Superposition as Lossy Compression – Measure with Sparse Autoencoders and Connect to Adversarial Vulnerability

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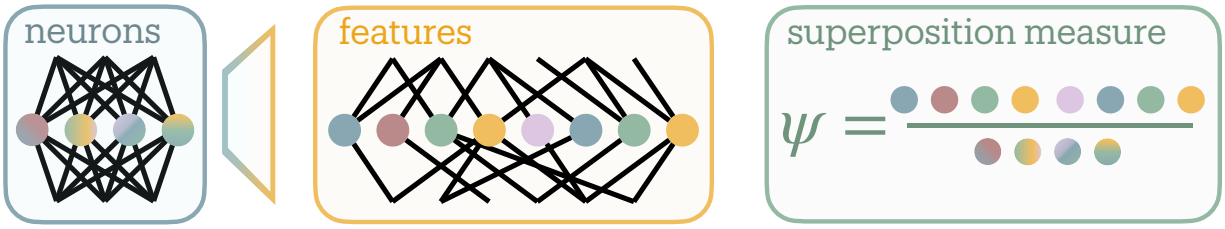
Abstract

Neural networks achieve remarkable performance through *superposition*—encoding multiple features as overlapping directions in activation space rather than dedicating individual neurons to each feature. This phenomenon challenges interpretability: when neurons respond to multiple unrelated concepts, understanding network behavior becomes difficult. Yet despite its importance, we lack principled methods to measure superposition. We present an information-theoretic framework measuring a neural representation's *effective degrees of freedom*. We apply the Shannon entropy to sparse autoencoder activations to compute the number of effective features as the the minimum number of neurons needed for interference-free encoding. Equivalently, this measures how many "virtual neurons" the network simulates through superposition. When networks encode more effective features than they have actual neurons, they must accept interference as the price of compression. Our metric strongly correlates with ground truth in toy models, detects minimal superposition in algorithmic tasks (effective features approximately equal neurons), and reveals systematic reduction under dropout. Layer-wise patterns of effective features mirror studies of intrinsic dimensionality on Pythia-70M. The metric also captures developmental dynamics, detecting sharp feature consolidation during the grokking phase transition. Surprisingly, adversarial training can increase effective features while improving robustness, contradicting the hypothesis that superposition causes vulnerability. Instead, the effect of adversarial training on superposition depends on task complexity and network capacity: simple tasks with ample capacity allow feature expansion (abundance regime), while complex tasks or limited capacity force feature reduction (scarcity regime). By defining superposition as lossy compression, this work enables principled, practical measurement of how neural networks organize information under computational constraints, in particular, connecting superposition to adversarial robustness.

1 Introduction

Interpretability and adversarial robustness could be two sides of the same coin (Räuker et al., 2023). Adversarially trained models learn more interpretable features (Engstrom et al., 2019; Ilyas et al., 2019), develop representations that transfer better (Salman et al., 2020), and align more closely with human perception (Santurkar et al., 2019). Conversely, interpretability-enhancing techniques improve robustness: input gradient regularization (Ross & Doshi-Velez, 2017; Boopathy et al., 2020), attribution smoothing (Etmann et al., 2019), and feature disentanglement (Augustin et al., 2020) all defend against adversarial attacks. Even architectural choices that promote interpretability—lateral inhibition (Eigen & Sadovnik, 2021) and second-order optimization (Tsiligkaridis & Roberts, 2020)—yield more robust models. This pervasive duality demands mechanistic explanation.

The superposition hypothesis offers a potential mechanism. Elhage et al. (2022) showed that neural networks compress information through *superposition*: encoding multiple features as overlapping activation patterns. When features share dimensions, their interference creates attack surfaces that adversaries might exploit. If, by this mechanism, superposition caused adversarial vulnerability, this would explain *i.)* adversarial transferability as shared feature correlations (Liu et al., 2017), *ii.)* the robustness-accuracy trade-off



(a) Observed network (b) Hypothetical disentangled model (c) Superposition as features per neuron

Figure 1: Defining superposition for a neural network layer. **(a)** Observed network with compressed representation where multiple features share neuronal dimensions. **(b)** Hypothetical disentangled model where each effective feature occupies its own neuron without interference (Elhage et al., 2022). **(c)** Superposition measure ψ quantifies effective features per neuron. Here, the network simulates twice as many effective features as it has neurons. Figure adapted from (Bereska & Gavves, 2024).

as models sacrificing representational capacity for orthogonality (Tsipras et al., 2019), and *iii.)* robust models becoming more interpretable by reducing feature entanglement (Engstrom et al., 2019). Also, this superposition-vulnerability hypothesis predicts that *adversarial training should reduce superposition*.

Testing this prediction requires measuring superposition in real networks. While Elhage et al. (2022) used weight matrix Frobenius norms, this approach requires ground truth features; available only in toy models. We need principled methods to quantify superposition without knowing the true features.

We solve this through information theory applied to sparse autoencoders (SAEs). SAEs extract interpretable features from neural activations (Cunningham et al., 2024; Bricken et al., 2023), decomposing them into sparse dictionary elements. We measure each feature's share of the network's representational budget through its activation magnitude across samples.

The exponential of the Shannon entropy reveals how many interference-free channels would transmit this feature distribution—the network's *effective degrees of freedom*. We call this count *effective features* F (Figure 1b): the minimum neurons needed to encode the observed features without interference. One can think of it as the "virtual neurons" the network simulates through superposition. The feature distribution compresses losslessly down to F neurons. Compress further and features must share dimensions, introducing interference.

We measure superposition as $\psi = F/N$ (Figure 1c)—effective features per physical neuron. When $\psi = 1$, the network operates at the interference-free boundary. When $\psi = 2$, it packs two effective features per neuron, achieving $2\times$ lossy compression. Superposition is thus compression beyond the lossless limit.

Our findings reveal a capacity-mediated bifurcation that challenges the simple superposition-vulnerability hypothesis. Rather than universally reducing superposition, adversarial training's effect depends critically on the ratio of task complexity to network capacity (Section 6.5). In an *abundance regime* (simple tasks, ample capacity), networks can afford to decompress—expanding effective features for robustness without sacrificing capability. In a *scarcity regime* (complex tasks, limited capacity), networks must compress further, accepting more interference to maintain performance.

Beyond adversarial robustness, our framework reveals systematic patterns of neural information organization. Layer-wise analysis shows compression factors peaking at $20\times$ neurons in early MLP layers of Pythia-70M before declining (Section 6.4), mirroring intrinsic dimensionality studies (Ansuini et al., 2019). Algorithmic tasks resist superposition entirely ($\psi \approx 1$), likely due to uniform activation patterns that prevent sparse feature packing (Section **??**). During grokking, sharp compression reduction at the generalization transition suggests algorithmic discovery corresponds to finding efficient encodings (Section 6.3).

By reframing superposition as measurable compression and revealing its nuanced relationship with robustness, this work enables quantitative study of how neural networks balance information capacity against interference under computational constraints.

2 Related Work

Superposition and polysemanticity. Neural networks employ distributed representations, encoding information across multiple units rather than in isolated neurons (Hinton, 1984; Olah, 2023). The discovery that semantic relationships manifest as directions in embedding space—exemplified by vector arithmetic like "king - man + woman = queen" (Mikolov et al., 2013)—established the *linear representation hypothesis* (Park et al., 2023). Building on this geometric insight, Elhage et al. (2022) formulated the *superposition hypothesis*: networks encode more features than dimensions by representing features as nearly orthogonal directions. Their toy models revealed phase transitions between monosemantic neurons (one feature per neuron) and polysemantic neurons (multiple features per neuron), governed by feature sparsity. Recent theoretical work proves networks can compute accurately despite the interference inherent in superposition (Vaintrob et al., 2024; Hänni et al., 2024).

While superposition (more effective features than neurons) inevitably creates polysemantic neurons through feature interference, polysemanticity (multiple features sharing a neuron) also emerges by other means: rotation of features relative to the neuron basis, incidentally Lecomte et al. (2023) (e.g. via regularization), or forced by noise (such as dropout) as redundant encoding Marshall & Kirchner (2024) (as we show in Section 6.2, dropout shows the opposite effect on superposition). Scherlis et al. (2023) analyzed how features compete for limited neuronal capacity, showing that importance-weighted feature allocation can explain which features become polysemantic under resource constraints.

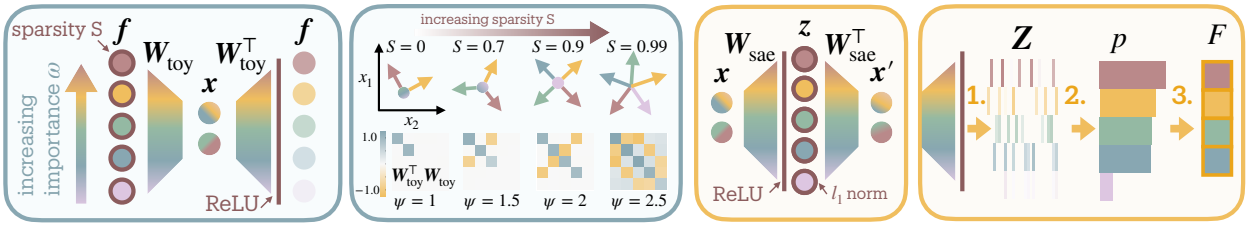
Sparse autoencoders for feature extraction. Sparse autoencoders (SAEs) tackle the challenge of extracting interpretable features from polysemantic representations by recasting it as sparse dictionary learning (Sharkey et al., 2022; Cunningham et al., 2024). SAEs decompose neural activations into sparse combinations of learned dictionary elements, effectively reversing the superposition process. Recent architectural innovations—gated SAEs (Rajamanoharan et al., 2024), TopK variants (Gao et al., 2024; Bussmann et al., 2024), and Matryoshka SAEs (Bussmann et al., 2025)—improve feature recovery. While our experiments employ vanilla SAEs for conceptual clarity, our entropy-based framework remains architecture-agnostic: improved feature extraction yields more accurate measurements without invalidating the theoretical foundation.

