi}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}+\frac{\alpha_0^2c^2Mt\xi}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}+\frac{\alpha_0 ca}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\\&+\frac{\alpha_0^2c^2a}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}-\frac{\alpha_0 c}{S_{n-2}^{\frac{1}{2}+\varepsilon_0}}\left(\mathbb{E}\left(\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\middle|\mathscr{F}_{n-2}\right)-\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\right)\\&-\frac{\alpha_0^2c^2}{S_{n-2}^{1+\varepsilon_0}}\left(\mathbb{E}\left(\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\middle|\mathscr{F}_{n-2}\right)-\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\right).\end{aligned}$$

Thus, we have

$$\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_n^{\varepsilon_0}}-\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-1}^{\varepsilon_0}}\leq\frac{\zeta}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}-K_{n-1}, \tag{126}$$

where

$$\begin{aligned}&\zeta=\alpha_0 ct\xi(M+1+\alpha_0 c)+\alpha_0 c(a+\alpha_0 c+1)\\&K_{n-1}=\frac{\alpha_0 c}{S_{n-2}^{\frac{1}{2}+\varepsilon_0}}\left(\mathbb{E}\left(\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\middle|\mathscr{F}_{n-2}\right)-\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\right)\\&+\frac{\alpha_0 c}{S_{n-2}^{1+\varepsilon_0}}\left(\mathbb{E}\left(\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\middle|\mathscr{F}_{n-2}\right)-\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2\right).\end{aligned}$$

It follows that

$$\mathbb{E}\left(\frac{1}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)\geq\frac{1}{a}\mathbb{E}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2-M\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)$$

By Lemma 15, we have that

$$\sum_{n=4}^{+\infty}\frac{1}{a}\mathbb{E}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)$$
$$>\lim_{n\to+\infty}\frac{1}{a}\mathbb{E}\left(\int_{S_2}^{S_{n-1}}\frac{1}{x^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}dx\right)=\lim_{n\to+\infty}\frac{6}{2-3\varepsilon_0}\frac{1}{a}\mathbb{E}\left(S_{n-1}^{\frac{1}{2}-\frac{1}{3}\varepsilon_0}-S_2^{\frac{1}{2}-\frac{1}{3}\varepsilon_0}\right).$$

Due to $S_{n-1}\to+\infty\ \ a.s.$, it follows that

$$\sum_{n=4}^{+\infty}\frac{1}{a}\mathbb{E}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)=+\infty. \tag{127}$$

From Lemma 12, we get

$$\sum_{n=4}^{+\infty}\frac{1}{a}\mathbb{E}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)<+\infty. \tag{128}$$

Combine equation 127 and equation 128, then

$$\begin{aligned}&\sum_{n=4}^{+\infty}E\left(\frac{1}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)>\sum_{n=4}^{+\infty}\frac{1}{a}\mathbb{E}\left(\frac{\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)\\&-\sum_{n=4}^{+\infty}\frac{1}{a}\mathbb{E}\left(\frac{M\left\|\nabla_{\theta_{n-1}}g(\theta_{n-1})\right\|^2}{S_{n-2}^{\frac{1}{2}+\frac{1}{3}\varepsilon_0}}\right)=+\infty\end{aligned} \tag{129}$$

is divergent a.s. From Lemma 12, we get that $\sum_{n=4}^{+\infty}\mathbb{E}\left(\|K_{n-1}\|^2\right)<+\infty$ and K_{n-1} is a martingale difference sequence. Thus, $\sum_{n=2}^{+\infty}K_{n-1}$ is convergent a.s.. Combine equation 126, equation 129 and Lemma 2, then we have

$$\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon_0}}\to 0\ \ a.s.. \tag{130}$$

C.6 The proof of Theorem 3

We consider the proof under two conditions, namely, $S_n<+\infty\ \ a.s.$ or $S_n=+\infty\ \ a.s.$.

First, if $S_n<+\infty\ \ a.s.$, from Lemma 12, we get that $\forall\varepsilon\in(0,\frac{1}{2})$, it holds that

$$\sum_{k=3}^{n}\frac{\left\|\nabla_{\theta_k}g(\theta_k)\right\|^2}{S_{k-1}^{\frac{1}{2}+\varepsilon}}<+\infty\ \ a.s.$$

Thus, we conclude that

$$\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}\to 0\ \ a.s..$$

Due to $S_{n-1}<+\infty\ \ a.s.$, we get

$$\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2\to 0\ \ a.s., \tag{131}$$

