Warm-Start Actor-Critic: From Approximation Error to Sub-optimality Gap

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Abstract

Warm-Start reinforcement learning (RL), aided by a prior policy obtained from offline training, is emerging as a promising RL approach for practical applications. Recent empirical studies have demonstrated that the performance of Warm-Start RL can be improved *quickly* in some cases but become *stagnant* in other cases, especially when the function approximation is used. To this end, the primary objective of this work is to build a fundamental understanding on "*whether and when online learning can be significantly accelerated by a warm-start policy from offline RL?*". Specifically, we consider the widely used Actor-Critic (A-C) method with a prior policy. We first quantify the approximation errors in the Actor update and the Critic update, respectively. Next, we cast the Warm-Start A-C algorithm as Newton's method with perturbation, and study the impact of the approximation errors on the finite-time learning performance with inaccurate Actor/Critic updates. Under some general technical conditions, we derive the upper bounds, which shed light on achieving the desired finite-learning performance in the Warm-Start A-C algorithm. In particular, our findings reveal that it is essential to reduce the algorithm bias in online learning. We also obtain lower bounds on the sub-optimality gap of the Warm-Start A-C algorithm to quantify the impact of the bias and error propagation.

1. Introduction

Online reinforcement learning (RL) (Kaelbling et al., 1996; Sutton & Barto, 2018) often faces the formidable challenge of high sample complexity and intensive computational cost (Kumar et al., 2020; Xie et al., 2021), which hinders its applicability in real-world tasks. Indeed, this is the case in portfolio management (Choi et al., 2009), vehicles control (Wu et al., 2017; Shalev-Shwartz et al., 2016) and other time-sensitive settings (Li, 2017; Garcıa & Fernández, 2015). To tackle this challenge, Warm-Start RL has recently garnered much attention (Nair et al., 2020; Gelly & Silver, 2007; Uchendu et al., 2022), by enabling online policy adaptation from an initial policy pre-trained using offline data (e.g., via behavior cloning or offline RL). One main insight of Warm-Start RL is that online learning can be significantly accelerated, thanks to the bootstrapping by an initial policy.

Despite the encouraging empirical successes (Silver et al., 2017; 2018; Uchendu et al., 2022), a fundamental understanding of the learning performance of Warm-Start RL is lacking, especially in the practical settings with function approximation by neural networks. In this work, we focus on the widely used Actor-Critic (A-C) method (Grondman et al., 2012; Peters & Schaal, 2008), which combines the merits of both policy iteration and value iteration approaches (Sutton & Barto, 2018) and has great potential for RL applications (Uchendu et al., 2022). Notably, in the framework of abstract dynamic programming (ADP) (Bertsekas, 2022a), the policy iteration method (Sutton et al., 1999) has been studied extensively, for warm-start learning *under the assumption of accurate updates*. In such a setting, policy iteration can be regarded as a second-order method in convex optimization (Grand-Clément, 2021) from the perspective of ADP, and can achieve *super-linear* convergence rate (Santos & Rust, 2004; Puterman & Brumelle, 1979; Boyd et al., 2004). Nevertheless, when the A-C method is implemented in practical applications, *the approximation errors are inevitable in the Actor/Critic updates* due to many implementation issues, including function approximation using neural networks, the finite sample size, and the finite number of gradient iterations. Moreover, the error propagation from iteration to iteration may exacerbate the 'slowing down' of the convergence and have intricate impact therein. Clearly, the (stochastic) accumulated errors may throttle the convergence rate significantly and degrade the learning performance dramatically (Fujimoto et al., 2018; Farahmand et al., 2010; Dalal et al., 2020; Lazaric et al., 2010). Thus, it is of great importance to characterize the learning performance of Warm-Start RL in practical scenar-

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ios; and the primary objective of this study is to take steps to build a fundamental understanding of the impact of the approximation errors on the finite-time sub-optimality gap for the Warm-Start A-C algorithm, i.e.,

Whether and when online learning can be significantly accelerated by a warm-start policy from offline RL?

To this end, we address the question in two steps:

(1) We first focus on the characterization of the approximation errors via finite time analysis, based on which we quantify its impact on the sub-optimality gap of the A-C algorithm in Warm-Start RL. In particular, we analyze the A-C algorithm in a more realistic setting where the samples are Markovian in the rollout trajectories for the Critic update (different from the widely used i.i.d. assumption). Further, we consider that the Actor update and the Critic update take place on the *single-time scale*, indicating that the time-scale decomposition is not applicable to the finite-time analysis here. We tackle these challenges using recent advances on Bernstein's Inequality for Markovian samples (Jiang et al., 2018; Fan et al., 2021). By delving into the coupling due to the interleaved updates of the Actor and the Critic, we provide upper bounds on the approximation errors in the Critic update and the Actor update of online exploration, respectively, from which we pinpoint the root causes of the approximation errors.

(2) We analyze the impact of the approximation errors on the finite-time learning performance of Warm-Start A-C. Based on the approximation error characterization, we treat the Warm-Start A-C algorithm as Newton's method with perturbation, and study the impact of the approximation errors on the finite-time learning performance of Warm-Start A-C. We first establish the upper bound of the bias term in the perturbation. Then we derive the upper bounds on the learning performance gap for both biased and unbiased cases. Our findings reveal that it is essential to reduce the algorithm bias in online learning. When the approximation errors are biased, we derive lower bounds on the sub-optimality gap, which reveals that even with a sufficiently good warm-start, the performance gap of online policy adaptation to the optimal policy is still bounded away from zero when the biases are not negligible. We present the experiments results to further elucidate our findings in Appendix L. We remark that the primary objective of this work is to understand the convergence behavior, which is essential before answering further questions related to the convergence rate and sampling complexity.

Related Work. (Warm-Start RL) The Warm-start RL considered in our work has the same setup as in (Bertsekas, 2022a) and recent successful applications including AlphaZero (Silver et al., 2017), where the offline pretrained policy is utilized as the initialization for online learning and this policy is *updated* while interacting with the MDP online. In a line of very recent works (Gupta et al., 2020)(Ijspeert et al., 2002)(Kim et al., 2013) on Warm-Start RL, the policy is initialized via behavior cloning from offline data and then is fine-tuned with online reinforcement learning. A variant of this scheme is proposed in Advanced Weighted Actor Critic (Nair et al., 2020) which enables quick learning of skills across a suite of benchmark tasks. In the same spirit, Offline-Online Ensemble (Lee et al., 2022) leverages multiple Q-functions trained pessimistically offline as the initial function approximation for online learning. However, we remark the theoretical characterization of the finite-time performance of Warm-Start RL is still lacking. Our work aims to take steps to quantify the impact of approximation error on online RL with a warm-start policy.

Table 1. Related work in terms of (1) Warm-start setting, (2) Actor function approximation and (3) Critic function approximation.

PAPER	WARM-START	ACTOR	CRITIC
(MUNOS, 2003)			√
(FARAHMAND ET AL., 2010)		√	√
(LAZARIC ET AL., 2010)		√	√
(FU ET AL., 2020)		√	√
(XIE ET AL., 2021)	√		
(BERTSEKAS, 2022B)	√		√
THIS WORK	√	√	√

In particular, it is worth to mention that some works (Bagnell et al., 2003)(Uchendu et al., 2022)(Xie et al., 2021) consider a different warm-start setting from ours. For instance, (Xie et al., 2021) considers the case where the reference policy is used to collect samples but remains *fixed* during the online learning. Under this setting, (Xie et al., 2021) provides a quantitative understanding on the policy fine-tuning problem in episodic Markov Decision Processes (MDPs) and establishes the lower bound for the sample complexity, where function approximation is not used. Jump-start RL (Uchendu et al., 2022) utilizes a guided-policy to initialize online RL in the early phase with a separate online exploration-policy.

Meanwhile, we remark the major differences from "offline-focus" works, which aim to derive conditions on the quality of the offline part in the warm-start RL, e.g., coverage. Notably, the focus of (Wagenmaker & Pacchiano, 2022)(Song et al., 2022) is on the offline policy quality while requiring the online learning part to satisfy certain conditions (either through delicate design or assumptions), e.g., (Song et al., 2022) requires the Bellman error to be upper bounded and (Wagenmaker & Pacchiano, 2022) requires the online exploration to satisfy certain conditions. In (Xie et al., 2021), the online algorithm needs to output a lower value estimate which is not available in standard online RL algorithms. On the contrary, motivated by recent empirical studies, which

have demonstrated that a "good" warm-start policy does not necessary improve the online learning performance, especially when the function approximation is used (Nair et al., 2020)(Uchendu et al., 2022), we consider the widely used Actor-Critic (A-C) method for online learning and aim to build a deep understanding on how the approximation errors in the *online* Actor and Critic step has impact on the learning performance. Furthermore, we summarize the comparison between our work and related work in Table 1. The detailed comparison in terms of the assumptions on the MDP and the function approximation is available in Appendix B.

(Actor-Critic as Newton's Method) The intrinsic connection between the A-C method and Newton's method can be traced back to the convergence analysis of policy iteration in MDPs with continuous action spaces (Puterman & Brumelle, 1979). The connection is further examined later in a special MDP with discretized continuous state space (Santos & Rust, 2004). Recent work (Bertsekas, 2022b) points out that the success of Warm-Start RL, e.g., AlphaZero, can be attributed to the equivalence between policy iteration and Newton's method in the ADP framework, which leads to the superlinear convergence rate for online policy adaptation. Under the generalized differentiable assumption, it has also been proved theoretically that policy iteration is the instances of semi-smooth Newton-type methods to solve the Bellman equation (Gargiani et al., 2022). While some prior works (Grand-Clément, 2021) have provided theoretical investigation of the connections between policy iteration and Newton's Method, the studies are carried out in the abstract dynamic programming (ADP) framework, assuming accurate updates in iterations. Departing from the ADP framework, this work treats the A-C algorithm as Newton's method in the presence of approximation errors, and focuses on the finite-time learning performance of Warm-Start RL.

(Finite-time analysis for Actor-Critic methods) Among the existing works on the finite time analysis of A-C methods with function approximation, (Yang et al., 2019) establishes the global convergence under the linear quadratic regulator. (Kumar et al., 2023) considers the sample complexity under i.i.d. assumptions where the Actor update and Critic update can be 'decoupled'. (Khodadadian et al., 2022) considers the two-timescale setting with Markovian samples. (Fu et al., 2020) focuses on the more general single-time scale setting but constrains the policy function approximation in the energy based function class. While the analysis in approximate policy/value iteration (Lazaric et al., 2010)(Munos, 2003)(Farahmand et al., 2010) present the error propagation in the upper bound, it is unclear how the error from each update step behave. In this work, we provide the analysis on the approximation error for each learning step explicitly and based on which we establish the error propagation in both the upper bound and lower bound.

2. Background

Markov Decision Processes. We consider a MDP defined by a tuple $(\mathcal{S}, \mathcal{A}, P, r, \gamma)$, where $\mathcal{S} = \{1, 2, \cdots, n\}$, $n < \infty$ and $\mathcal{A} = \{1, 2, \cdots, A\}$, $A < \infty$ represent the finite state space and finite action space, respectively. $P(s'|s, a) : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to [0, 1]$ is the probability of the transition from state s to state s' by applying action a and $r(s, a) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ is the corresponding reward. $\gamma \in (0, 1)$ is the discount factor. At each step t, an agent moves from the current state s_t to next state s_{t+1} by taking an action a_t following the policy $\pi \in \Pi : \mathcal{S} \to \mathcal{A}$ and receives the reward r_t. In the Warm-Start RL, we assume that the initial policy π_0 is given, e.g., in the form of a neural network (Li, 2017), and obtained by offline training. For brevity, we use bold symbols $\boldsymbol{r}_\pi \in \mathbb{R}^n : [r_\pi]_s = r(s, \pi(s))$ and $\boldsymbol{P}_\pi \in \mathbb{R}^{n\times n} : [P^\pi]_{s,s'} \triangleq P(s'|s, \pi(s))$ to denote the reward vector and the transition matrix induced by policy π. We further denote by $d^\pi : \mathcal{S} \to [0, 1]$ and $\rho^\pi : \mathcal{S} \times \mathcal{A} \to [0, 1]$ the stationary state distribution and state-action transition distribution induced by policy π. We use ρ_0 to represent the initial state distribution. We use $\|\cdot\|$ or $\|\cdot\|_2$ to represent the Euclidean norm.

Value Functions. For any policy π, define the value function $v^\pi(s) : \mathcal{S} \to \mathbb{R}$ as $v^\pi(s) = \mathbf{E}_{a_t\sim\pi(\cdot|s_t), s_{t+1}\sim P(\cdot|s_t,a_t)} [\sum_{t=0}^\infty \gamma^t r_t | s_0 = s]$ to measure the average accumulative reward staring from state s by following policy π. We define Q-function $Q^\pi(s, a) : \mathcal{S}\times\mathcal{A} \to \mathbb{R}$ as $Q^\pi(s, a) = \mathbf{E}[\sum_{t=0}^\infty \gamma^t r_t | s_0 = s, a_0 = a]$ to represent the expected return when the action a is chosen at the state s. By using the transition matrix and reward vector defined above, we have the compact form of the value function $\boldsymbol{v}^\pi = (\boldsymbol{I} - \gamma \boldsymbol{P}_\pi)^{-1} \boldsymbol{r}_\pi$, where $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ is the identity matrix and $\boldsymbol{v}^\pi \in \mathbb{R}^n$ is the value vector with the component-wise values $[v^\pi]_s \triangleq v^\pi(s)$, with

$$v^\pi(s) \triangleq \mathbf{E}_{a\sim\pi(\cdot|s)}[Q^\pi(s, a)].$$

The main objective is to find an optimal policy π^* such that the value function is maximized, i.e.,

$$\max_\pi \mathbf{E}_{s\sim\rho_0}[v^\pi(s)] \triangleq \max_\pi \mathbf{E}_{s\sim\rho_0, a\sim\pi(\cdot|s)}[Q^\pi(s, a)]. \quad (1)$$

In what follows, we use both Q-function and value function $v(s)$ for convenience, and the relation between the two is given in Eqn. (1).

Bellman Operator. For $\boldsymbol{v} \in \mathbb{R}^n$, define the Bellman evaluation operator $T^\pi : \mathbb{R}^n \to \mathbb{R}^n$ and the Bellman operator $T : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\begin{aligned} T^\pi(\boldsymbol{v}) =& \boldsymbol{r}_\pi + \gamma \boldsymbol{P}_\pi \boldsymbol{v}, \\ T(\boldsymbol{v}) =& \max_\pi \{\boldsymbol{r}_\pi + \gamma \boldsymbol{P}_\pi \boldsymbol{v}\} = \max_\pi T^\pi(\boldsymbol{v}). \end{aligned}$$

It is well known that the Bellman operator T is a contraction mapping and has order-preserving property. Note that the

Bellman operator T may not be differentiable everywhere due to the max operator, and the value $\boldsymbol{v}^*$ of the optimal policy π^* is the only fixed point of the Bellman operator T (Puterman, 2014). From the definition of the Bellman Evaluation Operator T^π, we have $\boldsymbol{v}^\pi$ to be the fixed point of T^π, i.e., $\boldsymbol{v}^\pi = T^\pi(\boldsymbol{v}^\pi)$.

2.1. Policy Iteration as Newton's Method in Abstract Dynamic Programming

Policy iteration carries out policy learning by alternating between two steps: policy improvement and policy evaluation. At time t, the policy evaluation step seeks to learn the value function $\boldsymbol{v}^{\pi_t}$ for the current policy π_t by solving the fixed point equation of the Bellman evaluation operator:

$$\boldsymbol{v} = T^{\pi_t}(\boldsymbol{v}).$$

Denote $\boldsymbol{v}_t = \boldsymbol{v}^{\pi_t}$ for simplicity. Then in the policy improvement step, a new policy π_{t+1} is obtained by maximizing the learnt value function $\boldsymbol{v}_t$ in the policy evaluation step, in a greedy manner, i.e.,

$$\pi_{t+1} = \arg\max_{\pi} T^\pi(\boldsymbol{v}_t). \tag{2}$$

To introduce the connection between policy iteration and Newton's Method, we first define operator $F : \boldsymbol{v} \to \boldsymbol{v} - T(\boldsymbol{v})$ for convenience. As in (Grand-Clément, 2021; Puterman, 2014), F can be treated as the "gradient" of an unknown function. Under the assumption that $F(\boldsymbol{v})$ is differentiable at $\boldsymbol{v}$, the Jacobian $\boldsymbol{J}_{\boldsymbol{v}}$ of F at $\boldsymbol{v}$ can be obtained as $\boldsymbol{J}_{\boldsymbol{v}} = I - \gamma \boldsymbol{P}_{\pi(\boldsymbol{v})}$, where $\pi(\boldsymbol{v}) \triangleq \arg\max_\pi T^\pi(\boldsymbol{v})$. Note that $\boldsymbol{J}_{\boldsymbol{v}}^{-1} = \sum_{i=1}^{\infty}(\gamma \boldsymbol{P}_{\pi(\boldsymbol{v})})^i$ is invertible (Puterman, 2014). Since it can be shown that $\boldsymbol{v}^{\pi_{t+1}} = (\boldsymbol{I} - \gamma \boldsymbol{P}_{\pi_{t+1}})^{-1}\boldsymbol{r}_{\pi_{t+1}} = \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1}\boldsymbol{r}_{\pi_{t+1}}$ for the policy evaluation of π_{t+1}, we have that,

$$\boldsymbol{v}^{\pi_{t+1}} = \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} F(\boldsymbol{v}^{\pi_t}), \tag{3}$$

which indicates that the analytic representation of policy iteration in the abstract dynamic programming framework reduces to Newton's Method. It is worth mentioning that the convergence behavior of policy iteration near the optimal value $\boldsymbol{v}^*$ cannot be directly obtained by using the results from convex optimization (Boyd et al., 2004) since the Bellman operator T may not be differentiable at any given value vector $\boldsymbol{v}$. The full proof is included in Appendix A.

