ROBUST CONFORMAL PREDICTION WITH A SINGLE BINARY CERTIFICATE

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ABSTRACT

Conformal prediction (CP) converts any model's output to prediction sets with a guarantee to cover the true label with (adjustable) high probability. Robust CP extends this guarantee to worst-case (adversarial) inputs. Existing baselines achieve robustness by bounding randomly smoothed conformity scores. In practice, they need expensive Monte-Carlo (MC) sampling ($\sim 10^4$ samples per point) to maintain an acceptable set size. We propose a robust conformal prediction that produces smaller sets even with significantly lower MC samples (e.g. 150 for CIFAR10) Our approach binarizes samples with an adaptive bin selected to preserve the coverage guarantee. Remarkably, we prove that robustness can be achieved by computing *only one* binary certificate, unlike previous methods that certify each calibration (or test) point. Thus, our method is faster and returns smaller robust sets. We also eliminate a previous limitation that requires a bounded score function.

1 INTRODUCTION

Despite their extensive applications, modern neural networks lack reliability as their output probability estimates are uncalibrated (Guo et al., 2017). Many uncertainty quantification methods are computationally expensive, lack compatibility with black-box models, and offer no formal guarantees. Alternatively, conformal prediction (CP) is a statistical post-processing approach that returns prediction *sets* with a guarantee to cover the true label with high adjustable probability. CP only requires a held-out calibration set and offers a distribution-free model-agnostic coverage guarantee (Vovk et al., 2005; Angelopoulos & Bates, 2021). The model is used as a black box to compute conformity scores which capture the agreement between inputs $\boldsymbol{x}$ and labels y. These prediction sets are shown to improve human decision-making both in terms of response time and accuracy (Cresswell et al., 2024). CP assumes exchangeability between the calibration and the test set (a relaxation of the i.i.d. assumption), making it broadly applicable, including e.g. node classification (Zargarbashi et al., 2023; Huang et al., 2023) where uncertainty quantification methods are limited. However, exchangeability, and therefore the conformal guarantee, easily breaks when the test data is noisy or subjected to adversarial perturbations.

Robust conformal prediction extends this guarantee to worst-case inputs $\tilde{\boldsymbol{x}}$ within a maximum radius around the clean point $\boldsymbol{x}$, e.g. $\forall \tilde{\boldsymbol{x}}$ s.t. $\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq r$. In the evasion setting, we assume that the calibration set is clean, and test datapoints can be perturbed. Building on the rich literature of robustness certificates (Kumar et al., 2020), recent robust CP baselines (Gendler et al., 2021; Zargarbashi et al., 2024; Jeary et al., 2024) use a conservative score at test time that is a *certified* bound on the conformity score of the clean unseen input. This maintains the guarantee even for the perturbed input since "if CP covers $\boldsymbol{x}$, then robust CP certifiably covers $\tilde{\boldsymbol{x}}$". However, the average set size increases, especially if the bounds are loose. The certified bounds can be derived through model-dependent verifiers (Jeary et al., 2024) or smoothing-based black-box certificates (Zargarbashi et al., 2024).

For the robustness of black-box models, an established approach is to certify the confidence score through randomized smoothing (Kumar et al., 2020), obtaining bounds on the expected smooth score. The tightness of these bounds depends on the information about the smooth score around the given input, e.g. the mean Yan et al. (2024), or the CDF Zargarbashi et al. (2024). Such methods: (i) assume the conformity score function has a bounded range, (ii) compute several certificates for each calibration (or test) point, and (iii) need a large number of Monte-Carlo samples to get tight

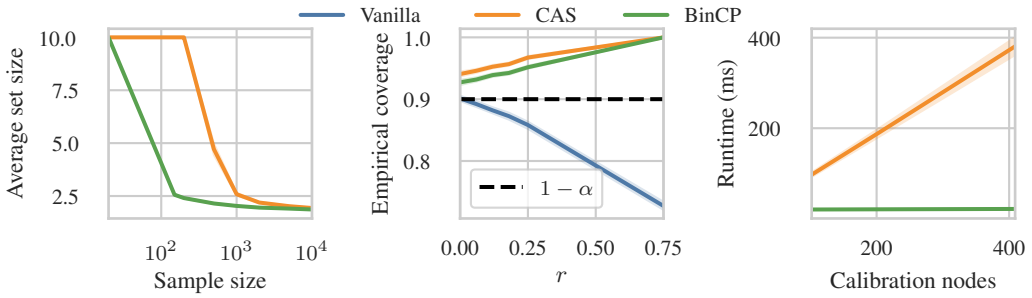

Figure 1: [Left] Average set size with different MC sample rates, [middle] empirical coverage of vanilla and robust CPs under attack, and [right] runtime of robust CP methods given the MC samples (single vs n lower bound computations).

confidence intervals. For the current SOTA method CAS (Zargarbashi et al., 2024), the effect of sample correction is highly problematic for sample rates below 2000 (see Fig. 1 [left]). In contrast, we obtain robust and small prediction sets with ~ 150 MC samples. This inefficiency increases to trivially returning $\mathcal{Y}$ as the prediction set when we run with higher coverage rates or higher radii (see § 6). Additionally, these methods require computing certified bounds for (at least) each calibration point which we further show is a wasteful computation.

BinCP. We observe that smooth inference is inherently more robust. Even without certificates randomized methods show a slower decrease in coverage under attack (see Fig. 7-left). Given any score function $s(\boldsymbol{x}, y)$ capturing conformity, Zargarbashi et al. (2024) and Gendler et al. (2021) define the smooth score as $\bar{s}(\boldsymbol{x}, y) = \mathbb{E}_{\boldsymbol{\epsilon}\sim\mathcal{N}(\mathbf{0},\sigma\boldsymbol{I})}[s(\boldsymbol{x}+\boldsymbol{\epsilon}, y)]$. Instead, we perform binarization via a threshold τ, i.e. $\bar{s}(\boldsymbol{x}, y) = \mathbb{E}_{\boldsymbol{\epsilon}\sim\mathcal{N}(\mathbf{0},\sigma\boldsymbol{I})}[\mathbb{I}[s(\boldsymbol{x}+\boldsymbol{\epsilon}, y) \geq \tau]] = \Pr[s(\boldsymbol{x}+\boldsymbol{\epsilon}, y) \geq \tau]$. Both are valid conformity scores, and both change slowly around any $\boldsymbol{x}$, however, our binarized CP (BinCP) method has several advantages. First, we define robust CP that only computes a single certificate. In comparison, SOTA requires at least one certificate per calibration (or test) point. Second, our method can effortlessly use many existing binary certificates out of the box without any additional assumptions or modifications. A direct consequence is that we can use de-randomization techniques (Levine & Feizi, 2021) that completely nullify the need for sample correction under ℓ_1 norm. Third, when we do need sample correction, working with Bernoulli parameters allows us to use tighter concentration inequalities (Clopper & Pearson, 1934). Thus, even with significantly lower MC samples, our method still produces small prediction sets (see Fig. 1 left). This improvement is even more pronounced for datasets with a large number of classes (e.g. ImageNet shown in Fig. 5). Finally, BinCP does not require the score function to be bounded which is a limitation in current methods.

2 Background

We assume a holdout set of labeled calibration datapoints $\mathcal{D}_{\text{cal}} = \{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$ which is exchangeable with future test points $(\boldsymbol{x}_{n+1}, y_{n+1})$, both sampled from some distribution $\mathcal{D}$. We have black-box access to a model from which we compute an arbitrary conformity score $s : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, e.g. score $s(\boldsymbol{x}, y) = \pi_y(\boldsymbol{x})$ where $\pi_y(\boldsymbol{x})$ is the predicted probability for class y (other scores in § A).[1]

Vanilla CP. For a user-specified nominal coverage level $1-\alpha$, let $q_\alpha = \mathbb{Q}\left(\alpha; \{s(\boldsymbol{x}_i, y_i)\}_{i=1}^n \cup \{\infty\}\right)$ where $\mathbb{Q}(\cdot;\cdot)$ is the quantile function. The sets defined as $\mathcal{C}(\boldsymbol{x}_{n+1}) = \{y : s(\boldsymbol{x}_{n+1}, y) \geq q_\alpha\}$ have $1-\alpha$ guarantee to include the true label y_{n+1}. Formally, $\Pr[y_{n+1} \in \mathcal{C}(\boldsymbol{x}_{n+1})] \geq 1-\alpha$ (Vovk et al., 2005) where the probability is over $\mathcal{D}_{\text{cal}} \sim \mathcal{D}, \boldsymbol{x}_{n+1} \sim \mathcal{D}$. This guarantee, and later our robust sets, are independent of the mechanics of the model and the score function – the model's accuracy or the quality of the score function is irrelevant. A score function that better reflects input-label agreement leads to more efficient (i.e., smaller) prediction sets. For noisy or adversarial inputs, the exchangeability between the test and

[1]Here $s(\boldsymbol{x}, y)$ quantifies agreement not the non-conformity between $\boldsymbol{x}$, and y. The setups are equivalent to a sign flip in scores.

calibration set breaks, making the coverage guarantee invalid. Fig. 1-middle (and Fig. 7-left) shows that an adversary (or bounded worst-case noise) can decrease the empirical coverage drastically with imperceptible perturbations on each test point. As a defense, *robust* CP extends this guarantee to the worst-case bounded perturbations.

Threat model. The adversary's goal is to decrease the empirical coverage probability by perturbing the input. Let $\mathcal{B} : \mathcal{X} \to 2^{\mathcal{X}}$ be a ball that returns all admissible perturbed points around an input. For images a common threat model is defined by the ℓ_2 norm: $\mathcal{B}_r(\boldsymbol{x}) = \{\tilde{\boldsymbol{x}} : \|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2 \leq r\}$ where r shows the magnitude of perturbation. Similarly, we can use the ℓ_1 norm. For binary data and graphs, Bojchevski et al. (2020) define $\mathcal{B}_{r_a,r_d}(\boldsymbol{x}) = \{\tilde{\boldsymbol{x}} : \sum_{i=1}^{d} \mathbb{I}[\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i - 1] \leq r_d, \sum_{i=1}^{d} \mathbb{I}[\tilde{\boldsymbol{x}}_i = \boldsymbol{x}_i + 1] \leq r_a\}$ where the adversary is allowed to toggle at most r_a zero bits, and r_d one bits.

Inverted ball $\mathcal{B}^{-1}$. At test time we are given a (potentially) perturbed $\tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x})$. However, to obtain robust sets, we need to reason about (the score) of the unseen clean $\boldsymbol{x}$. Naively, one might assume that $\boldsymbol{x} \in \mathcal{B}(\tilde{\boldsymbol{x}})$ – the clean point is in the ball around the perturbed point. However, this only holds in special cases such as the ball defined by the l_2 norm. For example, if a binary $\tilde{\boldsymbol{x}}$ was obtained by removing r_d bits and adding r_a bits, to able to reach the clean $\boldsymbol{x}$ from the perturbed $\tilde{\boldsymbol{x}}$ we need to add r_d bits and remove r_a bits instead since $\mathcal{B}_{r_a,r_d}$ unlike $\mathcal{B}_r$ is not symmetric. We define the inverted ball $\mathcal{B}^{-1}$ as the smallest ball centered at $\tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x})$ that includes the clean $\boldsymbol{x}$. Formally $\mathcal{B}^{-1}$ should satisfy $\forall \tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x}) \Rightarrow \boldsymbol{x} \in \mathcal{B}^{-1}(\tilde{\boldsymbol{x}})$. For symmetric balls like ℓ_p-norms, $\mathcal{B}^{-1} = \mathcal{B}$. For the binary ball $\mathcal{B}^{-1}_{r_a,r_d} = \mathcal{B}_{r_d,r_a}$ we need to swap r_a and r_d to ensure this condition. Zargarbashi et al. (2024) also discuss this subtle but important aspect without formally defining $\mathcal{B}^{-1}$.

Robust CP. Given a threat model, robust CP defines a *conservative* prediction set $\bar{\mathcal{C}}$ that maintains the conformal guarantee even for worst-case input. Formally

$$\Pr_{\mathcal{D}_{\text{cal}} \cup \{\boldsymbol{x}_{n+1}\} \sim \mathcal{D}} [y_{n+1} \in \bar{\mathcal{C}}(\tilde{\boldsymbol{x}}_{n+1}), \forall \tilde{\boldsymbol{x}}_{n+1} \in \mathcal{B}(\boldsymbol{x}_{n+1})] \geq 1 - \alpha \tag{1}$$

The intuition behind existing methods is as follows: (i) Vanilla CP covers $\boldsymbol{x}_{n+1}$ with $1-\alpha$ probability (ii) if $y \in \mathcal{C}(\boldsymbol{x}_{n+1})$ then $y \in \bar{\mathcal{C}}(\tilde{\boldsymbol{x}}_{n+1})$. Thus, robust CP covers $\tilde{\boldsymbol{x}}_{n+1}$ with the same probability. Here, (ii) is guaranteed via certified lower bounds $\mathrm{c}^{\downarrow}[s, \boldsymbol{x}, \mathcal{B}]$ or certified upper bounds $\mathrm{c}^{\uparrow}[s, \boldsymbol{x}, \mathcal{B}^{-1}]$.

Theorem 1 (Robust CP from Zargarbashi et al. (2024)). *With* $\mathrm{c}^{\uparrow}[s_y, \tilde{\boldsymbol{x}}, \mathcal{B}^{-1}] \geq \max_{\boldsymbol{x}' \in \mathcal{B}^{-1}(\tilde{\boldsymbol{x}})} s(\boldsymbol{x}', y)$*, let* $\bar{C}_{\text{test}}(\tilde{\boldsymbol{x}}_{n+1}) = \big\{y : \mathrm{c}^{\uparrow}[s_y, \tilde{\boldsymbol{x}}_{n+1}, \mathcal{B}^{-1}] \geq q\big\}$*, then* $\bar{C}_{\text{test}}$ *satisfies Eq. 1 (test-time robustness). Alternatively, with* $\mathrm{c}^{\downarrow}[s_y, \boldsymbol{x}, \mathcal{B}] \leq \min_{\boldsymbol{x}' \in \mathcal{B}(\boldsymbol{x})} s(\boldsymbol{x}', y)$*, define* $q^{\downarrow} = \mathbb{Q}\left(\alpha; \left\{\mathrm{c}^{\downarrow}[s_{y_i}, \boldsymbol{x}_i, \mathcal{B}]\right\}_{i=1}^{n}\right)$*. Then* $\bar{C}_{\text{cal}}(\tilde{\boldsymbol{x}}_{n+1}) = \left\{y : s(\tilde{\boldsymbol{x}}_{n+1}, y) \geq q^{\downarrow}\right\}$ *also satisfies Eq. 1 (calibration-time). Here* $s_y(\cdot) = s(\cdot, y)$*.*

In Theorem 1 test-time robustness uses $\mathcal{B}^{-1}$ since the clean test point is unseen, but calibration-time robustness uses $\mathcal{B}$ since the clean calibration point is given. We can obtain the $\mathrm{c}^{\downarrow}, \mathrm{c}^{\uparrow}$ bounds through neural network verifiers Jeary et al. (2024) or randomized smoothing (Cohen et al., 2019). We focus on the latter since we get model-agnostic certificates with black-box access. The coverage probability is theoretically proved in CP. Similarly, (adversarially) robust CP also comes with a theoretical guarantee. In both cases we can compute the empirical coverage as a sanity check. Another metric of interest in both cases is the average set size (the efficiency) of the conformal sets.

