Pseudo Clifford Bandpass Prolates

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***Abstract*—We introduce operators which generalise the classical modulation and translation operators, now acting on functions defined on $\mathbb{R}^m$ and taking values in the associated Clifford algebra $\mathbb{C}_m$. The modulation operators are used to map orthonormal bases for Paley-Wiener spaces associated with balls in $\mathbb{R}^m$ to orthonormal sets in Paley-Wiener spaces PW_A associated with annuli A in $\mathbb{R}^m$. The complementary spaces are characterised and an orthonormal basis for them is given. These bases are used to construct an orthonormal basis for PW_A composed of pseudo bandpass prolates.**

I. Introduction

Given $c > 0$, the Paley-Wiener space $PW_c(\mathbb{R})$ is the collection of functions $f \in L^2(\mathbb{R})$ whose Fourier transforms $\mathcal{F}f$ are supported on the interval $[-c, c]$. Let Q_c be the *time-limiting* projection operator $Q_c f(t) = \mathbf{1}_{[-c,c]}(t)f(t)$ where $\mathbf{1}_{[-c,c]}$ is the characteristic function of the interval $[-c, c]$. The *band-limiting* operator P_c is given by $P_c = \mathcal{F}Q_c\mathcal{F}$. The (one-dimensional) prolate spheroidal wavefunctions (PSWFs) are eigenfunctions of the self-adjoint Hilbert-Schmidt integral operator P_cQ_1. They are most efficiently computed by observing that they are also eigenfunctions of a second order differential operator $\mathcal{L}_c$ which commutes with P_cQ_1 [7], [4].

For $0 < c < c' < \infty$, bandpass prolates (BPPs) are eigenfunctions of $(P_{c'} - P_c)Q_1$. BPPs are members of $PW_{c',c}(\mathbb{R})$ – the space of square-integrable functions on the line whose Fourier transforms are supported on $[-c', -c] \cup [c, c']$. It was shown [6] that bandpass prolates are not eigenfunctions of a differential operator. Nevertheless [5] the construction of BPPs can be achieved by solving an appropriate matrix eigenvalue problem. The key observation in the construction is that if $\{\varphi_n\}_{n=0}^{\infty}$ is an orthonormal basis for PW_c and $t > c$, then $\{e^{-2\pi itx}\varphi_n(x)\}_{n=0}^{\infty} \cup \{e^{2\pi itx}\varphi_n(x)\}_{n=0}^{\infty}$ is an orthonormal basis for $PW_{t-c,t+c}$.

In this paper we investigate the multidimensional analogue of this construction in which intervals are replaced by balls and the union of intervals is replaced by spherically symmetric annuli in $\mathbb{R}^m$ ($m \geq 3$ odd). This requires the application of techniques from *Clifford analysis* [3].

We consider the right Clifford module $\mathcal{H} = L^2(\mathbb{R}^m, \mathbb{C}_m)$ of measurable $\mathbb{C}_m$-valued functions f defined on $\mathbb{R}^m$ for which $\int_{\mathbb{R}^m} |f(x)|^2\, dx < \infty$. The $\mathbb{C}_m$-valued inner product on $\mathcal{H}$ is defined by $\langle f, g\rangle = \int_{\mathbb{R}^m} \overline{f(x)}g(x)\, dx$ where $\bar{u}$ is the Clifford conjugate of $u \in \mathbb{C}^m$ (see [1]). The $\mathbb{C}_m$-valued inner product on the sequence space $\ell^2(\mathbb{N}, \mathbb{C}_m)$ is given by $\langle \mathbf{a}, \mathbf{b}\rangle_{\ell^2} = \sum_{n=1}^{\infty} \overline{a_n}b_n$. The Fourier transform $\mathcal{F}$ is defined on $L^1(\mathbb{R}^m, \mathbb{C}_m)$ by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^m} f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx$ and extends to a unitary operator on $\mathcal{H}$. The Paley-Wiener space PW_c^m associated with the ball $B(c)$ (centred at the origin and of radius $c > 0$) is the class of those functions $f \in \mathcal{H}$ for which $\mathcal{F}f(\xi) = 0$ for $|\xi| > c$. Similarly, if $0 \leq c < c' < \infty$, the Paley-Wiener space $PW_{c',c}^m$ associated with the annulus $A(c', c)$ is the class of those functions $f \in \mathcal{H}$ for which $\mathcal{F}f(\xi) = 0$ for $|\xi| < c$ or $|\xi| > c'$.