Second, if $S_n = +\infty \ \ a.s.$, let $\varepsilon \to 0$ on equation 111, then it holds that

$$
\begin{aligned}
& g(\theta_{n+1}) - g(\theta_n) \\
& \leq \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{\sqrt{S_{n-2}}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}}\right) \\
& + \frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{\sqrt{S_{n-1}}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}}\right) \\
& - \frac{\alpha_0}{20}\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}} + 4M^2\alpha_0^3 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}), \xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}}} + c\alpha_0^2 \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n} \\
& + \frac{\alpha_0^3 c^2 (M+1)^2}{2}\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2}{S_{n-1}} + \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2 \\
& + \frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{\sqrt{S_{n-1}}} - \frac{1}{\sqrt{S_n}}\right) + I_n^{\leq a} A_n + I_n^{\leq a} B_n + I_n^{>a} X_n + I_n^{>a} Y_n,
\end{aligned}
\tag{132}
$$

where

$$
\begin{aligned}
& I_n^{\leq a} A_n + I_n^{\leq a} B_n + I_n^{>a} X_n + I_n^{>a} Y_n = \lim_{\varepsilon \to 0}\left(I_n^{\leq a} A_n^{(\varepsilon)} + I_n^{\leq a} B_n^{(\varepsilon)} + I_n^{>a} X_n^{(\varepsilon)} + I_n^{>a} Y_n^{(\varepsilon)}\right) \\
& = \frac{\alpha_0 I_n^{\leq a}}{\sqrt{S_{n-1}}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right) \\
& + \frac{\alpha_0\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 I_n^{\leq a}}{2a(M+1)\sqrt{S_{n-1}}}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right) \\
& + \frac{\alpha_0}{2(M+1)}\frac{I_n^{>a}}{\sqrt{S_{n-1}}}\left(\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \middle| \mathscr{F}_{n-1}\right) - \left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2\right) \\
& + \frac{\alpha_0}{2}\frac{I_n^{>a}}{\sqrt{S_{n-1}}}\nabla_{\theta_n} g(\theta_n)^T\left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right).
\end{aligned}
$$

Make some transformations on $\frac{\left\|\nabla_{\theta_n} g(\theta_n), \xi_n\right\|^2}{S_n}$ to obtain that

$$
\begin{aligned}
& \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n} \leq \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_{n-1}} \\
& = \frac{\mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n), \xi_n\right\|^2 \middle| \mathscr{F}_{n-1}\right)}{S_{n-1}} + \frac{1}{S_{n-1}}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n), \xi_n\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right) \\
& \leq \frac{M\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}} + \frac{a}{S_{n-1}} + \frac{1}{S_{n-1}}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n), \xi_n\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right),
\end{aligned}
$$

where the last inequality is from Assumption 5 5). Let $0 < \varepsilon' < \frac{3}{8}$. It follows from Lemma 14 that $\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon'}} < \delta < +\infty$. Then we have $\frac{M\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}} \leq \frac{M\delta}{S_{n-1}^{1-\varepsilon'}} = \frac{M\delta}{S_{n-1}^{\frac{1}{2}+\varepsilon_1}} \quad \left(\varepsilon_1 = \frac{1}{2} - \varepsilon' \in (0, \frac{1}{4})\right)$ and

$$
\begin{aligned}
& \frac{\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2}{S_n} \leq \frac{M\delta + a}{S_n^{\frac{1}{2}+\varepsilon_1}} + \frac{1}{S_{n-1}}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n), \xi_n\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right) \\
& + (M\delta + a)\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon_1}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}}\right).
\end{aligned}
\tag{133}
$$

Similarly, we have

$$\begin{aligned}\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2}{S_{n-1}} &\leq \frac{M\delta+a}{S_n^{\frac{1}{2}+\varepsilon_1}} + (M\delta+a)\left(\frac{1}{S_{n-2}^{\frac{1}{2}+\varepsilon_1}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}}\right) \\ &+ \frac{1}{S_{n-2}}\left(\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}),\xi_{n-1}\right\|^2 \middle| \mathscr{F}_{n-2}\right)\right).\end{aligned} \tag{134}$$