Randomized smoothing. A smoothing scheme $\xi : \mathcal{X} \to \mathcal{X}$ maps any point to a random nearby point. For continuous data Gaussian smoothing $\xi(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{\epsilon}$ adds an isotropic Gaussian noise to the input $\boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma \boldsymbol{I})$. For sparse binary data Bojchevski et al. (2020) define sparse smoothing as $\xi(\boldsymbol{x}) = \boldsymbol{x} \oplus \boldsymbol{\epsilon}$ where $\oplus$ is the binary XOR, and $\boldsymbol{\epsilon}[i] \sim \text{Bernoulli}(p = p_{\boldsymbol{x}[i]})$, where p_1, and p_0 are two smoothing parameters to account for sparsity. Regardless of how rapidly a score function $s(\boldsymbol{x}, y)$ changes, the smooth score $\bar{s}(\boldsymbol{x}, y) = \mathbb{E}[s(\xi(\boldsymbol{x}), y]$ changes slowly near $\boldsymbol{x}$. This enables to compute tight $\mathrm{c}^{\downarrow}, \mathrm{c}^{\uparrow}$ bounds that depend on the smoothing strength. See § 4, § B, and § D for details.

3 BINARIZED CONFORMAL PREDICTION (BINCP)

We define conformal sets by binarizing randomized scores. We first show that this preserves the conformal guarantee for clean data. Then in § 4 we extends the guarantee to worst-case adversarial inputs. As we will see in § 6 our binarization approach has gains in terms of Monte-Carlo sampling budget, computational cost, and average set size.

Proposition 1. *For any two parameters $p \in (0,1), \tau \in \mathbb{R}$, given a smoothing scheme $\xi(\boldsymbol{x})$, define the boolean function* $\text{accept}[\cdot,\cdot;p,\tau]$ *and the prediction set $\mathcal{C}(\cdot;p,\tau)$ as*

$$\text{accept}[\boldsymbol{x}, y; p, \tau] = \mathbb{I}[\Pr_{\xi}[s(\xi(\boldsymbol{x}), y) \geq \tau] \geq p] \quad \textit{and} \quad \mathcal{C}(\boldsymbol{x}; p, \tau) = \{y : \text{accept}(\boldsymbol{x}, y; p, \tau)\}$$

For any fixed p, let

$$\tau_\alpha(p) = \sup_{\tau} \left\{ \tau : \sum_{i=1}^{n} \text{accept}(\boldsymbol{x}_i, y_i; p, \tau) \geq (1-\alpha) \cdot (n+1) \right\} \tag{2}$$

then the set $\mathcal{C}(\boldsymbol{x}_{n+1}; p, \tau_\alpha(p))$ has $1-\alpha$ coverage guarantee. Alternatively, for any fixed τ, let

$$p_\alpha(\tau) = \sup_{p} \left\{ p : \sum_{i=1}^{n} \text{accept}(\boldsymbol{x}_i, y_i; p, \tau) \geq (1-\alpha) \cdot (n+1) \right\} \tag{3}$$

again the prediction set $\mathcal{C}(\boldsymbol{x}_{n+1}; p_\alpha(\tau), \tau)$ has $1-\alpha$ coverage guarantee.

The correctness of Prop. 1 can be directly seen by noticing that we implicitly define new scores.

Quantile view. Let $S_i = s(\xi(\boldsymbol{x}_i), y_i)$ be the distribution of randomized scores for $\boldsymbol{x}_i$ and the true class y_i. Let $\tau_i(p) = \mathbb{Q}(p; S_i)$, we have that $\tau_\alpha(p) = \mathbb{Q}(\alpha; \{\tau_i(p)\}_{i=1}^n)$ is a quantile of quantiles. Similarly, define $p_i(\tau) = \mathbb{Q}^{-1}(\tau; S_i)$ then $p_\alpha(\tau) = \mathbb{Q}(\alpha; \{p_i(\tau)\}_{i=1}^n)$ is a quantile of inverse quantiles. Both $\tau_i(p)$ for a fixed p and $p_i(\tau)$ for a fixed τ are valid conformity scores for instance $\boldsymbol{x}_i$, since exchangeability is trivially preserved. Therefore, $\tau_\alpha(p)$ and $p_\alpha(\tau)$ are just the standard quantile thresholds from CP on some new score functions. This directly gives the $1-\alpha$ coverage guarantee. This view via the implicit scores is helpful for intuition, but we keep the original formulation since it is more directly amenable to certification as we show in § 4. We provide an additional formal proof of Prop. 1 via conformal risk control (Angelopoulos et al., 2022) in § C.

Using either variant from Prop. 1 let (p_α, τ_α) equal $(p, \tau_\alpha(p))$ or $(p_\alpha(\tau), \tau)$ as the final pair of parameters. For test points $\boldsymbol{x}_{n+1}$ we accept labels whose smooth score distribution has at least p_α proportion above the threshold τ_α, i.e. $\text{accept}(\boldsymbol{x}_{n+1}, y; p_\alpha, \tau_\alpha) = 1$. The term "binarization" refers to mapping each score sample above τ to 1 and all others 0. For distributions with a strictly increasing and continuous CDF (e.g. isotropic Gaussian smoothing) both variants are equivalent.

Lemma 1. *Given distributions $\{S_i\}_{i=1}^n$ with strictly increasing and continuous CDFs, let $\tau_\alpha(p)$ be obtained from Eq. 2 with fixed p and $p_\alpha(\cdot)$ be as defined in Eq. 3. We have $p_\alpha(\tau_\alpha(p)) = p$.*

We defer all proofs to § C. Let p be fixed, we get sets with $(p, \tau_\alpha(p))$. With Lemma 1 and fixing $\tau = \tau_\alpha(p)$ we get the sets with $(p_\alpha(\tau), \tau) = (p_\alpha(\tau_\alpha(p)), \tau_\alpha(p)) = (p, \tau_\alpha(p))$ which are equal. Fig. 2 shows the $\text{accept}(\boldsymbol{x}, y; p, \tau)$ function for several examples. This function is non-increasing in both parameters p and τ. In general, for any arbitrary p, and τ, the function $\text{accept}(\cdot, \cdot, p, \tau)$ results in a some expected coverage (Fig. 2-right). Intuitively, thresholds obtained from Prop. 1 are points on the $1-\alpha$ contour of this function. The expected coverage probability is close to the empirical coverage on the given calibration set due to exchangeability (Berti & Rigo, 1997).

Remarks. The scores $\tau_i(p)$ (and similarly $p_i(\tau)$ remain exchangeable whether the quantile over the smoothing distribution is computed exactly or estimated from any number of Monte-Carlo samples.

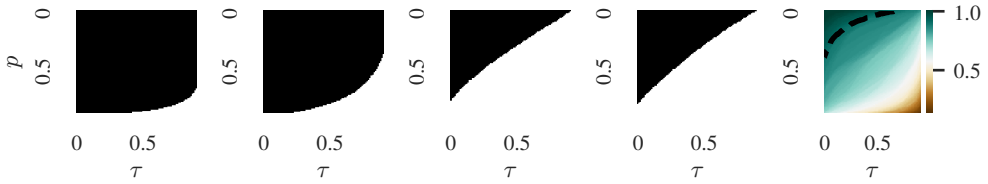

Figure 2: [Left] Function $\text{accept}(\cdot, \cdot; p, \tau)$ for different (p, τ) pairs for random CIFAR-10 instances. Black equals 1 and white equals 0. [Right] Empirical coverage for different (p, τ) pairs. Any (p, τ) pair on the dashed black line showing the 0.9 contour gives conformal sets with 90% coverage.

That is, Prop. 1 holds regardless. However, when need to be more careful when we consider the certified upper and lower bounds. In § 4 we first derive robust conservative sets that maintain worst-case coverage, assuming that we can compute probabilities and expectations exactly. Since this is not always possible, in § 5 we provide the appropriate sample correction that still preserves the robustness guarantee when using Monte-Carlo samples. We also discuss a de-randomized approach that does not need sample correction.

4 ROBUST BINCP

From Prop. 1 (either variant) we get a pair (p_α, τ_α). From the conformal guarantee, it follows that $\Pr[s(\boldsymbol{x}_{n+1}, y_{n+1}) \geq \tau_\alpha] \geq p_\alpha$ with probability $1-\alpha$ for clean $\boldsymbol{x}_{n+1}$. We will exploit this property. Define $f_y(\boldsymbol{x}) = \mathbb{I}[s(\boldsymbol{x}, y) \geq \tau_\alpha]$, we have $\bar{f}_y(\boldsymbol{x}) = \mathbb{E}_\xi[\mathbb{I}[s(\xi(\boldsymbol{x}), y) \geq \tau_\alpha]] = \Pr_\xi[s(\xi(\boldsymbol{x}), y) \geq \tau_\alpha]$.

Conventional robust CP. One way to attain robust prediction sets is to apply the same recipe as Zargarbashi et al. (2024) by finding upper or lower bounds on the new score function. Zargarbashi et al. (2024) use the smooth score $\bar{s}_y(\boldsymbol{x}) = \mathbb{E}_\xi[s(\xi(\boldsymbol{x}), y)]$. Instead, we can bound $\bar{f}_y(\boldsymbol{x})$ which is a smooth binary classifier. Following Theorem 1 the test-time, and calibration-time robust prediction sets are

$$\bar{\mathcal{C}}_{\text{test}}(\tilde{\boldsymbol{x}}_{n+1}) = \{y : \mathrm{c}^{\uparrow}[\bar{f}_y, \tilde{\boldsymbol{x}}_{n+1}, \mathcal{B}^{-1}] \geq p_\alpha\}, \quad \bar{\mathcal{C}}_{\text{cal}}(\tilde{\boldsymbol{x}}_{n+1}) = \{y : \bar{f}_y(\tilde{\boldsymbol{x}}_{n+1}) \geq q^{\downarrow}\} \tag{4}$$

where $q^{\downarrow} = \mathbb{Q}\left(\alpha; \{\mathrm{c}^{\downarrow}[\bar{f}_{y_i}, \boldsymbol{x}_i, \mathcal{B}]\}_{i=1}^n\right)$. In short, we replace the clean $\Pr[s(\boldsymbol{x}_{n+1}, y_{n+1}) \geq \tau_\alpha]$ with either its certified upper $\mathrm{c}^{\uparrow}$ or lower $\mathrm{c}^{\downarrow}$ bound. We elaborate on this approach before improving it.

Computing $\mathrm{c}^{\downarrow}$ and $\mathrm{c}^{\uparrow}$. Computing exact worst-case bounds on $\bar{f}$ ($\bar{f}_y$ for all y) is intractable and requires white-box access to the score function and therefore the model. Following established techniques in the randomized smoothing literature (Lee et al., 2019) we relax the problem. Formally,

$$\mathrm{c}^{\downarrow}[\bar{f}, \boldsymbol{x}, \mathcal{B}] = \min_{\substack{\tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x}) \\ h \in \mathcal{H}}} \Pr[h(\xi(\tilde{\boldsymbol{x}}))] \quad \text{s.t.} \quad \Pr[h(\xi(\boldsymbol{x}))] = \Pr[f(\xi(\boldsymbol{x}))] = \bar{f}(\boldsymbol{x}) \tag{5}$$

where $\mathcal{H}$ is the set of all measurable functions h. Since $f \in \mathcal{H}$ we have $\mathrm{c}^{\downarrow}[\bar{f}, \boldsymbol{x}, \mathcal{B}] \leq \bar{f}(\tilde{\boldsymbol{x}})$ for all $\tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x})$. The upper bound $\mathrm{c}^{\uparrow}[\bar{f}, \boldsymbol{x}, \mathcal{B}^{-1}]$ is the solution to a similar *maximization* problem.

Closed form. For ℓ_2 ball with Gaussian smoothing, Eq. 5 has a closed form solution $\Phi_\sigma(\Phi_\sigma^{-1}(\bar{f}_y(\boldsymbol{x})) - r)$ where Φ_σ is the CDF of the normal distribution $\mathcal{N}(\mathbf{0}, \sigma \boldsymbol{I})$(Cohen et al., 2019; Kumar et al., 2020). The upper bound is similarly computed by changing the sign of r. Yang et al. (2020) show the same closed-form applies solution for the ℓ_1 ball, and additionally, discuss other perturbation balls and smoothing schemes most of which are applicable. For sparse smoothing the bounds can with a simple algorithm with $O(r_a + r_d)$ runtime (Bojchevski et al., 2020), which we discuss in § C. For ℓ_1 ball and uniform smoothing the lower bound equals $\bar{f}_y(\boldsymbol{x}) - 1/(2\lambda)$ where $\boldsymbol{\epsilon} \sim \mathcal{U}[0, 2\lambda]^d$ (Levine & Feizi, 2021). This bound can also be de-randomized (see § 5).

Single Binary Certificate. From the closed-form solutions we see that the bounds are independent of the definition of f, and the test point $\boldsymbol{x}$; i.e. their output is a function of the scalar $p := \bar{f}_y(\boldsymbol{x})$. We defer the discussion for why this holds to § B, and § D; in short the solution for any $\boldsymbol{x}$ can be obtained from alternative canonical points $\boldsymbol{u}$, and $\tilde{\boldsymbol{u}}$. Therefore, we write $\mathrm{c}^{\downarrow}[p, \mathcal{B}] = \mathrm{c}^{\downarrow}[\bar{f}_y, \boldsymbol{x}, \mathcal{B}]$ to show that $\mathrm{c}^{\downarrow}$ depends only on p and $\mathcal{B}$, and the same for $\mathrm{c}^{\uparrow}$. We also notice that in common smoothing schemes and perturbation balls, it holds that $\mathrm{c}^{\downarrow}[\mathrm{c}^{\uparrow}[p, \mathcal{B}^{-1}], \mathcal{B}] = p$ which allows us to reduce both calibration-time and test-time robustness to solving a single binary certificate. We formalize this in Lemma 2.

Lemma 2. *If $\mathrm{c}^{\downarrow}[\mathrm{c}^{\uparrow}[p, \mathcal{B}^{-1}], \mathcal{B}] = p$ for all p, then $\bar{\mathcal{C}}_{\text{test}}(\tilde{\boldsymbol{x}}_{n+1}) = \bar{\mathcal{C}}_{\text{cal}}(\tilde{\boldsymbol{x}}_{n+1}) = \bar{\mathcal{C}}_{\text{bin}}(\tilde{\boldsymbol{x}}_{n+1})$ where $\bar{\mathcal{C}}_{\text{bin}}(\tilde{\boldsymbol{x}}_{n+1}) = \{y : \text{accept}(\tilde{\boldsymbol{x}}_{n+1}, y; \mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}], \tau_\alpha)\} = \{y : \Pr[s(\boldsymbol{x}_{n+1}, y_{n+1}) \geq \tau_\alpha] \geq \mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}]\}$.*

To see why, let $\tilde{p}_{n+1} = \bar{f}_{y_{n+1}}(\tilde{\boldsymbol{x}}_{n+1})$. The test-time robust coverage requires $\mathrm{c}^{\uparrow}[\tilde{p}_{n+1}, \mathcal{B}^{-1}] \geq p_\alpha$. Since both $\mathrm{c}^{\downarrow}$, and $\mathrm{c}^{\uparrow}$ are non-decreasing w.r.t. p, we have $\mathrm{c}^{\downarrow}[\mathrm{c}^{\uparrow}[\tilde{p}_{n+1}, \mathcal{B}^{-1}], \mathcal{B}] \geq \mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}]$. We have the equivalent condition $\tilde{p}_{n+1} \geq \mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}]$. This implies that we only need to compute a single certificate $\mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}]$ once with the single p_α value given by Prop. 1. This also allows us to seamlessly integrate other existing binary certificates in a plug and play manner. In contrast, with Theorem 1 for $\bar{\mathcal{C}}_{\text{test}}$ or $\bar{\mathcal{C}}_{\text{cal}}$ we need at least one certificate per test (or calibration) point. Notably, these prediction set are identical to our cheaper $\bar{\mathcal{C}}_{\text{bin}}$. For completeness, Fig. 3 shows the certified lower and upper bounds for various p_α values and various smoothing schemes.

