Online Convex Optimization with Hard Constraints: Towards the Best of Two Worlds and Beyond

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Abstract

This paper considers online convex optimization with hard constraints and analyzes achievable regret and cumulative hard constraint violation (violation for short). The problem distinguishes itself from online convex optimization with soft constraints, where a violation at one round can be compensated/cancelled by a conservative decision at a different round. We propose a RECtified Online Optimization algorithm (RECOO) and consider two settings: fixed (known) constraints and adversarial (unknown) constraints. Both settings have been considered in the literature. Compared with existing results, *RECOO achieves the best of two worlds and beyond.* For the fixed-constraints setting, RECOO achieves $O\left(\sqrt{T}\right)$ regret and $O(1)$ violation, where T is the learning horizon. The best known results in this case are $O(\sqrt{T})$ regret and $O\left(T^{1/4}\right)$ violation. For the adversarial-constraints setting, it guarantees $O(\sqrt{T})$ regret and $O(T^{3/4})$ violation, which match the best existing results. When the loss functions are strongly convex, RECOO can guarantee $O(\log T)$ regret and $O(1)$ violation for fixed constraints, and $O(\log T)$ regret and $O(\sqrt{T \log T})$ violation for adversarial constraints. Both these results are order-wise better than the existing bounds. The regret and violation bounds mentioned above use the best fixed decision in hindsight as the baseline. This paper further considers a dynamic baseline where the comparator sequence is time-varying. This paper shows that RECOO not only improves the existing results in the fixed-constraints setting but also *for the first time,* guarantees dynamic regret and violation bounds in the adversarial-constraints setting. Our experiment results confirm that RECOO outperforms several existing algorithms for both fixed and adversarial constraints.

1 Introduction

Online convex optimization (OCO) is a general framework for modelling and studying online decision-making in uncertain environments [21, 9, 19], where the learner adapts its decisions to minimize a loss function or to maximize a utility function when interacting with the environment in real-time. OCO has a broad range of applications such as online advertising [16, 2], resource allocation in network systems [8, 1, 29], load balancing in queueing systems [11, 13], and personalized healthcare [32, 23], etc. Specifically, given an unknown convex loss function $f_t(\cdot)$, the learner makes decision x_t at each round to minimize the total loss over T rounds, i.e. $\min \sum_{t=1}^{T} f_t(x_t)$, where the loss of decision x_t is revealed after x_t is executed. In practice, the decisions are often subject to a variety of operational constraints. For example, in online advertisement systems, users, who submit bids to advertise their items in order to maximize their click-through-rates, often have a weekly or monthly budget; and in a patient onboarding system, the hospital assigns an incoming patients to a medical unit to optimize the quality of treatment subject to personnel and medical resource constraints. To consider these applications, we study constrained online convex optimization (COCO) with $g_t(\cdot)$

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being a convex constraint function and $g_t(x_t) \leqslant 0$ being the constraint at round t. In COCO, the learner aims at minimizing the total loss while keeping the constraint violation to its minimum.

One possible method to satisfy the constraints is to project the decision x_t into the feasible set at each round [9]. However, such projection may not be possible when the constraints $g_t(\cdot)$ are adversarial and unknown before hand. It also may require heavy computation when $g_t(\cdot)$ is a complicated function [15, 30]. This motivates a sequence of studies on COCO with *soft* constraints, where the constraints are allowed to be violated as long as they are satisfied in the long term [15, 12, 18, 28, 22, 4, 3, 30, 24] (in other words, the severity of constraint violation is evaluated based on $\sum_t g_t(x_t)$). We call them soft constraints because the constraint violation at one round could be compensated in a different round. In other words, a decision sequence might have zero constraint violation based on the metric despite violating the constraint(s) at almost every round. For example, consider a decision sequence such that $\{g_t(x_t)\} = \{-1000, \underbrace{1, 1, 1, \ldots, 1, 1}_{1,000}\}$. For such a sequence, we have $\sum_{t=1}^{\tau} g_t(x_t) \leqslant 0$ for all $\tau \leqslant T$, but the constraint is violated at all rounds except the first round. While some constraints such as budget or fairness constraints are naturally soft constraints, a stronger notion of constraint and constraint violation is needed for other applications, in particular, for safety-critical applications. In this paper, we consider a stronger notion of constraint violation as in [31, 26, 27]

$$\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t), \tag{1}$$

where that the operator $(\cdot)^+ = \max(\cdot, 0)$ is imposed on each round and all violations during the decision process are added up. We call this *hard* constraint and the metric *cumulative hard constraint violation*. If a decision sequence has a small $\mathcal{V}(T)$, it implies the constraints are satisfied most of the time during the learning. For example, consider the same example, $\mathcal{V}(T) = 1000$ instead of 0 under this stronger notion of constraint violation. In this paper, we consider the following two settings:

- Fixed constraints $g_t(x) = g(x), \forall t$, where the constraint function is known (fixed) when making decision at round t as in [15, 12, 30, 26].
- Adversarial constraints $g_t(x)$, where the constraint function $g_t(x)$ is unknown when making decision at round t and can be arbitrarily and adversarially chosen, as in [22, 18, 27].

We will show that RECOO is a single algorithm that achieves small regret and violation in both settings. Next, we summarize our main contributions and compare them with the most related works.

1.1 Main contributions

Algorithm: This paper develops a RECtified Online Optimization algorithm, called RECOO, for COCO-Hard (COCO with hard constraints). RECOO is a *unifying* algorithm for both fixed and adversarial constraints. By introducing rectifiers in both "decision-making" and "penalty update" components, RECOO maintains a minimum penalty price for constraint violation and encourages conservative/pessimistic decisions, which is the key to minimizing the cumulative hard constraint violation.

Regret and violation bounds: We first consider the best fixed decision in hindsight as the baseline (also called the static baseline). The main theoretical results are summarized below.

- For fixed constraints, RECOO achieves $O(\sqrt{T})$ regret and $O(1)$ violation for convex loss functions, and $O(\log T)$ regret and $O(1)$ violations for strongly-convex loss functions.
- For adversarial constraints, RECOO achieves $O(\sqrt{T})$ regret and $O(T^{3/4})$ violation for convex loss functions, and $O(\log T)$ regret and $O(\sqrt{T \log T})$ violation for strongly convex loss functions.

We then consider a dynamic baseline where the comparator is time-varying with path length P_T and establish the following results.

- For fixed constraints, RECOO achieves $O(\sqrt{P_T T})$ regret and $O(\log T)$ violation, where P_T is the path length of the comparator sequence [33, 26].
- For adversarial constraints, RECOO achieves $O\left(P_T \sqrt{T}\right)$ regret and $O(T^{3/4})$ violation.

The comparison with most related works are summarized in Tables 1 and 2, from which we can see that RECOO achieves and improves the state-of-the-art results for both settings ("N/A" means no results are reported). Below are a few highlights we would like to mention:

- For fixed constraints and the static baseline, RECOO not only improves the results in [26] but also answers a conjecture in [26] that $O(1)$ violation may be achievable under Slater's condition. We proved that the conjecture is true even without Slater's condition.
- For adversarial constraints, the static baseline, and strongly convex loss functions, RECOO reduces the $O(T^c)$ regret and $O(T^{1-c/2})$ violation in [27] to $O(\log T)$ and $O(\sqrt{T\log T})$, respectively.
- For fixed constraints and the dynamic baseline, RECOO improves the violation from $O(\sqrt{T})$ in [26] to $O(\log T)$, while maintaining the same regret bound.
- For adversarial constraints and the dynamic baseline, RECOO achieves $O\left(P_T\sqrt{T}\right)$ regret and $O(T^{3/4})$ violation, which is the first regret and violation bounds in this setting.

Reference	Loss Function	Regret, Violation	Dynamic Regret, Violation
[31]	Convex	$O(\sqrt{T})$, $O(T^{3/4})$	N/A
[26]		$O(\sqrt{T})$, $O(T^{1/4})$	$O(\sqrt{P_T T})$, $O(\sqrt{T})$
RECOO		$O(\sqrt{T})$, $O(1)$	$O(\sqrt{P_T T})$, $O(\log T)$
[31]	Strongly Convex	$O(\log T)$, $O(\sqrt{T\log T})$	N/A
[26]		$O(\log T)$, $O(\log T)$	$O(\sqrt{P_T T})$, $O(\sqrt{T})$
RECOO		$O(\log T)$, $O(1)$	$O(\sqrt{P_T T})$, $O(\log T)$

Table 1: Our results and related work for fixed constraints.

Reference	Loss Function	Regret , Violation	Dynamic Regret , Violation
[27]	Convex	$O(\sqrt{T})$, $O(T^{3/4})$	N/A
RECOO		$O(\sqrt{T})$, $O(T^{3/4})$	$O(P_T\sqrt{T})$, $O(T^{3/4})$
[27]	Strongly Convex	$O(T^{c/2})$, $O(T^{1-c/2})$	N/A
RECOO		$O(\log T)$, $O(\sqrt{T\log T})$	$O(P_T\sqrt{T})$, $O(\sqrt{T\log T})$

Table 2: Our results and related work for adversarial constraints.

Analysis: The key method to establish these results is by imposing the rectified penalty when making decisions, which enable us to develop a simple yet effective self-bounding technique to quantify the trade-off between the regret and the cumulative hard constraints. These methods are tailored for the cumulative hard constraints and are different from the drift-plus-penalty method used in the related works [26, 27, 30], where the constraint violations are established by bounding the dual variables (also called virtual queues).

1.2 Related work

COCO has been studied in the literature [15, 22, 18, 14, 5, 20, 31, 26, 27]. We have already presented a detailed comparison with the existing results on COCO-Hard. We next review some of recent results on COCO-Soft, COCO with soft constraints.

