PARAMETER AND MEMORY EFFICIENT PRETRAINING VIA LOW-RANK RIEMANNIAN OPTIMIZATION

Anonymous authors
Paper under double-blind review

ABSTRACT

Pretraining large language models often requires significant computational resources and memory due to their vast parameter amount. An effective approach to enhance parameter efficiency in both training and inference is to parameterize each full-size weight as the product of two trainable low-rank factors. While low-rank fine-tuning has achieved great success, low-rank pretraining remains challenging as it requires learning extensive knowledge from scratch under the restrictive low-rank parameterization. During standard low-rank pretraining, separately optimizing the low-rank factors introduces redundant information from the full gradient, which hinders the learning process. To achieve efficient yet effective low-rank pretraining, we propose a **Lo**w-rank **R**iemannian **O**ptimizer (**LORO**). At each LORO update step, the low-rank factor pairs are jointly updated to ensure their full-size product moves along the steepest descent direction on the low-rank manifold, without the need to compute any memory-intensive full-size matrices or gradients. Hence, our LORO finds low-rank models that achieve high performance comparable to full-size pretrained models, while significantly reducing memory usage and accelerating both training and inference. A LLaMA 1B model pretrained with LORO achieves a perplexity score of 2% better than the full-size baseline, with a 54% reduction in model memory, a $\times 1.8$ speedup in training, and a $\times 2.2$ speedup in inference.

1 INTRODUCTION

In recent years, large language models (LLMs) (Brown et al., 2020; Touvron et al., 2023a;b; Team et al., 2024a;b), whose parameter sizes often reach hundreds of billions, have obtained remarkable performance across a wide range of applications. Typically, a pretraining and fine-tuning paradigm is adopted for building powerful LLMs from scratch: a model is first pretrained on large-scale unsupervised corpora (Devlin et al., 2019; Brown et al., 2020) to acquire general semantics and extensive knowledge, and then fine-tuned on a smaller downstream dataset to enhance its domain-specific capabilities (Hu et al., 2022; Team et al., 2024b; Ramesh et al., 2024).

An extensive amount of trainable parameters results in a substantial memory footprint and high computational costs during pretraining and fine-tuning. To alleviate these issues, various memory-efficient training techniques have been proposed to reduce memory usage and accelerate training. These approaches stem from either an engineering perspective, such as gradient checkpointing (Chen et al., 2016), memory offloading (Rajbhandari et al., 2020), or an optimization perspective, including low-rank optimizers (Hu et al., 2022; Zhang et al., 2023; Hayou et al., 2024; Lialin et al., 2024) and low-precision optimizers (Dettmers et al., 2021; Li et al., 2023; Zhang et al., 2024b). Since Hu et al. (2022), low-rank adaptation (LoRA) has gained increasing attention, becoming one of the most popular parameter-efficient fine-tuning methods. At each layer, LoRA replaces the full-size pretrained weight $\mathbf{W} \in \mathbb{R}^{d\times d}$ with $\mathbf{W} + \mathbf{BA}$, where $\mathbf{B} \in \mathbb{R}^{d\times r}$ and $\mathbf{A} \in \mathbb{R}^{r\times d}$ are two trainable low-rank factors, while $\mathbf{W}$ is frozen during fine-tuning to reduce training memory footprint. However, these methods depend on full-size pretrained weights, making them inapplicable to pretraining scenarios. Furthermore, they ultimately produce models with full-size weights, which do not result in any memory reduction or speedup in the inference phase.

In this work, we aim to enhance efficiency during both the training and inference by exploring the pretraining of low-rank parameterized language models, where each weight matrix is parameterized

as $\mathbf{W} \triangleq \mathbf{BA}$, the product of two rank-r matrices. In practice, such low-rank parameterization enjoys great potential in improving both training and inference efficiency, as it reduces the parameter size at each layer from d^2 to $2rd$ and decreases the FLOPS from $O(d^3)$ to $O(rd^2)$. Theoretically, a low-rank model can achieve a $\times(2r/d)$ memory reduction and a $\times(d/2r)$ inference speedup.

However, pre-training low-rank language models from scratch using standard stochastic gradient descent methods is much more challenging. As shown in (Kamalakara et al., 2022; Zhao et al., 2024), separately optimizing each low-rank factor of the model may result in a significant performance drop compared to full-size models. Therefore, as a compromise, recent works consider pretraining language models with full-size weights while utilizing low-rank or sparse updates to enhance training efficiency: ReLoRA (Lialin et al., 2024) periodically merges a low-rank update into the full-size weight after a full-size warm-up training phase; GaLore (Zhao et al., 2024) updates the full-size model weights using low-rank yet full-size gradients; SLTrain (Han et al., 2024) proposes sparse plus low-rank pretraining, where the full-size weights are decomposed into sum of sparse and low-rank matrices. However, these methods can merely produce full-size models, leading to limited or no memory reduction or acceleration during inference.

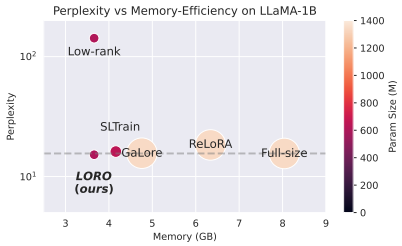


Figure 1: Visualization of the perplexity (log-scaled) and memory-efficiency trade-off of pretraining LLaMA-1B on C4 dataset (Table 1). Methods located closer to the bottom-left corner and associated with smaller bubble sizes exhibit superior performance and memory efficiency.

While recent studies (Zhao et al., 2024; Han et al., 2024) observe that the rank of the resulting weight matrix $\mathbf{W}$ obtained from full-size pretraining methods is typically high (close to d), this does not necessarily mean that low-rank pretraining approaches (learning $\mathbf{BA}$ instead of $\mathbf{W}$) are incapable of achieving satisfactory performance in learning foundation models. We argue that with careful design of the learning process for the low-rank factors $\mathbf{A}$ and $\mathbf{B}$, it is possible to train a model with performance comparable to full-size models. Therefore, this paper aims to address the research question: *Can we train a low-rank parameterized language model from scratch to achieve performance comparable to the full-size baseline?* In essence, low-rank pretraining is more challenging than fine-tuning. Given the pretrained full-size weights, fine-tuning only requires capturing the minor domain-specific knowledge through a low-rank adaptation. In contrast, learning extensive pretraining knowledge from scratch under the restrictive low-rank parameterization requires a more careful optimization process. As noted in the discussions of (3) in Section 3, optimizing the low-rank factors separately introduces redundant information from the full gradient, which hinders the feature learning process. In this scenario, the separate optimization of $\mathbf{B}$ and $\mathbf{A}$ fails to direct their product $\mathbf{BA}$ along the steepest descent direction of the loss on the rank-r matrix manifold. This necessitates considering the intricate geometry of the manifold, prompting us to optimize the low-rank factors jointly through Riemannian optimization. While recent works (Savostianova et al., 2023; Schotthöfer et al., 2022) have attempted to train low-rank vision classifiers using Riemannian gradient methods, they either require evaluating full-size gradients or rely on SVD-like parameterization, which introduces $O(r^2)$ additional parameters, making them less preferable for pretraining language models. To the best of our knowledge, this paper is the first to explore the pretraining of low-rank language models through the lens of Riemannian optimization.

Contributions. We propose the **Lo**w-rank **R**iemannian **O**ptimizer (**LORO**) for effective low-rank pretraining. At each LORO update step, each pair of low-rank factors is jointly updated to ensure their product is transported along the Riemannian gradient of the rank-r matrix manifold to pursue effective descent in the model loss. All LORO's updates are performed directly within the low-rank parameterization, without the need to evaluate the costly full-size weights or gradients. To our knowledge, we are the first to show that pretraining low-rank language models with Riemannian gradients achieve perplexity scores comparable to full-size pretraining, based on extensive experiments conducted on LLaMA models ranging from 60M to 1B in size. On LLaMA-1B, LORO achieves a perplexity score of 2% better than the full-size baseline, with 54% less model memory cost and offers a $\times 1.8$ speedup in training and a $\times 2.2$ speedup in inference. Within the same parameter budget, we further show that LORO outperforms its full-size counterparts, highlighting LORO's capacity to unblock the potential of low-rank parameterization for enhanced representation learning.

2 RELATED WORKS

Low-rank fine-tuning. Pioneered by LoRA (Hu et al., 2022), which achieves parameter-efficient fine-tuning by restricting optimization exclusively to the low-rank factors $\mathbf{BA}$, various low-rank fine-tuning methods have been proposed to enhance downstream adaptation (Zhang et al., 2023; Ding et al., 2023; Hayou et al., 2024; Zhang & Pilanci, 2024; Xia et al., 2024), improve cross-task generalization (Agiza et al., 2024; Huang et al., 2023; Buehler & Buehler, 2024), and increase fine-tuning efficiency (Chen et al., 2023; Dettmers et al., 2023; Gamal & Rabusseau, 2023; Koohpayegani et al., 2024; Valipour et al., 2023). However, all these low-rank fine-tuning methods require access to the full-size pretrained weights so the model can be adapted within a low-rank parameter space. When directly applied to pretraining, the absence of pretrained full-size weights—where most knowledge resides—often results in significant performance drops.

Low-rank pretraining. While a few works have attempted to train low-rank vision models Sui et al. (2023); Khodak et al. (2021); Savostianova et al. (2023), training low-rank parameterized language models from scratch has been observed to be challenging and remains an open problem (Kamalakara et al., 2022; Zhao et al., 2024). Existing methods circumvent these challenges by employing low-rank or sparse updates while still using full-size weights, such as updating full-size weights with low-rank gradients (Zhao et al., 2024; Zhang et al., 2024b), periodically merging low-rank factors into full-size weights (Lialin et al., 2024), or using a low-rank plus sparse weight parameterization (Han et al., 2024). As a result, these methods provide little to no improvement in inference efficiency. In contrast, LORO is the first to directly tackle language model pretraining using low-rank parameterized weights.

Optimization on low-rank matrix manifolds. Abundant theories and algorithms (Absil & Oseledets, 2015; Absil et al., 2009; Mishra et al., 2014) have been proposed to perform Riemannian optimization on low-rank matrix manifolds for traditional applications, such as matrix completion (Vandereycken, 2013; Bian et al., 2023) and regressions (Bleakley & Yamanishi, 2009; Yuan et al., 2007; Amit et al., 2007). Recent works (Savostianova et al., 2023; Schotthöfer et al., 2022) aim to train low-rank vision classifiers using Riemannian gradient flow (Koch & Lubich, 2007). However, these methods either require evaluating full-size gradient and weight matrices or rely on SVD-like parameterization, which introduces parameter overhead, making them incompatible with low-rank parameterized language models. In contrast, LORO operates under the low-rank parameterization, avoiding full-size matrix evaluations and enabling efficient pretraining on language models.

3 PRELIMINARIES

Low-rank pretraining. The rank-r parameterization of a d-by-d matrix is defined as $\mathbf{W} = \mathbf{BA}$, where $\mathbf{B} \in \mathbb{R}^{d\times r}$ and $\mathbf{A} \in \mathbb{R}^{r\times d}$ are the associated rank-r factors. When $r < d$, $\mathbf{BA}$ is called the low-rank parameterization of $\mathbf{W}$. In this work, we focus on training the decoder-only transformer-based language models (Vaswani et al., 2017; Touvron et al., 2023a) from scratch using low-rank parameterization. Recall that the two fundamental building block of the transformer model are the linear projection $f_{\text{proj}}(\cdot)$ and the self-attention function $f_{\text{attn}}(\cdot)$, which are defined as

$$f_{\text{proj}}(\mathbf{X}) \triangleq \mathbf{X}\mathbf{W}_p,\ f_{\text{attn}}(\mathbf{X}) \triangleq \text{softmax}\left(\mathbf{X}\mathbf{W}_q\mathbf{W}_k^\top\mathbf{X}^\top/\sqrt{d}\right)\mathbf{X}\mathbf{W}_v, \tag{1}$$

where $\mathbf{X} \in \mathbb{R}^{n\times d}$ is the hidden feature, n is the token number, d is the hidden dimension, and $\mathbf{W}_p$, $\mathbf{W}_q$, $\mathbf{W}_k$, $\mathbf{W}_v \in \mathbb{R}^{d\times d}$ denote the trainable projection, query, key, and value matrices, respectively. For notation simplicity, we assume that d remains constant across all layers, though all derivations in this work also apply to scenarios with varying dimensions. As the composition of multiple layers of linear projections, self-attention functions, and various operations (e.g., activation, normalization, and positional embedding), the target language model can be parameterized by the stack of trainable matrices across all N layers, denoted as $(\mathbf{W}_1, \cdots, \mathbf{W}_N)$. Therefore, the rank-$r$ parameterized language model is implemented by parameterizing each trainable matrix as a rank-r factorization $\mathbf{W}_i = \mathbf{B}_i\mathbf{A}_i$ with $\mathbf{B}_i \in \mathbb{R}^{d\times r}$ and $\mathbf{A}_i \in \mathbb{R}^{r\times d}$, and is represented by $(\mathbf{B}_1\mathbf{A}_1, \ldots, \mathbf{B}_N\mathbf{A}_N)$. In practice, we set $r < d$ for parameter efficiency. During pretraining, the rank-r model is trained from scratch on unsupervised corpus data for next token prediction, defined as:

$$\min L(\mathbf{W}_1, \cdots, \mathbf{W}_N),\ \text{s.t. } \mathbf{W}_i = \mathbf{B}_i\mathbf{A}_i, \mathbf{B}_i \in \mathbb{R}^{d\times r}, \mathbf{A}_i \in \mathbb{R}^{r\times d}, i = 1, ..., N, \tag{2}$$

where $L(\cdot) : \mathbb{R}^{d\times d} \times \cdots \times \mathbb{R}^{d\times d} \mapsto \mathbb{R}^{+}$ denotes the smooth next token prediction loss (i.e., the logarithm of perplexity score (Radford & Narasimhan, 2018)). At the t-th training step with learning rate $\eta > 0$, the full-size weight is updated by the full gradient $\mathbf{W}_i^{t+1} \leftarrow \eta \nabla_{\mathbf{W}_i^t} L$. In contrast, during the standard low-rank pretraining, the low-rank factors are updated separately, which leads to

$$\mathbf{W}_i^{t+1} \leftarrow (\mathbf{B}_i^t - \eta\, \nabla_{\mathbf{B}_i^t} L)(\mathbf{A}_i^t - \eta\, \nabla_{\mathbf{A}_i^t} L) \approx \mathbf{W}_i^t - \eta\, (\mathbf{B}_i^t \mathbf{B}_i^{t\top} \cdot \nabla_{\mathbf{W}_i^t} L + \nabla_{\mathbf{W}_i^t} L \cdot \mathbf{A}_i^{t\top} \mathbf{A}_i^t) \tag{3}$$

where the last line is derived from the chain-rule $\mathbf{B}_i^{t\top} \nabla_{\mathbf{W}_i^t} L = \nabla_{\mathbf{A}_i^t}$ and $\nabla_{\mathbf{W}_i^t} L \cdot \mathbf{A}_i^{t\top} = \nabla_{\mathbf{B}_i^t} L$. The update direction in (3) is the sum of two sketched versions of the full gradient, each projected onto the low-rank subspaces spanned by the columns of $\mathbf{B}_i^t$ and the rows of $\mathbf{A}_i^t$. When these two subspaces overlap, the sketched gradient introduces redundant information from the full gradient, hindering the learning of effective representations. From a geometric perspective, separately optimizing the low-rank factors overlooks the intricate structure of the low-rank parameterization and fails to guide their product toward the steepest loss descent on the low-rank matrix manifold [1]. To address this issue, we recast (2) into a Riemannian optimization problem:

$$\min_{\mathbf{B}_i \in \mathbb{R}^{d\times r},\ \mathbf{A}_i \in \mathbb{R}^{r\times d}} L(\mathbf{B}_1\mathbf{A}_1, \cdots, \mathbf{B}_N\mathbf{A}_N) \implies \min_{(\mathbf{B}_i\mathbf{A}_i) \in \mathcal{M}_r} L(\mathbf{B}_1\mathbf{A}_1, \cdots, \mathbf{B}_N\mathbf{A}_N), \tag{4}$$

where $\mathcal{M}_r$ denotes the manifold of all d-by-d rank-r matrices. In essence, this reformulation accounts for the geometric structure of $\mathcal{M}_r$ by **treating the product** $(\mathbf{B}_i\mathbf{A}_i)$ **as a whole**: it emphasizes that the principle for updating $\mathbf{B}_i$ and $\mathbf{A}_i$ is to guide the product $(\mathbf{B}_i\mathbf{A}_i)$ towards the steepest descent direction within the manifold, rather than optimizing them separately in the ambient space.

Optimization on low-rank matrix manifold. We briefly review the retraction-based Riemannian optimization on manifold, and refer interested readers to Lee (2019); Absil & Oseledets (2015) for rigorous definitions and a more comprehensive review. Imagine an ant on the Earth's surface seeking the highest point. Although it wants to climb straight up along the steepest path—an infeasible option that would send it into space—it must navigate the curved surface, choosing the best direction from the feasible tangents at its current location to remain on the Earth. This thought experiment helps illustrate the difference between optimization in the flat full-size matrix space (e.g., the ambient space) and on the curved rank-r matrix manifold $\mathcal{M}_r$ (e.g., the Earth).