SAEs scale to state-of-the-art models: Anthropic extracted millions of interpretable features from Claude 3 Sonnet (Templeton et al., 2024), while OpenAI achieved similar results with GPT-4 (Gao et al., 2024). Crucially, these features are causally relevant—activation steering produces predictable behavioral changes (Marks et al., 2024). Applications span attention mechanism analysis (Kissane et al., 2024), reward model interpretation (Marks et al., 2023), and automated feature labeling (Paulo et al., 2024), establishing SAEs as foundational for mechanistic interpretability (Bereska & Gavves, 2024).

Information theory and neural measurement. Information-theoretic principles provide rigorous foundations for understanding neural representations. The information bottleneck principle (Tishby et al., 2000), when applied to deep learning (Shwartz-Ziv & Tishby, 2017), reveals how networks balance compression with prediction. Each neural layer acts as a bandwidth-limited channel, forcing networks to develop efficient codes—including superposition—to transmit information forward (Goldfeld et al., 2019). This perspective recasts superposition as an optimal solution to rate-distortion constraints.

Most pertinent to our work, Ayonrinde et al. (2024) connected SAEs to minimum description length (MDL). By viewing SAE features as compression codes for neural activations, they showed that optimal SAEs balance reconstruction fidelity against description complexity. Our entropy-based framework extends this perspective, measuring the effective "alphabet size" networks use for internal communication.

Quantifying feature entanglement. Despite its theoretical importance, measuring superposition remains unexplored. Elhage et al. (2022) proposed a dimensions per feature metric for analyzing uniform importance settings in toy models, which when inverted could measure features per dimension—but this approach requires knowing the ground truth feature-to-neuron mapping matrix, limiting its applicability to controlled settings. Traditional disentanglement metrics from representation learning (Carbonneau et al., 2022; Eastwood & Williams, 2018) assess statistical independence rather than the representational compression characterizing superposition. Other dimensionality measures like effective rank (Roy & Vetterli, 2007)



(a) Toy model architecture (b) Effect of sparsity (c) SAE (d) Measurement

Figure 2: From toy model to practical superposition measurement. **(a)** Toy model bottlenecks features $\boldsymbol{f}$ through fewer neurons $\boldsymbol{x}$, with importance gradient determining allocation. **(b)** Sparsity enables interference-based compression: matrices $\boldsymbol{W}_{\text{toy}}^T\boldsymbol{W}_{\text{toy}}$ show off-diagonal terms growing as ψ increases from 1 to 2.5. **(c)** Sparse autoencoders learn sparse codes $\boldsymbol{z}$ reconstructing activations $\boldsymbol{x}$. **(d)** Measurement: extract activations $\boldsymbol{Z}$, derive probabilities p, compute $F = e^{H(p)}$, measure $\psi = F/N$.

and participation ratio (Gao et al., 2017) quantify the number of significant dimensions in a representation but do not directly measure feature-to-neuron compression ratios.

Entropy-based measures have proven effective across disciplines facing similar measurement challenges. Neuroscience employs participation ratios to quantify how many neurons contribute to population dynamics (Gao et al., 2017). Economics uses entropy to quantify portfolio concentration (Fontanari et al., 2021). Quantum physics applies von Neumann entropy to count effective pure states in entangled systems (Nielsen & Chuang, 2011). Recent work applies entropy measures to neural network analysis (Lee et al., 2023; Shin et al., 2024). Across fields, entropy naturally captures how information distributes across components: exactly what we need for measuring superposition.

3 Background on Superposition and Sparse Autoencoders

Neural networks must transmit information through layers with fixed dimensions. When neurons must encode information about many more features than available dimensions, networks employ *superposition*—packing multiple features into shared dimensions through interference. This compression mechanism enables representing more features than available neurons at the cost of introducing crosstalk between them. Superposition is compression beyond the lossless limit.

We examine toy models where superposition emerges under controlled bandwidth constraints, making interference patterns directly observable (Section 3.1). For real networks where ground truth remains unknown, we extract features through sparse autoencoders before measurement becomes possible (Section 3.2).

3.1 Observing Superposition in Toy Models

To understand how neural networks represent more features than they have dimensions, Elhage et al. (2022) introduced minimal models demonstrating superposition under controlled conditions. The toy model compresses a feature vector $\boldsymbol{f} \in \mathbb{R}^M$ through a bottleneck $\boldsymbol{x} \in \mathbb{R}^N$ where $M > N$ (Figure 2a):

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{W}_{\text{toy}}\boldsymbol{f}, \\ \boldsymbol{f}' &= \text{ReLU}(\boldsymbol{W}_{\text{toy}}^T\boldsymbol{x} + \boldsymbol{b}) \end{aligned} \tag{1}$$

Here M counts input features, N counts bottleneck neurons, and $\boldsymbol{W}_{\text{toy}} \in \mathbb{R}^{N\times M}$ maps between them. The model must represent M features using only N dimensions—impossible unless features share neuronal resources.

Each input feature f_i samples uniformly from $[0, 1]$ with sparsity S (probability of being zero) and importance weight ω_i. Training minimizes importance-weighted reconstruction error $\mathcal{L}(\boldsymbol{f}) = \sum_{i=1}^M \omega_i \|f_i - f_i'\|^2$, revealing how networks optimally allocate limited bandwidth.

As sparsity increases, the model packs features into shared dimensions through nearly-orthogonal arrangements (Figure 2b). The interference matrix $\boldsymbol{W}_{\text{toy}}^T \boldsymbol{W}_{\text{toy}}$ reveals this geometric solution: at low compression, strong diagonal with minimal off-diagonal terms; at high compression, substantial off-diagonal interference as features share space. These interference terms quantify the distortion networks accept for increased capacity. The ReLU nonlinearity proves essential, suppressing small interference to maintain reconstruction despite feature overlap.

Elhage et al. (2022) proposed measuring "dimensions per feature" as $D^* = N/\|\boldsymbol{W}_{\text{toy}}\|_{\text{Frob}}^2$ for analyzing uniform importance settings, where the Frobenius norm $\|\boldsymbol{W}_{\text{toy}}\|_{\text{Frob}}^2 = \sum_{i,j} W_{ij}^2$ aggregates weight magnitudes. While this metric was not intended for general superposition measurement, we nevertheless adopt its inverse as a baseline, as it provides the only existing weight-based comparison point for our toy model validation:

$$\psi_{\text{Frob}} = \frac{\|\boldsymbol{W}_{\text{toy}}\|_{\text{Frob}}^2}{N} \tag{2}$$

This weight-based approach requires knowing the true feature-to-neuron mapping (unavailable in real networks) and lacks scale invariance—multiplying weights by any constant arbitrarily changes the measure. We need a principled framework quantifying compression without ground truth features.

3.2 Extracting Features Through Sparse Autoencoders

Real networks do not reveal their features directly. Instead, we must untangle them from distributed neural activations. Sparse autoencoders (SAEs) decompose activations into sparse combinations of learned dictionary elements, effectively reverse-engineering the toy model's feature representation (Cunningham et al., 2024; Bricken et al., 2023).

Given layer activations $\boldsymbol{x} \in \mathbb{R}^N$, an SAE learns a higher-dimensional sparse code $\boldsymbol{z} \in \mathbb{R}^D$ where $D > N$ (Figure 2c):

$$\boldsymbol{z} = \text{ReLU}(\boldsymbol{W}_{\text{enc}}\boldsymbol{x} + \boldsymbol{b}) \tag{3}$$

The reconstruction combines these sparse features:

$$\boldsymbol{x}' = \boldsymbol{W}_{\text{dec}}\boldsymbol{z} = \sum_{i=1}^{D} z_i \boldsymbol{d}_i \tag{4}$$

where columns $\boldsymbol{d}_i$ of $\boldsymbol{W}_{\text{dec}}$ form the learned dictionary.

Training balances faithful reconstruction against sparse activation:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{z}) = \|\boldsymbol{x} - \boldsymbol{x}'\|_2^2 + \lambda \|\boldsymbol{z}\|_1 \tag{5}$$

The ℓ_1 penalty creates explicit competition: the bound on total activation $\sum_i |z_i|$ forces features to justify their magnitude by contributing to reconstruction. This implements resource allocation where larger $|z_i|$ indicates greater consumption of the network's limited representational budget (see Appendix A.2 for rate-distortion derivation).

SAE design choices. We tie encoder and decoder weights ($\boldsymbol{W}_{\text{dec}} = \boldsymbol{W}_{\text{enc}}^T$) to enforce features as directions in activation space, maintaining conceptual clarity at potential cost to reconstruction (Bricken et al., 2023). Weight tying can also prevent feature absorption artifacts (chanind et al., 2024). We omit decoder bias following Cunningham et al. (2024) for a transparent baseline, accepting slight performance degradation. The ℓ_1 regularization provides clean budget semantics, though alternatives like TopK (Gao et al., 2024) could work within our framework.

If networks truly employ superposition, SAEs should recover the underlying features enabling measurement. Recent work shows SAE features causally affect network behavior (Marks et al., 2024), suggesting they capture genuine computational structure. Our measurement framework remains architecture-agnostic—improved SAE variants enhance accuracy without invalidating the theoretical foundation.