2.2. An Illustrative Example of the Error Propagation in Actor-Critic Updates

The A-C method can be viewed as a generalization of policy iteration in ADP, where the Critic update corresponds to the policy evaluation of the current policy and the Actor update performs the policy improvement. In practice, function approximation (e.g., via neural networks) is often used to approximate both the Critic and the Actor, which inevitably incurs approximation errors for the policy update and evaluation. Moreover, the approximation errors could propagate along with the iterative updates in the A-C method. We have the illustrative example to get a more concrete sense of the impact of the approximation errors on the policy update.

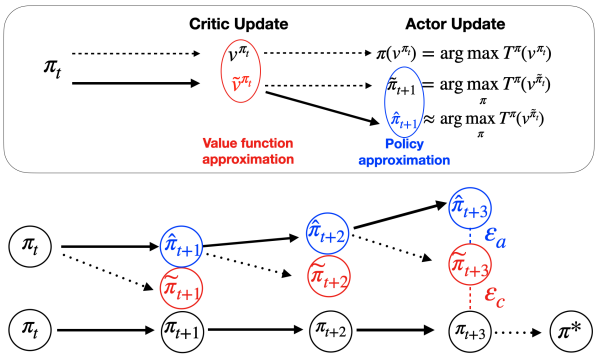

Figure 1. Illustration of error propagation effect in the A-C method: The approximation errors from Critic update ($\mathcal{E}_c$) and Actor update ($\mathcal{E}_a$) are carried forward and may get amplified due to accumulation. (To distinguish the approximation errors between Critic update and Actor update, we use tilde symbol ($\widetilde{}$) above variables, such as policy $\widetilde{\pi}$ and value vector $\widetilde{\boldsymbol{v}}$, to represent the policy and the value vector obtained in the presence of Critic update error. We use hat symbol ($\hat{}$) above the variables to represent the results with approximation error in Actor update.)

As illustrated in Figure 1, for a given policy π_t with the underlying true policy value $\boldsymbol{v}^{\pi_t}$, we denote $\widetilde{\boldsymbol{v}}^{\pi_t}$ as the learnt value estimation of $\boldsymbol{v}^{\pi_t}$ in the Critic step. We further denote π_{t+1} and $\widetilde{\pi}_{t+1}$ as the greedy policy obtained in the Actor update Eqn. (2) by using $\boldsymbol{v}^{\pi_t}$ and $\widetilde{\boldsymbol{v}}^{\pi_t}$, respectively. Let $\hat{\pi}_{t+1}$ be the policy estimation of $\widetilde{\pi}_{t+1}$ with function approximation in the Actor step. Intuitively, π_{t+1} is the underlying true policy update from π_t using one step policy iteration without any error, $\widetilde{\pi}_{t+1}$ is the policy update from π_t with approximation errors in the Critic update, and $\hat{\pi}_{t+1}$ is the policy update from π_t with approximation errors in both the Critic step and the Actor step. To characterize the impact of the approximation errors on the policy update, i.e., the difference between $\boldsymbol{v}^{\pi_{t+1}}$ and $\boldsymbol{v}^{\hat{\pi}_{t+1}}$, we evaluate the Critic error, i.e., the difference between $\boldsymbol{v}^{\pi_{t+1}}$ and $\boldsymbol{v}^{\widetilde{\pi}_{t+1}}$, and the Actor error, i.e., the difference between $\boldsymbol{v}^{\widetilde{\pi}_{t+1}}$ and $\boldsymbol{v}^{\hat{\pi}_{t+1}}$, in a separate manner. More specifically, to quantify the Critic error, we can first have the following update based on the same reasoning with Eqn. (3):

$$\begin{aligned}\boldsymbol{v}^{\widetilde{\pi}_{t+1}} =& \boldsymbol{v}_t - \boldsymbol{J}_{\widetilde{\boldsymbol{v}}_t}^{-1}\left(\boldsymbol{v}_t - (\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}\boldsymbol{v}_t)\right)\\ \triangleq& \boldsymbol{v}_t - \boldsymbol{J}_{\widetilde{\boldsymbol{v}}_t}^{-1}\left(\boldsymbol{v}_t - \widetilde{T}(\boldsymbol{v}_t)\right),\end{aligned}$$

where $\widetilde{T}(\boldsymbol{v}_t) = \boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}\boldsymbol{v}_t$ and $\boldsymbol{J}_{\widetilde{\boldsymbol{v}}_t} = \boldsymbol{I} - \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}$. Denote the approximation error (random variable) in the

Bellman operator and the Jacobian by $\mathcal{E}_{T,t}$ and $\mathcal{E}_{J,t}$, i.e.,

$$\widetilde{T}(\boldsymbol{v}_t) - T(\boldsymbol{v}_t) \triangleq \mathcal{E}_{T,t}, \;\; \boldsymbol{J}_{\widetilde{\boldsymbol{v}}_t}^{-1} - \boldsymbol{J}_{\boldsymbol{v}_t}^{-1} \triangleq \mathcal{E}_{J,t},$$

where it is clear that both error terms stem from the function approximation errors in the Critic update. To quantify the Actor error, we assume that

$$\boldsymbol{v}^{\hat{\pi}_{t+1}} = \boldsymbol{v}^{\widetilde{\pi}_{t+1}} + \mathcal{E}_{a,t},$$

where $\mathcal{E}_{a,t}$ is the error term. Therefore, by casting the A-C method as Newton's method with perturbation, we can characterize the approximation errors on the policy update:

$$\boldsymbol{v}^{\hat{\pi}_{t+1}} = \boldsymbol{v}^{\pi_{t+1}} + \mathcal{E}_{c,t} + \mathcal{E}_{a,t},$$

where $\mathcal{E}_{c,t} \triangleq -\mathcal{E}_{J,t}(\boldsymbol{v}_t - T(\boldsymbol{v}_t)) + (\boldsymbol{J}_{\boldsymbol{v}_t}^{-1} + \mathcal{E}_{J,t})\mathcal{E}_{T,t}$ and $\mathcal{E}_{a,t}$ capture the impact of the approximation error from Critic update step and Actor update step, respectively. Intuitively, as illustrated in Figure 1, both errors from the previous update in the A-C method may propagate to the next update and thus affect the convergence behavior of the algorithm substantially, in contrast to idealized policy iteration without approximation errors. This phenomenon has also been observed in the empirical results (Fujimoto et al., 2018; Thrun & Schwartz, 1993). In this work, we strive to systematically analyze the impact of the approximation errors, through (1) a detailed characterization of the approximation errors in the Critic update and the Actor update in Section 3 and (2) a thorough analysis of the error propagation effect and biases in Section 4. We also provide the illustration on our theoretical results in Fig. 2.

3. Characterization of Approximation Errors

Actor-Critic Methods with Function Approximation. In what follows, we consider that the policy is parameterized by $\theta \in \Theta$, which in general corresponds to a non-linear function class. Following (Konda & Tsitsiklis, 1999; Peters & Schaal, 2008; Kumar et al., 2023; 2020; Santos & Rust, 2004), the Q-function is parameterized by a linear function class with feature vector $\phi(s,a) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^d$ and parameter $\omega \in \Omega \subset \mathbb{R}^d$, i.e., $Q_\omega(s,a) = \omega^\top \phi(s,a)$. We note that the modeling of the Q-function via linear value function is often used to extract insight in the A-C method. Similar to the policy iteration, the update in the A-C method alternates between the following two steps [1].

Critic update: The Critic updates its parameter ω to evaluate the current policy π_t, e.g., through m-step ($m \geq 1$) Bellman evaluation operator T^π to the current Q-function estimator (namely, m-step return), which leads to the following update rule at time step t,

$$\begin{aligned} Q_{t+1}(s,a) \leftarrow \mathbf{E}_{\pi_t}[(1-\gamma)\cdot \textstyle\sum_{i=0}^{m-1} \gamma^i r(s_i,a_i) \\ + \gamma^m \cdot Q_{\omega_t}(s_m,a_m) \mid s_0 = s, a_0 = a], \\ \omega_{t+1} \leftarrow \arg\min_\omega \mathbf{E}_{(s,a)\sim\rho^{\pi_t}} \left[Q_{t+1} - \omega^\top \phi\right]^2 (s,a). \end{aligned} \quad (4)$$

Actor update: The Actor is updated through a greedy step to maximize Q-function $Q_{\omega_{t+1}}$, i.e.,

$$\pi_{t+1} \leftarrow \arg\max_\pi \mathbf{E}_{(s,a)\sim\rho^\pi} \left[Q_{\omega_{t+1}}(s,a)\right]. \quad (5)$$

3.1. Approximation Error in the Critic Update

Solving the minimization problem in Eqn. (4) involves the expectation over the stationary state-action distribution ρ^{π_t} induced by the current policy π_t, which can be approximated by sample average in practice. Therefore, we consider the Critic update below based on two groups of samples, $\{(s_l,a_l)\}_{l=1}^N$ and $\{\tau_l\}_{l=1}^N$ where $\tau_l = \{s_{l,t}, a_{l,t}, r_{l,t}\}_{t=0}^m$, which are collected by following π_t:

$$\begin{aligned} \omega_{t+1} = \Gamma_R\Big\{ \Big(\textstyle\sum_{l=1}^N \phi(s_l,a_l)\,\phi(s_l,a_l)^\top\Big)^{-1} \\ \cdot \textstyle\sum_{l=1}^N \Big((1-\gamma)\sum_{i=0}^{m-1}\gamma^i r_{l,i} + \gamma^m Q_{\omega_t}\Big)\,\phi(s_{l,m},a_{l,m})\Big\}, \end{aligned} \quad (6)$$

where Γ is the projection operator onto the Critic parameter space Ω with radius R in $\mathbb{R}^d$. Since the samples in each trajectory τ_l are obtained via rolllouts, in general the samples in each trajectory follow a Markovian process (Dalal et al., 2018; Kumar et al., 2023). We assume the samples are from the stationary distribution induced by the current policy.

In what follows, we use ω and $\widetilde{\omega}$ to distinguish the difference between the sample-based update and the solution from Eqn. (4), such that the approximation error in the Critic update can be quantified as $|Q_{\widetilde{\omega}_t} - Q_{\omega_t}|$. We first impose the following standard assumptions on the Bellman evaluation operator T^π, the feature vector $\phi(s,a)$ and the MDP.

Assumption 3.1. For given Critic parameter ω and policy parameter θ, the following condition holds:

$$\inf_{\bar{\omega}\in\Omega} \mathbf{E}_{\rho^{\pi_\theta}}[\big((T^{\pi_\theta})^m Q_\omega - \bar{\omega}^\top\phi\big)(s,a)] = 0,$$

where ρ^{π_θ} is the stationary state-action transition probability induced by policy π_θ.

Assumption 3.1 (Fu et al., 2020) indicates that the solution of the Critic update given in Eqn. (4) lies in the Critic parameter space Ω. We note that this assumption is used for ease of exposition, and our results can be modified by incorporating an additional constant term when this assumption does not hold. The proof sketch in this case can be found in Appendix D.

[1] We remark that our analysis framework and theoretical results are able to be applied to off-policy setting with the extra assumption on the behavior policy. We include the details in Appendix M.

Assumption 3.2. The feature vector $\phi(s,a)$ in the Critic satisfies the following two conditions: (1) $\|\phi(s,a)\|_2 \leq 1$, $\forall\ (s,a) \in \mathcal{S} \times \mathcal{A}$; and (2) the smallest singular value for $\mathbf{E}_{\rho^{\pi_\theta}}[\phi(s,a)\phi(s,a)^\top]$ is lower bounded by a positive constant σ^* for policy π_θ, where θ is the actor parameter obtained from the Actor update.

Assumption 3.2 is widely used in the A-C method to guarantee that the minimization in Eqn. (4) can be attained by a unique minimizer (Fu et al., 2020; Bhandari et al., 2018).

Assumption 3.3. The reward $r(s,a)$ satisfies the following two conditions: (1) The reward is upper bounded by a positive constant $r_{\max}$ for all $(s,a) \in \mathcal{S} \times \mathcal{A}$; and (2) the stationary state-action transition matrix $\boldsymbol{P}^\pi$ has non-zero spectral gap $1-\lambda > 0$ for all π.

The first condition in Assumption 3.3 is often used for discounted MDPs to ensure a finite value function (e.g., $Q(s,a) \leq Q_{\max}$)(Thrun & Schwartz, 1993; Fujimoto et al., 2018; Fu et al., 2020). Moreover, since the samples in the same trajectories are generally correlated, the second condition is adopted to guarantee the concentration properties of the Markov chain, which is generally true for the stationary Markov chain (Jiang et al., 2018; Ortner, 2020).

For any $\lambda \in (-1,1)$, let $\alpha_1(\lambda) = (1+\lambda)/(1-\lambda)$, $\alpha_2(\lambda) = 5/(1-\lambda)$ where $\alpha_2(0) = 1/3$ and $\alpha_3(\lambda) = \max\{\lambda, 0\}$. Define $\tilde{r}_m = \frac{\sqrt{\alpha_2^2 r_{\max}^2 \alpha_3^2 \ln^2 p - 2m\alpha_1\alpha_3 \ln p} - \alpha_2\alpha_3 \ln p}{m} + r_{\max}$ and then we can have the following main result on the approximation error in the Critic update step.

Proposition 3.4 (Approximation Error in Critic Update)**.** *Under Assumptions 3.1, 3.2, 3.3, the following inequality holds with probability at least $1-p$, for any $t > 0$, $(s,a) \in \mathcal{S} \times \mathcal{A}$:*

$$\begin{aligned}|Q_{\omega_t}(s,a) - Q_{\widetilde{\omega}_t}(s,a)| &\leq \tfrac{4((1-\gamma)\tilde{r}_m + \gamma^m R)}{\sqrt{N}(\sigma^*)^2} \\ &\cdot \left(-\tfrac{2}{3N}\log\tfrac{p}{4d} + \sqrt{\tfrac{4}{9N^2}\log^2\tfrac{p}{4d} - \tfrac{2}{N}\log\tfrac{p}{4d}}\right) \\ &:= \epsilon_p,\end{aligned}$$

where d is the dimension of the Critic parameter ω and R is the radius of Critic parameter space Ω as in Eqn. (6)*.*

Proposition 3.4 establishes the upper bound for the approximation error in the Critic update, which encapsulates the impact of the finite sample size and the finite-step rollout with Bellman evaluation operator T^π. It can be seen from Proposition 3.4 that in order to obtain an accurate evaluation of the policy, we can increase the sample size N in the update Eqn. (6) and have more steps of rollout with Bellman evaluation operator T^π. We remark that Proposition 3.4 considers the correlation across samples, and we appeal to the recent advances in Bernstein's Inequality for Markovian samples (Jiang et al., 2018)(Fan et al., 2021) to tackle this challenge. The proof of Proposition 3.4 can be found in Appendix C and Appendix D.

3.2. Approximation Error in the Actor Update

In practice, the greedy search step for solving Eqn. (5) is generally approximated by multiple (e.g., N_a) steps of policy gradient. Based on the policy gradient theorem (Silver et al., 2014; Sutton et al., 1999), we can have the following update at gradient step $k \in [1, N_a]$ in the t-th Actor update:

$$\begin{aligned}\theta_{t,k+1} = \theta_{t,k} + \alpha \mathbf{E}_{(s,a)\sim\rho^{\pi_{\theta_{t,k}}}}[Q_{\omega_{t+1}}(s,a)\nabla_\theta \pi_{\theta_{t,k}}(a|s)], \\ \theta_{t,1} = \theta_t,\ \ \theta_{t,N_a} = \theta_{t+1},\end{aligned} \tag{7}$$

where α is the learning rate. For simplicity, we drop the subscript t in $\theta_{t,k}$ when no confusion will arise and denote $\rho^k := \rho^{\pi_{\theta_k}}$. As in the Critic update, we sample a trajectory with length l by following the current policy π_{θ_k}, i.e., $\{s_1, a_1, s_2, a_2, \cdots, s_l, a_l\}$, to approximate the expectation in Eqn. (7). Then we can have that

$$\begin{aligned}\theta_{k+1} =& \theta_k + \alpha \tfrac{1}{l}\textstyle\sum_{i=1}^{l}[Q_{\omega_{t+1}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i)] \\ :=& \theta_k + \alpha(C_{k,t,1} + C_{k,t,2}) + \alpha f_{k,t},\end{aligned} \tag{8}$$

where $C_{k,t,1}$, $C_{k,t,2}$ and $f_{k,t}$ are defined as follows

$$\begin{aligned}C_{k,t,1} :=& 1/l \textstyle\sum_{i=1}^{l}(Q_{\omega_{t+1}} - Q_{\widetilde{\omega}_{t+1}})(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i), \\ C_{k,t,2} :=& 1/l \textstyle\sum_{i=1}^{l}(Q_{\widetilde{\omega}_{t+1}} - Q^{\pi_{\theta_t}})(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i), \\ f_{k,t} :=& 1/l \textstyle\sum_{i=1}^{l} Q^{\pi_{\theta_t}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i).\end{aligned}$$

Here $C_{k,t,1}$ captures the error resulted from using samples to estimate expectation in the Critic update. Based on our result in Proposition 3.4, with high probability, this term will go to 0 when we have infinite samples or infinite rollout length m. Note that $(T^{\pi_{\theta_t}})^m Q_{\omega_t} = Q_{\widetilde{\omega}_{t+1}}$ (Critic update) and $\lim_{m\to\infty}(T^{\pi_{\theta_t}})^m Q_{\omega_t} = Q^{\pi_{\theta_t}}$. And $C_{k,t,2}$ implies the approximation error when applying the Bellman operator limited (m) times. This term will go to 0 when $m \to \infty$. $f_{k,t}$ is an unbiased estimation of the gradient of $\mathbf{E}_{(s,a)\sim\rho^k}[Q^{\pi_{\theta_t}}(s,a)]$, i.e., $\mathbf{E}[f_{k,t}] = \mathbf{E}_{(s,a)\sim\rho^k}[Q^{\pi_{\theta_t}}(s,a)\nabla_\theta \pi_{\theta_k}(a|s)]$.