Substitute equation 133 and equation 134 into equation 132, then we get

$$g(\theta_{n+1}) - g(\theta_n) \leq \left(c\alpha_0^2 + \frac{c^3\alpha_0^2(M+1)}{2}\right)(M\delta+a)\frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}} + P_n + Q_n, \tag{135}$$

where

$$\begin{aligned}Q_n =& \frac{c\alpha_0^2}{S_{n-1}}\left(\left\|\nabla_{\theta_n} g(\theta_n,\xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n),\xi_n\right\|^2 \middle| \mathscr{F}_{n-1}\right)\right) \\ &+ \frac{\alpha_0^3 c^2 (M+1)^2}{2S_{n-2}}\left(\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}),\xi_{n-1}\right\|^2 \middle| \mathscr{F}_{n-2}\right)\right) \\ &+ I_n^{\leq a} A_n + I_n^{\leq a} B_n + I_n^{>a} X_n + I_n^{>a} Y_n.\end{aligned}$$

$$\begin{aligned}P_n =& \frac{\alpha_0}{20}\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{\sqrt{S_{n-2}}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}}\right) + \frac{\alpha_0}{2}(M+1)\left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{\sqrt{S_{n-1}}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}}\right) \\ &+ 4M^2\alpha_0^3 c^2 \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}),\xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}}} + \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}}}\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1},\xi_{n-1})\right\|^2 \\ &+ \frac{\alpha_0 a(M+1)}{2}\left(\frac{1}{\sqrt{S_{n-1}}} - \frac{1}{\sqrt{S_n}}\right) + \frac{\alpha_0^3 c^2 (M+1)^2 (M\delta+a)}{2}\left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon_1}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}}\right) \\ &+ c\alpha_0^2(M\delta+a)\left(\frac{1}{S_{n-2}^{\frac{1}{2}+\varepsilon_1}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}}\right).\end{aligned}$$

It follows from $S_n \to +\infty \ \ a.s.$ that

$$
\begin{aligned}
\sum_{n=3}^{+\infty} P_n =& \frac{\alpha_0}{20} \sum_{n=3}^{+\infty} \left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{\sqrt{S_{n-2}}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}} \right) \\
&+ \frac{\alpha_0}{2}(M+1) \sum_{n=3}^{+\infty} \left(\frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1})\right\|^2}{\sqrt{S_{n-1}}} - \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}} \right) \\
&+ 4M^2 \alpha_0^3 c^2 \sum_{n=3}^{+\infty} \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}), \xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}}} + \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}}} \sum_{n=3}^{+\infty} \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2 \\
&+ \sum_{n=3}^{+\infty} \frac{\alpha_0 a(M+1)}{2} \left(\frac{1}{\sqrt{S_{n-1}}} - \frac{1}{\sqrt{S_n}} \right) + \sum_{n=3}^{+\infty} \frac{\alpha_0^3 c^2 (M+1)^2 (M\delta + a)}{2} \left(\frac{1}{S_{n-1}^{\frac{1}{2}+\varepsilon_1}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}} \right) \\
&+ \sum_{n=3}^{+\infty} c\alpha_0^2 (M\delta + a) \left(\frac{1}{S_{n-2}^{\frac{1}{2}+\varepsilon_1}} - \frac{1}{S_n^{\frac{1}{2}+\varepsilon_1}} \right) \\
=& \frac{\alpha_0}{20} \left(\frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{\sqrt{S_1}} - \lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}} \right) \\
&+ \frac{\alpha_0}{2}(M+1) \left(\frac{\left\|\nabla_{\theta_2} g(\theta_2)\right\|^2}{\sqrt{S_2}} - \lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}} \right) + \frac{\alpha_0 a(M+1)}{2} \left(\frac{1}{\sqrt{S_2}} - \lim_{n\to+\infty} \frac{1}{\sqrt{S_n}} \right) \\
&+ 4M^2 \alpha_0^3 c^2 \sum_{n=2}^{+\infty} \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}), \xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}}} + \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}}} \sum_{n=2}^{+\infty} \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2 \\
&+ \frac{\alpha_0 a(M+1)}{2} \frac{1}{\sqrt{S_2}} + \frac{M\delta + a}{S_2^{\frac{1}{2}+\varepsilon_1}} + \frac{\alpha_0^3 c^2 (M+1)^2 (M\delta + a)}{2S_1^{\frac{1}{2}+\varepsilon_1}}.
\end{aligned}
$$

From Lemma 14, we get that

$$
\lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}} \le \lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon_0}} = 0,
$$

and

$$
\lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_n}} \le \lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{\sqrt{S_{n-1}}} \le \lim_{n\to+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon_0}} = 0.
$$

It follows that

$$
\begin{aligned}
\sum_{n=3}^{+\infty} P_n =& \frac{\alpha_0}{20} \frac{\left\|\nabla_{\theta_1} g(\theta_1)\right\|^2}{\sqrt{S_0}} + \frac{\alpha_0}{2}(M+1) \frac{\left\|\nabla_{\theta_1} g(\theta_1)\right\|^2}{\sqrt{S_1}} + \frac{\alpha_0 a(M+1)}{2} \frac{1}{\sqrt{S_1}} \\
&+ 4M^2 \alpha_0^3 c^2 \sum_{n=2}^{+\infty} \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}), \xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}}} + \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}}} \sum_{n=2}^{+\infty} \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2 \\
&+ \frac{\alpha_0 a(M+1)}{2} \frac{1}{\sqrt{S_2}} + \frac{M\delta + a}{S_2^{\frac{1}{2}+\varepsilon_1}} + \frac{\alpha_0^3 c^2 (M+1)^2 (M\delta + a)}{2S_1^{\frac{1}{2}+\varepsilon_1}}.
\end{aligned} \tag{136}
$$