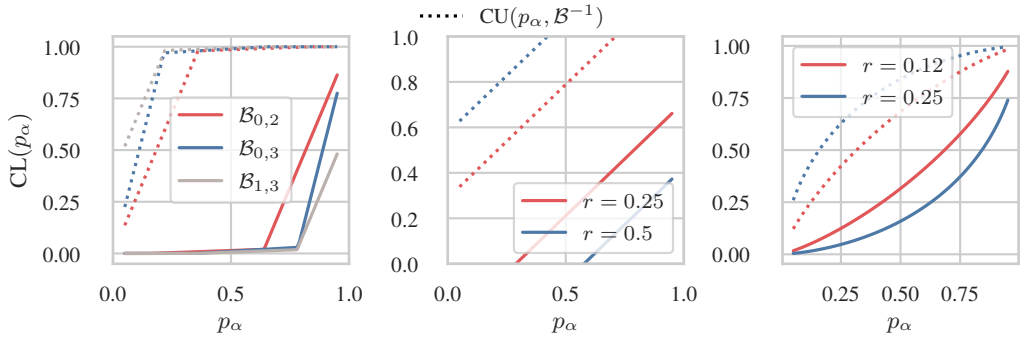

Figure 3: $c^{\downarrow}[p, \mathcal{B}]$, and $c^{\downarrow}[p, \mathcal{B}^{-1}]$ for [from left to right] sparse smoothing, ℓ_1 ball with de-randomized DSSN, and ℓ_2 (and ℓ_1) ball with Gaussian smoothing.

Intuitively $c^{\downarrow}[c^{\uparrow}[p, \mathcal{B}^{-1}], \mathcal{B}] = p$ holds due to symmetry of the smoothing scheme w.r.t. $\mathcal{B}$, and $\mathcal{B}^{-1}$ and is satisfied by most smoothing schemes. In Lemma 3 we prove that Gaussian, uniform, and sparse smoothing all have this property.

Lemma 3. *For Gaussian, and uniform smoothing under ℓ_1, and ℓ_2 balls $\mathcal{B}_r = \mathcal{B}_r^{-1}$. For sparse smoothing and $\mathcal{B}_{r_a,r_d}$ we have $\mathcal{B}_{r_a,r_d}^{-1} = \mathcal{B}_{r_d,r_a}$. In all three cases we have $c^{\downarrow}[c^{\uparrow}[p, \mathcal{B}^{-1}], \mathcal{B}] = p$.*

To summarize, for robust BinCP, we first compute conformal thresholds (p_α, τ_α) from Prop. 1. Then for a perturbation ball $\mathcal{B}$ that satisfies $c^{\downarrow}[c^{\uparrow}[p, \mathcal{B}^{-1}], \mathcal{B}] = p$, we compute $c^{\downarrow}[p_\alpha, \mathcal{B}]$ and compute the prediction sets with $(c^{\downarrow}[p_\alpha, \mathcal{B}], \tau_\alpha)$ instead. The resulting sets have $1 - \alpha$ robust coverage.

Corollary 1. *With (p_α, τ_α) from Prop. 1 on a calibration set $\mathcal{D}_{\text{cal}}$, let $\boldsymbol{x}_{n+1}$ be exchangeable with $\mathcal{D}_{\text{cal}}$ and $\tilde{\boldsymbol{x}}_{n+1} \in \mathcal{B}(\boldsymbol{x}_{n+1})$. If for the smoothing scheme ξ and the threat model $\mathcal{B}$ and for all p we have $c^{\downarrow}[c^{\uparrow}[p, \mathcal{B}^{-1}], \mathcal{B}] = p$, then the set $\bar{\mathcal{C}}_{\text{bin}}(\tilde{\boldsymbol{x}}_{n+1}) = \{y : \Pr[s(\boldsymbol{x}_{n+1}, y_{n+1}) \geq \tau_\alpha] \geq c^{\downarrow}[p_\alpha, \mathcal{B}]\}$ has $1 - \alpha$ coverage (the pseudocode is in § A).*

5 Robust BinCP with Finite Samples

The certificate in Corollary 1, needs the computation of exact probabilities $\Pr[s(\boldsymbol{x}_{n+1}, y_{n+1}) \geq \tau_\alpha]$ which is often intractable. Instead, we can either apply de-randomization techniques or estimate high-confidence bounds of these probabilities. We first describe the latter approach. For each calibration point $(\boldsymbol{x}_i, y_i)$ we compute $q_i = \frac{1}{m}\sum_{i=1}^{m} \mathbb{I}[s(\xi(\boldsymbol{x}_i), y_i) \geq \tau_\alpha]$ where m is the number of Monte-Carlo (MC) samples. For each label of the (potentially perturbed) test point we compute $\tilde{q}_{n+1,y} = \frac{1}{m}\sum_{i=1}^{m} \mathbb{I}[s(\xi(\tilde{\boldsymbol{x}}_{n+1}), y) \geq \tau_\alpha]$. We use the Clopper-Pearson confidence interval (Clopper & Pearson, 1934) to bound the exact probabilities via the MC estimates. To ensure the sets are conservative we compute a lower bound for calibration points and an upper bound for test points. Collectively, all bounds are valid with adjustable $1 - \eta$ probability. To account for this, we set the nominal coverage level to $1-\alpha+\eta$ such that we have $1-\alpha$ coverage in total. Similar to Zargarbashi et al. (2024), we compute each bound with $1 - \eta/(|\mathcal{D}_{\text{cal}} + k|)$ probability where k is the number of classes. Let $p_i = \Pr[s(\xi(\boldsymbol{x}_i), y_i) \geq \tau_\alpha]$ for $i \in \{1, \ldots, n+1\}$ be the exact probabilities. The final sample-corrected robust predictions sets are given in Prop. 2.

Proposition 2. *Let $q_i^{\downarrow} \leq p_i$ hold with $1 - \eta/(\mathcal{D}_{\text{cal}} + k)$ for each calibration point $i \in \{1, \ldots, n\}$ where k is the number of target classes. For a given test point $\tilde{\boldsymbol{x}}_{n+1}$ let $\tilde{q}_{n+1,y}^{\uparrow} \geq \tilde{p}_{n+1,y}$ with $1 - \eta/(\mathcal{D}_{\text{cal}} + k)$ where $\tilde{p}_{n+1,y} = \Pr[s(\tilde{\boldsymbol{x}}_{n+1}, y) \geq \tau_\alpha]$. With $p_\alpha^{\downarrow} = \mathbb{Q}\left(\alpha - \eta; \{q_i^{\downarrow}\}_{i=1}^{n}\right)$, we set the robust conformal threshold pair as $(c^{\downarrow}[p_\alpha^{\downarrow}, \mathcal{B}], \tau_\alpha)$. Then the prediction set defined as $\bar{\mathcal{C}}_{+}(\tilde{\boldsymbol{x}}_{n+1}; c^{\downarrow}[p_\alpha^{\downarrow}, \mathcal{B}], \tau_\alpha) = \{y : \tilde{q}_{n+1,y}^{\uparrow} \geq c^{\downarrow}[p_\alpha^{\downarrow}, \mathcal{B}]\}$ has $1 - \alpha$ coverage probability.*

Such sample correction is a crucial step for smoothing-based robust CP, since the robustness certificate is probabilistic. The failure of the certificate depends to the failure of the confidence intervals. In contrast, for deterministic and de-randomized certificates such as DSSN (Levine & Feizi, 2021),

we do not need sample correction since we can exactly compute p_i and $p_\alpha = \mathbb{Q}(\alpha; \{p_i\}_{i=1}^n)$. Note, vanilla (non-robust) BinCP does not need sample correction to maintain the guarantee (see § 3).

6 EXPERIMENTS

We show that: (i) We can return guaranteed and small sets for both image classification and node classification, with a significantly lower number of Monte Carlo samples. (ii) Our sets are computationally efficient. (iii) There is an inherent robustness in randomized methods. (iv) We can also use de-randomized smoothing-based certificates that do not require finite sample correction.

Setup. We evaluate our method on two image datasets: CIFAR-10 (Krizhevsky, 2009) and ImageNet (Deng et al., 2009), and for node-classification (graph) dataset we use Cora-ML McCallum et al. (2004). For the CIFAR-10 dataset we use `ResNet-110` and for the ImageNet dataset we use ResNet-50 pretrained models with noisy data augmentation from Cohen et al. (2019). For the graph classification task we trained a GCN model Kipf & Welling (2017) on CoraML dataset similarly with noise augmentation. The GNN is trained with 20 nodes per class with stratified sampling as the training set and similarly sampled validation set. The size of the calibration set is between 100 and 250 (sparsely labeled setting) unless specified explicitly. Our reported results on conformal prediction performance are averaged over 100 runs with different calibration set samples. We calibrated BinCP with a $p = 0.6$ fixed value, however small changes in p does not influence the result. For graph dataset we calibrated BinCP with $p = 0.9$ following the intuition from Fig. 3. While we report our results on mainly TPS (softmax), other score functions are reported in Fig. 7, and § E.

We conducted our experiment using three different smoothing schemes. (i) Smoothing with isotropic Gaussian noise, $\sigma = 0.12, 0.25$, and 0.15. Our reported results for BinCP are valid for both ℓ_1, and ℓ_2 perturbation balls. (ii) De-randomized smoothing with splitting noise (DSSN) from Levine & Feizi (2021) from which we attain ℓ_1 robustness. We examine two smoothing levels $\lambda = 0.25/\sqrt{3}$, and $0.5/\sqrt{3}$. (iii) Sparse smoothing from Bojchevski et al. (2020) with $p_+ = 0.01$, and $p_- = 0.6$ on node attributes. We report robustness results on $\mathcal{B}_{0,3}$, and $\mathcal{B}_{1,3}$. We compare our the result from BinCP to the SOTA method CAS (Zargarbashi et al., 2024). Previously it was shown that CAS significantly outperforms RSCP (Gendler et al., 2021) both with and without finite sample correction. In § 7 we discuss the other related works in detail. In the standard setup, we estimate the statistics (mean and CDF, or Bernoulli parameters) with 2×10^3 Monte-Carlo samples, and we set $1 - \alpha = 0.9$. This setup is picked in favor of the baseline since by increasing the nominal coverage or decreasing the sample size BinCP outperforms the baseline with an even higher margin. Throughout the paper we report different nominal coverages and MC sampling budgets.

Smaller set size. Fig. 4 shows that for all datasets, and both smoothing schemes (isotropic Gaussian and sparse smoothing), BinCP produces smaller prediction sets compared to CAS. We show our robustness results for ℓ_1 perturbation ball using derandomized DSSN in Fig. 7 (middle). Notably due to exactness, for randomized DSSN-based certificate we do not correct for finite MC samples. For ImageNet dataset Zargarbashi et al. (2024) report the set size only for asymptotically valid

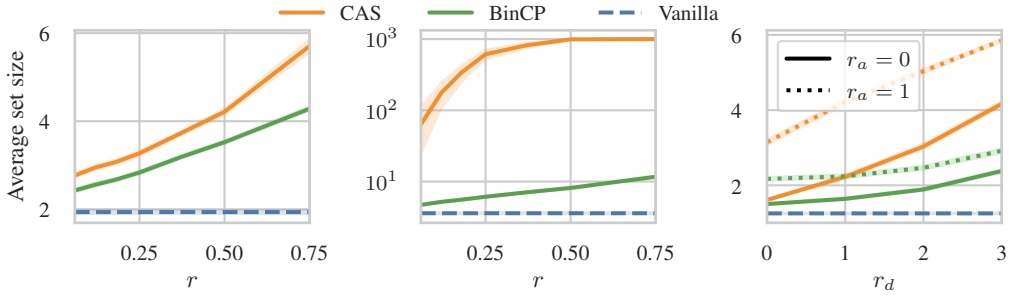

Figure 4: [Left to right] Average prediction set size of robust CP ($1 - \alpha = 0.9$) for CIFAR-10, ImageNet with Gaussian smoothing ($\sigma = 0.5$), and CoraML dataset with sparse smoothing. All results are for 2000 Monte-Carlo samples. For the ImageNet we show the results for ($1 - \alpha = 0.85$).

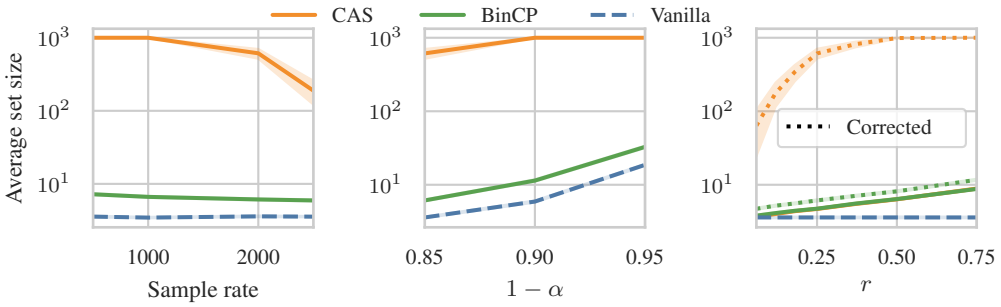

Figure 5: On ImageNet dataset, [Left] average set size for $1-\alpha = 0.85$ with various MC sampling budgets. [Middle] Set size across various levels of $1-\alpha$ for 2×10^3 samples. [Right] Set size without sample correction (asymptotically valid assumption). The sample-corrected variants are shown with a dotted line. In first two plots the y-axis is log-scaled.

setup. As depicted in Fig. 5, with sample correction, CAS produces trivial sets $\bar{C}(\boldsymbol{x}_{n+1}) = \mathcal{Y}$ for $1-\alpha = 0.9$. For sake of comparison, in Fig. 4 we selected $1-\alpha = 0.85$ for this dataset. We report the result on ImageNet dataset across various sampling budgets, and coverages in Fig. 5. Increasing the Monte Carlo sampling budget, the average set size of CAS and BinCP become closer – Fig. 1-left for CIFAR-10, Fig. 5-right for ImageNet, and Fig. 6-right for CoraML depict the impact of higher sampling budget. Additionally, we also show in § E that BinCP is consistently more efficient for smaller radii. Note that while Fig. 4 (right) shows the performance of BinCP on sparse smoothing, this result is evaluated on transductive node classification which according to a concurrent work (Gosch et al., 2023) perfect robustness is achievable for free through memorization. The result still compares the performance of BinCP to CAS for sparse smoothing, but returning robust conformal sets for GNN in a realistic setup is more challenging as discussed in § D.

Ignoring sample correction. While unrealistic in practice, Gendler et al. (2021) report results without applying finite sample correction. Zargarbashi et al. (2024) maintain small set sizes (with large MC sampling budget) for CIFAR-10. However, for ImageNet and CoraML they only reported results without correction. Such results only have an "asymptomatically valid" coverage guarantee. Here we show that CAS with sample correction fails for datasets like ImageNet, producing trivial sets, likely due to multiple testing on a large number of classes (see Fig. 5 (left)). On CIFAR-10 and ImageNet, both methods show similar prediction sets sizes without sample correction. Nevertheless, in practice we need sample correction. As we see in Fig. 5 (right) BinCP with sample correction is not far from the ideal setting (without correction), while CAS shows a large gap.