Based on Eqn. (8), it is clear that the Actor update with the approximation error resulted from the Critic update can be viewed as a stochastic gradient update with some perturbation $C_{k,t} = C_{k,t,1} + C_{k,t,2}$. For convenience, we define

$$h(\omega,\theta) := \mathbf{E}_{(s,a)\sim\rho^{\pi_\theta}}[Q_\omega(s,a)] = \mathbf{E}_{s\sim d^{\pi_\theta}}[v^{\pi_\omega}(s)].$$

Note that in the Actor update, the Critic parameter ω is fixed, and the Actor parameter θ is updated. Let θ^*_{t+1} denote the solution to Eqn. (5).

Denote the score function $\psi_\theta(a|s) := \nabla_\theta \pi_\theta(a|s)$. We have the following assumptions on ψ_θ.

Assumption 3.5. For any $\theta, \theta' \in \mathbb{R}^d$ and state-action pair $(s,a) \in \mathcal{S}\times\mathcal{A}$, there exist positive constants L_ψ, C_ψ and C_π such that the following holds: (1) $\|\psi_\theta - \psi_{\theta'}\| \le L_\psi\|\theta-\theta'\|$; (2) $\|\psi_\theta\| \le C_\psi$ and (3) $\|\pi_\theta(\cdot|s) - \pi_{\theta'}(\cdot|s)\|_{\mathrm{TV}} \le C_\pi\|\theta - \theta'\|$, where $\|\cdot\|_{TV}$ is the total-variation distance.

The smoothness and bounded property of the score function as stated in the (1) and (2) in Assumption 3.5 are widely adopted in the literature (Xu et al., 2020b; Zou et al., 2019; Agarwal et al., 2020; Kumar et al., 2023), and it has been shown (Xu et al., 2020a) that (3) in Assumption 3.5 can be satisfied for any smooth policy with bounded action space.

Let $L_0 = Q_{\max}L_\psi$, $\alpha \le \frac{1}{2L_0}$, $\kappa = C_\psi\frac{r_{\max}}{1-\gamma}$, $\sigma = 3\kappa$, $\mu = \frac{g_{\min}}{h^*_{\max}-h_{\max}}$, where $h_{\max} = \max_{\theta\neq\theta^*} h(\theta,\omega)$, $h^*_{\max} = \max_{\theta=\theta^*} h(\theta,\omega)$, $g_{\min} = \min_{\theta\neq\theta^*}\|\nabla h(\omega,\theta)\|$. Denote $\Upsilon = (1-\alpha\mu)^{N_a}$. Finally, we present the upper bound of the approximation error in the Actor update.

Proposition 3.6 (Approximation Error in Actor Update)**.** *Given Actor parameter θ_{t-1}, the following inequality holds:*

$$\begin{aligned}&\mathbf{E}_{\theta_t}[h(\omega,\theta_t^*) - h(\omega,\theta_t)|\theta_{t-1}]\\ &\le \Upsilon(h(\omega,\theta_t^*) - h(\omega,\theta_{t-1})) + \Xi_p,\end{aligned}$$

where $\Xi_p = ((C_\psi\epsilon_p + 2\kappa)^2 + 2\alpha L\sigma^2)/2\mu$.

It can be seen in Proposition 3.6 that the Critic approximation error has direct impact on the Actor update through Ξ_p. Proposition 3.6 reveals that due to the bias and noise induced by the Critic approximation error, running more gradient iterations (the first term on the RHS) do not necessarily guarantee the convergence to the optimal policy $\pi_{\theta_t^*}$. The proof can be found in Appendix H.

4. The Impact of Approximation Errors on Warm-Start Actor-Critic

We next quantify the impact of the approximations errors on the sub-optimality gap of the Warm-Start A-C method with inaccurate Actor/Critic updates. We first cast the A-C method as Newton's Method with perturbation, and then present both the finite-time upper bound and lower bound on the finite-time learning performance.

Actor-Critic Method as Newton's Method with Perturbation. As mentioned earlier, the Critic update follows Eqn. (6) with finite samples and finite step rollout with Bellman evaluation operator T^π and the Actor update follows Eqn. (8). Given the policy π_t at time t, we denote the resulting policy of one A-C update as $\hat{\pi}_{t+1}$. Recall that we use $\widetilde{\pi}_{t+1}$ to denote the policy attained the $\max$ in $T(\boldsymbol{v}^{\pi_t})$ as illustrated in Figure 1. Furthermore, we define the following notations for ease of our discussion: (1) Denote $\mathcal{E}_{v,t} = \boldsymbol{v}^{\hat{\pi}_{t+1}} - \boldsymbol{v}^{\widetilde{\pi}_{t+1}}$ as the approximation error in the Actor update; (2) Denote $\mathcal{E}_{r,t} = \boldsymbol{r}_{\widetilde{\pi}_{t+1}} - \boldsymbol{r}_{\hat{\pi}_{t+1}}$ as the error in the reward vector, which is induced by the approximation error in the Actor update step; (3) Denote $\mathcal{E}_{P,t} = \boldsymbol{P}_{\widetilde{\pi}_{t+1}} - \boldsymbol{P}_{\hat{\pi}_{t+1}}$ as the error in the transition matrix $\boldsymbol{P}$; (4) Denote $\mathcal{E}_{\hat{J},t} = \boldsymbol{J}^{-1}_{\widetilde{\boldsymbol{v}}_t} - \boldsymbol{J}^{-1}_{\hat{\boldsymbol{v}}_t}$ where $\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} = \boldsymbol{I} - \gamma\boldsymbol{P}_{\hat{\pi}_{t+1}}$ and $\boldsymbol{J}_{\widetilde{\boldsymbol{v}}_t} = \boldsymbol{I} - \gamma\boldsymbol{P}_{\widetilde{\pi}_{t+1}}$.

Following the same line as in Section 2.2, we treat the A-C algorithm as Newton's method with perturbation $\mathcal{E}_t$, i.e.,

$$\boldsymbol{v}^{\hat{\pi}_{t+1}} := \boldsymbol{v}^{\hat{\pi}_t} - \hat{\mathcal{L}}(t), \tag{9}$$

where $\hat{\mathcal{L}}(t) = \boldsymbol{J}^{-1}_{\hat{\boldsymbol{v}}_t}(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})) - \mathcal{E}_t$ is the stochastic estimator of Newton's update $\mathcal{L}(t) = \boldsymbol{J}^{-1}_{\hat{\boldsymbol{v}}^{\pi_t}}\left(\boldsymbol{v}^{\hat{\pi}_t} - T\left(\boldsymbol{v}^{\hat{\pi}_t}\right)\right)$, and

$$\begin{aligned}\mathcal{E}_t =& \mathcal{E}_{v,t} + \mathcal{E}_{\hat{J},t}(\boldsymbol{v}^{\hat{\pi}_{t+1}} - (\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma\boldsymbol{P}_{\widetilde{\pi}_{t+1}}\boldsymbol{v}^{\hat{\pi}_{t+1}}))\\ &- \boldsymbol{J}^{-1}_{\hat{\boldsymbol{v}}_t}(\mathcal{E}_{r,t} + \gamma\mathcal{E}_{P,t}\boldsymbol{v}^{\hat{\pi}_t}),\end{aligned}$$

which can be further decomposed into bias and Martingale difference noise as follows:

$$\begin{aligned}\mathcal{B}(t) \triangleq& \mathbf{E}[\hat{\mathcal{L}}(t)] - \mathcal{L}(t) = \mathbf{E}[\mathcal{E}_t],\\ \mathcal{N}(t) \triangleq& \hat{\mathcal{L}}(t) - \mathbf{E}[\hat{\mathcal{L}}(t)] = \mathcal{E}_t - \mathbf{E}[\mathcal{E}_t].\end{aligned}$$

We have a few observations in order. It can be seen that the perturbation $\mathcal{E}_t$ results from both Actor approximation error (e.g., $\mathcal{E}_{r,t}$, $\mathcal{E}_{P,t}$) and Critic approximation error (e.g., $\mathcal{E}_{v,t}$). Meanwhile, the learnt Q function in the Critic update Eqn. (6) is biased in general due to finite rollout steps m which further leads to the biased gradients in the Actor update Eqn. (8) (Kumar et al., 2023). More importantly, due to the error propagation effect (see Fig. 1), the approximation errors from previous step may get amplified. Clearly, the estimation bias plays an important role in affecting the learning performance, especially when deep neural networks are used as function approximations, which has been extensively investigated using empirical studies (Fujimoto et al., 2018; Elfwing et al., 2018; Van Hasselt et al., 2016).

Next, we examine the bias $\mathcal{B}(t)$ based on the approximation errors in the Actor/Critic updates. Combining the results in Proposition 3.4 and 3.6 on the approximation error in the Critic/Actor updates, we define

$$H_t \triangleq \textstyle\sum_{i=0}^t \Upsilon^i\Xi_p + \Upsilon^{t+1}(h(\omega,\theta_t^*) - h(\omega,\theta_0)).$$

Then we have the following result on the bias $B(t)$. The detailed derivation is given in Appendix I.

Proposition 4.1 (Upper Bound on the Bias)**.** *Suppose Assumption 3.5 holds. Let $S_\epsilon(\cdot)$ be an open ball of radius ϵ. There exist positive constants L_b, and ϵ, such that when $\theta_{t+1} \in S_\epsilon(\theta^*_{t+1})$, the following holds for any $t > 0$,*

$$\|\mathcal{B}(t)\| \le L_b H_t$$

4.1. Upper Bound on Sub-optimality Gap

In order to address the question "*Under what condition online learning can be significantly accelerated by a warm-start policy*?", we derive the upper bound on the sub-optimality gap.

Case 1: Unbiased Case. We first consider the finite-time upper bound in the unbiased case, i.e., $B(t) = 0,\ \forall t$. In this case, we introduce the following standard assumption on the Jacobian $\boldsymbol{J}_{\boldsymbol{v}}$.

Assumption 4.2 (Local Lipschitz Continuity)**.** For some $0 < q < 1$ there exist constant $0 < L_J < +\infty$ and constant $0 < M < +\infty$ such that starting from the warm-start policy π_0, the policies $\{\hat{\pi}_t,\ t = 1, 2, \cdots\}$ generated by the A-C algorithm satisfy

$$\|\boldsymbol{J}_{\boldsymbol{v}^*} - \boldsymbol{J}_{\boldsymbol{v}^{\hat{\pi}_t}}\| \leq L_J \|\boldsymbol{v}^{\hat{\pi}_t} - \boldsymbol{v}^*\|^q,$$

and $\|\boldsymbol{J}^{-1}_{\boldsymbol{v}^{\hat{\pi}_t}}\| \leq M$.

Intuitively, Assumption 4.2 means that the difference of Jacobian $\|\boldsymbol{J}_{\boldsymbol{v}^{\pi_t}} - \boldsymbol{J}_{\boldsymbol{v}^*}\|$ is small whenever the underlying value functions that induces the policies are close. We note that the conditions of this type are commonly used in the convergence analysis of policy iteration algorithms for exact dynamic programming (Puterman & Brumelle, 1979; Grand-Clément, 2021). In particular, we remark that the Jacobian function (of π or $\boldsymbol{v}^{\pi}$) is non-linear and this assumption implies the learned policy initialized with it is essential for the warm-start policy to be reasonably "close" to the optimal policy. Next, we present the finite-time upper bound in the unbiased case.

Proposition 4.3 (Unbiased Case)**.** *In the unbiased case, i.e., $\mathcal{B}_t = 0,\ \forall\, t \geq 0$, we have*

$$\|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}]\| \leq L \|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}]\|^{1+q} \tag{10}$$

where $L := ML_J$ with M and L_J defined in Assumption 4.2. By applying Eqn. (10) *recursively, we obtain,*

$$\|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}]\| \leq L^{\frac{(1+q)^{t+1}-1}{q}} \|\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}\|^{(1+q)^{(t+1)}}$$

In Proposition 4.3, as the Warm-start policy is close to the optimal policy, we establish the superlinear convergence of $\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}]$ in the presence of approximation error $\mathcal{E}(t)$ from both Actor update and Critic update. This observation corroborates the most recent empirically finding (Bertsekas, 2022b)(Silver et al., 2017), where the online RL can further improve the warm-start policy by only few adaptation steps.

Case 2: Bounded Bias. Next, We present the finite-time upper bound in the general case when the bias is upper bounded (as given in Proposition 4.1).

Corollary 4.4. *If Assumption 4.2 holds in the biased case, we have that for any $t > 0$,*

$$\|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}]\| \leq L \|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}]\|^{1+q} + L_b H_t. \tag{11}$$

By applying Eqn. (11) *recursively, we obtain,*

$$\|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}]\| \leq \|\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}\|^{(1+q)^{1+t}} \cdot (L \cdots ((L + u_1)^{1+q} + u_2)^{1+q} \cdots + u_t),$$

where $u_t := \frac{L_b H_t}{\|\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}\|^{(1+q)^{(1+t)}}}$ and $L_b H_t$ is the upper bound of the bias as in Proposition 4.1.

Implication on Reducing the Performance Gap. The upper bound in Corollary 4.4 sheds light on the impact of warm-start policy π_0 (the first term) and the bias $\{\mathcal{B}(t)\}$ (u_t) (the second term), thereby providing guidance on how to achieve desired finite-time learning performance. When the bias $\mathcal{B}(t) \neq \mathbf{0}$ $(u_t \neq 0)$, the upper bound hinges heavily on the biases in the approximation errors, even when the warm-start policy π_0 is close to the optimal policy (see the second term in Eqn. (11)). In this case, recall the result on the upper bound of the bias $\mathcal{B}(t)$ in proposition 4.1, where we establish the connection between the bias and the approximation error. As expected, in order to reduce the performance gap, it is essential to decrease the bias in the approximation error, which can be achieved by increasing gradient steps, rollout length and sample sizes.

"Wash-out" Phenomenon. In Corollary 4.4, the product structure between the warm-start term and bias term also implies that the imperfections of the Warm-start policy can be "washed out" by online learning when the bias is close to zero. For instance, when the value function $\boldsymbol{v}^{\pi_0}$ induced by the Warm-start policy π_0 is bounded away from $\boldsymbol{v}^*$, e.g., $\epsilon < \|\boldsymbol{v}^{\pi_0} - \boldsymbol{v}^*\| < L^{-q}$ and the bias is sufficiently small, e.g., $u_t \leq \epsilon^{-q} - L$, then we have $\|\mathbf{E}[\boldsymbol{v}^{\pi_1} - \boldsymbol{v}^*]\| \leq \|\boldsymbol{v}^{\pi_0} - \boldsymbol{v}^*\|$. We note that this result corroborates with the observation in the very recent literature (Bertsekas, 2022b) and this phenomenon has not been formalized by previous works on error propagation (Munos, 2003)(Lazaric et al., 2010). Furthermore, we clarify that the "Wash-out" phenomenon in Corollary 4.4 would not hold in the case when Assumption 4.2 is not satisfied, which may likely yield a policy far away from the optimal during the online learning. The proof of Corollary 4.4 is relegated to Appendix J.

Remark. In the case when the bias is pronounced, Assumption 4.2 can be stringent. Nevertheless, it is of more interest to find lower bounds on the sub-optimality gap, which we turn our attention to next.

4.2. Lower Bound on Sub-optimality Gap

Aiming to understand "*whether online learning can be accelerated by a warm-start policy*", we derive a lower bound

to quantify the impact of the bias and the error propagation. Let $(\pi_0, \hat{\pi}_1, \cdots, \hat{\pi}_t)$ be the sequence of policies generated by running t-step A-C algorithm in Eqn. (6) and Eqn. (8). Fro convenience, let filtration $\boldsymbol{\mathcal{F}}_t$ be the σ-algebra generated by $(\pi_0, \hat{\pi}_1, \cdots, \hat{\pi}_t)$. We obtain the lower bound by unrolling the recursion of the Newton update (with perturbation) Eqn. (9).

Theorem 4.5. *Conditioned on the filtration $\boldsymbol{\mathcal{F}}_t = \sigma(\pi_0, \hat{\pi}_1, \cdots, \hat{\pi}_t)$, the lower bound of $\|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}} | \boldsymbol{\mathcal{F}}_t]\|$ satisfies that*

$$\|\mathbf{E}\left[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}} | \boldsymbol{\mathcal{F}}_t\right]\| \geq \|\gamma^{t+1} \bar{\boldsymbol{P}}_{t+1}(\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}) + \textstyle\sum_{i=1}^{t} \gamma^i \bar{\boldsymbol{P}}_i \mathcal{B}(t-i) + \mathcal{B}(t)\|, \quad (12)$$

where $\bar{\boldsymbol{P}}_{t+1} = \mathbf{E}\left[\left(\prod_{i=0}^{t} \boldsymbol{P}_{\pi_{t+1-i}}\right)\right]$.

Error Propagation and Accumulation. It can be seen form Theorem 4.5 that the bias terms $\{\mathcal{B}(t)\}$ add up over time, and the propagation effect of the bias terms is encapsulated by the last two terms on the right side of Eqn. (12). Clearly, the first term on the right side, corresponding to the impact of the warm-start policy π_0, diminishes with A-C updates. To get a more concrete sense of Theorem 4.5, we consider the following special settings. (1) When the bias is always positive, i.e., $\mathcal{B}(t) > \mathbf{0}$ for all $t \geq 0$, the lower bound in Theorem 4.5 is always positive, i.e., $\|\mathbf{E}\left[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}\right]\| \geq \|\mathcal{B}(t)\| > 0$. In this case, the sub-optimal gap remains bounded away from zero. Similar conclusion can be made when the bias is always negative. (2) When the bias term can be either positive or negative, the lower bound is shown as Eqn. (12). In this case, the learning performance of the A-C algorithm largely depends on the behavior of the Bias term. It can be seen from Theorem 4.5 that even when the warm-start policy is near-optimal, it is still challenging to guarantee that online fine-tuning can improve the policy if the approximation error is not handled correctly. We note that this has also been observed empirically (Nair et al., 2020; Lee et al., 2022). The proof of Theorem 4.5 is provided in Appendix K.