4 Measuring Superposition Through Information Theory

We quantify superposition by determining how many neurons would be required to transmit the observed feature distribution without interference. Information theory provides a precise answer: Shannon's source coding theorem establishes that any distribution with entropy $H(p)$ can be losslessly compressed to $e^{H(p)}$ uniformly-allocated channels. This represents the minimum bandwidth for interference-free transmission—the network's *effective degrees of freedom*.

We formalize superposition as the compression ratio $\psi = F/N$, where N counts physical neurons and $F = e^{H(p)}$ measures effective degrees of freedom extracted from SAE activation statistics (Figure 2d). When $\psi = 1$, the network operates at the lossless boundary. When $\psi > 1$, features necessarily share dimensions through interference. For instance, in Figure 2b, 5 features represented in 2 neurons yields $\psi = 2.5$.

Feature probabilities from resource allocation. Consider a layer with N neurons whose activations have been processed by an SAE with dictionary size D. Across S samples, the SAE produces sparse codes $\boldsymbol{Z} = \mathrm{ReLU}(\boldsymbol{W}_{\mathrm{sae}}\boldsymbol{X}) \in \mathbb{R}^{D\times S}$ where $\boldsymbol{X} \in \mathbb{R}^{N\times S}$ contains the original activations. Each feature's probability reflects its share of total activation magnitude[1]:

$$p_i = \frac{\sum_{s=1}^{S} |z_{i,s}|}{\sum_{j=1}^{D}\sum_{s=1}^{S} |z_{j,s}|} = \frac{\text{budget allocated to feature } i}{\text{total representational budget}} \tag{6}$$

The SAE's ℓ_1 regularization ensures these allocations reflect computational importance. Features activating more frequently or strongly consume more capacity, with optimal $|z_i|$ proportional to marginal contribution to reconstruction quality (derivation in Appendix A.2).

Effective features as lossless compression limit. Shannon entropy quantifies the information content of this distribution: $H(p) = -\sum_i p_i \log p_i$. Its exponential:

$$F = e^{H(p)} \tag{7}$$

measures effective degrees of freedom—the minimum neurons needed to encode p without interference. This is the network's lossless compression limit: the feature distribution could be transmitted through F neurons with no information loss. Using fewer than F neurons guarantees interference as features must share dimensions; using exactly F achieves the interference-free boundary; the actual layer width N determines whether compression remains lossless ($N \geq F$) or becomes lossy ($N < F$). The ratio

$$\psi = \frac{F}{N} \tag{8}$$

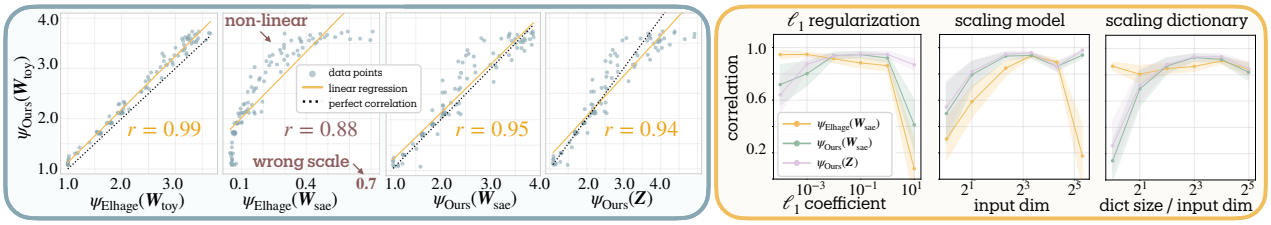
then measures superposition as lossy compression.

While the SAE extracts D *interpretable features*—semantic concepts humans might recognize, our measure quantifies F *effective features*—the interference-free channel capacity required for their activation distribution. A network might use $D = 1000$ interpretable features but need only $F = 50$ effective features if most activate rarely, as the compression from D to F happens losslessly through entropy coding.

Our measure inherits desirable properties from entropy: *i.)* Output remains bounded ($1 \leq F \leq D$) and interpretable as degrees of freedom. *ii.)* The measure satisfies additivity for independent feature sets via $H(p \otimes q) = H(p) + H(q)$, a desirable property ensuring consistent behavior under composition. *iii.)* It requires no arbitrary thresholds. Features contribute according to their information content—rare features matter less than common ones, weak features less than strong ones. Rare and weak features get naturally downweighted as $p_i \log(1/p_i) \to 0$ when $p_i \to 0$.

This framework enables quantifying superposition without ground truth by measuring each layer's compression ratio—how many virtual neurons it simulates relative to its physical dimension.

[1] Why not measure SAE weights instead of activations? Weight magnitude $\|\boldsymbol{w}_i\|$ indicates potential representation but misses actual usage: "dead features" may exist in the dictionary without ever activating. Empirically, weight-based measures succeed only in toy models (Figure 3a); and even simple Tracr transformers require our activation-based approach (Section **??**).



(a) Correlation with observable patterns

(b) Robustness across hyperparameters

Figure 3: Validation of entropy-based superposition metrics. **(a)** Our measure maintains high correlation whether applied to toy weights ($r = 0.99$) or SAE activations ($r = 0.94$), while the Frobenius norm fails on SAE weights. **(b)** Performance remains stable across ℓ_1 regularization, model scale, and dictionary size variations. Shaded regions show 95% confidence intervals across 100 model-SAE pairs.

5 Validation of the Measurement Framework

5.1 Toy Model of Superposition

We validate our entropy-based measure using the toy model of superposition (Elhage et al., 2022), where interference patterns are directly observable. This controlled setting tests whether sparse autoencoders can recover accurate feature counts from superposed representations.

Following Elhage et al. (2022), we generate 100 toy models with sparsity $S \in [0.001, 0.999]$. Each model compresses 20 features through a 5-neuron bottleneck, with importance weights decaying as $\omega_i = 0.7^i$. After training to convergence, we extract 10,000 activation samples and train SAEs with 40-dimensional dictionaries (8× expansion) and ℓ_1 coefficient 0.1. This two-stage process mimics real-world measurement where ground truth remains unknown.

Tracking observable compression patterns. The interference matrix $\boldsymbol{W}_{\text{toy}}^T \boldsymbol{W}_{\text{toy}}$ directly reveals compression levels: diagonal dominance indicates orthogonal features; off-diagonal terms show interference. As sparsity increases, we observe the transition from 5 to 20 represented features. Both our entropy-based measure and a Frobenius norm baseline (Eq. 2) achieve near-perfect correlation ($r = 0.99 \pm 0.01$) when applied to toy model weights, confirming both track these observable patterns (Figure 3a).

Yet the Frobenius norm catastrophically fails on SAE weights, producing nonlinear relationships and incorrect scales (0.1–0.7 versus expected 1–4). The ℓ_1 regularization fundamentally alters weight statistics. Our activation-based approach maintains strong correlation ($r = 0.94 \pm 0.02$) even through the SAE bottleneck, enabling practical application to real networks.

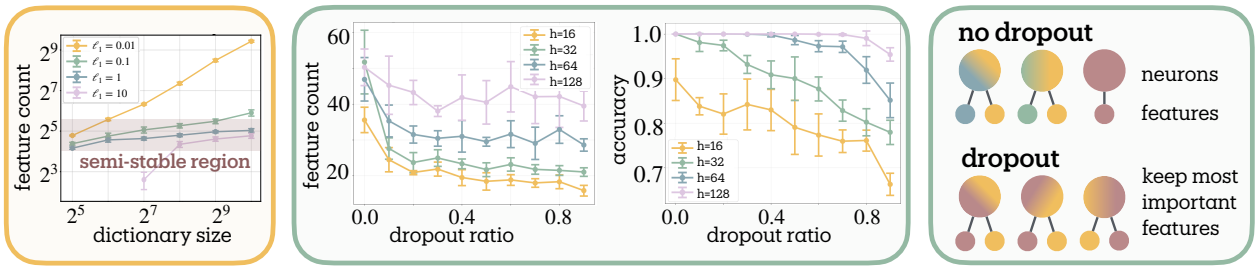
Hyperparameter stability. We test sensitivity across three axes: ℓ_1 strength (10^{-3} to 10^0), model scale (8–32 input dimensions), and dictionary expansion (2× to 32×). Figure 3b shows stable performance except at extremes: excessive regularization ($\ell_1 = 1.0$) or insufficient dictionary capacity (fewer elements than features). This stability confirms we capture intrinsic superposition properties rather than measurement artifacts.

5.2 Dictionary Scaling Convergence

Measuring a natural coastline with a finer ruler yields a longer measurement; potentially without bound (Mandelbrot, 1967). As SAE dictionaries grow, might we discover arbitrarily many features at finer scales?

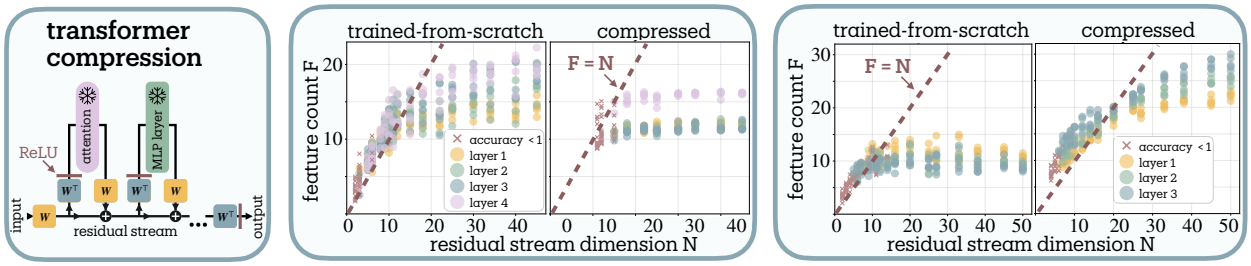
We test convergence using multi-task sparse parity (Michaud et al., 2023) (3 tasks, 4 bits each) where ground truth bounds meaningful features. Networks with 64 hidden neurons trained across dictionary scales (0.5× to 16× hidden dimension) and ℓ_1 strengths (0.01 to 10.0).