Number of samples. The upperbound in CAS is obtained through a two step process. First given the corrected CDF, we compute the worst case (adversarial) CDF. Then using upper bounded (or lower bounded) CDF, we apply the Anderson bound to obtain a bound on the mean from the CDF (Zargarbashi et al., 2024). Increasing the number of bins increases the computation slightly but produces tighter bounds. To observe that effect, in an "asymptomatically valid" setup we decrease the number of samples to a very low number (~ 10, however unrealistic) and in Fig. 6 (middle) we see that set size in CAS slightly increases even without sample correction.

Effect of σ. The strength of Gaussian smoothing is controlled by σ in $\xi(\boldsymbol{x}) = \boldsymbol{x}+\boldsymbol{\epsilon}, \boldsymbol{\epsilon} \sim \mathcal{N}(\boldsymbol{0}, \sigma^2\boldsymbol{I})$. Similar to CAS, we observe a trade-off in choosing σ. Higher smoothing intensity results in larger set sizes in the beginning, but by increasing the robustness radius the set size increases slowly (see Fig. 6 (left)). Still in all cases BinCP outperforms CAS. It is best practice to compute smooth prediction probabilities using a model trained with similar noise augmentation. We reported this result in § E (Table 2). Interestingly, BinCP shows less sensitivity to unmatching noise augmentation in training and inference time.

Benefits of smoothing. The guarantee of robust CP breaks for adversarial (or noisy) inputs. In Fig. 7 (left) we compare vanilla prediction and smooth prediction sets under adversarial attack. Notably, smooth models even without a conservative certificate show an inherent robustness. As illustrated, the non-smooth model quickly breaks to near 0 coverage guarantee for very small r. Relatedly,

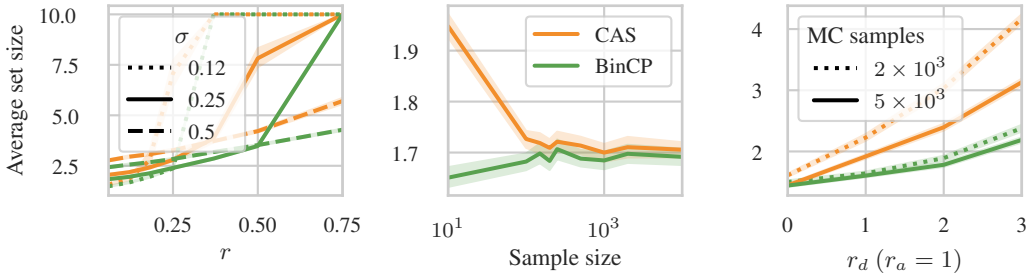

Figure 6: Comparison between BinCP and CAS for [left] various smoothing strengths σ, [middle] effect of low samples without finite samples correction for CIFAR-10 dataset ($\sigma = 0.25$), and [right] the effect of higher MC sample budget in CoraML dataset (sparse smoothing).

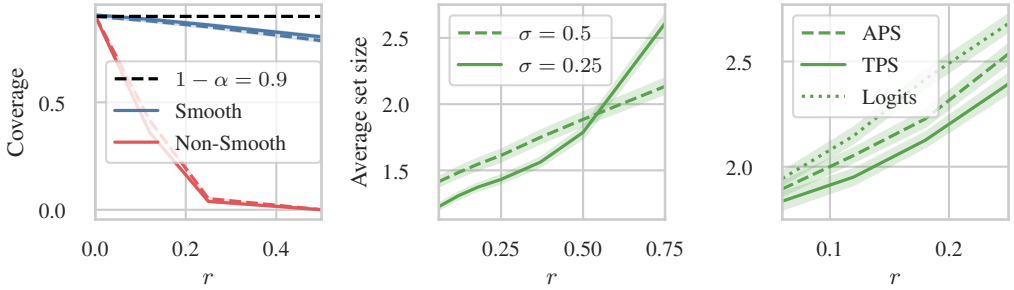

Figure 7: [Left] Vanilla non-smooth and smooth ($\sigma = 0.25$) prediction (solid and dashed colored lines show TPS and APS score function) under attack. [Middle] Set size of BinCP with ℓ_1 robustness and derandomized DSSN smoothing ($\sigma = \lambda/\sqrt{3}$). [Right] Performance of BinCP on different score functions under Gaussian smoothing with $\sigma = 0.25$. All results are on CIFAR-10.

recent verifier-based robust CP (Jeary et al., 2024) report comparably larger prediction sets even for one order of magnitude smaller radius (compared to the certified radii by BinCP). This intuitively suggests that for robust CP it seems that randomization is inherently beneficial.

Exact ℓ_1 robustness. We use the randomized DSSN, exact ℓ_1 certificate (Levine & Feizi, 2021) to derive the first smoothing-based de-randomized robust CP. Our prediction sets computed with Gaussian noise are robust to both ℓ_1, and ℓ_2 perturbation balls with the same radius (Yang et al., 2020). However, the de-randomized robust BinCP with uniform noise (Fig. 7-middle) shows a significantly smaller set size across all radii compared to Gaussian noise (Fig. 4-left). In addition to smaller Lipschitz constant, de-randomized DSSN allows us to bypass the finite samples correction due to the exactness of the computed statistics.

7 Related Work

Robust CP via smoothing. Gendler et al. (2021) introduced the problem and defined a baseline robust CP method, RSCP (randomly smoothed conformal prediction), which applies Theorem 1 in combination with the mean-constrained upper bound for ℓ_2 perturbations and Gaussian smoothing. This upper bound has a closed form solution: $\overline{s}(\tilde{\boldsymbol{x}}, y) = \Phi(\Phi^{-1}(p) + r)$ where $p = \mathbb{E}[s(\boldsymbol{x} + \boldsymbol{\epsilon}, y)]$. Originally, RSCP did not account for finite sample correction making its coverage guarantee only asymptotically valid. Yan et al. (2024) show that correcting for finite samples in RSCP leads to trivial prediction sets $\bar{\mathcal{C}}(\boldsymbol{x}) = \mathcal{Y}$. As a remedy, they define a new score function based on temperature scaling which in combination with conformal training (Stutz et al., 2021) improves the average set size. We compared with their method in § E. So far both methods use test-time robustness.

In contrast Zargarbashi et al. (2024) utilizes the CDF structure of the score and instead apply the tighter CDF-based constraint defining CDF aware sets (CAS). In combination with calibration-time robustness, they show that only $|\mathcal{D}_{\text{cal}}|$ certificate bounds should be computed to maintain a robust coverage guarantee as in Eq. 1. In addition to a gain in computational efficiency, they show that in the calibration-time robustness, the error correction budget can be used more efficiently. On CIFAR-10 they return a relatively small conformal set size. In all aforementioned methods, a large MC sampling budget (e.g. 10^4 samples) is assumed which is challenging for real-time applications. This issue is exacerbated for datasets like ImageNet where the large number of classes amplifies the effect of multiple testing corrections.

Robust CP via verifiers. Outside the scope of randomized smoothing Jeary et al. (2024) use neural network verification to compute upper (or lower) bounds. This requires white-box access to the model weights, while our proposed method works for any black-box model and randomized or exact smoothing-based certificate. Interestingly, in (Jeary et al., 2024) (Table 1) the empirical evaluation is for $r = 0.02$ which is smaller than the minimum radius we reported. For completeness, we evaluated BinCP on very small radii in § E (Fig. 9), and for the same r our sets are $2\times$ smaller. As discussed in § 6 (Fig. 7) in general, smooth prediction shows to have an inherent robustness.

Other robustness results. Alternatively Ghosh et al. (2023) introduce probabilistic robust coverage which intuitively accounts for average adversarial input. This is in contrast with our core assumption of worst-case adversarial input. In other words, instead of $1 - \alpha$ coverage for any point within the perturbation ball around $\boldsymbol{x}_{n+1}$, "probabilistically robust coverage" guarantees that the probability to cover the true label remains above $1 - \alpha$ over $(\boldsymbol{x}, y, \boldsymbol{\epsilon})$. Importantly, they average over all $\boldsymbol{\epsilon} \in \mathcal{B}$, while we consider the worst-case $\boldsymbol{\epsilon}$. Their "quantile of quantiles" method looks superficially similar to BinCP as they also compute $n+k+1$ quantiles. However there are two notable differences. Their first order of quantiles (on true calibration scores and the score for each class of the test point) is over random draws from the perturbation set. BinCP computes the first order quantiles ($\tau_i(p)$ in fixed τ setup) over the smooth score distribution. Their conservative quantile index is based on a user-specified hyperparameter that accounts for conservativeness while BinCP finds the certified probability $\mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}]$ for the worst case adversarial example. BinCP guarantees that any $\tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x})$ is covered if $\boldsymbol{x}$ is covered. Furthermore, there are other works addressing distribution or covariate shift in general beyond the score of worst-case noise robustness (Barber et al., 2022; Tibshirani et al., 2019).

8 Conclusion

We introduce BinCP, a robust conformal prediction method based on randomized smoothing that produces small prediction sets with a few Monte-Carlo samples. The key insight is that we binarize the distribution of smooth scores, by a threshold (or thresholds) that maintains the coverage guarantee. We show that both calibration and test-time robustness approaches are equivalent to computing a single binary certificate. This directly enables us to use any certificate that returns a certified lower-bound probability. The binarization enables us to use tighter Clopper-Pearson confidence intervals. This leads directly to faster computation of prediction sets with a Monte Carlo sampling budget that is significantly less than SOTA. In addition, our method in contrast to all previous smoothing-based robust CP approaches does not require the score function to be bounded.

Ethics Statement

In this paper, we study the robustness of conformal prediction. The main focus of our work is to increase the reliability of conformal prediction in presence of noise or adversarial perturbations. Therefore, we don't see any particular ethical concern to mention about this study.

Reproducibility Statement

To ensure the reproducibility of our results, we have provided the algorithm is in § A, and our anonymized code in supplementary materials available for download. The models we used are also pre-trained and all accessible from the cited works. We specified the setup including parameter selections in § 6.

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A ALGORITHM FOR ROBUST (AND VANILLA) BINCP

Here we provide the algorithm for BinCP in both p-fixed, and τ-fixed setups. The two setups differ in calibration and finite sample correction, while computing the certificate, and returning prediction sets is similar in both. Note that in p-fixed version after computing the quantile τ_α we correct for finite samples which results in a lower p_+.

Algorithm 1: BinCP with τ-fixed setup

Input: Score function $s : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$; Calibration set $\mathcal{D} = \{\boldsymbol{x}_i, y_i\}_{i=1}^n$; Smoothing scheme ξ; Threat model $\mathcal{B}$ satisfying the assumption in Lemma 2; Fixed threshold τ, and (potentially perturbed) test point $\tilde{\boldsymbol{x}}_{n+1}$

Output: A prediction set $\bar{\mathcal{C}}_{\text{bin}}(\tilde{\boldsymbol{x}}_{n+1})$ with $1-\alpha$ robust coverage probability

for *each calibration point* $(\boldsymbol{x}_i, y_i) \in \mathcal{D}_{\text{cal}}$ **do**
  Sample from $\xi(\boldsymbol{x}_i)$ for m times;
  Compute $q_i = \frac{1}{m}\sum_{j=1}^m \mathbb{I}[s(\xi(\boldsymbol{x}_i), y_i) \geq \tau]$;
  if *Exact Certificate* **then**
    $q_{i+} := q_i$;
  else
    $q_{i+} := \text{ClopperPearson}_{\text{low}}(q_i)$;
  end
end
Set $p_\alpha = \mathbb{Q}\left(\alpha; \{q_{i+}\}_{i=0}^n\right)$;
Compute $\text{c}^{\downarrow}[p_\alpha, \mathcal{B}]$ from Eq. 5 (Lower bound minimization);
for *each class* $y \in \mathcal{Y}$ **do**
  Sample from $\xi(\tilde{\boldsymbol{x}}_{n+1})$ for m times;
  Compute $q_{n+1,y} = \frac{1}{m}\sum_{j=1}^m \mathbb{I}[s(\xi(\tilde{\boldsymbol{x}}_{n+1}), y) \geq \tau]$;
  if *Exact Certificate* **then**
    $q_{n+1,y+} := q_{n+1}$;
  else
    $q_{n+1,y+} := \text{ClopperPearson}_{\text{up}}(q_{n+1,y})$;
  end
end
return $\bar{\mathcal{C}}\{y : q_{n+1,y+} \geq \text{c}^{\downarrow}[p_\alpha, \mathcal{B}]\}$

Algorithm 2: BinCP with p-fixed setup

Input: Data, score function, smoothing, and $\mathcal{B}$ same as algorithm 1. Fixed threshold τ

Output: Same as algorithm 1

for *each calibration point* $(\boldsymbol{x}_i, y_i) \in \mathcal{D}_{\text{cal}}$ **do**
  Sample from $\xi(\boldsymbol{x}_i)$ for m times;
  Compute $\tau_i = \mathbb{Q}\left(p; m\right)\{s(\xi(\boldsymbol{x}_i), y_i)\}_{j=1}^m$;
  if *Exact Certificate* **then**
    $p_{\alpha,+} := p_\alpha$;
  else
    $p_{\alpha,+} := \text{ClopperPearson}_{\text{low}}(p_\alpha)$;
  end
end
Set $\tau_\alpha = \mathbb{Q}\left(\alpha; \{\tau_i\}_{i=0}^n\right)$;
Compute $\text{c}^{\downarrow}[p_{\alpha,+}, \mathcal{B}]$ from Eq. 5 (Lower bound minimization);
for *each class* $y \in \mathcal{Y}$ **do**
  Sample from $\xi(\tilde{\boldsymbol{x}}_{n+1})$ for m times;
  Compute $q_{n+1,y+} = \frac{1}{m}\sum_{j=1}^m \mathbb{I}[s(\xi(\tilde{\boldsymbol{x}}_{n+1}), y) \geq \tau_\alpha]$;
  $q_{n+1,y+} :=$ same as algorithm 1;
end
return $\bar{\mathcal{C}}\{y : q_{n+1,y+} \geq \text{c}^{\downarrow}[p_\alpha^{\downarrow}, \mathcal{B}]\}$

Score functions. Throughout the paper we reported results for TPS score – directly setting the softmax values as the score $s(\boldsymbol{x}, y) = \pi(\boldsymbol{x}, y)$ (Sadinle et al., 2018). In vanilla CP ($r = 0$), TPS tends to over-cover easy examples and under-cover hard ones (Angelopoulos & Bates, 2021). Alternatively "adaptive prediction sets" (APS) aiming for conditional coverage uses the score function defined as $s(\boldsymbol{x}, y) := -\left(\rho(\boldsymbol{x}, y) + u \cdot \pi(\boldsymbol{x})_y\right)$ where $\rho(\boldsymbol{x}, y) := \sum_{c=1}^{K} \pi(\boldsymbol{x})_c 1\left[\pi(\boldsymbol{x})_c > \pi(\boldsymbol{x})_y\right]$ is the sum of all classes predicted as more likely than y, and $u \in [0, 1]$ is a uniform random value that breaks the ties between different scores to allow exact $1 - \alpha$ coverage (Romano et al., 2020). Another approach is to directly use the logits of the model as the score. This is not applicable in CAS and RCSP+ (Zargarbashi et al., 2024; Yan et al., 2024) as they work with bounded scores. BinCP can also work with unbounded score, hence, we also report the results on CP with logits. All three score functions are reported in Fig. 7 for BinCP. Interestingly we do not see any significant difference in set size between APS and TPS when smoothed. We report BinCP with APS score in § E over all σ, and radii. Orthogonally, while RSCP with either scores quickly breaks to returning trivial sets, RSCP+ refines the score function through a biased temperature scaling. This can also be considered as another score function (or transformation over any score function) tailored for robust setup. We also report this in Fig. 14 (§ E).

