Nonlinear Approximation with Subsampled Rank-1 Lattices

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***Abstract*—In this paper we approximate high-dimensional functions $f\colon \mathbb{T}^d \to \mathbb{C}$ by sparse trigonometric polynomials based on function evaluations. Recently it was shown that a dimension-incremental sparse Fourier transform (SFT) approach does not require the signal to be exactly sparse and is applicable in this setting. We combine this approach with subsampling techniques for rank-1 lattices. This way our approach benefits from the underlying structure in the sampling points making fast Fourier algorithms applicable whilst achieving the good sampling complexity of random points (logarithmic oversampling).**

In our analysis we show detection guarantees of the frequencies corresponding to the Fourier coefficients of largest magnitude. In numerical experiments we make a comparison to full rank-1 lattices and uniformly random points to confirm our findings.

I. Introduction

The recovery of sparse signals or compressed sensing is a thoroughly studied problem in signal processing. While many one-dimensional approaches exist [10], we have a look into the multivariate problem on the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. Given an s-sparse signal $f = \sum_{\boldsymbol{k}\in I} \hat{f}_{\boldsymbol{k}} \exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)$, $|I| = s$, the problem is to recover $I \subset \mathbb{Z}^d$ from function evaluations of the function f. Here, a sparse Fourier transform (SFT) approach can be generalized to work in higher dimensions, cf. [14]. However, when the signal is not exactly sparse and we approximate by $g = \sum_{\boldsymbol{k}\in I} \hat{g}_{\boldsymbol{k}} \exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)$ for some $I \subset \mathbb{Z}^d$, we obtain an additional error

$$\|f - g\|_{L_2}^2 = \|f - P_I f\|_{L_2}^2 + \|P_I f - g\|_{L_2}^2 \,.$$

In this setting we have to find

- a suitable frequency set $I \subset \mathbb{Z}^d$ to bound the first summand and
- $\hat{g}_{\boldsymbol{k}} \in \mathbb{C}$ approximating the true Fourier coefficients $\hat{f}_{\boldsymbol{k}} = \langle f, \exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\rangle_{L_2} \in \mathbb{C}$ in order to bound the second summand.

Given a frequency set I, i.e., a suitable linear approximation space, the corresponding Fourier coefficients can be computed via least squares where error bounds are known, cf. [1]. Thus, the main task is the detection of a frequency set I, which should optimally be the support of the Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ with largest magnitude like in the best m-term approximation, cf. [25, Section 1.7].

Recent approaches include [11] or [17] where arbitrary bounded orthonormal product basis were considered. As an application rank-1 lattices were used for the sampling similar to [14] in order to make use of fast Fourier algorithms. This was then compared to random points, which have a better sampling complexity but lack fast algorithms.

In this paper we modify the approach from [17] to work with subsampled rank-1 lattices utilizing recent subsampling techniques from [2] combining the good sampling complexity with the fast algorithms.

The SFT techniques from [17] work for other bounded orthonormal product basis as well and the subsampling methods from [2] for arbitrary L_2-Marcinkiewicz-Zygmund inequalities. Therefore, the presented theory can be generalized but for the sake of readability we restrict ourselves to the torus $\mathbb{T}^d$ and rank-1 lattices.

We will recap the ideas of an SFT approach in Section II-A followed by the subsampling techniques for rank-1 lattices in Section II-B, where we will give an L_2-error bound for least squares approximation. In Section III we will combine the SFT approach with the subsampled rank-1 lattices and show detection guarantees for the Fourier coefficients of largest magnitude in Theorem III.3. Finally, we conclude with a numerical experiment in Section IV comparing rank-1 lattices and random points with the subsampled rank-1 lattices with respect to sampling complexity and runtime. The proofs can be found in the supplementary material.

II. Prerequisites

A. Sparse Fourier Transform

We briefly recall the key idea of a sparse Fourier transform (SFT) approach. For a more detailed explanation see [22], [14], [16], or [17] for a more general version. As stated in the introduction, the goal is to find frequencies $I \subset \mathbb{Z}^d$ such that the target function $f\colon \mathbb{T}^d \to \mathbb{C}$ can be approximated well from $\operatorname{span}\{\exp(2\pi \mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$. In order to do so, we choose a suitable search space $\Gamma \subset \mathbb{Z}^d$ and proceed in a dimension-incremental way:

One-dimensional frequencies. We use the projections of Γ to its t-th component $\mathcal{P}_{\{t\}}(\Gamma) \coloneqq \{k_t : \boldsymbol{k} \in \Gamma\}$, $t = 1, \dots, d$ for the candidate sets. From these we construct frequency sets $I_{\{t\}} \subset \mathcal{P}_{\{t\}}(\Gamma)$, $t = 1, \dots, d$, consisting of the "most important", one-dimensional frequency components in the respective dimensions.