Figure 4a reveals two regimes. With appropriate regularization ($\ell_1 \geq 0.1$), feature counts plateau despite dictionary expansion; indication that we measure the network's representational structure, not arbitrary



(a) Dictionary scaling (b) Dropout reduces features and performance (c) Dropout effect on features

Figure 4: Measurements on multi-task sparse parity dataset. **(a)** Dictionary scaling plateaus with proper regularization ($\ell_1 \geq 0.1$), validating intrinsic structure measurement. Weak regularization ($\ell_1 = 0.01$) shows unbounded growth through arbitrary subdivision. **(b)** Dropout monotonically reduces effective features and accuracy. **(c)** Capacity-dependent response: larger networks show reduced sensitivity while narrow networks exhibit sharp feature reduction, distinguishing polysemanticity (neurons encoding multiple features) from superposition (compression beyond lossless limit).



(a) Architecture (b) Task: sequence reversal (c) Task: sequence sorting

Figure 5: Algorithmic tasks under compression. **(a)** Compression architecture projects activations through $\mathrm{ReLU}(\boldsymbol{W}^\top \boldsymbol{W}\boldsymbol{x})$ to force features into fewer dimensions. **(b)** Sequence reversal maintains constant feature count (12) despite 10× compression, with sharp failure when dimensions drop below requirements. **(c)** Sequence sorting also tracks the $F = N$ bound. Both tasks show minimal superposition likely due to lack of input sparsity.

decomposition. But weak regularization ($\ell_1 = 0.01$) shows unbounded growth, decomposing features into ever-finer components. While practical measurement requires standardized SAE training, convergence under proper regularization validates our approach captures genuine representational properties.

6 Applications and Findings

Our entropy-based superposition measure enables investigation of neural information organization across diverse contexts. We present four applications: algorithmic tasks that resist superposition (Section 6.1), capacity-constrained feature allocation under dropout (Section 6.2), developmental dynamics during learning transitions (Section 6.3), and layer-wise representational organization in language models (Section 6.4).

Each finding here is preliminary, an exploratory analysis on specific architectures and tasks; our primary contribution remains the measurement tool itself. These findings illustrate its potential while generating testable hypotheses for future systematic investigation across broader experimental conditions.

6.1 Algorithmic Tasks Resist Superposition

Tracr compiles human-readable programs into transformer weights with known computational structure (Lindner et al., 2023). We examine sequence reversal ("123" → "321") and sorting ("213" → "123"), comparing original compiled models against compressed variants and models trained from scratch.

Following Lindner et al. (2023), we compress models by projecting residual stream activations through learned compression matrices. Our compression scheme (Figure 5a) applies $\mathrm{ReLU}(\boldsymbol{W}^\top \boldsymbol{W}\boldsymbol{x})$ where $\boldsymbol{W} \in \mathbb{R}^{N' \times N}$, compressing from originally N' dimensions to N. The ReLU activation, which is absent in the original Tracr compression, allows small interference terms to cancel out, following the toy model rationale (Elhage et al., 2022).

Surprisingly, both compressed Tracr models and transformers trained from scratch maintain feature counts closely tracking the residual stream dimension $F \approx N$ (Figure 5b, 5c). Notably, these feature counts remain far below the original compiled dimensions, e.g. reversal models use only 12 features[2] despite originally having 45 dimensions, revealing substantial redundancy in Tracr's compilation process rather than genuine superposition. When dimensions drop below intrinsic task requirements, performance collapses ($\times$ markers).

This resistance to superposition likely stems from algorithmic tasks violating the sparsity assumption: The toy model of superposition (Elhage et al., 2022) requires features to activate sparsely across inputs: most features remain inactive on most samples; otherwise interference becomes costly. Algorithmic tasks break this assumption; sequence operations require consistent activation patterns across inputs. Without sparsity, interference that enables efficient superposition becomes destructive.

6.2 Dropout Reduces Features Through Redundant Encoding

Neural networks face capacity constraints that force feature reduction. We investigate how dropout affects feature organization using multi-task sparse parity (3 tasks, 4 bits each) with simple MLPs across hidden dimensions $h \in \{16, 32, 64, 128\}$ and dropout rates [0.0, 0.1, ..., 0.9].

Marshall & Kirchner (2024) showed dropout induces polysemanticity through redundancy: features must distribute across neurons to survive random deactivation. One might expect this redundancy to increase measured features. Instead, dropout monotonically reduces feature count by up to 50% (Figure 4b).

The resolution lies in distinguishing polysemanticity from superposition (Figure 4c). Dropout forces each feature to occupy multiple neurons for robustness, creating polysemantic neurons as a side effect. But this redundant encoding consumes capacity; fewer total features fit within the same dimensional budget. The network responds by pruning less essential features, aligning with Scherlis et al. (2023)'s framework where features compete for limited resources.

The capacity dependence validates this interpretation: larger networks show reduced dropout sensitivity while narrow networks exhibit sharp feature reduction, confirming that capacity constraints drive the observed dynamics.

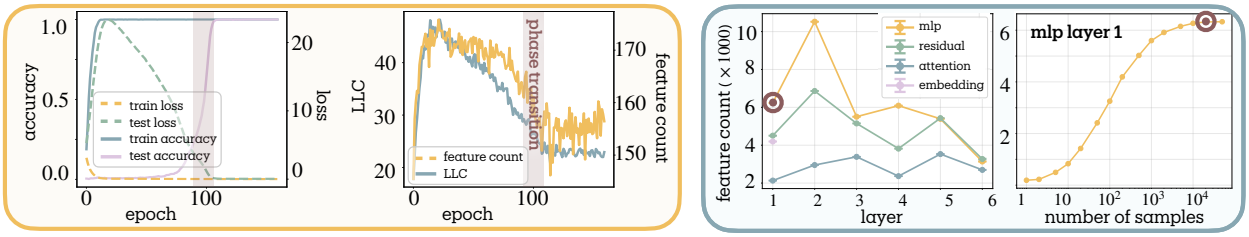
6.3 Capturing Grokking Phase Transition

Grokking—sudden perfect generalization after extended training on algorithmic tasks—provides an ideal testbed for developmental measurement (Power et al., 2022). We investigate whether feature count dynamics can detect this phase transition and how they relate to the Local Learning Coefficient (LLC) from singular learning theory (Hoogland et al., 2024).

We train a two-path MLP on modular arithmetic $(a + b) \bmod 53$. Figure 6a reveals distinct dynamics: while LLC shows initial proliferation followed by smooth decay throughout training, our feature count exhibits sharp consolidation precisely at the generalization transition (checkpoint 60).

This pattern suggests the measures capture different aspects of complexity evolution. During memorization, the model employs numerous superposed features to store input-output mappings. The sharp consolidation coincides with algorithmic discovery, where the model reorganizes from distributed lookup tables into compact representations that capture the modular arithmetic rule (Nanda et al., 2023). Strong correlation ($r = 0.908$, $p < 0.001$) between feature count and LLC positions superposition measurement as a developmental tool for detecting emergent capabilities through their information-theoretic signatures.

[2]While we generally recommend comparative interpretation due to measurement limitations (Section 7), the systematic $F = N$ tracking and performance decline when trespassing this boundary suggest our measure can provide meaningful absolute feature counts in sufficiently constrained computational settings.



(a) Grokking dynamics

(b) Pythia-70M layer analysis

Figure 6: **(a)** Non-monotonic feature organization across Pythia-70M. MLP layer 1 peaks at 10,000 features (20× neurons). Convergence analysis shows saturation after 2×10^4 samples. **(b)** Feature dynamics during grokking on modular arithmetic. Sharp consolidation at generalization transition (checkpoint 60) follows smoother LLC decay. Strong correlation ($r = 0.908$, $p < 0.001$) with LLC suggests feature count functions as a measure of model complexity.

6.4 Layer-wise Pattern Mirrors Intrinsic Dimensionality

We analyze Pythia-70M using pretrained SAEs from Marks et al. (2024), measuring feature counts across all layers and components. Convergence analysis (Figure 6b) shows saturation after 2×10^4 samples. Feature importance follows a power-law distribution (not shown). While 21,000 SAE features activate for MLP 1, our entropy-based measure yields 5,600 effective features, automatically downweighting rare activations.

MLPs store the most features, followed by residual streams, with attention maintaining minimal counts. This aligns with interpretations of MLPs as knowledge stores and attention as routing mechanisms (Geva et al., 2021). Features grow in early layers (MLP 1 achieves 20× compression), compress through middle layers, then re-expand before final consolidation.

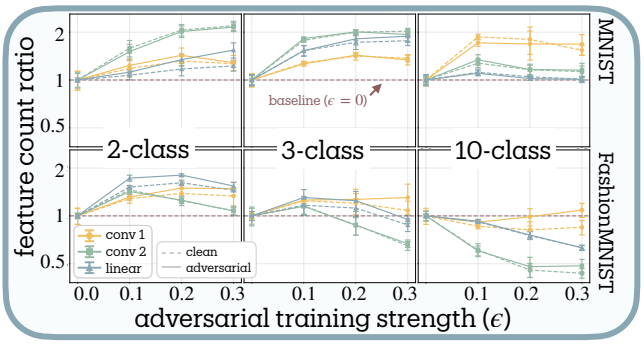
This trajectory parallels intrinsic dimensionality studies (Ansuini et al., 2019): both reveal non-monotonic "hunchback" patterns despite measuring complementary aspects—intrinsic dimensionality captures compression (minimal dimensions needed), while we measure expansion (features encoded). That both metrics reveal similar organizational structures suggests fundamental principles governing information flow through neural architectures.