A. Clifford-Legendre polynomials and Clifford Prolate Spheroidal Wavefunctions

In [1] and [2], the Clifford-Legendre polynomials (C-L polys) $\bar{C}_n^0(Y_k^\ell)$ are studied. Here $\{Y_k^\ell\}_{\ell=1}^{d_k^m}$ is an orthonormal basis for the space of *spherical monogenics* of homogeneous degree $k \geq 0$. Each C-L poly is an $\mathbb{R}_m$-valued function on $\mathbb{R}^m$, normalised so that $\int_{B(1)} |\bar{C}_n^0(Y_k^\ell)(x)|^2\, dx = 1$, and takes one of two forms, depending on whether n is even or odd:

$$\bar{C}_{2N}^0(Y_k^\ell)(x) = P_N(|x|^2)Y_k^\ell(x)$$
$$\bar{C}_{2N+1}^0(Y_k^\ell)(x) = xQ_N(|x|^2)Y_k^\ell(x)$$

where P_N and Q_N are real-valued polynomials of degree N on $[0, 1]$. The C-L polys satisfy the *Bonnet formula*:

$$\begin{aligned} x\bar{C}_{2N}^0(Y_k^\ell)(x) &= \alpha_N\bar{C}_{2N+1}^0(Y_k^\ell)(x) + \beta_N\bar{C}_{2N-1}^0(Y_k^\ell)(x) \\ x\bar{C}_{2N+1}^0(Y_k^\ell)(x) &= \alpha'_N\bar{C}_{2N+2}^0(Y_k^\ell)(x) + \beta'_N\bar{C}_{2N}^0(Y_k^\ell)(x) \end{aligned} \tag{1}$$

where α_N, β_N, α'_N, β'_N are explicit real constants which are bounded independent of N.

Given $0 \leq c < c' < \infty$ and $r > 0$, we define orthogonal projections P_c, $P_{c',c}$ and Q_r on $\mathcal{H}$ as follows: $Q_rf(x) = \mathbf{1}_{B(r)}(x)f(x)$ and

$$P_cf(x) = \mathcal{F}^{-1}Q_c\mathcal{F}; \quad P_{c',c} = \mathcal{F}^{-1}(Q_{c'} - Q_c)\mathcal{F}.$$

Here $\mathbf{1}_{B(r)}$ is the characteristic function of $B(r)$, Q_r is the *space-limiting operator* which truncates functions outside $B(r)$, P_c and $P_{c',c}$ are *frequency-limiting operators* which truncate the Fourier transforms of functions outside $B(c)$ and the annulus $A(c', c)$ respectively. By Q we mean the operator Q_1. Clifford-Prolate Spheroidal Wavefunctions (CPSWFs) are eigenfunctions of P_cQ. In [1] and [2], orthonormal bases $\{\psi_{n,k}^\ell : (n, k, \ell) \in \Lambda\}$ for PW_c^m consisting of CPSWFs are constructed and their properties developed. Here $\Lambda = \{(n, k, \ell) : n, k \geq 0, 1 \leq \ell \leq d_k^m\}$. We have $P_cQ\psi_{n,k}^\ell = \lambda_n^k\psi_{n,k}^\ell$ with $\lambda_n^k \geq 0$ and $\lambda_n^k \downarrow 0$ as $n \to \infty$. The functions

$\phi^\ell_{n,k} = (\lambda^k_n)^{-1/2} Q\psi^k_n$ $((n,k,\ell) \in \Lambda)$ form an orthonormal basis for $L^2(B(1), \mathbb{C}_m)$. The construction of CPSWFs given in [1] and [2] involves the numerical computation of real constants $d^{k,\ell}_{N,i}$ and $b^{k,\ell}_{N,i}$ for which

$$\phi^\ell_{2N,k} = \sum_{i=0}^{\infty} \bar{C}^0_{2i}(Y^\ell_k) d^{k,\ell}_{N,i};\ \phi^\ell_{2N+1,k} = \sum_{i=0}^{\infty} \bar{C}^0_{2i+1}(Y^\ell_k) b^{k,\ell}_{N,i}. \tag{2}$$

It is shown in [2] that the CPSWFs have the functional form

$$\begin{aligned} \phi^\ell_{2N,k}(x) &= p_N(|x|) Y^\ell_k(x) \\ \phi^\ell_{2N+1,k}(x) &= x q_N(|x|) Y^\ell_k(x) \end{aligned} \tag{3}$$

with p_N, q_N radial functions whose expansions in Jacobi polynomials are explicitly computed.

