RETHINKING THE ROLE OF FRAMES FOR SE(3)-INVARIANT CRYSTAL STRUCTURE MODELING

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ABSTRACT

Crystal structure modeling using geometric graph neural networks is important in various machine learning applications in materials science. In these applications, capturing SE(3)-invariant geometric features in crystal structures is a fundamental requirement for these networks. One approach is to model with orientation-standardized structures through structure-aligned coordinate systems called 'frames.' However, unlike molecules, determining frames for crystal structures is not trivial due to their infinite and highly symmetric nature. In the search for effective frames for crystals, we point out that existing work assumes a statically fixed frame for each structure based solely on its structural information, regardless of the task under consideration. Here, we rethink the role of frames, *questioning whether such simplistic alignment with the structure is sufficient*, and propose the concept of *dynamic frames*. While accommodating the infinite and symmetric nature of crystals, these frames give each atom its own dynamic view of the structure, focusing only on those atoms actively interacting with it. We demonstrate this concept by utilizing the attention mechanism in a recent transformer-based crystal encoder, developing a new encoder architecture called CrystalFramer. Extensive comparisons with conventional frames and crystal encoders show the superior performance of the proposed method in various crystal property prediction tasks.

1 INTRODUCTION

Geometric graph neural networks (Xie & Grossman, 2018; Chen et al., 2019; Choudhary & DeCost, 2021; Chen & Ong, 2022; Lin et al., 2023), including transformer variants (Yan et al., 2022; 2024; Taniai et al., 2024), play a central role in machine learning (ML)-based structural modeling of materials. This technology provides a powerful alternative to conventional simulation methods, such as density functional theory (DFT), for high-throughput prediction of material properties. Furthermore, it also serves as the basis for various ML applications in materials science, such as material representation learning (Suzuki et al., 2022) and generation (Jiao et al., 2023).

A key requirement for these networks is the ability to capture essential features of materials embedded in their crystal structures. Crystal structures are periodic, infinitely repeating arrangements of atoms in 3D space, typically represented by minimum repeatable patterns called unit cells. Material properties, such as formation energy and bandgap, are invariant under rigid transformations (*i.e.*, rotations and translations) in crystal structures, as well as under variations in their unit cells. This fact leads to the so-called periodic SE(3) invariance (Yan et al., 2022) as an essential property for crystal encoders. Therefore, recent studies have explored various forms of richer yet invariant structural information beyond the simplest interatomic distances (Chen & Ong, 2022; Duval et al., 2023; Yan et al., 2024).

One approach, which has shown promising results for molecules (Puny et al., 2022), is the use of 'frames.' A frame is a coordinate system aligned equivariantly to a given structure to provide an orientation-standardized view of the structure (see Fig. 1, left). Frames allow arbitrary networks to directly exploit rich 3D structural features, including the relative positions between atoms and their directions, without imposing any architectural constraints. However, determining frames for crystals is more challenging than for molecules, primarily due to the infinite and symmetric nature of crystals.

In this work, we study a new family of frames for crystal structures in rethinking the role of frames. We hypothesize that *the essential role of frames is not merely to provide a structure-aligned coordinate system for a given structure, but rather to align the coordinate system with the interatomic interactions*

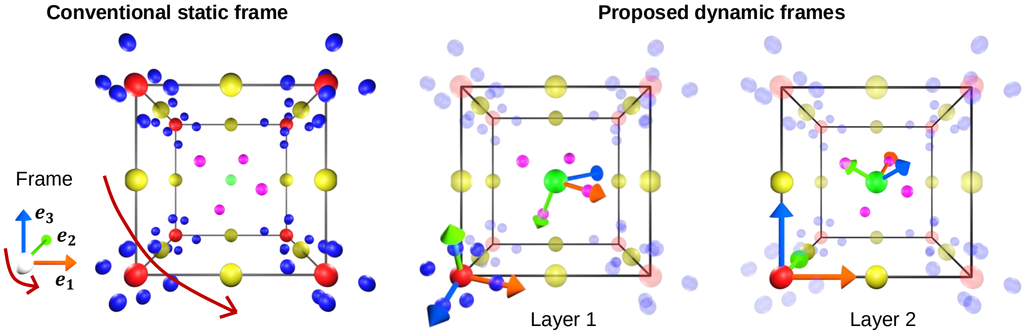

Figure 1: **Conventional static frame and proposed dynamic frames.** Conventional frames are determined statically to align with the structure, ensuring consistency under rotation and providing a canonical global representation of the structure. This is schematically illustrated by curved arrows. By contrast, the proposed dynamic frames are determined for each atom in each message-passing layer, by considering the local dynamic environment around that atom in that layer.

acting on the structure. Following this belief, we propose a novel concept of *dynamic frames*. These frames define local coordinate systems centered on individual atoms by dynamically accounting for the atoms actively engaged in learned interactions in each interatomic message-passing layer (Fig. 1, right). This concept challenges the conventional notion of 'static frames,' which are based on the premise of providing fixed views of structures (Puny et al., 2022). Thus, whether such a dynamic frame is effective or not is an unexplored non-trivial question, which we aim to answer.

To verify this concept, we develop several types of dynamic frames by utilizing the self-attention mechanism (Taniai et al., 2024) to quantify the interaction engagement. We perform extensive comparisons on datasets derived from the JARVIS, Materials Project (MP), and Open Quantum Materials Database (OQMD), and show that our method outperforms existing frame methods for crystals (Duval et al., 2023; Yan et al., 2024) and other state-of-the-art networks (Choudhary & DeCost, 2021; Chen & Ong, 2022; Yan et al., 2022; 2024; Lin et al., 2023; Taniai et al., 2024) for various crystal property prediction tasks. We will release our code upon acceptance.

2 PRELIMINARIES

2.1 CRYSTAL STRUCTURE

A crystal structure is described by its 3D unit cell slice, denoted as (A, P, L) following Yan et al. (2022). A unit cell is a parallelepipedal structure containing a finite number, say N, of atoms. The species (atomic numbers) and 3D Cartesian coordinates of these atoms are provided as $A = [a_i, a_2, ..., a_N] \in \mathbb{N}^{1\times N}$ and $P = [\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_N] \in \mathbb{R}^{3\times N}$. The parallelepipedal cell shape is given by three vectors: $L = [\boldsymbol{\ell}_1, \boldsymbol{\ell}_2, \boldsymbol{\ell}_3] \in \mathbb{R}^{3\times 3}$, called lattice vectors. By tiling the parallelepiped unit cell to fill 3D space, the species and positions of all the atoms in the crystal structure are determined as

$$\hat{A} = \{a_{i(\boldsymbol{n})} | a_{i(\boldsymbol{n})} = a_i, \boldsymbol{n} \in \mathbb{Z}^3, 1 \le i \le N\}, \tag{1}$$

$$\hat{P} = \{\boldsymbol{p}_{i(\boldsymbol{n})} | \boldsymbol{p}_{i(\boldsymbol{n})} = \boldsymbol{p}_i + L\boldsymbol{n}, \boldsymbol{n} \in \mathbb{Z}^3, 1 \le i \le N\}. \tag{2}$$

Following Taniai et al. (2024), we use i to denote the i-th atom in a unit cell, and $i(\boldsymbol{n})$ to denote its duplicate by a unit-cell translation: $L\boldsymbol{n} = n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 + n_3\boldsymbol{\ell}_3$. We use j and $j(\boldsymbol{n})$ similarly.