Remark. The primary goal of this work is to make a first attempt to quantify the learning performance of Warm-start RL by studying its convergence behavior. It can be seen from Corollary 4.4 and Theorem 4.5 that the bounds are in terms of the biases $\{B(t)\}$, and the structure of $\{B(t)\}$ remains open and is highly nontrivial. Hence, we submit that the convergence rate and the sampling complexity are of great interest but it is beyond the scope of this work.

Remark. We clarify the connection between our work and previous works on the "coverage" requirements (e.g., Assumption A (Xie et al., 2021)). The concentrability condition (Xie et al., 2021) characterizes the distance between the visitation distributions of the warm-start policy and some optimal policy for *every* state-action pair. Hence, this "coverage" assumption requires the state-action point-wise distance between the optimal policy and the policy to be upper bounded in the worst-case scenario, implying the bias is also bounded above since the worse-case distance is larger than average distance in general. While in our setting, we evaluate the sub-optimality gap in the average sense, i.e., $\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\pi_t}]$, by characterizing the upper bound of the bias from the Actor update and Critic update. Meanwhile, the performance requirements for online learning algorithms in the previous work (e.g., Bellman error is upper bounded by (Song et al., 2022)) correspond to the second term on the RHS of Proposition 4.3, Corollary 4.4 and Theorem 4.5, where we show that upper bound of the approximation error in the Actor update has direct impact on the sub-optimality.

5. Conclusion

We take a finite-time analysis approach to address the question "*whether and when online learning can be significantly accelerated by a warm-start policy from offline RL?*" in Warm-Start RL. By delving into the intricate coupling between the updates of the Actor and the Critic, we first provide upper bounds on the approximation errors in both the Critic update and Actor update of online adaptation, respectively, where the recent advances on Bernstein's Inequality are leveraged to deal with the sample correlation therein. Based on these results, we next cast the Warm-Start A-C method as Newton's method with perturbation, which serves as the foundation for characterizing the impact of the approximation errors on the finite-time learning performance of Warm-Start A-C. In particular, we provide upper bounds on the sub-optimality gap, which provides guidance on the design of Warm-Start RL for achieving desired finite-time learning performance. And we also derive lower bounds on the sub-optimality gap under biased approximation errors, indicating that the performance gap can be bounded away from zero even with a good prior policy. We note that as the biases structure remains open, the study on the efficiency of Warm-start RL calls for additional work. Finally, it is also worth to explore the setting beyond linear function approximation and further derive the practical warm-start RL algorithm utilizing the theoretical findings in this work.

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Appendix

A. Examples in Section 2.2

In this section, we elaborate further on the illustrative example in Section 2.2. We use the notation defined in Figure 1.

Policy Iteration as Newton's Method. Based on (Puterman & Brumelle, 1979)(Grand-Clément, 2021), we first build the relation between policy iteration and Newton's Method in the abstract dynamic programming (ADP) framework, assuming accurate updates.

From the definition of the value function $\boldsymbol{v}$, we have that for any policy π,

$$\boldsymbol{v}^{\pi} = \boldsymbol{r}_{\pi} + \gamma \boldsymbol{P}_{\pi} \boldsymbol{v}^{\pi}.$$

Recall the definition of Bellman evaluation operator $T^{\pi}(\cdot)$ and the Bellman operator $T(\cdot)$,

$$T^{\pi}(\boldsymbol{v}) = \boldsymbol{r}_{\pi} + \gamma \boldsymbol{P}_{\pi} \boldsymbol{v},\ T(\boldsymbol{v}) = \max_{\pi} \{\boldsymbol{r}_{\pi} + \gamma \boldsymbol{P}_{\pi} \boldsymbol{v}\} = \max_{\pi} T^{\pi}(\boldsymbol{v}).$$

It follows that

$$\begin{aligned}
\boldsymbol{v}^{\pi_{t+1}} &= \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \boldsymbol{r}_{\pi_{t+1}} \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{v}^{\pi_t} + \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \boldsymbol{r}_{\pi_{t+1}} \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}} \boldsymbol{v}^{\pi_t} + \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \boldsymbol{r}_{\pi_{t+1}} \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \left(-\boldsymbol{r}_{\pi_{t+1}} + \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}} \boldsymbol{v}^{\pi_t}\right) \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \left(-\boldsymbol{r}_{\pi_{t+1}} + \left(\boldsymbol{I} - \gamma \boldsymbol{P}_{\pi_{t+1}}\right) \boldsymbol{v}^{\pi_t}\right) \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \left(\boldsymbol{v}^{\pi_t} - \boldsymbol{r}_{\pi_{t+1}} - \gamma \boldsymbol{P}_{\pi_{t+1}} \boldsymbol{v}^{\pi_t}\right) \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\pi_t}}^{-1} \left(\boldsymbol{v}^{\pi_t} - T\left(\boldsymbol{v}^{\pi_t}\right)\right),
\end{aligned} \tag{13}$$

where $\boldsymbol{J}_{\boldsymbol{v}} = \boldsymbol{I} - \gamma \boldsymbol{P}_{\pi(\boldsymbol{v})}$ and $\pi(\boldsymbol{v})$ attains the $\max$ in $T(\boldsymbol{v})$. Eqn. (13) establishes a connection between policy iteration under ADP and Newton's Method. Specifically, if we assume function $F : \boldsymbol{v} \to \boldsymbol{v} - T(\boldsymbol{v})$ is differentiable at any vector $\boldsymbol{v}$ visited by policy iteration, then we have $\boldsymbol{v}_{t+1} = \boldsymbol{v}_t + \boldsymbol{J}_{\boldsymbol{v}_t}^{-1} F(\boldsymbol{v}_t)$, which is exactly the update of the Newton's Method in convex optimization (Boyd et al., 2004). Due to the fact that $F(\cdot)$ may not be differentiable at all $\boldsymbol{v}$ in policy iteration, the assumptions on the Lipschitzness of $\boldsymbol{v} \to \boldsymbol{J}_{\boldsymbol{v}}$ is commonly used to prove the convergence of the policy iteration (see Assumption 4.2). Following the same line, next we show the case when function approximation is used in the A-C algorithm.

A-C Updates with Function Approximation. Consider the illustration example in Section 2.2. Next we outline the main differences between the A-C update with function approximation and the policy iteration in the ADP framework, and cast A-C based policy iteration with function approximation as Newton's Method with perturbation. Specifically,

$$\begin{aligned}
\boldsymbol{v}^{\widetilde{\pi}_{t+1}} &= \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \boldsymbol{r}_{\widetilde{\pi}_{t+1}} \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{v}^{\pi_t} + \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \boldsymbol{r}_{\widetilde{\pi}_{t+1}} \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}} \boldsymbol{v}^{\pi_t} + \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \boldsymbol{r}_{\widetilde{\pi}_{t+1}} \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \left(-\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}} \boldsymbol{v}^{\pi_t}\right) \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \left(-\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \left(\boldsymbol{I} - \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}\right) \boldsymbol{v}^{\pi_t}\right) \\
&= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \left(\boldsymbol{v}^{\pi_t} - \left(\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}} \boldsymbol{v}^{\pi_t}\right)\right) \\
&\triangleq \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \left(\boldsymbol{v}^{\pi_t} - \widetilde{T}\left(\boldsymbol{v}^{\pi_t}\right)\right),
\end{aligned}$$

where $\boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}} = \boldsymbol{I} - \gamma \boldsymbol{P}_{\pi(\boldsymbol{v}^{\widetilde{\pi}_t})}$ and $\pi(\boldsymbol{v})$ attains the $\max$ in $T(\boldsymbol{v})$ (not $\widetilde{T}(v)$), with the following two operators defined as

$$\begin{aligned}
T(\boldsymbol{v_t}) &\triangleq \boldsymbol{r}_{\pi_{t+1}} + \gamma \boldsymbol{P}_{\pi_{t+1}} \boldsymbol{v}_t, \\
\widetilde{T}(\boldsymbol{v_t}) &\triangleq \boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}} \boldsymbol{v}_t.
\end{aligned}$$

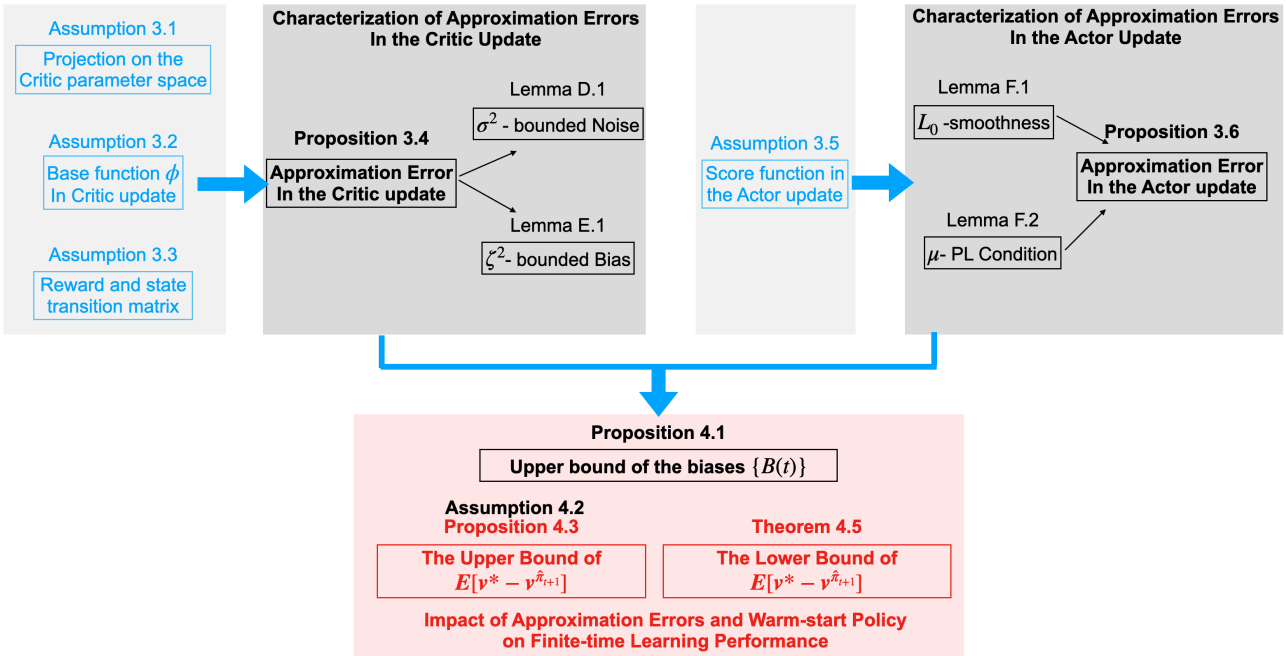


Figure 2. Illustration of the theoretical analysis.

For convenience, let $\mathcal{E}_{T,t}$ and $\mathcal{E}_{J,t}$ denote the approximation errors in the Bellman operator T and the Jacobian $\boldsymbol{J}_{\boldsymbol{v}}$, i.e.,

$$\begin{aligned}\widetilde{T}(\boldsymbol{v}_t) - T(\boldsymbol{v}_t) =& (\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}} \boldsymbol{v}_t) - (\boldsymbol{r}_{\pi_{t+1}} + \gamma \boldsymbol{P}_{\pi_{t+1}} \boldsymbol{v}_t) \triangleq \mathcal{E}_{T,t},\\ \boldsymbol{J}_{\widetilde{\boldsymbol{v}}_t}^{-1} - \boldsymbol{J}_{\boldsymbol{v}_t}^{-1} =& (\boldsymbol{I} - \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}})^{-1} - (\boldsymbol{I} - \gamma \boldsymbol{P}_{\pi_{t+1}})^{-1} \triangleq \mathcal{E}_{J,t},\end{aligned}$$

and define

$$\boldsymbol{v}^{\hat{\pi}_{t+1}} \triangleq \boldsymbol{v}^{\widetilde{\pi}_{t+1}} + \mathcal{E}_{a,t},$$

where $\mathcal{E}_{a,t}$ capture the error induced by inaccurate policy improvement (the greedy step, e.g., Eqn. (5)) in the Actor update. Then we have that

$$\begin{aligned}\boldsymbol{v}^{\widetilde{\pi}_{t+1}} &= \boldsymbol{v}^{\pi_t} - \boldsymbol{J}_{\boldsymbol{v}^{\widetilde{\pi}_t}}^{-1} \left(\boldsymbol{v}^{\pi_t} - \widetilde{T}\left(\boldsymbol{v}^{\pi_t}\right)\right)\\ &= \boldsymbol{v}_t - (\boldsymbol{J}_{\boldsymbol{v}_t}^{-1} + \mathcal{E}_{J,t})(\boldsymbol{v}_t - T(\boldsymbol{v}_t) - \mathcal{E}_{T,t})\\ &= \underbrace{\boldsymbol{v}_t - \boldsymbol{J}_{\boldsymbol{v}_t}^{-1}(\boldsymbol{v}_t - T(\boldsymbol{v}_t))}_{\text{Exact Newton Step}} \underbrace{-\mathcal{E}_{J,t}(\boldsymbol{v}_t - T(\boldsymbol{v}_t)) + (\boldsymbol{J}_{\boldsymbol{v}_t}^{-1} + \mathcal{E}_{J,t})\mathcal{E}_{T,t}}_{\text{Perturbation}}\\ &\triangleq \underbrace{\boldsymbol{v}_t - \boldsymbol{J}_{\boldsymbol{v}_t}^{-1}(\boldsymbol{v}_t - T(\boldsymbol{v}_t))}_{\text{Exact Newton Step}} + \mathcal{E}_t\\ &= \boldsymbol{v}^{\pi_{t+1}} + \mathcal{E}_t.\end{aligned}$$

In a nutshell, we have that

$$\boldsymbol{v}^{\hat{\pi}_{t+1}} = \boldsymbol{v}^{\pi_{t+1}} + \mathcal{E}_{c,t} + \mathcal{E}_{a,t},$$

where

$$\mathcal{E}_{c,t} \triangleq -\mathcal{E}_{J,t}(\boldsymbol{v}_t - T(\boldsymbol{v}_t)) + (\boldsymbol{J}_{\boldsymbol{v}_t}^{-1} + \mathcal{E}_{J,t})\mathcal{E}_{T,t}.$$

Table 2. Detailed Comparison with related work in terms of MDP and function approximation settings.

Reference	Previous Work	Our Work
(Xie et al., 2021)	Episodic MDP setting and no function approximation error during the policy finetuning.	We consider general MDP with the linear value function (Critic) approximation and general Actor function approximation.
(Bagnell et al., 2003)(Uchendu et al., 2022)	(Bagnell et al., 2003)(Uchendu et al., 2022) gives the results with either the approximation error from policy update (Theorem 1 (Bagnell et al., 2003)) or value function update (Section 4.2 (Bagnell et al., 2003), Assumption A.6 (Uchendu et al., 2022)) through term ϵ.	Our work first characterizes ϵ explicitly (which is not available in (Bagnell et al., 2003)(Uchendu et al., 2022)) and also studies how the approximation errors from "both Actor update and Critic update" affect the learning performance at the same time (through bias term $\mathcal{B}(t)$).
(Song et al., 2022)	(Song et al., 2022) considers Q-function approximation and assume the greedy policy can be obtained exactly (line 3, Algorithm 1)	We consider Actor-Critic and consider the approximation error in the Actor and Critic, respectively.
(Wagenmaker & Pacchiano, 2022)	(Wagenmaker & Pacchiano, 2022) requires the underlying MDP structure to be linear and only considers linear Softmax Policy (Actor)	We consider general MDP and general Actor approximation.

B. Detailed Comparison with Previous Work

In this work, we consider the same warm-start RL setup as in (Bertsekas, 2022a) and recent successful applications including AlphaZero, where the offline pretrained policy is utilized as the initialization for online learning and this policy is updated while interacting with the MDP online. Policy improvement via online adaptation (finetuning) plays a critical role in addressing the notorious challenge of "distribution shift" between offline training and online learning, and this is one main motivation for our study on Warm-start RL. In stark contrast, the reference policy in (Xie et al., 2021)(Bagnell et al., 2003) is used to collect samples but remains fixed during the online learning. It is clear that if one queries 0 samples from the reference policy, Algorithm 2 (Xie et al., 2021) would NOT reduce to the proposed warm-start learning algorithm in our setting. Meanwhile, Algorithm 1 and Algorithm 2 (Uchendu et al., 2022) assume the episodic MDP setting, which is different from the MDP setting in our study. On the other hand, the hybrid RL setting (Song et al., 2022)(Wagenmaker & Pacchiano, 2022) mainly focuses on the usage of the offline dataset while the initial policy is not initialized by any warm-start policy (e.g., Algorithm 1 (Song et al., 2022), Section 6 (Wagenmaker & Pacchiano, 2022)).

Moreover, the finite time analysis with function approximation errors in both Actor and Critic updates has not been studied before under this warm-start RL setting. From a theoretic perspective, our work has contributed to developing a fundamental understanding of the impact of the function approximation errors in the general MDP settings (ref. Table 2), beyond the references listed.

C. Proof of Bernstein's Inequality with General Makovian samples

In this section, we provide the proof of Bernstein's Inequality with General Makovian samples following the proof in Theorem 2 (Jiang et al., 2018).

With a bit abuse of notation,let π denote the stationary distribution of the Markov chain $\{X_i\}_{i\geq 1}$. We define $\pi(h) := \int h(x)\pi(dx)$ to be the integral of function h with respect to π. Let $\mathcal{L}_2(\pi) = \{h : \pi(h^2) < \infty\}$ be the Hilbert space of square-integrable functions and $\mathcal{L}_2^0(\pi) = \{h \in \mathcal{L}_2(\pi) : \pi(h) = 0\}$ be the subspace of mean zero functions. Let P be the Markov transition matrix of its underlying (state space) graph and P^* be its adjoint in the Hilbert space. Let $\lambda(P) \in [0, 1]$ be the operator norm of P on $\mathcal{L}_2^0(\pi)$ and $\lambda_r(P) \in [-1, 1]$ be the rightmost spectral value of $(P + P^*)/2$.

