HIGH PROBABILITY GENERALIZATION BOUNDS FOR MINIMAX PROBLEMS WITH FAST RATES

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ABSTRACT

Minimax problems are receiving an increasing amount of attention in a wide range of applications in machine learning (ML), for instance, reinforcement learning, robust optimization, adversarial learning, and distributed computing, to mention but a few. Current studies focus on the fundamental understanding of general minimax problems with an emphasis on convergence behavior. As a comparison, there is far less work to study the generalization performance. Additionally, existing generalization bounds are almost all derived in expectation, and the high probability bounds are all presented in the slow order $\mathcal{O}\left(1/\sqrt{n}\right)$, where n is the sample size. In this paper, we provide improved generalization analyses for almost all existing generalization measures of minimax problems, which enables the minimax problems to establish sharper bounds of order $\mathcal{O}\left(1/n\right)$, significantly, with high probability. We then use the improved learning bounds to establish $\mathcal{O}\left(1/n\right)$ high probability generalization bounds for classical empirical saddle point (ESP) solution and several popular gradient-based optimization algorithms, including gradient descent ascent (GDA), stochastic gradient descent ascent (SGDA), proximal point method (PPM), extra-gradient (EG), and optimistic gradient descent ascent (OGDA). Overall, we provide a comprehensive understanding of sharper generalization bounds of minimax problems.

1 INTRODUCTION

Minimax learning problems have achieved great success over a broad range of learning tasks in machine learning, with examples including reinforcement learning (Du et al., 2017; Dai et al., 2018), robust optimization (Chen et al., 2017; Namkoong & Duchi, 2017), adversarial learning (Goodfellow et al., 2014), distributed computing (Razaviyayn et al., 2020; Shamma, 2008; Mateos et al., 2010), and AUC maximization (Lei & Ying, 2021b), to just name a few. This framework is formulated as a zero-sum game characterized as two groups of decision variables, one for minimization and one for maximization. The coupling of the two groups of variables makes analysis of minimax problems more complex than the standard statistical learning theory setting, with only one minimization operator (Lei et al., 2021). Researchers have designed various optimization algorithms, for instance, gradient descent ascent (GDA), stochastic gradient descent ascent (SGDA), proximal point method (PPM), extra-gradient (EG), and optimistic gradient descent ascent (OGDA), to solve the minimax optimization problem (Farnia & Ozdaglar, 2021). Current theoretical research in ML literature is mainly devoted to the convergence rate and optimality of these minimax optimization algorithms in different setting, such as convex-concave settings (Nemirovski et al., 2008), nonconvex-concave setting (Rafique et al., 2018), strongly convex-strongly-concave setting (Balamurugan & Bach, 2016), and nonconvex-nonconcave setting (Liu et al., 2021; Yang et al., 2020). In contrast, there is far less work on the generalization performance analysis, which is an important measure to indicate the performance of the learned model based on training samples when generalized to the test data.

To the best of our knowledge, there is only three work on the generalization bounds of minimax optimization algorithms (Zhang et al., 2021a; Farnia & Ozdaglar, 2021; Lei et al., 2021). Among them, (Zhang et al., 2021a) studies the generalization bounds for ESP solution to minimax problems, (Farnia & Ozdaglar, 2021) analyzes the generalization properties of several gradient-based optimization algorithms: GDA, SGDA, GDmax and PPM, and (Lei et al., 2021) provides a comprehensive generalization analysis of SGDA. However, in the above-mentioned papers, almost all

generalization bounds are derived in expectation. Only two high probability bounds exist, proposed in (Lei et al., 2021). Unfortunately, they are of slow order $\mathcal{O}\left(1/\sqrt{n}\right)$.

It is known that the high probability bound is beneficial to understand the robustness of optimization algorithms (Bousquet et al., 2020; Klochkov & Zhivotovskiy, 2021) and is much more challenging to be derived (Bousquet et al., 2020; Lei et al., 2021). In this paper, our goal is to provide the sharper high probability generalization bounds for minimax learning problems. We leverage the lens of algorithmic stability, which is also served as an important tool in (Zhang et al., 2021a; Farnia & Ozdaglar, 2021; Lei et al., 2021). Our contributions are summarized below.

1. In view of the coupling between the minimization variable and the maximization variable, minimax learning problems have many generalization measures (Lei et al., 2021; Farnia & Ozdaglar, 2021; Zhang et al., 2021a). In this paper, we provide improved stability analyses for almost all existing generalization measures. Sharper high probability generalization bounds are established, which can be used to derive $\mathcal{O}\left(1/n\right)$ order bounds for stable minimax learning algorithms.

2. The generalization performance of the ESP solution and gradient-based optimization algorithms stands a central place in the learning theory of minimax problems (Lei et al., 2021). In this paper, we develop high probability generalization bounds of the order $\mathcal{O}\left(1/n\right)$ for ESP solution and several popular gradient-based optimization algorithms: GDA, SGDA, PPM, EG, and OGDA. Overall, we provide a comprehensive analysis of sharper generalization bounds for minimax learning problems.

The rest of this paper is organized as follows. In Section 2, we briefly discuss the related work. Preliminaries about minimax problems are introduced in Section 3. We provide the main theorem in Section 4. Applications are shown in Section 5. In the last, Section 6 concludes the paper.

2 Related Work

Algorithmic stability. Algorithmic stability is a fundamental concept in learning theory (Bousquet & Elisseeff, 2002), which has a deep connection with learnability (Rakhlin et al., 2005; Shalev-Shwartz & Ben-David, 2014; Shalev-Shwartz et al., 2010). A training algorithm is stable if small changes in the training set lead to small differences in the output predictions of the trained model. Different algorithmic stability measures have been developed, including uniform stability (Bousquet & Elisseeff, 2002; Feldman & Vondrak, 2018; 2019; Klochkov & Zhivotovskiy, 2021; Hardt et al., 2016; Lei et al., 2020), uniform argument stability (Liu et al., 2017; Bassily et al., 2020), hypothesis stability (Bousquet & Elisseeff, 2002; Charles & Papailiopoulos, 2018), hypothesis set stability (Foster et al., 2019), on average stability (Shalev-Shwartz et al., 2010; Lei & Ying, 2020; Kuzborskij & Lampert, 2018; Zhang et al., 2021b; Lei & Ying, 2021a), locally elastic stability (Deng et al., 2021), collective stability (London, 2017), and PAC-Bayesian stability (Li et al., 2020). These stability measures have been extensively studied in the generalization analysis of the standard statistical learning theory setting (Chen et al., 2018; Zhang et al., 2021b). Several stability measures have also been extended to minimax learning problems, for instance, weak stability, argument stability, and uniform stability (Farnia & Ozdaglar, 2021; Zhang et al., 2021a; Lei et al., 2021). In related work (Farnia & Ozdaglar, 2021; Zhang et al., 2021a; Lei et al., 2021), they mostly focus on the expectation form of these stability measures since they are to derive bounds in expectation. In this paper, we will focus on the last two measures, which are often used when establishing high probability generalization bounds (Feldman & Vondrak, 2018; 2019; Klochkov & Zhivotovskiy, 2021).

Convergence analysis. Convergence analysis has been widely studied in different settings, including convex-concave learning (Nemirovski, 2005; Nedic & Ozdaglar, 2009; Mokhtari et al., 2020; Cherukuri et al., 2017; Mokhtari et al., 2019; Balamurugan & Bach, 2016; Hsieh et al., 2019; Yan et al., 2020; Lin et al., 2020b; Wang & Li, 2020; Yoon & Ryu, 2021), nonconvex-concave learning (Rafique et al., 2018; Kong & Monteiro, 2019; Luo et al., 2020; Grnarova et al., 2017; Thekumparampil et al., 2019; Lu et al., 2020; Namkoong & Duchi, 2016; Sanjabi et al., 2018; Nouiehed et al., 2019; Lin et al., 2020a; Sinha et al., 2017; Chen et al., 2021), and nonconvex-nonconcave learning (Heusel et al., 2017; Balduzzi et al., 2018; Daskalakis & Panageas, 2018; Mertikopoulos et al., 2019; Loizou et al., 2020; Yang et al., 2020; Liu et al., 2021; Lin et al., 2018; Diakonikolas et al., 2021; Wang et al., 2020; Loizou et al., 2021; Fiez & Ratliff, 2021). There are so many studies on convergence, considering the length limit, the references listed here are not complete. Please refer to the related references concerning the above work. We investigate the generalization performance

of minimax problems instead of the convergence behavior. Note that the convergence analysis also plays a key role in this paper, formalized as strong PD empirical risk (please refer to Definition 1), which is defined on the function value difference and referred to as optimization error or primal-dual gap in some convergence literature (Lei et al., 2021; Nemirovski, 2005; Mokhtari et al., 2019).

3 PRELIMINARIES

Let $\mathcal{X}$ and $\mathcal{Y}$ be two parameter spaces in $\mathbb{R}^d$. Let $\mathbb{P}$ be a probability measure defined on a sample space $\mathcal{Z}$. We define $f : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \mapsto \mathbb{R}$ and consider the following minimax optimization problem

$$\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} F(\mathbf{x}, \mathbf{y}) := \mathbb{E}_{z\sim\mathbb{P}}[f(\mathbf{x}, \mathbf{y}; z)]. \tag{1}$$

The above minimax objective represents a expectation of a cost function $f(\mathbf{x}, \mathbf{y}; z)$ for minimization variable $\mathbf{x}$, maximization variable $\mathbf{y}$ and data variable z. Unfortunately, we typically are not available to the underlying distribution $\mathbb{P}$. In practice, F is approximated by the corresponding empirical risk. Let $S = \{z_1, ..., z_n\}$ be a dataset which are independent drawn according to $\mathbb{P}$, the empirical risk is defined as

$$F_S(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\sum_{i=1}^{n} f(\mathbf{x}, \mathbf{y}; z_i). \tag{2}$$

Let the output of a (randomized) algorithm A on a dataset S be $A(S) := (A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \in \mathcal{X} \times \mathcal{Y}$. Since $A(S)$ is just an empirical approximated solution of the true minimax optimization problem, we are interested in studying how well $A(S)$ generalizes to the unseen data. As claimed in (Farnia & Ozdaglar, 2021; Lei et al., 2021), the coupling between the minimization variable and the maximization variable in (1) makes minimax problems have many different generalization performance measures. These measures are collected in (Lei et al., 2021). For better readability, we use their symbols. Let $\mathbb{E}$ be the expectation with respect to (w.r.t.) the randomness of algorithm A and the dataset S. These generalization measures are listed below.

Definition 1. *(Lei et al., 2021) There are four groups of generalization measures.*

1 (Primal Measures.) The primal population risk of a model $\mathbf{x}$ is defined as $R(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{Y}} F(\mathbf{x}, \mathbf{y})$, and the corresponding primal empirical risk is defined as $R_S(\mathbf{x}) = \sup_{\mathbf{y}\in\mathcal{Y}} F_S(\mathbf{x}, \mathbf{y})$. Then, when using empirical risk $R_S(\mathbf{x})$ to bound $R(\mathbf{x})$, we call this error of the model $\mathbf{x}$ the primal generalization error. While using optimal $\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x})$ to bound $R(\mathbf{x})$, we call this error of the model $\mathbf{x}$ the excess primal population risk.

2 (Plain Measure.) When using $F_S(\mathbf{x}, \mathbf{y})$ to bound $F(\mathbf{x}, \mathbf{y})$, we call the this error of a model $(\mathbf{x}, \mathbf{y})$ the plain generalization error.

3 (Strong Measures.) The strong primal-dual (PD) population risk of a model $(\mathbf{x}, \mathbf{y})$ is defined as

$$\triangle^s(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{y}'\in\mathcal{Y}} F(\mathbf{x}, \mathbf{y}') - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', \mathbf{y}),$$

and the corresponding strong PD empirical risk is defined as

$$\triangle^s_S(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{y}'\in\mathcal{Y}} F_S(\mathbf{x}, \mathbf{y}') - \inf_{\mathbf{x}'\in\mathcal{X}} F_S(\mathbf{x}', \mathbf{y}).$$

Then, the strong PD generalization error $\triangle^s(\mathbf{x}, \mathbf{y}) - \triangle^s_S(\mathbf{x}, \mathbf{y})$ of the model $(\mathbf{x}, \mathbf{y})$ is defined as

$$\Big(\sup_{\mathbf{y}'\in\mathcal{Y}} F(\mathbf{x}, \mathbf{y}') - \sup_{\mathbf{y}'\in\mathcal{Y}} F_S(\mathbf{x}, \mathbf{y}')\Big) + \Big(\inf_{\mathbf{x}'\in\mathcal{X}} F_S(\mathbf{x}', \mathbf{y}) - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', \mathbf{y})\Big).$$

4 (Weak Measures.) The weak PD population risk of a (randomized) model $(\mathbf{x}, \mathbf{y})$ is defined as

$$\triangle^w(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{y}'\in\mathcal{Y}} \mathbb{E}[F(\mathbf{x}, \mathbf{y}')] - \inf_{\mathbf{x}'\in\mathcal{X}} \mathbb{E}[F(\mathbf{x}', \mathbf{y})],$$

and the corresponding weak PD empirical risk is defined as

$$\triangle^w_S(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{y}'\in\mathcal{Y}} \mathbb{E}[F_S(\mathbf{x}, \mathbf{y}')] - \inf_{\mathbf{x}'\in\mathcal{X}} \mathbb{E}[F_S(\mathbf{x}', \mathbf{y})].$$

Then, the weak PD generalization error $\triangle^w(\mathbf{x}, \mathbf{y}) - \triangle^w_S(\mathbf{x}, \mathbf{y})$ of the model $(\mathbf{x}, \mathbf{y})$ is defined as

$$\Big(\sup_{\mathbf{y}'\in\mathcal{Y}} \mathbb{E}[F(\mathbf{x}, \mathbf{y}')] - \sup_{\mathbf{y}'\in\mathcal{Y}} \mathbb{E}[F_S(\mathbf{x}, \mathbf{y}')]\Big) + \Big(\inf_{\mathbf{x}'\in\mathcal{X}} \mathbb{E}[F_S(\mathbf{x}', \mathbf{y})] - \inf_{\mathbf{x}'\in\mathcal{X}} \mathbb{E}[F(\mathbf{x}', \mathbf{y})]\Big).$$

Denote the L_p norm of a random variable $z \in \mathcal{Z}$ as $\|z\|_p = (\mathbb{E}_z|z|)^{1/p}$. Let $\|\cdot\|$ be the Euclidean norm and $\langle \cdot, \cdot \rangle$ be the inner product. A differentiable function $g : \mathcal{W} \mapsto \mathbb{R}$ is called μ-strongly-convex in $\mathbf{w}$ if the following inequality holds for every $\mathbf{w}_1, \mathbf{w}_2$:

$$g(\mathbf{w}_1) - g(\mathbf{w}_2) \geq \langle \nabla g(\mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2 \rangle + \frac{\mu}{2}\|\mathbf{w}_1 - \mathbf{w}_1\|^2,$$

where ∇ is the gradient operator. We say g is μ-strongly-concave if $-g$ is μ-strongly-convex.

Definition 2. *Let $g : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are convex feasible sets. Then*

1. g is μ-strongly-convex-strongly-concave (μ-SC-SC) if $g(\cdot, \mathbf{y})$ is μ-strongly-convex for any $\mathbf{y} \in \mathcal{Y}$ and $g(\mathbf{x}, \cdot)$ is μ-strongly-concave for any $\mathbf{x} \in \mathcal{X}$.

2. g is convex-concave (C-C) if g is 0-SC-SC.

We then introduce the definition of algorithmic stability this paper used. Algorithmic stability plays an important role in studying the generalization behavior of a learning algorithm. Intuitively, an algorithm $A : \mathcal{Z}^n \mapsto (\mathcal{X}, \mathcal{Y})$ is said to be stable if the output model $(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$ is insensitive to perturbations. Let S' be a neighboring dataset that differs at most one single example to S.

Definition 3 (Algorithmic Stability)**.** *Let A be a learning algorithm and $\epsilon > 0$.*

1. We say A is ϵ-uniformly-stable if for any training datasets $S, S' \in \mathcal{Z}^n$ we have

$$\sup_z \left[f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); z) - f(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'); z)\right] \leq \epsilon.$$

2. We say A is ϵ-argument-stable if for any training datasets $S, S' \in \mathcal{Z}^n$ we have

$$\|A_{\mathbf{x}}(S) - A_{\mathbf{x}}(S')\| + \|A_{\mathbf{y}}(S) - A_{\mathbf{y}}(S')\| \leq \epsilon.$$

From Definition 3, one can see that the uniform stability measures the sensitivity of the function values, while the argument stability measures the sensitivity of the arguments.

We finally introduce two standard assumptions in minimax problems. Assumption 1 implies f is Lipschitz continuous w.r.t. both $\mathbf{x}$ and $\mathbf{y}$, while Assumption 2 implies f is smooth w.r.t. $(\mathbf{x}, \mathbf{y})$.

Assumption 1 (Lipschitz continuity)**.** *Let $L > 0$. Assume that for any $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} \in \mathcal{Y}$ and $z \in \mathcal{Z}$, $f(\mathbf{x}, \mathbf{y}; z)$ satisfies*

$$\|\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}; z)\| \leq L \quad \textit{and} \quad \|\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}; z)\| \leq L.$$

Assumption 2 (Smoothness)**.** *Let $\beta > 0$. Assume that for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$ and $z \in \mathcal{Z}$, $f(\mathbf{x}, \mathbf{y}; z)$ satisfies*

$$\left\| \begin{pmatrix} \nabla_{\mathbf{x}} f(\mathbf{x}_1, \mathbf{y}_1; z) - \nabla_{\mathbf{x}} f(\mathbf{x}_2, \mathbf{y}_2; z) \\ \nabla_{\mathbf{y}} f(\mathbf{x}_1, \mathbf{y}_1; z) - \nabla_{\mathbf{y}} f(\mathbf{x}_2, \mathbf{y}_2; z) \end{pmatrix} \right\| \leq \beta \left\| \begin{pmatrix} \mathbf{x}_1 - \mathbf{x}_2 \\ \mathbf{y}_1 - \mathbf{y}_2 \end{pmatrix} \right\|.$$

Under Assumption 1, the argument stability implies the uniform stability. Therefore, the argument stability is the main stability measure that we will focus on.

4 Main Results

In this section, we provide sharper high probability bounds for almost all existing generalization measures of the minimax problem, as shown in Theorem 1.

Theorem 1. *Let A be a learning algorithm and $\epsilon > 0$. Suppose $|f(\mathbf{x}, \mathbf{y}; z)| \leq M$ for some $M > 0$ and $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}, z \in \mathcal{Z}$. Fixed any $\eta > 0$. There exists an absolute positive constant C.*

(a.) If the algorithm A is ϵ-uniformly stable, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq (1+\eta) F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + \epsilon \log_2 n \log\frac{1}{\delta}\Big).$$

(b.) Assume that for all $\mathbf{x}$, the function $\mathbf{y} \mapsto F(\mathbf{x}, \mathbf{y})$ is μ-strongly-concave. If the algorithm A is ϵ-argument stable and Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$R(A_{\mathbf{x}}(S)) \quad \leq \quad (1 + \eta) R_S(A_{\mathbf{x}}(S)) + C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big) L\epsilon \log_2 n \log\frac{1}{\delta}\Big).$$

(c.) Assume that for all $\mathbf{x}$ *and* $\mathbf{y}$*, the function* $F(\mathbf{x},\mathbf{y})$ *is* μ*-SC-SC. If the algorithm* A *is* ϵ*-argument stable and Assumptions 1 and 2 hold, then for any* $\delta > 0$*, with probability at least* $1-\delta$*,*

$$\triangle^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) \leq \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \eta\mathbb{E}_S\,\triangle_S^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)\epsilon L\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(d.) Assume that for all $\mathbf{x}$ *and* $\mathbf{y}$*, the function* $F(\mathbf{x},\mathbf{y})$ *is* μ*-SC-SC. If the algorithm* A *is* ϵ*-argument stable and Assumptions 1 and 2 hold, then for any* $\delta > 0$*, with probability at least* $1-\delta$*,*

$$\triangle^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) - \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq \eta\mathbb{E}_S\,\triangle_S^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)\epsilon L\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(e.) Assume that for all $\mathbf{x}$*, the function* $\mathbf{y} \mapsto F(\mathbf{x},\mathbf{y})$ *is* μ*-strongly-concave. If the algorithm* A *is* ϵ*-argument stable and Assumptions 1 and 2 hold, then for any* $\delta > 0$*, with probability at least* $1-\delta$*,*

$$R(A_{\mathbf{x}}(S)) \leq (1+\eta)\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) + C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)L\epsilon\log_2 n\log\frac{1}{\delta} + \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\Big).$$

According to Definition 1 and Jensen's inequality, we know that $\triangle^w(\mathbf{x},\mathbf{y}) \leq \mathbb{E}[\triangle^s(\mathbf{x},\mathbf{y})]$ and $\triangle_S^w(\mathbf{x},\mathbf{y}) \leq \mathbb{E}[\triangle_S^s(\mathbf{x},\mathbf{y})]$. By this connection, we have $\triangle^w(\mathbf{x},\mathbf{y}) - \triangle_S^w(\mathbf{x},\mathbf{y}) \leq \mathbb{E}[\triangle^s(\mathbf{x},\mathbf{y})] + |\mathbb{E}[\triangle_S^s(\mathbf{x},\mathbf{y})]|$. We therefore obtain the following results for $\triangle^w(\mathbf{x},\mathbf{y})$ and $\triangle^w(\mathbf{x},\mathbf{y}) - \triangle_S^w(\mathbf{x},\mathbf{y})$.