2.2 TRANSFORMERS FOR CRYSTAL STRUCTURES

Geometric graph neural networks are used as crystal encoders in various materials-related tasks. These encoders typically represent the state of a given crystal structure by a set of atom-wise abstract state features, $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_N] \in \mathbb{R}^{d\times N}$. These states are initially provided as atom embeddings, $X^{(0)} \leftarrow \text{AtomEmbedding}(A)$, which only symbolically represent atomic species. The encoders then evolve these states through interatomic message-passing layers, $X^{(t+1)} \leftarrow f^t(X^{(t)}, P, L)$, to eventually reflect the atomic states in the given structure appropriate for a target task. Since the seminal work of Xie & Grossman (2018) and Schütt et al. (2018), graph neural networks (GNNs) have long been the standard for crystal encoders until the advent of transformer-based networks by recent work (Yan et al., 2022; 2024; Taniai et al., 2024).

In particular, Taniai et al. (2024) have developed simple physics-informed formalism for crystal encoders using a self-attention mechanism. By imitating interatomic potential summations for energy calculations in physics simulations, they model the evolution of current state $\boldsymbol{x}$ using *infinitely connected distance-decay attention*. This attention mechanism models the interactions between each unit-cell atom i and all the infinitely repeating atoms $j(\boldsymbol{n})$ in the entire crystal structure as

$$\boldsymbol{x}'_i = \frac{1}{Z_i}\sum_{j=1}^{N}\sum_{\boldsymbol{n}\in\mathbb{Z}^3} \exp\left(\frac{\boldsymbol{q}_i^T\boldsymbol{k}_j}{\sqrt{d_K}} - \frac{\|\boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i\|^2}{2\sigma_i^2}\right)\left(\boldsymbol{v}_j + \boldsymbol{\psi}_{ij(\boldsymbol{n})}\right). \tag{3}$$

Here, query $\boldsymbol{q}$, key $\boldsymbol{k}$, and value $\boldsymbol{v}$ are linear projections of current state $\boldsymbol{x}$. Scalar σ_i is a tail-length variable of Gaussian distance-decay attention adaptively derived from $\boldsymbol{x}_i$. Vector $\boldsymbol{\psi}_{ij(\boldsymbol{n})}$ is a geometric position embedding that encodes the distance, $\|\boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i\|$, between atoms i and $j(\boldsymbol{n})$. Scalar $Z_i = \sum_j\sum_{\boldsymbol{n}} \exp(\boldsymbol{q}_i^T\boldsymbol{k}_j/\sqrt{d_K} - \|\boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i\|^2/2\sigma_i^2)$ is the normalizer of softmax attention weights. The exponential distance-decay factor in Eq. 3 provably ensures its rapid convergence within a finite range of cell shifts $\boldsymbol{n}$ (Taniai et al., 2024).

Their method, called Crystalformer, enjoys a good balance between a strong physically-motivated inductive bias and the flexibility of abstract feature representations, and is considered the state of the art with other GNN-based (Lin et al., 2023) and transformer-based (Yan et al., 2024) methods.

We utilize Crystalformer as a baseline in this work. This is because its architecture closely follows the standard softmax attention (Vaswani et al., 2017) and is suitable to demonstrate our concept of dynamic frames, while other existing transformers (Yan et al., 2022; 2024) use distinct channel-wise sigmoid attention. We discuss this more in Sec. 6. Our method in Sec. 3 will extend position embedding $\boldsymbol{\psi}_{ij(\boldsymbol{n})}$ in Eq. 3 to incorporate richer yet invariant information than distance $\|\boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i\|$.

2.3 FRAMES FOR SE(3)-INVARIANT STRUCTURAL MODELING

Frame averaging. Puny et al. (2022) have introduced Frame Averaging (FA) as a general framework to adapt networks to become invariant (or equivariant) to certain symmetries of the input data. Although FA is originally explained by group representation theory, we provide its high-level review specifically focused on SE(3)-invariant modeling of 3D point clouds. Given a point cloud as P, FA computes a frame, $F \in \mathcal{F}(P)$, as a coordinate system inherent to and aligned with P (Fig. 1, left). For example, $\mathcal{F}$ is principal component analysis (PCA) applied to P. Each frame F thus provides a geometric transformation that maps P to a canonical, rotation-invariant representation as FP. However, $\mathcal{F}(P)$ may not uniquely provide a single frame due to algorithmic ambiguities in $\mathcal{F}$ or symmetries in P. Even in such cases, FA allows us to derive rotation-invariant (*i.e.*, SO(3)-invariant) networks $\bar{f}_{\mathcal{F}}$ from arbitrary networks f, by averaging f's outputs over all possible finite frames as

$$\bar{f}_{\mathcal{F}}(X, P) = \frac{1}{|\mathcal{F}(P)|}\sum_{F\in\mathcal{F}(P)} f(X, FP). \tag{4}$$

The translation invariance is further attained by using relative positions (*e.g.*, $F\boldsymbol{p}_j - F\boldsymbol{p}_i$) in f, bringing the SE(3) invariance to $\bar{f}_{\mathcal{F}}$. FA can powerfully adapt arbitrary networks to be SE(3) invariant without constraining the architectural design. However, it hinders efficiency as the computation increases with the number of possible frames. Stochastic FA by Duval et al. (2023) mitigates this issue by randomly selecting a single frame from $\mathcal{F}(S)$ during training, enforcing networks f to learn the invariance to frame variations and approximately achieving the SE(3) invariance.

PCA frames. Puny et al. (2022) originally applied FA for molecules using PCA-based frames, and Duval et al. (2023) later extended it for crystals by simply treating unit cell structures P as finite-sized point clouds. These PCA frames compute three orthogonal eigenvectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the covariance matrix of P, corresponding to eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$, as the frame axes: $F = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]^T$. Because of the sign ambiguity of the eigenvectors, PCA produces eight frames for O(3)/E(3) invariance and four frames for SO(3)/SE(3) invariance with the restriction of $\det(F) = 1$.

Although PCA is well-established, it suffers from eigenvalue degeneration for highly symmetric data, such as crystal structures. For example, PCA for cubes produces identity covariance matrices up to a constant scale, whose eigenvectors are arbitrary vectors $\boldsymbol{e} \in \mathbb{R}^3$. The crystal frame construction by Duval et al. (2023) is thus vulnerable to this degeneration issue and, moreover, sensitive to unit-cell variations of the same crystal structure.

Lattice frames. Yan et al. (2024) have proposed frames based on the lattice vectors of crystals, as similar to *reduced cells* (*i.e.*, uniquely determined minimum cells) (Niggli, 1928). Specifically, their method selects a lattice point, $\boldsymbol{e} = n_1\boldsymbol{\ell}_1 + n_2\boldsymbol{\ell}_2 + n_3\boldsymbol{\ell}_3$, with the minimum non-zero norm $\|\boldsymbol{e}\|_2$ as first axis $\boldsymbol{e}_1$, and selects the second and third smallest ones as axes $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ while ensuring $\mathrm{rank}(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is full. The signs of these axes are adjusted so that the angles between $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and between $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ become acute and the coordinate system is right-handed (*i.e.*, $\det(F) > 0$).

Notice that these existing frame methods for crystals, specifically PCA and lattice frames, all provide a statically fixed frame for each crystal structure. Also, both rely on unit cell representations (either points P or lattice vectors L), which are rather artificially-introduced crystal descriptions that may not necessarily reflect the physical properties of materials (see Appendix A for more discussion). These observations motivate us to propose the concept of dynamic frames, as we discuss next.

3 DYNAMIC FRAMES

In the search for effective frames for crystals, we challenge the conventional notion of frames, which implicitly follows the simple premise of representing structures in a canonical manner (Puny et al., 2022; Duval et al., 2023; Yan et al., 2024). Let us reconsider how frames work in GNNs, whose interatomic message-passing layers are assumed to include the following general operation:

$$\boldsymbol{x}'_i = \sum_{j=1}^{N} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} w_{ij(\boldsymbol{n})} \boldsymbol{f}_{i \leftarrow j(\boldsymbol{n})}(X, \hat{P}). \tag{5}$$

This equation describes that state $\boldsymbol{x}_i$ of each unit-cell atom i is evolved through abstract influences or *messages*, $f_{i \leftarrow j(\boldsymbol{n})}$, from atoms $j(\boldsymbol{n})$ in the crystal structure, with scaling weights $w_{ij(\boldsymbol{n})}$. In standard GNNs (Xie & Grossman, 2018), these weights are pre-defined as neighborhood graphs with a cut-off radius. In recent transformer architectures, the weights are determined dynamically via self-attention, with (Yan et al., 2022; 2024) or without (Taniai et al., 2024) relying on an explicit cut-off radius.