6.5 Adversarial Robustness: Capacity-Constrained Defense Strategies

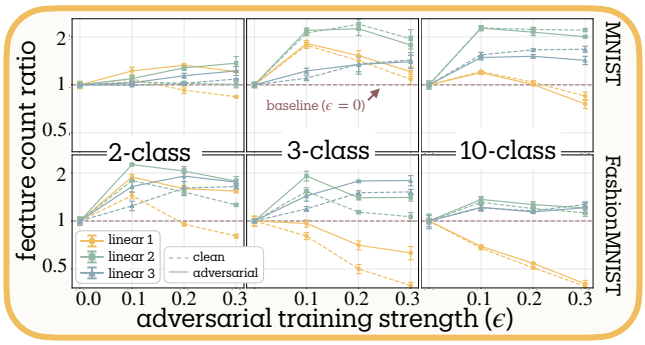
Testing the superposition-vulnerability hypothesis. The superposition-vulnerability hypothesis proposed by Elhage et al. (2022) predicts that adversarial training should universally reduce superposition, as networks trade representational efficiency for orthogonal, robust features. We test this prediction systematically across diverse architectures and conditions, finding that the relationship between adversarial robustness and superposition depends critically on task complexity and network capacity.

We employ PGD adversarial training (Madry et al., 2018) across architectures ranging from single-layer to deep networks (MLPs, CNNs, ResNet-18) on multiple datasets (MNIST, Fashion-MNIST, CIFAR-10). Task complexity varies through both classification granularity (2, 3, 5, 10 classes) and dataset difficulty. Network capacity varies through hidden dimensions (8–512 for MLPs), filter counts (8–64 for CNNs), and width scaling (1/4×–2× for ResNet-18). For convolutional networks, we measure superposition across channels by reshaping activation tensors to treat spatial positions as independent samples (see Appendix A.7 for details). All SAEs use 4× dictionary expansion with $\ell_1 = 0.1$. Measurements on adversarial examples match the training distribution—models trained with $\epsilon = 0.2$ are evaluated on $\epsilon = 0.2$ attacks.

Statistical methodology. To quantify adversarial training effects rigorously, we extract normalized slopes representing how feature counts change per unit increase in adversarial training strength ($\epsilon \in \{0.0, 0.1, 0.2, 0.3\}$). For each experimental condition, we fit linear regressions to feature counts across epsilon

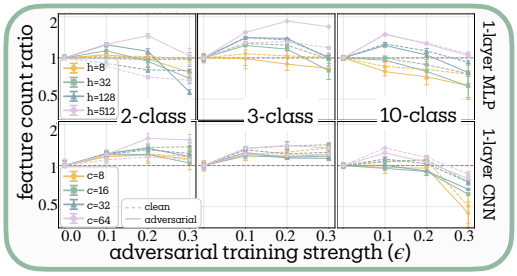


(a) 3-layer CNN on (Fashion-)MNIST

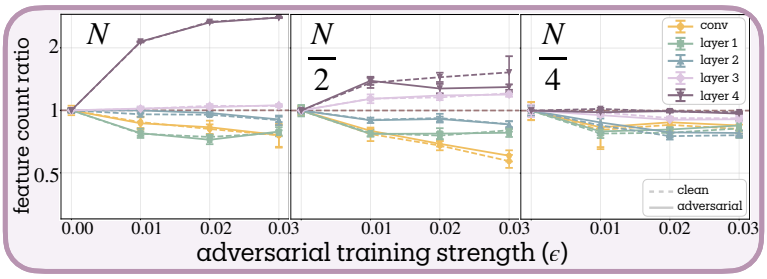


(b) 3-layer MLP on (Fashion-)MNIST

Figure 7: Task complexity modulates adversarial training's effect on superposition. Each panel shows results varying dataset (MNIST top, Fashion-MNIST bottom) and number of classes (2, 3, 10). **(a)** CNNs show clear complexity-dependent transitions: simple tasks enable feature expansion while complex tasks (10 classes) force reduction below baseline. **(b)** MLPs exhibit similar patterns with more pronounced layer-wise variation. Fashion-MNIST consistently amplifies the reduction effect compared to MNIST, suggesting that representational complexity drives defensive strategies beyond mere class count. Dashed lines: clean data; solid lines: adversarial examples. Feature count ratios normalized to $\epsilon = 0$ baseline. Error bars show standard error across 3 seeds.



(a) Widening 1-layer NNs on MNIST



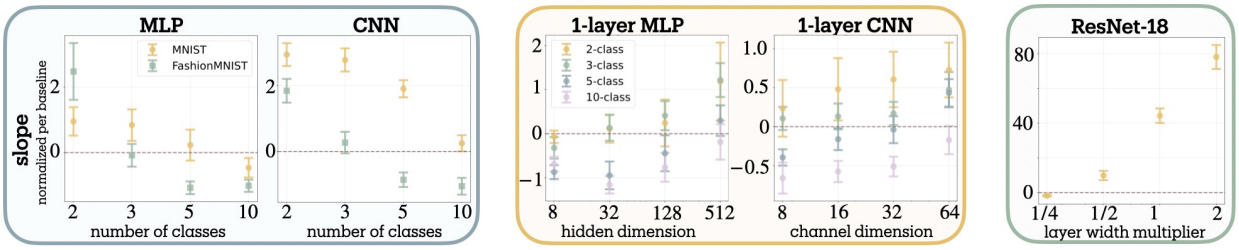
(b) Narrowing ResNet-18 on CIFAR-10

Figure 8: Network capacity modulates adversarial training responses across architectures. **(a)** Single-layer networks on MNIST demonstrate capacity-dependent phase transitions: MLPs with hidden dimensions $h \in \{8, 32, 128, 512\}$ (top) and CNNs with filter counts $c \in \{8, 16, 32, 64\}$ (bottom) show that narrow networks reduce features while wide networks maintain expansion across task complexities. **(b)** ResNet-18 on CIFAR-10 with width scaling ($1\times$, $1/2\times$, $1/4\times$) reveals layer-wise specialization: early layers reduce features while deeper layers (layer 3-4) expand dramatically, with this pattern dampening as width decreases. Dashed lines: clean data; solid lines: adversarial examples. Feature count ratios normalized to baseline.

values, pooling clean and adversarial observations to increase statistical power. These slopes are normalized by baseline ($\epsilon = 0$) feature counts, making effects comparable across layers with different absolute scales.

Since networks contain multiple layers, we aggregate layer-wise measurements using parameter-weighted averaging, where layers with more parameters receive proportionally higher weight. This reflects the assumption that computationally intensive layers better represent overall network behavior. For simple architectures, parameter counts include all weights and biases; for ResNet-18, we implement detailed counting that accounts for convolutions, batch normalization, and skip connections.

We test three formal hypotheses: **H1 (Universal Reduction)**: Adversarial training uniformly reduces superposition across all conditions, directly testing Elhage et al. (2022)'s original prediction. **H2 (Task Complexity Modulation)**: Task complexity systematically modulates adversarial training effects, with complex tasks showing greater feature reduction. We encode complexity ordinally (2-class=1, 3-class=2,



(a) Complexity effects (b) Capacity (1-layer networks) (c) Capacity effect

Figure 9: Statistical analysis of adversarial training effects on superposition. **(a)** Task complexity systematically modulates adversarial responses across architectures. Normalized slopes quantify feature count changes per unit adversarial strength, showing negative linear trends (CNN-MNIST: -0.895 ± 0.236, $p = 0.063$; CNN-Fashion-MNIST: -0.983 ± 0.220, $p = 0.047$). Fashion-MNIST (green) produces consistently lower slopes than MNIST (yellow), with significant paired difference (-1.467 ± 0.156, $p = 0.047$). **(b)** Single-layer networks show capacity-dependent phase transitions with significant positive effects (meta-analytic slope $= 0.220 \pm 0.037$, $p < 0.001$). Narrow networks (8–32 dimensions) reduce features while wide networks (128–512) maintain expansion. **(c)** ResNet-18 demonstrates dramatic log-linear scaling (31.0 ± 2.0 per log(width), $p = 0.004$, $R^2 = 0.929$), with 8-fold width increases yielding 65-unit slope changes. Error bars show standard errors across conditions.

5-class=3, 10-class=4) and test for linear trends. **H3 (Capacity Enabling)**: Network capacity enables different defensive strategies, with higher capacity permitting feature expansion. We test log-linear relationships between capacity measures and adversarial training slopes.

All statistical tests use inverse-variance weighting to account for measurement uncertainty, with random-effects meta-analysis when significant heterogeneity exists across conditions.

Task complexity determines defensive strategy. Contrary to H1's prediction of universal reduction, adversarial training produces bidirectional effects that depend systematically on task complexity (Figures 7 and 9a). Our meta-analysis reveals significant heterogeneity across conditions ($Q = 8.047$, $df = 3$, $p = 0.045$), necessitating random-effects modeling that yields an overall negative relationship between task complexity and feature expansion (combined slope $= -0.745 \pm 0.122$, $z = -6.14$, $p < 0.001$).