Corollary 1. *Suppose the same conditions as Theorem 1 hold.*

(f.) If the assumptions of Part (c) in Theorem 1 hold, then with probability at least $1-\delta$*,*

$$\triangle^w\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) \leq (1+\eta)\mathbb{E}\,\triangle_S^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)\epsilon L\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(g.) If the assumptions of Part (d) in Theorem 1 hold, then with probability at least $1-\delta$*,*

$$\triangle^w\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) - \triangle_S^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq |\mathbb{E}\,\triangle_S^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right)| + (1+\eta)\mathbb{E}\,\triangle_S^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)\epsilon L\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

Remark 1. In Theorem 1, we have established a quantitative connection between the generalization measure and the algorithmic stability measure. The complete proof of Theorem 1 is provided in Appendix A. Part (a) provides the relationship between the uniform stability and the plain generalization error of $(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$. If the uniform stability of algorithm A is of fast order $\mathcal{O}(1/n)$, then $F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$ is bounded by $\mathcal{O}\big(F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \frac{\log n\log(1/\delta)}{n}\big)$, which means the plain generalization error is of fast order $\mathcal{O}\big(\frac{\log n\log(1/\delta)}{n}\big)$. It is $\mathcal{O}(1/n)$ when we hide the logarithmic order. In related work, (Lei et al., 2021) establish the plain generalization error bound under the same assumptions via uniform stability, but their bound is of slow order $\mathcal{O}\big(\epsilon\log n\log(1/\delta) + Mn^{\frac{1}{2}}\sqrt{\log(1/\delta)}\big)$. Even if they get a sharper bound for uniform stability ϵ, the influence of $\mathcal{O}\big(n^{\frac{1}{2}}\sqrt{\log(1/\delta)}\big)$ can not disappear. Part (b) provides the connection between the argument stability and the primal generalization error. If the argument stability of algorithm A is of fast order $\mathcal{O}(1/n)$, then the primal generalization error is of the fast rate $\mathcal{O}(1/n)$. In (Lei et al., 2021), they also establish a bound for primal generalization error under the same assumptions via argument stability. However, their bound is $\mathcal{O}\big(L\beta\mu^{-1}\epsilon logn\log(1/\delta) + Mn^{\frac{1}{2}}\sqrt{\log(1/\delta)}\big)$, which is looser than ours. (Farnia & Ozdaglar, 2021) studies the expected primal generalization error, i.e., bounding $\mathbb{E}_{S,A}[R(A_{\mathbf{x}}(S))]$ by $\mathbb{E}_{S,A}[R_S(A_{\mathbf{x}}(S))]$. They establish the connection between the stability measure and the expected error under the same assumptions as Part (b), which is then used to

derive bounds of order $\mathcal{O}(1/n)$ for (S)GDA, (S)GDmax and PPM algorithms. By comparison, our results are derived in high probability. High probability bounds are beneficial to understand the generalization performance of the learned model when restricted to samples as compared to the rates in expectation (Bousquet et al., 2020; Klochkov & Zhivotovskiy, 2021). Part (c) provides the relationship between the argument stability and the strong PD population risk. If both the argument stability of algorithm A and the strong PD empirical risk are all of the fast order $\mathcal{O}(1/n)$, the strong PD population risk will be of the fast order $\mathcal{O}(1/n)$. To the best of our knowledge, this is the first high probability strong PD population risk bound. The expected version of this risk is studied for the ESP solution in (Zhang et al., 2021a). However, the discussion there does not establish the connection between stability and generalization. Their analysis is restricted to the specific ESP problem. Under the same assumptions as ours, they provide the upper bound of order $\mathcal{O}\left(1/n\right)$. Compared with their result, our result is derived with high probability. Additionally, our strong PD population risk bound is applicable for any stable minimax optimization algorithms. Part (d) provides the connection between the argument stability and the strong PD generalization error. Similarly, if both the argument stability of algorithm A and the strong PD empirical risk are all of the fast order $\mathcal{O}(1/n)$, the strong PD generalization error will be of the fast order $\mathcal{O}(1/n)$. To our best knowledge, this is also the first high probability strong PD generalization error bound. The expected version of this generalization error is studied in (Lei et al., 2021), that is $\mathbb{E}_{S,A}\left[\triangle^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\right]$. Under the same assumptions, their expected strong PD generalization error is bounded by $(1+\beta/\mu)L\sqrt{2}\epsilon$, which can also be used to obtain $\mathcal{O}\left(1/n\right)$ order rate when ϵ is of order $\mathcal{O}(1/n)$. However, this bound is provided for the expected error, while our bound is presented with high probability. Part (e) provides the relationship between the argument stability and the excess primal population risk. If both the argument stability of algorithm A and the strong PD empirical risk are all of the fast order $\mathcal{O}(1/n)$, the excess primal population risk will be of the fast order $\mathcal{O}(1/n)$. High probability excess primal population risk bound is also studied for SGDA in (Lei et al., 2021). Their bound, however, is of slow order $\mathcal{O}\big((\beta/\mu)n^{\frac{1}{2}}\log n \log^2(1/\delta)\big)$ and is restricted to the SGDA algorithm. By comparison, our result in Part (e) enables $\mathcal{O}(1/n)$ bounds for any stable minimax learning algorithms.

Remark 2. As can see from Remark 1, compared with (Zhang et al., 2021a; Farnia & Ozdaglar, 2021; Lei et al., 2021), we have established sharper and high probability generalization bounds under the same assumptions. Once both the stability bound and the strong PD empirical risk are established, we can obtain sharper bounds for minimax learning algorithms. Furthermore, we provide the connection between the argument stability and the strong PD empirical risk in Lemma 7 in Appendix C, which implies that we will only need to prove the strong PD empirical risk bound. Also, according to Definition 1 and Jensen's inequality, we know that $\triangle^w_S(\mathbf{x},\mathbf{y}) \leq \mathbb{E}[\triangle^s_S(\mathbf{x},\mathbf{y})]$. Meanwhile, we provide the strong PD empirical risk bounds for several important optimization algorithms in Section 5. Thus, it implies that we also establish bounds of fast order for $\triangle^w_S(\mathbf{x},\mathbf{y})$.

Remark 3. Different measures quantify different degrees of the generalization error. Thus, deriving different generalization measures requires different assumptions (Lei et al., 2021; Zhang et al., 2021a; Farnia & Ozdaglar, 2021). Strong measures require stronger assumptions compared with the weak (Lei et al., 2021). For instance the term $\sup_{\mathbf{y}\in\mathcal{Y}} F(A_{\mathbf{x}}(S),\mathbf{y})$ in $\triangle^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$, one has to consider the fact that for different $A_{\mathbf{x}}(S)$, $\mathbf{y}$ is different, which makes the proof more challenging. While in $\triangle^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$ and $\triangle^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \triangle^w_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$, both the supremum over $A_{\mathbf{x}}(S)$ and $A_{\mathbf{y}}(S)$ are outside the expectation operator, thus one does not need to consider the coupling between $A_{\mathbf{x}}(S)$ and $\mathbf{y}$. The upper bounds shown in Corollary 1 directly derived from Theorem 1 are sub-optimal since $\triangle^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$ and $\triangle^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \triangle^w_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$ are pretty weak generalization measures (Lei et al., 2021). We list Corollary 1 here to suggest that, when Theorem 1 is established, the fast order $\mathcal{O}(1/n)$ is easy to be achieved for $\triangle^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$ and $\triangle^w(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \triangle^w_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$. On the other hand, the two weak generalization measures in (Zhang et al., 2021a) is studied for the specific ESP solution, while Corollary 1 is applicable for any stable minimax learning algorithms, it thus may be useful in some applications.

Remark 4. To prove sharper high probability bounds than (Lei et al., 2021), the concentration inequality for a summation of weakly-dependent random variables proposed in (Bousquet et al., 2020) (Lemma 2 in Appendix A) plays a key role in our analysis. However, the direct use of this inequality will inevitably lead to a slow order bound since it contains a sampling error of slow order $\mathcal{O}(1/\sqrt{n})$. We exploit the idea in (Klochkov & Zhivotovskiy, 2021) to make new constructions of $g_i(S)$ so that the parameter M in Lemma 2 is 0. To make this idea work in the minimax learning framework, we must proceed with novel decompositions for the generalization measures. Note that different decompositions are required for different generalization measures. A pretty technical

decomposition is exploited in the proof of Part (c). Additionally, the analysis of excess primal population risk in Part (e) is different from the excess risk analysis in (Klochkov & Zhivotovskiy, 2021). The reason is that the supremum operator in $\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x})$ makes the Bernstein condition used in (Klochkov & Zhivotovskiy, 2021) not applicable for excess primal population risk. For more proof details of Theorem 1, please refer to Appendix A.

5 Applications

We now apply Theorem 1 to the ESP solution and several gradient-based optimization algorithms: GDA, SGDA, PPM, EG, and OGDA. Since our goal is to prove bounds with better rates, that is, which decrease fast with $n \mapsto \infty$, thus for brevity, we defer the explicit results of all theorems to Appendix. Considering the length limit, we also postpone the stability bounds and strong PD empirical risk bounds to Appendix. Here, we only show the main theorems.

5.1 Empirical Saddle Point

Empirical saddle point (ESP) problem refers to problem (2), which is also known as sample average approximation (SAA) (Zhang et al., 2021a). We denote $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$ as the ESP solution to (2), which is analogy to the ERM in stochastic optimization (Shalev-Shwartz et al., 2010). Theorem 2 provides sharper generalization bounds of order $\mathcal{O}(1/n)$ with high probability for the ESP solution $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$.

Theorem 2. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Suppose $|f(\mathbf{x}, \mathbf{y}; z)| \leq M$ for some $M > 0$ and $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}, z \in \mathcal{Z}$. Denote $A_{\mathbf{x}}(S) = \hat{\mathbf{x}}_S^*$ and $A_{\mathbf{y}}(S) = \hat{\mathbf{y}}_S^*$ for $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$. Fixed any $\eta > 0$.*

(a) If Assumption 1 holds, then for any $\delta > 0$, with probability at least $1-\delta$, the plain generalization error of $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$ is of the order $\mathcal{O}\Big(\frac{\log_2 n}{n}\log(1/\delta)\Big)$.

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, the primal generalization error, strong PD population risk, strong PD generalization error and excess primal population risk of $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$ are all of the order $\mathcal{O}\Big(\frac{\log_2 n}{n}\log(1/\delta)\Big)$.

Remark 5. (Zhang et al., 2021a) also studies the generalization bound of the ESP solution. They provide $\mathcal{O}(1/n)$ order bounds for weak PD population risk and expected strong PD population risk. Their proofs also show that the expected strong PD population risk is more difficult to analyze than the former. They have to consider the fact that different $\hat{\mathbf{x}}_S^*$ corresponds to different $\mathbf{y}$, as discussed in Remark 3. Moreover, the expectation operator in expected strong PD population risk also relaxes the difficulty of proof. Specifically, define $S^{(i)}$ be a dataset by replacing the i-th sample in S with another i.i.d. sample z_i' and $\mathbf{y}^*(\mathbf{x}) = \arg\max_{\mathbf{y}\in\mathcal{Y}} F(\mathbf{x}, \mathbf{y})$, there holds the following important property $\mathbb{E}\left[\sup_{\mathbf{y}\in\mathcal{Y}} F(\hat{\mathbf{x}}_S^*, \mathbf{y})\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[F(\hat{\mathbf{x}}_{S^{(i)}}^*, \mathbf{y}^*(\hat{\mathbf{x}}_{S^{(i)}}^*))\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[f(\hat{\mathbf{x}}_{S^{(i)}}^*, \mathbf{y}^*(\hat{\mathbf{x}}_{S^{(i)}}^*); z_i)\right]$ because $(\hat{\mathbf{x}}_S^*, \mathbf{y}^*(\hat{\mathbf{x}}_S^*))$ and $(\hat{\mathbf{x}}_{S^{(i)}}^*, \mathbf{y}^*(\hat{\mathbf{x}}_{S^{(i)}}^*))$ are identically distributed and the independence between z_i and $S^{(i)}$. On the contrary, when there is no expectation operator, we do not have this property and the proof is much more challenging. Please refer to Appendix A for the proof of strong PD population risk.

5.2 Gradient Descent Ascent

We need some notations to state results on GDA. Specifically, assume the initial point satisfies $\mathbf{x}_1 = 0$ and $\mathbf{y}_1 = 0$. Let $\{\eta_t\}$ be a sequence of positive step sizes. At the t-th iteration, GDA updates

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{x}_t - \eta_t \nabla_{\mathbf{x}} F_S(\mathbf{x}_t, \mathbf{y}_t), \\ \mathbf{y}_{t+1} &= \mathbf{y}_t + \eta_t \nabla_{\mathbf{y}} F_S(\mathbf{x}_t, \mathbf{y}_t). \end{aligned} \tag{3}$$

We denote the average of iterates by

$$\bar{\mathbf{x}}_T = \frac{\sum_{t=1}^T \eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t} \quad \text{and} \quad \bar{\mathbf{y}}_T = \frac{\sum_{t=1}^T \eta_t \mathbf{y}_t}{\sum_{t=1}^T \eta_t}. \tag{4}$$

Theorem 3 provides sharper bounds of order $\mathcal{O}(1/n)$ with high probability for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of GDA.

Reference	Algorithm	Assumption	Generalization Measure	Learning Rate
Zhang	ESP	SC-SC, Lip	Weak PD Risk	$\mathcal{O}(1/n)$
		SC-SC, Lip, S	(E.) Strong PD Risk	$\mathcal{O}(1/n)$
	R-ESP	C-C, Lip	Weak PD Risk	$\mathcal{O}(1/\sqrt{n})$
Farnia	SGDA	SC-SC, Lip, S	(E.) Primal generalization	$\mathcal{O}(1/n)$
	SGDmax	SC-SC, Lip, S	(E.) Primal generalization	$\mathcal{O}(1/n)$
	GDA	SC-SC, Lip, S	(E.) Primal generalization	$\mathcal{O}(1/n)$
	GDmax	SC-SC, Lip, S	(E.) Primal generalization	$\mathcal{O}(1/n)$
	PPM	SC-SC, Lip, S	(E.) Primal generalization	$\mathcal{O}(1/n)$
	PPM	C-C, Lip, S	(E.) Primal generalization	$\mathcal{O}(1/\sqrt{n})$
	SGDA	Lip, S	(E.) Primal generalization	$\mathcal{O}\big(T^{\frac{\beta c}{\beta c+1}}/n\big)$
	SGDmax	NC-SC, Lip, S	(E.) Primal generalization	$\mathcal{O}\big(T^{\frac{(k+1)\beta c}{(k+10L\beta+1}}/n\big)$
Lei	SGDA	C-C, Lip	Weak PD Risk	$\mathcal{O}(1/\sqrt{n})$
		C-C, Lip, S	Weak PD Risk	$\mathcal{O}(1/\sqrt{n})$
		SC-SC, Lip	Weak PD Risk	$\mathcal{O}(\sqrt{\log n}/n)$
		SC-SC, Lip, S	Weak PD Risk	$\mathcal{O}(\log n/n)$
		C-SC, Lip, S	(E.) Excess Primal Risk	$\mathcal{O}(1/\sqrt{n})$
		C-SC, Lip, S	(H.P.) Excess Primal Risk	$\mathcal{O}(\log n/\sqrt{n})$
		C-C, Lip	(H.P.) Plain Generalization	$\mathcal{O}(\log n/\sqrt{n})$
		WC-WC, Lip	Weak PD Generalization	$\mathcal{O}\big(T^{\frac{2c\mu}{2c\mu+3}}/n^{\frac{2c\mu+1}{2c\mu+3}}\big)$
		V-WC-WC, Lip, S	Weak PD Generalization	$\mathcal{O}\big(1/\sqrt{n}\big)$
	AGDA	NC-SC, PL, Lip, S	(E.) Excess Primal Risk	$\mathcal{O}\big(n^{-\frac{c\beta+1}{2c\beta+1}}\big)$
Ours	ESP	SC-SC, Lip	Plain Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Primal Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Risk	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Excess Primal Risk	$\mathcal{O}(\log n/n)$
	GDA	SC-SC, Lip	Plain Generalization	$\mathcal{O}((\log n)^{3/2}/n)$
		SC-SC, Lip, S	Primal Generalization	$\mathcal{O}((\log n)^{3/2}/n)$
		SC-SC, Lip, S	Strong PD Risk	$\mathcal{O}((\log n)^{3/2}/n)$
		SC-SC, Lip, S	Strong PD Generalization	$\mathcal{O}((\log n)^{3/2}/n)$
		SC-SC, Lip, S	Excess Primal Risk	$\mathcal{O}((\log n)^{3/2}/n)$
	SGDA	SC-SC, Lip	Plain Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Primal Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Risk	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Excess Primal Risk	$\mathcal{O}(\log n/n)$
	PPM	SC-SC, Lip, S	Plain Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Primal Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Risk	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Excess Primal Risk	$\mathcal{O}(\log n/n)$
	EG	SC-SC, Lip, S	Plain Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Primal Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Risk	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Excess Primal Risk	$\mathcal{O}(\log n/n)$
	OGDA	SC-SC, Lip, S	Plain Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Primal Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Risk	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Strong PD Generalization	$\mathcal{O}(\log n/n)$
		SC-SC, Lip, S	Excess Primal Risk	$\mathcal{O}(\log n/n)$

Table 1: Summary of Results. Here, "Zhang" means reference (Zhang et al., 2021a), "Farnia" means reference (Farnia & Ozdaglar, 2021), and "Lei" means reference (Lei et al., 2021). The bounds are established by choosing an optimal iterate number T. Auxiliary descriptions of Table 1 is shown in Section 5.4 and Appendix H.

Theorem 3. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Suppose $\sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{y}\| \leq R_Y$. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be produced by (3) with $\eta_t = 1/(\mu(t+t_0))$. Assume $t_0 \geq 0$. Denote $A_{\mathbf{x}}(S) = \bar{\mathbf{x}}_T$ and $A_{\mathbf{y}}(S) = \bar{\mathbf{y}}_T$ for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (4). Fixed any $\eta > 0$.*

(a) If Assumption 1 holds, then for any $\delta > 0$, with probability at least $1-\delta$, the plain generalization error of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of GDA is of the order $\mathcal{O}\Big(\Big(\frac{1}{n} + \sqrt{\frac{\log T}{T}}\Big)\log_2 n \log(1/\delta)\Big)$. We can take $\mathcal{O}(n^2)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\Big(\frac{\log^{3/2} n}{n}\log(1/\delta)\Big)$.

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, the primal generalization error, strong PD population risk, strong PD generalization error and excess primal population risk of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of GDA are all of the order $\mathcal{O}\Big(\Big(\frac{1}{n} + \sqrt{\frac{\log T}{T}}\Big)\log_2 n \log(1/\delta)\Big)$. We can take $\mathcal{O}(n^2)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\Big(\frac{\log^{3/2} n}{n}\log(1/\delta)\Big)$.

Remark 6. In studying the stability bound of gradient-based optimization algorithms, a popular approach is to use the property of smoothness to establish the nonexpansiveness of gradient mapping, proposed in the seminal work (Hardt et al., 2016). (Farnia & Ozdaglar, 2021; Lei et al., 2021) extend this approach to the minimax problems and use it to analyze the stability bound of SGDA, GDA, PPM, etc. However, their stability bounds are often derived in expectation. In (Lei et al., 2021), the authors also use the Chernoff bounds of Bernoulli variables to establish high probability stability bounds when they are to derive high probability generalization bounds. Unfortunately, these stability bounds are mostly of slow order $\mathcal{O}(1/\sqrt{n})$. To obtain sharper stability bounds, we need to adopt new proof ideas. We provide the proof of stability bounds in Appendix B and C.