B. Clifford translations and modulations

In the Clifford algebra $\mathbb{R}_m$, the space vector $x = \sum_{j=1}^m x_j e_j$ satisfies $x^2 = -|x|^2$. As a consequence, we have for each $t \in \mathbb{R}$ and $x \in \mathbb{R}^m$,

$$e^{tx} = \sum_{n=0}^{\infty} \frac{t^n x^n}{n!} = \cos(t|x|) + \frac{x}{|x|}\sin(t|x|). \tag{4}$$

Since $(\frac{x}{|x|})^2 = -1$, we may view equation (4) as a generalisation of Euler's formula for complex exponentials. Given $\xi, y \in \mathbb{R}^m$, the classical modulation operator M_ξ and classical translation operator T_y act on $\mathcal{H}$ by $M_\xi f(x) = e^{-2\pi i\langle x,\xi\rangle} f(x)$ and $T_y f(x) = f(x-y)$ and are intertwined by the Fourier transform: $\mathcal{F}T_y = M_y\mathcal{F}$. Analogously, given $t \in \mathbb{R}$, we define the *Clifford modulation operator* m_t on $L^2(\mathbb{R}^m, \mathbb{C}_m)$ by

$$m_t f(x) = e^{tx} f(x)$$

and the *Clifford translation operator* τ_t by $\tau_t = \mathcal{F}^{-1} m_t \mathcal{F}$. The multiplication operator S given by $Sf(x) = xf(x)$ and the Dirac operator $D = \sum_{i=1}^m e_i \dfrac{\partial}{\partial x_i}$ (see [1]) are intertwined by the Fourier transform: $\mathcal{F}D = 2\pi i S\mathcal{F}$. Since $D^2 = -\Delta$ (the space Laplacian), we see that $u(\xi,t) = e^{itD_\xi} f(\xi)$ satisfies the following initial value problem for the wave equation in $\mathbb{R}^m$:

$$\begin{aligned} \frac{\partial^2 u(\xi,t)}{\partial t^2} &= \Delta_x u(\xi,t) \quad (\xi \in \mathbb{R}^m,\ t > 0) \\ u(\xi,0) &= f(\xi) \qquad (\xi \in \mathbb{R}^m) \\ \left.\frac{\partial u(\xi,t)}{\partial t}\right|_{t=0} &= iDf(\xi) \qquad (\xi \in \mathbb{R}^m). \end{aligned}$$

When the ambient dimension m is odd, the solution of this problem is obtained by taking mean-values of the initial data over spheres in $\mathbb{R}^m$: if $\mu = \frac{m-3}{2}$ (an integer) then

$$\begin{aligned} u(\xi,t) = \frac{1}{\gamma_m |S^{m-1}|}\Bigg[&\partial_t \left(\frac{1}{t}\partial_t\right)^\mu \left(\frac{1}{t}\int_{\partial B_t(\xi)} f(y)\, d\sigma(y)\right) \\ &+ \left(\frac{1}{t}\partial_t\right)^\mu \left(\frac{1}{t}\int_{\partial B_t(\xi)} iDf(y)\, d\sigma(y)\right)\Bigg] \end{aligned} \tag{5}$$

where $\gamma_m = 1.3.5\ldots(m-2)$, S^{m-1} is the unit sphere in $\mathbb{R}^m$ and $|S^{m-1}|$ is its surface measure. From (5) we see that if f is supported on $B(c)$ and $t > c$, then $\tau_t f$ is supported on the annulus $A(t-c, t+c)$. Hence if $f \in PW_c$ and $t > c$, then $e^{2\pi t x} f \in PW^m_{t-c,t+c}$.

C. Clifford translates of C-L polys

We now investigate the action of Clifford modulations on the C-L polys. Since $\bar{C}^0_n(Y^\ell_k)$ is supported on $B(1)$, if $t > 1$ the Clifford translates $\tau_{\varepsilon t}\bar{C}^0_n(Y^\ell_k)$ $(\varepsilon \in \{\pm 1\})$ will be supported on $A(t-1, t+1)$. From the unitarity of $\tau_{\varepsilon t}$ and the orthonormality of the C-L polys on $B(1)$, we see that if $t > 1$, the Clifford translates $\{\tau_{\varepsilon t}\bar{C}^0_n(Y^\ell_k) : (n,k,\ell) \in \Lambda,\ \varepsilon \in \{\pm 1\}\}$ form an orthonormal collection in $L^2(A(t-1,t+1), \mathbb{C}_m)$, or equivalently, the Clifford modulates

$$\mathcal{B} = \{m_{\varepsilon t}\mathcal{F}\bar{C}^0_n(Y^\ell_k) :\ (n,k,\ell) \in \Lambda,\ \varepsilon \in \{\pm 1\}\}$$