By Lemma 15, we have

$$
4M^2 \alpha_0^3 c^2 \sum_{n=3}^{+\infty} \frac{\left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}), \xi_{n-1}\right\|^2}{S_{n-1}^{\frac{3}{2}}} + \sum_{n=3}^{+\infty} \frac{(M+1)\alpha_0^3 c^2}{2S_{n-1}^{\frac{3}{2}}} \left\|\nabla_{\theta_{n-1}} g(\theta_{n-1}, \xi_{n-1})\right\|^2 < +\infty. \tag{137}
$$

Thus, $\sum_{n=2}^{+\infty} P_n$ is convergent. In addition, it holds that

$$\begin{aligned}
&\sum_{n=3}^{+\infty} \mathbb{E}\left(\left\|I_n^{\leq a} A_n\right\|^2 \Big| \mathscr{F}_{n-1}\right) \\
&\leq \sum_{n=3}^{+\infty} \mathbb{E}\left(\left\|A_n\right\|^2 \Big| \mathscr{F}_{n-1}\right) \leq \sum_{n=3}^{+\infty} \mathbb{E}\left(\left\| \frac{\alpha_0}{\sqrt{S_{n-1}}} \nabla_{\theta_n} g(\theta_n)^T \left(\nabla_{\theta_n} g(\theta_n) - \nabla_{\theta_n} g(\theta_n, \xi_n)\right)\right\|^2 \Big| \mathscr{F}_{n-1}\right) \\
&\leq 2 \sum_{n=3}^{+\infty} \frac{\alpha_0^2 \left\|\nabla_{\theta_n} g(\theta_n)\right\|^4}{S_{n-1}} + 2 \sum_{n=3}^{+\infty} \mathbb{E}\left(\frac{\alpha_0^2}{S_{n-1}} \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \Big| \mathscr{F}_{n-1}\right).
\end{aligned}$$

It follows from Lemma 14 that $\exists 0 < \varepsilon_0 < \frac{3}{8}$, such that

$$\lim_{n \to +\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon_0}} = 0 \quad a.s.,$$

meaning that there is an almost surely bounded random variable δ_0, such that

$$\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\varepsilon_0}} < \delta_0 < \infty \quad a.s.$$

It follows that

$$\frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^4}{S_{n-1}} < \delta_0 \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+(\frac{1}{2}-\varepsilon_0)}} \quad a.s.,$$

Then we derive that

$$\begin{aligned}
&\sum_{n=3}^{+\infty} \mathbb{E}\left(\left\|I_n^{\leq a} A_n\right\|^2 \Big| \mathscr{F}_{n-1}\right) \\
&\leq 2 \sum_{n=3}^{+\infty} \frac{\alpha_0^2 \left\|\nabla_{\theta_n} g(\theta_n)\right\|^4}{S_{n-1}} + 2 \sum_{n=3}^{+\infty} \mathbb{E}\left(\frac{\alpha_0^2}{S_{n-1}} \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \Big| \mathscr{F}_{n-1}\right) \\
&\leq 2 \sum_{n=3}^{+\infty} \frac{\alpha_0^2 \left\|\nabla_{\theta_n} g(\theta_n)\right\|^4}{S_{n-1}} + 2 \sum_{n=3}^{+\infty} \frac{\alpha_0^2}{S_{n-1}} \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 \left(M \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + a\right) \\
&= 2(M+1) \sum_{n=3}^{+\infty} \frac{\alpha_0^2 \left\|\nabla_{\theta_n} g(\theta_n)\right\|^4}{S_{n-1}} + 2 \sum_{n=3}^{+\infty} \frac{\alpha_0^2 a \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}} \\
&< 2\big((M+1)\delta_0 + a\big) \sum_{n=3}^{+\infty} \frac{\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+(\frac{1}{2}-\varepsilon_0)}} < +\infty \quad a.s..
\end{aligned} \tag{138}$$