As visualized in Figure 2, when optimizing in the ambient space, we evaluate the negative Euclidean gradient (blue vector) at each step by backpropagation, then update the current weight by moving it a small step towards the blue vector. However, this update is infeasible on $\mathcal{M}_r$, as it moves the current weight off the manifold. Instead, we aim to update along the black dashed curve on the manifold, which guides us toward smaller loss values. To this end, we need to identify the Riemannian gradient (red solid vector) within the tangent space (gray plane) that aligns most closely with the Euclidean gradient. This can be done by projecting the blue vector onto the tangent space. We then update the weight for a small step along the red solid vector and retract it back to the manifold (via the red dashed vector) to avoid violating the rank constraint. This process is known as the retraction-based Riemannian optimization step, involving: 1. Updating with the projected gradient and 2. Retracting back to the manifold.

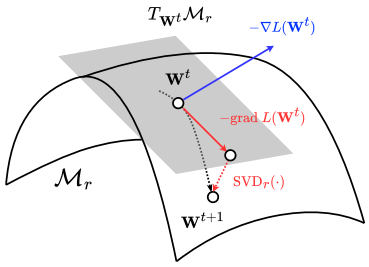


Figure 2: Retraction-based Riemannian optimization, where the grey plane is the tangent space, $\nabla L(\mathbf{W}^t)$ and $\text{grad}\, L(\mathbf{W}^t)$ denote the Euclidean and Riemannian gradient at $\mathbf{W}^t$, respectively.

Thanks to the embedded manifold theories, Proposition 1 (see Lee (2019, Example 8.14) and Absil et al. (2009, Equation 3.37)) shows how to compute the Riemannian gradient via orthogonal projection onto the tangent space, while Proposition 2 (see Lewis & Malick (2008, Lemma 2.1)) shows that retraction to $\mathcal{M}_r$ can be done by rank-r singular value decomposition (SVD).

Proposition 1 (Tangent Space and Riemannian Gradient on $\mathcal{M}_r$)**.** *$\mathcal{M}_r$ is a smooth submanifold embedded in the ambient space $\mathbb{R}^{d\times d}$. For any base point $\mathbf{W} \in \mathcal{M}_r$ with singular value decomposition $\mathbf{W} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ with $\mathbf{U}, \mathbf{V} \in \mathbb{R}^{d\times r}$ being orthogonal, the tangent space at $\mathbf{W}$ is given by*

$$T_{\mathbf{W}}\mathcal{M}_r \triangleq \left\{ \mathbf{U}\mathbf{Z}\mathbf{V}^\top + \mathbf{U}'\mathbf{V}^\top + \mathbf{U}\mathbf{V}'^\top : \mathbf{Z} \in \mathbb{R}^{r\times r}, \mathbf{U}', \mathbf{V}' \in \mathbb{R}^{d\times r}, \mathbf{U}^\top\mathbf{U}' = \mathbf{V}^\top\mathbf{V}' = \mathbf{0} \right\}, \tag{5}$$

[1] See Appendix C for more detailed discussions on the issues in standard low-rank pretraining.

and the orthogonal projection to $T_{\mathbf{W}}\mathcal{M}_r$ is given by $\mathcal{P}(\mathbf{G}) = \mathbf{U}\mathbf{U}^\top\mathbf{G} + \mathbf{G}\mathbf{V}\mathbf{V}^\top - \mathbf{U}\mathbf{U}^\top\mathbf{G}\mathbf{V}\mathbf{V}^\top$. When equipped with the matrix inner-product metric, $\mathcal{M}_r$ becomes a Riemannian manifold. Then, the Riemannian gradient of any smooth function $L(\cdot) : \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ evaluated at $\mathbf{W}$, denoted as $\operatorname{grad} L(\mathbf{W})$, can be calculated by projecting the Euclidean gradient $\nabla L(\mathbf{W})$ to $T_{\mathbf{W}}\mathcal{M}_r$:

$$\operatorname{grad} L(\mathbf{W}) = \mathcal{P}(\nabla L(\mathbf{W})) = \mathbf{U}\mathbf{U}^\top\nabla L(\mathbf{W}) + \nabla L(\mathbf{W})\mathbf{V}\mathbf{V}^\top - \mathbf{U}\mathbf{U}^\top\nabla L(\mathbf{W})\mathbf{V}\mathbf{V}^\top. \tag{6}$$

Proposition 2 (Retraction to $\mathcal{M}_r$ via SVD)**.** *Informally, for any base point $\mathbf{W} \in \mathcal{M}_r$ and tangent vector $\mathbf{G} \in T_{\mathbf{W}}\mathcal{M}_r$, a retraction is a function $R(\cdot)$ that maps the shifted base point $(\mathbf{W}+\mathbf{G})$ back to $\mathcal{M}_r$. A feasible retraction on $\mathcal{M}_r$ is $R(\mathbf{W}+\mathbf{G}) \triangleq \operatorname{SVD}_r(\mathbf{W}+\mathbf{G})$, where $\operatorname{SVD}_r(\cdot)$ is the rank-r singular value decomposition, projecting the updated base point from $\mathbb{R}^{d\times d}$ back to $\mathcal{M}_r$.*

4 Methodology

4.1 Riemannian Update under Low-rank Parameterization

Proposition 1 and Proposition 2 enable us to perform Riemannian low-rank pretraining on $\mathcal{M}_r$ via

$$\mathbf{W}_i^{t+1} \leftarrow R(\mathbf{W}_i^t - \eta \operatorname{grad} L(\mathbf{W}_i^t)) = \operatorname{SVD}_r\left(\mathbf{W}_i^t - \eta\, \mathcal{P}_i\left(\nabla_{\mathbf{W}_i^t} L\right)\right), \tag{7}$$

where $\mathcal{P}_i(\cdot)$ denotes the orthogonal projection to the tangent space $T_{\mathbf{W}_i}\mathcal{M}_r$ as defined in Proposition 1. For each trainable weight, we: 1. evaluate the Euclidean gradient $\nabla_{\mathbf{W}_i} L$, and derive the tangent space projection $P_i(\cdot)$ from the SVD of base point $\mathbf{W}_i = \mathbf{U}_i\mathbf{\Sigma}_i\mathbf{V}_i^\top$; 2. find the Riemannian gradient $\mathcal{P}_i(\nabla_{\mathbf{W}_i} L)$ by projecting $\nabla_{\mathbf{W}_i} L$ to the tangent space; 3. transport the base point along the negative Riemannian gradient; 4. retract the updated base point back to $\mathcal{M}_r$ via the $\operatorname{SVD}_r(\cdot)$ projection.

However, these update steps encounter several practical issues that render them incompatible with low-rank parameterization: First, the full-size gradient $\nabla_{\mathbf{W}_i} L$ is inaccessible during backpropagation Rumelhart et al. (1988); we can only obtain $\nabla_{\mathbf{B}_i} L$ and $\nabla_{\mathbf{A}_i} L$, the gradients of the low-rank factors. Second, in steps 1, 2, and 4, we need to manipulate the memory-intensive full-size matrix $\mathbf{W}_i$, either by performing SVD or matrix multiplication, which compromises memory efficiency.

This motivates us to revise the update steps of (7) into a "**low-rank parameterization friendly**" form by avoiding any computations with full-size d-by-d matrices (e.g., the weight $\mathbf{W}_i$ and gradient $\nabla_{\mathbf{W}_i} L$) and ensuring all operations can be performed under the low-rank parameterization. To achieve this objective, our focus is on the following two goals: 1. Eliminating the need for the full-size gradient when calculating the Riemannian gradient through tangent space projection; 2. Keeping the Riemannian update in its factorized form to avoid the necessity of performing SVD on full-size matrices. To achieve the 1st goal, we propose Proposition 3, which shows that the Riemannian gradient on $\mathcal{M}_r$ can be computed from $\nabla_{\mathbf{B}_i} L$ and $\nabla_{\mathbf{A}_i} L$.

Proposition 3 (Riemannian Gradient under Low-rank Parameterization)**.** *Following the previous notation, for any low-rank parameterized weight $\mathbf{W} = \mathbf{B}\mathbf{A} \in \mathcal{M}_r$, where $\mathbf{B} \in \mathbb{R}^{d\times r}$ and $\mathbf{A} \in \mathbb{R}^{r\times d}$, the Riemannian gradient of any smooth function $L(\cdot) : \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ evaluated at $\mathbf{W}$ is*

$$\operatorname{grad} L(\mathbf{W}) = \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\cdot\nabla_{\mathbf{A}} L + \nabla_{\mathbf{B}} L\cdot(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A} - \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\cdot\nabla_{\mathbf{A}} L\cdot\mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}. \tag{8}$$

The proof of Proposition 3 can be obtained from Proposition 1 based on two key observations: 1. Let $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ be the SVD of $\mathbf{B}\mathbf{A}$, the column space of $\mathbf{B}$ coincides with that of $\mathbf{U}$, and the row space of $\mathbf{A}$ coincides with the column space of $\mathbf{V}$. 2. Backpropagation gives $\nabla_{\mathbf{W}} L\cdot\mathbf{A}^\top = \nabla_{\mathbf{B}} L$ and $\mathbf{B}\cdot\nabla_{\mathbf{W}} L = \nabla_{\mathbf{A}} L$. The proof details are postponed to the Appendix B. In practice, for each weight matrix $\mathbf{W}_i$, (8) involves only multiplications of r-by-d and r-by-r matrices, as well as the inversion of r-by-r matrices, eliminating the need to compute any expensive d-by-d matrices or use the d-by-d gradient $\nabla_{\mathbf{W}_i} L$. Additionally, all terms in (8) can be efficiently computed from the readily available variables—$\mathbf{B}_i$, $\mathbf{A}_i$, $\nabla_{\mathbf{B}_i} L$, and $\nabla_{\mathbf{A}_i} L$—obtained during backpropagation.

Although Proposition 3 provides a memory-efficient approach for calculating the Riemannian gradient, the resulting $\operatorname{grad} L(\mathbf{W})$ remains a full-size d-by-d matrix, which compromises the feasibility of the subsequent Riemannian update and retraction. To address this issue and achieve the 2nd goal, we propose Proposition 4, which shows that the successive update steps can be achieved within a factorized form, avoiding the need to compute any full-size weight or gradient.

Proposition 4 (Riemannian Update under Low-rank Parameterization)**.** *Following the previous notations, for any low-rank parameterized weight* $\mathbf{W} = \mathbf{B}\mathbf{A} \in \mathcal{M}_r$*, let* $\mathbf{B} = \mathbf{Q}_B\mathbf{R}_B$ *and* $\mathbf{A}^\top = \mathbf{Q}_A\mathbf{R}_A$ *be the QR-decomposition of* $\mathbf{B}$ *and* $\mathbf{A}^\top$*, such that* $\mathbf{Q}_B, \mathbf{Q}_A \in \mathbb{R}^{d\times r}$ *are column-wise orthogonal, and* $\mathbf{R}_B, \mathbf{R}_A \in \mathbb{R}^{r\times r}$ *are upper-triangular. Then the Riemannian updated weight, defined by* $\mathbf{W}' \triangleq \mathrm{SVD}_r(\mathbf{W} - \eta \operatorname{grad} L(\mathbf{W}))$*, admits a low-rank factorization* $\mathbf{W}' = \mathbf{B}'\mathbf{A}'$*, with* $\mathbf{B}' = \left(\mathbf{Q}_B, \overline{\mathbf{Q}}_B\right)\mathbf{U}_*\mathbf{\Sigma}_*^{1/2}$ *and* $\mathbf{A}' = \mathbf{\Sigma}_*^{1/2}\mathbf{V}_*^\top\left(\mathbf{Q}_A, \overline{\mathbf{Q}}_A\right)^\top$*. Specifically,* $\mathbf{U}_*\mathbf{\Sigma}_*\mathbf{V}_*^\top$ *is determined by the rank-*r *SVD of the following* $2r$*-by-*$2r$ *matrix, i.e.,*

$$\mathbf{U}_*\mathbf{\Sigma}_*\mathbf{V}_*^\top = \mathrm{SVD}_r\begin{pmatrix}\mathbf{S} & -\eta\,\overline{\mathbf{R}}_A^\top \\ -\eta\,\overline{\mathbf{R}}_B & \mathbf{0}\end{pmatrix} \in \mathbb{R}^{2r\times 2r},\ \mathbf{S} \triangleq \mathbf{R}_B\mathbf{R}_A^\top - \eta\,\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L\cdot\mathbf{Q}_A \in \mathbb{R}^{r\times r}, \tag{9}$$

and $\overline{\mathbf{Q}}_B\overline{\mathbf{R}}_B$ *is the QR decomposition of the* d*-by-*r *matrix* $(\nabla_{\mathbf{B}}L\cdot\mathbf{R}_A^{-1} - \mathbf{Q}_B\mathbf{Q}_B^\top\nabla_{\mathbf{B}}L\cdot\mathbf{R}_A^{-1})$*, while* $\overline{\mathbf{Q}}_A\overline{\mathbf{R}}_A$ *is the QR decomposition of the* d*-by-*r *matrix* $(\nabla_{\mathbf{A}}L^\top\mathbf{R}_B^{-1} - \mathbf{Q}_A\mathbf{Q}_A^\top\nabla_{\mathbf{A}}L^\top\mathbf{R}_B^{-1})$*.*

The proof of Proposition 4 can be derived from Proposition 3 using the tangent space decomposition (Vandereycken, 2013; Bian et al., 2023) and the proof details are postponed to Appendix B. The key to avoid SVD on full-size d-by-d matrices lies in factorizing the d-by-d tangent vector into a product of a d-by-$2r$ orthogonal matrix, a $2r$-by-$2r$ full-rank matrix, and a $2r$-by-d orthogonal matrix, enabling SVD to be applied to the smaller $2r$-by-$2r$ matrix instead. Notably, the revised Riemannian update in Proposition 4 only involves QR decomposition of a d-by-r matrix, inversion of an r-by-r matrix, SVD on a $2r$-by-$2r$ matrix, and multiplication of d-by-$2r$ matrices.

4.2 Low-rank Riemannian Optimizer

In Proposition 4, we have derived the "low-rank parameterization friendly" Riemannian update:

$$\mathbf{B}_i^{t+1} \leftarrow (\mathbf{Q}_B, \overline{\mathbf{Q}}_B)\mathbf{U}_*\mathbf{\Sigma}_*^{1/2},\ \mathbf{A}_i^{t+1} \leftarrow \mathbf{\Sigma}_*^{1/2}\mathbf{V}_*^\top(\mathbf{Q}_A, \overline{\mathbf{Q}}_A)^\top, \tag{10}$$

where $\mathbf{Q}_B$, $\overline{\mathbf{Q}}_B$, $\mathbf{Q}_A$, $\overline{\mathbf{Q}}_A$, $\mathbf{U}_*$, $\mathbf{\Sigma}_*$, and $\mathbf{V}_*$ can be directly computed from $\mathbf{B}_i^t$, $\mathbf{A}_i^t$, $\nabla_{\mathbf{B}_i^t}L$, and $\nabla_{\mathbf{A}_i^t}L$, without evaluating any full-size weight or gradient. Although being memory-efficient, the QR decomposition, SVD, and matrix inversion involved in (10) impede the computational efficiency of low-rank pretraining. This motivates us to develop a rough but efficient approximation of the exact Riemannian update, which enables us to amortize the computational cost across multiple steps by periodically applying the exact update while primarily using the approximated update.