form an orthonormal collection in $PW^m_{t-1,t+1}$. For $c > 0$, let D_c be the unitary isotropic dilation on $\mathcal{H}$, i.e., $D_c f(x) = c^{-m/2} f(\frac{x}{c})$. Since $\tau_t D_c = D_c \tau_{t/c}$, for $t > c$ the collection

$$\mathcal{B}_c = \{m_{\varepsilon t}\mathcal{F}D_c\bar{C}^0_n(Y^\ell_k) :\ (n,k,\ell) \in \Lambda,\ \varepsilon \in \{\pm 1\}\}$$

forms an orthonormal collection in $PW^m_{t-c,t+c}$. As we shall see, this collection does not span $PW^m_{t-c,t+c}$.

We have the following explicit description of the Clifford translates of the C-L polys. In what follows, let m be odd and $\nu = k + \frac{m}{2}$.

Theorem 1. *Let $C^{(\nu)}_n$ be the Gegenbauer polynomial of degree n on the real line. Then for each pair of integers $N, k \geq 0$, there are real constants a^k_N, b^k_N, c^k_N, d^k_N for which*

$$\begin{aligned} &\tau_t\bar{C}^0_{2N}(Y_k)(\xi) = \\ &a^k_N Y_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{3}{2}} \frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{3}{2}} C^{(\nu-1)}_{2N+1}(s-t)] \\ &+ b^k_N \xi Y_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{1}{2}} \frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{1}{2}} C^{(\nu)}_{2N}(s-t)], \\ &\tau_t\bar{C}^0_{2N+1}(Y_k) = \\ &c^k_N \xi Y_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{1}{2}} \frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{1}{2}} C^{(\nu)}_{2N+1}(s-t)] \\ &+ d^k_N Y_k(\xi)\left(\frac{1}{s}\frac{d}{ds}\right)^{\nu-\frac{3}{2}} \frac{1}{s}[(1-(s-t)^2)^{\nu-\frac{3}{2}} C^{(\nu-1)}_{2N+2}(s-t)] \end{aligned}$$

where $s = |\xi|$.

Let $t > c$ and

$$X_t = \overline{\mathrm{sp}}\{\tau_{\varepsilon t} D_c \bar{C}^0_n(Y^\ell_k) :\ (n,k,\ell) \in \Lambda,\ \varepsilon \in \{\pm 1\}\}.$$

X_t is a closed submodule of $L^2(A(t-c,t+c), \mathbb{C}_m)$. Let $X_t^\perp$ be the collection of those $g \in L^2(A(t-c,t+c), \mathbb{C}_m)$ which are orthogonal to all functions in X_t. Then $\tilde{X}_t := \mathcal{F}(X_t)$ is a closed submodule of $PW^m_{t-c,t+c}$.

D. An orthonormal basis for the complementary space $\tilde{X}_t^\perp$

As a consequence of Theorem 1, we have the following description of $X_t^\perp$:

Corollary 2. *Let $t > c$. Then*

$$X_t^\perp = \left\{ \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k^m} [p_{2\nu-2}^{(\ell)}(|x|) + \omega q_{2\nu-3}^{(\ell)}(|x|)] \frac{Y_k^\ell(\omega)}{|x|^{2\nu-1}} \right\} \quad (6)$$

where $\omega = \dfrac{x}{|x|}$, $p_{2\nu-2}$ is the restriction of an odd polynomial of degree less than or equal to $2\nu-2$ to the interval $[t-c, t+c]$ and $q_{2\nu-3}$ is the restriction of an even polynomial of degree less than or equal to $2\nu-3$ to the interval $[t-c, t+c]$.

For $t > c$ fixed let I_t be the interval $I_t = [t-c, t+c]$. For each non-negative integer k, consider the Hilbert space $L_k^2(I_t)$ consisting of measurable functions $f : I_t \to \mathbb{R}$ for which

$$\int_{t-c}^{t+c} |f(s)|^2 \frac{ds}{s^{2\nu-1}} < \infty.$$

The inner product in $L_k^2(I_t)$ is $\langle f, g\rangle_k = \int_{t-c}^{t+c} f(s)g(s) \frac{ds}{s^{2\nu-1}}$. Within $L_k^2(I_t)$, we identify the closed subspaces $\pi_n(I_t)$ of restrictions to I_t of polynomials of degree less than or equal to n, $\pi_n^e(I_t)$ of restrictions to I_t of even polynomials of degree less than or equal to n and $\pi_n^o(I_t)$ of restrictions of odd polynomials of degree less than or equal to n.