Dimension-incremental step. We construct the next frequency set $I_{\{1,\dots,t+1\}}$ as a subset of the candidate set

$(I_{\{1,\dots,t-1\}} \times I_{\{t\}}) \cap \mathcal{P}_{\{1,\dots,t\}}(\Gamma)$ consisting of the "most important", t-dimensional frequency components.

The output is the final frequency set $I := I_{\{1,\dots,d\}}$ and it is left to refine the formulation of "most important".

Let $I^\star$ be the set of frequencies corresponding to the s Fourier coefficients $\hat{f}_{\boldsymbol{k}}$ of largest magnitude for some sparsity $s \in \mathbb{N}$. Ideally, in the step $t-1 \to t$ we want to find the frequencies $\mathcal{P}_{\{1,\dots,t\}}(I^\star)$. The idea is to use an approximation of so-called projected Fourier coefficients

$$\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}) = \int_{\mathbb{T}^t} f(\boldsymbol{x},\boldsymbol{\xi}) \exp(-2\pi \mathrm{i}\langle \boldsymbol{k},\boldsymbol{x}\rangle)\,\mathrm{d}\boldsymbol{x}\,, \quad (1)$$

where $\boldsymbol{x} = (x_1,\dots,x_t) \in \mathbb{T}^t$, $\boldsymbol{\xi} = (\xi_1,\dots,\xi_{d-t}) \in \mathbb{T}^{d-t}$, and $f(\boldsymbol{x},\boldsymbol{\xi}) = f(x_1,\dots,x_t,\xi_1,\dots,\xi_{d-t})$. The name is based on the fact, that those values may be seen as the Fourier coefficients of the function $f(\cdot,\boldsymbol{\xi}) \in L_2(\mathbb{T}^t)$ with a fixed anchor $\boldsymbol{\xi} \in \mathbb{T}^{d-t}$. By the orthonormality of the Fourier basis we have

$$\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}) = \sum_{\boldsymbol{l}\in\mathbb{Z}^{d-t}} \hat{f}_{(\boldsymbol{k},\boldsymbol{l})} \exp(2\pi\mathrm{i}\langle \boldsymbol{l},\boldsymbol{\xi}\rangle)\,,$$

i.e., the projected Fourier coefficient $\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})$ contains information on the Fourier coefficients with $\boldsymbol{k} \in \mathbb{Z}^t$ in the first t components of their frequencies.

The frequency $\boldsymbol{k}$ is likely to be important and should be included in $I_{\{1,\dots,t\}}$, if the absolute value $|\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})|$ is larger than some detection threshold δ'. In the algorithm, we carry out r detection iterations with different, randomly drawn anchors $\boldsymbol{\xi}^i$, $i = 1,\dots,r$, to avoid cases where the factors $\exp(2\pi\mathrm{i}\langle \boldsymbol{l},\boldsymbol{\xi}\rangle)$ cause an annihilation (which results in small projected Fourier coefficients, even though the corresponding frequency components $\boldsymbol{k}$ are important). The detection of the most important one-dimensional components k_t in the first step of the SFT approach works analogously.

Finally, we need to discuss the approximation of (1). A favorable method $\mathcal{A}$ should combine the following properties:

- have small sample complexity;
- computationally fast, that is, both the construction of the sampling points $\boldsymbol{\xi}$ and the evaluation of the projected Fourier coefficients $\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}$ using the samples $f(\boldsymbol{x},\boldsymbol{\xi})$ can be performed efficiently.
- small error, such that the relative magnitude of the projected Fourier coefficients stays unharmed.

Note, that $\mathcal{A}$ has to be performed several times throughout the SFT approach in different dimensions up to d. It is favorable to use different methods in the one- and multivariate steps using advantages of the respective methods.