B Computing Certificate Optimization

Canonical view. Turns out that for isotropic Gaussian and sparse smoothing, we can always attain this minimum at canonical points – for any test point $\boldsymbol{x}_{n+1}$ we can translate the function, and the points to the origin, and the worst case to the border of $\mathcal{B}(\boldsymbol{x}_{n+1})$. Formally, there is a pair $(\boldsymbol{u}, \tilde{\boldsymbol{u}})$ such that $\rho_{\boldsymbol{u},\tilde{\boldsymbol{u}}} = \rho_{\boldsymbol{x}_{n+1},\tilde{\boldsymbol{x}}_{n+1}}$ for any $\boldsymbol{x}_{n+1}$, and $\tilde{\boldsymbol{x}}_{n+1} \in \mathcal{B}(\boldsymbol{x}_{n+1})$. Namely for the continuous ball $\mathcal{B}_r$ the canonical vectors are $\boldsymbol{u} = \boldsymbol{0}$ and $\tilde{\boldsymbol{u}} = [r, 0, 0, \dots]$. For the binary $\mathcal{B}_{r_a,r_d}$ we have the canonical $\boldsymbol{u} = [0, \dots, 0, 1, \dots, 1]$ and $\tilde{\boldsymbol{u}} = \boldsymbol{1} - \boldsymbol{u}$ where $\|\boldsymbol{u}\|_0 = r_d$ and $\|\tilde{\boldsymbol{u}}\|_0 = r_a$. Intuitively it is due to the symmetry of the ball and the smoothing distribution. To avoid many notations, we again use the $\boldsymbol{x}$, and $\tilde{\boldsymbol{x}}$ in the rest of the discussion that refers to the canonical points.

To obtain an upper or lower bound (Eq. 5 as maximization or minimization) we partition the space $\mathcal{X}$ to regions where the likelihood ratio between $(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ is constant; formally $\mathcal{X} = \cup_{i=1}^{k} \mathcal{R}_i$ where $\forall \boldsymbol{z} \in \mathcal{R}_i : \Pr[\xi(\boldsymbol{x}) = \boldsymbol{z}] / \Pr[\xi(\tilde{\boldsymbol{x}}) = \boldsymbol{z}] = c_i$. For any function h we can find an equivalent piecewise-constant $\hat{h}$ where inside each region it is assigned to the expected value of h in that region. Let $t_i = \Pr[\xi(\boldsymbol{x}) = \boldsymbol{z}]$, and $\tilde{t}_i = \Pr[\xi(\tilde{\boldsymbol{x}}) = \boldsymbol{z}]$ then Eq. 5 simplifies to the following linear programming

$$\min_{\boldsymbol{h} \in [0,1]^k} \boldsymbol{h}^\top \tilde{\boldsymbol{t}} \quad \text{s.t.} \quad \boldsymbol{h}^\top \boldsymbol{t} = p_\alpha \tag{6}$$

Where $\boldsymbol{h}$, $\boldsymbol{t}$, and $\tilde{\boldsymbol{t}}$ are vectors that include the values h_i, t_i, $\tilde{t}_i$ for each region. The optimum solution to the simplified linear programming is obtained by sorting regions based on the likelihood ratio and greedily assigning h to the possible maximum in each region until the budget $\boldsymbol{h}^\top \boldsymbol{t} = p_\alpha$ is met. The rest of the regions are similarly assigned to zero. This problem is equivalent to fractional knapsack. For isotropic Gaussian smoothing Cohen et al. (2019) show that the optimal solution has a closed form $\rho_\alpha = \Phi_\sigma(\Phi_\sigma^{-1}(p_\alpha) - r)$ where Φ_σ is the Gaussian CDF function of the Gaussian distribution with standard deviation σ. For sparse smoothing, following Bojchevski et al. (2020) we solve the greedy program on at most $r_a + r_d + 1$ distinct regions. The runtime is linear w.r.t. to the add and delete budget. For more detailed explanation see (Lee et al., 2019; Bojchevski et al., 2020).

C Supplementary to Theory

C.1 Vanilla BinCP

Conformal risk control. We use conformal risk control (CRC) (Angelopoulos et al., 2022) to prove the coverage guarantee in BinCP. Here we succinctly recall it before the proof of Prop. 1.

Theorem 2 (Conformal Risk Control - rephrased)**.** *Let λ be a parameter (larger λ yields more conservative output), and $L_i : \Lambda \to (-\infty, b]$ for $i = 1, \dots, n+1$ be exchangeable random functions. If (i) L_is are non-increasing right-continuous w.r.t. λ, (ii) for $\lambda_{\max} = \sup \Lambda$ we have $L_i(\lambda_{\max}) \leq \alpha$,*

and (iii) $\sup_\lambda L_i \leq b < \infty$*, then we have:*

$$\mathbb{E}[L_{n+1}(\hat{\lambda})] \leq \alpha \quad \textit{for} \quad \hat{\lambda} = \inf\left\{\lambda : \frac{\sum_{i=1}^n L_i(\lambda)}{n+1} + \frac{B}{n+1} \leq \alpha\right\} \tag{7}$$

In case that $B = 1$, by simplifying Eq. 7, we have $\hat{\lambda} = \inf\{\lambda : \sum_{i=1}^n L_i(\lambda) \leq \alpha(n+1) - 1\}$ We use this framework to prove the guarantee in BinCP;

Proof to Prop. 1. We prove the theorem through re-parameterizing of the conservativeness variable in each case. For fixed p we set $\tau = 1 - \lambda$; similarly for fixed τ we set $p = -\lambda$. In both cases, the risk is defined as

$$L_i(\tau, p) = 1 - \text{accept}(\boldsymbol{x}_i, y_i; p, \tau)$$

which for simplicity we define $\text{reject}(\boldsymbol{x}_i, y_i; p, \tau) = 1 - \text{accept}(\boldsymbol{x}_i, y_i; p, \tau)$ and by definition we have $\text{reject}(\boldsymbol{x}_i, y_i; p, \tau) = \mathbb{I}[\Pr[s(\xi(\boldsymbol{x}), y) < \tau] > 1 - p] = \mathbb{I}[\Pr[s(\xi(\boldsymbol{x}), y) \geq \tau] < p]$. We show that the risk function satisfies the properties for a risk function feasible to the setup in Theorem 2.

1. **Non-increasing to λ.** In both cases the risk L_i is non-inscreasing to λ; for fixed p we have

$$\begin{aligned} \lambda_1 < \lambda_2 &\Rightarrow 1 - \lambda_1 > 1 - \lambda_2 \\ \Rightarrow \Pr[s(\xi(\boldsymbol{x}), y) < 1 - \lambda_1] &\geq \Pr[s(\xi(\boldsymbol{x}), y) < 1 - \lambda_2] \\ \Rightarrow \text{reject}(\boldsymbol{x}, y; p, 1 - \lambda_1) &\geq \text{reject}(\boldsymbol{x}, y; p, 1 - \lambda_2) \end{aligned}$$

   Now for fixed τ, let $p_{\boldsymbol{x}} = \Pr[s(\xi(\boldsymbol{x}), y) \geq \tau]$ then we have

$$\begin{aligned} \lambda_1 < \lambda_2 \Rightarrow p_1 > p_2 &\text{ means that } \mathbb{I}[p_{\boldsymbol{x}} \leq p_1] \geq \mathbb{I}[p_{\boldsymbol{x}} \leq p_2] \\ \Rightarrow \text{reject}(\boldsymbol{x}, y; -\lambda_1, \tau) &\geq \text{reject}(\boldsymbol{x}, y; -\lambda_2, \tau) \end{aligned}$$

   Intuitively by adapting the definition of the rejection (risk) function $\text{reject}(\boldsymbol{x}_i, y_i; p, \tau) = \mathbb{I}[\Pr[s(\xi(\boldsymbol{x}), y) \geq \tau] < p]$, if we increase λ which means decreasing p, the chance of rejecting a label decreases. This is because, we require the same probability mass to be lower than a smaller value.

2. **Right continuous.** Formally the function accept is

$$\text{accept}(\boldsymbol{x}, y; p, \tau) = \begin{cases} 1 & \text{if } \Pr[s(\xi(\boldsymbol{x}), y) \geq \tau] \geq p \\ 0 & \text{otherwise} \end{cases}$$

   Across the domain (for either p or τ) this function has two values and it is just non-continuous in the jump between the values. For both p and τ this function is left continuous due to the $\geq$ comparison. Therefore for fixed p the function $\text{reject}(\boldsymbol{x}, y; p, 1 - \lambda)$ is right continuous to $\lambda = 1 - \tau$. Similar argument follows for fixed τ.

3. **Feasibility of risks less than α.** For fixed $p > 0$ if we set $\lambda = 1 - \tau$ to ∞ ($\tau = -\infty$), for all $\boldsymbol{x}_i$, we have $\text{accept}(\boldsymbol{x}_i, y_i; p, 0) = 1$; i.e. the risk is 0 for every data. Similarly by approaching p to zero in fixed τ setup, we decrease the risk to 0 for everyone. To avoid corner cases we can restrict τ to $\max s(\xi(\boldsymbol{x}), y)$ for $x \in \mathcal{X}$ from avive.

4. **Limited upperbound risk.** For any parameter and any input the highest possible risk is in case of rejection which is 1 ($b = 1$).

Fixed p. The risk function $L_i(\lambda) = \text{reject}(\boldsymbol{x}_i, y_i; p, 1 - \lambda)$ which means that the prediction set $\mathcal{C}(\boldsymbol{x}_i; p, 1 - \lambda)$ excludes y_i. We have

$$\mathbb{E}[\text{reject}(\boldsymbol{x}_{n+1}, y_{n+1}; p, 1 - \hat{\lambda})] \leq \alpha \text{ for } \hat{\lambda} = \inf_\lambda \left\{\lambda : \sum_{i=1}^n \text{reject}(\boldsymbol{x}_i, y_i; p, 1 - \lambda) \leq \alpha(n+1) - 1\right\}$$

Setting back the $\tau = 1 - \lambda$, and rewriting the expectation as a probability form, we have

$$\Pr[y_{n+1} \in \mathcal{C}(\boldsymbol{x}_{n+1}; p, \tau_p)] \geq 1 - \alpha \;\text{ for }\; \tau_p = \sup_\tau \left\{\tau : \sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau) \geq (1 - \alpha)(n+1)\right\}$$

In the above, we used the fact that if a test fails on $\alpha(n+1)-1$ variables among the total of n variables, it passes on $n-[\alpha(n+1)-1]$ and $(1-\alpha)(n+1)=n-[\alpha(n+1)-1]$.

Fixed τ. Similarly, we define the risk function as $L_i(\lambda)=\text{reject}(\boldsymbol{x}_i, y_i; -\lambda, \tau)$. We have

$$\mathbb{E}[\text{reject}(\boldsymbol{x}_{n+1}, y_{n+1}; p_\tau, \tau)] \leq \alpha \quad \text{for} \quad p_\tau = \inf_\lambda \left\{ \lambda : \sum_{i=1}^n \text{reject}(\boldsymbol{x}_i, y_i; -\lambda, \tau) \leq \alpha(n+1)-1 \right\}$$

□

Proof to Lemma 1. The function $\text{accept}(\boldsymbol{x}, y; p, \tau)$ is non-increasing in both p and τ. Therefore the term $\sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau)$ is also non-increasing in p and τ and its range is the integer numbers between 0 and n (or $[n]$). For a fixed p, let $\tau_\alpha(p)$ be the solution to Eq. 2, then by definition it satisfies that

$$\sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)) \geq (1-\alpha)(n+1)$$

This implies that p satisfies the same condition for $p_\alpha(\tau_\alpha(p))$. Therefore $p_\alpha(\tau_\alpha(p)) \geq p$ as p is a feasible solution in Eq. 3. The supremum search for $\tau_\alpha(p)$ directly implies that for any positive δ we have

$$\sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)) \geq \sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)+\delta) - 1$$

which intuitively means that increasing the $\tau(p)$ by any small margin fails at least in one more accept for calibration points. Since $\sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p))$ is the sum of n non-increasing functions, there is one index i for which

$$\text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)) = 1 \quad \text{and} \quad \text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)+\delta) = 0$$

For any small positive δ. Using the definition of the accept function we have

$$\Pr[s(\boldsymbol{x}_i + \boldsymbol{\epsilon}, y_i) \geq \tau_\alpha(p)] \geq p \quad \text{and} \quad \Pr[s(\boldsymbol{x}_i + \boldsymbol{\epsilon}, y_i) \geq \tau_\alpha(p)+\delta] < p$$

Due to the continuous strictly increasing CDF for S_i we have $\Pr[s(\boldsymbol{x}_i + \boldsymbol{\epsilon}, y_i) \geq \tau(p)] = p$. Therefore for any small positive δ

$$\text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)) = 1 \quad \text{and} \quad \text{accept}(\boldsymbol{x}_i, y_i; p+\delta, \tau_\alpha(p)) = 0$$

which means that the accept function for $\boldsymbol{x}_i$ fails by adding a small number to p. Since all other accept functions are also non-increasing we have $\sum_{i=1}^n \text{accept}(\boldsymbol{x}_i, y_i; p, \tau_\alpha(p)) \leq (1-\alpha)(n+1)-1$. This implies that p is also the supremum for $Eq.$ 2 with parameter $p_\alpha(\tau)$. □

C.2 Robust BinCP

Proof to Lemma 2. With $f_{\text{true}}(\boldsymbol{x}_i) = \mathbb{I}[s(\boldsymbol{x}_i, y_i) \geq \tau_\alpha]$ for true y_i, the calibration-time robust prediction set is defined as $\bar{\mathcal{C}}_{\text{cal}}(\tilde{\boldsymbol{x}}_{n+1}) = \{p(\tilde{\boldsymbol{x}}_{n+1}, y; \tau_\alpha) \geq \mathbb{Q}\left(\alpha; \{\text{c}^\downarrow[\bar{f}_{\text{true}}(\boldsymbol{x}_i), \mathcal{B}]\}_{i=1}^n\right)\}$. By definition we have $p_\alpha = \mathbb{Q}\left(\alpha; \{\bar{f}_{\text{true}}(\boldsymbol{x})\}_{i=1}^n\right)\}$. Both lower bound and upper bound functions are non-decreasing. As a result, the ranks, and hence the quantile index in $\{\bar{f}_{\text{true}}(\boldsymbol{x}_i)\}_{i=1}^n$ and $\{\text{c}^\downarrow[\bar{f}_{\text{true}}(\boldsymbol{x}_i), \mathcal{B}]\}_{i=1}^n$ are the same. Therefore, $\mathbb{Q}\left(\alpha; \{\text{c}^\downarrow[\bar{f}_{\text{true}}(\boldsymbol{x}_i), \mathcal{B}]\}_{i=1}^n\right)\} = \text{c}^\downarrow[p_\alpha, \mathcal{B}]$.