5.3 Other Optimization Algorithms

Considering the length limit, we postpone the introductions and theorems of SGDA, PPM, EG, and OGDA to Appendix. We provide the results of these optimization algorithms in Table 1. From Table 1, one can see that we have established sharper high probability bounds of order $\mathcal{O}(1/n)$ for the generalization measures of SGDA, PPM, EG, and OGDA.

5.4 Auxiliary Descriptions of Table 1

In Table 1, we provide almost all existing generalization bounds in minimax learning, including bounds in (Zhang et al., 2021a; Farnia & Ozdaglar, 2021; Lei et al., 2021) and the results this paper obtained. In Table 1, (E.) denotes that the bound is derived in expectation, while (H.P.) denotes high probability. Since our results are all established with high probability, we thus omit (H.P.) for brevity. Lip means Lipschitz continuity and S means smoothness. The descriptions of other notations are provided in Appendix H. From Table 1, one can see that (Zhang et al., 2021a) and (Farnia & Ozdaglar, 2021) focus on the expected generalization measures. Compared with (Zhang et al., 2021a), we have derived high probability bounds of strong PD population risk under the same assumptions. The results on GDA, SGDA, and PPM of (Farnia & Ozdaglar, 2021) are also improved to high probability guarantees under the same assumptions. Additionally, we establish high probability guarantees for other generalization measures. (Lei et al., 2021) focus on the SGDA algorithm and mainly provide guarantees for weak generalization measures (weak PD risk and weak PD generalization). Although they provide two high probability bounds, however, in slow order. Note that except for classical GDA, SGDA, and PPM, we also provide sharper bounds for EG and OGDA optimization methods in that their widespread use in training GANs (Mokhtari et al., 2019; Daskalakis et al., 2017; Liang & Stokes, 2019).

6 Conclusion

In this paper, we provide a comprehensive analysis of sharper generalization bounds for minimax problems. We first establish sharper high probability bounds for almost all existing generalization measures via algorithmic stability and then apply these bounds to several important applications. We believe that our research can provide in-depth insights into minimax learning problems. For future work, it would be important to relax the assumptions in this paper. Also, it would be interesting to investigate how well other theoretical tools perform on the generalization of minimax problems.

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A Proof of Theorem 1

We now provide proofs of Theorem 1. For better readability, we redisplay Theorem 1 in this appendix, as shown below.

Theorem 4. *Let A be a learning algorithm and $\epsilon > 0$. Suppose $|f(\mathbf{x}, \mathbf{y}; z)| \leq M$ for some $M > 0$ and $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}, z \in \mathcal{Z}$. Fixed any $\eta > 0$. There exists an absolute positive constant C.*

(a.) If the algorithm A is ϵ-uniformly stable, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq (1+\eta) F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + \epsilon \log_2 n \log\frac{1}{\delta}\Big).$$

(b.) Assume that for all $\mathbf{x}$, the function $\mathbf{y} \mapsto F(\mathbf{x}, \mathbf{y})$ is μ-strongly-concave. If the algorithm A is ϵ-argument stable and Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$R(A_{\mathbf{x}}(S)) \;\leq\; (1+\eta) R_S(A_{\mathbf{x}}(S)) + C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big) L\epsilon \log_2 n \log\frac{1}{\delta}\Big).$$

(c.) Assume that for all $\mathbf{x}$ and $\mathbf{y}$, the function $F(\mathbf{x}, \mathbf{y})$ is μ-SC-SC. If the algorithm A is ϵ-argument stable and Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$\begin{aligned}\triangle^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) \leq \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \eta \mathbb{E}_S \triangle^s_S\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) \\ + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)\epsilon L \log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).\end{aligned}$$

(d.) Assume that for all $\mathbf{x}$ and $\mathbf{y}$, the function $F(\mathbf{x}, \mathbf{y})$ is μ-SC-SC. If the algorithm A is ϵ-argument stable and Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$\begin{aligned}\triangle^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) - \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq \eta \mathbb{E}_S \triangle^s_S\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) \\ + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)\epsilon L \log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).\end{aligned}$$

(e.) Assume that for all $\mathbf{x}$, the function $\mathbf{y} \mapsto F(\mathbf{x}, \mathbf{y})$ is μ-strongly-concave. If the algorithm A is ϵ-argument stable and Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$,

$$\begin{aligned}R(A_{\mathbf{x}}(S)) \leq (1+\eta) \inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) \\ + C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big) L\epsilon \log_2 n \log\frac{1}{\delta} + \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\Big).\end{aligned}$$

The proof is partly inspired by (Klochkov & Zhivotovskiy, 2021). However, due to the minimax structure, the proof needs novel decompositions and refined analyses as discussed in Remark 4. To begin, we first introduce some key lemmas on concentration inequalities. The first lemma translates a moment bound into a high probability bound.

Lemma 1. *(Bousquet et al., 2020) Let Z be a random variable with*

$$\|Z\|_p \leq \sqrt{p}a + pb$$

for some $a, b > 0$ and for any $p \geq 2$. Then for any $\delta \in (0, 1)$ we have, with probability at least $1 - \delta$,

$$|Z| \leq e\Big(a\sqrt{\log\Big(\frac{e}{\delta}\Big)} + b\log\Big(\frac{e}{\delta}\Big)\Big),$$

where e is the base of the natural logarithm.

The second lemma establishes a concentration inequality for a summation of weakly-dependent random variables.

Lemma 2. *(Bousquet et al., 2020) Let $S = \{z_1, ..., z_n\}$ be a set of independent random variables each taking values in $\mathcal{Z}$ and $M > 0$. Denote $[n]$ as the set $\{1, ..., n\}$. Define $S \backslash \{z_i\}$ be set $\{z_1, ..., z_{i-1}, z_{i+1}, ..., z_n\}$. Let $g_1, ..., g_n$ be some functions $g_i : \mathcal{Z}^n \mapsto \mathbb{R}$ such that the following inequalities hold for any $i \in [n]$,*

- $\left|\mathbb{E}_{S\backslash\{z_i\}}[g_i(S)]\right| \leq M$ *almost surely (a.s.),*
- $\mathbb{E}_{z_i}[g_i(S)] = 0$ *a.s.,*
- *for any $j \in [n]$ with $j \neq i$, and $z_j'' \in \mathcal{Z}$*

$$\left|g_i(S) - g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)\right| \leq \beta.$$

Then, for any $p \geq 2$

$$\Big\|\sum_{i=1}^{n} g_i(S)\Big\|_p \leq 12\sqrt{2}pn\beta\lceil\log_2 n\rceil + 4M\sqrt{pn}.$$

The following definition and lemma give the concentration inequality for non-negative weakly self-bounded functions.

Definition 4. *(Weakly Self-Bounded Function) Assume that $a, b > 0$. A function $f : \mathcal{Z}^n \mapsto [0, +\infty)$ is said to be (a, b)-weakly self-bounded if there exist functions $f_i : \mathcal{Z}^{n-1} \mapsto [0, +\infty)$ that satisfies for all $Z^n \in \mathcal{Z}^n$,*

$$\sum_{i=1}^{n}(f(Z^n) - f_i(Z^n))^2 \leq af(Z^n) + b.$$

Lemma 3. *(Klochkov & Zhivotovskiy, 2021) Suppose that $z_1, ..., z_n$ are independent random variables and the function $f : \mathcal{Z}^n \mapsto [0, +\infty)$ is (a, b)-weakly self-bounded and the corresponding function f_i satisfy $f_i(Z^n) \geq f(Z^n)$ for $i = 1, ..., n$ and any $Z^n \in \mathcal{Z}^n$. Then, for any $t > 0$,*

$$Pr(\mathbb{E}f(z_1, ..., z_n) \geq f(z_1, ..., z_n) + t) \leq \exp\Big(-\frac{t^2}{2a\mathbb{E}f(z_1, ..., z_n) + 2b}\Big).$$

The following lemma is the classical Bernstein concentration inequality.

Lemma 4. *(Boucheron et al., 2013) Let $z_1, ..., z_n$ be i.i.d. random variables and assume that $\mathbb{E}[z_i] = \mu$. Suppose $|z_i| \leq c$ for any i. Then for any $\delta \in (0, 1)$, with probability at least $1 - \delta$,*

$$\Big|\frac{1}{n}\sum_{i=1}^{n} z_i - \mu\Big| \leq \sqrt{\frac{2\sigma^2\log(1/\delta)}{n}} + \frac{2c\log(1/\delta)}{3n},$$

where σ^2 is the variance of z_i.

A.1 Proof of Part (a)

We first prove the plain generalization error bound.

Proof. Let $S = \{z_1, ..., z_n\}$ be a set of independent random variables each taking values in $\mathcal{Z}$ and $S' = \{z_1', ..., z_n'\}$ be its independent copy. For any $i \in [n]$, define $S^{(i)} = \{z_1, ..., z_{i-1}, z_i', z_{i+1}, ..., z_n\}$ be a dataset by replacing the i-th sample in S with another i.i.d. sam-

ple z_i'. We first have the following decomposition

$$\begin{aligned}
&nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
=&\sum_{i=1}^{n} \mathbb{E}_Z[f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z) - \mathbb{E}_{z_i'}[f(A_{\mathbf{x}}(S^{(i)}), A_{\mathbf{y}}(S^{(i)}); Z)]] \\
+&\sum_{i=1}^{n} \mathbb{E}_{z_i'}[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), A_{\mathbf{y}}(S^{(i)}); Z)] - f(A_{\mathbf{x}}(S^{(i)}), A_{\mathbf{y}}(S^{(i)}); z_i)] \\
+&\sum_{i=1}^{n} \mathbb{E}_{z_i'}[f(A_{\mathbf{x}}(S^{(i)}), A_{\mathbf{y}}(S^{(i)}); z_i)] - \sum_{i=1}^{n} f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); z_i).
\end{aligned}$$

According to the definition of uniform stability (Part 1 of Definition 3), we have

$$nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq 2n\epsilon + \sum_{i=1}^{n} g_i(S),$$

where we have introduced $g_i(S) = \mathbb{E}_{z_i'}[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), A_{\mathbf{y}}(S^{(i)}); Z)] - f(A_{\mathbf{x}}(S^{(i)}), A_{\mathbf{y}}(S^{(i)}); z_i)]$. Thus, by a rearrangement, we have

$$\left| nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \sum_{i=1}^{n} g_i(S) \right| \leq 2n\epsilon. \tag{5}$$

Then, for any $i = 1, ..., n$, we define $h_i(S) = g_i(S) - \mathbb{E}_{S\setminus\{z_i\}}[g_i(S)]$. It is easy to verify that $\mathbb{E}_{S\setminus\{z_i\}}[h_i(S)] = 0$ and $\mathbb{E}_{z_i}[h_i(S)] = \mathbb{E}_{z_i}[g_i(S)] - \mathbb{E}_{z_i}\mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] = 0 - 0 = 0$. Also, for any $j \in [n]$ with $j \neq i$, and $z_j'' \in \mathcal{Z}$, we have the following inequality

$$\begin{aligned}
\left| h_i(S) - h_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n) \right| \leq& \left| g_i(S) - g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n) \right| \\
&+ \left| \mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] - \mathbb{E}_{S\setminus\{z_i\}}[g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)] \right|.
\end{aligned}$$

For the first term $|g_i(S) - g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)|$, it can be bounded by 2ϵ according to the definition of uniform stability. Similar result holds for the second term $\left|\mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] - \mathbb{E}_{S\setminus\{z_i\}}[g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)]\right|$ according to the definition of uniform stability. By a combination of the above analysis, we get $|h_i(S) - h_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)| \leq 4\epsilon$.

We thus have verified that the three conditions in Lemma 2 are satisfied for $h_i(S)$. There will hold the following result for any $p \geq 2$

$$\left\| \sum_{i=1}^{n} h_i(S) \right\|_p \leq 48\sqrt{2}\epsilon p n \lceil \log_2 n \rceil. \tag{6}$$

Furthermore, we can derive that

$$\begin{aligned}
&nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \sum_{i=1}^{n} g_i(S) + \sum_{i=1}^{n} h_i(S) \\
=&nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \sum_{i=1}^{n} \mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] \\
=&nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - n\mathbb{E}_{S'}F(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S')) \\
&+ n\mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'))
\end{aligned}$$

Due to the i.i.d. property between S and S', we know that $\mathbb{E}_{S'}F(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S')) = \mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$.

Thus, combined (5) with (6), we have

$$\begin{aligned}
&\|nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - n\mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + n\mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'))\|_p \\
&\leq \Big\|nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \sum_{i=1}^n g_i(S)\Big\|_p \\
&+ \Big\|\sum_{i=1}^n g_i(S) - n\mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + n\mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'))\Big\|_p \\
&= \Big\|nF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - nF_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - \sum_{i=1}^n g_i(S)\Big\|_p + \Big\|\sum_{i=1}^n h_i(S)\Big\|_p \\
&\leq 2n\epsilon + 48\sqrt{2}\epsilon p n\lceil \log_2 n\rceil \\
&\leq 50\sqrt{2}\epsilon p n\lceil \log_2 n\rceil.
\end{aligned}$$

According to Lemma 1, for any $\delta \in (0,1)$, with probability at least $1-\delta/3$, we have

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
\leq &|\mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S')) - \mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))| + 50\sqrt{2}e\epsilon\lceil \log_2 n\rceil \log(3e/\delta). \qquad (7)
\end{aligned}$$

We now begin to bound the term $\mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S')) - \mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$. There holds that $\mathbb{E}_S\mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S')) = \mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$. We first consider the variance of $\mathbb{E}_{S'} f(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'); z_i)$. By the Jensen's inequality, we have

$$\begin{aligned}
\mathbb{E}_{z_i}[(\mathbb{E}_{S'} f(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'); z_i))^2] &\leq \mathbb{E}_{z_i}\mathbb{E}_{S'}[(f(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'); z_i))^2] \\
&= \mathbb{E}_Z\mathbb{E}_{S'}[(f(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S'); Z))^2] \\
&= \mathbb{E}_Z\mathbb{E}_S[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2].
\end{aligned}$$

Then, by the Bernstein inequality in Lemma 4, we obtain the following inequality with probability at least $1-\delta/3$,

$$\begin{aligned}
&|\mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), A_{\mathbf{y}}(S')) - \mathbb{E}_S F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))| \qquad (8) \\
\leq &\sqrt{\frac{2\mathbb{E}_Z\mathbb{E}_S[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2]\log(3/\delta)}{n}} + \frac{2M\log(3/\delta)}{3n}.
\end{aligned}$$

Combined (7) with (8), we finally obtain that with probability at least $1-2\delta/3$,

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
\leq &\sqrt{\frac{2\mathbb{E}_Z\mathbb{E}_S[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2]\log(3/\delta)}{n}} + \frac{2M\log(3/\delta)}{3n} + 50\sqrt{2}e\epsilon\lceil \log_2 n\rceil \log(3e/\delta). \qquad (9)
\end{aligned}$$

In the following, we define $q = q(z_1, ..., z_n) = \mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2]$ and $q_i = q_i(z_1, ..., z_n) = \sup_{z_i\in\mathcal{Z}} q(z_1, ..., z_n)$. So there holds $q_i \geq q$ for any $i = 1, .., n$ and any $\{z_1, ..., z_n\} \in \mathcal{Z}^n$. Also, there holds that

$$\begin{aligned}
&\sum_{i=1}^n (q - q_i)^2 \\
= &\sum_{i=1}^n \Big(\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] - \sup_{z_i\in\mathcal{Z}} \mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2]\Big)^2 \\
\leq &\sum_{i=1}^n \epsilon^2\Big(\mathbb{E}_Z\Big[f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z) + \sup_{z_i\in\mathcal{Z}} f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z)\Big]\Big)^2 \\
\leq &n\epsilon^2\left(2\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))] + \epsilon\right)^2 \\
\leq &8n\epsilon^2 q + 2n\epsilon^4, \qquad (10)
\end{aligned}$$

where the first inequality follows from the Jensen's inequality and the definition of uniform stability, and where the second inequality also follows from the definition of uniform stability.

From (10), we know that q is $(8n\epsilon^2, 2n\epsilon^4)$ weakly self-bounded. Thus, by Lemma 3, we obtain that with probability at least $1-\delta/3$,

$$\begin{aligned}
&\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] - \mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] \\
\leq &\sqrt{(16n\epsilon^2\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] + 4n\epsilon^4)\log(3/\delta)} \\
= &\sqrt{\Big(\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] + \frac{1}{4}\epsilon^2\Big)16n\epsilon^2\log(3/\delta)} \\
\leq &\frac{1}{2}\Big(\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] + \frac{1}{4}\epsilon^2\Big) + 8n\epsilon^2\log(3/\delta),
\end{aligned}$$

where the last inequality follows from that $\sqrt{ab} \leq \frac{a+b}{2}$ for all $a, b > 0$.

Since $\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] \leq MF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$, we have

$$\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S); Z))^2] - 2MF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \leq \frac{1}{4}\epsilon^2 + 16n\epsilon^2\log(3/\delta). \tag{11}$$

Substituting (11) into (9), we finally obtain that with probability at least $1-\delta$,

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
\leq &\sqrt{\frac{(4MF(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \frac{1}{2}\epsilon^2 + 32n\epsilon^2\log(3/\delta))\log(3/\delta)}{n}} \\
&+ \frac{2M\log(3/\delta)}{3n} + 50\sqrt{2}e\epsilon\lceil\log_2 n\rceil\log(3e/\delta).
\end{aligned} \tag{12}$$

According to inequalities $\sqrt{ab} \leq \eta a + \frac{1}{\eta}b$ and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b, \eta > 0$, we have the following inequality with probability at least $1-\delta$

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
\leq &\sqrt{\frac{(\frac{1}{2}\epsilon^2 + 32n\epsilon^2\log(3/\delta))\log(3/\delta)}{n}} + \frac{\eta}{1+\eta}F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \frac{1+\eta}{\eta}\frac{4M\log(3/\delta)}{n} \\
&+ \frac{2M\log(3/\delta)}{3n} + 50\sqrt{2}e\epsilon\lceil\log_2 n\rceil\log(3e/\delta),
\end{aligned}$$

which implies that

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) - (1+\eta)F_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
\leq &(1+\eta)\Big(\sqrt{\frac{(\frac{1}{2}\epsilon^2 + 32n\epsilon^2\log(3/\delta))\log(3/\delta)}{n}} + \frac{1+\eta}{\eta}\frac{4M\log(3/\delta)}{n} \\
&+ \frac{2M\log(3/\delta)}{3n} + 50\sqrt{2}e\epsilon\lceil\log_2 n\rceil\log(3e/\delta)\Big) \\
\leq &C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + \epsilon\log_2 n\log(1/\delta)\Big),
\end{aligned}$$

where C is an absolute constant. The proof is complete. □

A.2 Proof of Part (b)

We then prove the primal generalization error bound. Before presenting the proof, we first introduce a lemma that quantifies the sensitivity of the optimal $\mathbf{y}$ and $\mathbf{x}$ w.r.t the perturbation of $\mathbf{x}$ and $\mathbf{y}$ respectively.

Lemma 5. *(Zhang et al., 2021a) Let $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Assume that f is μ-strongly-convex-strongly-concave. Suppose that for any $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ and $\mathbf{y}, \mathbf{y}' \in \mathcal{Y}$ we have*

$$\|\nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{y}} f(\mathbf{x}', \mathbf{y})\| \leq \beta\|\mathbf{x} - \mathbf{x}'\| \quad \textit{and} \quad \|\nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y}')\| \leq \beta\|\mathbf{y} - \mathbf{y}'\|.$$

Define $\mathbf{x}^*(\mathbf{y}) = \arg\min_{\mathbf{x}\in\mathcal{X}} h(\mathbf{x},\mathbf{y})$ *and* $\mathbf{y}^*(\mathbf{x}) = \arg\max_{\mathbf{y}\in\mathcal{Y}} h(\mathbf{x},\mathbf{y})$ *for any* $\mathbf{y}$ *and* $\mathbf{x}$ *respectively. Then, for any* $\mathbf{x},\mathbf{x}' \in \mathcal{X}$ *and* $\mathbf{y},\mathbf{y}' \in \mathcal{Y}$ *there holds that*

$$\|\mathbf{y}^*(\mathbf{x}) - \mathbf{y}^*(\mathbf{x}')\| \leq \frac{\beta}{\mu}\|\mathbf{x}-\mathbf{x}'\| \quad \textit{and} \quad \|\mathbf{x}^*(\mathbf{y}) - \mathbf{x}^*(\mathbf{y}')\| \leq \frac{\beta}{\mu}\|\mathbf{y}-\mathbf{y}'\|.$$

The proof of Part (b) shares similar proof techniques with Part (a), but requires a novel decomposition and several important changes. For instance, Lemma 5 must be needed to quantify the fact that for different $A_{\mathbf{x}}(S)$, the optimal $\mathbf{y}$ is different in $R(A_{\mathbf{x}}(S))$.