From Lemma 5, $\sum_{n=3}^{+\infty} I_n^{\leq a} A_n$ is convergent almost surely. Similarly, we can prove other parts of $\sum_{n=3}^{+\infty} Q_n$ are also convergent almost surely. Thus, $\sum_{n=3}^{+\infty} Q_n$ is convergent almost surely. By summary, we get

$$\sum_{n=1}^{+\infty} \big(P_n + Q_n\big) \textit{ is convergent almost surely.} \tag{139}$$

It is easy to find that J_i is a bounded closed set. So $\forall \varepsilon > 0$ we can construct an open cover $H_\varepsilon^{(i)} = \{U(\theta, \varepsilon)\}$ $(\theta \in J_i)$ of J_i. Through the *HeineBorel theorem*, we can get a finite open subcover $\{U(\theta_k, \varepsilon)$ $(k = 0, 1, ..., n)$ from $H_\varepsilon^{(i)}$. Then we assign $U_\varepsilon^{(i)} = \bigcup_{k=0}^{n} U(\theta_k, \varepsilon)$. We can get $U_\varepsilon^{(i)}$ is a open set. Under Assumption 5, $J = \{\theta | \nabla_\theta g(\theta)\}$ has only finite connected components $J_1, J_2, ..., J_m$. So $\inf_{i \neq j} d(J_i, J_j) = \min_{i \neq j} d(J_i, J_j)$. Let $\delta_0 = \min_{i \neq j} d(J_i, J_j)$. It follows from Lemma 4 that $\exists \varepsilon_0 > 0$, when $d(\theta, J_i) < \varepsilon_0$, there is

$$\|\nabla_\theta g(\theta)\|^2 \leq 2c|g(\theta) - g_i|, \tag{140}$$

where g_i denotes $g(\theta)$ $(\theta \in J_i)$. Let $c = min\{\varepsilon_{0,0}/4\}$ and construct $U_c^{(1)}, U_c^{(2)}, ..., U_c^{(m)}$. It is obvious that $\forall\, U_c^{(i)}, U^{(j)c}$ $(i \neq j)$, $d(U_c^{(i)}, U^{(j)}) > \delta_0/2$, and $\|\nabla_\theta g(\theta)\|^2 \leq 2c|g(\theta) - g_i|$ $(\theta \in U_c^{(i)})$. Since J is a bounded set, $\exists N > 0$, such that $J \subset K$ (K is the closure of $U(0,N)$). Then we construct a set $M = K/\bigcup_{i=1}^m U_c^{(i)}$. Since $U_c^{(i)}$ is a open set and K is a closed set, we conclude M is a closed set. Since $\|\nabla_\theta g(\theta)\|$ is a continuous function, $\exists \theta_0 \in M$, $\|\nabla_{\theta_0} g(\theta_0)\| = \min_{\theta \in M} \|\nabla_\theta g(\theta)\|$. Let $r = \|\nabla_{\theta_0} g(\theta_0)\| > 0$.

Then we prove that $\forall u > 0, \theta \in K$, $\exists \delta > 0$, if $\|\nabla_\theta g(\theta)\| < \delta$, makes $d(\theta, J) < u$. We prove it by contradiction. Assume $\exists u_0 > 0$, $\forall \delta_1 > 0$, $\exists\ \theta_{\delta_1}$ holds $\|\nabla_{\theta_{\delta_1}} g(\theta_{\delta_1})\| < \delta_1$ and $d(\theta, J) \geq u_0$. We make $\delta_1 = 1, 1/2, 1/3...$, and we form a sequence $\{\theta_{1/n}\}$. It is obvious that $\|\nabla_{\theta_{1/n}} g(\theta_{1/n}) \to 0\|$. Since $\{\theta_{1/n}\}$ is bounded, through the *Accumulation point theorem*, there exists a convergent subsequence $\{\theta_{1/k_n}\} \subset \{\theta_{1/n}\}$. We defined $\theta^{(0)} = \lim_{n\to+\infty} \theta_{1/k_n}$. Through the continuity of $d(\theta, J)$ and $\|\nabla_\theta g(\theta)\|$, we get $d(\theta^{(0)}) \geq u_0$ and $\|\nabla_{\theta^{(0)}} g(\theta^{(0)})\| = 0$. It is contradiction by the definition of J. So $\forall u > 0, \theta \in K$, $\exists \delta > 0$, $\|\nabla_\theta g(\theta)\| < \delta$, makes $d(\theta, J) < u$ ($\exists$, makes $d(\theta, J) < u$). And furthermore, due to the continuity of $g(\theta)$, we can get $\forall \varepsilon_1 > 0$, $\exists\ \delta' > 0$, if $d(\theta, J_i) < \delta'$, there is $|g(\theta) - g_i| < \varepsilon_1$. So combine these two consequences. We can prove $\forall \varepsilon_1 > 0$, $\exists b > 0$, if $\theta \in U_c^{(i)}$ and $\|\nabla_\theta g(\theta)\| < b$, there is $\left|g(\theta) - g_i\right| < \varepsilon_1$.