The test-time robust prediction set is defined as $\bar{\mathcal{C}}_{\text{test}} = \{y : \text{c}^\uparrow[\bar{f}_y(\tilde{\boldsymbol{x}}_{n+1}), \mathcal{B}^{-1}] \geq p_\alpha\}$, let $\tilde{p}_y = \bar{f}_y(\tilde{\boldsymbol{x}}_{n+1})$ then it follows

$$\begin{aligned} \text{c}^\uparrow[\bar{f}_y(\tilde{\boldsymbol{x}}_{n+1}), \mathcal{B}^{-1}] \geq p_\alpha \Leftrightarrow \text{c}^\downarrow[\text{c}^\uparrow[\bar{f}_y(\tilde{\boldsymbol{x}}_{n+1}), \mathcal{B}^{-1}], \mathcal{B}] &\geq \text{c}^\downarrow[p_\alpha, \mathcal{B}] \\ \Leftrightarrow \bar{f}_y(\tilde{\boldsymbol{x}}_{n+1}) &\geq \text{c}^\downarrow[p_\alpha, \mathcal{B}] \end{aligned}$$

By definition $\text{accept}(\tilde{\boldsymbol{x}}_{n+1}, y; \text{c}^\downarrow[p_\alpha, \mathcal{B}], \tau_\alpha) = \mathbb{I}[\bar{f}_y(\tilde{\boldsymbol{x}}_{n+1}) \geq \text{c}^\downarrow[p_\alpha, \mathcal{B}]]$. □

In the above, we proved that BinCP results in a valid conformal prediction. Here we prove the validity of robust BinCP to adversarial data within the bounded threat model.

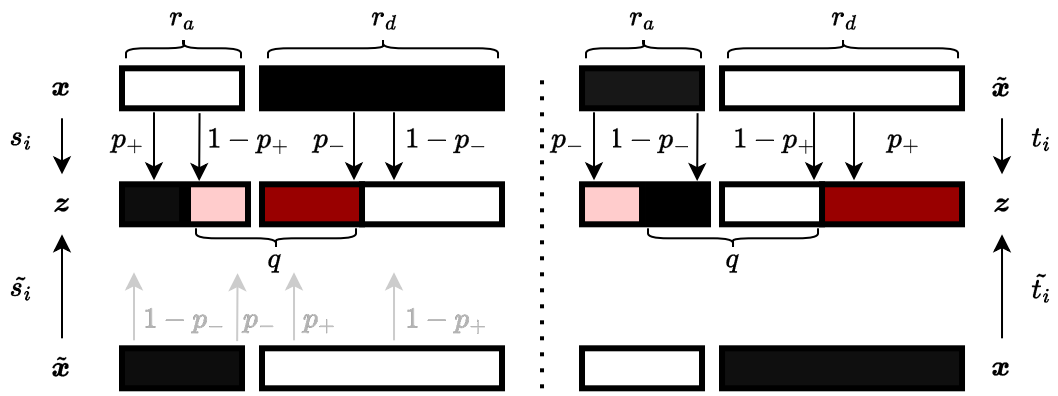

Figure 8: Illustration of likelihood ratio in sparse smoothing for both $\mathcal{B}_{r_a,r_d}$, and $\mathcal{B}_{r_d,r_a}$

Proof to Lemma 3. For each of the mentioned smoothing schemes we have:

Gaussian smoothing. In both cases since p norm is symmetric for any point $\tilde{\boldsymbol{x}}$ it holds that $\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_p \leq r$. In other words, from any perturbed point the clean point is within $\mathcal{B}_r(\tilde{\boldsymbol{x}})$. Therefore $\mathcal{B}_r^{-1} = \mathcal{B}_r$.

Given the closed from solution $\mathrm{c}^{\downarrow}[p, \mathcal{B}_r] = \Phi_\sigma(\Phi_\sigma^{-1}(p) - r)$, and $\mathrm{c}^{\uparrow}[p, \mathcal{B}_r] = \Phi_\sigma(\Phi_\sigma^{-1}(p) + r)$ we have

$$\begin{aligned}\overline{p} = \Phi_\sigma(\Phi_\sigma^{-1}(p) + r) \Leftrightarrow \Phi_\sigma^{-1}(\overline{p}) = \Phi_\sigma^{-1}(p) + r \Leftrightarrow \Phi_\sigma^{-1}(\overline{p}) - r = \Phi_\sigma^{-1}(p) \\ \Leftrightarrow \Phi_\sigma(\Phi_\sigma^{-1}(\overline{p}) - r) = p \Leftrightarrow \mathrm{c}^{\downarrow}[\overline{p}, \mathcal{B}] = p\end{aligned}$$

Uniform smoothing. For the uniform smoothing from Levine & Feizi (2021) we have that the smooth classifier is $1/(2\lambda)$-Lipschitz continuous. Therefore

$$\mathrm{c}^{\downarrow}[\mathrm{c}^{\uparrow}[p, \mathcal{B}_r^{-1}], \mathcal{B}_r] = \mathrm{c}^{\downarrow}[p + \frac{r}{2\lambda}, \mathcal{B}_r] = p + \frac{r}{2\lambda} - \frac{r}{2\lambda} = p$$

A similar argument can be applied to any Lipschitz continuous smoothing scheme.

Sparse smoothing. Any $\tilde{\boldsymbol{x}} \in \mathcal{B}_{r_a,r_d}(\boldsymbol{x})$ has at most r_a zero bits, and r_d one bits toggled from $\boldsymbol{x}$. By toggling those bit back we can reconstruct $\boldsymbol{x}$. The maximum needed toggles is therefore r_d zero bits and r_a one bits which is the definition of $\mathcal{B}_{r_d,r_a}$.

As discussed in § B, canonical points for $\mathcal{B}_{r_a,r_d}$ are $\boldsymbol{x} = [0, \ldots, 0, 1, \ldots, 1]$ and $\tilde{\boldsymbol{x}} = \mathbf{1} - \boldsymbol{x}$ where $\|\boldsymbol{x}\|_0 = r_d$ and $\|\tilde{\boldsymbol{x}}\|_0 = r_a$. For $\mathcal{B}_{r_a,r_d}^{-1}$ the canonical points are $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$ where $\|\boldsymbol{u}\|_0 = r_a$. By applying a permutation over $\boldsymbol{u}$, $\tilde{\boldsymbol{x}}$ and every other point in all regions we can set $\boldsymbol{u} = \tilde{\boldsymbol{x}}$, and $\tilde{\boldsymbol{u}} = \boldsymbol{x}$. For computing both $\mathcal{B}_{r_a,r_d}$, and $\mathcal{B}_{r_a,r_d}^{-1}$ there are $r_a + r_d + 1$ regions of constant likelihood ratio, each including all points that have the same number of total flips from the source $\boldsymbol{x}$, or $\boldsymbol{u}$; formally $\mathcal{R}_q = \{\boldsymbol{z} : \|\boldsymbol{x} - \boldsymbol{z}\|_0 = q\}$. The same region can also defined to preserve $r_d + r_a - q$ bits from $\tilde{\boldsymbol{x}}$. With $\frac{s_q}{\tilde{s}_q}$ as the likelihood ratio of a point $\boldsymbol{z}$ in $\mathcal{B}_{r_a,r_d}$ and $q = q_a + q_d$ as the number of changes in 1 and 0 bit, we have $s_q = (p_+)^{q_a}(1-p_+)^{r_a-q_a}(p_-)^{q_d}(1-p_-)^{r_d-q_d}$, and similarly $\tilde{s}_q = (p_-)^{q_a}(1-p_-)^{r_a-q_a}(p_-)^{q_d}(1-p_-)^{r_d-q_d}$. Then the likelihood ratio is simplified to

$$\frac{s_q}{\tilde{s}_q} = \left[\frac{p_+}{1-p_-}\right]^{q-r_d} \left[\frac{p_-}{1-p_+}\right]^{q-r_a} \tag{8}$$

As illustrated in Fig. 8 regions for $\mathcal{B}_{r_a,r_d}^{-1}$ are same as $\mathcal{B}_{r_a,r_d}$ only with reverse order. In other word, let t_i, $\tilde{t}_i$ be the probability of visiting region $\mathcal{R}_q$ from $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$, then $t_i = \tilde{s}_{r_a+r_d+1-q}$, and $\tilde{t}_i = s_{r_a+r_d+1-q}$. For a fixed $\boldsymbol{z}$ the probability to visit $\boldsymbol{z}$ from $\boldsymbol{x}$ is the probability of toggling $q = \|\boldsymbol{z} - \boldsymbol{x}\|_0$ bits which is the same as toggling q bits from $\tilde{\boldsymbol{u}}$ as $\tilde{\boldsymbol{u}} = \boldsymbol{x}$.

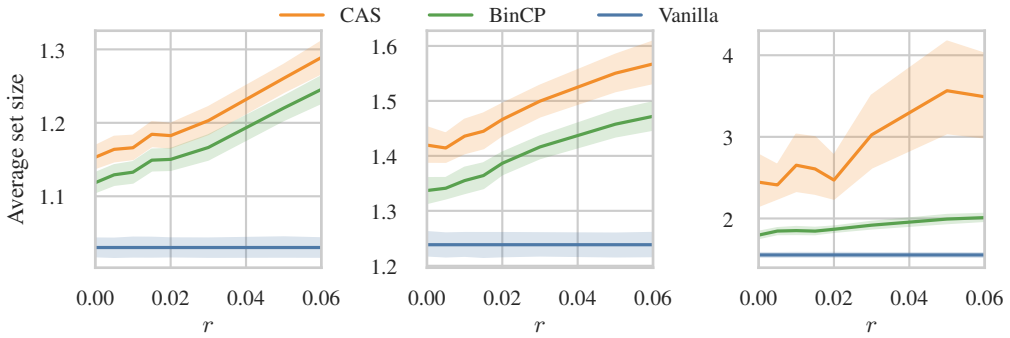

Figure 9: Comparison between BinCP and CAS on CIFAR-10 dataset with $\sigma = 0.25$ and small values of r. The nominal coverage $1 - \alpha$ is set to [from left to right] 85%, 90%, and 95%.

Solutions to $\mathrm{c}^{\downarrow}[p, \mathcal{B}_{r_a,r_d}]$ and $\mathrm{c}^{\uparrow}[p, \mathcal{B}^{-1}_{r_a,r_d}]$ are obtained from the following optimization functions:

$$\mathrm{c}^{\downarrow}[p, \mathcal{B}_{r_a,r_d}] = \min_{\boldsymbol{h} \in [0,1]^{r_a+r_d+1}} \boldsymbol{h}^\top \tilde{\boldsymbol{t}} \quad \text{s.t.} \quad \boldsymbol{h}^\top \boldsymbol{t} = p$$

$$\mathrm{c}^{\uparrow}[p, \mathcal{B}^{-1}_{r_a,r_d}] = \max_{\boldsymbol{h} \in [0,1]^{r_a+r_d+1}} \boldsymbol{h}^\top \tilde{\boldsymbol{s}} \quad \text{s.t.} \quad \boldsymbol{h}^\top \boldsymbol{s} = p$$

The solution to the lower bound optimization is obtained by a greedy algorithm. We visit each in increasing order w.r.t. $\frac{s_q}{\tilde{s}_q}$, we assign $h_q = 1$ until the budget $\boldsymbol{h}^\top \boldsymbol{s}$ is met and we set $h_q = 0$ for the remaining regions (fractional knapsack problem). For the maximization we do the same but in a decreasing order.

We want to prove $\mathrm{c}^{\downarrow}[\mathrm{c}^{\uparrow}[p, \mathcal{B}^{-1}_{r_a,r_d}], \mathcal{B}_{r_a,r_d}] = p$. This is the solution to

$$\min_{\boldsymbol{h} \in [0,1]^{r_a+r_d+1}} \boldsymbol{h}^\top \tilde{\boldsymbol{t}} \quad \text{s.t.} \quad \boldsymbol{h}^\top \boldsymbol{t} = \mathrm{c}^{\uparrow}[p, \mathcal{B}^{-1}_{r_a,r_d}] = \boldsymbol{h}'^\top \tilde{\boldsymbol{s}}$$

This directly means that $\boldsymbol{h}'$ (the solution from maximization problem) is a feasible solution. Let $\overleftarrow{\tilde{\boldsymbol{s}}}$ be the vector $\tilde{\boldsymbol{s}}$ in reverse order. Then $\boldsymbol{t} = \overleftarrow{\tilde{\boldsymbol{s}}}$ and therefore $\overleftarrow{\boldsymbol{h}'}$ is the solution to the maximization greedy problem. So the optimal solution is $\overleftarrow{\boldsymbol{h}'}^\top \tilde{\boldsymbol{t}} = \overleftarrow{\boldsymbol{h}'}^\top \overleftarrow{\boldsymbol{s}} = p$. □

For any ℓ_p with the same argument as ℓ_2 ball we have $\mathcal{B}_r^{-1} = \mathcal{B}_r$. Similar to isotropic Gaussian smoothing, the Lipschitz continuity in DSSN-smoothed distribution shows that Lemma 3 applies to ℓ_1 ball and this distribution as well.

C.3 Correction for Finite Sample Monte-Carlo Estimation

Proof to Prop. 2. With $p_i = \Pr[s(\xi(\boldsymbol{x}_i), y_i) \geq \tau_\alpha]$ as the true probability of crossing τ_α for each true score distribution in calibration set. We have $p_\alpha = \mathbb{Q}(\alpha; \{p_i\}_{i=1}^n)$. For all i we have $q_i^{\downarrow} \leq p_i$ which follows $p_\alpha^{\downarrow} \leq p_\alpha$. The probability of failure in each calibration datapoint is $\eta/(|\mathcal{D}_{\text{cal}}| + k)$; as a result, from the union bound the probability of failure in $q_i^{\downarrow} \leq p_i$ is $|\mathcal{D}_{\text{cal}}|\eta/(|\mathcal{D}_{\text{cal}}| + k)$.

For all classes of the test point we have $\tilde{q}_{n+1,y}^{\uparrow} \geq \tilde{p}_{n+1,y}$ with $\eta/(|\mathcal{D}_{\text{cal}}| + k)$. Therefore, for the true class we have $\tilde{q}_{n+1} \geq \tilde{p}_{n+1}$ with $k\eta/(|\mathcal{D}_{\text{cal}}| + k)$.

Conformal guarantee implies that with $1 - \alpha + \eta$ probability we have $p_{n+1} \geq p_\alpha$. The robustness certificate follows that $\tilde{p}_{n+1} \geq \mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}]$. Following holds by using the mentioned inequality:

$$q_{n+1}^{\uparrow} \underset{1-\frac{k\eta}{n+k}}{\geq} \tilde{p}_{n+1} \underset{1-\alpha+\eta}{\geq} \mathrm{c}^{\downarrow}[p_\alpha, \mathcal{B}] \underset{1-\frac{n\eta}{n+k}}{\geq} \mathrm{c}^{\downarrow}[p_\alpha^{\downarrow}, \mathcal{B}]$$

From the union bound it follows that the total failure probability is less than α. □

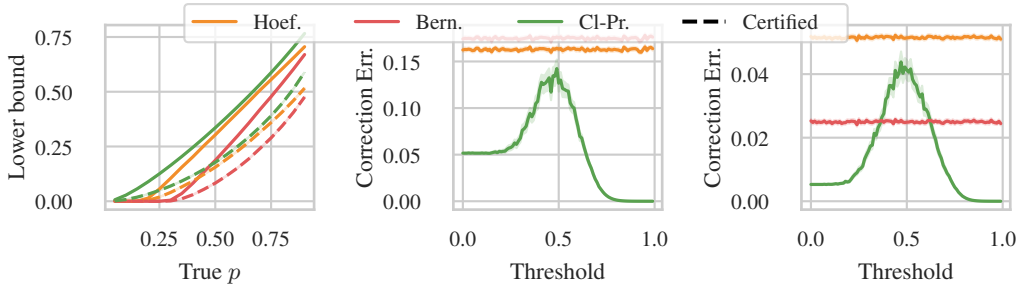

Figure 10: [Left] Confidence lower bound and the corresponding certified lower bound for scores derived from Beta and Bernoulli distributions. [Middle and right] Correction error (lower bound subtracted from the theoretical mean) of the scores distributed from the Gaussian distribution both in continuous case (mean lower bound) and binarized case (lower bound on the Bernoulli parameter) for [middle] 100 and [right] 1000 samples. Details of the experiments are in § D.