COCO-Soft with fixed constraints: [15] studies COCO-Soft and proposed an algorithm that achieves $O(\sqrt{T})$ regret and $O(T^{3/4})$ violation. This result has been extended in [12] to characterize the regret-violation tradeoff, in particular, the proposed algorithm achieves $O(T^{\max\{c,1-c\}})$ regret and $O(T^{1-c/2})$ violation, where c controls the trade-off. Assuming Slater's condition holds, the algorithm in [30] achieves $O(\sqrt{T})$ regret and $O(1)$ violation and is designed based on a powerful "drift-plus-penalty" method, which inspires some of the design in RECOO.

COCO-Soft with adversarial constraints: Adversarial constraints are more difficult to satisfy but have been considered in the literature [22, 18, 14, 5, 20]. For COCO-Soft with adversarial constraints, the authors in [22] developed an online mirrored descent type algorithm that achieves $O(\sqrt{T})$ regret and $O(T^{3/4})$ violation. Later, [5, 14] generalized the baseline in [22] and still achieve $O(\sqrt{T})$ regret

and $O(T^{3/4})$ violation. With Slater's condition, [18] presents an online gradient descent algorithm based on the drift-plus-penalty method [17], which achieves $O(\sqrt{T})$ regret and $O(\sqrt{T})$ violation and [20] extended it to an online optimization with sub-modular losses. It remains open that whether $O(T^{3/4})$ violation can be reduced with adversarial constraints (soft or hard) while keeping $O(\sqrt{T})$ regret without Slater' condition.

2 COCO-Hard

In this section, we formally define COCO-Hard. Consider the following online convex optimization problem: At each round $t \in [T]$, the learner makes decision x_t and then observes the loss $f_t(x_t)$ and constraint function $g_t(\cdot)$ after the decision is executed. The goal of the learner is to generate a decision sequence $\{x_1, x_2, \ldots, x_{T-1}, x_T\}$ to minimize the total loss $\sum_{t=1}^{T} f_t(x_t)$ and the hard constraint violation $\sum_{t=1}^{T} g_t^+(x_t)$.

To quantify the performance of an online convex optimization algorithm for COCO-Hard, we will first compare it a static baseline, called the best fixed decision in hindsight, which is the solution to the following offline COCO.

Offline COCO The offline COCO is formulated as follows

$$\min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_t(x) \text{ subject to: } g_t(x) \leqslant 0, \forall 1 \leqslant t \leqslant T, \tag{2}$$

where $\mathcal{X}$ is a "simple" convex set (e.g., positive quadrant or probability simplex), $\{f_t(\cdot)\}_t$ are convex loss functions and $\{g_t(\cdot)\}_t$ are convex constraint functions. The optimal solution x^* to offline COCO is called the best single decision in hindsight, a widely used baseline for COCO [9, 15, 22, 28].

Based on offline COCO, the regret and constraint violation of an online algorithm are defined below.

Regret and cumulative hard constraint violation (or violation for short) We define the regret and cumulative constraint violation with respect to the baseline as follows

$$\mathcal{R}(T) := \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*), \tag{3}$$

$$\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t). \tag{4}$$

3 RECOO: a rectified online optimization algorithm

We present RECOO, an online optimization algorithm with the rectified decision and penalty update.

RECOO — A Rectified Online Optimization Algorithm

Initialization: $x_0 \in \mathcal{X}$, $Q(0) = 0$, $f_0(x) = g_0(x) = 0$, $\forall x \in \mathcal{X}$ and learning rates $\alpha_t, \eta_t, \gamma_t$.

For $t = 1, \cdots, T$,

- **Set:** $\hat{g}_{t-1}^+(x) = \gamma_{t-1} g_{t-1}^+(x)$.
- **Rectified decision:** find the optimal solution of x_t by solving:

$$x_t = \arg\min_{x \in \mathcal{X}} \langle \nabla f_{t-1}(x_{t-1}), x - x_{t-1} \rangle + Q(t-1)\hat{g}_{t-1}^+(x) + \alpha_{t-1}\|x - x_{t-1}\|^2$$

- **Observe:** value $\nabla f_t(x_t)$ and constraint function $g_t(\cdot)$.
- **Rectified penalty update:** update the estimates of penalty variables $Q(t)$ as follows:

$$Q(t) = \max\left(Q(t-1) + \hat{g}_t^+(x_t), \eta_t\right).$$

In RECOO, the learning rates $\alpha_t, \eta_t, \gamma_t$ are chosen differently depending on the loss function is convex or strongly convex, but are oblivious to the type of constraints (fixed or adversarial). The choices of learning rates can be found in the theorem statements.

We explain the intuition behind RECOO and the importance of the "rectifiers". The Lagrange function of the offline optimization problem (2) is defined to be

$$L(\lambda, x) := \sum_{t=1}^{T} L_t(\lambda_t, x) := \sum_{t=1}^{T} f_t(x) + \lambda_t g_t(x),$$

where $\{\lambda_t\}$ are the dual variables associated with the constraints in (2).

Since we have no prior knowledge of $f_t(\cdot)$ when making decision x_t, we estimate it with the first-order approximation at x_{t-1} based on the historical information as follows

$$\hat{f}_t(x) = f_{t-1}(x_{t-1}) + \langle \nabla f_{t-1}(x_{t-1}), x - x_{t-1} \rangle.$$

Moreover, we replace the original constraint function $g_t(\cdot)$ with $\hat{g}_{t-1}^{+}(\cdot)$ and the dual variables λ_t with $Q(t-1)$ such that $Q(t-1)\hat{g}_{t-1}^{+}(x)$ is a rectified approximator of $\lambda_t g_t(x)$. We also added the the regularization (or smooth) term $\alpha_t \|x - x_t\|^2$ that helps the stability of the algorithm.

In the penalty update of $Q(t)$, we first rectify the original constraint function $g_t(\cdot)$ with $\hat{g}_t^{+}(\cdot)$ and add it to $Q(t-1)$ such that penalty increases when violation occurs in each round. Further we rectify $Q(t)$ with a round-dependent constant η_t to impose a "minimum" penalty price. The design of rectified penalty update induces conservative decisions in the decision-making step to minimize constraint violation. Note that it is different with the traditional primal-dual algorithm that does not rectify the constraint violation and impose a minimum penalty price, and when the price (dual variable) is zero, the algorithm can take very aggressive decisions that lead to large hard violation. This is not a problem when primal-dual algorithm is used as a numerical method for solving a constrained optimization problem, but leads to overly aggressive decisions and large violation when applying it to COCO. For a similar reason, RECOO rectifies the amount of violation $g_t(x)$ in the decision-making step so the violation does not become negative to prevent overly optimistic decisions. We will see that the rectifiers in both decision-making and penalty update lead to a small cumulative hard constraint violation $\sum_{t=1}^{T} g_t^{+}(x_t)$.

We next analyze the regret and violation of RECOO as defined in (3) and (4) based on the following standard technical assumptions on the feasible set, loss and constraint functions.

Assumption 1 *The feasible set $\mathcal{X}$ is convex with diameter D such that $\|x - x'\| \leqslant D, \forall x, x' \in \mathcal{X}$.*

Assumption 2 *The loss function is Lipschitz continuous with Lipschitz constant F such that $|f_t(x) - f_t(x')| \leqslant F\|x - x'\|, \forall x, x' \in \mathcal{X}, \forall t$.*

Assumption 3 *The constraint function is Lipschitz continuous with Lipschitz constant G such that $|g_t(x) - g_t(x')| \leqslant G\|x - x'\|, \forall x, x' \in \mathcal{X}, \forall t$. For ease of exposition, assume $g_1(0) \leqslant 0$.*

Note Assumption 1 implies the feasible set $\mathcal{X}$ is bounded. Assumptions 2 and 3 mean the loss and constraint functions have bounded gradients. With these assumption, we are ready to present our main result of RECOO for both fixed and adversarial constraints.

Theorem 1 *Choose $\alpha_t = \sqrt{t}$, $\eta_t = \sqrt{t}$, and $\gamma_t = t^{\frac{1}{2}+\varepsilon}$, where $\varepsilon > 0$. Under Assumptions 1-3, RECOO algorithm achieves the following regret and constraint violation bounds:*

$$\mathcal{R}(T) \leqslant \left(\frac{F^2}{4} + D^2\right)\sqrt{T} \text{ (fixed or adversarial constraints)},$$

$$\mathcal{V}(T) \leqslant F^2 + FD\left(1 + \frac{1}{\varepsilon}\right) + D^2 \text{ (fixed constraints), and}$$

$$\mathcal{V}(T) \leqslant \left(F^2 + \frac{G^2}{4} + FD\left(5 + \frac{1}{\varepsilon}\right) + 2D^2\right) T^{\frac{3}{4}} \text{ (adversarial constraints).}$$

Remark 1 *For the fixed constraints, Theorem 1 establishes the optimal order-wise results of $O(\sqrt{T})$ regret and $O(1)$ violation since $\Omega(T)$ regret is the lower bound for OCO with general convex loss and $O(1)$ is the best one can have. For the adversarial constraints, by carefully choosing the learning rates in RECOO algorithm, we can establish a trade-off of $O(T^c)$ regret and $O(T^{1-c/2})$ cumulative violation where $c \in [1/2, 1)$ (see details of Corollary 1 in Appendix C).*

Next, we show Theorem 1 can be further improved when the loss function is strongly convex.

Assumption 4 *The loss functions $f_t(\cdot), \forall t$, are μ-strongly convex in $\mathcal{X}$ for $\mu > 0$, i.e. $f_t(x') \geqslant f_t(x) + \langle x' - x, \nabla f_t(x) \rangle + \frac{\mu}{2}\|x - x'\|^2,\ \forall x, x' \in \mathcal{X}, \forall t$.*

With Assumption 4, we are able to establish a $O(\log T)$ regret and violation bounds at the same or a smaller order.