Through equation 119 and Lemma 6 we get there is a subsequence $\{\|\nabla_{\theta_{k_n}} g(\theta_{k_n})\|^2\}$ of $\{\|\nabla_{\theta_n} g(\theta_n)\|^2\}$ which satisfies that

$$\lim_{n\to+\infty} \left\|\nabla_{\theta_{k_n}} g(\theta_{k_n})\right\|^2 = 0 \quad a.s.. \tag{141}$$

Next we aim to prove $\lim_{n\to+\infty} \|\nabla_{\theta_n} g(\theta_n)\|^2 = 0$. It is equivalent to prove that $\{\|\nabla_n g(\theta_n)\|^2\}$ has no positive accumulation points, that is to say, $\forall e_0 > 0$, there are only finite values of $\|\nabla_{\theta_n} g(\theta)\|$ larger than e_0. And obviously, we just need to prove $\forall 0 < e_0 < r$, there are only finite values of $\|\nabla_{\theta_n} g(\theta)\|$ larger than e. We prove this by contradiction. We suppose $\exists 0 < e < a$, making the set $S = \{\|\nabla_{\theta_n} g(\theta_n)\|^2\}$ be an infinite set. Then we assign $\varepsilon_1 = e/8c$ and define $o = min\{b, e/4\}$. Due to equation 141, we get there exists a subsequence $\{\theta_{p_n}\}$ of $\{\theta_n\}$ which satisfies $\|\nabla_{\theta_{pn}} g(\theta_{p_n})\| < o$. We rank S as a subsequence $\{\|\nabla_{m_n} g(\theta_{m_n})\|^2\}$ of $\{\|\nabla_n g(\theta_n)\|^2\}$. Then there is an infinite subsequence $\{\|\nabla_{m_{i_n}} g(\theta_{m_{i_n}})\|^2\}$ of $\{\|\nabla_{m_n} g(\theta_{m_n})\|^2\}$ such that $\forall n \in \mathbb{N}_+$, $\exists l$, $n_{p_n} \in (m_{i_l}, m_{i_{l+1}})$. For convenient, we abbreviate $\{m_{i_n}\}$ as $\{i_n\}$. And we construct another infinite sequence $\{q_n\}$ as follows

$$\begin{aligned}
&q_1 = \max\left\{n : p_1 < n < \min\{m_{i_l : m_{i_l} > p_1}\}, \left\|\nabla_{\theta_n} g(\theta_n)\right\| \leq o\right\},\\
&q_2 = \min\left\{n : n > q_1, \left\|\nabla_{\theta_n} g(\theta_n)\right\| > e\right\},\\
&q_{2n-1} = \max\left\{n : \min\{m_{i_l} : m_{i_l} > q_{2n-3}\} < n < \min\{m_l : m_l > \min\{m_{i_l} : m_{i_l} > q_{2n-3}\},\right.\\
&\left.\left\|\nabla_{\theta_n} g(\theta_n)\right\| \leq o\right\},\\
&q_{2n} = \min\left\{n : n > q_{2n-1}, \left\|\nabla_{\theta_n} g(\theta_n)\right\| > e\right\}.
\end{aligned}$$

Now we prove that $\exists N_0$, when $q_{2n} > N_0$, it has $e < \left\|\nabla_{\theta_{q_{2n}}} g(\theta_{q_{2n}})\right\| < r$. The left side is obvious (the definition of q_{2n}). And for the right side, we know $\left\|\nabla_{\theta_{q_{2n}-1}} g(\theta_{q_{2n}-1})\right\| \leq e$. It follows from equation 5 that

$$\begin{aligned}
\|\theta_{n+1} - \theta_n\|^2 &= \frac{\alpha_0^2}{S_n} \left\|\nabla_{\theta_n} g(\theta_{q_n}, \xi_n)\right\|^2\\
&\leq \frac{\alpha_0^2}{S_{n-1}} \left(\left\|\nabla_{\theta_{n-1}} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \middle| \mathscr{F}_n\right)\right)\\
&+ \frac{\alpha_0^2}{S_{n-1}} \left(M\left\|\nabla_{\theta_n} g(\theta_n)\right\|^2 + a\right).
\end{aligned}$$