For a fixed positive integer k, we apply Gram-Schmidt orthogonalisation within $L_k^2(I_t)$ to the even monomials $\{1, s^2, \ldots, s^{2\nu-3}\}$ to produce an orthonormal basis $\{P_{2j}^k\}_{j=0}^{\nu-\frac{3}{2}}$ for $\pi_{2\nu-3}^e(I_t)$. Similarly, we apply Gram-Schmidt orthogonalisation within $L_k^2(I_t)$ to the odd monomials $\{s, s^3, \ldots, s^{2\nu-2}\}$ to produce an orthonormal basis $\{P_{2j+1}^k\}_{j=0}^{\nu-\frac{3}{2}}$ for $\pi_{2\nu-2}^o(I_t)$. Note that the collection $\{P_{2j}^k\}_{j=0}^{\nu-\frac{3}{2}}$ need not be orthogonal to the collection $\{P_{2j+1}^k\}_{j=0}^{\nu-\frac{3}{2}}$.

Define functions $\varphi_{j,k,l}^\varepsilon$ ($k \geq 0$, $1 \leq \ell \leq d_k^m$, $0 \leq j \leq \nu - \frac{3}{2}$, $\varepsilon \in \{\pm 1\}$) by

$$\varphi_{j,k,l}^\varepsilon(x) = [P_{2j+1}^k(|x|) + \varepsilon\omega P_{2j}^k(|x|)] \frac{Y_k^\ell(\omega)}{\sqrt{2}|x|^{k+m-1}}.$$

Theorem 3. *The collection*

$$\mathcal{B}' = \left\{ \mathcal{F}\varphi_{j,k,\ell}^\varepsilon : k \geq 0,\, 1 \leq \ell \leq d_k^m,\, 0 \leq j \leq \nu - \frac{3}{2},\, \varepsilon \in \{\pm 1\} \right\}$$

is an orthonormal basis for $\tilde{X}_t^\perp$.

II. Pseudo Clifford Bandpass Prolates (PCBPs)

From Theorem 3 we see that the collection $\mathcal{B} \cup \mathcal{B}'$ is an orthonormal basis for $PW_{t-c,t+c}^m$ provided $t > 1$. By a *pseudo prolate* we mean an eigenfunction of either $P_{\tilde{X}_t} Q$ or $P_{\tilde{X}_t^\perp} Q$.

A. Matrix formulation of the eigenvalue problem for PBCPs

Let $\{\psi_{n,k}^\ell : (n,k,\ell) \in \Lambda\}$ be the orthonormal basis for $L^2(B(1), \mathbb{C}_m)$ consisting of the CPSWFs of section IA. Since $\mathcal{H}_{k,\ell} = \overline{\text{sp}}\{e^{2\pi tx}\psi_{n,k}^\ell\}_{n=0}^\infty \oplus \overline{\text{sp}}\{e^{-2\pi tx}\psi_{n,k}^\ell\}_{n=0}^\infty$ is invariant under $P_{\tilde{X}} Q$ for each $k \geq 0$ and $1 \leq \ell \leq d_k^m$, we may seek eigenfunctions of $P_{\tilde{X}} Q$ within each $\mathcal{H}_{k,\ell}$. Suppose

$$f = \sum_{n=0}^\infty e^{2\pi tx}\psi_{n,k}^\ell a_n^+ + \sum_{n=0}^\infty e^{-2\pi tx}\psi_{n,k}^\ell a_n^- \in \mathcal{H}_{k,\ell}$$

is an eigenfunction of $P_{\tilde{X}_t} Q$. Then

$$\begin{aligned}
P_{\tilde{X}_t} Qf &= \sum_{m=0}^\infty e^{2\pi tx}\psi_{m,k}^\ell \langle e^{2\pi tx}\psi_{m,k}^\ell, Qf\rangle \\
&\quad + \sum_{m=0}^\infty e^{-2\pi tx}\psi_{m,k}^\ell \langle e^{-2\pi tx}\psi_{m,k}^\ell, Qf\rangle \\
&= \sum_{m,n} e^{2\pi tx}\psi_{m,k}^\ell [\langle Q\psi_{m,k}^\ell, \psi_{n,k}^\ell\rangle a_n^+ + \langle Q\psi_{m,k}^\ell, e^{-4\pi tx}\psi_{n,k}^\ell\rangle a_n^-] \\
&+ \sum_{m,n} e^{-2\pi tx}\psi_{m,k}^\ell [\langle Q\psi_{m,k}^\ell, e^{4\pi tx}\psi_{n,k}^\ell\rangle a_n^+ + \langle Q\psi_{m,k}^\ell, \psi_{n,k}^\ell\rangle a_n^-] \\
&= \sum_m \psi_{m,k}^\ell [e^{2\pi tx}[D\mathbf{a}^+ + G\mathbf{a}^-]_m + e^{-2\pi tx}[G^*\mathbf{a}^+ + D\mathbf{a}^-]_m]
\end{aligned}$$