Then the right spectral gap of P is defined as $1-\lambda_r$ (Levin & Peres, 2017) (We remark that in Assumption 3.3, we assume the absolute spectral gap is non-zero, which implies the right spectral gap is also non-zero. This is true since $-1 \le \lambda_r \le \lambda \le 1$.). Let E^h denote the multiplication operator of function $e^h : x \mapsto e^{h(x)}$. In the Hilbert space $\mathcal{L}_2(\pi)$, we define the norm of a function h to be $\|h\|_\pi = \sqrt{\langle h, h\rangle_\pi}$. Furthermore, we introduce the norm of a linear operator T on $\mathcal{L}_2(\pi)$ as $\|\|T\|\|_\pi = \sup\{\|Th\|_\pi : \|h\|_\pi = 1\}$.

We first restate Bernstein's Inequality with General Makovian Samples (Jiang et al., 2018) in the following theorem. Let $\alpha_1(\lambda) = (1+\lambda)/(1-\lambda)$, $\alpha_2(\lambda) = 5/(1-\lambda)$ and $\alpha_2(0) = 1/3$.

Theorem C.1 (Bernstein's Inequality with General Makovian Samples)**.** *Suppose $\{X_i\}_{i\ge1}$ is a stationary Markov chain with invariant distribution π and non-zero right spectral gap $1-\lambda_r > 0$, and $f :\mapsto x[-c,c]$ is a function with $\pi(f) = 0$. Let $\sigma^2 = \pi(f^2)$. Then, for any $0 \le t < (1-\max\{\lambda_r, 0\})/5c$ and any $\epsilon > 0$,*

$$\mathbf{P}_\pi\left(\frac{1}{n}\sum_{i=1}^n f(X_i) > \epsilon\right) \le \exp\left(-\frac{n\epsilon^2/2}{\alpha_1(\max\{\lambda_r,0\})\cdot\sigma^2 + \alpha_2(\max\{\lambda_r,0\})\cdot c\epsilon}\right). \tag{14}$$

Proof. **Step 1.** Establish the upper bound of $\mathbf{E}\left[e^{t\sum_i^n f_i(X_i)}\right]$.

Let $I : x \mapsto 1$ be the function mapping x to 1 and let Π be the projection operator onto 1, i.e., $\Pi : g \mapsto \langle h, I\rangle_\pi I = \pi(h)I$. Define the León-Perron operator to be $\widehat{P}_\gamma = \gamma I + (1-\gamma)\Pi$, $\gamma \in [0,1)$. Then we recall the following lemma (Lemma 2, (Jiang et al., 2018)) on the stationary Markov chain (Fan et al., 2021).

Lemma C.2. *Let $\{X_i\}$ be a stationary Markov chain with invariant measure π and non-zero right spectral gap $1-\lambda_r > 0$. For any bounded function f and any $t \in \mathbb{R}$,*

$$\mathbf{E}_\pi\left[e^{t\sum_{i=1}^n f(X_i)}\right] \le \left\|\left|E^{tf/2}\widehat{P}_{\max\{\lambda_r,0\}}E^{tf/2}\right|\right\|_\pi^n.$$

Lemma C.3 indicates that it is sufficient to prove the upper bound of $\mathbf{E}\left[e^{t\sum_i^n f_i(X_i)}\right]$ by proving the upper bound of $\left\|\left|E^{tf/2}\widehat{P}_{\max\{\lambda_r,0\}}E^{tf/2}\right|\right\|_\pi^n$.

To this end, we first invoke the following lemma (Lemma 6, (Jiang et al., 2018)) to construct $\widehat{f}_k \approx f$ such that for any $\lambda \in [0,1)$, $\left\|\left|E^{tf/2}\widehat{P}_\lambda E^{tf/2}\right|\right\|_\pi = \lim_{k\to\infty}\left\|\left|E^{t\widehat{f}_k/2}\widehat{P}_\lambda E^{t\widehat{f}_k/2}\right|\right\|_\pi$.

Lemma C.3. *For function $f : x \in \mathcal{X} \mapsto [-c,c]$ such that $\pi(f) = c$, $\pi(f^2) = \sigma^2$. Let $\lceil\cdot\rceil$ be the ceiling function and $\widetilde{f}_k(x) = \left\lceil\frac{f(x)+c}{c/3k}\right\rceil \times \frac{c}{3k} - c$. Let $\widehat{f}_k = \frac{\widetilde{f}_k - \pi(\widetilde{f}_k)}{1+1/3k}$. Then $\widetilde{f}_k$ takes at most $6k+1$ possible values and satisfies that for any bounded linear operator T acting on the Hilbert Space $\mathcal{L}_2(\pi)$ and any $t \in \mathbb{R}$,*

$$\left\|\left|E^{tf/2}TE^{tf/2}\right|\right\|_\pi = \lim_{k\to\infty}\left\|\left|E^{t\widehat{f}_k/2}TE^{t\widehat{f}_k/2}\right|\right\|_\pi.$$

Assume that the Markov chain $\{\widehat{X}_i\}_{i\ge1}, \widehat{X}_i \in \mathcal{X}$ is generated by the León-Perron operator $\hat{P}_\lambda$. It follows that $\{\widehat{Y}_i\}_{i\ge1} = \{\hat{f}_k(\widehat{X}_i)\}_{i\ge1}$ is a Markov chain in the state space $\mathcal{Y} = \hat{f}_k(\mathcal{X})$. We recall the following lemma (Lemma 7, (Jiang et al., 2018)) on the relation between the two Markov chains.

Lemma C.4. *Let $\widehat{P}_\lambda$ be the León-Perron operator with $\lambda \in [0,1)$ on state space $\mathcal{X}$. Let f be a function on $\mathcal{X}$. On the finite state space $\mathcal{Y} = \{y \in f(\mathcal{X}) : \pi(\{x : f(x) = y\}) > 0\}$, define a transition matrix $\widehat{Q}_\lambda = \lambda I + (1-\lambda)I\mu^\top$, with transition vector μ consisting of elements $\pi(\{x : f(x) = y\})$ for $y in \mathcal{Y}$. Let $E^{t\mathcal{Y}}$ denote the diagonal matrix with elements $e^{ty} : y \in \mathcal{Y}$. Then we have,*

$$\left\|\left|E^{tf/2}\widehat{P}_\lambda E^{tf/2}\right|\right\|_\pi = \left\|\left|E^{t\mathcal{Y}/2}\widehat{Q}_\lambda E^{t\mathcal{Y}/2}\right|\right\|_\mu.$$

Next, we bound the term $\left|\left|\left|E^{t\mathcal{Y}/2}\widehat{Q}_\lambda E^{t\mathcal{Y}/2}\right|\right|\right|_\mu$ by the expansion of the largest eigenvalue of the perturbed Markov operator $E^{tf/2}PE^{tf/2}$ as a series in t. Specifically, we recall the following result (Lezaud, 1998).

Lemma C.5. *Consider a reversible, irreducible Markov chain on finite state space $\mathcal{X}$. Let D be the diagonal matrix with $\{f(x) : x \in \mathcal{X}\}$ and $T^{(m)} = PD^m/m!$ for any $m \geq 0$ with $D^0 = I$. Assume the invariant distribution of the Markov chain is π and the second largest eigenvalue of the transition matrix P is $\lambda_r < 1$. Let $t_0 = \left(2\left|\left|\left|T^{(1)}\right|\right|\right|_\pi (1-\lambda_r)^{-1} + c_0\right)^{-1}$ for some c_0 such that*

$$\left|\left|\left|T^{(m)}\right|\right|\right|_\pi \leq \left|\left|\left|T^{(1)}\right|\right|\right|_\pi c_0^{m-1}, \forall m \geq 1.$$

Denote the largest eigenvalue of PE^{tf} by $\beta(t)$ and $Z = (I - P + \Pi)^{-1} - \Pi$. Let $Z^0 = -\Pi$, $Z^{(j)} = Z^j,\ j \geq 1$, $\beta(0) = 1$ and $\beta(m),\ m \geq 1$ be

$$\beta^{(m)} = \sum_{p=1}^{m} \frac{-1}{p} \sum_{v_1+\cdots+v_p=m, v_i\geq 1, k_1+\cdots+k_p=p-1, k_j\geq 0} \operatorname{trace}\left(T^{(v_1)}Z^{(k_1)}\ldots T^{(v_p)}Z^{(k_p)}\right),$$

Then we have the following expansion on $\beta(t)$,

$$\beta(t) = \sum_{m=0}^{\infty} \beta^{(m)} t^m,\ |t| < t_0.$$

Follow the same line as in (Lezaud, 1998) (Page 854-856), denote $\sigma^2 = \|f\|_\pi^2$ and $c = c_0 \geq \left|\left|\left|D\right|\right|\right|_\pi$ (defined in Lemma C.5), then we have the following upper bound of $\beta(t)$.

$$\begin{aligned}
\beta(t) &= \beta^{(0)} + \beta^{(1)}t + \sum_{m=2} \beta^{(m)} t^m \\
&\leq 1 + 0 + \sum_{m=2}^{\infty} \frac{\pi\left(f^m\right)t^m}{m!} + \sum_{m=2}^{\infty} \frac{\sigma^2\lambda t}{5c}\left(\frac{5ct}{1-\lambda_{\mathrm{r}}}\right)^{m-1} \\
&\leq \exp\left(\sum_{m=2}^{\infty} \frac{\pi\left(f^m\right)t^m}{m!} + \sum_{m=2}^{\infty} \frac{\sigma^2\lambda t}{5c}\left(\frac{5ct}{1-\lambda_{\mathrm{r}}}\right)^{m-1}\right) \\
&\leq \exp\left(\frac{\sigma^2}{c^2}\left(e^{tc} - 1 - tc\right) + \frac{\sigma^2\lambda t^2}{1-\lambda_{\mathrm{r}} - 5ct}\right) \\
&:= \exp(g_1(t) + g_2(t))
\end{aligned} \tag{15}$$

Now we are ready to derive the bound for the term $\mathbf{E}\left[e^{t\sum_i^n f_i(X_i)}\right]$. Following the results in Lemma C.3, we consider a sequence of f_k such that,

$$\left|\left|\left|E^{tf/2}\widehat{P}_\lambda E^{tf/2}\right|\right|\right|_\pi = \lim_{k\to\infty} \left|\left|\left|E^{t\widehat{f}_k/2}\widehat{P}_\lambda E^{t\widehat{f}_k/2}\right|\right|\right|_\pi.$$

Next, we construct the finite state space counterpart of each pair of $E^{t\widehat{f}_k/2}\widehat{P}_\lambda E^{t\widehat{f}_k/2}$ and π by Lemma C.4, i.e.,

$$\left|\left|\left|E^{t\widehat{f}_k/2}\widehat{P}_\lambda E^{t\widehat{f}_k/2}\right|\right|\right|_\pi := \left|\left|\left|E^{t\mathcal{Y}_k/2}\widehat{Q}_\lambda E^{t\mathcal{Y}_k/2}\right|\right|\right|_{\mu_k}$$

Let the random variable in the state space $\mathcal{Y}_k$ be Y_k, then the mean and variance of Y_k is $\sum_{y\in\mathcal{Y}_k} \pi\left(\left\{x : \widehat{f}_k(x) = \mathcal{Y}\right\}\right) y = \pi\left(\widehat{f}_k\right) = 0$ and $\sum_{y\in\mathcal{Y}_k} \pi\left(\left\{x : \widehat{f}_k(x) = y\right\}\right) y^2 = \pi\left(\widehat{f}_k^2\right)$.

For each k, applying Eqn. (15) gives us,

$$\left|\left|\left|E^{t\mathcal{Y}_k/2}\widehat{Q}_\lambda E^{t\mathcal{Y}_k/2}\right|\right|\right|_{\mu_k} \leq \exp\left(\frac{\pi\left(\widehat{f}_k^2\right)}{c^2}\left(e^{tc}-1-tc\right)+\frac{\pi\left(\widehat{f}_k^2\right)\lambda t^2}{1-\lambda_r-5ct}\right)$$

Note that as $k \to \infty$, we have $\pi\left(\widehat{f}_k^2\right) \to \pi(f^2) = \sigma^2$. Then we have the upper bound for each operator $\left|\left|\left|E^{tf_i/2}PE^{tf_i/2}\right|\right|\right|_\pi$, i.e., for any $\lambda \in [0,1)$,

$$\left|\left|\left|E^{tf/2}P_\lambda E^{tf/2}\right|\right|\right|_\pi \leq \exp(g_1(t)+g_2(t))$$

where g_1 and g_2 are defined in Eqn. (15).

Consequently, we obtain the upper bound for $\mathbf{E}\left[e^{t\sum_i^n f_i(X_i)}\right]$ as follows, $\mathbf{E}\left[e^{t\sum_i^n f_i(X_i)}\right]$,

$$\mathbf{E}_\pi\left[e^{t\sum_{i=1}^n f_i(X_i)}\right] \leq \exp\left(\frac{n\sigma^2}{c^2}\left(e^{tc}-1-tc\right)+\frac{n\sigma^2\max\{\lambda_r,0\}t^2}{1-\max\{\lambda_r,0\}-5ct}\right)$$

Step 2 Use the convex analysis argument to derive the Bernstein's Inequality.

We first restate the following lemma (Lemma 9, (Jiang et al., 2018)) on the terms g_1 and g_2.

Lemma C.6. *For $\lambda \in [0,1)$, let $g_1(t) = \frac{n\sigma^2}{c^2}\left(e^{tc}-1-tc\right)$ and $g_2(t) = \frac{n\sigma^2\max\{\lambda_r,0\}t^2}{1-\max\{\lambda_r,0\}-5ct}$, then for any $0 \leq t < (1-\gamma)/5c$, the Frechet conjugates $(g_1+g_2)^*$ satisfy the following inequalities.*

$$\begin{aligned}
\text{if } \lambda \in (0,1): \quad & (g_1+g_2)^*(\epsilon) := \sup_{0\leq t<(1-\lambda)/5c}\{t\epsilon - g_1(t)-g_2(t)\} \geq \frac{\epsilon^2}{2}\left(\frac{1+\lambda}{1-\lambda}\sigma^2+\frac{5c\epsilon}{1-\lambda}\right)^{-1} \\
\text{if } \lambda = 0: \quad & (g_1+g_2)^*(\epsilon) = g_1^*(\epsilon) \geq \frac{\epsilon^2}{2}\left(\sigma^2+\frac{c\epsilon}{3}\right)^{-1}.
\end{aligned}$$

By the Chernoff bound, we have,

$$-\log\mathbf{P}\left(\frac{1}{n}\sum_{i=1}^n f_i(X_i) > \epsilon\right) \geq n \times \sup_{t\in\mathbb{R}}\{t\epsilon - g_1(t)-g_2(t)\}$$

Notice that $g_1(t) = O(t^2)$ and $g_2(t) = O(t^2)$ as $t \to 0$, then for some $t > 0$, we have $t\epsilon - g_1(t) - g_2(t) > 0$. Meanwhile, when $t \leq 0$, we have $t\epsilon - g_1(t) - g_2(t) \leq 0$. Thus, we can obtain that,

$$\sup\{t\epsilon - g_1(t)-g_2(t) : t > 0\} = \sup\{t\epsilon - g_1(t)-g_2(t) : t \in \mathbb{R}\} = (g_1+g_2)^*(\epsilon).$$

Letting $\lambda = \max\{\lambda_r, 0\}$, $\alpha_1(\lambda) = (1+\lambda)/(1-\lambda)$, $\alpha_2(\lambda) = 5/(1-\lambda)$ and $\alpha_2(0) = 1/3$ yields,

$$\mathbf{P}_\pi\left(\frac{1}{n}\sum_{i=1}^n f(X_i) > \epsilon\right) \leq \exp\left(-\frac{n\epsilon^2/2}{\alpha_1(\max\{\lambda_r,0\})\cdot\sigma^2+\alpha_2(\max\{\lambda_r,0\})\cdot c\epsilon}\right). \tag{16}$$

This concludes the proof.

□

D. Proof of Proposition 3.4

Let $\bar{\omega}_{t+1} = \Gamma_R(\tilde{\omega}_{t+1})$, and assume $\|\phi(s,a)\| \leq 1$ uniformly (see Assumption 3.1). Based on the approach in Appendix G.1 (Fu et al., 2020), it suffices to upper bound $\|\omega_{t+1} - \tilde{\omega}_{t+1}\|_2$. Observe that

$$\|\omega_{t+1} - \bar{\omega}_{t+1}\|_2 \leq \|\widehat{\Phi}\widehat{v} - \Phi v\|_2 \leq \|\Phi\|_2 \cdot \|\widehat{v} - v\|_2 + \|\widehat{\Phi} - \Phi\|_2 \cdot \|\widehat{v}\|_2,$$

where Φ and v are given as follows:

$$\begin{aligned}
\widehat{\Phi} &= \left(\frac{1}{N}\sum_{l=1}^{N} \phi\left(s_l, a_l\right)\phi\left(s_l, a_l\right)^\top\right)^{-1}, \\
\Phi &= \left(\mathbf{E}_{\rho_{t+1}}\left[\phi(s,a)\phi(s,a)^\top\right]\right)^{-1}, \\
\widehat{v} &= \frac{1}{N}\sum_{l=1}^{N}\left((1-\gamma)\sum_{i=0}^{m-1}\gamma^i r_{l,i} + \gamma^m Q_{\omega_t}\left(s_{l,m}, a_{l,m}\right)\right)\cdot\phi\left(s_{l,m}, a_{l,m}\right), \\
v &= \mathbf{E}_{\rho_{t+1}}\left[(1-\gamma)\sum_{i=0}^{m-1}\left(\gamma^i r_{l,i} + \gamma^m \boldsymbol{P}_{\pi_{\theta_{t+1}}} Q_{\omega_t}(s_m, a_m)\right)\cdot\phi(s_m, a_m)\right].
\end{aligned}$$

Recall that the following assumptions are in place: (1) Spectral norm $\|\phi(s,a)\|_2 \leq 1,\ \phi(s,a) \in \mathbb{R}^d$; (2) $|r(s,a)| \leq r_{\max}$ and $\bar{r} = \mathbf{E}_{s,a} r(s,a)$; (3) $\|\omega_t\|_2 \leq R$ and (4) the minimum singular value of the matrix $\mathbf{E}_{\rho_t}[\phi(s,a)\phi(s,a)^\top],\ t \geq 1$ is uniformly lower bounded by σ^*. It can be shown that $\|\Phi\|_2 \leq \frac{1}{\sigma^*}$.