Theorem 2 *Choose the learning rates to be $\alpha_t = \frac{\mu t}{2}, \eta_t = \sqrt{t}, \gamma_t = t^{\frac{1}{2}+\varepsilon}, \forall t \in [T]$, where $\varepsilon > 0$. Under Assumptions 1-4, RECOO achieves the following regret and violation bounds:*

$$\mathcal{R}(T) \leqslant \frac{F^2}{2\mu}(1 + \log T) \ \text{(fixed or adversarial constraints)},$$
$$\mathcal{V}(T) \leqslant \frac{F^2}{\mu} + FD\left(1 + \frac{1}{\epsilon}\right) \ \text{(fixed constraints), and}$$
$$\mathcal{V}(T) \leqslant \left(\frac{F^2}{\mu} + \frac{G^2}{4} + FD\left(1 + \frac{1}{\varepsilon} + \frac{4}{\mu}\right) + D^2\right)\sqrt{T(1 + \log T)} \ \text{(adversarial constraints)}.$$

Remark 2 *The $O(\log T)$ regret is a well-known result in OCO (unconstrained) for a strongly convex loss function [10]. Therefore, for fixed constraints, our results are order-wise sharp because $O(1)$ violation is the best one can achieve.*

Extension to a dynamic baseline

In this paper, we establish the main results against a static baseline $x_t^* = x^*, \forall t$, to convey the idea of RECOO algorithm that can improve the violation in Theorems 1 and 2. In fact, RECOO can be generalized to a dynamic baseline as in [33, 26], where x_{t+1}^* could be different with x_t^* but with limited variation defined by $P_T = \sum_{t=1}^{T-1} \|x_{t+1}^* - x_t^*\|$. Define $\mathcal{R}^{\text{dynamic}}(T) := \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_t^*)$. Without any modifications, we show that RECOO algorithm can achieve $O(\sqrt{T}(1 + P_T))$ regret and $O(1)$ violation in the following corollary.

Theorem 3 *Under Assumptions 1-3, let the learning rates be $\alpha_t = \sqrt{t}, \eta_t = \sqrt{t}, \gamma_t = t^{\frac{1}{2}+\varepsilon}, \forall t \in [T]$, where $\varepsilon > 0$. Let $\{x_t^*\}$ be the optimal solution to (2) with $P_T = \sum_{t=1}^{T-1} \|x_{t+1}^* - x_t^*\|$. RECOO algorithm achieves the following bounds on the regret and cumulative constraint violations:*

$$\mathcal{R}^{\textit{dynamic}}(T) \leqslant \left(\frac{F^2}{2} + D^2 + 2DP_T\right)\sqrt{T+1} \ \text{(fixed or adversarial constraints)},$$
$$\mathcal{V}(T) \leqslant F^2 + FD\left(1 + \frac{1}{\varepsilon}\right) + D^2 \ \text{(fixed constraints), and}$$
$$\mathcal{V}(T) \leqslant \left(F^2 + \frac{G^2}{4} + FD\left(5 + \frac{1}{\varepsilon}\right) + 2D^2\right)T^{\frac{3}{4}} \ \text{(adversarial constraints)}.$$

Remark 3 *RECOO algorithm can achieve $O\left(P_T\sqrt{T}\right)$ regret and $O(1)$ violation for fixed constraints, which improves $O\left(P_T\sqrt{T}\right)$ regret and $O(T^{1/4})$ violation in [26]. Moreover, RECOO algorithm achieve $O\left(P_T\sqrt{T}\right)$ regret and $O(T^{3/4})$ violation for adversarial constraints, which is new in the literature. As shown in Corollary 2 in Appendix F, combining RECOO with expert-tracking techniques proposed in [33], we achieve $O\left(\sqrt{P_T T}\right)$ regret and $O(\log T)$ violation for the case of fixed constraints, which improves $O\left(\sqrt{P_T T}\right)$ regret and $O(\sqrt{T})$ violation in [26].*

Next, we provide the detailed analysis on Theorem 1. The proofs of Theorem 2 and Theorem 3 can be found in Appendix.

3.1 Proof of Theorem 1

We first introduce a key lemma that establishes an upper bound on "regret plus violation" at each round t, which we called self-bounding property.

Lemma 1 (Self-Bounding Property) *Recall that x^* denotes the optimal solution to offline COCO* (2) *and x_t be the optimal solution returned by RECOO algorithm. We have*

$$f_t(x_t) - f_t(x^*) + Q(t)\hat{g}_t^+(x_{t+1}) \leqslant \frac{F^2}{4\alpha_t} + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2 \tag{5}$$

Based on Lemma 1, we establish the regret and the constant violation bounds in Theorem 1.

Proof of Theorem 1: fixed constraints

Based on (5), we have

$$f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{4\alpha_t} + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{6}$$

$$Q(t)\hat{g}_t^+(x_{t+1}) \leqslant \frac{F^2}{4\alpha_t} + |f_t(x_t) - f_t(x^*)| + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{7}$$

which we will use next to establish the bounds on regret and violation.

Regret bound: We sum up Inequality (6) for $t = 1, \cdots, T$ and have

$$\begin{aligned}\sum_{t=1}^{T}(f_t(x_t) - f_t(x^*)) \leqslant & \frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + \sum_{t=1}^{T}(\alpha_t - \alpha_{t-1})\|x^* - x_t\|^2 \\ \leqslant & \frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + D^2\sum_{t=1}^{T}(\alpha_t - \alpha_{t-1})\end{aligned}$$

where the first inequality holds by dropping the last negative term and the last inequality holds by Assumption 1. Choose $\alpha_t = \sqrt{t}$, we have

$$\sum_{t=1}^{T}(f_t(x_t) - f_t(x^*)) \leqslant \left(\frac{F^2}{2} + D^2\right)\sqrt{T}.$$

Violation bound: We substitute the definition of $\hat{g}_t^+(\cdot)$ in (7) and have

$$\begin{aligned}g_t^+(x_{t+1}) \leqslant & \frac{F^2}{4Q(t)\alpha_t\gamma_t} + \frac{|f_t(x_t) - f_t(x^*)|}{Q(t)\gamma_t} + \frac{\alpha_t}{Q(t)\gamma_t}\|x_t - x^*\|^2 - \frac{\alpha_t}{Q(t)\gamma_t}\|x_{t+1} - x^*\|^2 \\ \leqslant & \frac{F^2}{4t^{\frac{3}{2}+\varepsilon}} + \frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} + \frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{\frac{1}{2}+\varepsilon}},\end{aligned}$$

where the second inequality holds by $Q(t) \geqslant \eta_t$ according to the rectified penalty update $Q(t) = \max\left(Q(t-1) + \hat{g}_t^+(x_t), \eta_t\right)$. The inequality implies that

$$\begin{aligned}\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant & \sum_{t=1}^{T}\frac{F^2}{4t^{\frac{3}{2}+\varepsilon}} + \sum_{t=1}^{T}\frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} + \sum_{t=1}^{T}\frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{\frac{1}{2}+\varepsilon}} \\ \leqslant & F^2 + FD + \frac{FD}{\varepsilon} + D^2\end{aligned} \tag{8}$$

where the second inequality holds by Lemma 6, which includes the detailed calculations of the three terms above. Note that the left-hand-side of inequality (8) is not the violation because the index mismatch (the violation at round t is $g_t^+(x_t)$ not $g_t^+(x_{t+1})$).

For fixed constraints, i.e. $g_t(x) = g(x)$, the inequality above implies that $\mathcal{V}(T) := \sum_{t=1}^{T} g^+(x_t) \leqslant F^2 + FD\left(1 + \frac{1}{\varepsilon}\right) + D^2$. We have proved the first part of Theorem 1 for the fixed constraints. Let us continue with (8) to prove the second part of Theorem 1 for adversarial constraints.

Proof of Theorem 1: adversarial constraints

To quantify $\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t)$ for the adversarial constraints, we need to establish the relationship between $g_t^+(x_t)$ and $g_t^+(x_{t+1})$ to address the index mismatch, which can be bounded by $\|x_{t+1} - x_t\|^2$ as shown in the following lemma.

Lemma 2 *Under Assumptions 1-3, RECOO achieves for any $\beta > 0$*

$$g_t^+(x_t) - g_t^+(x_{t+1}) \leqslant \frac{G^2}{4\beta} + \beta\|x_t - x_{t+1}\|^2.$$

The next lemma further quantifies $\sum_{t=1}^T \|x_{t+1} - x_t\|^2$.

Lemma 3 *Under Assumptions 1-3, RECOO achieves*

$$\sum_{t=1}^T \|x_{t+1} - x_t\|^2 \leqslant 4FD\sqrt{T} + D^2.$$

Take summation of this equality in Lemma 2 from $t = 1$ to T, and we have

$$\begin{aligned}\sum_{t=1}^T \left(g_t^+(x_t) - g_t^+(x_{t+1})\right) \leqslant & \frac{G^2 T}{4\beta} + \beta \sum_{t=1}^T \|x_t - x_{t+1}\|^2 \\ \leqslant & \left(\frac{G^2}{4} + 4FD\right) T^{3/4} + D^2 T^{1/4}, \end{aligned} \tag{9}$$

where the last inequality holds by letting $\beta = T^{1/4}$. Combining (8) and (9) completes the proof.

4 Experiments

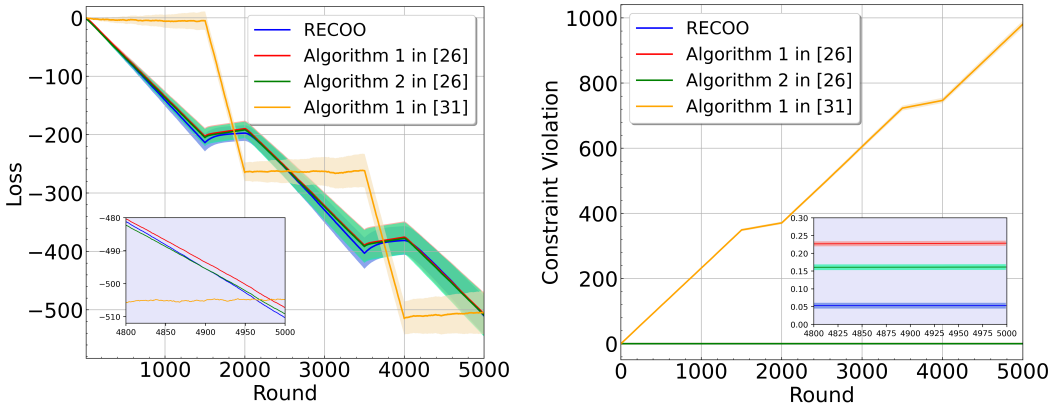
In this section, we present numerical experiments for evaluating the performance of RECOO with fixed and adversarial constraints. We compared RECOO with the algorithms in [31, 26]for the fixed-constraints setting; and with the algorithm in [27] for the adversarial-constraints setting. All results are obtained by averaging over 500 trials and reported with 95% confidence interval.