Through previous consequences we can easily find that

$$\begin{aligned}
&\sum_{n=2}^{+\infty} \left(\frac{\alpha_0^2}{S_{n-1}} \left(\left\|\nabla_{\theta_{n-1}} g(\theta_n, \xi_n)\right\|^2 - \mathbb{E}\left(\left\|\nabla_{\theta_n} g(\theta_n, \xi_n)\right\|^2 \middle| \mathscr{F}_n\right)\right) + \frac{\alpha_0^2 M \left\|\nabla_{\theta_n} g(\theta_n)\right\|^2}{S_{n-1}}\right)\\
&< +\infty \quad a.s..
\end{aligned}$$

Note that $\alpha_0^2 a/S_{n-1} \to 0$ *a.s.*. We conclude

$$\|\theta_{n+1}-\theta_n\| \to 0 \quad a.s.. \tag{142}$$

Through Assumption 5 2) we get $\left|\|\nabla_{\theta_{n+1}} g(\theta_{n+1})\|^2 - \|\nabla_{\theta_n} g(\theta_n)\|^2\right| \le \left|\|\nabla_{\theta_{n+1}} g(\theta_{n+1})\| - \|\nabla_{\theta_n} g(\theta_n)\|\right|^2 \le \|\nabla_{\theta_{n+1}} g(\theta_{n+1}) - \nabla_{\theta_n} g(\theta_n)\|^2 \le c\|\theta_{n+1}-\theta_n\| \to 0 \quad a.s.$, So $\exists N_0$, when $n > N_0$, there is $\left|\|\nabla_{\theta_{n+1}} g(\theta_{n+1})\|^2 - \|\nabla_{\theta_n} g(\theta_n)\|\right| < r-e$. Then we can get that when $q_{2n} > N_0+1$, there is $\left\|\nabla_{\theta_{q_{2n}}} g(\theta_{q_n})\right\| \le \|\nabla_{\theta_{q_{2n}-1}} g(\theta_{q_{2n}-1})\| + \left|\|\nabla_{\theta_{q_{2n}}} g(\theta_{q_{2n}})\| - \|\nabla_{\theta_{q_{2n}-1}} g(\theta_{q_{2n}-1})\|\right| \le e+r-e = r$. That means that $\theta_{q_{2n}} \in \bigcup_{i=1}^m U_\tau^{(i)}$, so we can prove $\exists i_0$, such that $\theta_{q_{2n}} \in U_\tau^{(i_0)}$. And due to $\|\nabla_{\theta_n} g(\theta_n)\| \le e < r \quad (n \in [q_{2n-1}, q_{2n}))$, we get $\forall k \in [q_{2n-1}, q_{2n})$, $\exists i_k$, such that $\theta_n \in U_\tau^{(i_k)}$. Due to $\|\theta_{n+1}-\theta_n\| \to 0 a.s.$, we know $i_0 = i_k \quad (\forall\ j \in [q_{2n-1}, q_{2n}))$. For convenient, we sign $i_0 = i_{q_{2n-1}} = ... = i_{q_{2n}-1} = i_{q_{2n}}$. And then we can conclude that

$$\|\nabla_\theta g(\theta_n)\|^2 \le 2c|g(\theta_n) - g_{i_{q_{2n}}}| \quad (n \in [q_{2n-1}, q_{2n}]).$$

Due to locally sign-preserving property, we get

$$\|\nabla_\theta g(\theta_n)\|^2 \le 2c(g(\theta_n) - g_{i_{q_{2n}}}) \quad (g(\theta_n) \ge g_{i_{q_{2n}}}) \quad or$$
$$\|\nabla_\theta g(\theta_n)\|^2 \le -2c(g(\theta_n) - g_{i_{q_{2n}}}) \quad (g(\theta_n) \le g_{i_{q_{2n}}}) \quad (n \in [q_{2n-1}, q_{2n}]).$$

Ways to dispose these two cases is same, so we just show how to prove the first case. We get

$$\begin{aligned} e-o &< \left\|\nabla_{\theta_{q_{2n}}} g(\theta_{q_{2n}})\right\|^2 - \left\|\nabla_{\theta_{q_{2n-1}}} g(\theta_{q_{2n-1}})\right\|^2 < 2c\big(g(\theta_{q_{2n}}) - g_{i_{q_{2n}}}\big) - \left\|\nabla_{\theta_{q_{2n-1}}} g(\theta_{q_{2n-1}})\right\|^2 \\ &= \left(2c \sum_{i=0}^{q_{2n}-q_{2n-1}-1} g(\theta_{q_{2n-1}+i+1}) - g(\theta_{q_{2n-1}+i})\right) + 2c\Big(g(\theta_{q_{2n-1}}) - g_{i_{q_{2n}}}\Big) - \left\|\nabla_{\theta_{q_{2n-1}}} g(\theta_{q_{2n-1}})\right\|^2. \end{aligned}$$