Next, we derive the bound by appealing to Bernstein's Inequality with General Makovian samples. Following Theorem 2 (Jiang et al., 2018) (The proof of Bernstein's Inequality can be found in Appendix C), let π_r be the invariant distribution (which is relevant to the current policy π_k) of the stationary Markov chain $\{r_t\}_{t=1}^m$. Suppose that it has non-zero right spectral gap $1 - \lambda_r > 0$. Let $\sigma_r^2 = \int (r - \bar{r})^2 \pi_r(dr)$. Then, we have that for any $\epsilon > 0$:

$$\mathbf{P}_{\pi_r}\left(\frac{1}{m}\sum_{i=1}^{m}(r_i - \bar{r}) > \epsilon\right) \leq \exp\left(-\frac{m\epsilon^2/2}{\alpha_1\left(\max\{\lambda_r, 0\}\right)\cdot\sigma^2 + \alpha_2\left(\max\{\lambda_r, 0\}\right)\cdot r_{\max}\epsilon}\right),$$

where $\alpha_1(\lambda) = \frac{1+\lambda}{1-\lambda}, \quad \alpha_2(\lambda) = \begin{cases} \frac{1}{3} & \text{if } \lambda = 0 \\ \frac{5}{1-\lambda} & \text{if } \lambda \in (0,1) \end{cases}.$

We conclude that with probability at least $1 - p$,

$$\sum_{i=0}^{m-1} r_i \leq \frac{\sqrt{\alpha_2^2\left(\max\{\lambda_r, 0\}\right)^2 \ln p^2 - 2m\alpha_1\left(\max\{\lambda_r, 0\}\right)\ln p} - \alpha_2\left(\max\{\lambda_r, 0\}\right)\ln p}{m} + \bar{r} := \tilde{r}_m.$$

It follows that with probability at least $1 - p$,

$$\|\hat{v}\|_2 \leq (1-\gamma)\tilde{r}_m + \gamma^m R,$$

Further, note that

$$\|v\|_2 \leq (1-\gamma)\bar{r} + \gamma^m R,$$

Since the minimum singular value of $\hat{\Phi}^{-1}$ is no less than $\frac{\sigma^*}{2}$ w.h.p. when N is large enough, we have that

$$\|\hat{\Phi}\|_2 \leq \frac{2}{\sigma^*}.$$

For convenience, define

$$\hat{X} \triangleq \left(\frac{1}{N}\sum_{l=1}^{N}\phi\left(s_l,a_l\right)\phi\left(s_l,a_l\right)^\top\right), \;\; X \triangleq \left(\mathbf{E}_{\rho_{t+1}}\left[\phi(s,a)\phi(s,a)^\top\right]\right),$$

and define

$$Z \triangleq \hat{X} - X = \sum_{k=1}^{N} S_k, \tag{17}$$

$$S_k \triangleq \frac{1}{N}(\phi_k\phi_k^\top - X), \tag{18}$$

where $S_k, k = 1, \cdots, N$ are independent.

Next, we derive the uniform bound on the spectral norm of each summand as follows:

$$\|S_k\|_2 = \frac{1}{N}\|\phi_k\phi_k^\top - X\| \leq \frac{1}{N}(\|\phi_k\phi_k^\top\| + \|X\|) \leq \frac{2}{N}.$$

To this end, we bound the matrix variance statistic $V(Z)$:

$$V(Z) := \|\mathbf{E}[Z^2]\| = \|\sum_{k=1}^{N}\mathbf{E}[S_k^2]\|.$$

Note that the variance of each summand is given by

$$\begin{aligned}
\mathbf{E}[S_k^2] =& \frac{1}{N^2}\mathbf{E}[(\phi_k\phi_k^\top - X)^2] \\
=& \frac{1}{N^2}\mathbf{E}[\|\phi_k\|^2 \cdot \phi\phi^\top - \phi\phi^\top X - X\phi\phi^\top + X^2] \\
\preccurlyeq& \frac{1}{N^2}[\mathbf{E}[\phi\phi^\top] - X^2] \\
\preccurlyeq& \frac{1}{N^2}X.
\end{aligned}$$

Combining the above, we conclude that

$$0 \preccurlyeq \sum_{k=1}^{N}\mathbf{E}[S_k^2] \preccurlyeq \frac{1}{N}X.$$

Observe that

$$\|X\| = \|\mathbf{E}[\phi\phi^\top]\|_2 \leq \mathbf{E}[\|\phi\phi^\top\|] = \mathbf{E}\|\phi\|^2 \leq 1.$$

Since the spectral norm is the variance statistic given by

$$V(Z) \leq \frac{1}{N}\|X\|,$$

appealing to Bernstein's Inequality, we have that

$$\begin{aligned}
\mathbf{P}\{\|Z\| \geq t\} \leq& 2d\exp\left(\frac{\frac{-t^2}{2}}{\frac{1}{N}\|X\| + \frac{2t}{3N}}\right), \\
\mathbf{E}[\|Z\|] \leq& \sqrt{\frac{2}{N}\|X\|\log(2d)} + \frac{2}{3N}\log(2d) \\
\leq& \sqrt{\frac{2}{N}\log(2d)} + \frac{2}{3N}\log(2d).
\end{aligned}$$

This is to say, with probability at least $1-p/2$, the following holds:

$$\|X-\hat{X}\| \leq -\frac{2}{3N}\log\frac{p}{4d}+\sqrt{\frac{4}{9N^2}\log^2\frac{p}{4d}-\frac{2}{N}\log\frac{p}{4d}}.$$

In a nutshell, we have that

$$\begin{aligned}\|\widehat{\Phi}-\Phi\|_2 =&\|\hat{X}^{-1}-X^{-1}\|_2\\ =&\|\hat{X}^{-1}(\hat{X}-X)X^{-1}\|_2\\ =&\|\hat{\Phi}(\hat{X}-X)\Phi\|_2\\ \leq&\frac{2}{(\sigma^*)^2}\|\hat{X}-X\|_2\\ \leq&\frac{4}{\sqrt{N}\,(\sigma^*)^2}\cdot\left(-\frac{2}{3N}\log\frac{p}{4d}+\sqrt{\frac{4}{9N^2}\log^2\frac{p}{4d}-\frac{2}{N}\log\frac{p}{4d}}\right).\end{aligned}$$

Similarly, the following inequality holds with probability at least $1-p/2$:

$$\|\widehat{v}-v\|_2 \leq -\frac{\delta_1}{3}\log\frac{p}{2(d+1)}+\sqrt{\frac{\delta_1^2}{9}\log^2\frac{p}{2(d+1)}-2\delta_2\log\frac{p}{2(d+1)}},$$

where d is the dimension of vector φ, $\delta_1=\frac{1}{N}\left((1-\gamma)(\tilde{r}_m+\bar{r})+2\gamma^m R\right)$ and $\delta_2=\|\mathbf{E}[\hat{v}-v]\|_2$ satisfying

$$\begin{aligned}\delta_2 \leq&\frac{1}{N}\left[(1-\gamma)(|\tilde{r}_m|(|\tilde{r}_m-\bar{r}|+\gamma^m R|\tilde{r}_m-\bar{r}|))\right]\\ \leq&\frac{1-\gamma}{N}[r_{\max}+\gamma^m R]|\tilde{r}_m-\bar{r}|.\end{aligned}$$

Summarizing, we have that

$$\begin{aligned}\|\omega_{t+1}-\bar{\omega}_{t+1}\|_2 \leq&\|\Phi\|_2\cdot\|\widehat{v}-v\|_2+\|\widehat{\Phi}-\Phi\|_2\cdot\|\widehat{v}\|_2\\ \leq&-\frac{\delta_1}{3\sigma^*}\log\frac{p}{2(d+1)}+\sqrt{\frac{\delta_1^2}{9}\log^2\frac{p}{2(d+1)}-2\delta_2\log\frac{p}{2(d+1)}}\\ &+\frac{4((1-\gamma)\tilde{r}_m+\gamma^m R)}{\sqrt{N}(\sigma^*)^2}\left(-\frac{2}{3N}\log\frac{p}{4d}+\sqrt{\frac{4}{9N^2}\log^2\frac{p}{4d}-\frac{2}{N}\log\frac{p}{4d}}\right),\end{aligned}$$

which indicates that with probability at least $1-p$,

$$\begin{aligned}|Q_{\omega_{t+1}}-Q_{\bar{\omega}_{t+1}}| \leq&\left(\frac{4((1-\gamma)\tilde{r}_m+\gamma^m R)}{\sqrt{N}(\sigma^*)^2}\left(-\frac{2}{3N}\log\frac{p}{4d}+\sqrt{\frac{4}{9N^2}\log^2\frac{p}{4d}-\frac{2}{N}\log\frac{p}{4d}}\right)\right)\\ \triangleq&\epsilon_Q.\end{aligned}\tag{19}$$

Remark. In the case when Assumption 3.1 does not hold, i.e., we have

$$\inf_{\bar{\omega}\in\Omega}\mathbf{E}_{\rho^{\pi_\theta}}\left[\left((T^{\pi_\theta})^m Q_\omega-\bar{\omega}^\top\phi\right)(s,a)\right]=c_1,$$

where $c_1>0$ is a constant. Let $\bar{\omega}_{t+1}=\Gamma_R(\tilde{\omega}_{t+1})$, recall that $\tilde{\omega}$ denotes the solution of Eqn. (4) and ω denotes the sample-based solution, then we have

$$|Q_{\bar{\omega}_{t+1}}-Q_{\tilde{\omega}_{t+1}}|=c_1$$

From Eqn. (19), we obtain that,

$$|Q_{\omega_{t+1}} - Q_{\bar{\omega}_{t+1}}| \leq \epsilon_Q$$

Then the difference between the sample-based solution and the underlying true solution of Eqn. (4) is,

$$|Q_{\omega_{t+1}} - Q_{\tilde{\omega}_{t+1}}| \leq \epsilon_Q + c_1.$$

Note that when Assumption 3.1 holds,

$$Q_{\tilde{\omega}_{t+1}} = Q_{\bar{\omega}_{t+1}}.$$

E. Proof of Bounded Noise in the Actor Update

Based on Proposition 3.4, we have the following two lemmas for the upper bounds on the bias term $b = \mathbf{E}[C_{k,t}]$ and the error term $\beta = f_{k,t} + C_{k,t} - \mathbf{E}[C_{k,t}] - \mathbf{E}[f_{k,t}]$ in the stochastic gradient update Eqn. (8), respectively. The proof of Lemmas E.1 and F.1 can be found in Appendix E and F, respectively.

Lemma E.1 (σ^2-bounded noise)**.** *Suppose Assumptions 3.1, 3.2, 3.3 hold. Then with probability at least* $1-p$*,* $\mathbf{E}[\|\beta\|^2] \leq \|\nabla_\theta h(\omega,\theta) + b\|^2 + \sigma^2,\ \forall\theta$*, where* $\sigma^2 \geq 0$ *is a constant and depends on* p*.*

Recall $\beta = f_{k,t} + C_{k,t} - \mathbf{E}[C_{k,t}] - \mathbf{E}[f_{k,t}]$. We also have the following definitions:

$$\begin{aligned}
C_{k,t,1} \triangleq& 1/l \sum_{i=1}^{l} (Q_{\omega_{t+1}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i) - Q_{\tilde{\omega}_{t+1}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i)),\\
C_{k,t,2} \triangleq& 1/l \sum_{i=1}^{l} (Q_{\tilde{\omega}_{t+1}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i) - Q^{\pi_{\theta_t}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i)),\\
f_{k,t} \triangleq& 1/l \sum_{i=1}^{l} Q^{\pi_{\theta_t}}(s_i,a_i)\nabla_\theta \pi_{\theta_k}(a_i|s_i),\\
C_{k,t} \triangleq& C_{k,t,1} + C_{k,t,2}.
\end{aligned}$$

Next we evaluate $\mathbf{E}[\|f_{k,t} + C_{k,t} - \mathbf{E}[C_{k,t}] - \mathbf{E}[f_{k,t}]\|^2]$ as follows:

$$\begin{aligned}
&\|f_{k,t} + C_{k,t} - \mathbf{E}[C_{k,t}] - \mathbf{E}[f_{k,t}]\|^2\\
=&(f_{k,t} + C_{k,t})(f_{k,t} + C_{k,t})^\top + (\mathbf{E}[C_{k,t}] + \mathbf{E}[f_{k,t}])(\mathbf{E}[C_{k,t}] + \mathbf{E}[f_{k,t}])^\top\\
&- 2(f_{k,t} + C_{k,t})(\mathbf{E}[C_{k,t}] + \mathbf{E}[f_{k,t}])^\top\\
\leq&(f_{k,t} + C_{k,t})(f_{k,t} + C_{k,t})^\top + (\mathbf{E}[C_{k,t}] + \mathbf{E}[f_{k,t}])(\mathbf{E}[C_{k,t}] + \mathbf{E}[f_{k,t}])^\top. \qquad (20)
\end{aligned}$$

Note that $C_{k,t}$ and $f_{k,t}$ are both bounded above since Q-function is bounded and $\nabla_\theta \pi_\theta(a|s)$ is bounded (see Assumption 3.5), i.e.,

$$\begin{aligned}
\|\nabla \pi(a|s)\| \leq& C_\psi,\\
\|Q(s,a)\| \leq& \sum_{t=1}^{\infty} \gamma^t r_{\max} = \frac{r_{\max}}{1-\gamma}.
\end{aligned}$$

Then we have the following bounds for $C_{k,t}$ and $f_{k,t}$:

$$\begin{aligned}
\|C_{k,t}\| &\leq 2C_\psi \frac{r_{\max}}{1-\gamma},\\
\|f_{k,t}\| &\leq C_\psi \frac{r_{\max}}{1-\gamma}.
\end{aligned}$$

Then we have

$$\begin{aligned}(f_{k,t}+C_{k,t})(f_{k,t}+C_{k,t})^\top \leq & \|f_{k,t}\|^2+\|C_{k,t}\|^2+2\|f_{k,t}\|\|C_{k,t}\| \\ \leq & 9C_\psi^2(\frac{r_{\max}}{1-\gamma})^2\end{aligned}$$

Taking expectation over both sides of the inequality (20), we have that

$$\mathbf{E}[\|\beta\|^2] \leq 1\cdot\|\mathbf{E}[C_{k,t}]+\mathbf{E}[f_{k,t}]\|^2+\mathbf{E}[(f_{k,t}+C_{k,t})(f_{k,t}+C_{k,t})^\top].$$

Let $M_n=1$ and $\sigma^2=9C_\psi^2(\frac{r_{\max}}{1-\gamma})^2$. Then we have that

$$\mathbf{E}[\|\beta\|^2] \leq M_n\cdot\|\nabla_\theta h(\omega,\theta)+\mathbf{E}[C_{k,t}]\|+\sigma^2.$$

F. Proof of Bounded Bias in the Actor Update

Lemma F.1 (ζ^2-bounded bias)**.** *Suppose Assumptions 3.1, 3.2, 3.3 hold. Then with probability at least* $1-p$, $\|b\|^2\leq\zeta^2,\ \forall\theta$, *where* $\zeta^2\geq 0$ *is a constant and depends on* p.

Recall that $b=\mathbf{E}[C_{k,t}]$ and

$$\begin{aligned}C_{k,t}:=&C_{k,t,1}+C_{k,t,2}\\ =&1/l\sum_{i=1}^{l}\Big(Q_{\omega_{t+1}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)-Q_{\widetilde{\omega}_{t+1}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)+\\ &(Q_{\widetilde{\omega}_{t+1}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)-Q^{\pi_{\theta_t}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)\Big).\end{aligned}$$

Next, we evaluate $\|b\|^2$. Observe that (see also Appendix E)

$$\|Q_{\widetilde{\omega}_{t+1}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)-Q^{\pi_{\theta_t}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)\|\leq 2C_\psi\frac{r_{\max}}{1-\gamma}.$$

Meanwhile, recall the results from Proposition 3.4 Eqn. 19, we have that, for any $(s,a)\in\mathcal{S}\times\mathcal{A}$,

$$\|Q_\omega-Q_{\tilde{\omega}}\|\leq\epsilon_Q.$$

Then we have,

$$\|Q_{\omega_{t+1}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)-Q_{\widetilde{\omega}_{t+1}}(s_i,a_i)\nabla_\theta\pi_{\theta_k}(a_i|s_i)\|\leq c_\psi\epsilon_Q,$$

where ϵ_Q depends on p.

Let $\zeta=c_\psi\epsilon_Q+2C_\psi\frac{r_{\max}}{1-\gamma}$. Then we have

$$\|b\|^2=\|\mathbf{E}[C_{k,t}]\|^2\leq\mathbf{E}[\|C_{k,t}\|^2]\leq\zeta^2.$$

G. Proof of the Smoothness and PL Condition of h in Actor Update

For the sake of tractability, we next give the following two lemmas about the smoothness and Polyak-Lojasiewicz Condition on the objective function $h(\cdot,\theta)$.

Lemma G.1 (L-smoothness)**.** *Suppose Assumption 3.5 hold. Then function* $h(\cdot,\theta)$ *is bounded from below by an infimum* $h^{\inf}\in\mathbb{R}$, *differentiable and* ∇h *is L-Lipschitz, i.e.,* $\|\nabla h(\omega,\theta)-\nabla h(\omega,\theta')\|\leq L\|\theta-\theta'\|,\ \forall\,\omega,\theta,\theta'$.

Lemma G.2 (μ-PL)**.** *If* $\nabla h(\omega,\theta)\neq 0$, *then we have* $\frac{1}{2}\|\nabla h(\omega,\theta)\|\geq\mu(h(\omega,\theta^*)-h(\omega,\theta))\geq 0,\forall\,\theta,\omega$.