Fixed constraints Similar to [26], we considered loss functions $f_t(x) = \langle \theta(t), x\rangle$, where $\theta(t)$ is time varying and unknown at round t; and the fixed constraint function $Ax \leqslant b$, where $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$. We chose the number of total rounds to be $T = 5000$, and chose $\theta(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]$, where $\theta_1(t) \sim U(-t^{1/10}, +t^{1/10})$, $\theta_2(t) \sim U(-1, 0)$ when $t \in [1, 1500] \cup [2000, 3500] \cup [4000, 5000]$ and $\theta_2(t) \in U(0, 1)$ otherwise; $\theta_3(t) = (-1)^{\mu(t)}$ where $\mu(t)$ is a random permutation of vector $[1 : 5000]$. Let $\mathcal{X} = \{x | 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$, $A_{i,j} \sim U(0.1, 0.5), 1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 2$, and $b_i \sim U(0, 0.3), 1 \leqslant i \leqslant 3$, respectively.

We compared RECOO with Algorithm in [31] and Algorithm 1 and Algorithm 2 in [26], where learning rates are summarized in Table 3 as suggested in [31] and [26]. Figure 1a illustrates the trajectories of the cumulative losses $\sum_{t=1}^T f_t(x_t)$, where we observe all of these algorithms have similar trends while RECOO achieves the best cumulative losses. Figure 1b shows RECOO achieves the smallest cumulative violation. In particular, the (mean, variance) pair of RECOO for loss and violation are $(-510.23, 32.78)$ and $(0.05, 0.008)$ at the end of learning horizon, respectively; that of Algorithm 1 in [26] are $(-507.17, 36.46)$ and $(0.23, 0.007)$; that of Algorithm 2 in [26] are $(-509.09, 36.73)$ and $(0.16, 0.008)$;that of Algorithm 1 in [31] are $(-504.81, 36.82)$ and $(980.64, 10.49)$. These results and Figure 1 confirm that RECOO outperforms the existing algorithms w.r.t. both cumulative loss and violation.

Algorithm	Parameters
RECOO Algorithm	$\alpha_t = 3\sqrt{t}, \eta_t = \sqrt{t}, \gamma_t = t^{1/2+\varepsilon}, \varepsilon = 0.01$
Algorithm 1 in [31]	$\eta_t = \frac{1}{G\sqrt{R(m+1)T}}, \sigma_t = \frac{(m+1)R}{2(1-\alpha)}, G = 4, m = 3, R = 2, \alpha = 3/4$
Algorithm 1 in [26]	$\alpha_t = 2/T^c, \gamma_t = T^{c/2}, c = 0.5$
Algorithm 2 in [26]	$\alpha_{i,t} = 2^{i-1}/T^c, \gamma_{i,t} = T^{c/2}, c = \kappa = 0.5$

Table 3: Parameters in each algorithm.



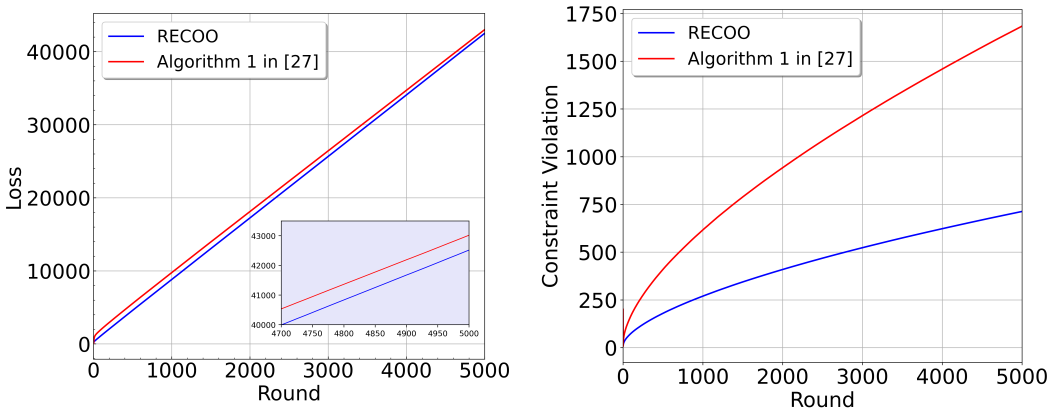
(a) Loss

(b) Cumulative Violation

Figure 1: Experiment with fixed constraints

Experiment with adversarial constraints We considered an OCO with adversarial constraints similar as the one in [27]. The loss function was chosen to be $f_t(x) = \frac{1}{2}\|H(t) \cdot x - y(t)\|^2$ with $H(t) \in R^{4\times 10}$, $x \in \mathbb{R}^{10}$ and $y(t) \in \mathbb{R}^4$ where $H_{i,j}(t) \sim U(-1,1), 1 \leqslant i \leqslant 4, 1 \leqslant j \leqslant 10$, and $y_i(t) = \sum_{j=1}^{i,j} H_{i,j}(t) + \varepsilon_i$ with ε_i being the standard normal random variable for any i. The constraint functions were chosen to be $g_t(x) = A(t) \cdot x - b(t)$ with $A(t) \in \mathbb{R}^{2\times 10}$ and $b(t) \in \mathbb{R}^2$, where $A_{i,j}(t) \sim U(0,2), \forall i,j,t$ and $b_i(t) \sim U(0,1), \forall i,t$, respectively.

We compared RECOO with Algorithm 1 in [27], with $\alpha_t = \eta_t = \sqrt{t}$, $\gamma_t = t^{1/2+0.01}$ in RECOO; and $\alpha_t = 0.8/\sqrt{t}$, $\beta_t = 5/\sqrt{t}$ and $\gamma_t = 0.5/\sqrt{t}$ for Algorithm 1 in [27] (these are the optimized learning rates). Figure 2 includes the cumulative losses and violations. In particular, The (mean, variance) pair of RECOO for loss and violation are $(42510.14, 38.05)$ and $(713.45, 1.60)$ at the end of learning horizon, respectively while that of Algorithm 1 in [27] are $(43011.29, 56.17)$ and $(1684.17, 2.85)$. It is justified RECOO performs better, especially in terms of the cumulative violation, which further shows the "rectified" design reduces the cumulative hard constraint violation.



(a) Loss

(b) Cumulative Violation

Figure 2: Experiment with adversarial constraints

5 Conclusions

In this paper, we studied online convex optimization with hard constraints (COCO-Hard) under two settings: fixed (known) constraints and adversarial (unknown) constraints. We proposed a RECtified Online Optimization algorithm (RECOO) and proved it achieves the best of two worlds and beyond. The algorithm improves the best-known results for the fixed-constraints, matches the best results for the adversarial-constraints, outperforms the state-of-art results when the loss functions are strongly-convex. The experiments confirmed our theoretical results.

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Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper's contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [No] We didn't find any major limitations in applying our results to constrained online convex programming.
   (c) Did you discuss any potential negative societal impacts of your work? [N/A]
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]
2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes]
3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes]
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [N/A]
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes]
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [N/A]
4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [N/A]
   (b) Did you mention the license of the assets? [N/A]
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you're using/curating? [N/A]
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]
5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]

A Auxiliary lemmas

The following lemma provides a useful upper bound on the optimal value of strongly convex function [28, 7, 25].

Lemma 4 *Let $\mathcal{X}$ be a convex set. Let $h : \mathcal{X} \to \mathcal{R}$ be α-strongly convex function on $\mathcal{X}$ and x_{opt} be an optimal solution of h, i.e., $x_{opt} = \arg\min_{x\in\mathcal{X}} h(x)$. Then, $h(x_{opt}) \leqslant h(x) - \frac{\alpha}{2}\|x - x_{opt}\|^2$ for any $x \in \mathcal{X}$.*

Proof *The proof of the lemma is based on the definition of α-strongly convex functions and the first-order optimality condition. Define the subgradient of $h(x)$ to be $\nabla h(x)$, According to the definition of strong convexity, we have*

$$h(x) \geqslant h(y) + \langle \nabla h(y), x - y\rangle + \frac{\alpha}{2}\|x - y\|^2. \tag{10}$$

Define $x_{opt} = \arg\min_{x\in\mathcal{X}} h(x)$. Let $y = x_{opt}$ in (10), we have

$$h(x) \geqslant h(x_{opt}) + \langle \nabla h(x_{opt}), x - x_{opt}\rangle + \frac{\alpha}{2}\|x - x_{opt}\|^2.$$

We then conclude the proof based on the first-order optimality condition that for any $x \in \mathcal{X}$,

$$\langle \nabla h(x_{opt}), x - x_{opt}\rangle \geqslant 0.$$

The following lemma is the key to bridge the regret and the constraint violation.

Lemma 5 *Define*

$$h_t(x) := \langle \nabla f_t(x_t), x - x_t\rangle + Q(t)\hat{g}_t^+(x) + \alpha_t\|x - x_t\|^2.$$

Let x_{t+1} be the optimal solution returned by RECOO, i.e., $x_{t+1} = \arg\min_{x\in\mathcal{X}} h_t(x)$. We have for any $x \in \mathcal{X}$ that

$$\begin{aligned}&\langle \nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2\\ \leqslant&\langle \nabla f_t(x_t), x - x_t\rangle + Q(t)\hat{g}_t^+(x) + \alpha_t\|x - x_t\|^2 - \alpha_t\|x - x_{t+1}\|^2.\end{aligned} \tag{11}$$

Proof *The proof is a direct application of Lemma 4. Note that $h_t(x)$ is $2\alpha_t$-strongly convex because $\langle \nabla f_t(x_t), x - x_t\rangle + Q(t)\hat{g}_t^+(x)$ is convex in x and $\alpha_t\|x - x_t\|^2$ is $2\alpha_t$-strongly convex.*

The following lemma is to provide the detailed calculations required for obtaining inequality (8).