where D is the (doubly-infinite) diagonal matrix with diagonal entries $D_{n,n} = \lambda_n^k$ and G is the (doubly-infinite) matrix with entries $G_{n,m} = \langle Q\psi_{m,k}^\ell, e^{4\pi tx}\psi_{n,k}^\ell\rangle$. We conclude that $P_{\tilde{X}_t} Qf = f\lambda$ for some Clifford constant λ if and only if the vector $\mathbf{a} = \begin{pmatrix}\mathbf{a}^+ & \mathbf{a}^-\end{pmatrix}^T$ satisfies the matrix equation

$$C\mathbf{a} = \begin{pmatrix} D & G \\ G^* & D \end{pmatrix} \begin{pmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \end{pmatrix} = \begin{pmatrix} \mathbf{a}^+ \\ \mathbf{a}^- \end{pmatrix} \lambda. \quad (7)$$

It can be shown that the entries of G are real. The matrix C on the left hand side of (7) is self-adjoint on $\ell^2 \times \ell^2$ and by the orthonormality of $\{\phi_{n,k}^\ell\}_{n=0}^\infty$ on $B(1)$ we have

$$\begin{aligned}
\sum_{m,n} |C_{m,n}|^2 &= 2\sum_n (\lambda_n^k)^2 + 2\sum_{m,n} |G_{m,n}|^2 \\
&= 2\sum_n (\lambda_n^k)^2 + 2\sum_{m,n} \lambda_n^k \lambda_m^k |\langle \phi_{m,k}^\ell, e^{4\pi tx} Q\phi_{n,k}^\ell\rangle|^2 \\
&\leq 2\sum_n (\lambda_n^k)^2 + 2\left(\sum_n \lambda_n^k\right)^2 < \infty.
\end{aligned}$$

We conclude that C has a complete system of eigenvectors $\mathbf{a}_n = \begin{pmatrix}\mathbf{a}_n^+ & \mathbf{a}_n^-\end{pmatrix}^T$ with real eigenvalues θ_n. Let $\mathbf{a}_n$ be such an eigenvector and

$$\Psi_{n,k}^\ell = \sum_m e^{2\pi tx}\psi_{m,k}^\ell (\mathbf{a}_n^+)_m + \sum_m e^{-2\pi tx}\psi_{m,k}^\ell (\mathbf{a}_n^-)_m.$$

Then $\Psi_{n,k}^\ell$ is an eigenfunction of $P_{\tilde{X}_t} Q$ with eigenvalue θ_n and

$$\langle \Psi_{n,k}^\ell, \Psi_{m,k}^\ell\rangle = \langle \mathbf{a}_n^+, \mathbf{a}_m^+\rangle + \langle \mathbf{a}_n^-, \mathbf{a}_m^-\rangle = \langle \mathbf{a}_n, \mathbf{a}_m\rangle = \delta_{n,m}.$$

We have:

Theorem 4. *The collection $\tilde{\mathcal{B}} = \{\Psi_{n,k}^\ell : (n,k,\ell) \in \Lambda\}$ is an orthonormal basis for $\tilde{X}_t$ consisting of eigenfunctions of $P_{\tilde{X}_t} Q$.*

B. Matrix formulation of the eigenvalue problem for the complementary space

The elements of the orthonormal basis $\mathcal{B}'$ for $\tilde{X}_t^\perp$ may be computed as follows:

$$\mathcal{F}\phi_{j,k}^{\ell,\varepsilon}(x) = \int_{A_{t-c,t+c}} \frac{P_{2j+1}^k(|\xi|) + \varepsilon\omega P_{2j}^k(|\xi|)}{\sqrt{2}|\xi|^{k+m-1}} Y_k^\ell(\omega) e^{-2\pi i\langle x,\xi\rangle}\, d\xi$$
$$= A_{j,k}^\ell(x) + \varepsilon B_{j,k}^\ell(x)$$

where

$$A_{j,k}^\ell(x) = \frac{2\pi}{\sqrt{2}}(-i)^k \frac{Y_k^\ell(\eta)}{|x|^{\nu-1}} \int_{t-c}^{t+c} \frac{P_{2j+1}^k(r)}{r^{\nu-1}} J_{\nu-1}(2\pi r|x|)\, dr$$
$$B_{j,k}^\ell(x) = \frac{2\pi}{\sqrt{2}}(-i)^{k+1}\eta \frac{Y_k^\ell(\eta)}{|x|^{\nu-1}} \int_{t-c}^{t+c} \frac{P_{2j}^k(r)}{r^{\nu}} J_{\nu}(2\pi r|x|)\, dr,$$