Binary classification consistently expands feature counts—up to $2\times$ baseline—suggesting networks develop additional defensive features when task demands are simple. Ten-class problems show the opposite pattern, with feature counts decreasing by up to 60%, particularly in early layers. Three-class tasks exhibit intermediate behavior with inverted-U curves: moderate adversarial training ($\epsilon = 0.1$) initially expands features before stronger training ($\epsilon = 0.3$) triggers reduction.

The dataset difficulty amplifies these effects significantly. While both datasets show negative complexity trends, Fashion-MNIST produces systematically lower slopes than MNIST (mean difference $= -1.467 \pm 0.156$, $t(7) = -2.405$, $p = 0.047$, Cohen's $d = -0.85$). CNN architectures demonstrate particularly clear patterns, with statistically significant negative trends for both MNIST (slope $= -0.895 \pm 0.236$, $p = 0.063$) and Fashion-MNIST (slope $= -0.983 \pm 0.220$, $p = 0.047$). This suggests that representational demands, beyond mere class count, drive defensive strategies.

Layer-wise analysis reveals architectural differences in how complexity effects manifest. In MLPs, first layers typically show strongest reduction (down to 30% of baseline on Fashion-MNIST 10-class), while middle layers often expand even under complex tasks. CNNs display an inverted pattern: second convolutional layers reduce most dramatically while first layers remain relatively stable. Both architectures transition through similar inverted-U curves, suggesting common underlying dynamics despite different inductive biases.

Network capacity enables defensive elaboration. Network capacity systematically modulates adversarial training responses, strongly supporting H3 (Figures 8, 9b, and 9c). Single-layer networks demonstrate

clear capacity thresholds with a significant positive relationship between capacity and feature expansion (meta-analytic slope $= 0.220 \pm 0.037$, $z = 5.90$, $p < 0.001$). Networks with minimal capacity (8 hidden units for MLPs, 8 filters for CNNs) reduce features across all task complexities, while high-capacity networks (512 units/64 filters) maintain expansion even for 10-class problems.

This capacity dependence scales dramatically in deep architectures. ResNet-18 on CIFAR-10 exhibits a strong log-linear relationship between width multiplier and adversarial training slopes (slope $= 31.0 \pm 2.0$ per log(width), $t(2) = 15.7$, $p = 0.004$, $R^2 = 0.929$). An 8-fold width increase ($0.25\times$ to $2\times$) produces a 65-unit change in normalized slope—orders of magnitude larger than simple network effects. At minimal width ($0.25\times$), adversarial training barely affects feature counts, while at double width, networks show massive feature expansion with slopes approaching 80.

The layer-wise progression in ResNet-18 reveals hierarchical specialization: early layers (conv1, layer1) reduce features by up to 50%, middle layers remain stable, while deep layers (layer3, layer4) expand up to $4\times$. Systematically narrowing the network dampens this pattern—at 1/4 width, late-layer expansion vanishes while early-layer reduction persists but weakens. This could reflect vulnerability hierarchies, where early layers processing low-level statistics are easily exploited by imperceptible perturbations, necessitating feature reduction, while late layers encoding semantic information can safely expand their representational repertoire.

Reconciling with the original hypothesis. Our findings reveal a more nuanced relationship between superposition and adversarial vulnerability than originally theorized. Rather than universal feature reduction, adversarial training operates in two distinct regimes:

Abundance regime: When task complexity is low relative to network capacity, adversarial training expands the feature repertoire. Networks appear to maintain original capabilities while adding defensive variants—achieving robustness through redundancy rather than orthogonalization. This regime dominates in high-capacity networks facing simple tasks.

Scarcity regime: When task demands approach capacity limits, networks economize by reducing to fewer, potentially more orthogonal features as predicted by the vulnerability hypothesis. This regime emerges in capacity-constrained networks or when facing complex classification tasks.

Several patterns in our data remain unexplained. We observe non-monotonic inverted-U curves where moderate adversarial training ($\epsilon = 0.1$) expands features while stronger training ($\epsilon = 0.3$) reduces them below baseline. The gap between clean and adversarial feature counts varies unpredictably—sometimes negligible, sometimes substantial—without clear pattern. MLPs and CNNs show opposing layer-wise responses: MLP first layers reduce most while middle layers expand; CNN second layers reduce most while first layers stay stable. Some results contradict our complexity hypothesis, with 2-class MLPs occasionally showing lower feature counts than 3-class. CNN experiments consistently yield stronger statistical significance ($p < 0.02$) than equivalent MLP experiments ($p \approx 0.09$) for unknown reasons. These patterns suggest our understanding of how adversarial training affects neural representations remains incomplete despite the systematic trends we identify.

This framework might explain why robust models often appear more interpretable (Engstrom et al., 2019): not necessarily because they have fewer features universally, but potentially because resource constraints force more organized, essential representations in the scarcity regime.

The bidirectional relationship between robustness and superposition suggests that achieving robustness without capability loss may require ensuring sufficient capacity for defensive elaboration. While our experiments demonstrate that increased robustness can coincide with either increased or decreased superposition depending on the regime, establishing the exact causal connection between superposition and robustness remains an important direction for future work.

7 Limitations

Our superposition measurement framework is limited by its dependence on sparse autoencoder quality, theoretical assumptions about neural feature representation, and should be interpreted as proxy for representational complexity rather than literal feature count:

Sparse autoencoder quality. Our approach inherently depends on sparse autoencoder feature extraction quality. While recent architectural advances—gated SAEs (Rajamanoharan et al., 2024), TopK variants (Gao et al., 2024), and end-to-end training (Braun et al., 2024)—have substantially improved feature recovery, fundamental challenges remain. SAE training exhibits sensitivity to hyperparameters, particularly ℓ_1 regularization strength and dictionary size, with different initialization or training procedures potentially yielding different feature counts for identical networks. Ghost features—SAE artifacts without computational relevance (Gao et al., 2024)—can artificially inflate measurements, while poor reconstruction quality may deflate them.

Assumptions on feature representation. Our framework rests on several assumptions that real networks systematically violate. The linear representation assumption—that features correspond to directions in activation space—has been challenged by recent discoveries of circular feature organization for temporal concepts (Engels et al., 2024) and complex geometric structures beyond simple directions (Black et al., 2022). Our entropy calculation assumes features contribute independently to representation, but neural networks exhibit extensive feature correlations, synergistic information where feature combinations provide more information than individual contributions, and gating mechanisms where some features control others' activation. The approximation that sparse linear encoding captures true computational structure breaks down in hierarchical representations where low-level and high-level features are not substitutable, and in networks with substantial nonlinear feature interactions that cannot be decomposed additively.

Comparative rather than absolute count. Our measure quantifies effective representational diversity under specific assumptions rather than providing literal feature counts. This creates several interpretational limitations. The measure exhibits sensitivity to the activation distribution used for measurement—SAE training distributions must match the network's operational regime to avoid systematic bias. Feature granularity remains fundamentally ambiguous: broader features may decompose into specific ones in wider SAEs, creating uncertainty about whether we're discovering or creating features. Our single-layer analysis potentially misses features distributed across layers through residual connections or attention mechanisms. Most critically, we measure the effective alphabet size of the network's internal communication channel rather than counting distinct computational primitives, making comparative rather than absolute interpretation most appropriate.

The limitations largely reflect active research areas in sparse dictionary learning and mechanistic interpretability. Each advance in SAE architectures, training procedures, or theoretical understanding directly benefits measurement quality. Within its scope—comparative analysis of representational complexity under sparse linear encoding assumptions—the measure enables systematic investigation of neural information structure previously impossible.

8 Conclusion

We introduced an information-theoretic framework to measure superposition in neural networks through sparse autoencoder activations. The entropy-based measure $F = e^{H(p)}$ quantifies effective feature counts without requiring ground truth, enabling systematic study of representational organization.

Our key empirical finding contradicts theoretical prediction: adversarial training often increases rather than decreases feature counts while improving robustness. This suggests robust models expand their feature repertoires rather than orthogonalizing them, challenging the superposition-vulnerability hypothesis. In interpretability research, the tools are only as good as the assumptions they force us to abandon.

The measurement framework reveals general principles of neural organization. Language models show non-monotonic feature patterns across layers, algorithmic tasks resist superposition entirely, and grokking exhibits sharp representational transitions. While our approach depends on SAE quality and assumes linear feature combination, it provides the first principled method for quantifying superposition at scale. This enables comparative analysis across architectures and training regimes.

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A Theoretical Foundations

A.1 Networks as Resource-Constrained Communication Channels

Neural networks must transmit information through layers with limited dimensions. Each layer acts as a communication bottleneck where multiple features compete for neuronal bandwidth. When a network needs to represent F features using only $N < F$ dimensions, it uses lossy compression—superposition.

This resource scarcity creates a natural analogy to communication theory. Just as telecommunications systems multiplex multiple signals through shared channels, neural networks multiplex multiple features through shared dimensions. Our measurement framework formalizes this intuition by quantifying how efficiently networks allocate their limited representational budget across competing features.

A.2 L1 Norm as Optimal Budget Allocation

The sparse autoencoder's ℓ_1 regularization creates an explicit budget constraint on feature activations:

$$\mathcal{L}_{\text{SAE}} = \|\boldsymbol{h} - \boldsymbol{W}_{\text{sae}}^T \boldsymbol{z}\|_2^2 + \lambda \|\boldsymbol{z}\|_1 \tag{9}$$

The penalty term $\lambda\|\boldsymbol{z}\|_1 = \lambda \sum_i |z_i|$ enforces that the total activation budget $\sum_i |z_i|$ remains bounded. This creates competition where features must justify their budget allocation by contributing to reconstruction quality.