Lemma 6 *Under Assumptions 1-3, we have*

$$\sum_{t=1}^{T}\frac{1}{t^{\frac{3}{2}+\varepsilon}} \leqslant 3,\quad \sum_{t=1}^{T}\frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} \leqslant FD\left(1 + \frac{1}{\varepsilon}\right),\quad \sum_{t=1}^{T}\frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{\frac{1}{2}+\varepsilon}} \leqslant D^2.$$

Proof *Under Assumptions 1-3, we calculate these three terms as follows:*

$$\sum_{t=1}^{T}\frac{1}{t^{\frac{3}{2}+\varepsilon}} \leqslant 1 + \int_1^T \frac{1}{t^{\frac{3}{2}+\varepsilon}}dt \leqslant 3,$$

$$\sum_{t=1}^{T}\frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} \leqslant \sum_{t=1}^{T}\frac{FD}{t^{1+\varepsilon}} \leqslant FD + \int_1^T \frac{FD}{t^{1+\varepsilon}}dt \leqslant \frac{FD(1+\varepsilon)}{\varepsilon},$$

$$\sum_{t=1}^{T}\frac{\|x_t - x^*\|^2 - \|x_{t+1} - x^*\|^2}{t^{\frac{1}{2}+\varepsilon}} \leqslant D^2 + \sum_{t=2}^{T}\left(\frac{1}{t^{\frac{1}{2}+\varepsilon}} - \frac{1}{(t-1)^{\frac{1}{2}+\varepsilon}}\right)\|x_t - x^*\|^2 \leqslant D^2.$$

B Proof of lemmas in Theorem 1

B.1 Proof of Lemma 1

We prove the key self-bounding property in this section. Since (11) holds for any $x \in \mathcal{X}$ in Lemma 5, let $x = x^*$ such that

$$\begin{aligned}&\langle \nabla f_t(x_t), x_{t+1} - x_t\rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2\\ \leqslant&\langle \nabla f_t(x_t), x^* - x_t\rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2.\end{aligned} \tag{12}$$

We add $f_t(x_t)$ to both sides of (12) that

$$\begin{aligned}
&f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t \|x_{t+1} - x_t\|^2 \\
\leqslant & f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t \rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2 \\
\leqslant & f_t(x^*) + Q(t)\hat{g}_t^+(x^*) + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2 \\
\leqslant & f_t(x^*) + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2 \qquad (13)
\end{aligned}$$

where the second inequality holds because $f_t(\cdot)$ is a convex function; the third inequality holds because x^* is an feasible point $g_t(x^*) \leqslant 0$ such that $\hat{g}_t^+(x^*) = 0$ holds.

By moving $f_t(x^*)$ to the left-hand side and $\alpha_t \|x_{t+1} - x_t\|^2$ to the right-hand side of (13), respectively, we have

$$\begin{aligned}
&f_t(x_t) - f_t(x^*) + Q(t)\hat{g}_t^+(x_{t+1}) \\
\leqslant & \langle \nabla f_t(x_t), x_t - x_{t+1} \rangle - \alpha_t \|x_{t+1} - x_t\|^2 + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2 \\
\leqslant & \frac{F^2}{4\alpha_t} + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2
\end{aligned}$$

where the last inequality holds by Assumption 2 that

$$\begin{aligned}
&\langle \nabla f_t(x_t), x_t - x_{t+1} \rangle - \alpha_t \|x_{t+1} - x_t\|^2 \\
= & \langle \nabla f_t(x_t), x_t - x_{t+1} \rangle - \alpha_t \|x_{t+1} - x_t\|^2 - \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} + \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} \\
= & -\left\| \frac{\nabla f_t(x_t)}{2\sqrt{\alpha_t}} - \sqrt{\alpha_t}(x_{t+1} - x_t) \right\|^2 + \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} \\
\leqslant & \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} \leqslant \frac{F^2}{4\alpha_t}.
\end{aligned}$$

B.2 Proof of Lemma 2

Since $g_t(x)$ is convex function, we have $g_t^+(x)$ to be convex because $\max$ over convex functions is convex. Denoting by $\nabla g_t^+(x)$ the subgradient of $g_t^+(x)$, we have

$$g_t^+(x) \geqslant g_t^+(y) + \langle \nabla g_t^+(y), x - y \rangle.$$

Let $y = x_t, x = x_{t+1}$, we have

$$\begin{aligned}
g_t^+(x_t) - g_t^+(x_{t+1}) &\leqslant \langle \nabla g_t^+(x_t), x_t - x_{t+1} \rangle \\
&\leqslant \|\nabla g_t^+(x_t)\| \|x_t - x_{t+1}\| \\
&\leqslant G\|x_t - x_{t+1}\| - \frac{G^2}{4\beta} - \beta\|x_t - x_{t+1}\|^2 + \frac{G^2}{4\beta} + \beta\|x_t - x_{t+1}\|^2 \\
&\leqslant \frac{G^2}{4\beta} + \beta\|x_t - x_{t+1}\|^2
\end{aligned}$$

where the second inequality holds because of Cauchy-Schwarz inequality; the third inequality holds because of Assumption 3. Take summation of the equality above from 1 to T, we have

$$\sum_{t=1}^{T} (g_t^+(x_t) - g_t^+(x_t + 1)) \leqslant \frac{TG^2}{4\beta} + \beta \sum_{t=1}^{T} \|x_t - x_{t+1}\|^2.$$

B.3 Proof of Lemma 3

According to Lemma 5, we have

$$\begin{aligned}
&f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t \|x_{t+1} - x_t\|^2 \\
\leqslant & f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t \rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2 \\
\leqslant & f_t(x^*) + Q(t)\hat{g}_t^+(x^*) + \alpha_t \|x^* - x_t\|^2 - \alpha_t \|x^* - x_{t+1}\|^2
\end{aligned}$$

where the last inequality holds because of the convexity of $f_t(\cdot)$. By $g_t^+(x_{t+1}) \geqslant 0$ and $g_t^+(x^*) = 0$, we rearrange the inequality above and have

$$\|x_{t+1} - x_t\|^2 \leqslant \frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t)) + \frac{1}{\alpha_t}\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle + \|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2$$

Take summation of the inequality above from $t = 1$ to T, we have

$$\begin{aligned}
&\sum_{t=1}^{T} \|x_{t+1} - x_t\|^2 \\
\leqslant &\sum_{t=1}^{T} \frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t)) + \sum_{t=1}^{T} \frac{1}{\alpha_t}\langle \nabla f_t(x_t), x_t - x_{t+1}\rangle + \sum_{t=1}^{T}(\|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2) \\
\leqslant &\sum_{t=1}^{T} \frac{2FD}{\alpha_t} + \|x^* - x_1\|^2 \\
\leqslant &4FD\sqrt{T} + D^2
\end{aligned}$$

where the second inequality holds because

$$f_t(x) - f_t(y) \leqslant \langle \nabla f_t(x), x - y\rangle \leqslant \|\nabla f_t(x)\|\|x - y\| \leqslant FD;$$

the last inequality holds because of $\alpha_t = \sqrt{t}$ and Assumption 1. The proof is completed.

C Corollary 1

Corollary 1 *Under Assumptions 1-3, let the learning rates be $\alpha_t = t^c, \eta_t = t^c, \gamma_t = t^{c+\varepsilon}, \forall t \in [T]$, where $c \in [1/2, 1)$ and $\varepsilon > 0$. RECOO algorithm achieves the following bounds on the regret and cumulative constraint violations:*

$$\mathcal{R}(T) \leqslant \left(\frac{F^2}{4(1-c)} + D^2\right) T^c, \text{ for both types of constraints,}$$

$$\mathcal{V}(T) \leqslant F^2 + FD + \frac{FD}{\varepsilon} + D^2 \text{ for fixed constraints, and}$$

$$\mathcal{V}(T) \leqslant \left(F^2 + \frac{G^2}{4} + FD\left(1 + \frac{2}{1-c} + \frac{1}{\varepsilon}\right) + 2D^2\right) T^{1-c/2} \text{ for adversarial constraints.}$$

The proof of corollary follows almost the same steps as in Theorem 1 where $c = 1/2$. Based on (5) in Lemma 1, we still have

$$f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{4\alpha_t} + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{14}$$

$$Q(t)\hat{g}_t^+(x_{t+1}) \leqslant \frac{F^2}{4\alpha_t} + |f_t(x_t) - f_t(x^*)| + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{15}$$

which is used to establish the bounds of regret and violation.

Regret Bound: we take summation of the inequality (14) from $t = 1, \cdots, T$ and have

$$\begin{aligned}
\sum_{t=1}^{T}(f_t(x_t) - f_t(x^*)) \leqslant &\frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + \sum_{t=1}^{T}(\alpha_t - \alpha_{t-1})\|x^* - x_t\|^2 \\
\leqslant &\frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + D^2\sum_{t=1}^{T}(\alpha_t - \alpha_{t-1})
\end{aligned}$$

where the last inequality holds by Assumption 1. Choose $\alpha_t = t^c$, we have

$$\sum_{t=1}^{T}\frac{1}{\alpha_t} = \sum_{t=1}^{T} t^{-c} \leqslant \frac{T^{1-c}}{1-c}, \quad \sum_{t=1}^{T}(\alpha_t - \alpha_{t-1}) = \sum_{t=1}^{T}(t^c - (t-1)^c) \leqslant T^c$$

It implies that

$$\sum_{t=1}^{T}(f_t(x_t)-f_t(x^*)) \leqslant \left(\frac{F^2}{4(1-c)}+D^2\right)T^{\max\{c,1-c\}},$$

which gives the regret bound with $O(T^{\max\{c,1-c\}})$.