$\eta = \frac{x}{|x|}$ and J_μ is a Bessel function of the first kind. Closed forms for $A_{j,k}^\ell(x)$ and $B_{j,k}^\ell(x)$ may be written in terms of the values of P_{2j}^k and P_{2j+1}^k and their derivatives evaluated at $(t \pm c)$ and the values of various Bessel functions J_μ at $2\pi(t \pm c)|x|$. The details will appear elsewhere.

If $f = \sum_j \mathcal{F}\varphi_{j,k}^{\ell,+} b_j^+ + \sum_j \mathcal{F}\varphi_{j,k}^{\ell,-} b_j^- \in \tilde{X}_t^\perp$ is an eigenfunction of $P_{\tilde{X}_t^\perp} Q$, then the coefficients b_j^+ and b_j^- satisfy the matrix equation

$$\begin{pmatrix} H^{++} & H^{+-} \\ H^{-+} & H^{--} \end{pmatrix} \begin{pmatrix} \mathbf{b}^+ \\ \mathbf{b}^- \end{pmatrix} = \begin{pmatrix} \mathbf{b}^+ \\ \mathbf{b}^- \end{pmatrix} \lambda \tag{8}$$

where H^{++}, H^{+-}, H^{-+}, H^{--} are (doubly-infinite) matrices given by $H_{j',j}^{\varepsilon_1,\varepsilon_2} = \langle \mathcal{F}\varphi_{j,k}^{\ell,\varepsilon_1}, Q\mathcal{F}\varphi_{j',k}^{\ell,\varepsilon_2}\rangle$. The matrix on the left hand side of (8) is self-adjoint. Before invoking the spectral theory, we must show that this matrix is Hilbert-Schmidt. Let

$$K_k(r,s) = \int_0^1 u J_{k+\nu-1}(2\pi r u) J_{\nu-1}(2\pi s u)\, du$$
$$= \frac{2\pi s J_{\nu-1}(2\pi s) J_{\nu-1}(2\pi r) - 2\pi r J_{\nu-2}(2\pi r) J_{\nu-1}(2\pi s)}{r^2 - s^2}.$$

We find that

$$\frac{\langle A_{j,k}^\ell, QA_{j',k}^\ell\rangle}{2\pi^2}$$
$$= \int_{t-c}^{t+c} \frac{P_{2j+1}^k(r)}{r^{\nu-1}} \int_{t-c}^{t+c} \frac{P_{2j'+1}^k(s)}{s^{\nu-1}} K_k(r,s)\, ds\, dr$$
$$= \mathcal{G}_1^k \mathcal{G}_2^k (K_k)(j,j')$$

where $\mathcal{G}_1^k\mathcal{G}_2^k F(j,j')$ is the generalised (j,j')-th Fourier coefficient of $F(r,s)$ $(r,s \in [t-c,t+c])$ relative to the orthonormal collection $\{P_{2j}^k\}_{j=0}^{\nu-\frac{3}{2}}$. By Bessel's inequality we have

$$\sum_{j,j'} |\langle A_{j,k}^\ell, QA_{j',k}^\ell\rangle|^2 \le \int_{t-c}^{t+c} |K_k(r,s)|^2 \frac{dr\, ds}{(sr)^{\nu-1}} < \infty.$$

A similar estimate may be made for $\sum_{j,j'} |\langle B_{j,k}^\ell, QB_{j',k}^\ell\rangle|^2$. Since $\langle A_{j,k}^\ell, B_{j',k}^\ell\rangle = 0$, we have

$$\sum_{j,j'} |H_{j,j'}^{\varepsilon_1,\varepsilon_2}|^2 = \sum_{j,j'} |\langle A_{j,k}^\ell + \varepsilon_1 B_{j,k}^\ell, Q(A_{j',k}^\ell + \varepsilon_2 B_{j',k}^\ell\rangle|^2$$
$$= \sum_j [|\langle A_{j,k}^\ell, QA_{j,k}^\ell\rangle|^2 + |\langle B_{j,k}^\ell, QB_{j,k}^\ell\rangle|^2$$
$$+ 2\varepsilon_1\varepsilon_2 \Re(\langle A_{j,k}^\ell, QA_{j,k}^\ell\rangle \overline{\langle B_{j,k}^\ell, QB_{j,k}^\ell\rangle})$$