B. Subsampling of rank-1 lattices

Rank-1 lattices $\boldsymbol{X}_M = \{\boldsymbol{x}^1,\dots,\boldsymbol{x}^M\} \subset \mathbb{T}^d$ consist of equispaced points on a line which wraps around the d-dimensional torus $\mathbb{T}^d$, more precisely, for a generating vector $\boldsymbol{z} \in \mathbb{R}^d$ and a lattice size $M \in \mathbb{M}$ they are defined via

$$\boldsymbol{X}_M := \left\{ \frac{1}{M}(i\boldsymbol{z} \bmod M\mathbb{1}) \in \mathbb{T}^d : i = 0,\dots,M-1 \right\},$$

where the modulus operation is used entry-wise. We will use them in the least squares approximation

$$S_{\boldsymbol{X}_M} f = \operatorname*{arg\,min}_{g\in V} \sum_{i=1}^{M} |g(\boldsymbol{x}^i) - f(\boldsymbol{x}^i)|^2\,,$$

where $V = \operatorname{span}\{\exp(2\pi\mathrm{i}\langle \boldsymbol{k},\cdot\rangle)\}_{k\in I}$. By simple calculus we have for the Fourier coefficients $\hat{\boldsymbol{g}} = (\hat{g}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$ of $S_{\boldsymbol{X}_M} f = \sum_{\boldsymbol{k}\in I} \hat{g}_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\langle \boldsymbol{k},\cdot\rangle)$ the equation $\hat{\boldsymbol{g}} = (\boldsymbol{L}^*\boldsymbol{L})^{-1}\boldsymbol{L}^*\boldsymbol{f}$, where

$$\boldsymbol{L} = (\exp(2\pi\mathrm{i}\langle \boldsymbol{k},\boldsymbol{x}^i\rangle))_{i=1,\dots,M,\boldsymbol{k}\in I}\,.$$

We will solve this system of equations iterative only using matrix-vector multiplications. Because of their one-dimensional structure of the rank-1 lattices, a one-dimensional FFT can be used to compute the matrix-vector product with the corresponding Fourier matrix $\boldsymbol{L}$ in $\mathcal{O}(M\log M + d|I|)$ instead of the naïve $\mathcal{O}(M\cdot|I|)$, where $I \subset \mathbb{Z}^d$ is an arbitrary frequency index set. For approximating functions with rank-1 lattices we suppose the following feature: We say a rank-1 lattice $\boldsymbol{X}_M$ has the *reconstructing property* for a frequency index set I, if

$$\frac{1}{M}\sum_{i=1}^{M} \exp(2\pi\mathrm{i}\langle \boldsymbol{k},\boldsymbol{x}^i\rangle) = \delta_{\boldsymbol{0},\boldsymbol{k}} \quad \text{for all} \quad \boldsymbol{k} \in \mathcal{D}(I)\,, \quad (2)$$

where $\mathcal{D}(I) = \{\boldsymbol{k}-\boldsymbol{l} : \boldsymbol{k},\boldsymbol{l} \in I\}$. Approximation bounds and further resources can be found in [23], [20], [12], [13], [21], [7].

Example II.1. *When approximating functions from Sobolev spaces with mixed smoothness H^s_{mix} for $s > 1/2$ the best frequency index sets I for approximation are so called are hyperbolic crosses, cf. [6]. We consider the following two scenarios:*

1. *When approximating with samples from a reconstructing rank-1 lattice $\boldsymbol{X}_M$ the following error bound was shown for the least squares approximation in [3, Theorem 2]:*

$$M^{-s} \lesssim \sup_{\|f\|_{H^s_{\mathrm{mix}}}\le 1} \|f - S_{\boldsymbol{X}_M} f\|^2_{L_2} \lesssim M^{-s}(\log M)^{(d-2)s+d-1},$$

where the lower bound holds for all rank-1 lattices and there exists a rank-1 lattice satisfying the upper one.

2. *In contrast to that, for the same frequencies from the hyperbolic cross I and using uniformly drawn points $\boldsymbol{X} = \{\boldsymbol{x}^1,\dots,\boldsymbol{x}^n\}$ we obtain by [15, Corollary 2]*

$$\sup_{\|f\|_{H^s_{mix}}\le 1} \|f - S_{\boldsymbol{X}} f\|^2_{L_2} \lesssim n^{-2s}(\log n)^{2ds}\,.$$

Example II.1 demonstrates that the sample complexity loses half the rate of convergence when approximating with rank-1 lattices compared to uniformly random points. The reason for that lies in the reconstructing requirement (2) which are $|\mathcal{D}(I)| \approx |I|^2$ conditions blowing up the size M of the rank-1 lattice. However, when we use the uniformly random points with the better approximation rate, the lack of structure in the uniformly random points prevents the implementation

of a fast and efficient matrix-vector multiplication with the corresponding Fourier matrix.