To this end, a simple approach is to skip the costly Riemannian SVD retraction and directly update the weight along the Riemannian gradient. Unfortunately, the Riemannian gradient in (8) has a maximum rank of $2r$, making it incompatible with the rank-r parameterization. To establish an approximate update that can be applied to $\mathbf{B}_i$ and $\mathbf{A}_i$ respectively, we highlight the differences between the standard low-rank update (3) and Riemannian update (8) in blue:

$$\begin{aligned}
&\text{Standard low-rank update } = -\eta\left(\mathbf{B}_i^t\cdot\nabla_{\mathbf{A}_i^t}L + \nabla_{\mathbf{B}_i^t}L\cdot\mathbf{A}_i^t\right) + O(\eta^2),\\
&\text{Riemannian update } = -\eta\big(\mathbf{B}_i^t(\mathbf{B}_i^{t\top}\mathbf{B}_i^t)^{-1}\nabla_{\mathbf{A}_i^t}L\cdot\underbrace{(\mathbf{I} - \mathbf{A}_i^{t\top}(\mathbf{A}_i^t\mathbf{A}_i^{t\top})^{-1}\mathbf{A}_i^t)}_{\text{Orthogonal projection}} + \nabla_{\mathbf{B}_i^t}L\cdot(\mathbf{A}_i^t\mathbf{A}_i^{t\top})^{-1}\mathbf{A}_i^t\big).
\end{aligned}$$

In the Riemannian update, the gradients are additionally normalized by multiplying them with either $(\mathbf{B}_i^{t\top}\mathbf{B}_i^t)^{-1}$ or $(\mathbf{A}_i^t\mathbf{A}_i^{t\top})^{-1}$. Additionally, $\nabla_{\mathbf{A}_i^t}L$ is projected onto the orthogonal space of the row space of $\mathbf{A}_i^t$ (the middle blue term), thereby removing the redundant gradient components. This motivates us to approximate this normalization by scaling the gradient terms in proportion to the spectral norm of either $(\mathbf{B}_i^{t\top}\mathbf{B}_i^t)^{-1}$ or $(\mathbf{A}_i^t\mathbf{A}_i^{t\top})^{-1}$. The orthogonal projection can be omitted, as it does not affect the gradient scale. We derive a heuristic scaling factor of $(\mathbf{B}_i^{t\top}\mathbf{B}_i^t)^{-1}$ as follows:

$$\left\|(\mathbf{B}_i^{t\top}\mathbf{B}_i^t)^{-1}\right\|_2 = \sigma_1\left((\mathbf{B}_i^{t\top}\mathbf{B}_i^t)^{-1}\right) = \sigma_r\left(\mathbf{B}_i^{t\top}\mathbf{B}_i^t\right)^{-1} = \sigma_r\left(\mathbf{B}_i^t\right)^{-2} \tag{11}$$

where $\|\cdot\|_2$ is the spectral norm and $\sigma_k(\cdot)$ denotes the k-th largest singular value of a matrix. We estimate $\sigma_r(\mathbf{B}_i^t)^2$ by the average of the squared singular values, i.e., $(1/r)\sum_{k=1}^r\sigma_k(\mathbf{B}_i^t)^2$, which equals $(1/r)\|\mathbf{B}_i^t\|_{\mathrm{F}}^2$, where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm. Assuming each row of $\mathbf{B}_i^t$ contains no extreme values and its norm is upper-bounded by a constant, we obtain $\|\mathbf{B}_i^t\|_{\mathrm{F}}^2 = O(d)$. Thus, we approximate the scaling factor as r/d to avoid explicitly computing $\|\mathbf{B}_i^t\|_{\mathrm{F}}^2$. By repeating a similar argument on $(\mathbf{A}_i^t\mathbf{A}_i^{t\top})^{-1}$, the approximated Riemannian update step is given by

$$\mathbf{B}_i^{t+1} \leftarrow \mathbf{B}_i^t - \eta\cdot(r/d)\cdot\nabla_{\mathbf{B}_i^t}L,\ \mathbf{A}_i^{t+1} \leftarrow \mathbf{A}_i^t - \eta\cdot(r/d)\cdot\nabla_{\mathbf{A}_i^t}L. \tag{12}$$

We now propose the **Lo**w-rank **R**iemannian **O**ptimizer): for every K training steps, LORO periodically employs an exact low-rank Riemannian update step (equation 10), while in between, it uses the approximated low-rank Riemannian update (equation 12). In practice, LORO accumulates the gradients of $\mathbf{B}_i^t$ and $\mathbf{A}_i^t$ using the momentum strategy as the Adam optimizer (Kingma & Ba, 2015). A pseudo-code of LORO is outlined in Algorithm 1, and a PyTorch code is attached in Algorithm 2. We refer the readers to Appendix C for more detailed discussions on how LORO benefits low-rank pretraining.

Algorithm 1 **Lo**w-rank **R**iemannian **O**ptimizer

Input: current update step t, trainable low-rank factors $\mathbf{B}_i^t, \mathbf{A}_i^{t\top} \in \mathbb{R}^{d\times r}$, backward gradients $\nabla_{\mathbf{B}_i^t} L$, $\nabla_{\mathbf{A}_i^t} L$, learning rate η, exact update frequency K.
Output: updated low-rank factors $\mathbf{B}_i^{t+1}$, $\mathbf{A}_i^{t+1}$.
if $t \bmod K \neq 0$ **then**
  $\mathbf{B}_i^{t+1} \leftarrow \mathbf{B}_i^t - \eta(r/d)\nabla_{\mathbf{B}_i^t} L$
  $\mathbf{A}_i^{t+1} \leftarrow \mathbf{A}_i^t - \eta(r/d)\nabla_{\mathbf{A}_i^t} L$
else
  $(\mathbf{Q}_B, \mathbf{R}_B) \leftarrow \mathrm{QR}(\mathbf{B}_i^t)$, $(\mathbf{Q}_A, \mathbf{R}_A) \leftarrow \mathrm{QR}(\mathbf{A}_i^{t\top})$
  $\mathbf{Z}_B \leftarrow \nabla_{\mathbf{B}_i^t} L \cdot \mathbf{R}_A^{-1}$, $\mathbf{Z}_A \leftarrow \nabla_{\mathbf{A}_i^t} L^\top \mathbf{R}_B^{-1}$
  $(\overline{\mathbf{Q}}_B, \overline{\mathbf{R}}_B) \leftarrow \mathrm{QR}(\mathbf{Z}_B - \mathbf{Q}_B\mathbf{Q}_B^\top \mathbf{Z}_B)$
  $(\overline{\mathbf{Q}}_A, \overline{\mathbf{R}}_A) \leftarrow \mathrm{QR}(\mathbf{Z}_A - \mathbf{Q}_A\mathbf{Q}_A^\top \mathbf{Z}_A)$
  $\mathbf{S} \leftarrow \mathbf{R}_B\mathbf{R}_A^\top - \eta\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}_i^t} L \cdot \mathbf{Q}_A$
  $\mathbf{U}_*\mathbf{\Sigma}_*\mathbf{V}_*^\top \leftarrow \mathrm{SVD}_r \begin{pmatrix} \mathbf{S} & -\eta\overline{\mathbf{R}}_A^\top \\ -\eta\overline{\mathbf{R}}_B & \mathbf{0} \end{pmatrix}$
  $\mathbf{B}_i^{t+1} \leftarrow (\mathbf{Q}_B, \overline{\mathbf{Q}}_B)\, \mathbf{U}_*\mathbf{\Sigma}_*^{1/2}$
  $\mathbf{A}_i^{t+1} \leftarrow \mathbf{\Sigma}_*^{1/2}\mathbf{V}_*^\top\, (\mathbf{Q}_A, \overline{\mathbf{Q}}_A)^\top$
end if
return $\mathbf{B}_i^{t+1}$, $\mathbf{A}_i^{t+1}$

Complexity Analysis. As shown in Algorithm 1, the algorithmic complexity of the approximate LORO update step is $O(rd)$ as it only involves element-wise operations. On the other hand, the exact update step includes QR decomposition of d-by-r matrices with a complexity of $O(r^2d)$, inversion of r-by-r triangular matrices with a complexity of $O(r^2)$, SVD of $2r$-by-$2r$ matrices at $O(r^3)$, and matrix multiplications with a complexity of $O(r^2d)$. Suppose the frequency of the exact update is K, then the average complexity of LORO is $O(r^2(r+d)/K + rd)$. In practice, we set the frequency K in the same order as r, which simplifies the LORO complexity to $O(r^2 + rd)$. Therefore, when comparing with the standard Euclidean low-rank pretraining with a $O(rd)$ complexity, LORO does not introduce evident computational overhead.

5 Experiment Results

5.1 Efficient Low-rank Pretraining

Experiment setup. In this section, we evaluate the effectiveness of our proposed LORO in training low-rank language models from scratch. All the experiments are implemented in PyTorch (Paszke et al., 2019) and conducted on NVIDIA 40G A100 GPUs. For a fair comparison with existing baselines, we follow a standard experiment setup used in Zhao et al. (2024); Lialin et al. (2024). Specifically, we adopt the LLaMA-based language model (Touvron et al., 2023b) equipped with RMSNorm (Zhang & Sennrich, 2019) and SwiGLU activations (Shazeer, 2020). We follow the same model configurations as (Zhao et al., 2024, Table 5) for the LLaMA-60M, -130M, -350M, and -1B models[2] and we conduct all the pretraining experiments using BF16 (bfloat16) format. We train all the models on the C4 (Colossal Clean Crawled Corpus) dataset (Raffel et al., 2019), a large-scale cleaned dataset designed for language models pretraining. The models are trained on a sufficiently large volume of data without repetition to simulate the pretraining setting in practice.

LORO configurations. We consider the low-rank parameterization that represents each target weight matrix as the product of a rank-r tall matrix and a rank-r wide matrix as defined in equation 2. In this paper, we apply low-rank parameterization to all weights in attention functions (e.g. query-, key-, value-, and out-projection) and all linear projection layers (e.g. down-, up-, and gate-projection) with the same rank r. Following the setup in Zhao et al. (2024); Han et al. (2024), we keep the embedding layer and the output language model head to be full-size. We fix the LORO update frequency as $K = 500$ step for all pretraining experiments, and we only tune the learning rate. We refer the readers to Appendix A for implementation details of the experiments in this subsection.

Baselines. We compare our LORO against multiple baselines that involve low-rank structures. 'Full-size' denotes pretraining the full-size model with the Adam optimizer (Kingma & Ba, 2015). 'Low-rank' is a traditional low-rank training approach (Kamalakara et al., 2022). 'LoRA' (Hu et al., 2022) is a low-rank fine-tuning method that reparameterize each weight as $\mathbf{W} + \mathbf{BA}$, where $\mathbf{W}$ is a full-size matrix that remains fixed during training, and $\mathbf{B}$ and $\mathbf{A}$ are trainable low-rank factors. In pretraining scenarios, the full-size matrix $\mathbf{W}$ is randomly initialized. 'ReLoRA' (Lialin et al., 2024) is a variant of LoRA that are adapted to pretraining by periodically absorbing the updated

[2]Since this study primarily focuses on comparing performance using BF16 across various pertaining methods, we did not conduct experiments on larger models due to limited computational resources.

$\mathbf{BA}$ into $\mathbf{W}$. 'GaLore' (Zhao et al., 2024) leverages a low-rank gradient to update the full-size weights. 'SLTrain' (Han et al., 2024) parameterizes each weight as the sum of low-rank factors and a sparse matrix, with both components being jointly optimized. However, all baselines, except for 'Low-rank', work with full-size models and do not engage in the challenging low-rank pretraining.

LORO finds high-performing low-rank Models. Table 1 shows that, our LORO consistently outperforms all other memory-efficient baselines in terms of both perplexity and total memory cost (including the memory usage of model parameters and optimizer states, as estimated in Han et al. (2024, Appendix C)) across all settings. For LLaMA-1B model, our LORO achieves a perplexity score that is 2% lower than the full-size baseline. These results validate the existence of competitive low-rank models within the low-rank matrix manifold, which exhibit comparable or superior performance compared to the full-size baselines. Moreover, our LORO offers a practical approach for training low-rank models from scratch without sacrificing the performance.

Table 1: Results of low-rank pretraining on C4 dataset, including validation perplexity (PPL), the number of parameters in millions (Param), the estimated total memory cost in GB (Mem), where 1GB contains 10^9 bytes. r, d_1, and d_2 denote the rank, the hidden- and intermediate-dimension of the LLaMA model. Results of the baselines are reported from Zhao et al. (2024); Han et al. (2024).

Model # Token r / d_1 / d_2	LLaMA-60M 1.1 B 128 / 512 / 1376			LLaMA-130M 2.2 B 256 / 768 / 2048			LLaMA-350M 6.4 B 256 / 1024 / 2736			LLaMA-1B 13.1 B 512 / 2048 / 5461		
Method	PPL (↓)	Param	Mem (↓)	PPL (↓)	Param	Mem (↓)	PPL (↓)	Param	Mem (↓)	PPL (↓)	Param	Mem (↓)
Full-size	34.06	58	0.35	24.36	134	0.81	18.80	368	2.21	15.56	1339	8.04
Low-rank	78.18	43	0.24	45.51	94	0.57	37.41	185	1.11	142.53	609	3.66
LoRA	34.99	58	0.36	33.92	134	0.84	25.58	368	1.85	19.21	1339	6.34
ReLoRA	37.04	58	0.36	29.37	134	0.84	29.08	368	1.85	18.33	1339	6.34
GaLore	34.88	58	0.28	25.36	134	0.61	18.95	368	1.59	15.64	1339	4.76
SLTrain	34.15	44	0.26	26.04	97	0.60	19.42	194	1.24	16.14	646	4.16
LORO	**33.96**	43	0.24	**24.59**	94	0.57	**18.84**	185	1.11	**15.19**	609	3.66

LORO enhances training efficiency. Section 1 shows that a low-rank model with $d_1 = d_2 = d$ is expected to achieve a $(2r/d)$-times reduction in memory and a $(d/2r)$-times speedup in inference. In practice, we set $r = d_1/4$ across all runs to pursue an approximate $\times 0.5$ FLOPS reduction and about $\times 0.5$ model size reduction. As shown in Table 2, for a LLaMA-1B model, LORO achieves $\times 1.82$ training speedup and a 54% reduction in memory compared to the full-size baseline, while GaLore does not exhibit practical speedup as it still works on the full-size model. We observe that the training speedup of LORO is less evident on LLaMA-350M model. This is because the data loading process introduces additional overhead and execution time, which overshadow the efficiency gains provided by LORO on smaller models.

Table 2: Training efficiency metrics evaluated on $1\times$40G A100 GPU, including ratio of memory cost reduction (MemRd), token batch size (TBSZ), and throughput (Tokens / sec).

Model	Method	PPL (↓)	Mem (↓)	MemRd (↑)	TBSZ	Tokens / sec (↑)	Speedup (↑)
LLaMA 350M	Full-size	18.80	2.21	0%	16K	35264	×1
	GaLore	18.95	1.59	28%	16K	35427	×1
	LORO	18.84	1.11	**50%**	16K	40640	**×1.15**
LLaMA 1B	Full-size	15.56	8.04	0%	8K	7890	×1
	GaLore	15.64	4.76	41%	8K	7886	×0.98
	LORO	15.19	3.66	**54%**	8K	14351	**×1.82**

LORO enhances inference efficiency. In Table 3, we empirically evaluate the inference efficiency of the low-rank LLaMA-1B model obtained by LORO with varying batch sizes and sequence lengths. Compared to the full-size baseline, our LORO achieves a $\times 2.08 \sim \times 4.04$ inference speedup and a $5\% \sim 17\%$ reduction in maximum memory allocation, across all settings. The speedup of LORO is evident for small batch sizes, and it converges to $\times 2$ as the batch size increases. This occurs as full-size baselines benefit more from GPU parallelization at larger batch sizes.

LORO finds flatter minima. As Table 1 shows that LORO is able to find minima within the low-rank matrix manifold with high performance comparable to the full-size models, we aim to compare between the low-rank minima found by LORO, denoted by $(\mathbf{B}_i^*\mathbf{A}_i^*)$, and the full-size minima, denoted by $(\mathbf{W}_i^*)$. To this end, we probe the loss landscape of these minima by via linear model interpolation (Lucas et al., 2021; Gueta et al., 2023). Specifically, for $\alpha \in \{0, 0.1, ..., 1\}$, we visualize the perplexity score of the model with weights $(1-\alpha)\mathbf{B}_i^*\mathbf{A}_i^* + \alpha\mathbf{W}_i^*$. As shown in Figure 3, a steep loss barrier is observed between the low-rank LORO-pretrained models and the full-size pretrained models, indicating that the LORO minima do not reside in the same basin as the full-size minima. Furthermore, the loss barrier exhibits an asymmetry, with the side closer to the full-size models being significantly steeper than the side closer to the low-rank models. This implies that LORO can find flatter low-rank minima, which may explain why it sometimes outperforms the full-size baselines as shown in Table 2.

Table 3: Inference efficiency metrics on 1×40G A100 GPU, including maximum memory allocation in GB (MaxMem), memory reduction (MemRd), and total time of processing 500 batches.

LLaMA 1B	Batch size = 512					Batch size = 256				
	Seq len	MaxMem (↓)	MemRd (↑)	Time (↓)	Speedup (↑)	Seq len	MaxMem (↓)	MemRd (↑)	Time (↓)	Speedup (↑)
Full-size	128	26.47	-	4.31	-	256	26.50	-	6.67	-
LORO	128	25.07	5%	1.57	×2.14	256	25.10	5%	3.07	×2.17
Full-size	64	14.49	-	1.79	-	128	14.52	-	4.69	-
LORO	64	13.08	10%	0.79	×2.27	128	13.11	10%	1.54	×3.04
Full-size	32	8.49	-	1.41	-	64	8.52	-	3.28	-
LORO	32	7.09	17%	0.43	×3.32	64	7.12	16%	0.81	×4.04
LLaMA 1B	Batch size = 128					Batch size = 64				
	Seq len	MaxMem (↓)	MemRd (↑)	Time (↓)	Speedup (↑)	Seq len	MaxMem (↓)	MemRd (↑)	Time (↓)	Speedup (↑)
Full-size	512	26.52	-	13.21	-	1024	26.53	-	26.63	-
LORO	512	25.12	5%	6.33	×2.09	1024	25.12	5%	12.79	×2.08
Full-size	256	14.53	-	6.67	-	512	14.54	-	14.68	-
LORO	256	13.13	10%	3.1	×2.15	512	13.14	10%	6.31	×2.33
Full-size	128	8.54	-	3.73	-	256	8.55	-	7.59	-
LORO	128	7.14	16%	1.61	×2.32	256	7.15	16%	3.23	×2.35

LORO explores effective low-rank structures. We compare the performance of LORO low-rank pretrained models against various full-size counterparts within the same parameter budget. Specifically, we evaluate three full-size variants of the original model: 1) 'Full-size Shallow,' with fewer layers than the full model, 2) 'Full-size Slim,' with a reduced hidden dimension but the same number of layers, and 3) 'LORO + Slim MLP,' where LORO is applied only to the attention weights, while the MLP layers are slimmer but retain full-size matrices. Their configurations are detailed in Appendix A. As reported in Table 4, the LORO low-rank pretrained models outperform full-size variants, demonstrating that LORO unlocks the potential of low-rank structures as a more effective parameterization scheme. We attribute this to the fact that, within the same parameter budget, low-rank structures benefit from a larger hidden dimension, which enhances their ability in learning high-dimensional features.