Proof. Let $S = \{z_1, ..., z_n\}$ be a set of independent random variables each taking values in $\mathcal{Z}$ and $S' = \{z_1', ..., z_n'\}$ be its independent copy. For any $i \in [n]$, define $S^{(i)} = \{z_1, ..., z_{i-1}, z_i', z_{i+1}, ..., z_n\}$ be a dataset by replacing the i-th sample in S with another i.i.d. sample z_i'. Denote $\mathbf{y}_S^* = \arg\max_{\mathbf{y}\in\mathcal{Y}} F(A_{\mathbf{x}}(S),\mathbf{y})$ and $\hat{\mathbf{y}}_S^* = \arg\max_{\mathbf{y}\in\mathcal{Y}} F_S(A_{\mathbf{x}}(S),\mathbf{y})$. We have the following decomposition

$$\begin{aligned}
&nR(A_{\mathbf{x}}(S)) - nR_S(A_{\mathbf{x}}(S)) \\
=&nF(A_{\mathbf{x}}(S),\mathbf{y}_S^*) - nF_S(A_{\mathbf{x}}(S),\hat{\mathbf{y}}_S^*) \\
=&\sum_{i=1}^n \mathbb{E}_Z[f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z) - \mathbb{E}_{z_i'}[f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;Z)]] \\
&+\sum_{i=1}^n \mathbb{E}_{z_i'}[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;Z)] - f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;z_i)] \\
&+\sum_{i=1}^n \mathbb{E}_{z_i'}[f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;z_i)] - \sum_{i=1}^n f(A_{\mathbf{x}}(S),\hat{\mathbf{y}}_S^*;z_i).
\end{aligned} \tag{13}$$

Firstly, we have

$$\begin{aligned}
&f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z) - f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;Z) \\
=&f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z) - f(A_{\mathbf{x}}(S),\mathbf{y}_{S^{(i)}}^*;Z) + f(A_{\mathbf{x}}(S),\mathbf{y}_{S^{(i)}}^*;Z) - f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;Z) \\
\leq&L\|\mathbf{y}_S^* - \mathbf{y}_{S^{(i)}}^*\| + L\|A_{\mathbf{x}}(S) - A_{\mathbf{x}}(S^{(i)})\| \\
\leq&\Big(1+\frac{\beta}{\mu}\Big)L\|A_{\mathbf{x}}(S) - A_{\mathbf{x}}(S^{(i)})\| \\
\leq&\Big(1+\frac{\beta}{\mu}\Big)L\epsilon,
\end{aligned} \tag{14}$$

where the second inequality follows from Lemma 5 with the fact that F is smooth and μ-SC-SC.

Secondly, we have

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}_{z_i'}[f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;z_i)] \\
=&\sum_{i=1}^n \mathbb{E}_{z_i'}[f(A_{\mathbf{x}}(S^{(i)}),\mathbf{y}_{S^{(i)}}^*;z_i) - f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;z_i) + f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;z_i)] \\
\leq&\sum_{i=1}^n \mathbb{E}_{z_i'}\Big(1+\frac{\beta}{\mu}\Big)L\|A_{\mathbf{x}}(S) - A_{\mathbf{x}}(S^{(i)})\| + \sum_{i=1}^n f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;z_i) \\
\leq&\Big(1+\frac{\beta}{\mu}\Big)Ln\epsilon + \sum_{i=1}^n f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;z_i),
\end{aligned}$$

where the first and the last inequalities follow from (14). Substituting the above two results into (13), we obtain that

$$\begin{aligned}
&nF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - nF_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) \\
\leq& 2\Big(1+\frac{\beta}{\mu}\Big)Ln\epsilon + \sum_{i=1}^n f(A_{\mathbf{x}}(S), \mathbf{y}_S^*; z_i) + \sum_{i=1}^n g_i(S) - \sum_{i=1}^n f(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*; z_i) \\
\leq& 2\Big(1+\frac{\beta}{\mu}\Big)Ln\epsilon + \sum_{i=1}^n g_i(S),
\end{aligned}$$

where the last inequality follows from the facts that $\sum_{i=1}^n f(A_{\mathbf{x}}(S), \mathbf{y}_S^*; z_i) - \sum_{i=1}^n f(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*; z_i) \leq 0$ and that we have introduced $g_i(S) = \mathbb{E}_{z_i'}[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; Z)] - f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z_i)]$.

Now we get

$$nF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - nF_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) - \sum_{i=1}^n g_i(S) \leq 2\Big(1+\frac{\beta}{\mu}\Big)Ln\epsilon. \tag{15}$$

For any $i = 1, ..., n$, we define $h_i(S) = g_i(S) - \mathbb{E}_{S\setminus\{z_i\}}[g_i(S)]$.

We also get that $\mathbb{E}_{S\setminus\{z_i\}}[h_i(S)] = 0$ and $\mathbb{E}_{z_i}[h_i(S)] = \mathbb{E}_{z_i}[g_i(S)] - \mathbb{E}_{z_i}\mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] = 0 - 0 = 0$. Moreover, for any $j \in [n]$ with $j \neq i$, and $z_j'' \in \mathcal{Z}$, we have the following inequality

$$\begin{aligned}
|h_i(S) - h_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)| \leq& |g_i(S) - g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)| \\
&+ |\mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] - \mathbb{E}_{S\setminus\{z_i\}}[g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)]|.
\end{aligned}$$

Denote $S_j^{(i)}$ as the set collected by replacing the j-th element of $S^{(i)}$ with z_j''. For the first term $|g_i(S) - g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)|$, we have

$$\begin{aligned}
&\Big|g_i(S) - g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)\Big| \\
=&\Big|\mathbb{E}_{z_i'}\Big[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; Z)] - f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z_i)\Big] \\
&- \mathbb{E}_{z_i'}\Big[\mathbb{E}_Z[f(A_{\mathbf{x}}(S_j^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; Z)] - f(A_{\mathbf{x}}(S_j^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; z_i)\Big]\Big| \\
\leq&\Big|\mathbb{E}_{z_i'}\Big[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; Z)] - f(A_{\mathbf{x}}(S_j^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; Z)\Big]\Big| \\
&+ \Big|\mathbb{E}_{z_i'}\Big[f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z_i) - f(A_{\mathbf{x}}(S_j^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; z_i)\Big]\Big|.
\end{aligned} \tag{16}$$

Furthermore, for any z, we have the following result which can help to bound the above inequality.

$$\begin{aligned}
&\Big|f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z) - f(A_{\mathbf{x}}(S_j^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; z)\Big| \\
\leq&\Big|f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z) - f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; z)\Big| + \Big|f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; z) - f(A_{\mathbf{x}}(S_j^{(i)}), \mathbf{y}_{S_j^{(i)}}^*; z)\Big| \\
\leq& L\Big\|\mathbf{y}_{S^{(i)}}^* - \mathbf{y}_{S_j^{(i)}}^*\Big\| + L\Big\|A_{\mathbf{x}}(S^{(i)}) - A_{\mathbf{x}}(S_j^{(i)})\Big\| \\
\leq& \Big(\frac{\beta}{\mu}+1\Big)L\Big\|A_{\mathbf{x}}(S^{(i)}) - A_{\mathbf{x}}(S_j^{(i)})\Big\| \\
\leq& \Big(\frac{\beta}{\mu}+1\Big)L\epsilon,
\end{aligned} \tag{17}$$

where the third inequality follows from Lemma 5. Thus, we can bound the first term by $2\Big(\frac{\beta}{\mu}+1\Big)L\epsilon$. By a similar analysis, the second term $|\mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] - \mathbb{E}_{S\setminus\{z_i\}}[g_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)]|$ can also be bounded by $2\Big(\frac{\beta}{\mu}+1\Big)L\epsilon$.

We now have verified that the three conditions in Lemma 2 are satisfied for $h_i(S)$. We obtain that for any $p \geq 2$, there holds

$$\Big\| \sum_{i=1}^{n} h_i(S) \Big\|_p \leq 48\sqrt{2}pn\Big(\frac{\beta}{\mu}+1\Big)L\epsilon\lceil \log_2 n \rceil. \tag{18}$$

Thus, combined (15) with (18), we derive that

$$\begin{aligned}
&\|nF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - nF_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) - n\mathbb{E}_{S'}F(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) + n\mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*)\|_p \\
=&\Big\| nF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - \sum_{i=1}^{n} g_i(S) + \sum_{i=1}^{n} g_i(S) - \sum_{i=1}^{n} \mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] \Big\|_p \\
\leq&\Big\| nF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - nF_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) - \sum_{i=1}^{n} g_i(S) \Big\|_p + \Big\| \sum_{i=1}^{n} h_i(S) \Big\|_p \\
\leq&2\Big(1+\frac{\beta}{\mu}\Big)Ln\epsilon + 48\sqrt{2}pn\Big(\frac{\beta}{\mu}+1\Big)L\epsilon\lceil \log_2 n \rceil \\
\leq&50\sqrt{2}\Big(1+\frac{\beta}{\mu}\Big)\epsilon Lpn\lceil \log_2 n \rceil.
\end{aligned} \tag{19}$$

Then, according to Lemma 1, for any $\delta \in (0,1)$, with probability at least $1-\delta/3$, we have

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) \\
\leq&|\mathbb{E}_{S'}F(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) - \mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*)| + 50\sqrt{2}\epsilon eL\Big(1+\frac{\beta}{\mu}\Big)\lceil \log_2 n \rceil \log(3e/\delta).
\end{aligned} \tag{20}$$

With a similar analysis to the proof of Part (a), we now begin to bound the variance of $\mathbb{E}_{S'}f(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*; z_i)$.

$$\begin{aligned}
\mathbb{E}_{z_i}[(\mathbb{E}_{S'}f(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*; z_i))^2] &\leq \mathbb{E}_{z_i}\mathbb{E}_{S'}[(f(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*; z_i))^2] \\
&= \mathbb{E}_{Z}\mathbb{E}_{S'}[(f(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*; Z))^2] \\
&= \mathbb{E}_{Z}\mathbb{E}_{S}[(f(A_{\mathbf{x}}(S), \mathbf{y}_{S}^*; Z))^2].
\end{aligned}$$

There holds that $\mathbb{E}_S\mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) = \mathbb{E}_{S'}F(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*)$. Then, by the Bernstein inequality in Lemma 4, we obtain that with probability at least $1-\delta/3$,

$$\begin{aligned}
&|\mathbb{E}_{S'}F(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) - \mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*)| \\
\leq&\sqrt{\frac{2\mathbb{E}_Z\mathbb{E}_S[(f(A_{\mathbf{x}}(S), \mathbf{y}_S^*; Z))^2]\log(3/\delta)}{n}} + \frac{2M\log(3/\delta)}{3n}.
\end{aligned} \tag{21}$$

Combined (20) with (21), we finally obtain that with probability at least $1-2\delta/3$,

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) \\
\leq&\sqrt{\frac{2\mathbb{E}_Z\mathbb{E}_S[(f(A_{\mathbf{x}}(S), \mathbf{y}_S^*; Z))^2]\log\left(\frac{3}{\delta}\right)}{n}} + \frac{2M\log\left(\frac{3}{\delta}\right)}{3n} + 50\sqrt{2}\epsilon eL\frac{\beta+\mu}{\mu}\lceil \log_2 n \rceil \log\Big(\frac{3e}{\delta}\Big).
\end{aligned} \tag{22}$$

In the following, we define $q = q(z_1, ..., z_n) = \mathbb{E}_Z[(f(A_{\mathbf{x}}(S), \mathbf{y}_S^*; Z))^2]$ and $q_i = q_i(z_1, ..., z_n) = \sup_{z_i \in \mathcal{Z}} q(z_1, ..., z_n)$. So there holds $q_i \geq q$ for any $i = 1, .., n$ and any $\{z_1, ..., z_n\} \in \mathcal{Z}^n$. Also,

there holds that

$$\begin{aligned}
&\sum_{i=1}^{n}(q-q_i)^2\\
=&\sum_{i=1}^{n}\Big(\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]-\sup_{z_i\in\mathcal{Z}}\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]\Big)^2\\
\leq&\sum_{i=1}^{n}\Big(\mathbb{E}_Z\Big[\sup_{z_i\in\mathcal{Z}}(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2-(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2\Big]\Big)^2\\
\leq&n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\left(2\mathbb{E}_Z[f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z)]+\Big(\frac{\beta}{\mu}+1\Big)L\epsilon\right)^2\\
\leq&8n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2q+2n\Big(\frac{\beta}{\mu}+1\Big)^4L^4\epsilon^4,
\end{aligned}$$

where the first inequality follows from the Jensen's inequality and the second inequality follows from a similar analysis to (14) or (17). Now, we know that q is $\Big(8n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2, 2n\Big(\frac{\beta}{\mu}+1\Big)^4L^4\epsilon^4\Big)$-weakly self-bounded. Thus, by Lemma 3, we obtain the following inequality with probability at least $1-\delta/3$,

$$\begin{aligned}
&\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]-\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]\\
\leq&\sqrt{\Big(16n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]+4n\Big(\frac{\beta}{\mu}+1\Big)^4L^4\epsilon^4\Big)\log(3/\delta)}\\
=&\sqrt{\Big(\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]+\frac{1}{4}\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\Big)16n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\log(3/\delta)}\\
\leq&\frac{1}{2}\Big(\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]+\frac{1}{4}\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\Big)+8n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\log(3/\delta),
\end{aligned}$$

where the last inequality follows from $\sqrt{ab}\leq\frac{a+b}{2}$ for all $a,b>0$.

Since $\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]\leq MF(A_{\mathbf{x}}(S),\mathbf{y}_S^*)$, we have

$$\begin{aligned}
&\mathbb{E}_S\mathbb{E}_Z[(f(A_{\mathbf{x}}(S),\mathbf{y}_S^*;Z))^2]-2MF(A_{\mathbf{x}}(S),\mathbf{y}_S^*)\\
\leq&\frac{1}{4}\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2+16n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\log(3/\delta).
\end{aligned}\tag{23}$$

Plugging (23) into (22), we finally obtain that with probability at least $1-\delta$,

$$\begin{aligned}
&F(A_{\mathbf{x}}(S),\mathbf{y}_S^*)-F_S(A_{\mathbf{x}}(S),\hat{\mathbf{y}}_S^*)\\
\leq&\sqrt{\frac{\Big(4MF(A_{\mathbf{x}}(S),\mathbf{y}_S^*)+\frac{1}{2}\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2+32n\Big(\frac{\beta}{\mu}+1\Big)^2L^2\epsilon^2\log(3/\delta)\Big)\log(3/\delta)}{n}}\\
&+\frac{2M\log(3/\delta)}{3n}+50\sqrt{2}\epsilon eL\frac{\beta+\mu}{\mu}\lceil\log_2 n\rceil\log(3e/\delta).
\end{aligned}\tag{24}$$

By the elementary inequalities $\sqrt{ab}\leq\eta a+\frac{1}{\eta}b$ and $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ for any $a,b,\eta>0$, we have the following inequality with probability at least $1-\delta$

$$F(A_{\mathbf{x}}(S),\mathbf{y}_S^*)-(1+\eta)F_S(A_{\mathbf{x}}(S),\hat{\mathbf{y}}_S^*)\leq C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta}+\Big(\frac{\beta}{\mu}+1\Big)L\epsilon\log_2 n\log\frac{1}{\delta}\Big),$$

where C is an absolute constant. The proof is complete. □

A.3 Proof of Part (C)

We then prove the strong PD population risk bound. To begin, we introduce the following concentration inequality, which is a moment version of the Bernstein inequality.

Lemma 6. *(Boucheron et al., 2013) If $z_1, ..., z_n$ are i.i.d., zero mean and $|z_i| \leq M$ almost surely. Then, for any $p \geq 2$,*

$$\Big\| \sum_{i=1}^n z_i \Big\|_p \leq 6\sqrt{\Big(\sum_{i=1}^n \mathbb{E} z_i^2\Big)p} + 4pM.$$

Proof. Let $S = \{z_1, ..., z_n\}$ be a set of independent random variables each taking values in $\mathcal{Z}$ and $S' = \{z_1', ..., z_n'\}$ be its independent copy. For any $i \in [n]$, define $S^{(i)} = \{z_1, ..., z_{i-1}, z_i', z_{i+1}, ..., z_n\}$ be a dataset by replacing the i-th sample in S with another i.i.d. sample z_i'. Denote $\mathbf{y}_S^* = \arg\max_{\mathbf{y}\in\mathcal{Y}} F(A_{\mathbf{x}}(S), \mathbf{y})$, $\hat{\mathbf{y}}_S^* = \arg\max_{\mathbf{y}\in\mathcal{Y}} F_S(A_{\mathbf{x}}(S), \mathbf{y})$, $\mathbf{x}_S^* = \arg\min_{\mathbf{x}\in\mathcal{X}} F(\mathbf{x}, A_{\mathbf{y}}(S))$ and $\hat{\mathbf{x}}_S^* = \arg\min_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, A_{\mathbf{y}}(S))$.

The proof of Part (c) requires a pretty technical error decomposition, i.e.,

$$\begin{aligned}
&\triangle^s\left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right)\\
=&F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S))\\
=&F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) + \mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) - \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}_S^*)\\
&+ \mathbb{E}_S F(\mathbf{x}_S^*, A_{\mathbf{y}}(S)) - \mathbb{E}_{S'} F_S(\mathbf{x}_{S'}^*, A_{\mathbf{y}}(S')) + F_S(\hat{\mathbf{x}}_S^*, A_{\mathbf{y}}(S)) - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S))\\
&- \mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) + \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) + \mathbb{E}_{S'} F_S(\mathbf{x}_{S'}^*, A_{\mathbf{y}}(S')) - \mathbb{E}_S F(\mathbf{x}_S^*, A_{\mathbf{y}}(S))\\
&+ F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) - F_S(\hat{\mathbf{x}}_S^*, A_{\mathbf{y}}(S)).
\end{aligned}$$

Let's first consider the term $F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) + \mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) - \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}_S^*)$. It can be then decomposed into

$$\begin{aligned}
&F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) + \frac{1}{n}\sum_{i=1}^n \mathbb{E}_{z_i'}\Big[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; Z)] - f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z_i)\Big]\\
&- \frac{1}{n}\sum_{i=1}^n \mathbb{E}_{z_i'}\Big[\mathbb{E}_Z[f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; Z)] - f(A_{\mathbf{x}}(S^{(i)}), \mathbf{y}_{S^{(i)}}^*; z_i)\Big] + \mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*)\\
&- \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}_S^*),
\end{aligned}$$

which can be bounded by the proof techniques in the proof of Part (b). According to (19), we know that

$$\begin{aligned}
&\|F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}_S^*) + \mathbb{E}_{S'} F_S(A_{\mathbf{x}}(S'), \mathbf{y}_{S'}^*) - \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}_S^*)\|_p\\
\leq& 50\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon L p \lceil \log_2 n \rceil. \qquad (25)
\end{aligned}$$

For the second term $\mathbb{E}_S F(\mathbf{x}_S^*, A_{\mathbf{y}}(S)) - \mathbb{E}_{S'} F_S(\mathbf{x}_{S'}^*, A_{\mathbf{y}}(S')) + F_S(\hat{\mathbf{x}}_S^*, A_{\mathbf{y}}(S)) - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S))$, we have the following decomposition

$$\begin{aligned}
&nF_S(\hat{\mathbf{x}}_S^*, A_{\mathbf{y}}(S)) - n\inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S))\\
=&nF_S(\hat{\mathbf{x}}_S^*, A_{\mathbf{y}}(S)) - \sum_{i=1}^n \mathbb{E}_Z\left[f(\mathbf{x}_S^*, A_{\mathbf{y}}(S), Z) - \mathbb{E}_{z_i'}[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); Z)]\right]\\
&+ \sum_{i=1}^n \mathbb{E}_{z_i'}\left[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); z_i) - \mathbb{E}_Z[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); Z)]\right] - \sum_{i=1}^n \mathbb{E}_{z_i'}\left[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); z_i)\right].
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}_{z_i'}\left[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); z_i)\right]\\
=&\sum_{i=1}^n \mathbb{E}_{z_i'}\left[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); z_i) - f(\mathbf{x}_S^*, A_{\mathbf{y}}(S); z_i) + f(\mathbf{x}_S^*, A_{\mathbf{y}}(S); z_i)\right]\\
\geq& nF_S(\hat{\mathbf{x}}_S^*, A_{\mathbf{y}}(S)) + \sum_{i=1}^n \mathbb{E}_{z_i'}\left[f(\mathbf{x}_{S^{(i)}}^*, A_{\mathbf{y}}(S^{(i)}); z_i) - f(\mathbf{x}_S^*, A_{\mathbf{y}}(S); z_i)\right]. \qquad (26)
\end{aligned}$$