From the first-order optimality conditions of SAE training, the magnitude $|z_i|$ for any active feature satisfies:

$$|z_i| = \frac{1}{\lambda} |\boldsymbol{w}_i^T (\boldsymbol{h} - \boldsymbol{W}_{-i}^T \boldsymbol{z}_{-i})| \tag{10}$$

where $\boldsymbol{W}_{-i}$ excludes feature i. This reveals that $|z_i|$ measures the marginal contribution of feature i to reconstruction quality—exactly the budget allocation that optimally balances reconstruction accuracy against sparsity. Our probability distribution therefore has meaning as "relative feature strength":

$$p_i = \frac{\mathbb{E}[|z_i|]}{\sum_j \mathbb{E}[|z_j|]} = \frac{\text{expected budget allocation to feature } i}{\text{total representational budget}} \tag{11}$$

This fraction represents how much of the network's limited representational resources are optimally allocated to feature i under the SAE's constraints. Alternative norms fail to preserve this budget interpretation. The ℓ_2 norm $\mathbb{E}[z_i^2]$ overweights outliers and breaks the linear connection to reconstruction contributions through squaring. The ℓ_∞ norm captures only peak activation while ignoring frequency of use. The ℓ_0 norm provides binary active/inactive information but loses the magnitude data essential for measuring resource allocation intensity.

A.3 Shannon Entropy as Information Capacity Measure

Given the budget allocation distribution p, the exponential of Shannon entropy provides the theoretically optimal feature count. The exponential of Shannon entropy, $\exp(H)$, is formally known as perplexity in information theory and the Hill number (order-1 diversity index) in ecology (Hill, 1973; Jost, 2006):

$$\text{PP}(p) = \exp\left(-\sum_i p_i \log p_i\right) = \prod_{i=1}^{n} p_i^{-p_i} \tag{12}$$

This quantifies the effective number of outcomes in a probability distribution—how many equally likely outcomes would yield identical uncertainty. In information theory, it represents the effective alphabet size of a communication system (Jelinek et al., 1977). In ecology, it quantifies the effective number of species in

an ecosystem (Jost, 2006). In statistical physics, it relates to the number of accessible states in a system (Jaynes, 1957). In quantum mechanics, it corresponds to the effective number of pure quantum states in a mixed state (Schrödinger, 1935).

Shannon entropy uniquely satisfies the mathematical properties required for principled feature counting (Anand et al., 2011). The measure exhibits coding optimality, equaling the minimum expected code length for optimal compression. It satisfies additivity for independent feature sets through $H(p \otimes q) = H(p) + H(q)$. Small changes in feature importance yield small changes in measured count through continuity. Uniform distributions where all features are equally important maximize the count. Adding features with positive probability monotonically increases the count. These axioms uniquely characterize Shannon entropy up to a multiplicative constant, making $\exp(H(p))$ the theoretically principled choice for aggregating feature importance into an effective count.

In quantum systems, von Neumann entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ measures entanglement, with $e^{S(\rho)}$ representing effective pure states participating in a mixed quantum state (Nielsen & Chuang, 2011). Neural superposition exhibits parallel structure: just as quantum entanglement creates non-separable correlations that cannot be decomposed into independent subsystem states, neural superposition creates feature representations that cannot be cleanly separated into individual neuronal components. Both phenomena involve compressed encoding of information—quantum entanglement distributes correlations across subsystems resisting local description, while neural superposition distributes features across neurons resisting individual interpretation. Our measure $e^{H(p)}$ captures this compression by quantifying the effective number of features participating in the neural representation, analogous to how $e^{S(\rho)}$ quantifies effective pure states in an entangled quantum mixture.

Higher-order Hill numbers provide different sensitivities to rare versus common features:

$$^{q}\mathrm{D} = \left(\sum_{i=1}^{n} p_i^q \right)^{1/(1-q)} \tag{13}$$

where $q = 1$ gives our exponential entropy measure (via L'Hôpital's rule), $q = 0$ counts non-zero components, and $q = 2$ gives the inverse Simpson concentration index (participation ratio in statistical mechanics).

A.4 Rate-Distortion Theoretical Foundation

Our measurement framework emerges from two nested rate-distortion problems that formalize the intuitive resource allocation perspective. The neural network layer itself solves:

$$R_{\mathrm{NN}}(D) = \min_{p(\boldsymbol{h}|\boldsymbol{x}):\mathbb{E}[d(\boldsymbol{y}, f(\boldsymbol{h}))] \leq D} I(\boldsymbol{X}; \boldsymbol{H}) \tag{14}$$

where the layer width N constrains the mutual information $I(\boldsymbol{X}; \boldsymbol{H})$ that can be transmitted, while D represents acceptable task performance degradation. When the optimal solution requires representing $F > N$ features, superposition emerges naturally as the rate-optimal encoding strategy.

The sparse autoencoder solves a complementary problem:

$$R_{\mathrm{SAE}}(D) = \min_{p(\boldsymbol{z}|\boldsymbol{h}):\mathbb{E}[\|\boldsymbol{h} - \hat{\boldsymbol{h}}\|_2^2] \leq D} \mathbb{E}[\|\boldsymbol{z}\|_1] \tag{15}$$

where sparsity $\|\boldsymbol{z}\|_1$ acts as the rate constraint and reconstruction error as distortion. This dual structure justifies SAE-based measurement: we quantify the effective rate required to represent the network's compressed internal information under sparsity constraints.

The SAE optimization can be viewed as an information bottleneck problem balancing information preservation $\mathbb{E}[\|\boldsymbol{h} - g(\boldsymbol{z})\|_2^2]$ against information cost $\lambda \mathbb{E}[\|\boldsymbol{z}\|_1]$. Under this interpretation, $\mathbb{E}[|z_i|]$ represents the

information cost of including feature i in the compressed representation, making our probability distribution a natural measure of information allocation across features.

A.5 Critical Assumptions and Failure Modes

Our method measures effective representational diversity under sparse linear encoding, which approximates but does not exactly equal the number of distinct computational features. We must carefully assess the conditions under which this approximation holds.

Feature Correspondence Assumption. We assume SAE dictionary elements correspond one-to-one with genuine computational features. This assumption fails through feature splitting where one computational feature decomposes into multiple SAE features, artificially inflating counts. Feature merging combines multiple computational features into one SAE feature, deflating counts. Ghost features represent SAE artifacts without computational relevance (Gao et al., 2024). Incomplete coverage occurs when SAEs miss computationally relevant features entirely.

Linear Representation Assumption. We assume features combine primarily through linear superposition in activation space. Real networks violate this through hierarchical structure where low-level and high-level features aren't interchangeable. Gating mechanisms allow some features to control whether others activate (Elhage et al., 2022). Combinatorial interactions emerge when meaning comes from feature combinations rather than individual contributions (Black et al., 2022).

Magnitude-Importance Correspondence. We assume $|z_i|$ reflects feature i's computational importance. This breaks when SAE reconstruction preserves irrelevant details while missing computational essentials, when features interact nonlinearly in downstream processing (Engels et al., 2024), or when feature importance depends heavily on context rather than magnitude.

Independent Information Assumption. We assume Shannon entropy correctly aggregates information across features. This fails when correlated features don't contribute independent information, when synergistic information means feature pairs provide more information together than separately, or when redundant encoding has multiple features encoding identical computational factors.

The approximation captures genuine signal about representational complexity under specific conditions. The measure works best when features combine primarily through linear superposition, activation patterns are sparse with balanced importance, SAEs achieve high reconstruction quality on computationally relevant information, and representational structure is relatively flat rather than hierarchical. The approximation degrades with highly hierarchical representations, dense activation patterns with complex feature interactions, poor SAE reconstruction quality, or extreme feature importance skew. Despite these limitations, the measure provides principled approximation rather than exact counting, with primary value in comparative analysis across networks and training regimes.

A.6 Why Eigenvalue Decomposition Fails for SAE Analysis

Following the quantum entanglement analogy, one might consider eigenvalue decomposition of the covariance matrix:

$$\boldsymbol{\Sigma} = \frac{1}{n}\boldsymbol{A}\boldsymbol{A}^T \tag{16}$$

where $\boldsymbol{A}$ represents the activation matrix. Eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ represent explained variance along principal components, normalized to form a probability distribution:

$$p_i = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} \tag{17}$$

This approach faces fundamental rank deficiency when applied to SAEs. Expanding from lower dimension (N neurons) to higher dimension ($D > N$ dictionary elements) yields covariance matrices with rank at most N, making detection of more than N features impossible regardless of SAE capacity.

Our activation-based approach circumvents this limitation by directly measuring feature utilization through activation magnitude distributions rather than intrinsic dimensionality. This enables superposition quantification with overcomplete SAE dictionaries.

A.7 Adaptation to Convolutional Networks

Convolutional neural networks organize features across channels rather than spatial locations. For CNN layers with activations $\mathcal{X} \in \mathbb{R}^{B \times C \times H \times W}$, we measure superposition across the channel dimension while accounting for spatial structure.

We extract features from each spatial location's channel vector independently, then aggregate when computing feature probabilities:

$$p_i = \frac{\sum_{b,h,w} |z_{b,i,h,w}|}{\sum_{j=1}^{D} \sum_{b,h,w} |z_{b,j,h,w}|} \tag{18}$$

where $z_{b,i,h,w}$ represents feature i's activation at spatial position (h, w) in sample b.

This aggregation treats the same semantic feature activating at different spatial locations (e.g., edge detectors firing everywhere) as evidence for a single feature's importance rather than separate features.

B Experimental Details

B.1 Tracr Compression

We compile RASP programs using Tracr's standard pipeline with vocabulary $\{1, 2, 3, 4, 5\}$ and maximum sequence length 5. The sequence reversal program uses position-based indexing, while sorting employs Tracr's built-in sorting primitive with these parameters.