Summing over $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ gives

$$\sum_{\varepsilon_1,\varepsilon_2} \sum_{j,j'} |H_{j,j'}^{\varepsilon_1,\varepsilon_2}|^2$$
$$= 4\sum_j [|\langle A_{j,k}^\ell, QA_{j,k}^\ell\rangle|^2 + |\langle B_{j,k}^\ell, QB_{j,k}^\ell\rangle|^2]$$
$$\le 16\pi^4 \int_{t-c}^{t+c} [|K_k(r,s)|^2 + |K_{k+1}(r,s)|^2]\, dr\, ds < \infty.$$

C. Computation of G

The construction of bandpass prolates within X still requires the computation of the matrix G in (7). One scheme involves another eigenfunction property of the CPSWFs:

$$\int_B \psi_n^{k,\ell}(x) e^{2\pi i c\langle x,y\rangle}\, dx = \mu_m^k \psi_{n,k}^\ell(x) \tag{9}$$

where $\mu_n^k = \pm i^{k+n}\sqrt{\frac{\lambda_k^k}{c^m}}$. From (9) we have the following:

Theorem 5. *Let* $\alpha_{n,n';k} = \frac{\mu_n^k \overline{\mu_{n'}^k}(-1)^{n+k+1}}{2\pi i c((\mu_n^k)^2 + (\overline{\mu_{n'}^k})^2)}$. *Then*

$$\int_B \overline{\psi_{n',k'}^{\ell'}(x)} x \psi_{n,k}^\ell(x)\, dx = \delta_{k,k'}\delta_{\ell,\ell'}\sqrt{\lambda_n^k \lambda_{n'}^k}\gamma_{n,n,;k}$$

where

$$\gamma_{n,n';k} = \alpha_{n,n';k} \begin{cases} -\overline{p_{\frac{n'}{2}}^k(1)} q_{\frac{n-1}{2}}^k(1) & \text{if } n' \text{ even and } n \text{ odd} \\ \overline{q_{\frac{n'-1}{2}}^k(1)} p_{\frac{n}{2}}^k(1) & \text{if } n' \text{ odd and } n \text{ even} \\ 0 & \text{if } n - n' \text{ even} \end{cases} \tag{10}$$

with p_N *and* q_N *as in (3).*

The orthonormality of the functions $\varphi_{n,k}^\ell$ on $B(1)$ now allows for the following description of the matrix G:

Theorem 6. *Let* $\Gamma^{(k)}$ *be the doubly-infinite matrix with entries* $\Gamma_{n,n'}^{(k)} = \gamma_{n,n';k}$ *with* $\gamma_{n,n';k}$ *as in (10). Then* $G = \exp(4\pi t\Gamma^{(k)})$.

Finally, we provide another method of computation of the matrix G of (7). We recall the expansions (2) of $L^2(B)$-normalised CPSWFs as linear combinations of C-L polys. By iterating the Bonnet formulae (1) we find negative definite, tri-diagonal, symmetric, doubly-infinite matrices M^e and M^o for which

$$x^2 \sum_{i=0}^\infty \bar{C}_{2i}^0(Y_k^\ell) a_i = \sum_{i=0}^\infty \bar{C}_{2i}^0(Y_k^\ell)(M^e\mathbf{a})_i$$
$$x^2 \sum_{i=0}^\infty \bar{C}_{2i+1}^0(Y_k^\ell) b_i = \sum_{i=0}^\infty \bar{C}_{2i+1}^0(Y_k^\ell)(M^o\mathbf{b})_i$$

As a consequence, we have the following result:

Theorem 7. *The entries* $G_{2N,2N'}$ *of the matrix* G *are given by*

$$\int_B \overline{\psi_{2N,k}^\ell(x)} e^{4\pi t x} \psi_{2N',k'}^{\ell'}(x)\, dx$$
$$= \sqrt{\lambda_{2N}^k \lambda_{2N'}^k}\delta_{k,k'}\delta_{\ell,\ell'}\langle \mathbf{d}_N^{k,\ell}, \cos(4\pi t\sqrt{-M^e})\mathbf{d}_{N'}^{k,\ell}\rangle_{\ell^2}$$

where the sequence $\mathbf{d}_N^{k,\ell}$ *with* i*-th entry* $d_{N,i}^{k,\ell}$ *is as in (2). Similar formulae hold for other entries of* G*.*

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