D Supplementary Discussion

High-level understanding of robustness certificates. A certificate of robustness is a formal guarantee that the model predicts the same class for any perturbation within the specified threat model - within the ball around the input. In other words, if the function f is certified to be robust for the point $\boldsymbol{x}$ w.r.t. $\mathcal{B}$, for any $\tilde{\boldsymbol{x}} \in \mathcal{B}(\boldsymbol{x})$ we have $\arg\max_y f_y(\boldsymbol{x}) = \arg\max_y f_y(\tilde{\boldsymbol{x}})$. This certificate ensures that the top label remains the same within the threat model (binary certificate). A similar (confidence, or soft) certificate guarantees a lower (and upper) bound on the model confidence within the threat model given the predictive probability. One way to attain such certificates is through verifiers. Verifiers need white-box access (knowledge about the model structure and weights) and they work efficiently only on a limited class of models. However, our robust conformal guarantee is black-box.

A common approach for black-box certification is through randomized smoothing. A randomly smoothed classifier results from inference given the input augmented with random noise. For example $g(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{\epsilon}\sim\mathcal{N}(\mathbf{0},\sigma^2\boldsymbol{I})}[f(\boldsymbol{x}+\boldsymbol{\epsilon})]$ – model g returns the expected output of f given randomly augmented $\boldsymbol{x}$ where the noise comes from an isotropic Gaussian distribution with scale σ. The randomization function is smooth even if the original function changes rapidly, which is the effect of the expectation. It is also Lipschitz continuous, meaning that we can bound the output based on the distance of $\tilde{\boldsymbol{x}}$ from $\boldsymbol{x}$. The latter allows us to provide formal guarantees that the top class probability (or confidence) remains high (changes slowly) even if $\tilde{\boldsymbol{x}} \in \mathcal{B}$ is passed to the model instead of $\boldsymbol{x}$.

Ultimately a randomized smoothing-based certificate returns a lower (or upper) bound probability (or score) on the expected output (given the randomized $\boldsymbol{x}$). In robust CP we use these bounds to answer "if instead of the clean input $\boldsymbol{x}_{n+1}$ which is already exchangeable with the calibration set, the model received the worst case $\tilde{\boldsymbol{x}}_{n+1} \in \mathcal{B}(\boldsymbol{x}_{n+1})$ how much lower the conformity score has become". Or in other words "if the model is queried with $\tilde{\boldsymbol{x}}_{n+1} \in \mathcal{B}(\boldsymbol{x}_{n+1})$ (which has a lower conformity score in order not to be covered) how much higher the conformity score of the clean input can be". Technical details of smoothing-based certificates are mentioned in § 4. For a more detailed discussion see (Cohen et al., 2019; Kumar et al., 2020).

Comparison of confidence intervals. As discussed in § 5, BinCP, CAS, and RSCP, all require true probability, CDF, and mean from the distribution of scores which is intractable to compute (except in the case of de-randomized BinCP). Therefore we use confidence bounds that are lower (or higher) than the true values with collective probability $1-\eta$ (which is taken into account while calibrating). CAS, and RSCP are defined on continuous scores that are bounded by Hoeffding, Bernstein, or DKW inequalities. BinCP is defined through binarized scores, and the final parameter is the success probability of a Bernoulli distribution which can bounded by the Clopper-Pearson interval which is exact (Clopper & Pearson, 1934). The width of all mentioned confidence intervals is decreasing w.r.t. the sample size. Therefore a tighter interval can result in the same or better efficiency (correction error) with fewer samples; e.g. For scores sampled from a Gaussian $\mathcal{N}(0.5, 0.1)$ Clopper Pearson error (for $z \geq 0.6$) with 100 samples is still lower than Bernstein's error with 250 samples.

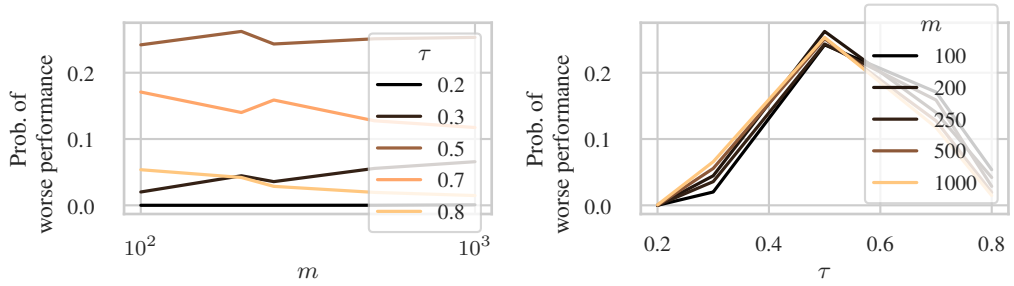

Figure 11: Probability of observing higher upper bound from Clopper Pearson confidence interval in comparison with Hoeffding's interval. The result is for $\mathrm{Beta}(2, 2)$, and $\eta = 0.01$.

To illustrate this we conducted two experiments. First, to compare the tightness of each concentration inequality, we sampled from a Beta distribution with mean p to have continuous score values between $[0, 1]$. The distribution for a fixed β is $\mathrm{Beta}(\frac{p}{\beta(1-p)}, \beta)$. Then for the continuous score, we computed both Hoeffding's and Bernstein's lower bound on the mean, alongside the Clopper-Pearson bound for the given parameter p and the same sample size. As shown in Fig. 10 (left, with $\beta = 1$) the binary lower bound is always higher (better). Since the certified lower bound is an increasing function of the given probability, the certified lower bound for the binary values is again higher.

In another experiment shown in Fig. 10 (middle and right) we sample scores from a Gaussian distribution $\mathcal{N}(0.5, 0.1)$, and computed the lower-bound mean given both Hoeffding and Bernstein's inequalities. Then for various thresholds, we computed the probability of scores passing that threshold and lower bounded this probability by Clopper Pearson concentration inequality. As shown in the figure for lower sample rates, binarization results in less error compared to the theoretical mean. Even with higher sample rates Clopper Pearson interval is significantly tighter than the other two for low and high thresholds (which is a parameter of BinCP).

Proposition 3. *Let $X \sim \mathrm{Beta}(a, b)$ and $x_1, \ldots, x_m$ be m i.i.d. samples of X. Given the empirical mean $\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$ the upper bound for the true mean $\mu = \mathbb{E}[X]$ is given by the Hoeffding's inequality as $\mu \leq \bar{x} + b_{\mathrm{hoef}}$, where $b_{\mathrm{hoef}} = \sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}}$. For any user-specified $\tau \in (0, 1)$, let $Y = \mathbb{I}[X > \tau]$. The Clopper-Pearson (CP) upper bound p_u for the true $p = \mathbb{E}[Y] = \Pr[X \geq \tau]$ is:*

$$p_u = \Phi^{-1}_{\mathrm{Beta}}(1 - \eta; 1 + \sum_{i=1}^{m} \mathbb{I}[x_m > \tau], m - \sum_{i=1}^{m} \mathbb{I}[x_m > \tau_\mu])$$

Each upper bound holds with probability $1 - \eta$. For any number of samples m, and any significance level η, the probability that the CP bound is tighther is $\Pr[p_u - \mathbb{E}[Y] \leq b_{\mathrm{hoef}}]$ and equals:

$$\Pr[p_u - \mathbb{E}[Y] \leq b_{\mathrm{hoef}}] = \Phi_{\mathrm{Binom}}(\hat{m}; m, \mathbb{E}[Y]) \tag{9}$$

where $\hat{m}$ is defined in Eq. 11.

Proof. The variable Y is distributed as $Y \sim \mathrm{Bernoulli}(p)$ where $p = \mathbb{E}[Y] = 1 - \Phi_{\mathrm{Beta}}(\tau; a, b)$ and Φ_{Beta} is the CDF of the beta distribution with parameters a and b.

Let $m_+ = \sum_{i=1}^{m} \mathbb{I}[x_m > \tau]$. We will compute the probability that the inequality

$$p_u - \mathbb{E}[Y] \leq \sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}} \tag{10}$$

holds. Substituting the definition of p_u and $\mathbb{E}[Y]$ we get:

$$\Phi^{-1}_{\text{Beta}}(1-\eta; 1+m_+, m-m_+) - (1-\Phi_{\text{Beta}}(\tau; a, b)) \leq \sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}} \Leftrightarrow$$

$$1-\eta \leq \Phi_{\text{Beta}}\left(\sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}} + 1 - \Phi_{\text{Beta}}(\tau; a, b); 1+m_+, m-m_+\right) \Leftrightarrow$$

$$\eta \geq 1 - \Phi_{\text{Beta}}\left(\sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}} + 1 - \Phi_{\text{Beta}}(\tau; a, b); 1+m_+, m-m_+\right)$$

Define $\hat{m}$ as the break-point after which the CP bound becomes looser than the Hoeffding bound:

$$\hat{m} = \sup\left\{m_+ : \mathbb{I}\left[\eta \geq 1 - \Phi_{\text{Beta}}\left(\sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}} + 1 - \Phi_{\text{Beta}}(\tau; a, b); 1+m_+, m-m_+\right)\right]\right\} \tag{11}$$

In other words, $m_+ > \hat{m} \Leftrightarrow p_u - \mathbb{E}[E] > b_{\text{hoef}}$. Since Φ is monotonic, it follows that:

$$\Pr[p_u - \mathbb{E}[Y] > b_{\text{hoef}}] = \Pr[m_+ > \hat{m}] = 1 - \Phi_{\text{Binom}}(\hat{m}; m, \mathbb{E}[Y]) \tag{12}$$

Where Φ_{Binom} is the CDF of the Binomial distribution with the specified parameters.

□

Similarly, we can compare CP with the Bernstein bound we use $\mu \leq \bar{x} + b_{\text{bern}}$ where

$$b_{\text{bern}} = \sqrt{2\sigma_m^2 \frac{\ln\left(\frac{2}{\eta}\right)}{m}} + \frac{7\ln\left(\frac{2}{\eta}\right)}{3(m-1)}$$

By replacing b_{hoef} with b_{bern} in Prop. 3 we can derive a similar result. We choose a Beta distribution to simulate the fact that conformity scores such as TPS and APS are bounded. Moreover, we need bounded scores to be able to apply Hoeffding's inequality. Any other distribution (after some transformation that ensures bounded scores) could be used, as long as we can compute its CDF.

In Fig. 11 we show the probability defined in Eq. 12 for $X \sim \text{Beta}(2, 2)$ for different values of m and τ. There is a choice of τ such that the probability is effectively 0 for all values of m, i.e. the CP bound is always better. Interestingly, at worst, both in terms of the number of samples m and τ, we see that it is less than 25%. That is, CP is better on average for all configurations.

To get some additional intuition, instead of the exact CP bound for p we can use the following bound derived from a Normal approximation which approximately holds with probability $1-\eta$:

$$p \leq \hat{p} + \frac{z_\eta}{\sqrt{m}}\sqrt{\hat{p}(1-\hat{p})}$$

where $\hat{p} = \frac{1}{m}\sum_{i=1}^{m}\mathbb{I}[x_m > \tau]$ and z_η is the $1-\eta$ quantile of the standard normal distribution. It is not difficult to verify that for all values of $\hat{p} \in [0, 1]$ we have that

$$\frac{z_\eta}{\sqrt{m}}\sqrt{\hat{p}(1-\hat{p})} \leq \sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2m}}$$

To see this, note that the $\sqrt{m}$ term cancels, and for $\eta = 0.05$ $z_\eta \approx 1.64$, $\sqrt{\frac{\ln\left(\frac{1}{\eta}\right)}{2}} \approx 1.22$. Since $\sqrt{\hat{p}(1-\hat{p})} \in [0, 0.5]$, even in the worst-case $1.64 \cdot 0.5 \leq 1.22$. This analysis again confirms that CP gives tighter bounds. Prop. 3 can be analogues extended to lower bounds.

Finite sample correction for fixed τ setup. What we showed in Prop. 2 adds MC sample correction to BinCP with fixed τ computation. We can correct for finite samples in a fixed p setup in a similar way. First, we compute the $\tau_i(p)$ for each of the calibration points. In an asymptomatically valid setup this implies that for $\tau_i(p)$ we have $\Pr[s(\xi(\boldsymbol{x}_i), y_i) \geq \tau_i(p)] \geq p$. To account for finite samples we reduce p to $p^{\downarrow}$ ($p^{\downarrow} \leq p$). Again this holds for each calibration with $1/(|\mathcal{D}_{\text{cal}}| + k)$ probability, and the conformal threshold is $(\tau_p, p^{\downarrow})$. In the test time the setup is identical to the fixed-τ setup.

Realistic setup to evaluate GNN robustness. Concurrent to our work Gosch et al. (2023) shows that transductive setup is flawed to evaluate GNN robustness. This is since assuming that the clean graph is accessible to the defender during training, can lead to perfect robustness by simply remembering the clean graph. Many robust- and self-training GNNs proposed for robustness also exploit this flaw.

To evaluate GNN node-classification for robust CP, following prior works, we assumed the transductive setup where the clean graph is given during the training and calibration. Specifically, we assume that the defender trains, and computes the calibration scores on the clean graph. Perturbations in test nodes are then applied after calibration (in evasion setup). This again is based on similar assumption made by many other GNN robustness works. Therefore our robust node-classification results are only representing the comparison of BinCP and other approaches on a sparse binary certificate.

A more realistic setup is the inductive setup where the defender is given a clean subgraph $\mathcal{G}_{\text{tr}}$ for training (and calibration), and the adversary perturbs the rest of the graph upon arrival; i.e. the defender does not know the clean test graph. In vanilla setup, the conformal guarantee is still valid in the inductive setting via re-computing calibration nodes (Zargarbashi & Bojchevski, 2024). However in evasion, robust CP is not as easily applicable by recomputing the quantile and computing upper bounds on scores, since the calibration nodes are also affected by the adversarial test nodes through the message passing. Therefore robustness in GNNs under realistic setups is challenging which we leave it to future works.

E Additional Experiments

Various models and smoothing maginitudes (σ). In Table 2 we compare the result between SOTA CAS and BinCP for CIFAR-10 dataset. The results are reported across various data smoothing σ values, and models trained with different noise augmentations (data augmented during training with different σ values). We call them smoothing σ, and model σ respectively. In the robustness certificate for classification, it is considered best practice to use the same σ in both model's noise augmentation, and the smoothing process. Similarly, in robust conformal prediction mismatching smoothing and model σ results in a larger prediction set. Interestingly this adverse effect is much less observed in BinCP although it remains present. Overall, across all smoothing parameters, model σ values, coverage rates, and perturbation radii, BinCP consistently outperforms CAS.

Performance on small radii. For completeness, in Fig. 9, we report the performance of BinCP on small values of r. As Jeary et al. (2024) reports ~ 4.45 average set size for $r = 0.02$ (Table 1 in (Jeary et al., 2024)) our report shows more than twice smaller sets for the same r. As in Table 2 we observe the same average set size for $r \sim 0.5$ ($\geq 20\times$ higher radius) for smallest $\sigma = 0.12$. As we discussed, one effect of this eye-catching difference is the inherent robustness of the randomized smooth prediction. As shown in Fig. 7, the empirical coverage of non-smooth prediction drastically decreases to 0 for small radii, while in smooth prediction the coverage decreases slowly.

APS score function. Although we observe similar comparison between CAS and BinCP given APS score function, for completeness we report the performance of both methods in Table 1.