Figure 3: Linear interpolation between LORO pretrained models and full-size baselines.

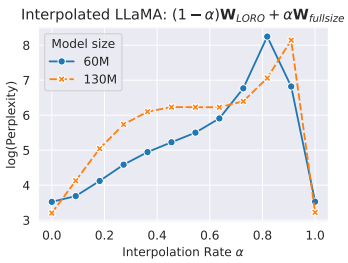


Table 4: Comparison between LORO and other pretrained full-size architectures within the same parameter budget.

Model # Token r / d_1 / d_2	LLaMA-130M 2.2 B 256 / 768 / 2048			LLaMA-350M 6.4 B 256 / 1024 / 2736		
Method	PPL (↓)	Param	Mem (↓)	PPL (↓)	Param	Mem (↓)
Full-size	24.36	134	0.81	18.80	368	2.21
Low-rank	45.51	94	0.57	37.41	185	1.11
Full-size Shallow	28.49	94	0.57	21.67	185	1.11
Full-size Slim	25.34	94	0.57	21.40	185	1.11
LORO + Slim MLP	25.16	94	0.57	18.95	185	1.11
LORO	**24.59**	94	0.57	**18.84**	185	1.11

5.2 Ablation Studies

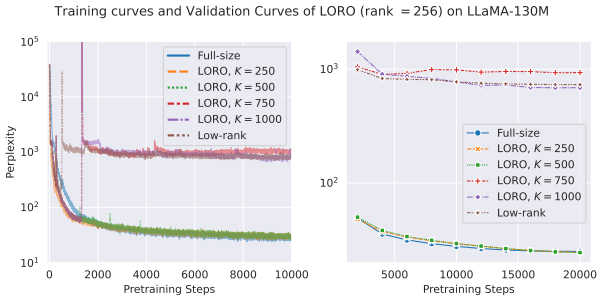
Ablation on the LORO rank r. In Table 5, we evaluate the perplexity scores of LORO pretrained LLaMA-60M models with varying ranks. As anticipated, the performance decays as r decreases. In practice, setting $r = d_1/4$ is preferable, as it achieves a 50% reduction in both memory and FLOPS while maintaining high performance.

Ablation on the LORO exact update frequency K. In Figure 4, we compare the convergence of different runs of LORO with varying exact update frequencies. It is observed that, for $K \geqslant 750$ when the LORO exact update is overly lazy, the training curve tends to explode and fail to converge to performing minima. In practice, we recommend to set $K = 500$, as it significantly reduces training overhead and results in competitive low-rank models with performance comparable to full-size baselines across LLaMA models of varying sizes, ranging from 60M to 1B parameters.

Table 5: Ablation on LORO rank r.

Model	LLaMA-60M			
# Token	1.1 B			
d_1 / d_2	512 / 1376			
Method	Rank	PPL (↓)	Param	Mem (↓)
Full-size	512	34.06	58	0.35
Low-rank	128	78.18	43	0.24
LORO	256	31.54	53	0.31
LORO	128	33.96	43	0.24
LORO	64	39.40	38	0.22
LORO	32	49.88	35	0.21

Figure 4: Ablation on LORO exact update frequency K.



5.3 Efficient Low-rank Fine-tuning

Following the experiment setup in Zhao et al. (2024, Section 5.4), we extend our LORO to fine-tune the pretrained RoBERTa-base model Liu et al., 2019 on GLUE datasets (Wang et al., 2019). Specifically, we reparameterized each weight in the query- and key-projections as $\mathbf{W} + \mathbf{BA}$, where $\mathbf{W}$ is the full-size pretrained weight and $\mathbf{B}$ and $\mathbf{A}$ are low-rank factors. Then, we apply LORO to $\mathbf{B}$ and $\mathbf{A}$ exclusively and freeze the remaining parameters. The hyperparameters are detailed in Appendix A.2. In Table 6, we compare the fine-tuning performance of LORO with three baselines, including 'LoRA' (Hu et al., 2022), 'GaLore' (Zhao et al., 2024), and 'SLTrain' (Han et al., 2024). Although LORO achieves decent performance, its outperformance over other baselines in fine-tuning is less significant than in pretraining. We argue that, with full-size pretrained weights provided, fine-tuning requires learning only a small amount of domain-specific knowledge, which can be easily captured through low-rank adjustments in each weight matrix. As a result, the potential for improvement with different low-rank techniques is limited, and the performance gains become less noticeable.

Table 6: Extending LORO to fine-tuning RoBERTa-base models on GLUE benchmark. Results of the baselines are reported from Zhao et al. (2024); Han et al. (2024).

RoBERTa-base	Memory	CoLA	STS-B	MRPC	RTE	SST2	MNLI	QNLI	QQP	Avg
Full-size	747M	62.24	90.92	91.3	79.42	94.57	87.18	92.33	92.28	86.28
LoRA, $r = 4$	257M	61.38	90.57	91.07	78.70	92.89	86.82	92.18	**91.29**	85.61
Galore, $r = 4$	253M	60.35	90.73	92.25	**79.42**	94.04	**87.00**	92.24	91.06	**85.89**
LORO, $r = 4$	257M	**61.58**	**90.75**	**92.60**	78.69	**94.14**	86.15	**92.58**	90.64	**85.89**
LoRA, $r = 8$	264M	61.83	90.80	91.90	79.06	93.46	86.94	92.25	91.22	85.93
Galore, $r = 8$	257M	60.06	90.82	92.01	**79.78**	94.38	**87.17**	92.2	91.11	85.94
SLTrain, $r = 8$	-	60.35	90.74	92.38	79.42	94.15	86.53	92.40	**91.27**	85.93
LORO, $r = 8$	257M	**63.10**	**90.97**	**93.12**	78.92	**94.61**	86.65	**92.75**	91.00	**86.39**

6 Conclusion

In this paper, we proposed LORO for efficient yet effective pretraining of low-rank parameterized language models. Specifically, LORO optimizes the full-size product of the low-rank factors toward the steepest gradient descent direction on the low-rank matrix manifold, without computing any expensive full-size weight or gradient. Extensive experiments demonstrate that LORO can discover competitive low-rank models with performance comparable to full-size baselines, while providing significant memory reduction and acceleration in both training and inference. This work demonstrates the potential of low-rank structures in language model pretraining, inspiring us to explore efficient low-rank pretraining methods for vision and multi-modal generative models.

REFERENCES

P. A. Absil and I. V. Oseledets. Low-rank retractions: a survey and new results. *Comput. Optim. Appl.*, 62(1):5–29, September 2015. ISSN 0926-6003. doi: 10.1007/s10589-014-9714-4. URL https://doi.org/10.1007/s10589-014-9714-4.

P.A. Absil, R. Mahony, and R. Sepulchre. *Optimization Algorithms on Matrix Manifolds*. Princeton University Press, 2009. ISBN 9781400830244. URL https://books.google.com.sg/books?id=NSQGQeLN3NcC.

Ahmed Agiza, Marina Neseem, and Sherief Reda. Mtlora: Low-rank adaptation approach for efficient multi-task learning. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, pp. 16196–16205, 2024.

Yonatan Amit, Michael Fink, Nathan Srebro, and Shimon Ullman. Uncovering shared structures in multiclass classification. In *Proceedings of the 24th International Conference on Machine Learning*, ICML '07, pp. 17–24, New York, NY, USA, 2007. Association for Computing Machinery. ISBN 9781595937933. doi: 10.1145/1273496.1273499. URL https://doi.org/10.1145/1273496.1273499.

Fengmiao Bian, Jian-Feng Cai, and Rui Zhang. A preconditioned riemannian gradient descent algorithm for low-rank matrix recovery, 2023. URL https://arxiv.org/abs/2305.02543.

Kevin Bleakley and Yoshihiro Yamanishi. Supervised prediction of drug–target interactions using bipartite local models. *Bioinformatics*, 25(18):2397–2403, September 2009. ISSN 1367-4803. doi: 10.1093/bioinformatics/btp433. URL https://doi.org/10.1093/bioinformatics/btp433.

Tom Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared D Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, Sandhini Agarwal, and et al. Language models are few-shot learners. In H. Larochelle, M. Ranzato, R. Hadsell, M.F. Balcan, and H. Lin (eds.), *Advances in Neural Information Processing Systems*, volume 33, pp. 1877–1901. Curran Associates, Inc., 2020. URL https://proceedings.neurips.cc/paper_files/paper/2020/file/1457c0d6bfcb4967418bfb8ac142f64a-Paper.pdf.

Eric L. Buehler and Markus J. Buehler. X-lora: Mixture of low-rank adapter experts, a flexible framework for large language models with applications in protein mechanics and molecular design, 2024. URL https://arxiv.org/abs/2402.07148.

Tianqi Chen, Bing Xu, Chiyuan Zhang, and Carlos Guestrin. Training deep nets with sublinear memory cost. *ArXiv*, abs/1604.06174, 2016. URL https://api.semanticscholar.org/CorpusID:15865278.

Yukang Chen, Shengju Qian, Haotian Tang, Xin Lai, Zhijian Liu, Song Han, and Jiaya Jia. Longlora: Efficient fine-tuning of long-context large language models. *ArXiv*, abs/2309.12307, 2023. URL https://api.semanticscholar.org/CorpusID:262084134.

Tim Dettmers, Mike Lewis, Sam Shleifer, and Luke Zettlemoyer. 8-bit optimizers via block-wise quantization. *ArXiv*, abs/2110.02861, 2021. URL https://api.semanticscholar.org/CorpusID:238408308.

Tim Dettmers, Artidoro Pagnoni, Ari Holtzman, and Luke Zettlemoyer. Qlora: Efficient finetuning of quantized llms. *arXiv preprint arXiv:2305.14314*, 2023.

Jacob Devlin, Ming-Wei Chang, Kenton Lee, and Kristina Toutanova. BERT: Pre-training of deep bidirectional transformers for language understanding. In Jill Burstein, Christy Doran, and Thamar Solorio (eds.), *Proceedings of the 2019 Conference of the North American Chapter of the Association for Computational Linguistics: Human Language Technologies, Volume 1 (Long and Short Papers)*, pp. 4171–4186, Minneapolis, Minnesota, June 2019. Association for Computational Linguistics. doi: 10.18653/v1/N19-1423. URL https://aclanthology.org/N19-1423.

Ning Ding, Xingtai Lv, Qiaosen Wang, Yulin Chen, Bowen Zhou, Zhiyuan Liu, and Maosong Sun. Sparse low-rank adaptation of pre-trained language models. In *The 2023 Conference on Empirical Methods in Natural Language Processing*, 2023. URL https://openreview.net/forum?id=jxgz7FEqWq.

Marawan Gamal and Guillaume Rabusseau. ROSA: Random orthogonal subspace adaptation. In *Workshop on Efficient Systems for Foundation Models @ ICML2023*, 2023. URL https://openreview.net/forum?id=4P9vOFpb63.

Xavier Glorot and Yoshua Bengio. Understanding the difficulty of training deep feedforward neural networks. In *International Conference on Artificial Intelligence and Statistics*, 2010. URL https://api.semanticscholar.org/CorpusID:5575601.

Almog Gueta, Elad Venezian, Colin Raffel, Noam Slonim, Yoav Katz, and Leshem Choshen. Knowledge is a region in weight space for fine-tuned language models. In *The 2023 Conference on Empirical Methods in Natural Language Processing*, 2023. URL https://openreview.net/forum?id=vq4BnrPyPb.

Andi Han, Jiaxiang Li, Wei Huang, Mingyi Hong, Akiko Takeda, Pratik Jawanpuria, and Bamdev Mishra. Sltrain: a sparse plus low-rank approach for parameter and memory efficient pretraining, 2024. URL https://arxiv.org/abs/2406.02214.

Soufiane Hayou, Nikhil Ghosh, and Bin Yu. LoRA+: Efficient low rank adaptation of large models. In *Forty-first International Conference on Machine Learning*, 2024. URL https://openreview.net/forum?id=NEv8YqBROO.

Edward J Hu, yelong shen, Phillip Wallis, Zeyuan Allen-Zhu, Yuanzhi Li, Shean Wang, Lu Wang, and Weizhu Chen. LoRA: Low-rank adaptation of large language models. In *International Conference on Learning Representations*, 2022. URL https://openreview.net/forum?id=nZeVKeeFYf9.

Chengsong Huang, Qian Liu, Bill Yuchen Lin, Chao Du, Tianyu Pang, and Min Lin. Lorahub: Efficient cross-task generalization via dynamic loRA composition. In *R0-FoMo:Robustness of Few-shot and Zero-shot Learning in Large Foundation Models*, 2023. URL https://openreview.net/forum?id=lyRpY2bpBh.

Siddhartha Rao Kamalakara, Acyr Locatelli, Bharat Venkitesh, Jimmy Ba, Yarin Gal, and Aidan N. Gomez. Exploring low rank training of deep neural networks, 2022. URL https://arxiv.org/abs/2209.13569.

Mikhail Khodak, Neil A. Tenenholtz, Lester Mackey, and Nicolo Fusi. Initialization and regularization of factorized neural layers. In *International Conference on Learning Representations*, 2021. URL https://openreview.net/forum?id=KTlJT1nof6d.

Diederik P. Kingma and Jimmy Ba. Adam: A method for stochastic optimization. In Yoshua Bengio and Yann LeCun (eds.), *3rd International Conference on Learning Representations, ICLR 2015, San Diego, CA, USA, May 7-9, 2015, Conference Track Proceedings*, 2015. URL http://arxiv.org/abs/1412.6980.

Othmar Koch and Christian Lubich. Dynamical low-rank approximation. *SIAM Journal on Matrix Analysis and Applications*, 29(2):434–454, 2007. doi: 10.1137/050639703. URL https://doi.org/10.1137/050639703.

Soroush Abbasi Koohpayegani, Navaneet K L, Parsa Nooralinejad, Soheil Kolouri, and Hamed Pirsiavash. NOLA: Compressing loRA using linear combination of random basis. In *The Twelfth International Conference on Learning Representations*, 2024. URL https://openreview.net/forum?id=TjfXcDgvzk.

J.M. Lee. *Introduction to Riemannian Manifolds*. Graduate Texts in Mathematics. Springer International Publishing, 2019. ISBN 9783319917542. URL https://books.google.com.sg/books?id=UIPltQEACAAJ.

Adrian S. Lewis and Jérôme Malick. Alternating projections on manifolds. *Math. Oper. Res.*, 33 (1):216–234, February 2008. ISSN 0364-765X. doi: 10.1287/moor.1070.0291. URL https://doi.org/10.1287/moor.1070.0291.

Bingrui Li, Jianfei Chen, and Jun Zhu. Memory efficient optimizers with 4-bit states. In *Thirty-seventh Conference on Neural Information Processing Systems*, 2023. URL https://openreview.net/forum?id=nN8TnHB5nw.

Vladislav Lialin, Sherin Muckatira, Namrata Shivagunde, and Anna Rumshisky. ReloRA: High-rank training through low-rank updates. In *The Twelfth International Conference on Learning Representations*, 2024. URL https://openreview.net/forum?id=DLJznSp6X3.

Yinhan Liu, Myle Ott, Naman Goyal, Jingfei Du, Mandar Joshi, Danqi Chen, Omer Levy, Mike Lewis, Luke Zettlemoyer, and Veselin Stoyanov. Roberta: A robustly optimized BERT pretraining approach. *CoRR*, abs/1907.11692, 2019. URL http://arxiv.org/abs/1907.11692.

James Lucas, Juhan Bae, Michael R. Zhang, Stanislav Fort, Richard S. Zemel, and Roger Baker Grosse. Analyzing monotonic linear interpolation in neural network loss landscapes. *ArXiv*, abs/2104.11044, 2021. URL https://api.semanticscholar.org/CorpusID:233346934.

Bamdev Mishra, Gilles Meyer, Silvère Bonnabel, and Rodolphe Sepulchre. Fixed-rank matrix factorizations and riemannian low-rank optimization. *Comput. Stat.*, 29(3–4):591–621, June 2014. ISSN 0943-4062. doi: 10.1007/s00180-013-0464-z. URL https://doi.org/10.1007/s00180-013-0464-z.

Guillaume Olikier, André Uschmajew, and Bart Vandereycken. Gauss-southwell type descent methods for low-rank matrix optimization. *ArXiv*, abs/2306.00897, 2023. URL https://api.semanticscholar.org/CorpusID:258999971.

Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, Alban Desmaison, Andreas Köpf, Edward Yang, Zach DeVito, Martin Raison, Alykhan Tejani, Sasank Chilamkurthy, Benoit Steiner, Lu Fang, Junjie Bai, and Soumith Chintala. *PyTorch: an imperative style, high-performance deep learning library*. Curran Associates Inc., Red Hook, NY, USA, 2019.

Alec Radford and Karthik Narasimhan. In *Improving Language Understanding by Generative Pre-Training*, 2018. URL https://api.semanticscholar.org/CorpusID:49313245.

Colin Raffel, Noam Shazeer, Adam Roberts, Katherine Lee, Sharan Narang, Michael Matena, Yanqi Zhou, Wei Li, and Peter J. Liu. Exploring the limits of transfer learning with a unified text-to-text transformer. *CoRR*, abs/1910.10683, 2019. URL http://arxiv.org/abs/1910.10683.

Samyam Rajbhandari, Jeff Rasley, Olatunji Ruwase, and Yuxiong He. Zero: memory optimizations toward training trillion parameter models. In *Proceedings of the International Conference for High Performance Computing, Networking, Storage and Analysis*, SC '20. IEEE Press, 2020. ISBN 9781728199986.

Amrutha Varshini Ramesh, Vignesh Ganapathiraman, Issam H. Laradji, and Mark Schmidt. BlockLLM: Memory-efficient adaptation of LLMs by selecting and optimizing the right coordinate blocks, 2024. URL https://openreview.net/forum?id=L0pXYjtfE3.

David E. Rumelhart, Geoffrey E. Hinton, and Ronald J. Williams. *Learning representations by back-propagating errors*, pp. 696–699. MIT Press, Cambridge, MA, USA, 1988. ISBN 0262010976.

Dayana Savostianova, Emanuele Zangrando, Gianluca Ceruti, and Francesco Tudisco. Robust low-rank training via approximate orthonormal constraints. In *Thirty-seventh Conference on Neural Information Processing Systems*, 2023. URL https://openreview.net/forum?id=NJPSvv0u3R.

Steffen Schotthöfer, Emanuele Zangrando, Jonas Kusch, Gianluca Ceruti, and Francesco Tudisco. Low-rank lottery tickets: finding efficient low-rank neural networks via matrix differential equations. In Alice H. Oh, Alekh Agarwal, Danielle Belgrave, and Kyunghyun Cho (eds.), *Advances in Neural Information Processing Systems*, 2022. URL https://openreview.net/forum?id=IILJ0KWZMy9.

Noam Shazeer. GLU variants improve transformer. *CoRR*, abs/2002.05202, 2020. URL https://arxiv.org/abs/2002.05202.

Yang Sui, Miao Yin, Wanzhao Yang, Yu Gong, Jinqi Xiao, Huy Phan, Ding Ding, Xiaozhong Xu, Shan Liu, Zhenzhong Chen, and Bo Yuan. ELRT: Towards efficient low-rank training for compact neural networks, 2023. URL https://openreview.net/forum?id=TC39w69m8bB.

Gemma Team, Thomas Mesnard, Cassidy Hardin, Robert Dadashi, Surya Bhupatiraju, Shreya Pathak, Laurent Sifre, Morgane Rivière, Mihir Sanjay Kale, Juliette Love, and et al. Gemma: Open models based on gemini research and technology, 2024a. URL https://arxiv.org/abs/2403.08295.

Gemma Team, Morgane Riviere, Shreya Pathak, Pier Giuseppe Sessa, Cassidy Hardin, Surya Bhupatiraju, Léonard Hussenot, Thomas Mesnard, Bobak Shahriari, Alexandre Ramé, and et al. Gemma 2: Improving open language models at a practical size, 2024b. URL https://arxiv.org/abs/2408.00118.

Hugo Touvron, Thibaut Lavril, Gautier Izacard, Xavier Martinet, Marie-Anne Lachaux, Timothée Lacroix, Baptiste Rozière, Naman Goyal, Eric Hambro, Faisal Azhar, and et al. Llama: Open and efficient foundation language models, 2023a. URL https://arxiv.org/abs/2302.13971.

Hugo Touvron, Louis Martin, Kevin Stone, Peter Albert, Amjad Almahairi, Yasmine Babaei, Nikolay Bashlykov, Soumya Batra, Prajjwal Bhargava, Shruti Bhosale, and et al. Llama 2: Open foundation and fine-tuned chat models, 2023b. URL https://arxiv.org/abs/2307.09288.

Mojtaba Valipour, Mehdi Rezagholizadeh, Ivan Kobyzev, and Ali Ghodsi. Dylora: Parameter efficient tuning of pre-trained models using dynamic search-free low-rank adaptation, 2023. URL https://arxiv.org/abs/2210.07558.

Bart Vandereycken. Low-rank matrix completion by riemannian optimization. *SIAM Journal on Optimization*, 23(2):1214–1236, 2013. doi: 10.1137/110845768. URL https://doi.org/10.1137/110845768.

Ashish Vaswani, Noam Shazeer, Niki Parmar, Jakob Uszkoreit, Llion Jones, Aidan N Gomez, Ł ukasz Kaiser, and Illia Polosukhin. Attention is all you need. In I. Guyon, U. Von Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett (eds.), *Advances in Neural Information Processing Systems*, volume 30. Curran Associates, Inc., 2017. URL https://proceedings.neurips.cc/paper_files/paper/2017/file/3f5ee243547dee91fbd053c1c4a845aa-Paper.pdf.

Alex Wang, Amanpreet Singh, Julian Michael, Felix Hill, Omer Levy, and Samuel R. Bowman. GLUE: A multi-task benchmark and analysis platform for natural language understanding. In *International Conference on Learning Representations*, 2019. URL https://openreview.net/forum?id=rJ4km2R5t7.

Wenhan Xia, Chengwei Qin, and Elad Hazan. Chain of lora: Efficient fine-tuning of language models via residual learning. *ArXiv*, abs/2401.04151, 2024. URL https://api.semanticscholar.org/CorpusID:266899736.

Ming Yuan, Ali Ekici, Zhaosong Lu, and Renato Monteiro. Dimension Reduction and Coefficient Estimation in Multivariate Linear Regression. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 69(3):329–346, 05 2007. ISSN 1369-7412. doi: 10.1111/j.1467-9868.2007.00591.x. URL https://doi.org/10.1111/j.1467-9868.2007.00591.x.

Biao Zhang and Rico Sennrich. *Root mean square layer normalization*. Curran Associates Inc., Red Hook, NY, USA, 2019.

Fangzhao Zhang and Mert Pilanci. Riemannian preconditioned loRA for fine-tuning foundation models. In *Forty-first International Conference on Machine Learning*, 2024. URL https://openreview.net/forum?id=IwqE4QqBew.

Qingru Zhang, Minshuo Chen, Alexander Bukharin, Pengcheng He, Yu Cheng, Weizhu Chen, and Tuo Zhao. Adaptive budget allocation for parameter-efficient fine-tuning. In *The Eleventh International Conference on Learning Representations*, 2023. URL https://openreview.net/forum?id=lq62uWRJjiY.

Yushun Zhang, Congliang Chen, Ziniu Li, Tian Ding, Chenwei Wu, Yinyu Ye, Zhi-Quan Luo, and Ruoyu Sun. Adam-mini: Use fewer learning rates to gain more. In *Workshop on Efficient Systems for Foundation Models II @ ICML2024*, 2024a. URL https://openreview.net/forum?id=TOnkMNfx9d.

Zhenyu Zhang, Ajay Jaiswal, Lu Yin, Shiwei Liu, Jiawei Zhao, Yuandong Tian, and Zhangyang Wang. Q-galore: Quantized galore with int4 projection and layer-adaptive low-rank gradients, 2024b. URL https://arxiv.org/abs/2407.08296.

Jiawei Zhao, Zhenyu Zhang, Beidi Chen, Zhangyang Wang, Anima Anandkumar, and Yuandong Tian. Galore: Memory-efficient llm training by gradient low-rank projection, 2024.

A IMPLEMENTATION DETAILS

A.1 PRETRAINING LLAMA MODELS ON C4 DATASET

The implementation details introduced in this subsection applies to the experiments in Table 1, Table 2, Table 3, Table 4, Table 5, Figure 3, and Figure 4. To achieve a fair comparison with other pretraining baselines, we follow the same model architecture and pretraining hyperparameters as in Zhao et al. (2024, Appendix C.1). For all runs, we set the max data sequence length as 256 with a batch size of 512 (i.e., a token batch size of 131K). For a fair comparisons in terms of the pretraining efficiency, we run all the experiments on $1\times$ NIVIDIA 40G A100 GPU, without using model parallelization or data parallelization. To achieve the 512 training batch size, we adjust the per-device batch size and vary the number of gradient accumulation steps according to the size of the LLaMA models. Based on Zhao et al. (2024, Appendix C.1), we employ a learning rate warmup starting from 0 during the first 10% of the pretraining steps, followed by a cosine annealing scheduler that decays to 10% of the maximum learning rate. We initialize the low-rank factors with Xavier initialization (Glorot & Bengio, 2010) across all experiments.

We implement LORO following the pseudo-code attached in 2. Following Zhao et al. (2024); Touvron et al. (2023b), we impose low-rank parameterization to all the weight matrices in the self-attention modules (including `q_proj`, `k_proj`, `v_proj`, and `o_proj`) and linear projection modules (including `down_proj`, `up_proj`, and `gate_proj`) and we update them with LORO, while we apply the standard Adam optimizer to update the `token_emb`, `lm_head`, and other layer normalization modules. All modules share the same learning rate. Specifically, for all LORO pretraining results reported in Section 5.1, we set the LORO exact update frequency to $K = 500$ and the learning rate to 0.01 for the LLaMA-60M, -130M, and -350M models, while for LLaMA-1B, we set $K = 200$ and the learning rate to 0.005. To stabilize the training process, at each LORO exact update step, we apply a 5-step linear learning rate warmup and we refresh the Adam statistics.

A.2 FINE-TUNING ROBERTA MODELS ON C4 DATASET

For a fair comparison on fine-tuning with other baselines, we follow the same model architecture, datasets and evaluation processes as in Zhao et al. (2024, Appendix D.1). We report the Matthew's correlation score for CoLA, the Pearson correlation score for STS-B, matched and mismatched accuracy for MNLI, F1 score for MRPC, and accuracy for other tasks. Specifically, we fine-tune the pretrained RoBERTa-base checkpoint from Huggingface[3] on the GLUE benchmark. For each weight matrices in `q_proj` and `k_proj`, we adopt the standard LoRA parameterization $\mathbf{W} + \lambda \mathbf{B}\mathbf{A}$, where $\mathbf{W}$ denotes the full-size pretrained weight, λ is the LoRA scale hyperparameter, and $\mathbf{B}$ and $\mathbf{A}$ are the trainable low-rank factors. During fine-tuning, we apply LORO to update the low-rank factors in the same manner as in pretraining, while keeping all other parameters unchanged. The training hyperparameters are attached in Table 8.

Table 7: Model architecture and pretraining settings. d_1 and d_2 denotes the hidden- and intermediate-dimension of the LLaMA model. '# Tokens' denotes the amount of training tokens, 'PD-BSZ' denotes the per-device batch size, and '# Grad Accum' denotes the number of gradient accumulation steps.

Model	Params	d_1	d_2	Heads	Layers	Steps	# Tokens	PD-BSZ	# Grad Accum
60M	58M	512	1376	8	8	10K	1.3B	256	2
130M	134M	768	2048	12	12	20K	2.6B	128	4
94M Shallow	94M	820	1720	12	6	20K	2.6B	128	4
94M Slim	94M	640	1250	12	12	20K	2.6B	128	4
94M Slim MLP	94M	768	928	12	12	20K	2.6B	128	4
368M	368M	1024	2736	16	24	60K	7.8B	64	8
185M Shallow	185M	1024	1880	16	12	60K	7.8B	64	8
185M Slim	185M	736	1640	16	24	60K	7.8B	64	8
185M Slim MLP	185M	1024	940	16	24	60K	7.8B	64	8
1B	1339M	2048	5461	24	32	100K	13.1B	32	16

[3] `https://huggingface.co/docs/transformers/model_doc/roberta`

Table 8: Hyperparameters of LORO in fine-tuning RoBERTa experiments.

		CoLA	STS-B	MRPC	RTE	SST-2	MNLI	QNLI	QQP
Rank $r = 4$	Batch Size	64	16	16	16	16	16	16	16
	# Epochs	30	40	20	30	40	30	30	30
	Learning Rate	4e-4	4e-4	2e-4	5e-4	5e-5	5e-5	4e-4	5e-5
	LoRA λ	32	8	32	8	8	8	8	8
	LORO Freq K	100	200	100	100	100	100	100	100
Rank $r = 8$	Batch Size	64	16	16	16	16	16	16	16
	# Epochs	30	40	20	30	40	30	30	30
	Learning Rate	4e-4	4e-4	2e-4	5e-4	5e-5	5e-5	4e-4	5e-5
	LoRA λ	16	16	8	8	8	16	8	16
	LORO Freq K	20	100	100	100	500	200	100	200

A.3 PyTorch Implementation of LORO

Algorithm 2 Low-rank **R**iemannian **O**ptimizer (**LORO**) in PyTorch (Paszke et al., 2019)

```
def LORO_step(B,A,lr,n_step,K):
    # B: [d, r], A: [d, r]
    if (n_step + 1) % K != 0:              # Approximate Riemannian update
        d, r = B.shape
        B = B - lr * (r / d) * B.grad
        A = A - lr * (r / d) * A.grad
    else:
        dB, dA = B.grad, A.grad # dB: [d, r], dA: [r, d]
        Qb, Rb = torch.qr(B) # Qb: [d, r], Rb: [r, r]
        Qa, Ra = torch.qr(A) # Qa: [d, r], Ra: [r, r]
        # dB_Ra_inv: [d, r], dA_Rb_inv: [d, r]
        dB_Ra_inv = torch.linalg.solve(Ra.T, dB.T).T
        dA_Rb_inv = torch.linalg.solve(Rb.T, dA).T
        # Qb_: [d, r], Rb_: [r, r], Qa_: [d, r], Ra_: [r, r]
        Qb_, Rb_ = torch.qr(dB_Ra_inv - Qb @ Qb.T @ dB_Ra_inv)
        Qa_, Ra_ = torch.qr(dA_Rb_inv - Qa @ Qa.T @ dA_Rb_inv)
        S = Rb @ Ra.T - lr * dA_Rb_inv.T @ Qa # [r, r]
        # SVD on [2r, 2r], U: [2r, r], Sig: [r,r], V: [2r,r]
        U, Sig, V = torch.svd(mat([[S, - lr * Ra_.T], [- lr * Rb, zeros]]))

        B = mat([Qb, Qb_]) @ U @ Sig.sqrt()
        A = Sig.sqrt() @ V.T @ mat([Qa, Qa_]).T  # Exact Riemannian update

    return B, A
```

B Proof Details

Proposition 3 (Riemannian Gradient under Low-rank Parameterization)**.** *Following the previous notation, for any low-rank parameterized weight* $\mathbf{W} = \mathbf{BA} \in \mathcal{M}_r$*, where* $\mathbf{B} \in \mathbb{R}^{d\times r}$ *and* $\mathbf{A} \in \mathbb{R}^{r\times d}$*, the Riemannian gradient of any smooth function* $L(\cdot) : \mathbb{R}^{d\times d} \mapsto \mathbb{R}$ *evaluated at* $\mathbf{W}$ *is*

$$\operatorname{grad} L(\mathbf{W}) = \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1} \cdot \nabla_{\mathbf{A}} L + \nabla_{\mathbf{B}} L \cdot (\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A} - \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1} \cdot \nabla_{\mathbf{A}} L \cdot \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}. \quad (13)$$

Proof of Proposition 3. We first need to show that, for any ambient point $\mathbf{G} \in \mathbb{R}^{d\times d}$, the orthogonal projection to the tangent space $T_{(\mathbf{BA})}\mathcal{M}_r$ is given by

$$\mathcal{P}(\mathbf{G}) = \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A} - \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top \cdot \mathbf{G} \cdot \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}.$$

Then, the proof can be completed by taking $\mathbf{G} = \nabla_{\mathbf{W}} L$, and showing that $\nabla_{\mathbf{W}} L \cdot \mathbf{A}^\top = \nabla_{\mathbf{B}} L$ and $\mathbf{B} \cdot \nabla_{\mathbf{W}} L = \nabla_{\mathbf{A}} L$.