Denote $g_i(S) = \mathbb{E}_{z_i'}\left[f(\mathbf{x}^*_{S^{(i)}}, A_{\mathbf{y}}(S^{(i)}); z_i) - \mathbb{E}_Z[f(\mathbf{x}^*_{S^{(i)}}, A_{\mathbf{y}}(S^{(i)}); Z)]\right]$. By (26), we now get

$$\begin{aligned}
&nF_S(\hat{\mathbf{x}}^*_S, A_{\mathbf{y}}(S)) - n\inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \sum_{i=1}^n g_i(S)\\
\leq& -\sum_{i=1}^n \mathbb{E}_Z\left[f(\mathbf{x}^*_S, A_{\mathbf{y}}(S); Z) - \mathbb{E}_{z_i'}[f(\mathbf{x}^*_{S^{(i)}}, A_{\mathbf{y}}(S^{(i)}); Z)]\right]\\
&-\sum_{i=1}^n \mathbb{E}_{z_i'}\left[f(\mathbf{x}^*_{S^{(i)}}, A_{\mathbf{y}}(S^{(i)}); z_i) - f(\mathbf{x}^*_S, A_{\mathbf{y}}(S); z_i)\right].
\end{aligned}$$

Furthermore, according to Lemma 5, we have

$$\left|f(\mathbf{x}^*_S, A_{\mathbf{y}}(S); Z) - f(\mathbf{x}^*_{S^{(i)}}, A_{\mathbf{y}}(S^{(i)}); Z)\right| \leq \Big(1+\frac{\beta}{\mu}\Big)L\|A_{\mathbf{y}}(S) - A_{\mathbf{y}}(S^{(i)})\|.$$

Similarly,

$$\left|f(\mathbf{x}^*_{S^{(i)}}, A_{\mathbf{y}}(S^{(i)}); z_i) - f(\mathbf{x}^*_S, A_{\mathbf{y}}(S); z_i)\right| \leq \Big(1+\frac{\beta}{\mu}\Big)L\|A_{\mathbf{y}}(S) - A_{\mathbf{y}}(S^{(i)})\|.$$

Thus, we obtain

$$\begin{aligned}
&\left|nF_S(\hat{\mathbf{x}}^*_S, A_{\mathbf{y}}(S)) - n\inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \sum_{i=1}^n g_i(S)\right|\\
\leq& \sum_{i=1}^n 2\Big(1+\frac{\beta}{\mu}\Big)L\|A_{\mathbf{y}}(S^{(i)}) - A_{\mathbf{y}}(S)\| \leq 2n\Big(1+\frac{\beta}{\mu}\Big)L\epsilon,
\end{aligned} \tag{27}$$

where the last inequality follows from the definition of argument stability (Part 2 of Definition 3).

Furthermore, we define $h_i(S) = g_i(S) - \mathbb{E}_{S\setminus\{z_i\}}[g_i(S)]$. For $h_i(S)$, We have $\mathbb{E}_{S\setminus\{z_i\}}[h_i(S)] = 0$ and $\mathbb{E}_{z_i}[h_i(S)] = \mathbb{E}_{z_i}[g_i(S)] - \mathbb{E}_{z_i}\mathbb{E}_{S\setminus\{z_i\}}[g_i(S)] = 0 - 0 = 0$. Moreover, for any $j \in [n]$ with $j \neq i$, and $z_j'' \in \mathcal{Z}$, we get

$$|h_i(S) - h_i(z_1, ..., z_{j-1}, z_j'', z_{j+1}, ..., z_n)| \leq 2\Big(1+\frac{\beta}{\mu}\Big)L\epsilon,$$

where this inequality follows from a similar analysis to (16) and (17) of the proof of Part (b). We thus can obtain that for any $p \geq 2$, there holds

$$\Big\|\sum_{i=1}^n h_i(S)\Big\|_p \leq 48\sqrt{2}pn\Big(\frac{\beta}{\mu}+1\Big)L\epsilon\lceil\log_2 n\rceil. \tag{28}$$

Combined (27) with (28), we finally get the bound of the second term:

$$\begin{aligned}
&\Big\|\mathbb{E}_S F(\mathbf{x}^*_S, A_{\mathbf{y}}(S)) - \mathbb{E}_{S'}F_S(\mathbf{x}^*_{S'}, A_{\mathbf{y}}(S')) + F_S(\hat{\mathbf{x}}^*_S, A_{\mathbf{y}}(S)) - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S))\Big\|_p\\
\leq&\Big\|F_S(\hat{\mathbf{x}}^*_S, A_{\mathbf{y}}(S)) - \inf_{\mathbf{x}'\in\mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \frac{1}{n}\sum_{i=1}^n g_i(S)\Big\|_p + \Big\|\frac{1}{n}\sum_{i=1}^n h_i(S)\Big\|_p\\
\leq&50\sqrt{2}\Big(1+\frac{\beta}{\mu}\Big)\epsilon Lp\lceil\log_2 n\rceil.
\end{aligned} \tag{29}$$

We then consider the third term $-\mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) + \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) + \mathbb{E}_{S'}F_S(\mathbf{x}^*_{S'}, A_{\mathbf{y}}(S')) - \mathbb{E}_S F(\mathbf{x}^*_S, A_{\mathbf{y}}(S))$. It is clear that $\mathbb{E}_S[\mathbb{E}_{S'}F_S(\mathbf{x}^*_{S'}, A_{\mathbf{y}}(S'))] = \mathbb{E}_S F(\mathbf{x}^*_S, A_{\mathbf{y}}(S))$ and $\mathbb{E}_S[\mathbb{E}_{S'}F_S(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'})] = \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}^*_S)$.

Moreover, we having the following important property due to the strong convexity and strong concavity of F,

$$\begin{aligned}
&\mathbb{E}\left[(\mathbb{E}_{S'}f(\mathbf{x}^*_{S'}, A_{\mathbf{y}}(S'); z_i) - \mathbb{E}_{S'}f(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}; z_i))^2\right]\\
\leq&\mathbb{E}\mathbb{E}_{S'}\left[L^2(\|\mathbf{x}^*_{S'} - A_{\mathbf{x}}(S')\| + \|A_{\mathbf{y}}(S') - \mathbf{y}^*_{S'}\|)^2\right]\\
\leq&2L^2\mathbb{E}\mathbb{E}_{S'}[\|\mathbf{x}^*_{S'} - A_{\mathbf{x}}(S')\|^2 + \|A_{\mathbf{y}}(S') - \mathbf{y}^*_{S'}\|^2]\\
\leq&4L^2\mu^{-1}\mathbb{E}\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))]\\
=&4L^2\mu^{-1}\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))],
\end{aligned} \tag{30}$$

where the first inequality follows from Jensen's inequality and the Lipschitz continuity of f (Assumption 1), the second inequality follows from that $(a+b)^2 \leq 2a^2 + 2b^2$ and the third inequality follows from the property of strong convexity and strong concavity of F and the optimality condition, derived as follows,

$$\begin{aligned}
&F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S')) \\
=&F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, \mathbf{y}^*_{S'}) + F(\mathbf{x}^*_{S'}, \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S')) \\
\geq&\frac{\mu}{2}\Big[\|A_{\mathbf{x}}(S') - \mathbf{x}^*_{S'}\|^2 + \|\mathbf{y}^*_{S'} - A_{\mathbf{x}}(S')\|\Big].
\end{aligned}$$

It is clear that $\mathbb{E}[(Z - \mathbb{E}Z)^2] \leq \mathbb{E}[Z^2]$. Therefore, by the variance bound in (30) and applying the moment Bernstein inequality in Lemma 6 to the sum of independent random variables $-\mathbb{E}_{S'} f(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}; z_i) + \mathbb{E}_{S'} f(\mathbf{x}^*_{S'}, A_{\mathbf{y}}(S'); z_i) + \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - \mathbb{E}_S F(\mathbf{x}^*_S, A_{\mathbf{y}}(S))$, we have the following inequality for all $p \geq 2$,

$$\begin{aligned}
&\Big\|\frac{1}{n}\sum_{i=1}^{n} -\mathbb{E}_{S'} f(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}; z_i) + \mathbb{E}_{S'} f(\mathbf{x}^*_{S'}, A_{\mathbf{y}}(S'); z_i) + \mathbb{E}_S F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - \mathbb{E}_S F(\mathbf{x}^*_S, A_{\mathbf{y}}(S))\Big\|_p \\
&\leq 6\sqrt{\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))]\frac{4L^2 p}{n\mu}} + \frac{16pM}{n}.
\end{aligned}$$

From Definition 1, we know that the last term $F_S(A_{\mathbf{x}}(S), \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, A_{\mathbf{y}}(S))$ is actually the strong PD empirical risk $\triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$.

Based on the above analysis, we have derived that for each $p \geq 2$,

$$\begin{aligned}
&\Big\|F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\Big\|_p \\
\leq&12\sqrt{\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))]\frac{pL^2}{n\mu}} + \frac{16pM}{n} + 100\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon Lp\lceil\log_2 n\rceil \\
\leq&\frac{\eta}{1+\eta}\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))] + \frac{1+\eta}{\eta}\frac{12pL^2}{n\mu} + \frac{16pM}{n} \\
&+ 100\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon Lp\lceil\log_2 n\rceil,
\end{aligned} \tag{31}$$

where the last inequality holds since for any $a, b, \eta > 0$, $\sqrt{ab} \leq \eta a + \frac{b}{\eta}$.

Taking $p = 2$ and using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}
&\mathbb{E}_S\Big[F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\Big] \\
\leq&\|F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \triangle^s_S(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\|_2 \\
\leq&\frac{\eta}{1+\eta}\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))] + \frac{1+\eta}{\eta}\frac{24L^2}{n\mu} + \frac{32M}{n} \\
&+ 200\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon L\lceil\log_2 n\rceil.
\end{aligned}$$

Since $\mathbb{E}_{S'}[F(A_{\mathbf{x}}(S'), \mathbf{y}^*_{S'}) - F(\mathbf{x}^*_{S'}, A_{\mathbf{x}}(S'))] = \mathbb{E}_S[F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - F(\mathbf{x}^*_S, A_{\mathbf{x}}(S))]$, we finally get

$$\begin{aligned}
&\mathbb{E}_S[F(A_{\mathbf{x}}(S), \mathbf{y}^*_S) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S))] \\
\leq&(1+\eta)\Big(\mathbb{E}_S \triangle^s_S (A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \frac{24L^2(1+\eta)}{n\mu\eta} + \frac{32M}{n} + 200\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon L\lceil\log_2 n\rceil\Big).
\end{aligned}$$

Plugging this inequality into (31), we thus have that for each $p \geq 2$,

$$\begin{aligned}
&\Big\| F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \Big\|_p \\
\leq & \eta \Big(\mathbb{E}_S \triangle_S^s \left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) + \frac{24L^2(1+\eta)}{n\mu\eta} + \frac{32M}{n} + 200\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon L \lceil \log_2 n \rceil \Big) \\
& + \frac{12pL^2(1+\eta)}{n\mu\eta} + \frac{16pM}{n} + 100\sqrt{2}\Big(1 + \frac{\beta}{\mu}\Big)\epsilon L p \lceil \log_2 n \rceil.
\end{aligned}$$

According to Lemma 1, for any $\delta > 0$, with probability at least $1 - \delta$, there holds that

$$\begin{aligned}
& F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) \leq \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + \eta \mathbb{E}_S \triangle_S^s \left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) \\
& + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)\epsilon L \log_2 n \Big) \log\Big(\frac{1}{\delta}\Big), \qquad (32)
\end{aligned}$$

where $C > 0$ is an absolute constant. The proof is complete. □

A.4 Proof of Part (d)

We now prove the strong PD generalization error bound.

Proof. From (32) in the proof of Part (c), we know that for any $\delta > 0$, with probability at least $1 - \delta$, there holds that

$$\begin{aligned}
& \triangle^s \left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) - \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
= & F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) - \inf_{\mathbf{x}' \in \mathcal{X}} F(\mathbf{x}', A_{\mathbf{y}}(S)) - \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) \\
\leq & \eta \mathbb{E}_S \triangle_S^s \left(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)\right) + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)\epsilon L \log_2 n \Big) \log\Big(\frac{1}{\delta}\Big).
\end{aligned}$$

Therefore, the proof is complete. □

A.5 Proof of Part (e)

We finally prove the excess primal population risk bound.

Proof. Denote $\mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathcal{X}} R(\mathbf{x})$ and $\mathbf{y}_S^* = \arg\max_{\mathbf{y} \in \mathcal{Y}} F(A_{\mathbf{x}}(S), \mathbf{y})$. Firstly, we have the following decomposition

$$\begin{aligned}
R(A_{\mathbf{x}}(S)) - \inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x}') = & R(A_{\mathbf{x}}(S)) - R_S(A_{\mathbf{x}}(S)) + R_S(A_{\mathbf{x}}(S)) - F_S(\mathbf{x}^*, A_{\mathbf{y}}(S)) \\
& + F_S(\mathbf{x}^*, A_{\mathbf{y}}(S)) - F(\mathbf{x}^*, A_{\mathbf{y}}(S)) + F(\mathbf{x}^*, A_{\mathbf{y}}(S)) - R(\mathbf{x}^*).
\end{aligned}$$

Consider the first term $R(A_{\mathbf{x}}(S)) - R_S(A_{\mathbf{x}}(S))$. From (24) of the Part (b), we know that with probability at least $1 - \delta$,

$$\begin{aligned}
R(A_{\mathbf{x}}(S)) - R_S(A_{\mathbf{x}}(S)) \leq & \frac{2M \log(3/\delta)}{3n} + 50\sqrt{2}\epsilon e L \frac{\beta + \mu}{\mu} \lceil \log_2 n \rceil \log(3e/\delta) \\
& + \sqrt{\frac{\Big(4MF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) + \frac{1}{2}\Big(\frac{\beta}{\mu} + 1\Big)^2 L^2\epsilon^2 + 32n\Big(\frac{\beta}{\mu} + 1\Big)^2 L^2\epsilon^2 \log(3/\delta)\Big)\log(3/\delta)}{n}}.
\end{aligned}$$

For the second term $R_S(A_{\mathbf{x}}(S)) - F_S(\mathbf{x}^*, A_{\mathbf{y}}(S))$, we have $R_S(A_{\mathbf{x}}(S)) - F_S(\mathbf{x}^*, A_{\mathbf{y}}(S)) \leq R_S(A_{\mathbf{x}}(S)) - \inf_{\mathbf{x}' \in \mathcal{Y}} F_S(\mathbf{x}', A_{\mathbf{y}}(S)) = \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))$.

Note that under Assumption 1, the argument stability implies the uniform stability. Therefore, for the third term $F_S(\mathbf{x}^*, A_{\mathbf{y}}(S)) - F(\mathbf{x}^*, A_{\mathbf{y}}(S))$, from (12) of Part (a), we know that with probability at least $1 - \delta$

$$\begin{aligned}
F_S(\mathbf{x}^*, A_{\mathbf{y}}(S)) - F(\mathbf{x}^*, A_{\mathbf{y}}(S)) \leq & \frac{2M \log(3/\delta)}{3n} + 50\sqrt{2}e\epsilon \lceil \log_2 n \rceil \log(3e/\delta) \\
& + \sqrt{\frac{(4MF(\mathbf{x}^*, A_{\mathbf{y}}(S)) + \frac{1}{2}\epsilon^2 + 32n\epsilon^2 \log(3/\delta))\log(3/\delta)}{n}}.
\end{aligned}$$

It is clear that $F(\mathbf{x}^*, A_{\mathbf{y}}(S)) - R(\mathbf{x}^*) \leq 0$.

Since $F(\mathbf{x}^*, A_{\mathbf{y}}(S)) \leq \sup_{\mathbf{y}' \in \mathcal{Y}} F(\mathbf{x}^*, \mathbf{y}') = R(\mathbf{x}^*) = \inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x})$, based on the above results, we have the following inequality with probability at least $1 - 2\delta$

$$
\begin{aligned}
& R(A_{\mathbf{x}}(S)) - \inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x}) \\
\leq & \sqrt{\frac{\Big(4MF(A_{\mathbf{x}}(S), \mathbf{y}_S^*) + \frac{1}{2}\Big(\frac{\beta}{\mu}+1\Big)^2 L^2\epsilon^2 + 32n\Big(\frac{\beta}{\mu}+1\Big)^2 L^2\epsilon^2 \log(3/\delta)\Big)\log(3/\delta)}{n}} \\
& + \frac{4M\log(3/\delta)}{3n} + \sqrt{\frac{(4M\inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x}) + \frac{1}{2}\epsilon^2 + 32n\epsilon^2\log(3/\delta))\log(3/\delta)}{n}} \\
& + 50\sqrt{2}\epsilon e L\frac{\beta+\mu}{\mu}\lceil \log_2 n \rceil \log(3e/\delta) + \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + 50\sqrt{2}e\epsilon\lceil \log_2 n \rceil \log(3e/\delta) \\
\leq & \sqrt{\frac{\Big(\frac{1}{2}\Big(\frac{\beta}{\mu}+1\Big)^2 L^2\epsilon^2 + 32n\Big(\frac{\beta}{\mu}+1\Big)^2 L^2\epsilon^2 \log(3/\delta)\Big)\log(3/\delta)}{n}} + \frac{\eta}{1+\eta}F(A_{\mathbf{x}}(S), \mathbf{y}_S^*) \\
& + \frac{1+\eta}{\eta}\frac{4M\log(3/\delta)}{n} + \frac{4M\log(3/\delta)}{3n} + \sqrt{\frac{(\frac{1}{2}\epsilon^2 + 32n\epsilon^2\log(3/\delta))\log(3/\delta)}{n}} \\
& + \frac{\eta}{1+\eta}\inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x}) + \frac{1+\eta}{\eta}\frac{4M\log(3/\delta)}{n} \\
& + 50\sqrt{2}\epsilon e L\frac{\beta+\mu}{\mu}\lceil \log_2 n \rceil \log(3e/\delta) + \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S)) + 50\sqrt{2}e\epsilon\lceil \log_2 n \rceil \log(3e/\delta),
\end{aligned}
$$

where the last inequality follows from the elementary inequalities $\sqrt{ab} \leq \eta a + \frac{1}{\eta}b$ and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b > 0$. Therefore, by a rearrangement, we have the following inequality with probability at least $1 - \delta$

$$
\begin{aligned}
& R(A_{\mathbf{x}}(S)) - (1+2\eta)\inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x}) \\
\leq & C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)L\epsilon\log_2 n\log\frac{1}{\delta} + \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\Big),
\end{aligned}
$$

that is

$$
\begin{aligned}
& R(A_{\mathbf{x}}(S)) - (1+\eta)\inf_{\mathbf{x}' \in \mathcal{X}} R(\mathbf{x}) \\
\leq & C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)L\epsilon\log_2 n\log\frac{1}{\delta} + \triangle_S^s(A_{\mathbf{x}}(S), A_{\mathbf{y}}(S))\Big),
\end{aligned}
$$

where C is an absolute constant. The proof is complete. □

Till here, the proof of Theorem 1 is complete.

B Empirical Saddle Point

We first provide the explicit results of Theorem 2, as shown below.

Theorem 5. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Suppose $|f(\mathbf{x}, \mathbf{y}; z)| \leq M$ for some $M > 0$ and $\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}, z \in \mathcal{Z}$. Denote $A_{\mathbf{x}}(S) = \hat{\mathbf{x}}_S^*$ and $A_{\mathbf{y}}(S) = \hat{\mathbf{y}}_S^*$ for $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$. Fixed any $\eta > 0$. There exists an absolute positive constant C.*

(a) If Assumption 1 holds, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$
F(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*) \leq (1+\eta)F_S(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*) + C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + \frac{4L}{n\mu}\log_2 n\log(1/\delta)\Big).
$$

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$R(\hat{\mathbf{x}}_S^*) \leq (1+\eta)R_S(\hat{\mathbf{x}}_S^*) + C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)\frac{4L^2}{n\mu}\log_2 n \log\frac{1}{\delta}\Big).$$

(c) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$\triangle^s(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*) \leq C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)\frac{4L^2}{n\mu}\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(d) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$\begin{aligned}&\triangle^s\left(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*\right) - \triangle_S^s(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)\\ \leq& C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)\frac{4L^2}{n\mu}\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).\end{aligned}$$

(e) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$R(\hat{\mathbf{x}}_S^*) \leq (1+\eta)\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) + C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)\frac{4L^2}{n\mu}\log_2 n \log\frac{1}{\delta}\Big).$$

Proof. To prove Theorem 5, we should derive the strong PD empirical risk bound and the stability bound of $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$. It is easy to verify that $\triangle_S^s(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*) = 0$ (Zhang et al., 2021a). We then investigate the stability bound of $(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*)$.