Conditional and class-conditional coverage. Following Romano et al. (2020), we approximated the conditional coverage gap as the worst coverage among n different slices. Each slice is defined as $\mathcal{X}_s = \{\boldsymbol{x}_i \in \mathcal{D} : a \leq \boldsymbol{x}_i \cdot \boldsymbol{v} \leq b\}$ for the random vector $\boldsymbol{v}$ and random scalers a, b (Romano et al., 2020). For that, we sampled 200 random vectors $\boldsymbol{v}$ and among all the scalers randomly sampled a, b from the set $\{\boldsymbol{x}_i \cdot \boldsymbol{v}, \boldsymbol{x}_i \in \mathcal{D}\}$. We report the result over 100 different calibration samplings. In each iteration of the experiment, we exclude the slices with less than 200 points of support.

BinCP has smoothing, binarization and accounting for perturbation radius. For a better intuition we observe the effect of each step separately: vanilla TPS (without smoothing) as the baseline score, vanilla smooth TPS (labeled as STPS) which is an average of TPS scores over randomized sampled

Table 1: Comparison of smoothing-based robust CP methods on APS score

		$1-\alpha=0.9$				$1-\alpha=0.95$			
		CAS		BinCP		CAS		BinCP	
σ	**r**	**Coverage**	**Set Size**	**Coverage**	**Set Size**	**Coverage**	**Set Size**	**Coverage**	**Set Size**
	0.06	0.954	1.635	0.946	1.529	0.990	4.022	0.980	2.151
	0.12	0.971	1.939	0.963	1.757	0.996	6.435	0.985	2.389
	0.18	0.987	2.879	0.978	2.076	1.000	9.745	0.991	2.876
0.12	0.25	0.998	7.454	0.986	2.510	1.000	10.000	0.995	3.405
	0.37	1.000	10.000	1.000	10.000	1.000	10.000	1.000	10.000
	0.50	1.000	10.000	1.000	10.000	1.000	10.000	1.000	10.000
	0.75	1.000	10.000	1.000	10.000	1.000	10.000	1.000	10.000
	0.06	0.955	2.108	0.944	1.894	0.986	3.316	0.976	2.677
	0.12	0.964	2.309	0.954	2.054	0.989	3.682	0.980	2.857
	0.18	0.970	2.495	0.961	2.227	0.993	5.038	0.986	3.181
0.25	0.25	0.980	2.900	0.972	2.537	0.997	6.004	0.989	3.444
	0.37	0.991	3.795	0.982	3.047	1.000	9.360	0.994	4.035
	0.50	0.999	7.430	0.991	3.729	1.000	10.000	0.997	4.850
	0.75	1.000	10.000	1.000	10.000	1.000	10.000	1.000	10.000
	0.06	0.956	2.738	0.942	2.479	0.981	3.864	0.975	3.342
	0.12	0.962	2.890	0.951	2.635	0.984	4.077	0.978	3.508
	0.18	0.968	3.078	0.959	2.801	0.986	4.277	0.981	3.658
0.50	0.25	0.973	3.304	0.966	2.994	0.989	4.546	0.984	3.899
	0.37	0.980	3.684	0.974	3.302	0.993	5.193	0.988	4.300
	0.50	0.986	4.153	0.979	3.663	0.996	5.868	0.991	4.733
	0.75	0.995	5.441	0.989	4.584	0.999	8.026	0.995	5.542

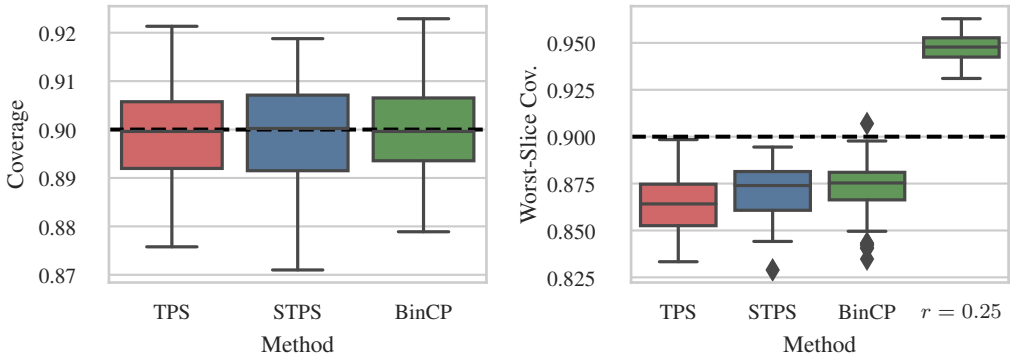

Figure 12: Comparison of coverage [left] and worst-slice coverage [right]. Here the STPS refers to the smooth TPS which is the average of 2000 randomly smooth inferences per point. The results are for CIFAR-10 dataset and r= 0 unless specified.

(same as CAS), BinCP without robustness (set $r = 0$) which reflects the effect of binarization (we did not correct for finite sample due to validity in vanilla setup), and robust CP via BinCP. As shown in Fig. 12, the smooth model has a better worst-slice coverage than vanilla TPS. Though binarization although the average worst-slice coverage remains the same, there is a slight decrease in the variance of this metric. Note that sampling correction and making CP robust increases the empirical coverage guarantee, therefore the worst slice coverage is increased due to the inherent increase in marginal coverage.

We also reported the result of the class-conditional coverage in Fig. 13. Empirically in almost all classes, BinCP is closer to the nominal guarantee compared to normal smoothing. Ultimately both smooth prediction and BinCP are not comparable with vanilla TPS.

Comparison with RSCP+. Yan et al. (2024) shows a flaw of RSCP (Gendler et al., 2021) indicating that the score function is not corrected for finite sample estimation. They show that by adding finite

			$1-\alpha=0.9$				$1-\alpha=0.95$			
			CAS		**BinCP**		**CAS**		**BinCP**	
Model σ	**Data** σ	r	Coverage	Set Size	Coverage	Set Size	Coverage	Set Size	Coverage	Ave Set Size
0.12	0.12	0.06	0.950	1.581	0.942	1.483	0.987	3.353	0.976	2.009
		0.12	0.968	1.839	0.959	1.671	0.996	6.731	0.986	2.387
		0.18	0.985	2.761	0.974	1.946	0.999	9.417	0.990	2.666
		0.25	0.997	7.078	0.985	2.369	1.000	10.000	0.994	3.213
	0.25	0.06	1.000	10.000	0.943	4.911	1.000	10.000	0.977	6.500
		0.12	1.000	10.000	0.953	5.328	1.000	10.000	0.984	6.880
		0.18	1.000	10.000	0.961	5.711	1.000	10.000	0.991	7.366
		0.25	1.000	10.000	0.974	6.323	1.000	10.000	0.994	7.628
		0.37	1.000	10.000	0.988	7.133	1.000	10.000	0.998	8.387
		0.50	1.000	10.000	0.996	7.990	1.000	10.000	0.999	9.049
		0.75	1.000	10.000	1.000	10.000	1.000	10.000	1.000	10.000
	0.50	0.06	1.000	10.000	0.950	8.836	1.000	10.000	0.980	9.310
		0.12	1.000	10.000	0.960	8.985	1.000	10.000	0.984	9.402
		0.18	1.000	10.000	0.965	9.075	1.000	10.000	0.986	9.450
		0.25	1.000	10.000	0.974	9.222	1.000	10.000	0.990	9.535
		0.37	1.000	10.000	0.983	9.403	1.000	10.000	0.996	9.656
		0.50	1.000	10.000	0.991	9.557	1.000	10.000	0.999	9.789
		0.75	1.000	10.000	0.999	9.820	1.000	10.000	1.000	9.947
0.25	0.12	0.06	0.954	2.570	0.937	2.232	0.991	7.016	0.968	2.992
		0.12	0.969	3.049	0.949	2.395	0.998	9.042	0.976	3.301
		0.18	0.984	4.573	0.956	2.562	1.000	9.845	0.981	3.567
		0.25	0.999	9.510	0.969	2.908	1.000	10.000	0.986	3.895
	0.25	0.06	0.953	2.051	0.941	1.836	0.984	3.307	0.974	2.551
		0.12	0.960	2.183	0.950	1.951	0.991	4.077	0.981	2.832
		0.18	0.969	2.411	0.959	2.126	0.994	5.242	0.984	3.054
		0.25	0.979	2.790	0.969	2.394	0.997	6.749	0.988	3.295
		0.37	0.991	3.867	0.981	2.858	1.000	9.660	0.994	3.888
		0.50	0.999	7.824	0.989	3.480	1.000	9.948	0.996	4.564
	0.50	0.06	1.000	10.000	0.947	6.762	1.000	10.000	0.979	7.837
		0.12	1.000	10.000	0.956	7.024	1.000	10.000	0.983	8.016
		0.18	1.000	10.000	0.960	7.212	1.000	10.000	0.988	8.221
		0.25	1.000	10.000	0.969	7.523	1.000	10.000	0.992	8.430
		0.37	1.000	10.000	0.981	7.935	1.000	10.000	0.996	8.763
		0.50	1.000	10.000	0.990	8.350	1.000	10.000	0.998	9.040
		0.75	1.000	10.000	0.997	9.008	1.000	10.000	0.999	9.494
0.50	0.12	0.06	0.948	3.701	0.923	3.060	0.994	8.485	0.965	4.196
		0.12	0.961	4.230	0.929	3.159	0.999	9.564	0.971	4.457
		0.18	0.980	5.843	0.937	3.330	1.000	9.960	0.973	4.538
		0.25	0.998	9.329	0.947	3.550	1.000	10.000	0.977	4.753
	0.25	0.06	0.943	3.152	0.925	2.792	0.990	7.254	0.969	4.013
		0.12	0.951	3.417	0.933	2.929	0.995	7.818	0.973	4.172
		0.18	0.961	3.771	0.942	3.112	0.996	8.476	0.976	4.276
		0.25	0.970	4.095	0.948	3.246	0.998	8.916	0.978	4.416
		0.37	0.987	5.773	0.959	3.586	1.000	9.958	0.984	4.753
		0.50	0.999	9.121	0.970	3.952	1.000	10.000	0.988	5.171
	0.50	0.06	0.957	2.767	0.943	2.428	0.984	3.995	0.974	3.288
		0.12	0.964	2.948	0.949	2.558	0.986	4.165	0.977	3.405
		0.18	0.968	3.071	0.955	2.673	0.988	4.351	0.980	3.542
		0.25	0.974	3.272	0.962	2.835	0.991	4.850	0.983	3.806
		0.37	0.982	3.721	0.973	3.182	0.993	5.160	0.986	4.044
		0.50	0.987	4.215	0.980	3.524	0.997	6.439	0.990	4.511
		0.75	0.996	5.708	0.987	4.285	1.000	9.287	0.994	5.316

Table 2: Comparison of CAS and BinCP for model trained with various smoothing σ, and input data with different smoothing σ. Results are for CIFAR-10 dataset.

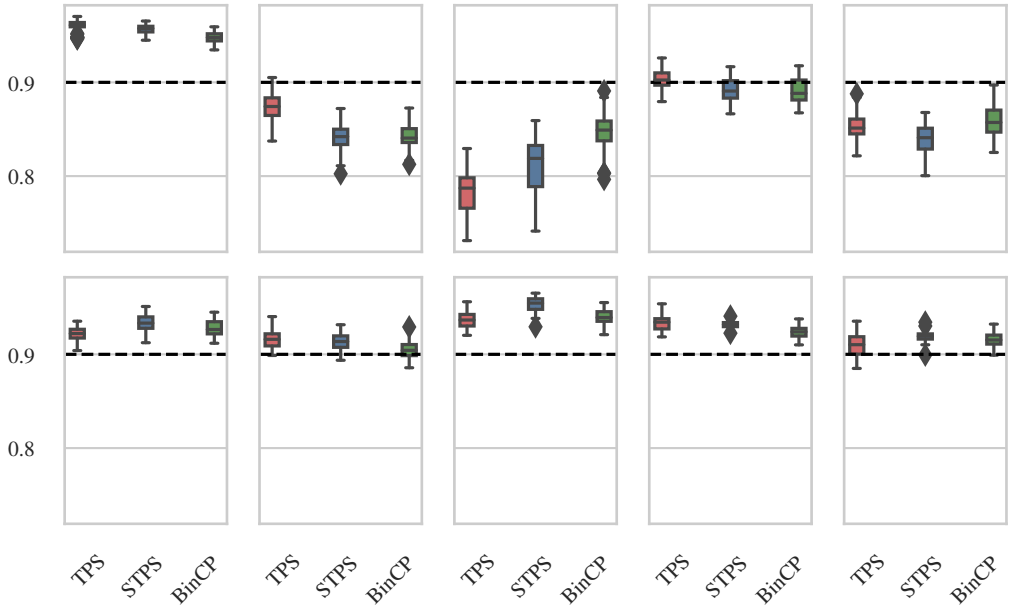

Figure 13: Comparison of methods in class-conditional coverage for all classes of CIFAR-10, note that here BinCP is used without sampling correction. That is because the correction slightly increases empirical coverage which can be misleading.

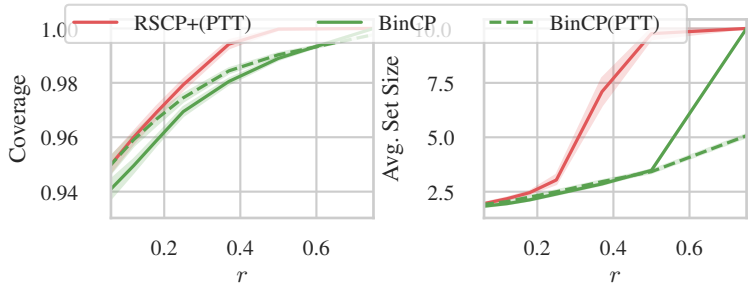

Figure 14: Comparison between BinCP and RSCP+ (PPT, Eq. 13) and BinCP with Eq. 13. The result is on CIFAR-10 dataset with $\sigma = 0.25$.

sample correction to RSCP, it becomes significantly inefficient and produces trivial sets $\mathcal{C}(\boldsymbol{x}_{n+1}) = \mathcal{Y}$. They remedy that by designing a ranking-based transformation on top of the given score function which defines a new score as

$$s_{\text{ppt}}(\boldsymbol{x}, y) = \sigma\left(\frac{1}{T|\mathcal{D}_{\text{tune}}|}\text{rank}(s(\boldsymbol{x}, y); \{s(\boldsymbol{x}_j, y_j)\}_{(\boldsymbol{x}_j, y_j)\in\mathcal{D}_{\text{tune}}}) - \frac{b}{T}\right) \tag{13}$$

Where $\mathcal{D}_{\text{tune}}$ is a holdout tuning index, T is the temperature parameter, b is a bias parameter, and σ is the sigmoid function. The original experiment from Yan et al. (2024) has several issues, which we resolved and compared with it: (i) The scores have possible ties; i.e. two different data points can have the same score value. To remedy that we added an unnoticeable random number $\delta \sim \text{Uniform}[0, 1/|\mathcal{D}_{\text{tune}}|]$ to the scores. (ii) The tuning set and the calibration set in the experiments are significantly large. Yan et al. (2024) use 5250/10000 test datapoints as a tuning and calibration set. This unrealistic holdout labeled set contradicts the sparse labeling assumption. In our reproduction of their results we used a total of ~ 380 datapoints where 200 of them are for tuning. As shown in Fig. 14, BinCP still outperforms RSCP+(PPT). As the score function in Eq. 13 is also a valid score, we can use BinCP on top which shows slightly better efficiency for larger radii compared to BinCP combined with TPS score. Here we set $b = 1 - \alpha$, and $T = 0.001$, and report the results on 2000 MC samples. The reported result is on CIFAR-10 dataset.