It was show in [2, Theorem 3.1] that the good approximation rates and the fast algorithms can be combined: The approach is to discretely subsample a rank-1 lattice to obtain points $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\}$ from a rank-1 lattice with $n \geq 12|I|(\log|I| + t)$. Since the underlying structure is preserved fast Fourier algorithms are applicable, cf. [2, Eq. 5.5]. Further, we have

$$A\|f\|_{L_2}^2 \leq \frac{1}{n}\sum_{i=1}^{n} |f(\boldsymbol{x}^i)|^2 \leq B\|f\|_{L_2}^2 \tag{3}$$

for all $f \in \operatorname{span}\{\exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)\}_{\boldsymbol{k}\in I}$ with $A = 1/2$, $B = 3/2$, and probability exceeding $1 - 2\exp(-t)$. The condition (3) is known as *L_2-Marcinkiewicz-Zygmund inequality* and is a relaxation of the reconstructing property (2), since for $A = B = 1$ (3) is equivalent to (2) which can be shown using the parallelogram law, cf. [2, Theorem 2.3]. It gives a relation of the continuous L_2-norm and the point evaluations and is used to show error bounds for least squares approximation. For continuously random points this was done for individual functions in [4], [5] and improved by [1]. Note, the existence of a probability density was shown such that (3) holds with merely linear oversampling, cf. [8].

The following result is a combination of the discrete subsampling techniques from [2] and the error bound from [1, Thm. 3.2] for individual function approximation.

Theorem II.2. *Let $f\colon \mathbb{T}^d \to \mathbb{C}$ be a fixed function and $\boldsymbol{X}_M = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^M\} \subset \mathbb{T}^d$ be a reconstructing rank-1 lattice for a frequency set $I_M \subset \mathbb{Z}^d$. Further, let $\emptyset \neq I \subset I_M$, $t > 0$, and n be such that $n \geq 12|I|(\log|I| + t)$. Drawing a set $\boldsymbol{X} = \{\boldsymbol{x}^i\}_{i\in J}$, $|J| = n$ of points i.i.d. and uniform from $\boldsymbol{X}_M$, we have*

$$\begin{aligned}
&\|f - S_{\boldsymbol{X}} f\|_{L_2}^2 \\
&\quad\leq \Big(3\|f - P_I f\|_{L_2} + \sqrt{\frac{2}{9|I|}}\|P_{I_M} f - P_I f\|_\infty\Big)^2 \\
&\qquad + 4\|f - P_{I_M} f\|_\infty^2 \\
&\quad\leq \Big(3 + \sqrt{\frac{2|I_M \setminus I|}{9|I|}}\Big)^2 \|f - P_I f\|_{L_2}^2 + 4\|f - P_{I_M} f\|_\infty^2
\end{aligned}$$

with probability exceeding $1 - 2\exp(-t)$.

Given the logarithmic oversampling and assuming $|I_M \setminus I| = c|I|$ for some constant $c > 0$, we obtain the projection error in the first summand, which is the best possible from the given approximation space. This has to be balanced with the second term which decreases for bigger I_M, which is a degree of freedom not affecting the sampling complexity. In the numerical experiments we will see that $I = I_M$ is sufficient in practice. Note, that in this case the corresponding rank-1 lattice will still be of size $M \approx |I|^2$ and the random subsampling with logarithmic oversampling will improve the sampling complexity.

III. SFT WITH SUBSAMPLED RANK-1 LATTICES

For a function $f\colon \mathbb{T}^d \to \mathbb{C}$ and a threshold $\delta > 0$, the final goal is to find $I_\delta := \{\boldsymbol{k} \in \mathbb{Z}^d : |\hat{f}_{\boldsymbol{k}}| \geq \delta\}$ or a superset of slightly bigger size. As described in Section II-A, our approach works in a dimension-incremental way and so will the analysis. The goal in the step from dimension $t-1$ to t is the detection of $\mathcal{P}_{\{1,\dots,t\}}(I_\delta) \subset \mathbb{Z}^t$. We first show that using the projected coefficients (1) yields the objective.