The role of frames in Eq. 5 is to offer more informative invariant edge features than distances through frame-projected coordinates $F\hat{P}$ in the design of $\boldsymbol{f}_{i \leftarrow j(\boldsymbol{n})}$. From this perspective, constructing a frame shared for the state updates of all atoms i, as done in conventional methods, is not preferable, because the frame construction can be influenced even by atoms $j(\boldsymbol{n})$ with zero weights in Eq. 5. In other words, particularly when the state of atom i is updated in Eq. 5, this atom has its own partial and local view of the entire crystal structure, $\hat{P}$, with weights $w_{ij(\boldsymbol{n})}$ acting as a mask on the structure.

This interpretation leads to a new concept of dynamic frames. That is, we define frames locally for each atom i to align them with its interatomic interactions acting dynamically on the structure, instead of directly aligning them with the structure itself. We denote these dynamic atom-wise frames as F_i. Each F_i is determined based on the masked view of structure $\hat{P}$ with weights $w_{ij(\boldsymbol{n})}$, by emphasizing or de-emphasizing the presence of atoms $j(\boldsymbol{n})$ with larger or smaller weights. Thus, these frames F_i change dynamically depending on target atoms i and also on the layers in a GNN, as shown in Fig. 1.

We hypothesize that dynamically adapting frames for each atom i in each message-passing layer (Eq. 5) provides better invariant edge features via projected coordinates $F_i\hat{P}$. We also point out that these frames are defined with the entire crystal structure, $\hat{P}$, reconstructed from (P, L). This fact highlights an advantage of our frames being invariant to unit cell variations of the same structure.

3.1 FRAME DEFINITIONS

We now present several instances of this new family of frames. These frames F_i are constructed for each target atom i in each message-passing layer (Eq. 5), by using coordinates $\hat{P}$ and weights $w_{ij(\boldsymbol{n})}$ of atoms $j(\boldsymbol{n})$ in the structure. We typically assume $w_{ij(\boldsymbol{n})} \geq 0$, but we can use real-valued weights, for example, by using their absolute values for frame construction. For brevity, we denote $r_{ij(\boldsymbol{n})} = \|\boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i\|_2$ and $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})} = (\boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i)/r_{ij(\boldsymbol{n})}$, both derived from $\hat{P}$.

Weighted PCA frames. The first instance of dynamic frames extends the original PCA frames (Puny et al., 2022; Duval et al., 2023). For each target atom i in each message-passing layer, we compute a 3×3 weighted covariance matrix, $\Sigma_i = \sum_j \sum_{\boldsymbol{n}} w_{ij(\boldsymbol{n})} \bar{\boldsymbol{r}}_{ij(\boldsymbol{n})} \bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}^T$, and computes its orthogonal eigenvectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ as the axes of a frame: $F_i = [\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]^T$. For the sign ambiguity of eigenvectors, we adopt the stochastic FA (Duval et al., 2023) and generate a single frame by randomly flipping the signs of these vectors while ensuring $\det(F_i) = 1$. However, there remains another possible ambiguity in this weighted PCA scheme owing to eigenvalue degeneration by symmetries[1].

Max frames. To avoid the degeneration of PCA, we also propose to directly select atoms $j(\boldsymbol{n})$ with large weights $w_{ij(\boldsymbol{n})}$ and use their directions $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$ to determine axes $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of F_i. Specifically, we select first axis $\boldsymbol{e}_1$ as $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$ with maximum weight $w_{ij(\boldsymbol{n})}$. For the second axis, we find $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$ with maximum adjusted-weight $(1 - |\boldsymbol{e}_1 \cdot \boldsymbol{r}_{ij(\boldsymbol{n})}|)w_{ij(\boldsymbol{n})}$, which avoids selecting a direction parallel to $\boldsymbol{e}_1$. The selected vector, denoted as $\bar{\boldsymbol{r}}_2$, is further orthogonalized by the Gram-Schmidt method as $\hat{\boldsymbol{e}}_2 \leftarrow \bar{\boldsymbol{r}}_2 - (\boldsymbol{e}_1 \cdot \bar{\boldsymbol{r}}_2)\boldsymbol{e}_1$, and normalized to a unit vector as $\boldsymbol{e}_2 \leftarrow \hat{\boldsymbol{e}}_2/\|\hat{\boldsymbol{e}}_2\|_2$. Finally, third axis $\boldsymbol{e}_3$ is simply obtained as $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$, which ensures the orthogonality and $\det(F_i) = 1$. In this process, multiple atoms may have the same weight. For this ambiguity, we add small perturbation noise to each weight $w_{ij(\boldsymbol{n})}$, resulting in randomly selecting a single frame from possible ones. This perturbation scheme is considered a type of stochastic FA (Duval et al., 2023) outlined in Sec. 2.3.

Since these frame construction processes are not stably differentiable, we omit the computation of the gradients from frames F_i to weights $w_{ij(\boldsymbol{n})}$ during training[2]. Still, weights $w_{ij(\boldsymbol{n})}$ receive gradients from $\boldsymbol{x}'$ in Eq. 5 to learn their main function of allowing or blocking messages $\boldsymbol{f}_{i \leftarrow j(\boldsymbol{n})}$ from $j(\boldsymbol{n})$ to i. Therefore, we can successfully train a network that includes the dynamic frame construction without using these frame gradients.

3.2 CRYSTALFRAMER ARCHITECTURE

We demonstrate the proposed concept using Crystalformer (Taniai et al., 2024) as the baseline architecture, as mentioned in Sec. 2.2, and consequently develop a new architecture called CrystalFramer (Fig. 2). We here regard Eq. 3 as Eq. 5. Thus, we regard the softmax self-attention weights (*i.e.*, exponential weights normalized by Z_i in Eq. 3) as dynamic scaling weights $w_{ij(\boldsymbol{n})}$ in each message-passing layer (Eq. 5). Likewise, we regard the position-augmented value vectors, $\boldsymbol{v}_j + \boldsymbol{\psi}_{ij(\boldsymbol{n})}$, as messages $\boldsymbol{f}_{i \leftarrow j(\boldsymbol{n})}$. In the process of updating each state $\boldsymbol{x}_i$ using Eq. 3, we first compute the attention weights as $w_{ij(\boldsymbol{n})}$. Then, we dynamically construct a local frame as matrix F_i, by following one of the procedures outlined in Sec. 3.1. Finally, we compute $\boldsymbol{\psi}_{ij(\boldsymbol{n})}$ using F_i and perform Eq. 3. The following explains how to derive invariant edge features $\boldsymbol{\psi}_{ij(\boldsymbol{n})}$ given frame F_i.

Invariant edge features using a dynamic frame. For invariant edge feature $\psi_{ij(\boldsymbol{n})}$, Crystalformer originally uses linearly projected Gaussian basis functions (GBFs) encoding distance $r_{ij(\boldsymbol{n})}$. Specifically, GBFs are provided as mapping $\boldsymbol{b}(x) = [b_1, b_2, \cdots, b_D]^T$ from scalar x to a vector of pre-defined dimension D, whose k-th component is computed as a Gaussian as

$$b_k(x; \mu_k, \sigma_k) = \exp\left(-(x - \mu_k)^2 / 2\sigma_k^2\right). \tag{6}$$

Here, μ_k and σ_k are pre-defined as $\mu_k = \mu_{\text{min}} + (k-1)(\mu_{\text{max}} - \mu_{\text{min}})/(D-1)$ and $\sigma_k = s(\mu_{\text{max}} - \mu_{\text{min}})/(D-1)$ with four hyperparameters $\{\mu_{\text{max}}, \mu_{\text{min}}, s, D\}$. Intuitively, $\boldsymbol{b}(x)$ encodes scalar x into a soft one-hot vector, using D Gaussians uniformly distributed between μ_{min} and μ_{max}. Widths σ_k of these Gaussians are given proportional to the interval distance, controlled by scaling factor s.