- [Lemma 3] Given Critic parameter ω in the objective function, it can be seen that $\|\nabla h(\omega,\theta)-\nabla h(\omega,\theta')\|\leq Q_{\max}\|\nabla\pi_\theta-\nabla\pi_{\theta'}\|$. Since value function is bounded (e.g., $Q_{\max}$) and the score function $\nabla\pi_\theta$ is L_ψ-smooth (ref. Assumption 6), the constant in Assumption 4 can be easily determined by $L=Q_{\max}L_\psi$.
- [Lemma 4] Since the objective function is finite, let $h_{\max}=\max_{\theta\neq\theta^*}h(\theta,\omega), h^*_{\max}=\max_{\theta=\theta^*}h(\theta,\omega)$,. In the case when the gradient is non-zero, let $g_{\min}=\min_{\theta\neq\theta^*}\nabla h$, then we can determine $\mu=\frac{g_{\min}}{h^*_{\max}-h_{\max}}\geq 0$.

H. Proof of Proposition 3.6

Observe that the Actor updates use the biased stochastic gradient methods (SGD). For simplicity, we adopt the following notations to study the Actor update:

$$\theta_{k+1} = \theta_k + \alpha(\nabla h(\omega, \theta_k) + b(t) + \beta(t)). \tag{21}$$

where $b(t) = \mathbf{E}[C_{k,t}]$ is the bias, α is the step size, and

$$\beta = f_{k,t} + C_{k,t} - \mathbf{E}[C_{k,t}] - \mathbf{E}[f_{k,t}]$$

is the zero-mean noise. Note that the objective function $h(\omega, \theta_k)$ is a function of θ. Denote the optimal value (in this iteration of the Actor update) by $h(\omega, \theta^*)$.

We prove the following lemma on the modified version of the descent lemma for smooth function (cf. (Ajalloeian & Stich, 2020; Nesterov, 2003)).

Lemma H.1. *Suppose Assumption G.1 and G.2 hold. Then, for any stepsize $\alpha \leq \frac{1}{(M_n+1)L}$, the following inequality holds:*

$$\mathbf{E}[h(\omega, \theta_{k+1}) - h(\omega, \theta_k)|\theta_k] \leq \frac{\alpha}{2}\zeta^2 + \frac{\alpha^2 L}{2}\sigma^2 - \frac{\alpha}{2}\|\nabla h(\omega, \theta_k)\|^2.$$

Observe that under the PL-condition (Assumption G.2), we have the following recursion:

$$\mathbf{E}[h(\omega, \theta_{k+1}) - h(\omega, \theta^*)|\theta_k] \leq (1 - \alpha\mu)\mathbf{E}[h(\omega, \theta_k) - h(\omega, \theta^*)] + \frac{\alpha}{2}\zeta_p^2 + \frac{\alpha^2 L}{2}\sigma^2, \tag{22}$$

where $\zeta_p = c_\psi \epsilon_p + 2C_\psi \frac{r_{\max}}{1-\gamma}$ is defined in Lemma F.1 and depends on p.

By applying Eqn. (22) recursively, we obtain the desired results in Proposition 3.6.

$$\mathbf{E}_\theta[\|h(\omega, \theta_t^*) - h(\omega, \theta_t)\||\theta_{t-1}] \leq (1 - \alpha\mu)^{N_a}(h(\omega, \theta_t^*) - h(\omega, \theta_{t-1})) + \frac{\zeta_p^2 + 2\alpha L\sigma^2}{2\mu},$$

I. Proof of Proposition 4.1

We first prove the following lemma on the relation between Actor parameter θ and the objective function $h(\omega, \theta)$.

Lemma I.1. *There exist a contant $L_h > 0$ and an open ball $S_\epsilon(\theta_t^*)$ such that for any $\theta_t \in B_\epsilon(\theta_t^*)$ the following holds for any $t > 0$.*

$$\mathbf{E}[\|\pi_{\theta_t} - \pi^*\|_{\mathrm{TV}}] \leq L_h \mathbf{E}[h(\omega, \theta_t^*) - h(\omega, \theta_t)].$$

Proof. By Taylor's expansion, we have

$$h(\omega, \theta^*) = h(\omega, \theta_t) + \nabla h(\omega, \theta_t)(\theta_t^* - \theta_t) + o(\|\theta_t^* - \theta_t\|).$$

Since $h(\omega, \cdot)$ satisfies Polyak-Lojasiewicz Condition, it follows that

$$\|\nabla h(\omega, \theta)\| \geq 2\mu(h(\omega, \theta^*) - h(\omega, \theta)) := L_g \text{ for all } \theta.$$

Note that $L_g > 0$ when $\theta \neq \theta^*$. Then we have that

$$\begin{aligned} h(\omega, \theta_t^*) - h(\omega, \theta_t) =& |\nabla h(\omega, \theta_t)(\theta^* - \theta_t) + o(\|\theta^* - \theta_t\|)| \\ \geq& |\nabla h(\omega, \theta_t)(\theta_t^* - \theta_t)| - |o(\|\theta^* - \theta_t\|)| \\ \geq& L_g\|\theta_t^* - \theta_t\| - L_o\|\theta_t^* - \theta_t\| \\ =& (L_g - L_o)\|\theta_t^* - \theta_t\|, \end{aligned}$$

where the last inequality uses the fact that there exists ϵ such that when $\|\theta_t - \theta_t^*\| \le \epsilon$,

$$|o(\|\theta_t^* - \theta_t\|)| \le L_o\|\theta_t^* - \theta_t\|, \;\; L_o < L_g.$$

Taking expectation over both sides gives

$$\begin{aligned}\mathbf{E}[h(\omega,\theta_t^*) - h(\omega,\theta_t)] =&(L_g - L_o)\mathbf{E}[\|\theta_t^* - \theta_t\|]\\ \ge&(L_g - L_o)\|\mathbf{E}[\theta_t^* - \theta_t]\|.\end{aligned}$$

Then we conclude that the parameter of interest L_h,

$$L_h = \frac{C_\pi}{L_g - L_o} > 0.$$

where C_π is defined in Assumption 3.5. □

We are ready to present the proof of Proposition 4.1. Based on the definition of $\mathcal{E}_{\hat{J},t}$ and $\mathcal{E}_{\hat{T},t}$, we derive the upper bound for each term respectively.

$$\begin{aligned}\mathcal{E}_{\hat{J},t} =&(\boldsymbol{I} - \gamma \boldsymbol{P}_{\hat{\pi}_{t+1}})^{-1} - (\boldsymbol{I} - \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}})^{-1}\\ =&(\boldsymbol{I} - \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}})^{-1}\left(\gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}} - \gamma \boldsymbol{P}_{\hat{\pi}_{t+1}}\right)(\boldsymbol{I} - \gamma \boldsymbol{P}_{\hat{\pi}_{t+1}})^{-1}.\end{aligned}$$

Observe that value function $\boldsymbol{v}$ is smooth and upper bounded. We denote the smoothness parameter by L_v, the upper bound by $\|\boldsymbol{v}\| \le V^{\max}$, and the smoothness of the reward function by L_r.

By taking the norm of both sides and applying Assumption 3.3, 3.5 and 4.2, we obtain

$$\|\mathcal{E}_{\hat{J},t}\| \le M^2 L_J L_v \|\widetilde{\pi}_{t+1} - \hat{\pi}_{t+1}\|_{\mathrm{TV}}.$$

Further, observe that

$$\begin{aligned}\mathcal{E}_{\hat{T},t} &= \boldsymbol{r}_{\hat{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\hat{\pi}_{t+1}}\boldsymbol{v}^{\hat{\pi}_t} - (\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}\boldsymbol{v}^{\hat{\pi}_t}),\\ &= \boldsymbol{r}_{\hat{\pi}_{t+1}} - \boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma(\boldsymbol{P}_{\hat{\pi}_{t+1}} - \boldsymbol{P}_{\widetilde{\pi}_{t+1}})\boldsymbol{v}^{\hat{\pi}_t}.\end{aligned}$$

By taking the norm of both sides and applying Assumption 4.2, we obtain

$$\begin{aligned}\|\mathcal{E}_{\hat{T},t}\| =& \|\boldsymbol{r}_{\hat{\pi}_{t+1}} - \boldsymbol{r}_{\widetilde{\pi}_{t+1}}\| + \|\gamma(\boldsymbol{P}_{\hat{\pi}_{t+1}} - \boldsymbol{P}_{\widetilde{\pi}_{t+1}})\boldsymbol{v}^{\hat{\pi}_t}\|\\ \le&(L_r + \gamma V^{\max})\|\widetilde{\pi}_{t+1} - \hat{\pi}_{t+1}\|_{\mathrm{TV}}\\ :=&L_T^{\max}.\end{aligned}$$

Recall the definition of $\mathcal{E}_t$ is given as

$$\mathcal{E}_t = -\left(\mathcal{E}_{\hat{J},t}(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})) + \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\mathcal{E}_{\hat{T},t} + \mathcal{E}_{\hat{T},t}\mathcal{E}_{\hat{J},t}\right).$$

Taking the norm and expectation on both sides yields that

$$\begin{aligned}\|\mathbf{E}[\mathcal{E}_t]\| \le \mathbf{E}[\|\mathcal{E}_t\|] =&\mathbf{E}\left[\|\mathcal{E}_{\hat{J},t}(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})) + \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\mathcal{E}_{\hat{T},t} + \mathcal{E}_{\hat{T},t}\mathcal{E}_{\hat{J},t}\|\right]\\ \le&L_{\mathcal{E}}\mathbf{E}[\|\widetilde{\pi}_{t+1} - \hat{\pi}_{t+1}\|_{\mathrm{TV}}],\end{aligned}$$

where $L_{\mathcal{E}} = (2V^{\max}K + L_T^{\max})M^2 L_v L_J + M(L_r + \gamma V^{\max}) > 0$ is a constant. Since $\widetilde{\pi}_{t+1} = \pi_{t+1}^*$ is the greedy solution, we thereby complete the proof of Proposition 4.1.

J. Proof of Corollary 4.4

Based on the update rule of the value function, we have

$$\begin{aligned}\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}} =& \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) + \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})\right) + \mathcal{E}_t \\ \leq& \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) - \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\boldsymbol{J}_{\boldsymbol{v}^*}(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) - \mathcal{E}_t \\ \leq& \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} - \boldsymbol{J}_{\boldsymbol{v}^*}\right](\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) + \mathcal{E}_t,\end{aligned}$$

which implies that

$$\mathbf{E}_{\hat{\pi}_{t+1}}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}|\boldsymbol{v}^{\hat{\pi}_t}] \leq \mathbf{E}_{\hat{\pi}_{t+1}}[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}]\left[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} - \boldsymbol{J}_{\boldsymbol{v}^*}\right](\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) + \mathcal{B}(t).$$

Then, taking expectation over $\hat{\pi}_t$ on both sides gives us,

$$\mathbf{E}_{\hat{\pi}_{t+1},\hat{\pi}_t}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}|\boldsymbol{v}^{\hat{\pi}_t}] \leq \mathbf{E}_{\hat{\pi}_{t+1},\hat{\pi}_t}[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} - \boldsymbol{J}_{\boldsymbol{v}^*}\right]]\mathbf{E}_{\hat{\pi}_t}[(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t})] + \mathcal{B}(t) \tag{23}$$

Let $J_t := \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} - \boldsymbol{J}_{\boldsymbol{v}^*}\right]$. It follows from Assumption 4.2 that

$$\|J_t\| \leq ML_J\|\boldsymbol{v}^{\hat{\pi}_t} - \boldsymbol{v}^*\|^q := L\|\boldsymbol{v}^{\hat{\pi}_t} - \boldsymbol{v}^*\|^q.$$

where $L = ML_J$ and L_J is defined in Assumption 4.2.

Meanwhile, we have,

$$\begin{aligned}\|\mathbf{E}[J_t]\| \leq& \mathbf{E}[\|J_t\|] \\ \leq& L\mathbf{E}[\|\boldsymbol{v}^{\hat{\pi}_t} - \boldsymbol{v}^*\|^q] \\ \leq& L\|\mathbf{E}[\boldsymbol{v}^{\hat{\pi}_t} - \boldsymbol{v}^*]\|^q,\end{aligned}$$

where the last inequality follows Jensen's inequality.

Then, taking norm on both sides of the inequality 23 gives

$$\begin{aligned}\|\mathbf{E}_{\hat{\pi}_{t+1},\hat{\pi}_t}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}|\boldsymbol{v}^{\hat{\pi}_t}]\| \leq& \|\mathbf{E}_{\hat{\pi}_{t+1},\hat{\pi}_t}[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} - \boldsymbol{J}_{\boldsymbol{v}^*}\right]]\mathbf{E}_{\hat{\pi}_t}[(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t})] + \mathcal{B}(t)\| \\ \leq& \|\mathbf{E}_{\hat{\pi}_{t+1},\hat{\pi}_t}[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left[\boldsymbol{J}_{\hat{\boldsymbol{v}}_t} - \boldsymbol{J}_{\boldsymbol{v}^*}\right]]\|\|\mathbf{E}_{\hat{\pi}_t}[(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t})]\| + \|\mathcal{B}(t)\| \\ =& \|\mathbf{E}_{\hat{\pi}_{t+1},\hat{\pi}_t}[J_t]\|\|\mathbf{E}_{\hat{\pi}_t}[(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t})]\| + \|\mathcal{B}(t)\| \\ \leq& L\|\mathbf{E}_{\hat{\pi}_t}[(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t})]\|^{1+q} + \|\mathcal{B}(t)\|\end{aligned}$$

Let $a_t = \|\mathbf{E}_{\hat{\pi}_t}[(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t})]\|$ and $b_t = \|\mathcal{B}(t)\|$. Then we have the following recursive inequality,

$$a_{t+1} \leq La_t^{1+q} + b_t, \quad t = 0, 1, \cdots \tag{24}$$

Staring from $t = 0$, we have,

$$a_1 \leq La_0^{1+q} + b_0$$

Let $b_0 = u_0 a_0^{1+q}$, where $u_0 = \frac{L_b H_t}{a_0^{1+q}}$, then we have,

$$a_1 \leq (L + u_0)a_0^{1+q}$$

Similarly, let $t = 1$ and $b_1 = u_1 a_0^{(1+q)^2}$ with $u_0 = \frac{L_b H_t}{a_0^{(1+q)^2}}$. Then we have,

$$a_2 \leq (L(L + u_0)^{1+q} + u_1)a_0^{(1+q)^2}$$

By applying Eqn. (24) recursively, we conclude that

$$\begin{aligned}\|\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}]\| \leq \|\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}\|^{(1+q)^{1+t}} \\ \cdot (L \cdots ((L+u_1)^{1+q} + u_2)^{1+q} \cdots + u_t),\end{aligned}$$

where $u_t := \frac{L_b H_t}{\|\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}\|^{(1+q)(1+t)}}$ and $L_b H_t$ is the upper bound of the bias as in

K. Proof of Theorem 4.5

Following the value function update rule, we have

$$\begin{aligned}\boldsymbol{v}^{\hat{\pi}_{t+1}} &= \boldsymbol{v}^{\hat{\pi}_t} - \left(\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})\right) + \mathcal{E}_t\right) \\ &= \boldsymbol{v}^{\hat{\pi}_t} - (L(t) + \mathcal{E}_t) \\ &:= \boldsymbol{v}^{\hat{\pi}_t} - \hat{\mathcal{L}}(t).\end{aligned}$$

Then, the difference between $\boldsymbol{v}^{\hat{\pi}_{t+1}}$ and $\boldsymbol{v}^*$ is given by

$$\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}} = \boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t} + \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})\right) + \mathcal{E}_t. \tag{25}$$

Observe the following result holds for any $\hat{\pi}_t$,

$$(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})) - \underbrace{(\boldsymbol{v}^* - T(\boldsymbol{v}^*))}_{=0} \geq \boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^2(\boldsymbol{v}^{\hat{\pi}_t} - \boldsymbol{v}^*). \tag{26}$$

Recall our decomposition of the value function update is given as

$$\hat{\mathcal{L}}(t) = \mathcal{L}(t) + \underbrace{\hat{\mathcal{L}}(t) - \mathbf{E}[\hat{\mathcal{L}}(t)]}_{\text{Martingale Difference Noise: } \mathcal{N}(t)} + \underbrace{\mathbf{E}[\hat{\mathcal{L}}(t)] - \mathcal{L}(t)}_{\text{Bias: } \mathcal{B}(t)}.$$

Plugging Eqn. (26) into Eqn. (25), we obtain

$$\begin{aligned}\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}} &= \boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t} + \left(\boldsymbol{J}_{\hat{\boldsymbol{v}}_t}^{-1}\left(\boldsymbol{v}^{\hat{\pi}_t} - T(\boldsymbol{v}^{\hat{\pi}_t})\right) + \mathcal{E}_t\right) \\ &\geq (\boldsymbol{I} - \boldsymbol{J}_{\boldsymbol{v}^{\hat{\pi}_t}}))\,(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) + \mathcal{B}(t) + \mathcal{N}(t) \\ &= \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) + \mathcal{B}(t) + \mathcal{N}(t).\end{aligned}$$

Taking expectation on both sides yields that

$$\mathbf{E}[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}} | \boldsymbol{v}^{\hat{\pi}_t}] \geq \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}}(\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_t}) + \mathcal{B}(t).$$

Applying the above inequality recursively gives that

$$\begin{aligned}\mathbf{E}\left[\boldsymbol{v}^* - \boldsymbol{v}^{\hat{\pi}_{t+1}}\right] \geq& \gamma^{t+1}\mathbf{E}\left[\left(\prod_{i=0}^{t} \boldsymbol{P}_{\widetilde{\pi}_{t+1-i}}\right)\right](\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}) \\ &+ \sum_{i=1}^{t} \gamma^i \mathbf{E}\left[\left(\prod_{j=0}^{i-1} \boldsymbol{P}_{\widetilde{\pi}_{t+1-j}}\right)\right](\mathcal{B}(t-i)) + \mathcal{B}(t) \\ :=& \gamma^{t+1}\bar{\boldsymbol{P}}_{t+1}(\boldsymbol{v}^* - \boldsymbol{v}^{\pi_0}) + \sum_{i=1}^{t} \gamma^i \bar{\boldsymbol{P}}_i \mathcal{B}(t-i) + \mathcal{B}(t),\end{aligned} \tag{27}$$

with $\bar{\boldsymbol{P}}_{t+1} = \mathbf{E}\left[\left(\prod_{i=0}^{t} \boldsymbol{P}_{\widetilde{\pi}_{t+1-i}}\right)\right]$. Taking norm on both sides of Eqn. (27) yields the desired results.