Let $S = \{z_1, ..., z_n\}$ be a set of independent random variables each taking values in $\mathcal{Z}$. For any $i \in [n]$, define $S^{(i)} = \{z_1, ..., z_{i-1}, z_i', z_{i+1}, ..., z_n\}$ be a dataset by replacing the i-th sample in S with another i.i.d. sample z_i'. We define $F_{S^{(i)}}$ be the empirical risk on dataset $S^{(i)}$ and define $(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_{S^{(i)}}^*)$ be the ESP solution on dataset $S^{(i)}$.

Then we have

$$\begin{aligned}
&F_S(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*) - F_S(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*)\\
=&\frac{1}{n}\sum_{j=1}^{n}\big(f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_j) - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_j)\big)\\
=&\frac{1}{n}\Big(\sum_{j=1, j\neq i}^{n}(f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_j) - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_j)) + f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_i') - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_i')\Big)\\
&+\frac{1}{n}\left(f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_i) - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_i)\right) - \frac{1}{n}\left(f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_i') - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_i')\right)\\
=&F_{S^{(i)}}(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*) - F_{S^{(i)}}(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*)\\
&+\frac{1}{n}\left(f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_i) - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*; z_i) + f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*; z_i) - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_i)\right)\\
&-\frac{1}{n}\left(f(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*; z_i') - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*; z_i') + f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_S^*; z_i') - f(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*; z_i')\right)\\
\leq&F_{S^{(i)}}(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*) - F_{S^{(i)}}(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*) + \frac{2L}{n}(\|\hat{\mathbf{x}}_{S^{(i)}}^* - \hat{\mathbf{x}}_S^*\| + \|\hat{\mathbf{y}}_S^* - \hat{\mathbf{y}}_{S^{(i)}}^*\|)\\
=&F_{S^{(i)}}(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_S^*) - F_{S^{(i)}}(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_{S^{(i)}}^*) + F_{S^{(i)}}(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_{S^{(i)}}^*) - F_{S^{(i)}}(\hat{\mathbf{x}}_S^*, \hat{\mathbf{y}}_{S^{(i)}}^*)\\
&+\frac{2L}{n}(\|\hat{\mathbf{x}}_{S^{(i)}}^* - \hat{\mathbf{x}}_S^*\| + \|\hat{\mathbf{y}}_S^* - \hat{\mathbf{y}}_{S^{(i)}}^*\|)\\
\leq&-\frac{\mu}{2}\|\hat{\mathbf{x}}_{S^{(i)}}^* - \hat{\mathbf{x}}_S^*\|^2 - \frac{\mu}{2}\|\hat{\mathbf{y}}_S^* - \hat{\mathbf{y}}_{S^{(i)}}^*\|^2 + \frac{2L}{n}(\|\hat{\mathbf{x}}_{S^{(i)}}^* - \hat{\mathbf{x}}_S^*\| + \|\hat{\mathbf{y}}_S^* - \hat{\mathbf{y}}_{S^{(i)}}^*\|),
\end{aligned}$$

where the first inequality follows from the Lipschitz continuous assumption, and where the second inequality follows from the facts that the μ-SC-SC property of $F_{S^{(i)}}$ and $(\hat{\mathbf{x}}_{S^{(i)}}^*, \hat{\mathbf{y}}_{S^{(i)}}^*)$ is the ESP solution of $F_{S^{(i)}}$.

Similarly, according to the μ-SC-SC property of F_S, we have

$$\begin{aligned}
&F_S(\hat{\mathbf{x}}^*_{S^{(i)}}, \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, \hat{\mathbf{y}}^*_{S^{(i)}}) \\
=&F_S(\hat{\mathbf{x}}^*_{S^{(i)}}, \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, \hat{\mathbf{y}}^*_S) + F_S(\hat{\mathbf{x}}^*_S, \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, \hat{\mathbf{y}}^*_{S^{(i)}}) \\
\geq&\frac{\mu}{2}\|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\|^2 + \frac{\mu}{2}\|\hat{\mathbf{y}}^*_S - \hat{\mathbf{y}}^*_{S^{(i)}}\|^2
\end{aligned} \tag{33}$$

Based on the above results, we have

$$\begin{aligned}
&\mu\|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\|^2 + \mu\|\hat{\mathbf{y}}^*_S - \hat{\mathbf{y}}^*_{S^{(i)}}\|^2 \\
\leq&\frac{2L}{n}(\|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\| + \|\hat{\mathbf{y}}^*_S - \hat{\mathbf{y}}^*_{S^{(i)}}\|) \\
\leq&\frac{2L}{n}\sqrt{2}\sqrt{\|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\|^2 + \|\hat{\mathbf{y}}^*_S - \hat{\mathbf{y}}^*_{S^{(i)}}\|^2},
\end{aligned}$$

where the last inequality uses the Caucy-Schwartz inequality. Therefore, we have

$$\begin{aligned}
&\|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\| + \|\hat{\mathbf{y}}^*_S - \hat{\mathbf{y}}^*_{S^{(i)}}\| \\
\leq&\sqrt{2(\|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\|^2 + \|\hat{\mathbf{y}}^*_S - \hat{\mathbf{y}}^*_{S^{(i)}}\|^2)} \\
\leq&\frac{4L}{n\mu}.
\end{aligned} \tag{34}$$

Now, plugging this stability bound into Theorem 1, we obtain generalization bounds of the ESP solution. The proof of Theorem 5 is complete. □

C GRADIENT DESCENT ASCENT

We also first provide the explicit results of Theorem 3, as shown below.

Theorem 6. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Suppose $\sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{y}\| \leq R_Y$. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be produced by (3) with $\eta_t = \frac{1}{\mu(t+t_0)}$. Assume $t_0 \geq 0$. Denote $A_{\mathbf{x}}(S) = \bar{\mathbf{x}}_T$ and $A_{\mathbf{y}}(S) = \bar{\mathbf{y}}_T$ for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (4). Let $M = \sup_{z\in\mathcal{Z}} f(0, 0; z) + L(R_X + R_Y)$. Fixed any $\eta > 0$. There exists an absolute positive constant C.*

(a) If Assumption 1 holds, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\begin{aligned}
&F(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1+\eta)F_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \\
&+ C\frac{1+\eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + \Big(\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{\mu t_0(R_X^2 + R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}}\Big)\log_2 n \log(1/\delta)\Big).
\end{aligned}$$

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\begin{aligned}
&R(\bar{\mathbf{x}}_T) \leq (1+\eta)R_S(\bar{\mathbf{x}}_T) + C\frac{1+\eta}{\eta} \\
&\quad\times\Big(\frac{M\log\frac{1}{\delta}}{n} + \frac{\beta+\mu}{\mu}L\Big(\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{\mu t_0(R_X^2 + R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}}\Big)\log_2 n \log\frac{1}{\delta}\Big).
\end{aligned}$$

(c) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\begin{aligned}
&\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1+\eta)\Big(\frac{\mu t_0(R_X^2 + R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}\Big) + C(1+\eta) \\
&\times\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \frac{\beta+\mu}{\mu}\Big(\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{\mu t_0(R_X^2 + R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}}\Big)L\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).
\end{aligned}$$

(d) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) - \triangle^s_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq \eta\Big(\frac{\mu t_0(R_X^2+R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}\Big) + C(1+\eta)$$
$$\times\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \frac{\beta+\mu}{\mu}\Big(\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{\mu t_0(R_X^2+R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}}\Big)L\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(e) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1+\eta)\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) + C\frac{2+\eta}{\eta}\Big(\frac{\mu t_0(R_X^2+R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}\Big)$$
$$+C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)L\Big(\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{\mu t_0(R_X^2+R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}}\Big)\log_2 n\log\frac{1}{\delta}\Big).$$

We now prove Theorem 6. Here, we first provide an important lemma to connect the argument stability with the strong PD empirical risk, which will also be used in the remaining applications.

Lemma 7. *For any $i \in [n]$, define $S^{(i)} = \{z_1, ..., z_{i-1}, z_i', z_{i+1}, ..., z_n\}$. Let $(\mathbf{x}_t, \mathbf{y}_t)$ be the output produced by F_S on dataset S in running a minimax learning algorithm. Let $(\mathbf{x}_t^i, \mathbf{y}_t^i)$ be the corresponding output produced by $F_{S^{(i)}}$ on dataset $S^{(i)}$, where $F_{S^{(i)}}$ is empirical risk on dataset $S^{(i)}$. Suppose Assumption 1 holds. Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. For any $S^{(i)}$ and S, we have*

$$\|\mathbf{x}_t^i - \mathbf{x}_t\| + \|\mathbf{y}_t^i - \mathbf{y}_t\| \leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle^s_S(\mathbf{x}_t, \mathbf{y}_t)}.$$

Proof. Define $(\hat{\mathbf{x}}^*_{S^{(i)}}, \hat{\mathbf{y}}^*_{S^{(i)}})$ be the ESP solution on dataset $S^{(i)}$ and $(\hat{\mathbf{x}}^*_S, \hat{\mathbf{y}}^*_S)$ be the ESP solution on dataset S. To prove the stability bound, we consider

$$\begin{aligned}
&\|\mathbf{x}_t^i - \mathbf{x}_t\| + \|\mathbf{y}_t^i - \mathbf{y}_t\| \\
=&\|\mathbf{x}_t^i - \hat{\mathbf{x}}^*_{S^{(i)}} + \hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S + \hat{\mathbf{x}}^*_S - \mathbf{x}_t\| + \|\mathbf{y}_t^i - \hat{\mathbf{y}}^*_{S^{(i)}} + \hat{\mathbf{y}}^*_{S^{(i)}} - \hat{\mathbf{y}}^*_S + \hat{\mathbf{y}}^*_S - \mathbf{y}_t\| \\
\leq&\|\mathbf{x}_t^i - \hat{\mathbf{x}}^*_{S^{(i)}}\| + \|\hat{\mathbf{x}}^*_{S^{(i)}} - \hat{\mathbf{x}}^*_S\| + \|\hat{\mathbf{x}}^*_S - \mathbf{x}_t\| + \|\mathbf{y}_t^i - \hat{\mathbf{y}}^*_{S^{(i)}}\| + \|\hat{\mathbf{y}}^*_{S^{(i)}} - \hat{\mathbf{y}}^*_S\| + \|\hat{\mathbf{y}}^*_S - \mathbf{y}_t\| \\
\leq&\frac{4L}{n\mu} + \|\mathbf{x}_t^i - \hat{\mathbf{x}}^*_{S^{(i)}}\| + \|\hat{\mathbf{x}}^*_S - \mathbf{x}_t\| + \|\mathbf{y}_t^i - \hat{\mathbf{y}}^*_{S^{(i)}}\| + \|\hat{\mathbf{y}}^*_S - \mathbf{y}_t\| \\
\leq&\frac{4L}{n\mu} + \sqrt{2}\sqrt{\|\mathbf{x}_t^i - \hat{\mathbf{x}}^*_{S^{(i)}}\|^2 + \|\mathbf{y}_t^i - \hat{\mathbf{y}}^*_{S^{(i)}}\|^2} + \sqrt{2}\sqrt{\|\hat{\mathbf{y}}^*_S - \mathbf{y}_t\|^2 + \|\hat{\mathbf{x}}^*_S - \mathbf{x}_t\|^2} \\
\leq&\frac{4L}{n\mu} + \sqrt{\frac{4}{\mu}}\sqrt{F_{S^{(i)}}(\mathbf{x}_t^i, \hat{\mathbf{y}}^*_{S^{(i)}}) - F_{S^{(i)}}(\hat{\mathbf{x}}^*_{S^{(i)}}, \mathbf{y}_t^i)} + \sqrt{\frac{4}{\mu}}\sqrt{F_S(\mathbf{x}_t, \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, \mathbf{y}_t)},
\end{aligned}$$

where the second inequality uses the result in (34), the third inequality uses the Caucy-Schwartz inequality, and the last inequality uses the strong convexity and strong concavity of $F_{S^{(i)}}$ and F_S and the optimality condition (please refer to (33)). As will see in the rest paper, we bound $F_{S^{(i)}}(\mathbf{x}_t^i, \hat{\mathbf{y}}^*_{S^{(i)}}) - F_{S^{(i)}}(\hat{\mathbf{x}}^*_{S^{(i)}}, \hat{\mathbf{y}}^*_{S^{(i)}})$ and $F_S(\mathbf{x}_t, \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, \mathbf{y}_t)$ with the same upper bound since they are all strong PD empirical risk. Thus, for brevity, we derive the following inequality

$$\begin{aligned}
&\|\mathbf{x}_t^i - \mathbf{x}_t\| + \|\mathbf{y}_t^i - \mathbf{y}_t\| \\
\leq&\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{F_S(\mathbf{x}_t, \hat{\mathbf{y}}^*_S) - F_S(\hat{\mathbf{x}}^*_S, \mathbf{y}_t)} \\
\leq&\frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle^s_S(\mathbf{x}_t, \mathbf{y}_t)}.
\end{aligned}$$

The proof is complete. □

Remark 7. Lemma 7 establishes the connection between the stability bound and the strong PD empirical risk. The subscript t here represents not only the output of an iterative optimization algorithm, but any output of the empirical risk of any minimax learning algorithm.

The following lemma shows the strong PD empirical risk of GDA.

Lemma 8. *Suppose Assumption 1 holds and $F_S(\cdot,\cdot)$ be μ-SC-SC with $\mu > 0$. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be the sequence produced by (3) with $\eta_t = \frac{1}{\mu(t+t_0)}$. Assume $t_0 \geq 0$. Suppose $\sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{y}\| \leq R_Y$. Then for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (4) we have*

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{\mu t_0 (R_X^2 + R_Y^2)}{T} + \frac{L^2 \log(eT)}{\mu T}.$$

If $t_0 = 0$, then

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{L^2 \log(eT)}{\mu T}.$$

Proof. Firstly, we have

$$\begin{aligned}
\|\mathbf{x}_{t+1} - \mathbf{x}\|^2 &= \|\mathbf{x}_t - \eta_t \nabla_{\mathbf{x}} F_S(\mathbf{x}_t, \mathbf{y}_t) - \mathbf{x}\|^2 \\
&= \|\mathbf{x}_t - \mathbf{x}\|^2 + \eta_t^2 \|\nabla_{\mathbf{x}} F_S(\mathbf{x}_t, \mathbf{y}_t)\|^2 + 2\eta_t \langle \mathbf{x} - \mathbf{x}_t, \nabla_{\mathbf{x}} F_S(\mathbf{x}_t, \mathbf{y}_t)\rangle \\
&\leq \|\mathbf{x}_t - \mathbf{x}\|^2 + \eta_t^2 L^2 + 2\eta_t \langle \mathbf{x} - \mathbf{x}_t, \nabla_{\mathbf{x}} F_S(\mathbf{x}_t, \mathbf{y}_t)\rangle,
\end{aligned}$$

where the first inequality holds because of Assumption 1. By the strong convexity of $F_S(\cdot, \mathbf{y}_t)$, we have

$$2\eta_t (F_S(\mathbf{x}_t, \mathbf{y}_t) - F_S(\mathbf{x}, \mathbf{y}_t)) \leq (1 - \eta_t \mu)\|\mathbf{x}_t - \mathbf{x}\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}\|^2 + \eta_t^2 L^2.$$

Since $\eta_t = \frac{1}{\mu(t+t_0)}$, we further get

$$\frac{2}{\mu(t+t_0)} (F_S(\mathbf{x}_t, \mathbf{y}_t) - F_S(\mathbf{x}, \mathbf{y}_t)) \leq \Big(1 - \frac{1}{(t+t_0)}\Big)\|\mathbf{x}_t - \mathbf{x}\|^2 - \|\mathbf{x}_{t+1} - \mathbf{x}\|^2 + \Big(\frac{1}{\mu(t+t_0)}\Big)^2 L^2.$$

Multiplying both sides by $t + t_0$, we have

$$\frac{2}{\mu} (F_S(\mathbf{x}_t, \mathbf{y}_t) - F_S(\mathbf{x}, \mathbf{y}_t)) \leq (t + t_0 - 1)\|\mathbf{x}_t - \mathbf{x}\|^2 - (t + t_0)\|\mathbf{x}_{t+1} - \mathbf{x}\|^2 + \frac{L^2}{\mu^2 (t+t_0)}.$$

Since $\mathbf{x}_1 = 0$ and $\sum_{t=1}^T t^{-1} \leq \log(eT)$, by taking a summation of the above inequality from $t = 1$ to T, we obtain

$$\sum_{t=1}^T (F_S(\mathbf{x}_t, \mathbf{y}_t) - F_S(\mathbf{x}, \mathbf{y}_t)) \leq \frac{\mu}{2} t_0 R_X^2 + \frac{L^2 \log(eT)}{2\mu}.$$

From the concavity of $F_S(\mathbf{x}, \cdot)$ we get

$$\sum_{t=1}^T (F_S(\mathbf{x}_t, \mathbf{y}_t) - F_S(\mathbf{x}, \bar{\mathbf{y}}_T)) \leq \frac{\mu}{2} t_0 R_X^2 + \frac{L^2 \log(eT)}{2\mu}.$$

Since this inequality holds for any $\mathbf{x}$, we get

$$\sum_{t=1}^T (F_S(\mathbf{x}_t, \mathbf{y}_t) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T)) \leq \frac{\mu}{2} t_0 R_X^2 + \frac{L^2 \log(eT)}{2\mu}.$$

This implies that

$$\frac{1}{T} \sum_{t=1}^T (F_S(\mathbf{x}_t, \mathbf{y}_t) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T)) \leq \frac{\mu t_0 R_X^2}{2T} + \frac{L^2 \log(eT)}{2\mu T}.$$

In a similar way, we have the following inequality

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \frac{1}{T} \sum_{t=1}^T (F_S(\mathbf{x}_t, \mathbf{y}_t)) \leq \frac{\mu t_0 R_Y^2}{2T} + \frac{L^2 \log(eT)}{2\mu T}.$$

Combined the above two inequalities together we get

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{\mu t_0 (R_X^2 + R_Y^2)}{T} + \frac{L^2 \log(eT)}{\mu T}.$$

□

For optimization algorithm GDA, substituting the strong PD empirical risk bound of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ into Lemma 7, we get the following stability bound,

$$\begin{aligned}\|\bar{\mathbf{x}}_T^i - \bar{\mathbf{x}}_T\| + \|\bar{\mathbf{y}}_T^i - \bar{\mathbf{y}}_T\| &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)} \\ &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{\mu t_0(R_X^2 + R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T}}. \end{aligned} \tag{35}$$

Furthermore, for any $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} \in \mathcal{Y}$ and $z \in \mathcal{Z}$,

$$f(\mathbf{x}, \mathbf{y}; z) - f(0, 0; z) \leq L\|\mathbf{x} - 0\| + L\|\mathbf{y} - 0\| \leq L(R_Y + R_Y),$$

which implies that

$$f(\mathbf{x}, \mathbf{y}; z) \leq \sup_{z\in\mathcal{Z}} f(0, 0; z) + L(R_Y + R_Y). \tag{36}$$

Till here, plugging (36), the stability bound in (35) and the strong PD empirical risk bound in Lemma 8 into Theorem 1, we obtain generalization bounds of GDA. The proof of Theorem 6 is complete.

D STOCHASTIC GRADIENT DESCENT ASCENT

We need some notations to state results on SGDA. Specifically, assume the initial point satisfies $\mathbf{x}_1 = 0$ and $\mathbf{y}_1 = 0$. Let $\{\eta_t\}$ be a sequence of positive step sizes. At the t-th iteration, SGDA first randomly select an index i_t form the uniform distribution over $[n] := \{1, ..., n\}$ and then do the update

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{x}_t - \eta_t \nabla_{\mathbf{x}} f(\mathbf{x}_t, \mathbf{y}_t, z_{i_t}), \\ \mathbf{y}_{t+1} &= \mathbf{y}_t + \eta_t \nabla_{\mathbf{y}} f(\mathbf{x}_t, \mathbf{y}_t, z_{i_t}).\end{aligned} \tag{37}$$

We denote the average of iterates by

$$\bar{\mathbf{x}}_T = \frac{\sum_{t=1}^T \eta_t \mathbf{x}_t}{\sum_{t=1}^T \eta_t} \quad \text{and} \quad \bar{\mathbf{y}}_T = \frac{\sum_{t=1}^T \eta_t \mathbf{y}_t}{\sum_{t=1}^T \eta_t}. \tag{38}$$

Let's first introduce two concentration inequalities for martingales, which are required in deriving the strong PD empirical risk bound of SGDA.