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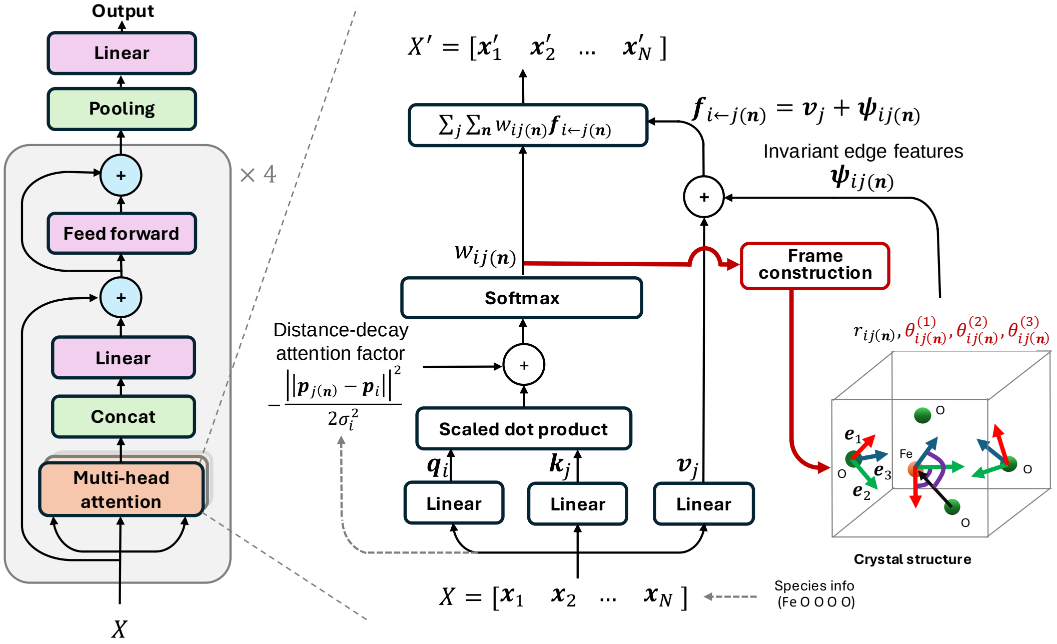
[1] We confirmed that covariance matrices Σ_i computed with a pretrained Crystalformer model suffered from eigenvalue degeneration at two degrees in about 10% of cases and at three degrees in about 1% of cases. These cases cause rotation ambiguities for two or three (all) axes of F_i. To mitigate this issue, we add small perturbation noise to $w_{ij(\boldsymbol{n})}$ in Σ_i, which stochastically breaks the symmetries in the structural data and empirically helps to compute non-degenerate eigenvalues and eigenvectors. This scheme is considered a type of stochastic FA.

[2] The gradients of the eigenvectors in PCA become numerically unstable when the eigenvalues are degenerate, as the gradients depend on the computation of $1/(\lambda_i - \lambda_j)$ for $i \neq j$. Also, the max-frame procedure is not differentiable due to the use of argmax operations. Although we tried approximating the gradients of argmax, for example, by using a straight-through estimator technique or temperature annealing of softmax, simply ignoring the frame gradients gave the best results.


Output
Linear
Pooling
+
Feed forward
+
Linear
Concat
Multi-head attention
× 4
X
$X' = [x'_1 \; x'_2 \; \dots \; x'_N]$
$\sum_j \sum_n w_{ij(n)} f_{i\leftarrow j(n)}$
$f_{i\leftarrow j(n)} = v_j + \psi_{ij(n)}$
Invariant edge features
$\psi_{ij(n)}$
+
$w_{ij(n)}$
Frame construction
Softmax
Distance-decay attention factor
$-\frac{\|p_{j(n)} - p_i\|^2}{2\sigma_i^2}$
+
Scaled dot product
q_i
k_j
v_j
Linear
Linear
Linear
$r_{ij(n)}, \theta^{(1)}_{ij(n)}, \theta^{(2)}_{ij(n)}, \theta^{(3)}_{ij(n)}$
O
e_1
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Crystal structure
$X = [x_1 \; x_2 \; \dots \; x_N]$
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Figure 2: **CrystalFramer architecture.** Dynamic frame construction and frame-based invariant edge features (highlighted in red) are introduced to a transformer for crystals (Taniai et al., 2024).

We retain their distance-based edge feature and further add frame-based edge features to $\psi_{ij(\boldsymbol{n})}$. Specifically, following existing work (Yan et al., 2024), we invariantly represent direction vector $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$ by projecting onto the frame coordinate system, as $\boldsymbol{\theta}_{ij(\boldsymbol{n})} = F_i \bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$. Its k-th component is calculated as $\boldsymbol{e}_k \cdot \bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$, the cosine value of the angle between k-th frame axis $\boldsymbol{e}_k$ and direction $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$. We convert each component to a vector via GBFs. By combining the distance-based and three angle-based features via linear projections, we obtain our geometric relative position encoding:

$$\psi_{ij(\boldsymbol{n})} = W_0 \boldsymbol{b}_{\text{dist}}\left(r_{ij(\boldsymbol{n})}\right) + \sum_{k=1,2,3} W_k \boldsymbol{b}_{\text{angl}}\left(\theta^{(k)}_{ij(\boldsymbol{n})}\right). \tag{7}$$

This $\psi_{ij(\boldsymbol{n})}$ as a whole essentially encodes the 3D relative position vector: $\boldsymbol{r}_{ij(\boldsymbol{n})} = \boldsymbol{p}_{j(\boldsymbol{n})} - \boldsymbol{p}_i$. Furthermore, its angle part can be interpreted to encode the absolute deviations of $\boldsymbol{r}_{ij(\boldsymbol{n})}$ in angle from the three primary directions of interatomic interactions (*i.e.*, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$) around target atom i. Here, four weight matrices $\{W_0, W_1, W_2, W_3\}$ are trainable parameters provided per layer. We also use two types of GBFs ($\boldsymbol{b}_{\text{dist}}$ and $\boldsymbol{b}_{\text{angl}}$) with different hyperparameters for the distance and angles. Specifically, we set $\{\mu_{\min}, \mu_{\max}, s, D\}$ to $\{\frac{14.0}{64}\text{Å}, 14.0\text{Å}, 1.0, 64\}$ for $\boldsymbol{b}_{\text{dist}}$, as suggested by Taniai et al. (2024). We also set to $\{-1.0, 1.0, 4.0, 64\}$ for $\boldsymbol{b}_{\text{angl}}$, using the range $[-1.0, 1.0]$ of cosine values and relatively larger width-scale s that empirically works better for angles. Note that if $\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}$ is undefined due to zero division (*i.e.*, $j(\boldsymbol{n}) = i$), we provide $\boldsymbol{b}_{\text{angl}}(\bar{\boldsymbol{r}}_{ij(\boldsymbol{n})}) = \boldsymbol{0}$.