Theorem III.1. *Let $f\colon \mathbb{T}^d \to \mathbb{C}$, $\varepsilon, \delta > 0$, $I_\delta := \{\boldsymbol{k} \in \mathbb{Z}^d : |\hat{f}_{\boldsymbol{k}}| \geq \delta\}$, and*

$$r \geq 4\Big(|I_\delta| + \frac{1}{\delta^2}\Big(\sum_{\boldsymbol{k}\notin I_\delta} |\hat{f}_{\boldsymbol{k}}|\Big)^2\Big)\Big(\log|I_\delta| + \log\frac{1}{\varepsilon}\Big).$$

Further let $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^r \in \mathbb{T}^{d-t}$ be drawn i.i.d. uniformly random. With probability $1-\varepsilon$ we detect all important frequencies in dimension t via the projected Fourier coefficients (1) with r detection iterations and threshold $\delta' \leq \delta/\sqrt{2}$, i.e.,

$$\max_{i=1,\dots,r} |\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)| \geq \delta' \quad \forall \boldsymbol{k} \in \mathcal{P}_{\{1,\dots,t\}}(I_\delta).$$

In practice we do not have the exact projected Fourier coefficients $\hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi})$. Rather, we will approximate them by approximating

$$f(\cdot, \boldsymbol{\xi}^i) = \sum_{\boldsymbol{k}\in\mathbb{Z}^t} \hat{f}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}) \exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle)$$

for fixed anchors $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^r \in \mathbb{T}^{d-t}$ in the last $d-t$ components and a subsampled rank-1 lattice $\boldsymbol{X} \subset \mathbb{T}^t$ in the first t components:

$$S_{\boldsymbol{X}} f(\cdot, \boldsymbol{\xi}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^t} \hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}) \exp(2\pi\mathrm{i}\langle \boldsymbol{k}, \cdot\rangle). \tag{4}$$

Theorem III.2. *Let the assumptions from Theorem III.1 hold and let $\mathcal{P}_{\{1,\dots,t\}}(I_\delta) \subset I_{\{1,\dots,t\}} \subset I^M_{\{1,\dots,t\}}$ be frequency index sets such that $|I^M_{\{1,\dots,t\}} \setminus I_{\{1,\dots,t\}}| \leq 9/2|I_{\{1,\dots,t\}}|$. Further, let $\boldsymbol{X}^M$ be a reconstructing rank-1 lattice for $I^M_{\{1,\dots,t\}}$ with probability $1-\varepsilon$ and $\boldsymbol{X} \subset \boldsymbol{X}^M$ an i.i.d. uniformly drawn subset with*

$$|\boldsymbol{X}| \geq 12|I_{\{1,\dots,t\}}|\Big(\log|I_{\{1,\dots,t\}}| + \log\Big(\frac{2r}{\varepsilon}\Big)\Big).$$

With probability $1-3\varepsilon$ we detect all important frequencies in dimension t via the approximated projected Fourier coefficients $\hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)$ from (4) with r detection iterations and threshold

$$\delta' \leq \frac{\delta}{\sqrt{2}} - 4\|f - P_{I_\delta} f\|_{L_2} - 2\|f - \mathcal{P}_{I^M_{\{1,\dots,t\}}\times\mathbb{T}^{d-t}} f\|_\infty,$$

i.e.,

$$\max_{i=1,\dots,r} |\hat{g}_{\{1,\dots,t\},\boldsymbol{k}}(\boldsymbol{\xi}^i)| \geq \delta' \quad \forall \boldsymbol{k} \in \mathcal{P}_{\{1,\dots,t\}}(I_\delta).$$

Note, choosing $I^M_{\{1,\dots,t\}}$ large does not affect the sampling complexity but only the initial rank-1 lattice from which we sample and diminishes the term $\|f - \mathcal{P}_{I^M_{\{1,\dots,t\}}\times\mathbb{T}^{d-t}} f\|_\infty$.

Having shown the successful detection of the important frequencies in one dimension-incremental step it is left to

apply Theorem III.2 iteratively to obtain our main theorem stating the successful detection of all important frequencies $\boldsymbol{k} \in I_\delta$ using samples in subsampled rank-1 lattices.