Lemma 9. *(Boucheron et al., 2013) Let $z_1, ..., z_n$ be a sequence of random variables such that z_k may depend the previous variables $z_1, ..., z_{k-1}$ for all $k = 1, ..., n$. Consider a sequence of functionals $\xi_k(z_1, ..., z_k)$, $k = 1, ..., n$. Assume $|\xi_k - \mathbb{E}_{z_k}[\xi_k]| \leq b_k$ for each k. Let $\delta \in (0, 1)$. With probability at least $1 - \delta$*

$$\sum_{k=1}^n \xi_k - \sum_{k=1}^n \mathbb{E}_{z_k}[\xi_k] \leq \Big(2\sum_{k=1}^n b_k^2 \log\frac{1}{\delta}\Big)^{\frac{1}{2}}.$$

Lemma 10. *(Tarres & Yao, 2014) Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a martingale difference sequence in $\mathbb{R}^d$. Suppose that almost surely $\|\xi_k\| \leq D$ and $\sum_{k=1}^t \mathbb{E}[\|\xi_k\|^2|\xi_1, ..., \xi_{k-1}] \leq \sigma_t^2$. Then, for any $0 < \delta < 1$, the following inequality holds with probability at least $1 - \delta$*

$$\max_{1\leq j\leq t}\Big\|\sum_{k=1}^j \xi_k\Big\| \leq 2\Big(\frac{D}{3} + \sigma_t\Big)\log\frac{2}{\delta}.$$

The following lemma shows the strong PD empirical risk bound of SGDA.

Lemma 11. *Suppose Assumption 1 holds and $F_S(\cdot, \cdot)$ be μ-SC-SC with $\mu > 0$. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be the sequence produced by (37) with $\eta_t = \frac{1}{\mu(t+t_0)}$. Assume $t_0 \geq 0$. Suppose $\sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{y}\| \leq R_Y$. Let $\delta > 0$. Then for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (38), with probability at least $1 - \delta$ we have*

$$\begin{aligned}\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T)) &\leq \frac{2\mu t_0(R_X^2 + R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T} \\ + \frac{2(R_X + R_Y)}{T}\Big(\frac{2L}{3} + 2L\sqrt{T}\Big)\log\frac{6}{\delta} &+ \frac{2L(R_X + R_Y)(2T\log(6/\delta))^{\frac{1}{2}}}{T}.\end{aligned}$$

If $t_0 = 0$, then with probability at least $1-\delta$ we have

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T)) \leq \frac{L^2\log(eT)}{\mu T} + \frac{2(R_X+R_Y)}{T}\Big(\frac{2L}{3} + 2L\sqrt{T}\Big)\log\frac{6}{\delta} + \frac{2L(R_X+R_Y)(2T\log(6/\delta))^{\frac{1}{2}}}{T}.$$

Proof. By a similar analysis to the proof of Lemma 8, we have

$$\begin{aligned}
&\|\mathbf{x}_{t+1}-\mathbf{x}\|^2 = \|\mathbf{x}_t - \eta_t\nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \mathbf{x}\|^2 \\
&= \|\mathbf{x}_t-\mathbf{x}\|^2 + \eta_t^2\|\nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\|^2 + 2\eta_t\langle \mathbf{x}-\mathbf{x}_t, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle \\
&\leq \|\mathbf{x}_t-\mathbf{x}\|^2 + \eta_t^2 L^2 + 2\eta_t\langle \mathbf{x}-\mathbf{x}_t, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle + 2\eta_t\langle \mathbf{x}-\mathbf{x}_t, \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle,
\end{aligned}$$

where the first inequality holds because of Assumption 1. By the strong convexity of $F_S(\cdot,\mathbf{y}_t)$, we have

$$\begin{aligned}
2\eta_t(F_S(\mathbf{x}_t,\mathbf{y}_t) - F_S(\mathbf{x},\mathbf{y}_t)) \leq (1-\eta_t\mu)\|\mathbf{x}_t-\mathbf{x}\|^2 - \|\mathbf{x}_{t+1}-\mathbf{x}\|^2 + \eta_t^2 L^2 \\
+2\eta_t\langle \mathbf{x}-\mathbf{x}_t, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle.
\end{aligned}$$

Since $\eta_t = \frac{1}{\mu(t+t_0)}$, we further get

$$\begin{aligned}
&\frac{2}{\mu(t+t_0)}(F_S(\mathbf{x}_t,\mathbf{y}_t) - F_S(\mathbf{x},\mathbf{y}_t)) \leq \Big(1-\frac{1}{(t+t_0)}\Big)\|\mathbf{x}_t-\mathbf{x}\|^2 - \|\mathbf{x}_{t+1}-\mathbf{x}\|^2 \\
&+\Big(\frac{1}{\mu(t+t_0)}\Big)^2 L^2 + \frac{2}{\mu(t+t_0)}\langle \mathbf{x}-\mathbf{x}_t, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle.
\end{aligned}$$

Multiplying both sides by $t+t_0$, we have

$$\begin{aligned}
\frac{2}{\mu}(F_S(\mathbf{x}_t,\mathbf{y}_t) - F_S(\mathbf{x},\mathbf{y}_t)) \leq (t+t_0-1)\|\mathbf{x}_t-\mathbf{x}\|^2 - (t+t_0)\|\mathbf{x}_{t+1}-\mathbf{x}\|^2 + \frac{L^2}{\mu^2(t+t_0)} \\
+\frac{2}{\mu}\langle \mathbf{x}-\mathbf{x}_t, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle.
\end{aligned}$$

Since $\mathbf{x}_1 = 0$ and $\sum_{t=1}^T t^{-1} \leq \log(eT)$, by taking a summation of the above inequality from $t=1$ to T, we obtain

$$\begin{aligned}
&\sum_{t=1}^T (F_S(\mathbf{x}_t,\mathbf{y}_t) - F_S(\mathbf{x},\mathbf{y}_t)) \leq \frac{\mu}{2}t_0 R_X^2 + \frac{L^2\log(eT)}{2\mu} \\
&+\sum_{t=1}^T \langle \mathbf{x}, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle + \sum_{t=1}^T \langle \mathbf{x}_t, \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t) - \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle.
\end{aligned}$$

From the concavity of $F_S(\mathbf{x},\cdot)$ we get

$$\begin{aligned}
&\sum_{t=1}^T (F_S(\mathbf{x}_t,\mathbf{y}_t) - F_S(\mathbf{x},\bar{\mathbf{y}}_T)) \leq \frac{\mu}{2}t_0 R_X^2 + \frac{L^2\log(eT)}{2\mu} \\
&+\sum_{t=1}^T \langle \mathbf{x}, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle + \sum_{t=1}^T \langle \mathbf{x}_t, \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t) - \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle.
\end{aligned}$$

Since this inequality holds for any $\mathbf{x}$, we get

$$\begin{aligned}
&\sum_{t=1}^T (F_S(\mathbf{x}_t,\mathbf{y}_t) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x},\bar{\mathbf{y}}_T)) \leq \frac{\mu}{2}t_0 R_X^2 + \frac{L^2\log(eT)}{2\mu} \\
&+\sum_{t=1}^T \sup_{\mathbf{x}\in\mathcal{X}} \langle \mathbf{x}, \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t}) - \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t)\rangle + \sum_{t=1}^T \langle \mathbf{x}_t, \nabla_{\mathbf{x}} F_S(\mathbf{x}_t,\mathbf{y}_t) - \nabla_{\mathbf{x}} f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle.
\end{aligned}$$

By Schwartz's inequality, we have

$$\sum_{t=1}^{T}(F_S(\mathbf{x}_t,\mathbf{y}_t)-\inf_{\mathbf{x}\in\mathcal{X}}F_S(\mathbf{x},\bar{\mathbf{y}}_T))\leq\frac{\mu}{2}t_0R_X^2+\frac{L^2\log(eT)}{2\mu}$$
$$+R_X\Big\|\sum_{t=1}^{T}\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})-\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)\Big\|+\sum_{t=1}^{T}\langle\mathbf{x}_t,\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)-\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle.$$

Denote $\xi_t=\langle\mathbf{x}_t,\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)-\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle$. Since $\mathbb{E}_{i_t}[\langle\mathbf{x}_t,\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)-\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle]=0$, so $\{\xi_t|t=1,...,T\}$ is a martingale difference sequence. By Schwartz's inequality and Assumption 1, we know that $|\langle\mathbf{x}_t,\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)-\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle|\leq 2LR_X$. Then, according to Lemma 9, we have the following inequality with probability at least $1-\delta/6$

$$\sum_{t=1}^{T}\langle\mathbf{x}_t,\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)-\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})\rangle\leq 2LR_X(2T\log(6/\delta))^{\frac{1}{2}}.$$

Define $\xi_t'=\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})-\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)$. Then we get $\|\xi_t'\|\leq 2L$ and

$$\sum_{t=1}^{T}\mathbb{E}[\|\xi_t'\|^2|\xi_1',...,\xi_{t-1}']\leq 4TL^2.$$

Applying Lemma 10 to the martingale difference sequence $\{\xi_t'\}$, we have the following inequality with probability at least $1-\delta/3$

$$\Big\|\sum_{t=1}^{T}\xi_t'\Big\|\leq 2\Big(\frac{2L}{3}+2L\sqrt{T}\Big)\log\frac{6}{\delta}.$$

That is, with probability at least $1-\delta/3$

$$\Big\|\sum_{t=1}^{T}\nabla_{\mathbf{x}}f(\mathbf{x}_t,\mathbf{y}_t;z_{i_t})-\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t)\Big\|\leq 2\Big(\frac{2L}{3}+2L\sqrt{T}\Big)\log\frac{6}{\delta}.$$

Combined with the above results, we finally have the following inequality with probability at least $1-\delta/2$

$$\frac{1}{T}\sum_{t=1}^{T}(F_S(\mathbf{x}_t,\mathbf{y}_t)-\inf_{\mathbf{x}\in\mathcal{X}}F_S(\mathbf{x},\bar{\mathbf{y}}_T))\leq\frac{\mu t_0R_X^2}{2T}+\frac{L^2\log(eT)}{2\mu T}$$
$$+\frac{2R_X}{T}\Big(\frac{2L}{3}+2L\sqrt{T}\Big)\log\frac{6}{\delta}+\frac{2LR_X(2T\log(6/\delta))^{\frac{1}{2}}}{T}.$$

In a similar way, we have the following inequality with probability at least $1-\delta/2$

$$\sup_{\mathbf{y}\in\mathcal{Y}}F_S(\bar{\mathbf{x}}_T,\mathbf{y})-\frac{1}{T}\sum_{t=1}^{T}(F_S(\mathbf{x}_t,\mathbf{y}_t))\leq\frac{\mu t_0R_Y^2}{2T}+\frac{L^2\log(eT)}{2\mu T}$$
$$+\frac{2R_Y}{T}\Big(\frac{2L}{3}+2L\sqrt{T}\Big)\log\frac{6}{\delta}+\frac{2LR_Y(2T\log(6/\delta))^{\frac{1}{2}}}{T}.$$

Combined the above two inequalities together we get the result with probability at least $1-\delta$

$$\sup_{\mathbf{y}\in\mathcal{Y}}F_S(\bar{\mathbf{x}}_T,\mathbf{y})-\inf_{\mathbf{x}\in\mathcal{X}}F_S(\mathbf{x},\bar{\mathbf{y}}_T))\leq\frac{\mu t_0(R_X^2+R_Y^2)}{T}+\frac{L^2\log(eT)}{\mu T}$$
$$+\frac{2(R_X+R_Y)}{T}\Big(\frac{2L}{3}+2L\sqrt{T}\Big)\log\frac{6}{\delta}+\frac{2L(R_X+R_Y)(2T\log(6/\delta))^{\frac{1}{2}}}{T}.$$

□

Denote $E = \frac{\mu t_0(R_X^2+R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T} + \frac{2(R_X+R_Y)(\frac{2L}{3}+2L\sqrt{T})\log\frac{6}{\delta}}{T} + \frac{2L(R_X+R_Y)(2T\log(6/\delta))^{\frac{1}{2}}}{T}$. Now, plugging Lemma 11 to Lemma 7, we know that the argument stability bound of SGDA is

$$\|\bar{\mathbf{x}}_T^i - \bar{\mathbf{x}}_T\| + \|\bar{\mathbf{y}}_T^i - \bar{\mathbf{y}}_T\| \leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)} \leq 4\sqrt{\frac{1}{\mu}}\sqrt{E} + \frac{4L}{n\mu}. \tag{39}$$

Furthermore, for any $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} \in \mathcal{Y}$ and $z \in \mathcal{Z}$,

$$f(\mathbf{x}, \mathbf{y}; z) \leq f(0, 0; z) + L\|\mathbf{x} - 0\| + L\|\mathbf{y} - 0\| \leq \sup_{z\in\mathcal{Z}} f(0, 0; z) + L(R_X + R_Y), \tag{40}$$

Therefore, plugging (40), the stability bound in (39) and the strong PD empirical risk bound in Lemma 11 into Theorem 1, we obtain generalization bounds of SGDA.

Now, We can write the main theorem of SGDA as follows.

Theorem 7. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Suppose $\sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{y}\| \leq R_Y$. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be produced by (37) with $\eta_t = 1/(\mu(t + t_0))$. Assume $t_0 \geq 0$. Denote $A_{\mathbf{x}}(S) = \bar{\mathbf{x}}_T$ and $A_{\mathbf{y}}(S) = \bar{\mathbf{y}}_T$ for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (38). Fixed any $\eta > 0$. Let $M = \sup_{z\in\mathcal{Z}} f(0, 0; z) + L(R_X + R_Y)$. Let $E = \frac{\mu t_0(R_X^2+R_Y^2)}{T} + \frac{L^2\log(eT)}{\mu T} + \frac{2(R_X+R_Y)}{T}\Big(\frac{2L}{3} + 2L\sqrt{T}\Big)\log\frac{6}{\delta} + \frac{2L(R_X+R_Y)(2T\log(6/\delta))^{\frac{1}{2}}}{T}$ and $B = \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{E}$. There exists an absolute positive constant C.*

(a) If Assumption 1 holds, then for any $\delta > 0$, with probability at least $1 - 2\delta$, we have

$$F(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)F_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + B\log_2 n\log(1/\delta)\Big).$$

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - 2\delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1 + \eta)R_S(\bar{\mathbf{x}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big)LB\log_2 n\log\frac{1}{\delta}\Big).$$

(c) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - 2\delta$, we have

$$\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(d) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - 2\delta$, we have

$$\begin{aligned}&\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) - \triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)\\ &\qquad\leq \eta E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).\end{aligned}$$

(e) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - 2\delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1 + \eta)\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) + C\frac{2 + \eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big)LB\log_2 n\log\frac{1}{\delta} + E\Big).$$

Remark 8. When conditions in Theorem 7 hold, we can see that

(1) If Assumption 1 holds, then for any $\delta > 0$, with probability at least $1-\delta$, the plain generalization error of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of SGDA is of the order $\mathcal{O}\Big(\Big(\frac{1}{n} + \sqrt{\frac{\log(1/\delta)}{T^{\frac{1}{2}}}}\Big)\log_2 n\log(1/\delta)\Big)$. We can take $\mathcal{O}(n^4)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\Big(\frac{\log n}{n}\log^{\frac{3}{2}}(1/\delta)\Big)$.

(2) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, the primal generalization error, strong PD population risk, strong PD generalization error and excess primal population risk of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of SGDA are all of the order $\mathcal{O}\left(\left(\frac{1}{n} + \sqrt{\frac{\log(1/\delta)}{T^{\frac{1}{2}}}}\right)\log_2 n\log(1/\delta)\right)$. We can take $\mathcal{O}(n^4)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\left(\frac{\log n}{n}\log^{\frac{3}{2}}(1/\delta)\right)$.

E PROXIMAL POINT METHOD

One of the classical algorithms studied for solving the minimax problem is the Proximal Point method (Rockafellar, 1976). We denote the t-th iterate of PPM as $(\mathbf{x}_t, \mathbf{y}_t)$. The averaged iterate is defined as

$$\bar{\mathbf{x}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t \quad \text{and} \quad \bar{\mathbf{y}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t. \tag{41}$$

Given stepsize parameter ν, the PPM generates the iterate $\{\mathbf{x}_{t+1}, \mathbf{y}_{t+1}\}$ by

$$\arg\min_{\mathbf{x}\in\mathcal{X}} \arg\max_{\mathbf{y}\in\mathcal{Y}} \left\{ F_S(\mathbf{x},\mathbf{y}) + \frac{1}{2\nu}\|\mathbf{x}-\mathbf{x}_t\| - \frac{1}{2\nu}\|\mathbf{y}-\mathbf{y}_t\| \right\}. \tag{42}$$

$\{\mathbf{x}_{t+1}, \mathbf{y}_{t+1}\}$ is the unique solution since the objective function of problem (42) is strongly convex in $\mathbf{x}$ and strongly concave in $\mathbf{y}$. From the discussion in (Mokhtari et al., 2019), the update of PPM can be written as

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{x}_t - \nu\nabla_{\mathbf{x}} F_S(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}), \\ \mathbf{y}_{t+1} &= \mathbf{y}_t + \nu\nabla_{\mathbf{y}} F_S(\mathbf{x}_{t+1}, \mathbf{y}_{t+1}), \end{aligned} \tag{43}$$

Assume that the initial point satisfies $\mathbf{x}_0 = 0$ and $\mathbf{y}_0 = 0$.

We now begin to prove the strong PD empirical risk. Firstly, two lemmas are introduced.

Lemma 12. *(Nemirovski, 2005) Define vector $\mathbf{v} = [\mathbf{x}, \mathbf{y}] \in \mathbb{R}^{2d}$ and the operator $P : \mathbb{R}^{2d} \mapsto \mathbb{R}^{2d}$ as*

$$P(\mathbf{v}) = [\nabla_{\mathbf{x}} F_S(\mathbf{x},\mathbf{y}); -\nabla_{\mathbf{y}} F_S(\mathbf{x},\mathbf{y})]. \tag{44}$$

Consider $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (41). Suppose the ESP solution exists. Assume the function $F_S(\mathbf{x},\mathbf{y})$ is continuously differentiable in $\mathbf{x}$ and $\mathbf{y}$. Assume that $F_S(\mathbf{x},\mathbf{y})$ is a convex function of $\mathbf{x}$ for any $\mathbf{y}$ and is a concave function of $\mathbf{y}$ for any $\mathbf{x}$. Then for any $\mathbf{v} = [\mathbf{x},\mathbf{y}] \in \mathbb{R}^{2d}$, we have

$$F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{1}{T}\sum_{t=1}^{T} P(\mathbf{v}_t)^T(\mathbf{v}_t - \mathbf{v}).$$

Lemma 13. *(Mokhtari et al., 2019) Consider the sequence of iterates $\{\mathbf{v}_t\} \in \mathbb{R}^{2d}$ generated by the following update*

$$\mathbf{v}_{t+1} = \mathbf{v}_t - \nu P(\mathbf{v}_{t+1}),$$

where P is a monotone and Lipschitz continuous operator, and ν is a positive constant. Then for any $\mathbf{v} \in \mathbb{R}^{2d}$ and for each $t \geq 1$ we have

$$P(\mathbf{v}_{t+1})^T(\mathbf{v}_{t+1} - \mathbf{v}) = \frac{1}{2\nu}\|\mathbf{v}_t - \mathbf{v}\|^2 - \frac{1}{2\nu}\|\mathbf{v}_{t+1} - \mathbf{v}\|^2 - \frac{1}{2\nu}\|\mathbf{v}_{t+1} - \mathbf{v}_t\|^2.$$

The following lemma is strong PD empirical risk bound of PPM.