Overall architecture. The proposed network precisely follows the Crystalformer architecture (Taniai et al., 2024) as shown in Fig. 2, except for the newly introduced frame construction (Sec. 3.1) and angular edge features (Eq. 7) highlighted in the figure. As we will see in Sec. 5, these simple extensions bring drastic performance improvements to the baseline method. Below we summarize the important network design aspects. The overall architecture consists of the input atom-embedding layer and the stack of four self-attention blocks, followed by the global mean pooling and the final feed-forward network with two linear layers. The self-attention blocks adopt the normalization-free architecture (left part of Fig. 2) by Huang et al. (2020) for better training stability. The infinite summation in self-attention (Eq. 3) is computed convergently and efficiently, by adaptively determining the range of unit-cell shifts $\boldsymbol{n}$ to sufficiently cover the neighbor radius of $3.5\sigma_i$ based on dynamic Gaussian tail-length σ_i. We also employ multi-head self-attention as in the original transformer (Vaswani et al., 2017) using eight heads. So we construct frames per unit-cell atom, per head, and per layer. For further architectural details, please refer to the original work (Taniai et al., 2024).

4 RELATED WORK

The notion of invariant structural modeling widely covers various invariance properties. The most elementary one is the invariance to the permutation of data-point indices i in structural data (*i.e.*, A and P in Sec. 2.1), which was first addressed by PointNet (Qi et al., 2017) and DeepSets (Zaheer et al., 2017) and is now inherited by GNNs and transformers. The ML community has then shifted its focus to invariance to geometric transformations, such as rotations with or without translations (*i.e.*, so-called SO(3)/O(3) or SE(3)/E(3) invariance). In particular, the periodicity of crystals introduces more complex invariance notions, such as periodic SE(3) invariance (Yan et al., 2022), which additionally require invariance to unit-cell variations of the same crystal structure. These geometric invariance properties have been studied in three main approaches using 1) invariant features, 2) equivariant features, and 3) frames. We briefly review them below, focusing primarily on crystal structures.

Invariant features. The most straightforward approach is to rely entirely on naturally invariant geometric quantities, such as the lengths of relative position vectors, throughout a model (Xie & Grossman, 2018; Chen et al., 2019; Ying et al., 2021; Yan et al., 2022; Taniai et al., 2024). However, such distance-based GNNs and transformers have the limited expressibility (Pozdnyakov & Ceriotti, 2022). Thus, recent studies have explored more advanced geometric features, such as the angles between triplets using 3-body interactions (Park & Wolverton, 2020; Choudhary & DeCost, 2021; Chen & Ong, 2022) at the cost of increased computational complexity. More recently, PotNet by Lin et al. (2023) used the sum of pre-defined interatomic scalar potentials as more physically-informed invariant edge features than distances.

Equivariant features. The so-called equivariant networks, based on group representation theory, make another active research area in 3D structural modeling and include invariant networks as special cases. While we refer readers to recent surveys (Gerken et al., 2023; Duval et al., 2024; Han et al., 2024) for more comprehensive reviews, the initial approach specifically using GNNs for 3D point clouds and atomic systems was proposed by Thomas et al. (2018). Subsequently, this approach has been extended, for example, to introduce better nonlinearity forms (Batzner et al., 2022; Brandstetter et al., 2022) or attention mechanisms (Fuchs et al., 2020), or to improve efficiency (Liao & Smidt, 2023; Liao et al., 2024) in molecular structure modeling. Essentially, these methods use spherical harmonic representations of unit direction vectors $\bar{\boldsymbol{r}}_{ij}$ as rotation-equivariant edge features, and then equivariantly transform them through specially designed networks. These equivariant features form type-L vectors, whose type-1 features can express 3D equivariant vectors such as forces, while type-0 can be used for invariant prediction. However, these equivariant networks are constrained by restricted nonlinearity forms and the increasing computational complexity to model higher frequency components. Because of these constraints, the use of equivariant networks for crystals rather than molecules is relatively limited. For example, eComFormer (Yan et al., 2024) has exploited equivariant features in part within each message-passing block for invariant crystal property prediction.

Frames. As explained in Sec. 2.3, Puny et al. (2022) introduced the FA and applied the PCA frames for molecules. Duval et al. (2023) further extended it in two ways, by proposing the stochastic FA to improve the efficiency and the PCA frames for crystals by simply treating their unit cell structures P as finite point clouds. Cheng et al. (2021) used plane waves in crystal structures as invariant positional features, which implicitly use reciprocal lattice vectors as a frame. Similarly, Yan et al. (2024) proposed iComFormer using transformed lattice vectors with reduced ambiguities as a frame.

Our work is in the line of research on invariant networks using frames (Puny et al., 2022; Duval et al., 2023; Cheng et al., 2021; Yan et al., 2024), where we provide a new perspective on the previous notion of frames with dynamic frames. These dynamic frames are incorporated into a simple distance-based transformer model for crystals (Taniai et al., 2024) to enhance its expressive power.

5 EXPERIMENTS

To validate the effectiveness of the proposed dynamic frames, we conducted extensive experiments on crystal property prediction, comparing them with conventional PCA frames (Duval et al., 2023), lattice frames (Yan et al., 2024), and other state-of-the-art architectures for property prediction.

Datasets. We use three datasets: the JARVIS (55,723 materials), MP (69,239 materials), and OQMD (817,636 materials), using snapshots available through a Python package (jarvis-tools). These datasets provide several material properties, such as formation energy and bandgap, simulated by DFT calculations. For further dataset descriptions, see Appendix B. Choudhary & DeCost (2021) and Yan et al. (2022) made great efforts to evaluate many methods on the JARVIS and MP datasets using consistent data splits. Following these and later studies (Lin et al., 2023; Yan et al., 2024; Taniai et al., 2024), we use the same data splits and cite their reported scores to reduce the computational burden. Additionally, we use the much larger-scale OQMD dataset to evaluate the scalability.

Training settings. To assess the pure effects of introducing the frames, we precisely follow the training settings of the baseline method, Crystalformer (Taniai et al., 2024). The only change is the number of epochs. We have increased it to account for the increased complexity of our edge feature design (*i.e.*, our method takes longer to converge, but reduces validation losses more rapidly.). Specifically for the JARVIS dataset, we train our model from scratch by optimizing the mean absolute loss function using Adam (Kingma & Ba, 2015) for a total of 2000 epochs, while enabling the frames from the beginning. A summary of detailed training settings, such as the number of epochs, batch size, and learning rate for the three datasets, can be found in Appendix C.

5.1 CRYSTAL PROPERTY PREDICTION

Tables 1 and 2 extensively compare the mean absolute errors of the proposed and existing methods for the JARVIS (5 tasks) and MP (4 tasks) datasets. Overall, our method with max frames achieves the best results in most tasks, significantly boosting the performance of the baseline Crystalformer model. Such improvements never fade even when feeding the much larger OQMD dataset, as shown in Table 3. Our weighted PCA frame method shows relatively limited improvements, which we will discuss in more detail in Appendix D. Nevertheless, it outperforms its conventional counterpart using PCA frames. These results successfully validate the effectiveness of our concept of dynamic frames. It is also worth noting that the current state-of-the-art, ComFormer, uses finely-tuned hyperparameters (*e.g.*, learning rate, loss function, number of layers, graph structure) for each individual task, whereas we simply adjust the number of epochs and batch size for each dataset.

5.2 EFFICIENCY COMPARISON

Table 4 compares the model efficiency of several top-performing architectures. Notably, despite the superior performance of the proposed method, it requires significantly fewer parameters than PotNet, Matformer, and iComFormer. Compared to Crystalformer, it introduces a small overhead of about 100K parameters owing to projection matrices $\{W_1, W_2, W_3\}$ in Eq. 7. Given the performance gains shown in Tables 1–3, this high cost-performance ratio also highlights the effectiveness of our feature design using dynamic frames. Meanwhile, the training and test times are more than double compared to Crystalformer, mainly due to the increased computation cost of $\sum_{\boldsymbol{n}} w_{ij(\boldsymbol{n})}\psi_{ij(\boldsymbol{n})}$. As noted by Taniai et al. (2024), this part is the main bottleneck in Crystalformer and also in our model. Since our current configuration uses largely overlapping GBFs ($s = 4.0$) for angular features, pruning GBFs (*i.e.*, reducing D) could improve runtime. Pre-training without frames to efficiently learn attention weights first may also accelerate overall training. Nonetheless, the test time is still faster than PotNet, Matformer, and iComFormer, which are hindered by relatively heavy data preprocessing. Scalability for large structures is further discussed in Appendix E.