According to Proposition 1 (Lee (2019, Example 8.14) and Absil et al. (2009, Equation 3.37)), the othorgonal operator is $\mathcal{P}(\mathbf{G}) \triangleq \mathbf{U}\mathbf{U}^\top \cdot \mathbf{G} + \mathbf{G} \cdot \mathbf{V}\mathbf{V}^\top - \mathbf{U}\mathbf{U}^\top \cdot \mathbf{G} \cdot \mathbf{V}\mathbf{V}^\top$, where $\mathbf{W} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ is the singular value decomposition (SVD) of $\mathbf{W}$. Suppose the QR decompositions of $\mathbf{B}$ and $\mathbf{A}^\top$ are

$\mathbf{Q}_B\mathbf{R}_B = \mathbf{B}$ and $\mathbf{Q}_A\mathbf{R}_A = \mathbf{A}^\top$, and the SVD of the full-rank r-by-r matrix $\mathbf{R}_B\mathbf{R}_A^\top$ is $\mathbf{U}_R\mathbf{\Sigma}_R\mathbf{V}_R^\top$. Then, it holds that $(\mathbf{Q}_B\mathbf{U}_R)\mathbf{\Sigma}_R(\mathbf{V}_R^\top\mathbf{Q}_A^\top) = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$, where $(\mathbf{Q}_B\mathbf{U}_R), (\mathbf{V}_R^\top\mathbf{Q}_A^\top) \in \mathbb{R}^{d\times r}$ are column-wise orthogonal and $\mathbf{\Sigma}_R \in \mathbb{R}^{r\times r}$ is diagonal. By the uniqueness of SVD, it holds that $\mathbf{Q}_B\mathbf{U}_R = \mathbf{U}$ and $\mathbf{Q}_A\mathbf{V}_R = \mathbf{V}$, which implies

$$\begin{aligned}\mathbf{U}\mathbf{U}^\top =& \mathbf{Q}_B\mathbf{U}_R\mathbf{U}_R^\top\mathbf{Q}_B^\top = \mathbf{Q}_B\mathbf{Q}_B^\top = (\mathbf{Q}_B\mathbf{R}_B)(\mathbf{R}_B^{-1}\mathbf{Q}_B^\top\mathbf{Q}_B\mathbf{R}_B^{-\top})(\mathbf{R}_B^\top\mathbf{Q}_B^\top) \qquad (14)\\ =&(\mathbf{Q}_B\mathbf{R}_B)(\mathbf{R}_B^\top\mathbf{Q}_B^\top\mathbf{Q}_B\mathbf{R}_B)^{-1}(\mathbf{R}_B^\top\mathbf{Q}_B^\top) = \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top. \qquad (15)\end{aligned}$$

Similarly, we have $\mathbf{V}\mathbf{V}^\top = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}$, which completes the proof. In intuition, the derivation above proves that the row-space of $\mathbf{B}$ coincides with that of $\mathbf{U}$, and the column-space of $\mathbf{A}^\top$ coincides with that of $\mathbf{V}$. Therefore, the othorgonal projection to the tangent space can be reparameterized under the basis of $\mathbf{B}$ and $\mathbf{A}$.

□

Proposition 4 (Riemannian Update under Low-rank Parameterization)**.** *Following the previous notations, for any low-rank parameterized weight $\mathbf{W} = \mathbf{B}\mathbf{A} \in \mathcal{M}_r$, let $\mathbf{B} = \mathbf{Q}_B\mathbf{R}_B$ and $\mathbf{A}^\top = \mathbf{Q}_A\mathbf{R}_A$ be the QR-decomposition of $\mathbf{B}$ and $\mathbf{A}^\top$, such that $\mathbf{Q}_B, \mathbf{Q}_A \in \mathbb{R}^{d\times r}$ are column-wise orthogonal, and $\mathbf{R}_B, \mathbf{R}_A \in \mathbb{R}^{r\times r}$ are upper-triangular. Then the Riemannian updated weight, defined by $\mathbf{W}' \triangleq \mathrm{SVD}_r(\mathbf{W} - \eta \operatorname{grad} L(\mathbf{W}))$, admits a low-rank factorization $\mathbf{W}' = \mathbf{B}'\mathbf{A}'$, with $\mathbf{B}' = \left(\mathbf{Q}_B, \overline{\mathbf{Q}}_B\right)\mathbf{U}_*\mathbf{\Sigma}_*^{1/2}$ and $\mathbf{A}' = \mathbf{\Sigma}_*^{1/2}\mathbf{V}_*^\top\left(\mathbf{Q}_A, \overline{\mathbf{Q}}_A\right)^\top$. Specifically, $\mathbf{U}_*\mathbf{\Sigma}_*\mathbf{V}_*^\top$ is determined by the rank-r SVD of the following $2r$-by-$2r$ matrix, i.e.,*

$$\mathbf{U}_*\mathbf{\Sigma}_*\mathbf{V}_*^\top = \mathrm{SVD}_r\begin{pmatrix}\mathbf{S} & -\eta\,\overline{\mathbf{R}}_A^\top\\ -\eta\,\overline{\mathbf{R}}_B & \mathbf{0}\end{pmatrix} \in \mathbb{R}^{2r\times 2r},\ \mathbf{S} \triangleq \mathbf{R}_B\mathbf{R}_A^\top - \eta\,\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L\cdot\mathbf{Q}_A \in \mathbb{R}^{r\times r}, \qquad (16)$$

and $\overline{\mathbf{Q}}_B\overline{\mathbf{R}}_B$ is the QR decomposition of the d-by-r matrix $(\nabla_{\mathbf{B}}L\cdot\mathbf{R}_A^{-1} - \mathbf{Q}_B\mathbf{Q}_B^\top\nabla_{\mathbf{B}}L\cdot\mathbf{R}_A^{-1})$, while $\overline{\mathbf{Q}}_A\overline{\mathbf{R}}_A$ is the QR decomposition of the d-by-r matrix $(\nabla_{\mathbf{A}}L^\top\mathbf{R}_B^{-1} - \mathbf{Q}_A\mathbf{Q}_A^\top\nabla_{\mathbf{A}}L^\top\mathbf{R}_B^{-1})$.

Proof of Proposition 4. This proof follow a similar spirit of orthogonal tangent vector decomposition as in Vandereycken (2013, Section 3) and Bian et al. (2023, Section 2.2.3). To complete the proof, we need to perform such decomposition under the unnormalized basis spanned by $\mathbf{B}$ and $\mathbf{A}$, such that the full-size weight and gradient terms can be cancelled out using the relation $\nabla_{\mathbf{W}}L\cdot\mathbf{A}^\top = \nabla_{\mathbf{B}}L$ and $\mathbf{B}\cdot\nabla_{\mathbf{W}}L = \nabla_{\mathbf{A}}L$.

As the low-rank representation of the Riemannian gradient is provided in Proposition 3, we only need to manipulate the terms in the updated weight $\mathbf{W} - \eta \operatorname{grad} L(\mathbf{W})$ such that it can be decomposed as $\widetilde{\mathbf{U}}\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^\top$, where $\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}} \in \mathbb{R}^{d\times 2r}$ are column-wise othorgonal matrices. Thereby, the Riemannian SVD retraction can be performed on a smaller $2r$-by-$2r$ matrix instead, since

$$\mathrm{SVD}_r(\widetilde{\mathbf{U}}\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^\top) = \widetilde{\mathbf{U}}\cdot\mathrm{SVD}_r(\widetilde{\mathbf{\Sigma}})\cdot\widetilde{\mathbf{V}}^\top. \qquad (17)$$

To complete the proof, we only need to identify the representation of $\widetilde{\mathbf{U}}, \widetilde{\mathbf{\Sigma}}, \widetilde{\mathbf{V}}$ by othorgonal tangent vector decomposition, under low-rank parameterization. Based on Proposition 3, it holds that

$$\begin{aligned}&\operatorname{grad} L(\mathbf{W})\\ =&\mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\cdot\nabla_{\mathbf{W}}L + \nabla_{\mathbf{W}}L\cdot\mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A} - \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top\cdot\nabla_{\mathbf{W}}L\cdot\mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}\\ =&\mathbf{Q}_B\mathbf{Q}_B^\top\cdot\nabla_{\mathbf{W}}L + \nabla_{\mathbf{W}}L\cdot\mathbf{Q}_A\mathbf{Q}_A^\top - \mathbf{Q}_B\mathbf{Q}_B^\top\cdot\nabla_{\mathbf{W}}L\cdot\mathbf{Q}_A\mathbf{Q}_A^\top\\ =&\mathbf{Q}_B\mathbf{Q}_B^\top\cdot\nabla_{\mathbf{W}}L\cdot\mathbf{Q}_A\mathbf{Q}_A^\top + (\mathbf{I} - \mathbf{Q}_B\mathbf{Q}_B^\top)\cdot\nabla_{\mathbf{W}}L\cdot\mathbf{Q}_A\mathbf{Q}_A^\top + \mathbf{Q}_B\mathbf{Q}_B^\top\cdot\nabla_{\mathbf{W}}L\cdot(\mathbf{I} - \mathbf{Q}_A\mathbf{Q}_A^\top),\end{aligned}$$

where the latter two terms are orthogonal to the column-space of $\mathbf{B}$ and $\mathbf{A}^\top$ respectively. To cancel out the full-size gradient terms, we leverage the relations $\mathbf{Q}_B^\top\cdot\nabla_{\mathbf{W}}L = \mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L$ and $\nabla_{\mathbf{W}}L\cdot\mathbf{Q}_A = \nabla_{\mathbf{B}}L\cdot\mathbf{R}_A^{-1}$, which shows that

$$\begin{aligned}&\operatorname{grad} L(\mathbf{W})\\ =&\mathbf{Q}_B\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L\cdot\mathbf{Q}_A\mathbf{Q}_A^\top + (\mathbf{I} - \mathbf{Q}_B\mathbf{Q}_B^\top)\cdot\nabla_{\mathbf{B}}L\cdot\mathbf{R}_A^{-1}\cdot\mathbf{Q}_A^\top + \mathbf{Q}_B\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L\cdot(\mathbf{I} - \mathbf{Q}_A\mathbf{Q}_A^\top).\end{aligned}$$

Therefore, the updated weight can be represented by

$$
\begin{aligned}
&\mathbf{W} - \eta \operatorname{grad} L(\mathbf{W}) \\
=&\mathbf{Q}_B(\mathbf{R}_B\mathbf{R}_A^\top)\mathbf{Q}_A^\top - \eta \left(\mathbf{Q}_B\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L \cdot \mathbf{Q}_A\mathbf{Q}_A^\top \right. \\
&\left. + (\mathbf{I} - \mathbf{Q}_B\mathbf{Q}_B^\top) \cdot \nabla_{\mathbf{B}}L \cdot \mathbf{R}_A^{-1} \cdot \mathbf{Q}_A^\top + \mathbf{Q}_B\mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L \cdot (\mathbf{I} - \mathbf{Q}_A\mathbf{Q}_A^\top)\right) \\
=&\mathbf{Q}_B\big(\mathbf{R}_B\mathbf{R}_A^\top - \eta\, \mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L \cdot \mathbf{Q}_A\big)\mathbf{Q}_A^\top - \eta\, \overline{\mathbf{Q}}_B \cdot \overline{\mathbf{R}}_B \cdot \mathbf{Q}_A^\top - \eta\, \mathbf{Q}_B \cdot \overline{\mathbf{R}}_A^\top \cdot \overline{\mathbf{Q}}_A^\top \\
=&\begin{pmatrix}\mathbf{Q}_B & \overline{\mathbf{Q}}_B\end{pmatrix}\begin{pmatrix}\mathbf{R}_B\mathbf{R}_A^\top - \eta\, \mathbf{R}_B^{-\top}\nabla_{\mathbf{A}}L \cdot \mathbf{Q}_A & -\eta\, \overline{\mathbf{R}}_A^\top \\ -\eta\, \overline{\mathbf{R}}_B & \mathbf{0}\end{pmatrix}\begin{pmatrix}\mathbf{Q}_A^\top \\ \overline{\mathbf{Q}}_B^\top\end{pmatrix}.
\end{aligned}
$$

Notice that $(\mathbf{Q}_B, \overline{\mathbf{Q}}_B)$ and $(\mathbf{Q}_A, \overline{\mathbf{Q}}_A)$ are column-wise othorgonal matrices, making them a desirable choice of the aforementioned $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$, and hence completes the proof.

□

C Further Discussions on the Mechanism of LORO

In this section, we provide supplementary analysis and discussion of the LORO mechanism. Specifically, Section C.1 analyzes the limitations of standard low-rank pretraining, while Section C.2 discusses how LORO effectively overcomes these limitations, leading to significant improvements.

C.1 Limitations in Standard Low-rank Optimizer

In this subsection, we present additional theoretical and empirical evidence demonstrating that the standard low-rank optimizer (i.e. independently optimizing low-rank factors) is inefficient for achieving satisfactory training performance.

In Section C.1.1, our empirical evidence demonstrates that: 1. the standard low-rank optimizer produces spiky training curves and high perplexity across various initialization schemes and learning rates, 2. it learns low-rank weights with high condition numbers (i.e., the ratio of the largest singular value to the smallest nonzero singular value), and 3. it continues to yield poor performance even when combined with optimizer state refreshing.

In Section C.1.2, our theoretical analysis indicates that: 1. the standard low-rank gradient in (3) fails to preserve adequate information from the full-size gradient within the tangent space, and 2. a nearly-stationary point obtained through standard low-rank gradient descent with a near-zero gradient norm may be significantly far away from a true stationary point on the manifold.

Based on these observations and analysis, we conjecture that the failure of the standard low-rank optimizer stems from the following factors: 1. the standard low-rank gradient contains redundant full-size gradient information, preventing it from accurately approximating the steepest descent direction on the manifold, 2. the lack of proper scaling in the standard low-rank gradient causes deviations from the steepest descent direction on the manifold, and 3. the standard low-rank gradient biases towards ill-conditioned weights, leading to instability during training.

C.1.1 Empirical Evidence

Standard low-rank optimizer generally leads to poor performance. To validate that pretraining low-rank language model is generally hard, we train a rank-128 LLaMA-60M models with standard low-rank optimizer under different initialization schemes and learning rates. Specifically, we initialize the low-rank factors with Xavier initialization (Glorot & Bengio, 2010), or standard Gaussian initialization with standard deviation varying in $\{0.1, 0.01, 0.001\}$, and we tried different learning rates in $\{0.01, 0.005, 0.001\}$. We report the training curve and the final perplexity of the models.

As shown in Figure 5, in all settings, the training curve of the vanilla low-rank optimizer is spiky and unstable, leading to unsatisfactory evaluation perplexity. This suggests that independent updates of the low-rank factors are insufficient for good performance across different configurations. Training a low-rank language model to match the full-size baseline is generally challenging.

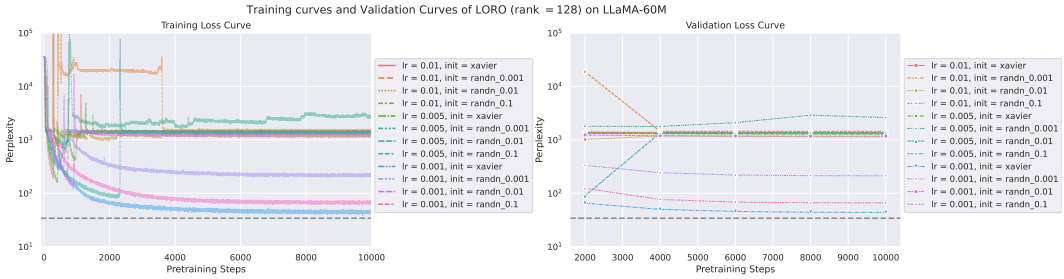


Figure 5: Ablation study on the impact of learning rate and initialization schemes on standard low-rank training for a rank-128 LLaMA-60M model. The y-axis (log-scale) represents perplexity, with the gray horizontal dashed line indicating the performance of full-size training.

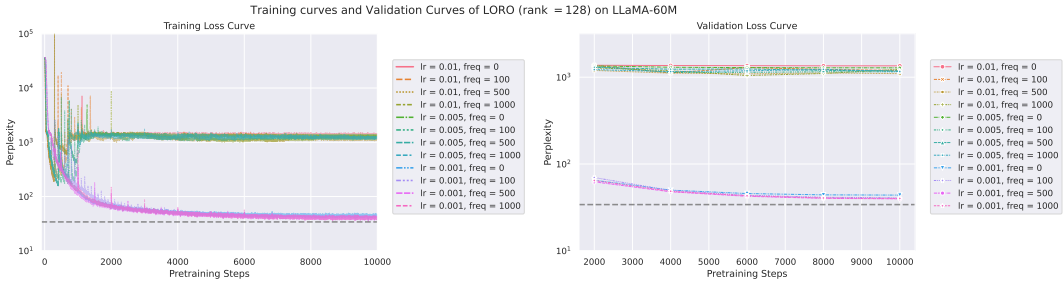


Figure 6: Ablation study on the impact of optimizer state refreshing on standard low-rank training for a rank-128 LLaMA-60M model. The y-axis (log-scale) represents perplexity, with the gray horizontal dashed line indicating the performance of full-size training.

Standard low-rank optimizer leads to poor performance even when combined with optimizer state refreshing. To assess whether optimizer state refreshing can enhance the standard low-rank optimizer, we perform an ablation study by varying the refreshing frequency in the standard low-rank optimizer. Following the setup in Section 5.1, we trained rank-128 LLaMA-60M models using vanilla low-rank gradient descent with different learning rates (chosen from $\{0.01, 0.005, 0.001\}$) and optimizer state refreshing frequencies (chosen from $\{0, 100, 500, 1000\}$).