Cumulative constraint violation bound: For constraint violation, we still have (15) and

$$g_t^+(x_{t+1}) \leqslant \frac{F^2}{4Q(t)\alpha_t\gamma_t}+\frac{|f_t(x_t)-f_t(x^*)|}{Q(t)\gamma_t}+\frac{\alpha_t}{Q(t)\gamma_t}\|x_t-x^*\|^2-\frac{\alpha_t}{Q(t)\gamma_t}\|x_{t+1}-x^*\|^2,$$

Set $\gamma_t = t^c$ and $\eta_t = t^{c+\varepsilon}$, where $c \in [1/2, 1)$ and $\varepsilon > 0$, this implies that

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant \sum_{t=1}^{T}\frac{F^2}{4t^{3c+\varepsilon}}+\sum_{t=1}^{T}\frac{|f_t(x_t)-f_t(x^*)|}{t^{2c+\varepsilon}}+\sum_{t=1}^{T}\frac{\|x_t-x^*\|^2-\|x_{t+1}-x^*\|^2}{t^{c+\varepsilon}}$$

By Lemma 6, we establish

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant F^2+FD+\frac{FD}{\varepsilon}+D^2, \tag{16}$$

which proves $\mathcal{V}(T) := \sum_{t=1}^{T} g^+(x_t) \leqslant F^2+FD+\frac{FD}{\varepsilon}+D^2$. Let's continue with (16) to prove the second part of Corollary 1 for the adversarial constraints. Recall Lemma 2 that under Assumptions 1-3, RECOO achieves for any $\beta > 0$

$$g_t^+(x_t)-g_t^+(x_{t+1}) \leqslant \frac{G^2}{4\beta}+\beta\|x_t-x_{t+1}\|^2.$$

From Lemma 2, it is required to quantify the stability term $\|x_t - x_{t+1}\|^2$ that is established in the following lemma.

Lemma 7 *Under Assumptions 1-3, let the learning rates be $\alpha_t = t^c, \eta_t = t^c, \gamma_t = t^{c+\varepsilon}, \forall t \in [T]$, where $c \in [1/2, 1)$ and $\varepsilon > 0$. RECOO achieves*

$$\sum_{t=1}^{T}\|x_{t+1}-x_t\|^2 \leqslant \frac{2FD}{1-c}T^{1-c}+D^2.$$

Proof *From Lemma 5,we still have:*

$$\begin{aligned}
&f_t(x_t)+\langle\nabla f_t(x_t), x_{t+1}-x_t\rangle+Q(t)\hat{g}_t^+(x_{t+1})+\alpha_t\|x_{t+1}-x_t\|^2\\
\leqslant& f_t(x_t)+\langle\nabla f_t(x_t), x^*-x_t\rangle+Q(t)\hat{g}_t^+(x^*)+\alpha_t\|x^*-x_t\|^2-\alpha_t\|x^*-x_{t+1}\|^2\\
\leqslant& f_t(x^*)+Q(t)\hat{g}_t^+(x^*)+\alpha_t\|x^*-x_t\|^2-\alpha_t\|x^*-x_{t+1}\|^2
\end{aligned}$$

According to $g_t^+(x_{t+1}) \geqslant 0$ and $g_t^+(x^*) = 0$ and rearrange the inequality, we have

$$\begin{aligned}
&\|x_{t+1}-x_t\|^2\\
\leqslant&\frac{1}{\alpha_t}(f_t(x^*)-f_t(x_t))+\frac{1}{\alpha_t}\langle\nabla f_t(x_t), x_t-x_{t+1}\rangle+\|x^*-x_t\|^2-\|x^*-x_{t+1}\|^2.
\end{aligned}$$

Take summation of the inequality above, we have

$$\sum_{t=1}^{T}\|x_{t+1}-x_t\|^2 \leqslant \sum_{t=1}^{T}\frac{1}{\alpha_t}(f_t(x^*)-f_t(x_t))+\sum_{t=1}^{T}\frac{1}{\alpha_t}\langle\nabla f_t(x_t), x_t-x_{t+1}\rangle+\sum_{t=1}^{T}(\|x^*-x_t\|^2-\|x^*-x_{t+1}\|^2).$$

Since $\alpha_t = t^c$, we have:

$$\sum_{t=1}^{T}\frac{1}{\alpha_t}(f_t(x^*)-f_t(x_t)+\langle\nabla f_t(x_t), x_t-x_{t+1}\rangle) \leqslant 2FD\sum_{t=1}^{T}t^{-c} \leqslant \frac{2FD}{1-c}T^{1-c},$$

$$\sum_{t=1}^{T}(\|x^*-x_t\|^2-\|x^*-x_{t+1}\|^2) \leqslant D^2,$$

which proves Lemma 7.

Recall in Lemma 2 and we have

$$\sum_{t=1}^{T} g_t^+(x_t) = \sum_{t=1}^{T} g_{t+1}^+(x_t) + \sum_{t=1}^{T} g_t^+(x_t) - \sum_{t=1}^{T} g_{t+1}^+(x_t)$$
$$\leqslant \left(F^2 + FD + \frac{FD}{\varepsilon} + D^2\right) + \frac{G^2 T}{4\beta} + \beta\left(D^2 + \frac{2FD}{1-c}T^{1-c}\right).$$

Let $\beta = T^{c/2}$, we establish

$$\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t) \leqslant \left(F^2 + \frac{G^2}{4} + FD\left(1 + \frac{2}{1-c} + \frac{1}{\varepsilon}\right) + 2D^2\right) T^{1-c/2},$$

which proves the corollary.

D Proof of Theorem 2

In this section, we prove Theorem 2 which considers strongly convex loss functions. According to the definition of μ-strongly convex, we have

$$f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t \rangle \leqslant f(x^*) - \frac{\mu}{2}\|x^* - x_t\|^2 \tag{17}$$

We start with inequality (12) by using Lemma 5:

$$\begin{aligned}
&f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2 \\
\leqslant &f_t(x_t) + \langle \nabla f_t(x_t), x^* - x_t \rangle + Q(t)\hat{g}_t^+(x^*) + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2 \\
\leqslant &f_t(x^*) - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2
\end{aligned} \tag{18}$$

where the last inequality holds according to the strongly convex condition of $f_t(\cdot)$ in (17). By following the same steps that lead to inequality (5), we have

$$\begin{aligned}
&f_t(x_t) - f_t(x^*) + Q(t)\hat{g}_t^+(x^*) \\
\leqslant &\frac{F^2}{4\alpha_t} - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2.
\end{aligned} \tag{19}$$

This is the key "self-bounding" property when $f_t(\cdot)$ is strongly convex function, from which we obtain

$$f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{4\alpha_t} - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2, \tag{20}$$

$$Q(t)\hat{g}_t^+(x_{t+1}) \leqslant \frac{F^2}{4\alpha_t} + |f_t(x_t) - f_t(x^*)| - \frac{\mu}{2}\|x^* - x_t\|^2 + \alpha_t\|x^* - x_t\|^2 - \alpha_t\|x^* - x_{t+1}\|^2. \tag{21}$$

Recall that $\alpha_t = \frac{\mu t}{2}$, $\eta_t = \sqrt{t}$, and $\gamma_t = t^{\frac{1}{2}+\varepsilon}, \varepsilon > 0$. We next establish the regret and violation bounds.

Regret bound: From (20), we immediately have

$$\begin{aligned}
\sum_{t=1}^{T} f_t(x_t) - f_t(x^*) \leqslant &\frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + \sum_{t=1}^{T}\left(\alpha_t - \alpha_{t-1} - \frac{\mu}{2}\right)\|x_t - x^*\|^2 \\
\leqslant &\frac{F^2}{4}\sum_{t=1}^{T}\frac{1}{\alpha_t} + D^2\sum_{t=1}^{T}\left(\alpha_t - \alpha_{t-1} - \frac{\mu}{2}\right)
\end{aligned}$$

By the choice of $\alpha_t = \frac{\mu t}{2}$, we have

$$\sum_{t=1}^{T} f_t(x_t) - f_t(x^*) \leqslant \frac{F^2}{2\mu}(1 + \log T).$$

Following the same steps as in the proof of Theorem 1, we establish

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant \frac{F^2}{2\mu} \sum_{t=1}^{T} \frac{1}{t^{2+\varepsilon}} + \sum_{t=1}^{T} \frac{|f_t(x_t) - f_t(x^*)|}{t^{1+\varepsilon}} \tag{22}$$

which implies that

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) \leqslant \frac{F^2}{\mu} + FD\left(1 + \frac{1}{\epsilon}\right). \tag{23}$$

D.1 Violation bound: fixed constraints

For fixed constraints, inequality (23) implies $\mathcal{V}(T) := \sum_{t=1}^{T} g^+(x_t) \leqslant \frac{F^2}{\mu} + FD\left(1 + \frac{1}{\epsilon}\right)$ because the constraint is fixed. We have proved the first part of Theorem 2 for the fixed constraints. Let's continue with (23) to prove the second part of Theorem 2 for the adversarial constraints.