6 DISCUSSION AND LIMITATIONS

Visual analysis. Figure 3 qualitatively compares four types of frames generated for a test material (JVASP-30609) in the JARVIS formation energy prediction task. While the PCA (Duval et al., 2023) and lattice (Yan et al., 2024) frames are static, the proposed weighted PCA and max frames exhibit dynamic behavior based on learned attention weights. A closer inspection of the input crystal structure reveals two distinct patterns: octahedra with a green central atom (magnesium) surrounded by blue atoms (fluorine), and tetrahedra with a red central atom (tin) surrounded by blue atoms (magnesium). These local structures are common in crystal structures and are sometimes distorted, as in this case. With these patterns in mind, the max frame in the third layer actually captures the blue atoms of an octahedron centered on the target atom. In the fourth layer, the attention shifts to relatively distant red atoms, whose tetrahedral structures share blue atoms with the central octahedron. The ability to capture these local structures and measure distortions via relative positions may contribute to the

Table 1: **Property prediction results on the JARVIS dataset.** Accuracies are in mean absolute error. The sizes of training, validation, and test splits are listed under each property name. **Bold** indicates the best results, underline the second best.

	Form. energy	Total energy	Bandgap (OPT)	Bandgap (MBJ)	E hull
	44578 / 5572 / 5572	44578 / 5572 / 5572	44578 / 5572 / 5572	14537 / 1817 / 1817	44296 / 5537 / 5537
Method	eV/atom	eV/atom	eV	eV	eV
CGCNN (Xie & Grossman, 2018)	0.063	0.078	0.20	0.41	0.17
SchNet (Schütt et al., 2018)	0.045	0.047	0.19	0.43	0.14
MEGNet (Chen et al., 2019)	0.047	0.058	0.145	0.34	0.084
GATGNN (Louis et al., 2020)	0.047	0.056	0.17	0.51	0.12
M3GNet Chen & Ong (2022)	0.039	0.041	0.145	0.362	0.095
ALIGNN (Choudhary & DeCost, 2021)	0.0331	0.037	0.142	0.31	0.076
Matformer (Yan et al., 2022)	0.0325	0.035	0.137	0.30	0.064
PotNet (Lin et al., 2023)	0.0294	0.032	0.127	0.27	0.055
eComFormer (Yan et al., 2024)	0.0284	0.032	0.124	0.28	**0.044**
iComFormer (Yan et al., 2024)	0.0272	0.0288	0.122	0.26	0.047
Crystalformer (Taniai et al., 2024)	0.0306	0.0320	0.128	0.274	0.0463
— w/ PCA frames (Duval et al., 2023)	0.0325	0.0334	0.144	0.292	0.0568
— w/ lattice frames (Yan et al., 2024)	0.0302	0.0323	0.125	0.274	0.0531
— w/ weighted PCA frames (proposed)	0.0287	0.0305	0.126	0.279	**0.0444**
— w/ max frames (proposed)	**0.0263**	**0.0279**	**0.117**	**0.242**	0.0471

Table 2: **Property prediction results on the MP dataset.**

	Formation energy	Bandgap	Bulk modulus	Shear modulus
	60000 / 5000 / 4239	60000 / 5000 / 4239	4664 / 393 / 393	4664 / 392 / 393
Method	eV/atom	eV	log(GPa)	log(GPa)
CGCNN	0.031	0.292	0.047	0.077
SchNet	0.033	0.345	0.066	0.099
MEGNet	0.030	0.307	0.060	0.099
GATGNN	0.033	0.280	0.045	0.075
M3GNet	0.024	0.247	0.050	0.087
ALIGNN	0.022	0.218	0.051	0.078
Matformer	0.021	0.211	0.043	0.073
PotNet	0.0188	0.204	0.040	0.065
eComFormer	0.0182	0.202	0.0417	0.0729
iComFormer	0.0183	0.193	0.0380	**0.0637**
Crystalformer	0.0186	0.198	0.0377	0.0689
— w/ PCA frames	0.0197	0.217	0.0424	0.0719
— w/ lattice frames	0.0194	0.212	0.0389	0.0720
— w/ weighted PCA frames (proposed)	0.0197	0.214	0.0423	0.0715
— w/ max frames (proposed)	**0.0172**	**0.185**	**0.0338**	0.0677

Table 3: **Property prediction results on the OQMD dataset.**

Method	Form. energy (eV/atom) 654108 / 81763 / 81763	Bandgap (eV) 653388 / 81673 / 81673	E hull (eV/atom) 654108 / 81763 / 81763
Crystalformer	0.02115	0.06028	0.06759
CrystalFramer (max frames)	**0.01871**	**0.05805**	**0.06607**

Table 4: **Efficiency comparison.** Per-epoch training time includes validation, and per-material test time includes preprocessing, such as graph construction. The runtimes are evaluated for the formation energy prediction in the JARVIS dataset using a single NVIDIA A6000 GPU with 48GB VRAM.

Model	Arch. type	Time/epoch	Test/mater.	#Params.	#Params./block
PotNet	GNN	43 s	313 ms	1.8 M	527 K
Matformer	Transformer	60 s	20.4 ms	2.9 M	544 K
iComFormer	Transformer	59 s	54.8 ms	5.0 M	855 K
Crystalformer	Transformer	32 s	6.6 ms	853 K	206 K
CrystalFramer	Transformer	74 s	16.8 ms	952 K	231 K

high performance observed. A more detailed analysis, including comparative discussions on the max and weighted PCA frames, additional visualizations, and an examination of frame evolution during training, is provided in Appendix D.

Baseline choice. This study used Crystalformer (Taniai et al., 2024) for demonstration, since its architecture using the standard multi-head softmax attention is suitable for dynamic frames. Other existing transformers for crystals (Yan et al., 2022; 2024) use distinct channel-wise sigmoid attention,

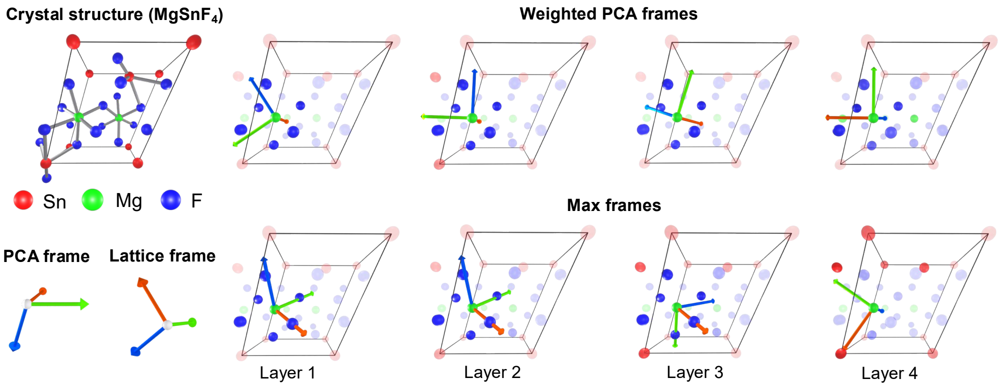

Figure 3: **Frame visualizations.** Conventional PCA and lattice frames provide a global coordinate system based solely on the structure. The proposed dynamic frames extract different structural information for each atom and layer using dynamic attention weights, shown as varying transparency.

similar to maximally multi-headed attention. Since we compute a frame and angular features per atom, per head, and per layer, such channel-wise attention is not preferable. However, we consider the Crystalformer model using Eq. 3 to be simply the original fully-connected self-attention (Vaswani et al., 2017), $\boldsymbol{x}'_i = Z_i^{-1}\sum_j \exp(\boldsymbol{q}_i^T\boldsymbol{v}_j/\sqrt{d_K}+\phi_{ij})(\boldsymbol{v}_j+\boldsymbol{\psi}_{ij})$, with two straightforward extensions: 1) relative position encoding (ϕ_{ij} and $\boldsymbol{\psi}_{ij}$) by Shaw et al. (2018) and 2) duplication of each atom j as $j(\boldsymbol{n})$ using $\sum_{\boldsymbol{n}}$ to account for crystal periodicity. Because of the widely proven practicability and versatility of the original transformer architecture in many fields (Lin et al., 2021), our demonstration can be thought to provide the basis for transformer-based crystal encoders using dynamic frames.