Following Lindner et al. (2023), we train compression matrices using a dual objective that ensures compressed models maintain both computational equivalence and representational fidelity:

$$\mathcal{L} = \lambda_{\text{out}} \mathcal{L}_{\text{out}} + \lambda_{\text{layer}} \mathcal{L}_{\text{layer}} \tag{19}$$

$$\mathcal{L}_{\text{out}} = \text{KL}(\text{softmax}(\mathbf{y}_c), \text{softmax}(\mathbf{y}_o)) \tag{20}$$

$$\mathcal{L}_{\text{layer}} = \frac{1}{L} \sum_{i=1}^{L} \|\mathbf{h}_i^{(o)} - \mathbf{h}_i^{(c)}\|_2^2 \tag{21}$$

where $\mathbf{y}_c$ and $\mathbf{y}_o$ denote compressed and original logits, and $\mathbf{h}_i^{(o)}$, $\mathbf{h}_i^{(c)}$ represent original and compressed activations at layer i.

Hyperparameters: $\lambda_{\text{out}} = 0.01$, $\lambda_{\text{layer}} = 1.0$, learning rate 10^{-3}, temperature $\tau = 1.0$, maximum 500 epochs with early stopping at 100% accuracy. We use Adam optimization and train separate compression matrices for each trial. For each compressed model achieving perfect accuracy, we extract activations from all residual stream positions across 5 trials. SAEs use fixed dictionary size 100, L1 coefficient 0.1, learning rate 10^{-3}, training for 300 epochs with batch size 128. We analyze the final layer activations (post-MLP) for consistency across compression factors.

B.2 Multi-Task Sparse Parity Experiments

Dataset Construction. We use the multi-task sparse parity dataset from Michaud et al. (2023) with 3 tasks and 4 bits per task. Each input consists of a 3-dimensional one-hot control vector concatenated with

12 data bits (total dimension 15). For each sample, the control vector specifies which task is active, and the label is computed as the parity (sum modulo 2) of the 4 bits corresponding to that task. This creates a dataset where ground truth bounds the number of meaningful features while maintaining task complexity.

Model Architecture. Simple MLPs with architecture Input(15) $\rightarrow$ Linear(h) $\rightarrow$ ReLU $\rightarrow$ Linear(1), where $h \in \{16, 32, 64, 128, 256\}$ for capacity experiments. We apply interventions (dropout) to hidden activations before the ReLU nonlinearity. Training uses Adam optimizer (lr=0.001), batch size 64, for 300 epochs with BCEWithLogitsLoss. Dataset split: 80% train, 20% test with stratification by task and label.

Intervention Protocols. Dropout experiments: Applied to hidden activations with rates [0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9]. **Dictionary scaling**: Expansion factors [0.5, 1.0, 2.0, 4.0, 8.0, 16.0] relative to hidden dimension, with L1 coefficients [0.01, 0.1, 1.0, 10.0], maximum dictionary size capped at 1024. Each configuration tested across 5 random seeds with 3 SAE instances per configuration for stability measurement.

SAE Architecture and Training. Standard autoencoder with tied weights: $\mathbf{z} = \text{ReLU}(\mathbf{W}_{\text{enc}}\mathbf{x} + \mathbf{b})$, $\mathbf{x}' = \mathbf{W}_{\text{dec}}\mathbf{z}$ where $\mathbf{W}_{\text{dec}} = \mathbf{W}_{\text{enc}}^T$. Dictionary size typically 4$\times$ layer width unless specified otherwise.

Loss function: $\mathcal{L} = ||\mathbf{x} - \mathbf{x}'||_2^2 + \lambda||\mathbf{z}||_1$ with L1 coefficient $\lambda = 0.1$ (unless testing λ sensitivity). Adam optimizer (lr=0.001), batch size 128, 300 epochs. For stability analysis, we train 3-5 SAE instances per configuration with different random seeds and report mean $\pm$ standard deviation.

B.3 Grokking

Task and Architecture. Modular arithmetic task: $(a + b) \bmod 53$ using sparse training data (40% of all possible pairs, 60% held out for testing). Model architecture: two-path MLP with shared embeddings.

$$\mathbf{e}_a = \text{Embedding}(a, \text{dim} = 12) \tag{22}$$

$$\mathbf{e}_b = \text{Embedding}(b, \text{dim} = 12) \tag{23}$$

$$\mathbf{h} = \text{GELU}(\mathbf{W}_1\mathbf{e}_a + \mathbf{W}_2\mathbf{e}_b) \tag{24}$$

$$\text{logits} = \mathbf{W}_3\mathbf{h} \tag{25}$$

where $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{R}^{48\times 12}$ and $\mathbf{W}_3 \in \mathbb{R}^{53\times 48}$.

Training Configuration. 25,000 training steps, learning rate 0.005, batch size 128, weight decay 0.0002. Model checkpoints saved every 250 steps (100 total checkpoints). Random seed 0 for reproducibility.

LLC Estimation Protocol. Local Learning Coefficient estimated using Stochastic Gradient Langevin Dynamics (SGLD) with hyperparameters: learning rate 3×10^{-3}, localization parameter $\gamma = 5.0$, effective inverse temperature $n_\beta = 2.0$, 500 MCMC samples across 2 independent chains. Hyperparameters selected via 5×5 grid search over epsilon range $[3 \times 10^{-5}, 3 \times 10^{-1}]$ ensuring $\varepsilon > 0.001$ for stability and $n_\beta < 100$ for β-independence.

B.4 Pythia-70M Analysis

Data Sampling and Preprocessing. 20,000 samples from Pile dataset (Gao et al., 2020), shuffled with seeds [42, 123, 456] for reproducibility. Text preprocessing: truncate to 512 characters before tokenization to prevent memory issues. Tokenization using model's native tokenizer with `max_length=512`, `truncation=True`, no padding. Samples with empty text or tokenization failures excluded.

Model and SAE Configuration. Pythia-70M model with layer specifications: embedding layer, and $\{\text{attn_out}, \text{mlp_out}, \text{resid_out}\}$ for layers 0–5. Pretrained SAEs from Marks et al. (2024) with dictionary size $64 \times 512 = 32,768$ features per layer. SAE weights loaded from subdirectories following pattern: `layer_type/10_32768/ae.pt`.

Activation Processing. Activations extracted using nnsight tracing with error handling for failed forward passes. Feature activations accumulated across all token positions and samples: $\text{feature_sum}_i = \sum_{\text{samples,positions}} |\mathbf{z}_i|$. Feature count computed from accumulated sums using entropy-based measure. Memory management: explicit cleanup of activation tensors and CUDA cache clearing between seeds.

B.5 Adversarial Robustness

B.5.1 Model Architectures

Simple Models (Single Hidden Layer)

- **SimpleMLP**: Input(784) → Linear(h) → ReLU → Linear(output)
  - Hidden dimensions $h \in 8, 32, 128, 512$
- **SimpleCNN**: Input → Conv2d(h, 5×5) → ReLU → MaxPool(2) → Linear(output)
  - Filter counts $h \in 8, 16, 32, 64$

Standard Models

- **StandardMLP**: Input(784) → Linear($4h$) → ReLU → Linear($2h$) → ReLU → Linear(h) → ReLU → Linear(output)
  - Base dimension $h = 32$, yielding layer widths [128, 64, 32]
- **StandardCNN**: LeNet-style architecture
  - Conv2d(1, h, 3×3) → ReLU → MaxPool(2)
  - Conv2d(h, $2h$, 3×3) → ReLU → MaxPool(2)
  - Linear($4h$) → ReLU → Linear(output)
  - Base dimension $h = 16$

CIFAR-10 Models

- **CIFAR10CNN**: Three-block CNN with batch normalization
  - Conv2d(3, h, 3×3) → BN → ReLU → MaxPool(2)
  - Conv2d(h, $2h$, 3×3) → BN → ReLU → MaxPool(2)
  - Conv2d($2h$, $4h$, 3×3) → BN → ReLU → MaxPool(2)
  - Dropout(0.2) → Linear(output)
  - Base dimension $h = 32$
- **ResNet-18**: Modified for CIFAR-10
  - Initial: Conv2d(3, 64, 3×3, stride=1, padding=1)
  - MaxPool replaced with Identity
  - Standard ResNet-18 blocks [2, 2, 2, 2]
- **WideResNet**: ResNet-18 with variable width
  - Width factors: $1/16, 1/8, 1/4, 1/2, 1, 2, 4, 8$
  - Initial channels: $16 \times$ width factor
  - Block channels: $16, 32, 64, 128 \times$ width factor

B.5.2 Training Protocols

MNIST/Fashion-MNIST:

- Optimizer: SGD with momentum 0.9
- Learning rate: 0.01, MultiStep decay at epochs [50, 75]
- Weight decay: 10^{-4}
- Epochs: 100
- Batch size: 128
- PGD: 40 steps, step size $\alpha = 0.01$
- FGSM: Single step, $\alpha = \epsilon$

CIFAR-10:

- Optimizer: SGD with momentum 0.9
- Learning rate: 0.1, MultiStep decay at epochs [100, 150]
- Weight decay: 5×10^{-4}
- Epochs: 200
- Batch size: 128
- PGD: 10 steps, step size $\alpha = 2/255$
- FGSM: Single step, $\alpha = \epsilon$

B.6 SAE Configuration

- Dictionary size: $4N$ ($4\times$ layer width)
- L1 coefficient: 0.1
- Optimizer: Adam, learning rate 10^{-3}
- Training: 800 epochs with early stopping (patience 50)
- Activation collection: 10,000 samples from test set
- Separate SAEs trained per layer