D.2 Violation bound: adversarial constraints

By Lemma 2, we have

$$\sum_{t=1}^{T} (g_t^+(x_t) - g_t^+(x_{t+1})) \leqslant \frac{TG^2}{4\beta} + \beta \sum_{t=1}^{T} \|x_t - x_{t+1}\|^2. \tag{24}$$

From (18), we again establish the bound on $\sum_{t=1}^{T} \|x_t - x_{t+1}\|^2$ as follows

$$\begin{aligned}
&\sum_{t=1}^{T} \|x_t - x_{t+1}\|^2 \\
\leqslant &\sum_{t=1}^{T} \frac{1}{\alpha_t}(f_t(x^*) - f_t(x_t) + \langle \nabla f_t(x_t), x_t - x_{t+1}\rangle) + \sum_{t=1}^{T}(\|x^* - x_t\|^2 - \|x^* - x_{t+1}\|^2) \\
\leqslant &\sum_{t=1}^{T} \frac{2FD}{\alpha_t} + \|x^* - x_1\|^2 \\
\leqslant &\frac{4FD}{\mu}(1 + \log T) + D^2,
\end{aligned} \tag{25}$$

where the second inequality holds because

$$f_t(x) - f_t(y) \leqslant \langle \nabla f_t(x), x - y\rangle \leqslant \|\nabla f_t(x)\|\|x - y\| \leqslant FD, \forall x, y \in \mathcal{X};$$

the last inequality holds because of $\alpha_t = \frac{\mu t}{2}$ and Assumption 1. We combine (24) and (25) with $\beta = \sqrt{T/(1 + \log T)}$ as follows

$$\begin{aligned}
\sum_{t=1}^{T} (g_t^+(x_t) - g_t^+(x_t + 1)) &\leqslant \frac{TG^2}{4\beta} + \beta \sum_{t=1}^{T} \|x_t - x_{t+1}\|^2 \\
&\leqslant \frac{TG^2}{4\beta} + \beta\left(\frac{4FD}{\mu}(1 + \log T) + D^2\right) \\
&= \left(\frac{G^2}{4} + \frac{4FD}{\mu}\right)\sqrt{T(1 + \log T)} + D^2\sqrt{T/(1 + \log T)}.
\end{aligned}$$

Combining the inequality above with inequality (23), we establish the second part of Theorem 2 for adversarial constraints.

E Proof of Theorem 3

We prove Theorem 3 with a dynamic baseline similar to that in [26], which is the solution to the following offline OCO with constraints:

$$\min_{x_t \in \mathcal{X}} \sum_{t=1}^{T} f_t(x_t) \tag{26}$$

$$\text{subject to: } g_t(x_t) \leqslant 0, \forall t \in [T], \tag{27}$$

$$\sum_{t=1}^{T} \|x_{t+1} - x_t\| \leqslant P_T. \tag{28}$$

where (28) imposes a path-length constraint on the baseline solution which limits change of $\{x_t\}$. Note that the solution to (26)-(28) with $P_T = 0$ reduces to the best fixed decision in hindsight. Let $\{x_t^*\}$ be the optimal solution to (26)-(28). We define the regret and cumulative constraint violation as follows

$$\mathcal{R}^{\text{dynamic}}(T) := \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_t^*), \tag{29}$$

$$\mathcal{V}(T) := \sum_{t=1}^{T} g_t^+(x_t). \tag{30}$$

Next, we state the key self-bounding property for establishing Theorem 3.

Lemma 8 *Let $\{x_t\}$ be the decision sequence generated by RECOO. Under Assumptions 1-3, the following inequality holds for any sequence $\{y_t\}$ with $y_t \in \mathcal{X}, \forall t$,*

$$\begin{aligned} & f_t(x_t) - f_t(y_t) + Q(t)\hat{g}_t^+(x_{t+1}) \\ \leqslant & \frac{F^2}{4\alpha_t} + Q(t)\hat{g}_t^+(y_t) + \alpha_t\|y_t - x_t\|^2 - \alpha_t\|y_t - x_{t+1}\|^2. \end{aligned} \tag{31}$$

Proof *According to Lemma 5, we have*

$$\begin{aligned} & \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2 \\ \leqslant & \langle \nabla f_t(x_t), y_t - x_t \rangle + Q(t)\hat{g}_t^+(y_t) + \alpha_t\|y_t - x_t\|^2 - \alpha_t\|y_t - x_{t+1}\|^2. \end{aligned}$$

We add $f_t(x_t)$ to both sides of the inequality above

$$\begin{aligned} & f_t(x_t) + \langle \nabla f_t(x_t), x_{t+1} - x_t \rangle + Q(t)\hat{g}_t^+(x_{t+1}) + \alpha_t\|x_{t+1} - x_t\|^2 \\ \leqslant & f_t(x_t) + \langle \nabla f_t(x_t), y_t - x_t \rangle + Q(t)\hat{g}_t^+(y_t) + \alpha_t\|y_t - x_t\|^2 - \alpha_t\|y_t - x_{t+1}\|^2 \\ \leqslant & f_t(y_t) + Q(t)\hat{g}_t^+(y_t) + \alpha_t\|y_t - x_t\|^2 - \alpha_t\|y_t - x_{t+1}\|^2, \end{aligned}$$

which proves the lemma because

$$\langle \nabla f_t(x_t), x_t - x_{t+1} \rangle - \alpha_t\|x_{t+1} - x_t\|^2 \leqslant \frac{\|\nabla f_t(x_t)\|^2}{4\alpha_t} \leqslant \frac{F^2}{4\alpha_t}.$$

Regret bound with dynamic baseline: Let the optimal sequence be $\{x_t^*\}$. Substitute $\{y_t\} = \{x_t^*\}$ in (31). By $g_t(x_t^*) \leqslant 0$, we have

$$\begin{aligned} & f_t(x_t) - f_t(x_t^*) + Q(t)\hat{g}_t^+(x_{t+1}) \\ \leqslant & \frac{F^2}{4\alpha_t} + \alpha_t\|x_t^* - x_t\|^2 - \alpha_t\|x_t^* - x_{t+1}\|^2. \end{aligned}$$

Take the summation of the inequality above from $t = 1$ to T, we have

$$\mathcal{R}^{\text{dynamic}}(T) \leqslant \sum_{t=1}^{T} \frac{F^2}{4\alpha_t} + \sum_{t=1}^{T} \alpha_t \left(\|x_t^* - x_t\|^2 - \|x_t^* - x_{t+1}\|^2\right) \tag{32}$$

Recall $\alpha_t = \sqrt{t}$, we have

$$\sum_{t=1}^{T} \frac{F^2}{4\alpha_t} \leqslant \frac{F^2\sqrt{T}}{2}.$$

For the second term, we have

$$\begin{aligned}
&\sum_{t=1}^{T} \alpha_t \left(\|x_t^* - x_t\|^2 - \|x_t^* - x_{t+1}\|^2\right) \\
=&\sum_{t=1}^{T} \sqrt{t} \left(\|x_t^* - x_t\|^2 - \|x_t^* - x_{t+1}\|^2\right) \\
=&\sum_{t=1}^{T} \sqrt{t}\|x_t^* - x_t\|^2 - \sqrt{t+1}\|x_{t+1}^* - x_{t+1}\|^2 + \sqrt{t+1}\|x_{t+1}^* - x_{t+1}\|^2 \\
&\quad - \sqrt{t}\|x_{t+1}^* - x_{t+1}\|^2 + \sqrt{t}\|x_{t+1}^* - x_{t+1}\|^2 - \sqrt{t}\|x_t^* - x_{t+1}\|^2 \\
\leqslant&\|x_1^* - x_1\|^2 + \sum_{t=1}^{T} \left(\sqrt{t+1} - \sqrt{t}\right) D^2 + \sum_{t=1}^{T} \sqrt{t} \left(\|x_{t+1}^* - x_{t+1}\|^2 - \|x_t^* - x_{t+1}\|^2\right) \\
\leqslant&D^2\sqrt{T+1} + 2DP_T\sqrt{T},
\end{aligned}$$

where the first inequality holds because of Assumption 1 and the last inequality holds because

$$\begin{aligned}
\sum_{t=1}^{T} |\|x_{t+1}^* - x_{t+1}\|^2 - \|x_t^* - x_{t+1}\|^2| =& \sum_{t=1}^{T} |\langle x_{t+1}^* - x_t^*, x_{t+1}^* - x_{t+1} + x_t^* - x_{t+1}\rangle| \\
\leqslant& \sum_{t=1}^{T} \|x_{t+1}^* - x_t^*\| \left(\|x_{t+1}^* - x_{t+1}\| + \|x_t^* - x_{t+1}\|\right) \\
\leqslant&2D \sum_{t=1}^{T} \|x_{t+1}^* - x_t^*\|.
\end{aligned}$$

Therefore, we prove the regret in Theorem 3 as follows

$$\mathcal{R}(T) \leqslant \frac{F^2\sqrt{T}}{2} + (D^2 + 2DP_T)\sqrt{T+1} \leqslant \left(\frac{F^2}{2} + D^2 + 2DP_T\right)\sqrt{T+1}.$$

Cumulative hard constraint violation bound: When $\alpha_t = \eta_t = \sqrt{t}, \gamma_t = t^{\frac{1}{2}+\varepsilon}$ with $\varepsilon > 0$, the proof follows the same steps in the proof of Theorem 1 in Section 3.1 because the definition of constraint violation is the same.

F Refined results of RECOO with expert-tracking under the dynamic baseline

Motivated by [33], we combine RECOO with expert tracking techniques in [6] to improve the performance bounds w.r.t. P_T, similar in [26]. The intuition of the algorithm is to set up N parallel experts (N RECOO algorithms) and to track the best one. We state RECOO with expert tracking algorithm as follows.

A Rectified Online Optimization with Expert-Tracking Algorithm

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Initialization (N Experts): $f_0(x) = g_0(x) = 0, \forall x \in \mathcal{X}$, $x_{i,0} \in \mathcal{X}$, $Q_i(0) = 0$. The learning rates $\alpha_{i,t}, \eta_{i,t}, \gamma_t, \kappa$ and $w_{i,1} = \frac{N+1}{i(i+1)N}, \forall i \in [N]$.