As shown in Figure 6, all trials exhibit spiky training curves and unsatisfactory performance. This validates that applying optimizer state refreshing to standard low-rank gradient descent still leads to bad training performance. We conjecture that, this is because the standard low-rank gradient does not actively explore new optimization subspace, and the training process biases towards low-rank weights with increasing condition numbers. In this case, refreshing the past momentum statistics resided does not help exploring beneficial optimization subspace.

Standard low-rank optimizer learns ill-conditioned weights. To understand the cause of the instability and spiky training curves in the standard low-rank optimizer, we inspect the condition number and norm of the low-rank factors throughout the training process. Specifically, we train rank-128 LLaMA-60M models with Xavier initialization and a learning rate of 0.01, using different optimizers: the standard low-rank optimizer, the full-size Adam optimizer (Kingma & Ba, 2015), and LORO. We visualize the evolution of the singular values, Frobenius norm, and condition numbers of the learned weights at various layers throughout the training process.

As shown in Figure 11, Figure 12, and Figure 13, the standard low-rank gradient results in model weights with significantly higher condition numbers compared to LORO or full-size Adam. We conjecture that the standard low-rank gradient update introduces imbalances among the singular values of the low-rank factors, thereby impairing the stability of the feature learning process.

C.1.2 THEORETICAL EVIDENCE

Standard low-rank gradient fails to approximate the full-size gradient well. To understand the difference between the training dynamic of the standard low-rank optimizer and its full-size

counterpart, we examine the gap between the standard low-rank gradient and the full-size gradient. Following (3), we define the standard low-rank gradient as $\mathbf{Z} \triangleq \mathbf{B}\mathbf{B}^\top \nabla_{\mathbf{W}} L + \nabla_{\mathbf{W}} L \mathbf{A}^\top \mathbf{A}$, where $\nabla_{\mathbf{W}} L$ is the full-size gradient and $\mathbf{W} = \mathbf{B}\mathbf{A}$ is the low-rank weight. We show that $\mathbf{Z}$ fails to adequately preserve the full-size gradient information. In contrast, LORO adopts the Riemannian gradient, which provides the best approximation of the full-size gradient within the tangent space $T_{\mathbf{W}}\mathcal{M}_r$, thereby preserving the full-size gradient information more effectively than the standard low-rank gradient.

According to Proposition 1 and Proposition 3, the orthogonal projection onto the tangent space is

$$\mathcal{P}(\mathbf{G}) = \mathbf{U}\mathbf{U}^\top \mathbf{G} + \mathbf{G}\mathbf{V}\mathbf{V}^\top - \mathbf{U}\mathbf{U}^\top \mathbf{G}\mathbf{V}\mathbf{V}^\top = \mathbf{P}_{\mathbf{B}}\mathbf{G} + \mathbf{G}\mathbf{P}_{\mathbf{A}} - \mathbf{P}_{\mathbf{B}}\mathbf{G}\mathbf{P}_{\mathbf{A}}, \tag{18}$$

where $\mathbf{P}_{\mathbf{B}} \triangleq \mathbf{B}(\mathbf{B}^\top\mathbf{B})^{-1}\mathbf{B}^\top$ and $\mathbf{P}_{\mathbf{A}} \triangleq \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}$ are the orthogonal projection matrices onto the column space of $\mathbf{B}$ and the row space of $\mathbf{A}$. We first study how well the full-size gradient can be approximated within the tangent space w.r.t Frobenius norm $\|\cdot\|_F$. For any tangent vector $\mathbf{G}$ resides in the tangent space, the full-size gradient approximation error equals to

$$\|\mathbf{G} - \nabla_{\mathbf{W}} L\|_F = \|\mathbf{G} - \mathcal{P}(\nabla_{\mathbf{W}} L)\|_F + \|\mathcal{P}^\perp(\nabla_{\mathbf{W}} L)\|_F, \tag{19}$$

where $\mathcal{P}^\perp(\cdot)$ is the orthogonal projection to the complemented subspace of the tangent space. This implies that, the full-size gradient approximation error is minimized at $\mathbf{G} = \mathcal{P}(\nabla_{\mathbf{W}} L)$, which is exactly the Riemannian gradient as shown in Proposition 3. Therefore, the Riemannian gradient represents the steepest gradient descent direction on the manifold, as it provides the best approximation of the full-size gradient within the tangent space.

Now, we can study how well the standard low-rank gradient $\mathbf{Z}$ approximates the full-size gradient. Since $\mathbf{Z}$ is an element in the tangent space (can be proven by showing $\mathcal{P}(\mathbf{Z}) = \mathbf{Z}$), the full-size gradient approximation error of $\mathbf{Z}$ is determined by $\|\mathbf{Z} - \mathcal{P}(\nabla_{\mathbf{W}} L)\|$, which equals

$$\|\mathbf{B}((\mathbf{B}^\top\mathbf{B})^{-1} - \mathbf{I})\mathbf{B}^\top \nabla_{\mathbf{W}} L + \nabla_{\mathbf{W}} L \mathbf{A}^\top((\mathbf{A}\mathbf{A}^\top)^{-1} - \mathbf{I})\mathbf{A} + \mathbf{P}_{\mathbf{B}} \nabla_{\mathbf{W}} L \mathbf{P}_{\mathbf{A}}\|_F. \tag{20}$$

This error arises from two main factors: 1. the scale of gradient components, $\mathbf{B}\mathbf{B}^\top \nabla_{\mathbf{W}} L$ and $\nabla_{\mathbf{W}} L \mathbf{A}^\top \mathbf{A}$, is not properly normalized to match that of $\mathbf{P}_{\mathbf{B}}\mathbf{G}$ and $\mathbf{G}\mathbf{P}_{\mathbf{A}}$, and 2. the redundant gradient information, $\mathbf{P}_{\mathbf{B}} \nabla_{\mathbf{W}} L \mathbf{P}_{\mathbf{A}}$—which represents the gradient component in the intersection of the column space of $\mathbf{B}$ and the row space of $\mathbf{A}$—is double-counted across the two sketched gradient terms. Our conclusions align with the findings in Zhang et al. (2024a), which suggests that down scaling the learning rates of some specific modules (e.g., the low-rank weights) is beneficial.

Therefore, instead of updating $\mathbf{B}$ and $\mathbf{A}$ independently, we should employ LORO to update the low-rank weight $\mathbf{W} = (\mathbf{B}\mathbf{A})$ in the direction of the negative Riemannian gradient, which represents the steepest descent direction on the manifold. This approach ensures that the full-size gradient information is maximally exploited.

Nearly-zero standard low-rank gradient norm does not indicate a true stationary point. To understand the performance gap between standard low-rank optimizer and its full-size counterpart, we study compare their stationary point conditions. As mentioned in Absil et al. (2009); Vandereycken (2013); Olikier et al. (2023), in Riemannian optimization, a nearly-zero Riemannian gradient norm indicates that the parameter is close to a stationary point on the manifold, showing that the training process nearing convergence.

When using the standard low-rank optimizer, however, while the norm of the standard low-rank gradient is nearly-zero, the learned parameter can significantly deviate from a true stationary point on the manifold. This is because

$$\|(\nabla_{\mathbf{B}} L, \nabla_{\mathbf{A}} L)\|_F^2 = \|\nabla_{\mathbf{W}} L \mathbf{A}^\top\|_F^2 + \|\mathbf{B}\nabla_{\mathbf{W}} L\|_F^2 \geqslant (\sigma_r(\mathbf{A})^2 + \sigma_r(\mathbf{B})^2)\|\nabla_{\mathbf{W}} L\|_F^2, \tag{21}$$

where $\sigma_r(\cdot)$ denotes the r-th largest singular value (i.e. the smallest nonzero singular value). This further implies

$$\|\mathcal{P}(\nabla_{\mathbf{W}} L)\|_F^2 \leqslant (\sigma_r(\mathbf{A})^2 + \sigma_r(\mathbf{B})^2)^{-1}\|(\nabla_{\mathbf{B}} L, \nabla_{\mathbf{A}} L)\|_F^2. \tag{22}$$

Consequently, when $\mathbf{B}$ and $\mathbf{A}$ are ill-conditioned (i.e., when the condition numbers $\sigma_1(\mathbf{A})/\sigma_r(\mathbf{A})$ and $\sigma_1(\mathbf{B})/\sigma_r(\mathbf{B})$ are large) or nearly singular (i.e., $\sigma_r(\mathbf{A})$ and $\sigma_r(\mathbf{B})$ are small), the Riemannian gradient norm can remain large even if the norm of the standard low-rank gradient is nearly-zero. This indicates the inferior convergence of standard low-rank optimizer. To mitigate this issue, we adopt Riemannian gradient to optimize the product of the low-rank factors.

C.2 EFFECTIVENESS OF LORO

In this section, we present additional theoretical and empirical evidence showing that LORO is able to overcome the limitations in standard low-rank training. Furthermore, we show that both the exact and approximate LORO updates are necessary and effective.

In Section C.2.1, our empirical evidence shows that: 1. the exact LORO update is necessary and effective, 2. the approximate LORO update is necessary and effective, 3. LORO benefits from optimizer state refreshing.

In Section C.2.2, our theoretical evidence shows that: 1. the exact LORO update balances the singular values of low-rank factors, and 2. the exact LORO update potentially reduce the full-size gradient approximation error in the successive approximate LORO updates.

Based on our observations and analysis, we argue that the performance improvement stems from the LORO update itself, because: 1. LORO approximates the Riemannian gradient better than standard low-rank gradient, and it benefits from utilizing more information from the full-size gradient, 2. LORO effectively explore new optimization subspace which reduces the condition number of the low-rank weights, leading to a more stable training process, and 3. LORO minimizes the accumulative error during successive approximate LORO steps.

C.2.1 EMPIRICAL EVIDENCE

The exact LORO update is necessary and effective. To validate the effectiveness of the exact LORO update, we conduct a finer ablation study on the exact LORO update frequency. Following Section 5.1, we train rank-256 LLaMA-130m models with $K \in \{10, 50, 100, 250, 500, 650, 750\}$. We visualize the training curve and report the final perplexity.

As shown in Figure 7, as the LORO exact update is getting more frequent, the loss curve is less spiky and the evaluation perplexity exhibit a slight improvement. However, when the LORO exact update is overly lazy, i.e. $K > 650$, the learning curve explodes. Our results show that while the exact LORO update is necessary, it does not need to be applied at every update step. In practice, performing the LORO update every 500 steps achieves satisfactory performance.

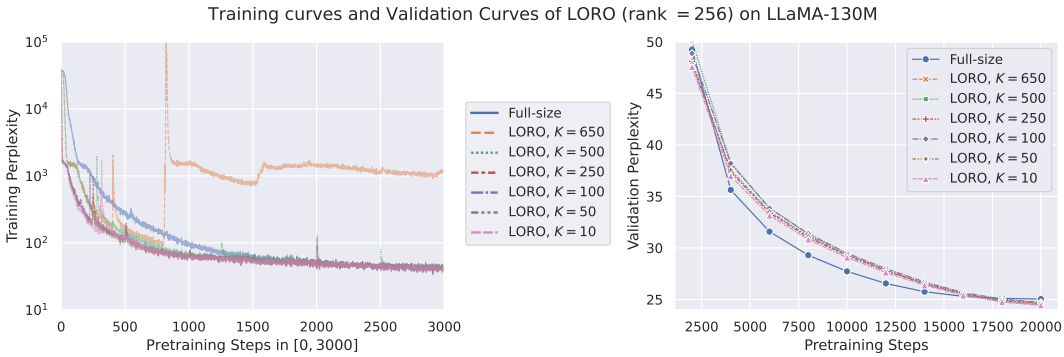


Figure 7: Ablation study on the impact of exact LORO update frequency K on LORO for training rank-256 LLaMA-130M models. The y-axis (log-scale) represents perplexity, with the gray horizontal dashed line indicating the performance of full-size training.

The approximate LORO update is necessary and effective. To validate the necessity of using the approximate LORO steps, we train rank-256 LLaMA-130M models with LORO, both with and without learning rate down-scaling between successive exact LORO updates.

As shown in Figure 8, the training curve either fails to converge or yields poor results when the approximate LORO steps are omitted. These results imply that the exact LORO update is necessary and effective. The performance improvement stems from the exact LORO update. In practice, LORO with $K = 500$ can achieve satisfactory training performance without the need to apply the exact LORO update at every step.

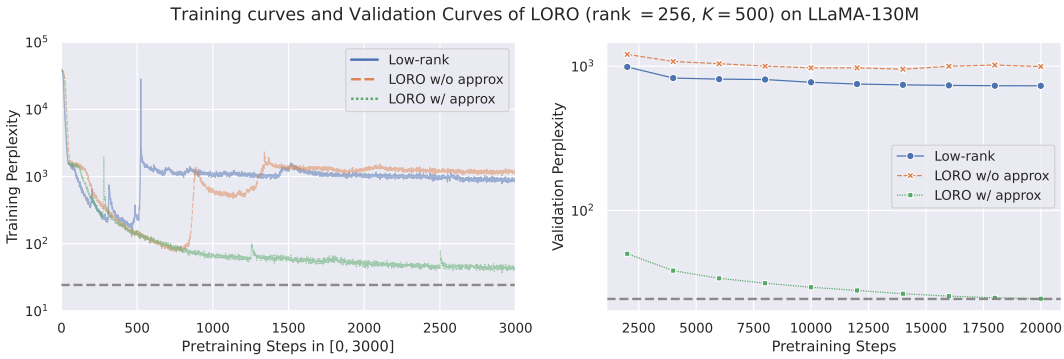


Figure 8: Ablation study on the impact of learning rate down-scaling in approximate LORO updates for training rank-256 LLaMA-130M models. The y-axis (log-scale) represents perplexity, with the gray horizontal dashed line indicating the performance of full-size training.

LORO benefits from optimizer state refreshing. To study how optimizer state refreshing effects LORO, we conduct ablation study on the optimizer state refreshing. Following the setup in Section 5.1, we compare the training curve and validation perplexity of rank-256 LLaMA-130M models trained with LORO, both with and without refreshing the optimizer states periodically. As shown in Figure 9, both configurations lead to satisfactory performance. While refreshing the optimizer states introduces spikes in the early stages of training, it eventually improves the validation loss of the LORO results. This suggests that LORO benefits from optimizer state refreshing.

We conjecture that the stability of LORO against optimizer state refreshing arises because the exact LORO step shifts the low-rank factors $(\mathbf{B}, \mathbf{A})$ to a new optimization subspace with a reduced condition number. This conjecture is partially supported by analysis in Section C.2.2. In this scenario, the optimizer state retains momentum statistics accumulated many steps ago, originating from an obsolete subspace that significantly deviates from the current one. This indicates that dropping the old momentum statistics can help reduce accumulated error during the approximate LORO steps and encourage exploration in the new subspace.

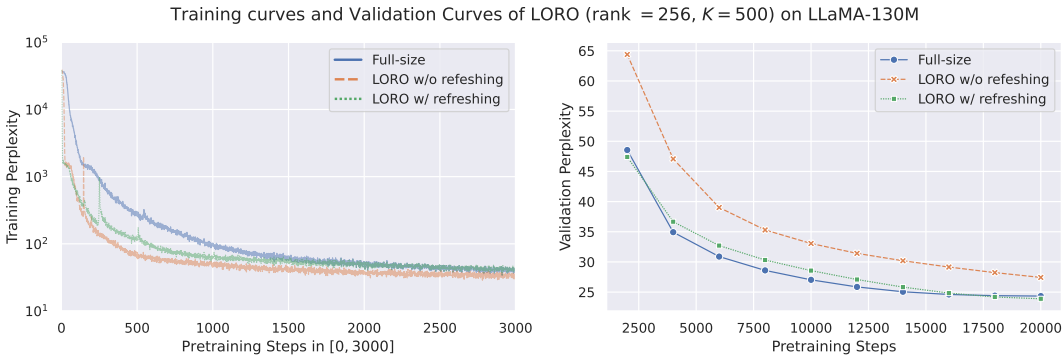


Figure 9: Ablation study on the impact of optimizer state refreshing on LORO updates for training rank-256 LLaMA-130M models. The y-axis (log-scale) represents perplexity.

C.2.2 Theoretical Evidence

Exact LORO update balances the singular values of low-rank factors. As discussed in Section C.1.2, in contrast to the standard low-rank gradient, LORO adopts Riemannian gradients, which provides the best approximation of the full-size gradient within the tangent space. To better understand the mechanism of the exact LORO update, we need to inspect how the exact LORO update affect the singular values of low-rank weights.

We show that the exact LORO update implicitly balances the singular values of the low-rank factors. According to (10) and Algorithm 1, the exact LORO update ensures $\sigma_i(\mathbf{B}) = \sigma_i(\mathbf{A}) = \sqrt{\sigma_i(\mathbf{W})}$,

for $i = 1, ..., r$. Therefore, the exact LORO update minimize each $\sigma_i(\mathbf{B})^2 + \sigma_i(\mathbf{A})^2$ among all the low-rank factors $(\mathbf{B}, \mathbf{A})$ that satisfy $\sigma_i(\mathbf{B})\sigma_i(\mathbf{A}) = \sigma_i(\mathbf{W})$. This implies LORO prevents $(\sigma_r(\mathbf{A})^2 + \sigma_r(\mathbf{B})^2)$ to be overly small, and it can potentially reduce the condition number of the weights and stabilize the training process. As discussed in Section C.1.2, smaller condition numbers of $\mathbf{B}$ and $\mathbf{A}$ also indicate better convergence property.