Lemma 14. *Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be the iterates generated by PPM in (43). Assume ν is a positive constant. Suppose the ESP solution exists. Assume that $F_S(\mathbf{x},\mathbf{y})$ is a convex function of $\mathbf{x}$ for any $\mathbf{y}$ and is a concave function of $\mathbf{y}$ for any $\mathbf{x}$. Suppose $\sup_{\mathbf{x}\in\mathcal{X}}\|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}}\|\mathbf{y}\| \leq R_Y$. If Assumption 2 holds, then for all $T \geq 1$, we have*

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{R_X^2 + R_Y^2}{2\nu T}.$$

Proof. The update of the PPM in (43) can be written as

$$\mathbf{v}_{t+1} = \mathbf{v}_t - \nu P(\mathbf{v}_{t+1}).$$

According to Lemma 1 in (Mokhtari et al., 2019), if $F_S(\mathbf{x},\mathbf{y})$ is convex-concave and Assumption 2 holds, then $P(\mathbf{v})$ defined in (44) is monotone and Lipschitz continuous. According to Lemma 13, we have

$$P(\mathbf{v}_{t+1})^T(\mathbf{v}_{t+1} - \mathbf{v}) = \frac{1}{2\nu}\|\mathbf{v}_t - \mathbf{v}\|^2 - \frac{1}{2\nu}\|\mathbf{v}_{t+1} - \mathbf{v}\|^2 - \frac{1}{2\nu}\|\mathbf{v}_{t+1} - \mathbf{v}_t\|^2.$$

Taking a summation of the above inequality from $t = 0$ to $T - 1$, we obtain

$$\sum_{t=0}^{T-1} P(\mathbf{v}_{t+1})^T(\mathbf{v}_{t+1} - \mathbf{v}) \leq \frac{1}{2\nu}\|\mathbf{v}_0 - \mathbf{v}\|^2 - \frac{1}{2\nu}\|\mathbf{v}_T - \mathbf{v}\|^2. \tag{45}$$

According to (45), we know that

$$\begin{aligned}\sum_{t=0}^{T-1} P(\mathbf{v}_{t+1})^T(\mathbf{v}_{t+1} - \mathbf{v}) &\leq \frac{1}{2\nu}\|\mathbf{v}_0 - \mathbf{v}\|^2 \\ &= \frac{\|\mathbf{x}_0 - \mathbf{x}\|^2 + \|\mathbf{y}_0 - \mathbf{y}\|^2}{2\nu}.\end{aligned}$$

Combined this result with Lemma 12, we can write

$$F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{\|\mathbf{x}_0 - \mathbf{x}\|^2 + \|\mathbf{y}_0 - \mathbf{y}\|^2}{2\nu T},$$

which implies that

$$\sup_{\mathbf{y}\in\mathcal{Y}} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}\in\mathcal{X}} F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{R_X^2 + R_Y^2}{2\nu T}.$$

The proof is complete. □

Combined Lemma 7 and Lemma 14, we know that the argument stability bound of PPM is

$$\begin{aligned}\|\bar{\mathbf{x}}_T^i - \bar{\mathbf{x}}_T\| + \|\bar{\mathbf{y}}_T^i - \bar{\mathbf{y}}_T\| &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)} \\ &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{R_X^2 + R_Y^2}{2\nu T}}.\end{aligned} \tag{46}$$

Furthermore, for any $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} \in \mathcal{Y}$ and $z \in \mathcal{Z}$,

$$f(\mathbf{x}, \mathbf{y}; z) \leq f(0, 0; z) + L\|\mathbf{x} - 0\| + L\|\mathbf{y} - 0\| \leq \sup_{z\in\mathcal{Z}} f(0, 0; z) + L(R_X + R_Y), \tag{47}$$

Therefore, plugging (47), the stability bound in (46) and the strong PD empirical risk bound in Lemma 14 into Theorem 1, we obtain generalization bounds of PPM, shown as below.

Theorem 8. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Suppose $\sup_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\| \leq R_X$ and $\sup_{\mathbf{y}\in\mathcal{Y}} \|\mathbf{y}\| \leq R_Y$. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be produced by (43). Assume ν is a positive constant. Denote $A_{\mathbf{x}}(S) = \bar{\mathbf{x}}_T$ and $A_{\mathbf{y}}(S) = \bar{\mathbf{y}}_T$ for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (41). Fixed any $\eta > 0$. Let $M = \sup_{z\in\mathcal{Z}} f(0, 0; z) + L(R_X + R_Y)$. Let $E = \frac{R_X^2 + R_Y^2}{2\nu T}$ and $B = \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{E}$. There exists an absolute positive constant C.*

(a) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$F(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)F_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + B\log_2 n \log(1/\delta)\Big).$$

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1 + \eta)R_S(\bar{\mathbf{x}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big)LB\log_2 n \log\frac{1}{\delta}\Big).$$

(c) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(d) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\begin{aligned}&\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) - \triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \\ &\qquad \leq \eta E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).\end{aligned}$$

(e) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$R(\bar{\mathbf{x}}_T) - (1+\eta)\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) \leq C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)LB\log_2 n\log\frac{1}{\delta} + E\Big).$$

Remark 9. When conditions in Theorem 8 hold, we can see that if Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, the plain generalization error, primal generalization error, strong PD population risk, strong PD generalization error and excess primal population risk of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of PPM are all of the order $\mathcal{O}\Big(\Big(\frac{1}{n}+\sqrt{\frac{1}{T}}\Big)\log_2 n\log(1/\delta)\Big)$. We can take $\mathcal{O}(n^2)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\Big(\frac{\log n}{n}\log(1/\delta)\Big)$.

F EXTRAGRADIENT METHOD

EG is a classical algorithm for solving minimax problems introduced by (Korpelevich, 1976). We now introduce some notations. Followed (Mokhtari et al., 2019), we consider the following update of EG: given stepsize parameter ν, we first compute a set of mid-point iterates $\{\mathbf{x}_{t+\frac{1}{2}}, \mathbf{y}_{t+\frac{1}{2}}\}$

$$\begin{aligned}\mathbf{x}_{t+\frac{1}{2}} &= \mathbf{x}_t - \nu\nabla_{\mathbf{x}}F_S(\mathbf{x}_t,\mathbf{y}_t),\\ \mathbf{y}_{t+\frac{1}{2}} &= \mathbf{y}_t + \nu\nabla_{\mathbf{y}}F_S(\mathbf{x}_t,\mathbf{y}_t),\end{aligned} \tag{48}$$

we then compute the next iterates $\{\mathbf{x}_{t+1}, \mathbf{y}_{t+1}\}$

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{x}_t - \nu\nabla_{\mathbf{x}}F_S(\mathbf{x}_{t+\frac{1}{2}},\mathbf{y}_{t+\frac{1}{2}}),\\ \mathbf{y}_{t+1} &= \mathbf{y}_t + \nu\nabla_{\mathbf{y}}F_S(\mathbf{x}_{t+\frac{1}{2}},\mathbf{y}_{t+\frac{1}{2}}).\end{aligned} \tag{49}$$

Consider the averaged iterate

$$\bar{\mathbf{x}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t \quad \text{and} \quad \bar{\mathbf{y}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t. \tag{50}$$

Assume that the initial point satisfies $\mathbf{x}_0 = \mathbf{x}_{-1/2}$ and $\mathbf{y}_0 = \mathbf{y}_{-1/2}$.

We now show the strong PD empirical risk bound for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of EG.

Lemma 15. *(Mokhtari et al., 2019) Let $\{\mathbf{x}_t, \mathbf{y}_t\}$, $\{\mathbf{x}_{t+1/2}, \mathbf{y}_{t+1/2}\}$ be the iterates generated by the EG updates in (48) and (49). Assume that the initial point satisfies $\mathbf{x}_0 = \mathbf{x}_{-1/2}$ and $\mathbf{y}_0 = \mathbf{y}_{-1/2}$. Suppose the ESP solution $(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*)$ exists. Assume that $F_S(\mathbf{x},\mathbf{y})$ is a convex function of $\mathbf{x}$ for any $\mathbf{y}$ and is a concave function of $\mathbf{y}$ for any $\mathbf{x}$. If Assumption 2 holds and the stepsize ν satisfies the condition $\nu = \frac{c}{2\beta}$ for any $c \in (0,1)$, then:*

(a) the iterates $\{\mathbf{x}_t, \mathbf{y}_t\}$, $\{\mathbf{x}_{t+1/2}, \mathbf{y}_{t+1/2}\}$ stay within the compact convex set

$$D := \Big\{(\mathbf{x},\mathbf{y}) | \|\mathbf{x}-\hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}-\hat{\mathbf{y}}^*\|^2 \leq \Big(2+\frac{2}{1-4\nu^2\beta^2}\Big)(\|\mathbf{x}_0-\hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0-\hat{\mathbf{y}}^*\|^2)\Big\}. \tag{51}$$

(b) for all $T \geq 1$, we have

$$\sup_{\mathbf{y}:(\bar{\mathbf{x}}_T,\mathbf{y})\in D} F_S(\bar{\mathbf{x}}_T,\mathbf{y}) - \inf_{\mathbf{x}:(\mathbf{x},\bar{\mathbf{y}}_T)\in D} F_S(\mathbf{x},\bar{\mathbf{y}}_T) \leq \frac{2\beta(16+\frac{33}{2(1-c^2)})(\|\mathbf{x}_0-\hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0-\hat{\mathbf{y}}^*\|^2)}{T}.$$

Combined Lemma 7 and Lemma 15, we know that the argument stability bound of EG is

$$\begin{aligned}\|\bar{\mathbf{x}}_T^i - \bar{\mathbf{x}}_T\| + \|\bar{\mathbf{y}}_T^i - \bar{\mathbf{y}}_T\| &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle_S^s(\bar{\mathbf{x}}_T,\bar{\mathbf{y}}_T)}\\ &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{2\beta(16+\frac{33}{2(1-c^2)})(\|\mathbf{x}_0-\hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0-\hat{\mathbf{y}}^*\|^2)}{T}}.\end{aligned} \tag{52}$$

Furthermore, for any $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} \in \mathcal{Y}$ and $z \in \mathcal{Z}$,

$$\begin{aligned} f(\mathbf{x}, \mathbf{y}; z) &\leq f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + L\|\mathbf{x} - \hat{\mathbf{x}}^*\| + L\|\mathbf{y} - \hat{\mathbf{y}}^*\| \\ &\leq \sup_{z \in \mathcal{Z}} f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + \sqrt{2}L\sqrt{\|\mathbf{x} - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y} - \hat{\mathbf{y}}^*\|^2} \\ &\leq \sup_{z \in \mathcal{Z}} f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + L\sqrt{\Big(4 + \frac{4}{1 - 4\nu^2\beta^2}\Big)(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)}, \end{aligned} \tag{53}$$

where the first inequality follows from Assumption 1, the second inequality follows from Caucy-Schwartz inequality and the last inequality follows from (51). Now, plugging (53), the stability bound in (52) and the strong PD empirical risk bound in Lemma 15 into Theorem 1, we obtain generalization bounds of EG. The main theorem are shown below.

Theorem 9. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ and $\{\mathbf{x}_{t+1/2}, \mathbf{y}_{t+1/2}\}$ be the iterates produced by (48)-(49). Assume the stepsize ν satisfies the condition $\nu = \frac{c}{2\beta}$ for any $c \in (0, 1)$. Denote $A_{\mathbf{x}}(S) = \bar{\mathbf{x}}_T$ and $A_{\mathbf{y}}(S) = \bar{\mathbf{y}}_T$ for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (50). Denote the ESP solution as $(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*)$. Consider the compact convex set in (51). Fixed any $\eta > 0$. Let $M = \sup_{z \in \mathcal{Z}} f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + L\sqrt{\Big(4 + \frac{4}{1-4\nu^2\beta^2}\Big)(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)}$. Let $B = \frac{2\beta(16 + \frac{33}{2(1-c^2)})(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)}{T}$ and $E = \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{B}$. There exists an absolute positive constant C.*

(a) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$F(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)F_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + B\log_2 n \log(1/\delta)\Big).$$

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1 + \eta)R_S(\bar{\mathbf{x}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big)LB\log_2 n \log\frac{1}{\delta}\Big).$$

(c) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(d) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\begin{aligned} &\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) - \triangle^s_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \\ &\qquad \leq \eta E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big). \end{aligned}$$

(e) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1 + \eta)\inf_{\mathbf{x} \in \mathcal{X}} R(\mathbf{x}) + C\frac{2 + \eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big)LB\log_2 n \log\frac{1}{\delta} + E\Big).$$

Remark 10. When conditions in Theorem 9 hold, we can see that if Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, the plain generalization error, primal generalization error, strong PD population risk, strong PD generalization error and excess primal population risk of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of EG are all of the order $\mathcal{O}\Big(\Big(\frac{1}{n} + \sqrt{\frac{1}{T}}\Big)\log_2 n \log(1/\delta)\Big)$. We can take $\mathcal{O}(n^2)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\Big(\frac{\log n}{n}\log(1/\delta)\Big)$.

G Optimistic Gradient Descent Ascent

OGDA is introduced by Popov (1980), as a variant of the EG method. We introduce some notations to state the result of OGDA. Given a stepsize parameter $\nu > 0$, OGDA do the following update for

each $t \geq 0$

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{x}_t - 2\nu\nabla_{\mathbf{x}} F_S(\mathbf{x}_t, \mathbf{y}_t) + \nu\nabla_{\mathbf{x}} F_S(\mathbf{x}_{t-1}, \mathbf{y}_{t-1}),\\ \mathbf{y}_{t+1} &= \mathbf{y}_t + 2\nu\nabla_{\mathbf{y}} F_S(\mathbf{x}_t, \mathbf{y}_t) - \nu\nabla_{\mathbf{y}} F_S(\mathbf{x}_{t-1}, \mathbf{y}_{t-1}).\end{aligned} \tag{54}$$

Assume that the initial point satisfies $\mathbf{x}_0 = \mathbf{x}_{-1}$ and $\mathbf{y}_0 = \mathbf{y}_{-1}$. Consider the averaged iterate

$$\bar{\mathbf{x}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t \quad \text{and} \quad \bar{\mathbf{y}}_T = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t. \tag{55}$$

We first provide a lemma on strong PD empirical risk of OGDA.

Lemma 16. *(Mokhtari et al., 2019) Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be the iterates generated by the OGDA updates in (54). Assume that the initial point satisfies $\mathbf{x}_0 = \mathbf{x}_{-1}$ and $\mathbf{y}_0 = \mathbf{y}_{-1}$. Suppose the ESP solution $(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*)$ exists. Assume that $F_S(\mathbf{x}, \mathbf{y})$ is a convex function of $\mathbf{x}$ for any $\mathbf{y}$ and is a concave function of $\mathbf{y}$ for any $\mathbf{x}$. If Assumption 2 holds and the stepsize ν satisfies $0 < \nu \leq \frac{1}{4\beta}$, then:*

(a) the iterates $\{\mathbf{x}_t, \mathbf{y}_t\}$ stay within the compact convex set

$$D := \left\{(\mathbf{x}, \mathbf{y}) | \|\mathbf{x} - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y} - \hat{\mathbf{y}}^*\|^2 \leq 2(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)\right\}. \tag{56}$$

(b) for all $T \geq 1$, we have

$$\sup_{\mathbf{y}:(\bar{\mathbf{x}}_T, \mathbf{y}) \in D} F_S(\bar{\mathbf{x}}_T, \mathbf{y}) - \inf_{\mathbf{x}:(\mathbf{x}, \bar{\mathbf{y}}_T) \in D} F_S(\mathbf{x}, \bar{\mathbf{y}}_T) \leq \frac{(16\beta + \frac{1}{2\nu})(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)}{T}.$$

Combined Lemma 7 and Lemma 16, we know that the argument stability bound of OGDA is

$$\begin{aligned}\|\bar{\mathbf{x}}_T^i - \bar{\mathbf{x}}_T\| + \|\bar{\mathbf{y}}_T^i - \bar{\mathbf{y}}_T\| &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)}\\ &\leq \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{\frac{(16\beta + \frac{1}{2\nu})(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)}{T}}.\end{aligned} \tag{57}$$

Moreover, similar to (53), we have

$$\begin{aligned}f(\mathbf{x}, \mathbf{y}; z) &\leq f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + L\|\mathbf{x} - \hat{\mathbf{x}}^*\| + L\|\mathbf{y} - \hat{\mathbf{y}}^*\|\\ &\leq \sup_{z \in \mathcal{Z}} f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + 2L\sqrt{\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2}.\end{aligned} \tag{58}$$

Therefore, plugging (58), the stability bound in (57) and the strong PD empirical risk bound in Lemma 16 into Theorem 1, we obtain generalization bounds of OGDA.

Theorem 10. *Assume for all z, the function $(\mathbf{x}, \mathbf{y}) \mapsto f(\mathbf{x}, \mathbf{y}; z)$ is μ-SC-SC. Let $\{\mathbf{x}_t, \mathbf{y}_t\}$ be produced by (54). Assume the stepsize parameter ν satisfies $0 < \nu \leq \frac{1}{4\beta}$. Denote $A_{\mathbf{x}}(S) = \bar{\mathbf{x}}_T$ and $A_{\mathbf{y}}(S) = \bar{\mathbf{y}}_T$ for $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ in (55). Denote the ESP solution as $(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*)$. Consider the compact convex set in (56). Fixed any $\eta > 0$. Let $M = \sup_{z \in \mathcal{Z}} f(\hat{\mathbf{x}}^*, \hat{\mathbf{y}}^*; z) + 2L\sqrt{\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2}$. Define $B = \frac{(16\beta + \frac{1}{2\nu})(\|\mathbf{x}_0 - \hat{\mathbf{x}}^*\|^2 + \|\mathbf{y}_0 - \hat{\mathbf{y}}^*\|^2)}{T}$ and $E = \frac{4L}{n\mu} + 4\sqrt{\frac{1}{\mu}}\sqrt{B}$. There exists an absolute positive constant C.*

(a) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$F(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)F_S(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log(1/\delta) + B\log_2 n \log(1/\delta)\Big).$$

(b) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1 + \eta)R_S(\bar{\mathbf{x}}_T) + C\frac{1 + \eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu} + 1\Big)LB\log_2 n \log\frac{1}{\delta}\Big).$$

(c) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1 - \delta$, we have

$$\triangle^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq (1 + \eta)E + C(1 + \eta)\Big(\frac{L^2(1 + \eta)}{n\mu\eta} + \frac{M}{n} + \Big(1 + \frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(d) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$\triangle^s\left(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T\right) - \triangle_S^s(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T) \leq \eta E + C(1+\eta)\Big(\frac{L^2(1+\eta)}{n\mu\eta} + \frac{M}{n} + \Big(1+\frac{\beta}{\mu}\Big)BL\log_2 n\Big)\log\Big(\frac{1}{\delta}\Big).$$

(e) If Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, we have

$$R(\bar{\mathbf{x}}_T) \leq (1+\eta)\inf_{\mathbf{x}\in\mathcal{X}} R(\mathbf{x}) + C\frac{2+\eta}{\eta}\Big(\frac{M}{n}\log\frac{1}{\delta} + \Big(\frac{\beta}{\mu}+1\Big)LB\log_2 n\log\frac{1}{\delta} + E\Big).$$

Remark 11. When conditions in Theorem 10 hold, we can see that if Assumptions 1 and 2 hold, then for any $\delta > 0$, with probability at least $1-\delta$, the plain generalization error, primal generalization error, strong PD population risk, strong PD generalization error and excess primal population risk of $(\bar{\mathbf{x}}_T, \bar{\mathbf{y}}_T)$ of OGDA are all of the order $\mathcal{O}\Big(\Big(\frac{1}{n}+\sqrt{\frac{1}{T}}\Big)\log_2 n\log(1/\delta)\Big)$. We can take $\mathcal{O}(n^2)$ stochastic gradient evaluations to get bound of the order $\mathcal{O}\Big(\frac{\log n}{n}\log(1/\delta)\Big)$.

H Auxiliary Descriptions of Table 1

In Table 1, (R)-ESP means the (regularized)-empirical risk saddle point (Zhang et al., 2021a). C-SC means convex-μ-strongly-concave, and NC-SC means nonconvex-μ-strongly-concave. A function $f(\mathbf{x},\mathbf{y})$ is called nonconvex-strongly-concave if $f(\mathbf{x},\cdot)$ is strongly-concave for every $\mathbf{x}$. Moreover, a function $f(\mathbf{x},\mathbf{y})$ is μ-weakly-convex-weakly-concave (WC-WC) if $f+\frac{\mu}{2}\left(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2\right)$ is convex-concave. V-WC-WC is a variant of WC-WC, please refer to (Lei et al., 2021). PL means the two-sided PL condition, which relaxes the convex-concavity requirement of the objective function (Yang et al., 2020) and is usually used in some nonconvex convergence literature to guarantee the linear convergence rate (Karimi et al., 2016; Yang et al., 2020). AGDA algorithm is variant of GDA with alternating updates of the primal-dual variables. c is a parameter in the step size, β is a parameter in Assumption 2 and $k := \beta/\mu$.