From equation 135, we obtain

$$\begin{aligned} & g(\theta_{q_{2n-1}+i+1}) - g(\theta_{q_{2n-1}+i}) \\ & \le \left(c\alpha_0^2 + \frac{c^3\alpha_0^2(M+1)}{2}\right)(M\delta + a)\frac{1}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}} + P_{q_{2n-1}+i} + Q_{q_{2n-1}+i}. \end{aligned}$$

So there is

$$\begin{aligned} e-o < & \sum_{i=0}^{q_{2n}-q_{2n-1}-1} \frac{L}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}} + \sum_{i=0}^{q_{2n}-q_{2n-1}-1} \Big(P_{q_{2n-1}+i} + Q_{q_{2n-1}+i}\Big) \\ & + 2c\Big(g(\theta_{q_{2n-1}}) - g_{i_{2n-1}}\Big) - \left\|\nabla_{\theta_{q_{2n-1}}} g(\theta_{q_{2n-1}})\right\|^2, \end{aligned} \tag{143}$$

which

$$L = 2c\left(c\alpha_0^2 + \frac{c^3\alpha_0^2(M+1)}{2}\right)(M\delta + a).$$

Due to $\|\nabla_{\theta_{q_{2n-1}}} g(\theta_{q_{2n-1}})\|^2 < o < b$, so through equation 139, we get that $g(\theta_{q_{2n-1}}) - g_{i_{2n-1}} < e/8c$. Substitute it into equation 144. We get

$$\sum_{i=0}^{q_{2n}-q_{2n-1}-1} \frac{1}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}} > \frac{e}{2L} - \sum_{i=0}^{q_{2n}-q_{2n-1}-1} \Big(P_{q_{2n-1}+i} + Q_{q_{2n-1}+i}\Big). \tag{144}$$

Through equation 139, we know $\sum_{n=1}^{+\infty}(P_n + Q_n)$ is convergence almost surely. So we get that $\sum_{i=0}^{q_{2n}-q_{2n-1}-1} \Big(P_{q_{2n-1}+i} + Q_{q_{2n-1}+i}\Big) \to 0$ *a.s.* by the *Cauchy's test for convergence*. Combining $1/S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1} \to 0 \quad a.s.$, we get

$$\begin{aligned} & \sum_{i=1}^{q_{2n}-q_{2n-1}-1} \frac{1}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}} \\ & > \frac{e}{2L} - \frac{1}{S_{q_{2n-1}}^{\frac{1}{2}+\varepsilon_1}} - \sum_{i=0}^{q_{2n}-q_{2n-1}-1} \Big(P_{q_{2n-1}+i} + Q_{q_{2n-1}+i}\Big) \to \frac{e}{2L} \quad a.s., \end{aligned} \tag{145}$$

so there is

$$\sum_{n=1}^{+\infty}\left(\sum_{i=1}^{q_{2n}-q_{2n-1}-1}\frac{1}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}}\right)=+\infty \quad a.s.. \tag{146}$$

But on the other hand, we know $\|\nabla_{\theta_{q_{2n-1}+i}}g(\theta_{q_{2n-1}+i})\|>o \quad (i>0)$. Together with equation 119, we get

$$\begin{aligned}\sum_{n=1}^{+\infty}\left(\sum_{i=1}^{q_{2n}-q_{2n-1}-1}\frac{1}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}}\right)&<\frac{1}{o}\sum_{n=1}^{+\infty}\left(\sum_{i=1}^{q_{2n}-q_{2n-1}-1}\frac{\left\|\nabla_{\theta_{q_{2n-1}+i}}g(\theta_{q_{2n-1}+i})\right\|^2}{S_{q_{2n-1}+i}^{\frac{1}{2}+\varepsilon_1}}\right)\\&<\frac{1}{o}\sum_{n=3}^{n}\frac{\left\|\nabla_{\theta_n}g(\theta_n)\right\|^2}{S_{n-1}^{\frac{1}{2}+\varepsilon}}<+\infty \quad a.s..\end{aligned} \tag{147}$$

It contradicts with equation 146, so we get that $\|\nabla_{\theta_n}g(\theta_n)\|\to 0 \quad a.s.$. Combining equation 131, we get $\|\nabla_{\theta_n}g(\theta_n)\|\to 0$ no matter $S_n<+\infty$ $a.s.$ or $S_n=+\infty$. Under Assumption 5 1), it is safe to conclude that there exists a connected component J^* of J such that $\lim_{n\to\infty}d(\theta_n,J^*)=0$.