L. Experiments

Empirical Results. We consider experiments over the Gridworld benchmark task. In particular, we consider the following sizes of the grid to represent different problem complexity, i.e., 10×10, 15×15 and 20×20. The goal of the agent is to find a way (policy) to travel from a specified start location, e.g., the red square in Fig. 3, to an assigned target location, e.g., the red hexagram in Fig. 3, such that the (discounted) accumulative reward along the way is maximized. Specifically, the action space contains 4 discrete actions, namely, up, down, left, right, which are represented as 1,2,3,4 in the algorithm, respectively. The reward in the goal state is defined as 10 and in the bad state , e.g., the black cube in Fig. 3, is -6. The rest of the states result in the reward -1. The discounting factor is set as $\gamma = 0.9$. We consider the grid with 10 rows and 10 columns such that the state space has 100 states. The transition properties of the environment is as follows: the agent will transfer to next state following the chosen action with probability 0.7; the agent will go left of the desired action with probability 0.15 and go right with with probability 0.15. For each experiment, the shaded area represents a standard deviation of the average evaluation over 5 training seeds.

Specifically, we consider the following A-C algorithm to solve the Gridworld benchmark task,

Critic Update: The Critic updates its value by applying the Bellman evaluation operator (T^π) for m-times ($m \geq 1$), i.e., given policy π, at the t-th step A-C update,

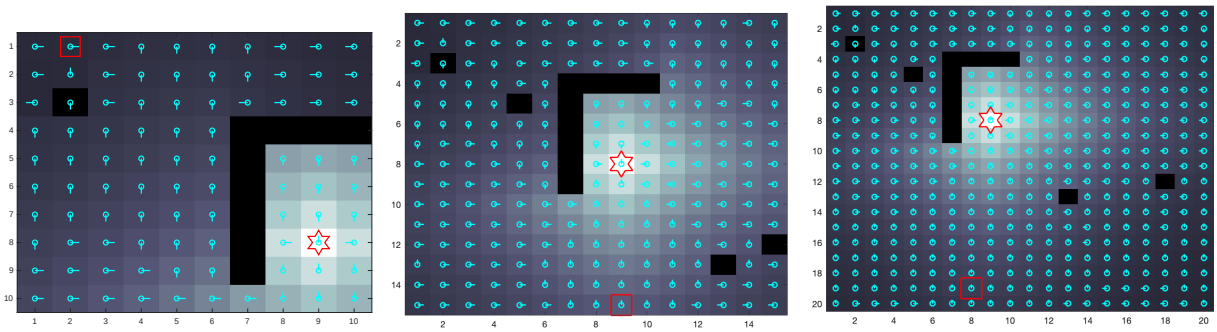
$$\boldsymbol{v}(t+1) = (T^\pi)^m(\boldsymbol{v}(t)). \tag{28}$$

Actor Update: The Actor updates the policy by a greedy step to maximize the learnt $\boldsymbol{v}$ value, i.e.,

$$\pi' = \arg\max_{\pi} T^\pi(\boldsymbol{v}(t+1)). \tag{29}$$

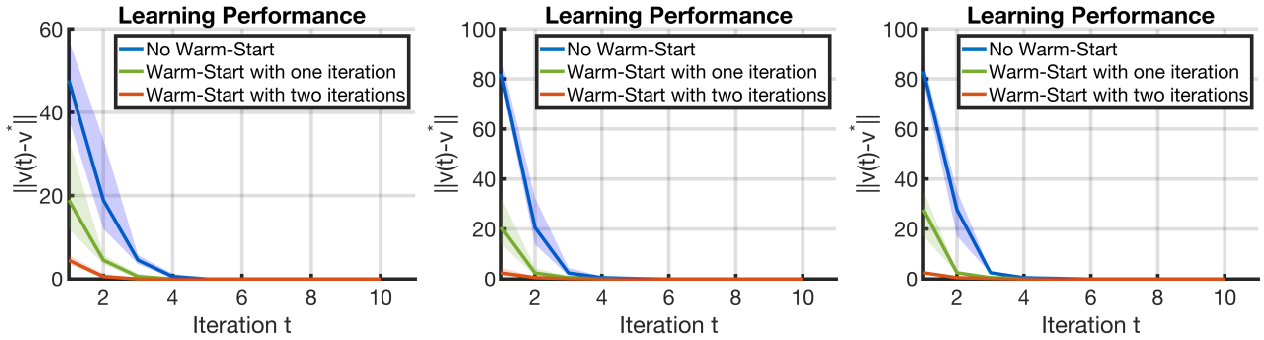
Impact of the Warm-Start Policy. We first consider the impact of the Warm-Start policy in the ideal setting, where both the Critic update and Actor update is nearly accurate as in ADP. In this case, we let m be large enough, e.g., $m = 1000$, in the Critic update Eqn. (28). As observed in Fig. 4, a 'good' Warm-Start policy can efficiently accelerate the learning process, e.g., it only takes two iterations to convergence with a Warm-Start policy. Meanwhile, in all three cases, the performance gap $\|\boldsymbol{v}(t) - \boldsymbol{v}^*\|$ decays over time which reflects our discovery in Corollary 4.4. Specifically, when the Warm-Start policy is not 'good' enough (or even no Warm-Start), the A-C algorithm can still be able to improve the learning performance overtime (see e.g., the first term on the right side of the upper bound in Corollary 4.4).

Impact of the Approximation Error in the Critic Update. We evaluate the impact of the approximation error in the Critic update on the convergence behavior by two approaches. (1) First, we study the Critic update with finite time Bellman evaluation, e.g., $m = 500, 50, 20, 5$. As shown in Fig. 5, the inaccurate Critic update impacts the convergence behavior as expected. The case when $m = 5$ shows that the finite time Bellman evaluation may contribute to the slower convergence. (2) Next, we consider the general case when there is approximation error in the Critic update. In particular, we add the uniform



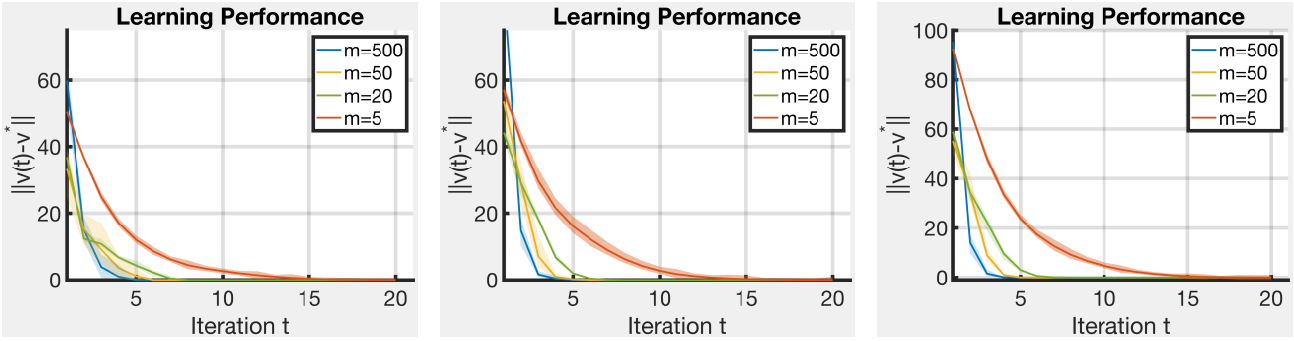
(a) 10×10 Gridworld. (b) 15×15 Gridworld. (c) 20×20 Gridworld.

Figure 3. Gridworld benchmark with different sizes. The colors specify the 'goodness' measure of the state, i.e., the darker color cubes are with lower $v(s)$ value and the agent should avoid those areas. The horizontal lines and vertical lines in each cube point to the direction the agent should take, i.e., policy at every state. Fig. 3(a), Fig. 3(b) and Fig. 3(c) show the learning results after 50 iterations of A-C update.



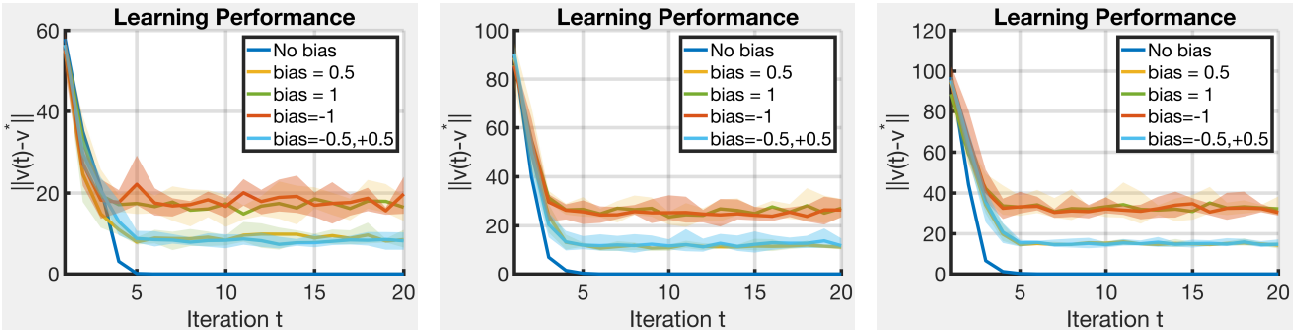
(a) 10×10 Gridworld. (b) 15×15 Gridworld. (c) 20×20 Gridworld.

Figure 4. The impact of the Warm-Start Policy when no approximation errors in Actor update and Critic update. The convergence behavior given different initial policy, i.e., a random policy (no Warm-Start), a Warm-Start policy obtained by running the A-C algorithm for one iteration and two iterations. The x-axis represents the A-C update step and y-axis is the value of the norm $\|\boldsymbol{v}(t) - \boldsymbol{v}^*\|$.



(a) 10×10 Gridworld. (b) 15×15 Gridworld. (c) 20×20 Gridworld.

Figure 5. Learning performance vs. rollout length.

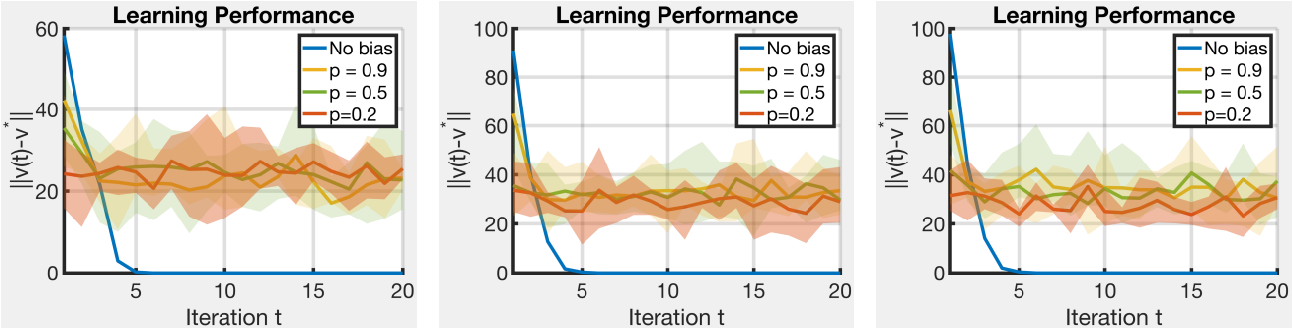


(a) 10×10 Gridworld. (b) 15×15 Gridworld. (c) 20×20 Gridworld.

Figure 6. Illustration of the lower bound in Theorem 3.4.

noise $e(t)$ in the value function with different bias, e.g.,$\mathbf{E}[e(t)] = 0, 0.5, 1, -1$. Meanwhile, we also consider the case when the bias can be either $+0.5$ or -0.5 in the learning process, e.g., $\mathbf{E}[e(t)] = 0.5$ with probability 0.5 and $\mathbf{E}[e(t)] = -0.5$ with probability 0.5. The resulting convergence behavior is presented in Fig. 6. Notably, it can be clearly seen that both the positive and negative bias may result in an error floor and 'prevent' the algorithm from converging to the optimal (e.g., the last two terms of the lower bound in Theorem 4.5). The experiment results in Fig. 6 corroborate our theoretical findings in Proposition 3.4, Corollary 4.4 and Theorem 4.5.

Impact of the Approximation Error in the Actor Update. We investigate the learning performance of the A-C algorithm



(a) 10×10 Gridworld. (b) 15×15 Gridworld. (c) 20×20 Gridworld.

Figure 7. Convergence behavior vs. Approximation Error in the Actor Update.

under inaccurate Actor update. In particular, we add the perturbation on the learnt policy in Eqn. (29) as follows,

$$\text{Policy}(s) = \begin{cases} \text{Policy}(s), & p, \\ \text{randi}([1,4]), & 1-p. \end{cases}$$

where $\text{Policy}(s)$ denotes the action should the agent take at the current state s following the learnt policy and $\text{randi}([1,4])$ is a random function to choose the action $1, 2, 3, 4$ uniformly. Thus, with probability p, the agent will choose the action follow the current policy while with probability $1-p$, the agent will choose a random action. By setting different p, we show in Fig. 7 that the approximation error in the Actor update may significantly degrade the learning performance. Meanwhile, Fig. 7 also indicates that decreasing bias can be helpful to improve the learning performance (see the red and green lines in Fig. 7). This observation also verifies our results in Theorem 4.5.

M. Off-policy A-C Algorithem as Newton's Method with Perturbation

We note that the actor and critic updates in Eqn. (9) and Eqn. (8) are a general template that admits both off- and on-policy method. More specifically, denote the target policy by π_{tar} and the behavior policy by π_{bhv}. When the off-policy menthod is used, then the updates in Eqn. (9) and Eqn. (8) are given by

$$\omega_{t+1} \leftarrow \arg\min_{\omega} \mathbf{E}_{(s,a)\sim\rho^{\pi_{\text{bhv}}}} \left[Q_{\omega,\pi_{\text{tar}_{t+1}}}(s,a) - \omega^\top \phi(s,a) \right]^2,$$

$$\pi_{t+1} \leftarrow \arg\max_{\pi} \mathbf{E}_{(s,a)\sim\rho^{\pi_{\text{bhv}}}} \left[Q_{\omega_{t+1},\pi_{\text{tar},t}}(s,a) \right].$$

This is in contrast to the updates given below when the on-policy method is used:

$$\omega_{t+1} \leftarrow \arg\min_{\omega} \mathbf{E}_{(s,a)\sim\rho^{\pi_{\text{tar}}}} \left[Q_{\omega,\pi_{\text{tar}_{t+1}}}(s,a) - \omega^\top \phi(s,a) \right]^2,$$

$$\pi_{t+1} \leftarrow \arg\max_{\pi} \mathbf{E}_{(s,a)\sim\rho^{\pi_{\text{tar}}}} \left[Q_{\omega_{t+1},\pi_{\text{tar},t}}(s,a) \right].$$

- One major challenge of the off-policy analysis lies in the fact that the behavior policy can be arbitrary (Sutton et al., 1999)(Sutton & Barto, 2018) and hence it is impossible to develop a unifying framework. For example, the behavior policy can be obtained by human demonstration (a similar idea is used in an early version of AlphaGo), deriving from the target policy as in Q-learning/DQN or from a previous behavior policy. Meanwhile, the key drawback of off-policy method is that it does not stably interact with the function approximation and is generally of greater variance and slower convergence rate (Sutton & Barto, 2018). In this regard, modern off-policy deep RL requires techniques such as growing batch learning, importance sampling or ensemble method to stabilize the algorithm. Thus, for ease of exposition, we only include the on-policy analysis in our work.

- Our framework and theoretical results are able to be applied to off-policy setting with the extra assumption on the behavior policy. In particular, we assume the behavior policy is in the neighborhood of the target policy, i.e., in each Actor and Critic update step,
$$\|\mathcal{E}_{\text{bhv-tar},t}\| := \|\pi_{\text{tar}}, t - \pi_{\text{bhv},t}\| \leq C_{bt},$$
where $C_{tb} \geq 0$ is a constant. In this way, we can write the A-C update in the off-policy setting as a Newton Method with perturbation, i.e.,
$$\boldsymbol{v}_{\pi_{\text{tar}},t+1} = \boldsymbol{v}_{\pi_{\text{tar}},t} - (\boldsymbol{J}^{-1}_{\boldsymbol{v}_{\pi_{\text{tar}},t}}(\boldsymbol{v}_{\pi_{\text{tar}},t} - T(\boldsymbol{v}_{\pi_{\text{tar}},t})) - \mathcal{E}_t),$$
where $\mathcal{E}_t$ is the perturbation which captures the approximation error from Actor update, Critic update and the behavior policy. Explicitly, we have the perturbation with the following form,
$$\mathcal{E}_t = \mathcal{E}_{v,t} + \mathcal{E}_{\hat{J},t}(\boldsymbol{v}^{\hat{\pi}_{t+1}} - (\boldsymbol{r}_{\widetilde{\pi}_{t+1}} + \gamma \boldsymbol{P}_{\widetilde{\pi}_{t+1}} \boldsymbol{v}^{\hat{\pi}_{t+1}})) - \boldsymbol{J}^{-1}_{\hat{\boldsymbol{v}}_t}(\mathcal{E}_{r,t} + \mathcal{E}_{bhv-tar,t} + \gamma(\mathcal{E}_{P,t} + \mathcal{E}_{bhv-tar,t})\boldsymbol{v}^{\hat{\pi}_t}).$$
Thus, the off-policy analysis is similar to the on-policy case but with the 'error' induced by the behavior policy.