Equivariant prediction. While this study focuses on SE(3) invariance, Puny et al. (2022) applied FA also to predict equivariant quantities, such as force vectors, by applying inverse mapping F^{-1} on $f(X, FP)$ in Eq. 4 before the averaging. In our case, one potential equivariant extension would thus invariantly output atom-wise geometric quantities $\boldsymbol{u}_i$ from $\boldsymbol{x}'_i$ (*e.g.*, via $\boldsymbol{u}_i = W\boldsymbol{x}'_i$) and inversely map them as $F_i^{-1}\boldsymbol{u}_i$. Another potential extension, similar to recent work (Shi et al., 2023), would equivariantly tie the outputs to the input structure, for example, by $\boldsymbol{u}_i = \sum_j \sum_{\boldsymbol{n}} w_{ij(\boldsymbol{n})}\boldsymbol{r}_{ij(\boldsymbol{n})}$. These equivariant extensions will enable force and relaxed structure prediction (Chanussot et al., 2021; Tran et al., 2023; Bihani et al., 2024), which are crucial for surface property analysis. Further investigation and detailed analysis of these equivariant extensions are left as future work.

Application to molecules. Transformers for molecular structures have been developed (Ying et al., 2021; Wang et al., 2023; Shi et al., 2023; Liu et al., 2024), and our dynamic frames could also be applied to them. However, crystal and molecular structures have very different characteristics. In particular, molecular structures sometimes have so few atoms that they locally take on a low-dimensional structure and are unlikely to form an effective frame. Extending our method to molecules is another interesting future direction of this research.

7 CONCLUSION

In this study, we revisited the challenge of determining effective frames for the SE(3)-invariant modeling of crystal structures. We proposed a new concept of dynamic frames based on the strengths of interatomic interactions, advocating that frames should consider the local dynamic environment around each atom rather than the static global structure. We integrated these frames into an existing transformer-based network for crystal property prediction (Taniai et al., 2024) and conducted comparative evaluations with conventional frame construction methods (Duval et al., 2023; Yan et al., 2024) and other state-of-the-art networks (Choudhary & DeCost, 2021; Yan et al., 2022; Lin et al., 2023). The results confirmed the hypothesis, demonstrating the superior performance of the proposed method. While the demonstration was limited to crystal structures, the underlying principle of using the strengths of interactions to determine frames has potential for diverse applications beyond crystal structures, such as molecular structure modeling and ML-based particle and liquid simulations.

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A LIMITATIONS OF UNIT-CELL-BASED CRYSTAL REPRESENTATIONS

The conventional PCA frames explained in Sec. 2.3 implicitly assume a unique lattice representation such as the Niggli reduced cell (Niggli, 1928). Similarly, the lattice frames assume a primitive cell and convert it to a cell similar to the reduced one. Otherwise, these frames are affected by the arbitrariness of unit cell representations, such as supercells and conventional cells.

Traditionally, primitive cells and conventional cells are used to represent periodic structures. Primitive cells are defined as the smallest repeating units of a lattice, having the minimum volume and containing only a single lattice point within each cell. By following a mathematical procedure on primitive cells, their unique representations called reduced unit cells can be obtained (Santoro & Mighell, 1970). On the other hand, conventional cells are defined as unit cells that are not necessarily primitive but are designed to exhibit symmetry in an easily understandable way. The notion of conventional cells is often illustrated by the face-centered cubic lattice and the body-centered cubic lattice. Figure A1

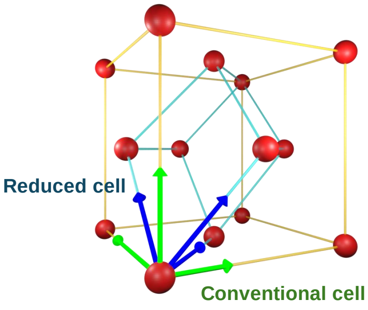

Figure A1: **Conventional cell (green) and Niggli reduced cell (blue) for a face-centered cube.**

compares a conventional cell and the Niggli reduced cell of a face-centered cubic structure. Examining the conventional unit cell easily reveals that it represents a cubic lattice, with atoms located at each corner and at each face center. However, this fact is obscured in the reduced cell. Therefore, reduced cells can be said to sacrifice the interpretability of physically important information, such as symmetry, in order to uniquely represent periodic structures.

B DATASET SPECIFICATIONS

We use the following three sources of materials data for evaluations. They are all publicly available through a Python package (jarvis-tools) created by Choudhary et al. (2020).

The JARVIS-DFT 3D 2021 is a collection of 55,723 materials provided by Choudhary et al. (2020) and is accessible as `dft_3d_2021` via jarvis-tools (or as `dft_3d` in older versions). These materials are annotated with various simulated properties using two DFT calculation methods, OptB88vdW (OPT) and TBmBJ (MBJ). Following recent studies (Yan et al., 2022; 2024; Lin et al., 2023; Taniai et al., 2024), we use formation energy (`formation_energy_peratom`), total energy (`optb88vdw_total_energy`), bandgap (`optb88vdw_bandgap` and `mbj_bandgap`), and energy above hull or E hull (`ehull`) as regression targets.

The Materials Project (MP) database (Jain et al., 2013) is an online public materials database providing various synthetic materials and their DFT-calculated properties. We specifically use its snapshot collected by Chen et al. (2019), which contains 69,239 materials and is accessible as `megnet` via jarvis-tools. Following recent studies (Yan et al., 2022; 2024; Lin et al., 2023; Taniai et al., 2024), we use formation energy (`e_form`), bandgap (`gap pbe`), bulk modulus (`bulk modulus`), and shear modulus (`shear modulus`) as regression targets. For bulk and shear modulus, we use the data splits provided by Yan et al. (2022).

The Open Quantum Materials Database (OQMD) is another online public materials database by Kirklin et al. (2015). We specifically use its snapshot provided as `oqmd_3d_no_cfid` in jarvis-tools, which contains 817,636 materials with three DFT-calculated properties: formation energy (`_oqmd_delta_e`), bandgap (`_oqmd_band_gap`), and energy above hull (`_oqmd_stability`). We use these properties as regression targets. We will release our data splits along with our codes in the future.