Exact LORO udpate potentially reduces error in approximate LORO updates. As discussed in Section C.1.1, the exact LORO plays a subtle yet crucial role in successful low-rank training. To understand how a few exact LORO steps can stabilize the whole training process, we study how an exact LORO step effect the successive approximate LORO steps.

Resuming from the derivations above, we show that the singular value balancing effect in the exact LORO step minimizes an upper bound of the full-size gradient approximation error. Specifically, the full-size gradient approximation error can be bounded by

$$\begin{aligned}&\|\mathbf{B}((\mathbf{B}^\top\mathbf{B})^{-1} - \mathbf{I})\mathbf{B}^\top\nabla_\mathbf{W}L + \nabla_\mathbf{W}L\mathbf{A}^\top((\mathbf{A}\mathbf{A}^\top)^{-1} - \mathbf{I})\mathbf{A} + \mathbf{P}_\mathbf{B}\nabla_\mathbf{W}L\mathbf{P}_\mathbf{A}\|_F + \|\mathcal{P}^\perp(\nabla_\mathbf{W}L)\|_F \\ \leqslant &(\|\mathbf{I} - \Lambda_\mathbf{B}^2\|_F + \|\mathbf{I} - \Lambda_\mathbf{A}^2\|_F)\|\nabla_\mathbf{W}L\|_2 + 2\|\nabla_\mathbf{W}L\|_F, \end{aligned} \tag{23}$$

where $\Lambda_\mathbf{B}, \Lambda_\mathbf{A}$ denote the singular value matrices of $\mathbf{B}$ and $\mathbf{A}$. Suppose $\Lambda_\mathbf{W}$ is the top-r singular values matrix of the low-rank weight $\mathbf{W} = (\mathbf{B}\mathbf{A})$, among all the choices of $(\mathbf{B}, \mathbf{A})$ satisfying $\Lambda_\mathbf{B}\Lambda_\mathbf{A} = \Lambda_\mathbf{W}$, the bound $(\|\mathbf{I} - \Lambda_\mathbf{B}^2\|_F + \|\mathbf{I} - \Lambda_\mathbf{A}^2\|_F)$ is minimized when $\Lambda_\mathbf{B} = \Lambda_\mathbf{A} = \Lambda_\mathbf{W}^{1/2}$. This condition is satisfied by the exact LORO steps, implying that the exact LORO update potentially reduce the accumulative full-size gradient approximation error in the successive approximate LORO updates.

In summary, we conjecture that LORO benefits from two key factors: 1. The exact LORO update explores new subspace by balancing the singular values of the low-rank factors. Thereby, it reduces the condition numbers, leading to smaller accumulative gradient approximation error in the successive approximate LORO updates. Section C.2.1 provides supportive empirical evidence on this conjecture. 2. The approximated LORO update down scales the standard low-rank gradients, preventing the rapid accumulation of redundant gradient information and avoiding undesired loss spikes before the next exact LORO update.

D FURTHER ABLATION STUDIES

Ablation study on the LORO scaling factor. To study how the learning rate scaler effects the approximate LORO steps, we conducted additional ablation study on the learning rate scaler scheme of low-rank factors in the approximated LORO steps. Following the same configuration as Section 5.1, and we train the rank-128 LLaMA-60M model with LORO, using learning rate scaler choosing from $\{1, 0.5, 0.25, 0.1, 0.01\}$, where $0.25 = r/n = 128/512$ is exactly the Reimannian down-scaling rate we used in (12). We report the final perplexity of the resulting models. As shown in Figure 10, our LORO exhibit robustness across a range of learning rate scalers (i.e. from 0.1 to 0.5).

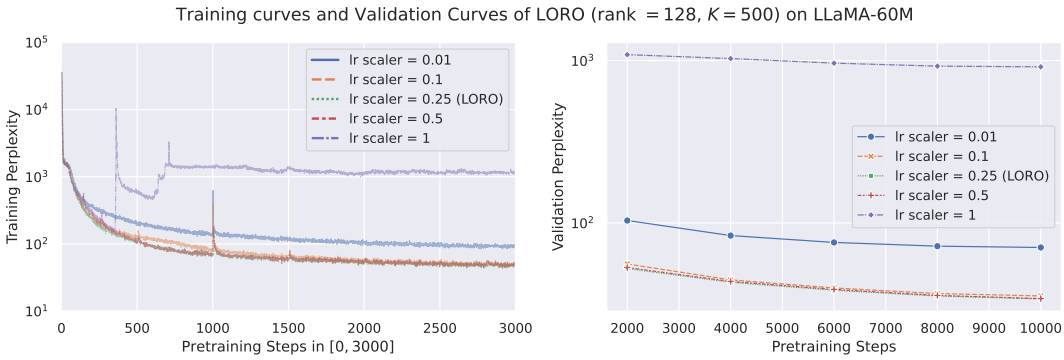


Figure 10: Ablation study on the impact of learning rate scaler on approximate LORO updates for training rank-128 LLaMA-60M models. The y-axis (log-scale) represents perplexity.

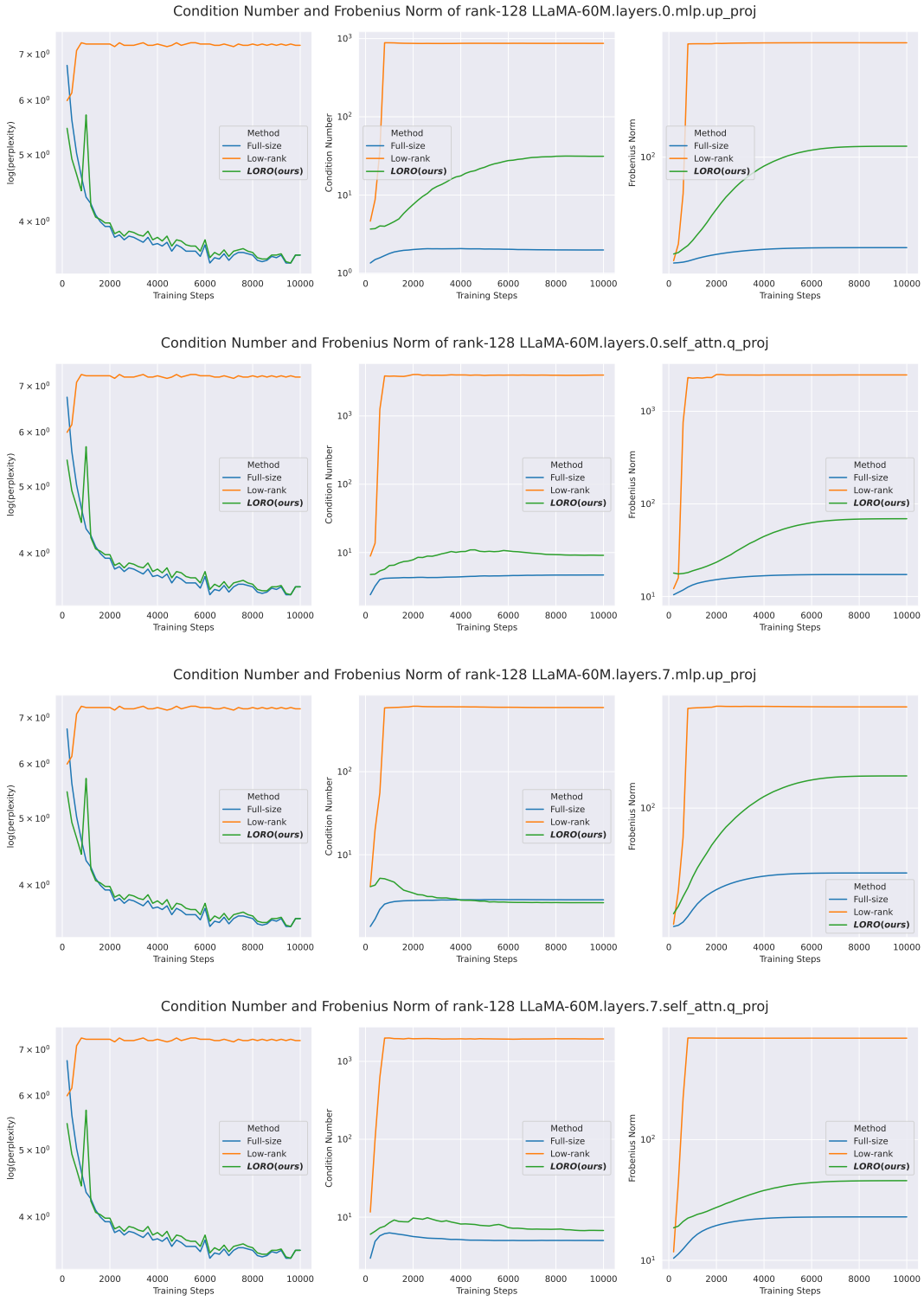


Figure 11: Visualization of the condition number and Frobenius norm for linear and self-attention layers in rank-128 LLaMA-60M model.

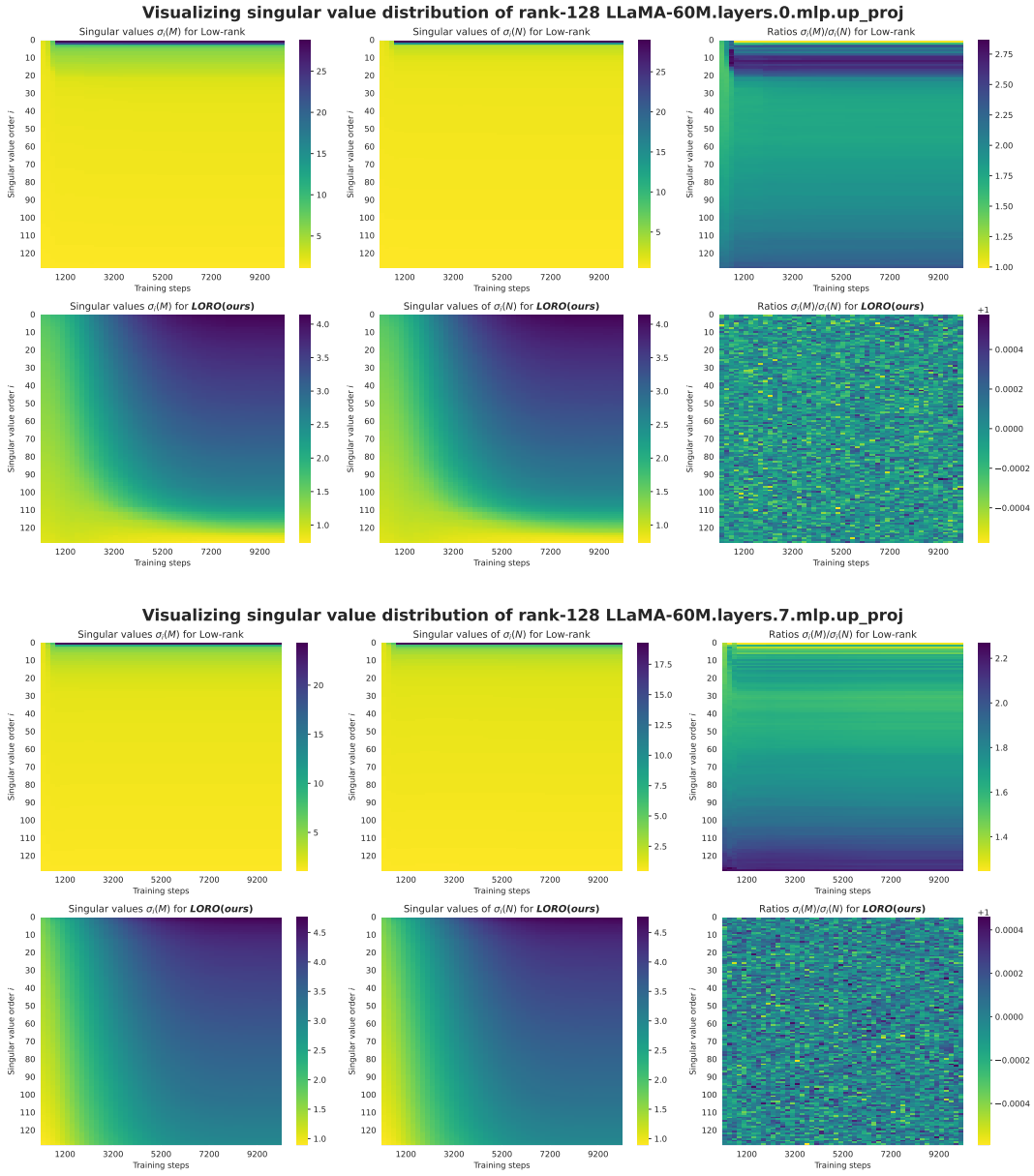


Figure 12: Visualization of the singular value distribution for the linear layers in rank-128 LLaMA-60M model. The y-axis represents the order of singular values, the x-axis represents the training steps. Deeper color indicates a larger singular value is observed.

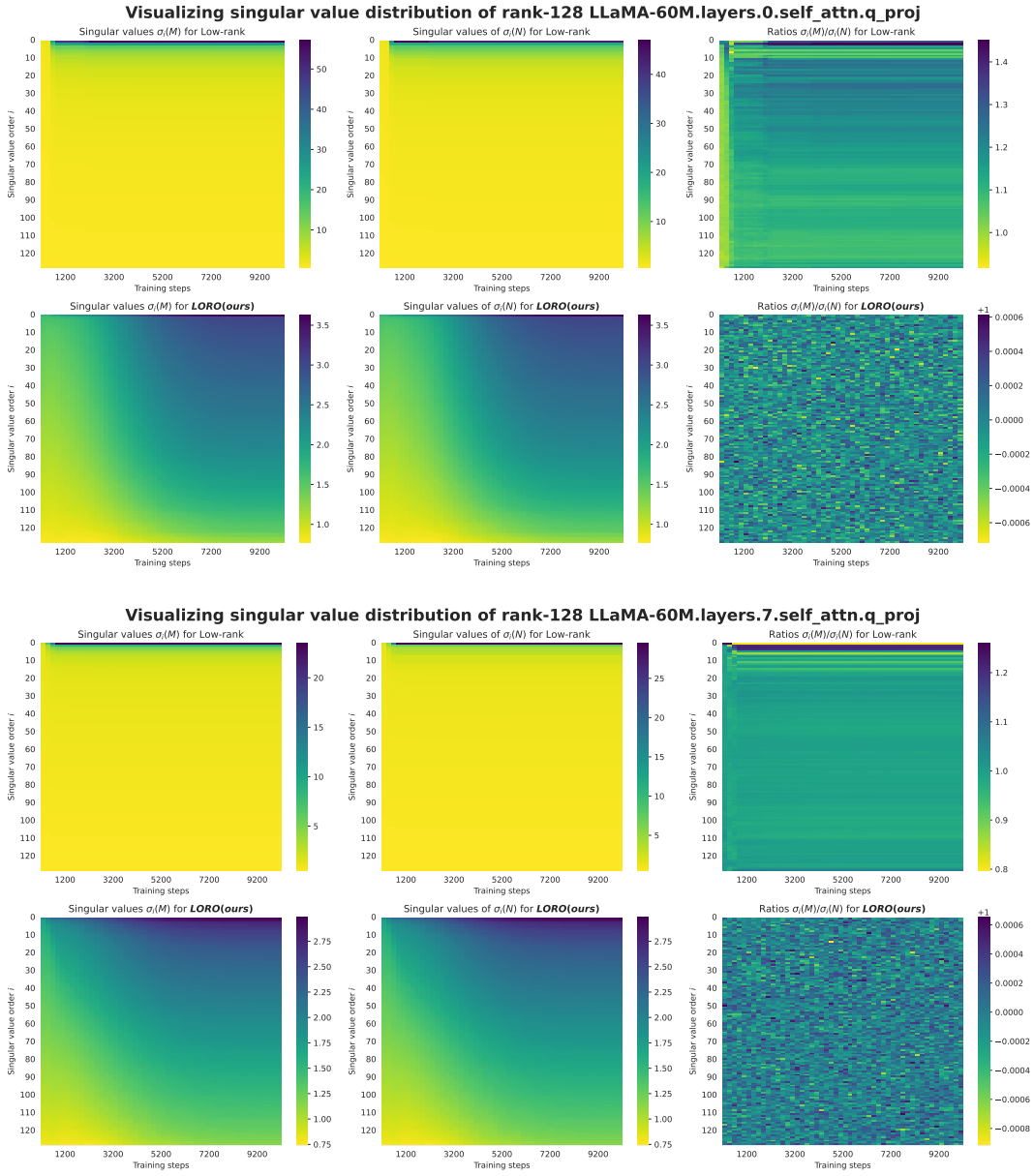


Figure 13: Visualization of the singular value distribution for the self-attention layers in rank-128 LLaMA-60M model. The y-axis represents the order of singular values, the x-axis represents the training steps. Deeper color indicates a larger singular value is observed.