For $t = 1, \cdots, T$,

- **Set:** $\hat{g}_{t-1}^{+}(x) = \gamma_{t-1} g_{t-1}^{+}(x)$.
- **Rectified decision:** find the optimal solution of $x_{i,t}$ for each expert and output x_t :

$$x_{i,t} = \arg\min_{x \in \mathcal{X}} \langle \nabla f_{t-1}(x_{i,t-1}), x - x_{i,t-1} \rangle + Q_i(t-1)\hat{g}_{t-1}^{+}(x) + \alpha_{i,t-1}\|x - x_{i,t-1}\|^2$$
$$x_t = \sum_{i=1}^{N} w_{i,t} x_{i,t}$$

- **Observe:** $\nabla f_t(\cdot)$ and $g_t(\cdot)$.
- **Rectified penalty update:** update $Q_i(t)$ and $w_{i,t}$ as follows:

$$Q_i(t) = \max\left(Q_i(t-1) + \hat{g}_t^{+}(x_t), \eta_{i,t}\right).$$
$$l_t(x) = \langle \nabla f_t(x_t), x - x_t \rangle$$
$$w_{i,t+1} = \frac{w_{i,t} e^{-\kappa l_t(x_{i,t})}}{\sum_{i=1}^{N} w_{i,t} e^{-\kappa l_t(x_{i,t})}}$$

Before presenting the main result of RECOO with expert-tracking algorithm, we impose an additional assumption on the loss functions as in [33].

Assumption 5 *The loss functions are bounded by a constant C such that $|f_t(x)| \leqslant C, \forall x \in \mathcal{X}, \forall t$.*

We are ready to show RECOO with expert-tracking algorithm improves Theorem 3 in the following.

Corollary 2 *Let $N = \left\lfloor \frac{1}{2}\log_2(1+T) \right\rfloor + 1$, $\kappa = 1/\sqrt{T}$. Let the learning rates be $\alpha_{i,t} = \sqrt{t}/2^{i-1}, \eta_{i,t} = 2^{i-1}\sqrt{t}, \gamma_t = t^{\frac{1}{2}+\varepsilon}$, $\forall i \in [N]$, where $\varepsilon > 0$. Let $\{x_t^*\}$ be the optimal solution to (2) with $P_T = \sum_{t=1}^{T-1} \|x_{t+1}^* - x_t^*\|$. Under Assumptions 1-3, RECOO with expert-tracking algorithm achieves the following bounds on the regret and cumulative constraint violations:*

$$\mathcal{R}^{dynamic}(T) \leqslant \left[(2F^2 + 4D^2)\sqrt{1 + \frac{P_T}{D}} + \frac{C^2}{2} + 2\ln\left(\left\lfloor \frac{1}{2}\log_2\left(1 + \frac{P_T}{D}\right) \right\rfloor + 2 \right) \right] \sqrt{T+1},$$
$$\mathcal{V}(T) \leqslant 2FD\left(1 + \frac{1}{\varepsilon}\right) + F^2(1 + \log(1+T)) + 2D^2.$$

We first introduce Lemma 1 in [33], which quantifies the difference between the weighted output of all experts with the best expert.

Lemma 9 (Lemma 1 in [33]) *Let $\{x_{i,t}\}$ and $\{x_t\}$ be the sequence generated by RECOO with expert-tracking algorithm, we have*

$$\sum_{t=1}^{T} l_t(x_t) - \min_{i \in [N]} \left\{ \sum_{t=1}^{T} l_t(x_{i,t}) + \frac{1}{\kappa}\ln\frac{1}{w_{i,1}} \right\} \leqslant \frac{\kappa C^2 T}{2}.$$

Substitute the learning rate of $\alpha_{i,t}$ in (32) in the proof of Theorem 3, we have

$$\sum_{t=1}^{T}(f_t(x_{i,t}) - f_t(x_t^*)) + \sum_{t=1}^{T} Q_i(t)\hat{g}_t^{+}(x_{i,t+1}) \leqslant 2^i F^2\sqrt{T} + \frac{D^2\sqrt{T+1}}{2^{i-1}} + \frac{2DP_T\sqrt{T}}{2^{i-1}}. \quad (33)$$

Recall $N = \left\lfloor \frac{1}{2}\log_2(1+T) \right\rfloor + 1$, there exists $i_0 = \left\lfloor \frac{1}{2}\log_2(1 + \frac{P_T}{D}) \right\rfloor + 1 \in [N]$ such that

$$2^{i_0 - 1} \leqslant \sqrt{1 + \frac{P_T}{D}} \leqslant 2^{i_0}. \quad (34)$$

Regret bound with dynamic baseline: Let $i = i_0$ in (33) and by Lemma 9, we have:

$$\begin{aligned}\sum_{t=1}^{T}(f_t(x_{i_0,t}) - f_t(x_t^*)) &\leqslant 2^{i_0}F^2\sqrt{T} + \frac{D^2\sqrt{T+1}}{2^{i_0-1}} + \frac{2DP_T\sqrt{T}}{2^{i_0-1}} \\ &\leqslant 2F^2\sqrt{T\left(1+\frac{P_T}{D}\right)} + \frac{D^2\sqrt{T+1}+2DP_T\sqrt{T}}{2^{i_0-1}} \\ &\leqslant 2F^2\sqrt{T\left(1+\frac{P_T}{D}\right)} + \frac{4}{2^{i_0}}\left(D^2\sqrt{T+1}\left(1+\frac{P_T}{D}\right)\right) \\ &\leqslant (2F^2+4D^2)\sqrt{(T+1)\left(1+\frac{P_T}{D}\right)}\end{aligned}$$

where the second and the last inequalities hold by using (34). Moreover, we have

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{i_0,t}) \leqslant \sum_{t=1}^{T} l_t(x_t) - \sum_{t=1}^{T} l_t(x_{i_0,t}) \leqslant \frac{\kappa C^2 T}{2} + \frac{1}{\kappa}\ln\frac{1}{w_{i_0,1}}$$

where the first inequality holds by the convexity of $f_t(\cdot)$ and the second inequality holds by Lemma 9. Recall $\kappa = \frac{1}{\sqrt{T}}$ and $w_{i,1} = \frac{N+1}{i(i+1)N}$, we have

$$\ln\frac{1}{w_{i_0,1}} \leqslant \ln(i_0(i_0+1)) \leqslant 2\ln(i_0+1) \leqslant 2\ln\left(\left\lfloor\frac{1}{2}\log_2\left(1+\frac{P_T}{D}\right)\right\rfloor + 2\right).$$

Combine all these inequalities, we have:

$$\begin{aligned}\mathcal{R}(T) &= \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{i_0,t}) + \sum_{t=1}^{T} f_t(x_{i_0,t}) - \sum_{t=1}^{T} f_t(x_t^*) \\ &\leqslant (2F^2+4D^2)\sqrt{(T+1)\left(1+\frac{P_T}{D}\right)} + \left(\frac{C^2}{2} + 2\ln\left(\left\lfloor\frac{1}{2}\log_2\left(1+\frac{P_T}{D}\right)\right\rfloor + 2\right)\right)\sqrt{T}\end{aligned}$$

which proves the regret in Corollary 2 and establish $O(\sqrt{P_T T})$ regret bound.

Violation bound: Since $g_t^+(x)$ is convex, we have:

$$\sum_{t=1}^{T} g_t^+(x_{t+1}) = \sum_{t=1}^{T} g_t^+\left(\sum_{i=1}^{N} w_{i,t+1}x_{i,t+1}\right) \leqslant \sum_{i=1}^{N}\sum_{t=1}^{T} w_{i,t+1}g_t^+(x_{i,t+1}) \leqslant \sum_{i=1}^{N}\sum_{t=1}^{T} g_t^+(x_{i,t+1}).$$

By the inequality (31), we have

$$Q_i(t)\hat{g}_t^+(x_{i,t+1}) \leqslant f_t(x_t^*) - f_t(x_{i,t}) + \frac{F^2}{4\alpha_t} + \alpha_{i,t}\|x_t^* - x_{i,t}\|^2 - \alpha_{i,t}\|x_t^* - x_{i,t+1}\|^2.$$

Let $\alpha_{i,t} = \sqrt{t}/2^{i-1}$, $\eta_{i,t} = 2^{i-1}\sqrt{t}$, $\gamma_t = t^{\frac{1}{2}+\varepsilon}$, $\varepsilon > 0$, we have

$$\begin{aligned}g_t^+(x_{i,t+1}) &\leqslant \frac{|f_t(x_t^*) - f_t(x_{i,t})|}{\eta_{i,t}\gamma_t} + \frac{F^2}{4\alpha_{i,t}\eta_{i,t}\gamma_t} + \frac{\alpha_{i,t}}{\eta_{i,t}\gamma_t}(\|x_t^* - x_{i,t}\|^2 - \|x_t^* - x_{i,t+1}\|^2) \\ &\leqslant \frac{1}{2^{i-1}}\frac{FD}{t^{1+\varepsilon}} + \frac{F^2}{4t^{\frac{3}{2}+\varepsilon}} + \frac{1}{4^{i-1}}\frac{1}{t^{\frac{1}{2}+\varepsilon}}(\|x_t^* - x_{i,t}\|^2 - \|x_t^* - x_{i,t+1}\|^2),\end{aligned}$$

which implies

$$\begin{aligned}\sum_{t=1}^{T} g_t^+(x_{i,t+1}) &\leqslant \frac{1}{2^{i-1}}\sum_{t=1}^{T}\frac{FD}{t^{1+\varepsilon}} + \sum_{t=1}^{T}\frac{F^2}{4t^{\frac{3}{2}+\varepsilon}} + \frac{1}{4^{i-1}}\sum_{t=1}^{T}\frac{1}{t^{\frac{1}{2}+\varepsilon}}(\|x_t^* - x_{i,t}\|^2 - \|x_t^* - x_{i,t+1}\|^2) \\ &\leqslant \frac{FD}{2^{i-1}}\left(1+\frac{1}{\varepsilon}\right) + F^2 + \frac{D^2}{4^{i-1}}\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{t=1}^{T} g_t^+(x_{t+1}) &\leqslant \sum_{i=1}^{N}\sum_{t=1}^{T} g_t^+(x_{i,t+1}) \\ &\leqslant \sum_{i=1}^{N}\left(\frac{FD}{2^{i-1}} + F^2 + \frac{D^2}{4^{i-1}}\right) \\ &\leqslant 2FD\left(1+\frac{1}{\varepsilon}\right) + F^2(1+\log(1+T)) + 2D^2\end{aligned}$$

which proves the violation in Corollary 2.