C TRAINING PARAMETERS

Table A1 summarizes the training settings for the JARVIS, MP, and OQMD datasets. Specifically for the JARVIS dataset, we optimize the mean absolute loss function using the Adam optimizer (Kingma & Ba, 2015) with $(\beta_1, \beta_2) = (0.9, 0.98)$, weight decay of 10^{-5} (Loshchilov & Hutter, 2019), and a batch size of 256 materials. We employ the warm-up-free inverse square root scheduling (Huang et al., 2020) for the learning rate, with the initial learning rate of 5.0×10^{-4} and decay factor of $\sqrt{4000/(4000+t)}$ according to the total train steps t. The model weights are initialized through the strategy for the normalization-free transformer architecture by Huang et al. (2020), which improves the training stability. The training is iterated for a total of 2000 epochs. Stochastic weight averaging

Table A1: **Detailed training settings.**

Hyperparameters	Settings (JARVIS, MP, OQMD)
Loss function	Mean absolute error
Optimizer	AdamW with $(\beta_1, \beta_2) = (0.9, 0.98)$
Weight decay	10^{-5}
Gradient norm clipping	1.0
Initial learning rate α	5.0×10^{-4}
Learning rate scheduling per step	$\alpha\sqrt{4000/(4000+t)}$
Warm-up steps	0 (no warm-up)
Batch size	256, 128, 1024
Number of epochs	2000, 800, 200
Dropout rate	0.0
SWA epochs	50, 50, 20

(SWA) (Izmailov et al., 2018) is adopted for model selection for testing and validation. Except for the increased number of epochs, we use the same settings with the baseline Crystalformer model (Taniai et al., 2024) to evaluate the pure effects of introducing the frames. For the OQMD dataset, which was not used by the baseline method, we use similar settings with a larger batch size of 1024 materials and fewer epochs of 200.

D DETAILED VISUAL ANALYSIS OF FRAMES

D.1 COMPARISON BETWEEN WEIGHTED PCA FRAMES AND MAX FRAMES

As shown in Tables 1 and 2, the max frame method performed very well, while the weighted PCA variant did not. In Fig. 3, the weighted PCA frames do not seem to capture the local structure very well compared to the max frames. This is because all the attention weights, even small ones, can influence the composition of the weighted PCA frames. In other words, the weighted PCA frames look at the structure over a broader area, while the max frames focus on relatively close neighbors. This difference seems to have a positive effect on the max frames and a negative effect on the weighted PCA frames in most tasks, except for the E hull in the MP dataset (Table 2).

For the E hull prediction, it is suggested by Taniai et al. (2024) that the inclusion of long-range interatomic interactions is a critical factor. This implication can reasonably explain the better performance of the weighted PCA frames for the E hull. That is, the weighted PCA frames emphasize distant atoms and help deliver more meaningful messages from these distant atoms that are important for the E hull prediction.

D.2 FRAME VISUALIZATIONS FOR A DIFFERENT MATERIAL

Figure. A2 shows the frame visualizations for another test material (JVASP-85272). This structure consists of carbon (red atoms) and nitrogen (blue atoms), forming a tetrahedral structure. Both dynamic models first attend to the central tetrahedral structure in the first two layers, and then increase the attention to relatively distant red atoms in the subsequent layers. However, the max frames capture these structures more clearly than the weighted PCA frames, as observed in the first example.

We have also noticed a general tendency for our models to attend to close neighbors in shallow layers and relatively distant neighbors in deeper layers. This tendency is also reasonable. Since the states of atoms are initialized as symbolic atomic species without rich information, they must gather information about their surroundings in shallow layers to configure their states. In deeper layers, these atoms become ready to engage in complex interactions with selected distant atoms.

D.3 EVOLUTION OF DYNAMIC FRAMES DURING TRAINING

We further examined how dynamic frames evolve throughout the training process, by visualizing frames using model checkpoints taken at 200-epoch intervals. Figure A3 compares the evolution of the weighted PCA frames and max frames for the same material as Fig. 3. We observed that the weighted PCA frames fluctuated throughout training, whereas the max frames stabilized quickly

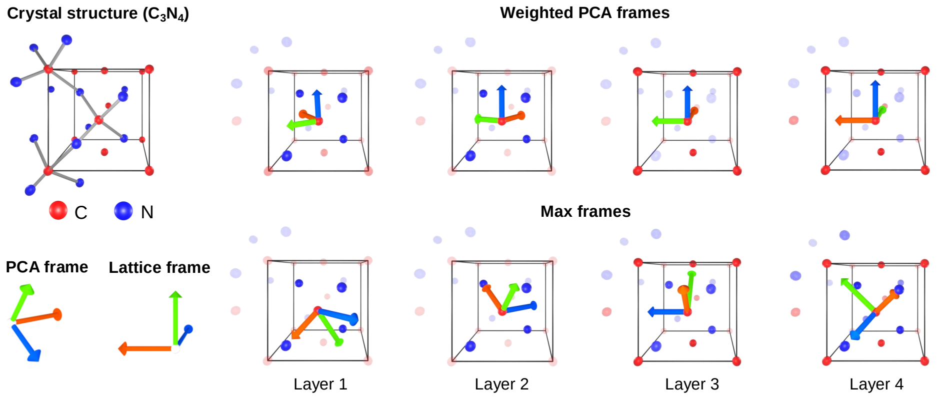

Figure A2: **Frame visualizations for a different material.**

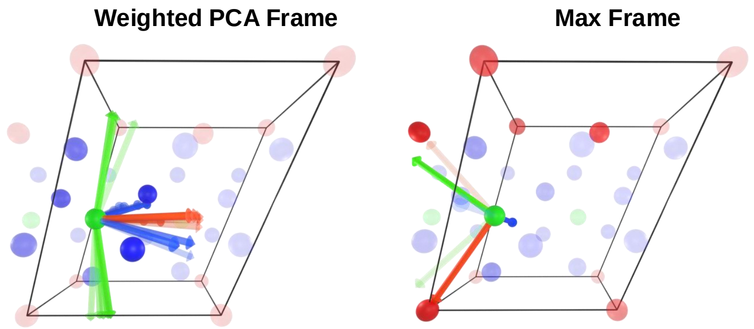

Figure A3: **Evolution of dynamic frames during training.** We visualize the weighted PCA frames and max frames using model checkpoints taken every 200 epochs, starting from epoch 100 until 2000. Frames from earlier checkpoints are overlaid with higher transparency. Notably, the max frames stabilize more quickly than the weighted PCA frames.

during the early stages. As frame fluctuations can introduce noise, the early stabilization of the max frames may explain their superior performance compared to the weighted PCA frames.

E SCALABILITY FOR LARGE STRUCTURES AND SUPERCELLS

Since the proposed CrystalFramer is based on a self-attention mechanism, its computational complexity is $O(Nk)$, where N is the number of atoms in the unit cell and k is the number of neighbors per unit-cell atom. Specifically, for the infinitely connected attention of Crystalformer (Taniai et al., 2024) shown in Eq. 3, the neighbors of each atom are identified by repeating unit cells within a finite range. This results in $O(k) = O(N)$, leading to an overall complexity of $O(N^2)$.

In practice, the training of CrystalFramer has successfully scaled to relatively large structures in the MP dataset, which features an average of 30 atoms per unit cell and a maximum of 296 atoms. For inference, the method can handle even larger structures than during training, as it requires significantly less memory and supports per-material (non-batched) processing.

Scalability for larger structures becomes crucial especially when processing supercells. Supercells are often utilized when structures deviate from perfect periodicity, such as in the presence of impurities, defects, or surfaces. We consider the following two potential approaches to improve efficiency with large supercells.

Mixed atom embedding. Structures with impurities or defects are often represented using site occupancy, which indicates the probabilities of different elements occupying an atomic site. Instead of modeling such structures with supercells, we can efficiently represent the site occupancy by mixing atomic embedding vectors. In this case, each a_i represents a probability distribution over elements rather than a single element. The corresponding atomic state can then be initialized as a linear blend of atom embeddings: $\boldsymbol{x}_i \leftarrow \sum_{\text{element}} a_i(\text{element})\text{AtomEmbedding}(\text{element})$. This approach can keep the structure size small without using a supercell, thereby maintaining overall efficiency.

Distance-based neighbor search. When unit cells are large, the current cell-based neighbor identification method will produce redundant neighbors, forcing $k \geq N$. By employing a more compact set of neighbors through nearest neighbor search, the complexity is reduced from $O(N^2)$ to $O(Nk)$, improving efficiency for larger structures.

Since structures with imperfect periodicity are common in realistic scenarios, developing scalable models for these structures is a important direction for future research.