Theorem III.3. *Let* $f\colon \mathbb{T}^d \to \mathbb{C}$, $\varepsilon, \delta > 0$, $\Gamma \supset I_\delta := \{\boldsymbol{k} \in \mathbb{Z}^d : |\hat{f}_{\boldsymbol{k}}| \geq \delta\}$, *and*

$$r \geq 4\Big(|I_\delta| + \frac{1}{\delta^2}\Big(\sum_{\boldsymbol{k}\notin I_\delta} |\hat{f}_{\boldsymbol{k}}|\Big)^2\Big)\Big(\log|I_\delta| + \log\frac{1}{\varepsilon}\Big).$$

1) *Let* $t = 1, \dots, t$ *and* $\boldsymbol{X}^M_{\{t\}}$ *be a reconstructing rank-1 lattice for* $J_{\{t\}} := \mathcal{P}_{\{t\}}(\Gamma)$ *with probability* $1-\varepsilon$ *and* $\boldsymbol{X}_{\{t\}} \subset \boldsymbol{X}^M_{\{t\}}$ *an i.i.d. uniformly drawn subset with*

$$|\boldsymbol{X}_{\{t\}}| \geq 12|J_{\{t\}}|\Big(\log|J_{\{t\}}| + \log\Big(\frac{2r}{\varepsilon}\Big)\Big).$$

Further, let $\boldsymbol{\Xi}_{\{t\}} = \{\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^r\} \subset \mathbb{T}^{d-1}$ *be drawn i.i.d. uniformly random.*
Using samples in $\boldsymbol{X}_{\{t\}} \times \boldsymbol{\Xi}_{\{t\}}$ *for* r *least squares approximations, we construct* $I_{\{t\}}$ *such that*

$$J_{\{t\}} \supset I_{\{t\}} \supset \mathcal{P}_{\{t\}}(I_\delta)$$

with probability exceeding $1-3\varepsilon$.

2) *Let* $t = 2, \dots, t$ *and* $\boldsymbol{X}^M_{\{1,\dots,t\}}$ *be a reconstructing rank-1 lattice for* $J_{\{1,\dots,t\}} := (I_{\{1,\dots,t-1\}} \times I_{\{t\}}) \cap \mathcal{P}_{\{1,\dots,t\}}(\Gamma)$ *with probability* $1-\varepsilon$ *and* $\boldsymbol{X}_{\{1,\dots,t\}} \subset \boldsymbol{X}^M_{\{1,\dots,t\}}$ *an i.i.d. uniformly drawn subset with*

$$|\boldsymbol{X}_{\{1,\dots,t\}}| \geq 12|J_{\{1,\dots,t\}}|\Big(\log|J_{\{1,\dots,t\}}| + \log\Big(\frac{2r}{\varepsilon}\Big)\Big).$$

Further, let $\boldsymbol{\Xi}_{\{1,\dots,t\}} = \{\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^r\} \subset \mathbb{T}^{d-t}$ *be drawn i.i.d. uniformly random.*
Using samples in $\boldsymbol{X}_{\{1,\dots,t\}} \times \boldsymbol{\Xi}_{\{1,\dots,t\}}$ *for* r *least squares approximations, we construct* $I_{\{1,\dots,t\}}$ *such that*

$$J_{\{1,\dots,t\}} \supset I_{\{1,\dots,t\}} \supset \mathcal{P}_{\{1,\dots,t\}}(I_\delta)$$

with probability exceeding $1-3\varepsilon$.

In particular, we have $I_{\{1,\dots,d\}} \supset I_\delta$ *with probability exceeding* $1-6d\varepsilon$.

Proof. The assertion follows from repeatedly applying Theorem III.2 and union bound. ■

IV. Numerical experiments

We consider the 10-dimensional test function $f\colon \mathbb{T}^{10} \to \mathbb{R}$,

$$f(\boldsymbol{x}) := \prod_{t\in\{1,3,8\}} N_2(x_t) + \prod_{t\in\{2,5,6,10\}} N_4(x_t) + \prod_{t\in\{4,7,9\}} N_6(x_t),$$

with $N_m\colon \mathbb{T} \to \mathbb{R}$ being the B-Spline of order $m \in \mathbb{N}$

$$N_m(x) := C_m \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{sinc}\Big(\frac{\pi k}{m}\Big)^m \exp(2\pi \mathrm{i} k x),$$

with a constant $C_m > 0$ such that $\|N_m\|_{L_2(\mathbb{T})} = 1$. This function was already used to test high-dimensional algorithms in [26], [22], [14], [16], [17]. It is not a sparse signal with respect to the trigonometric system, i.e., $|\hat{f}_{\boldsymbol{k}}| \neq 0$ for infinitely many $\boldsymbol{k} \in \mathbb{Z}^{10}$.

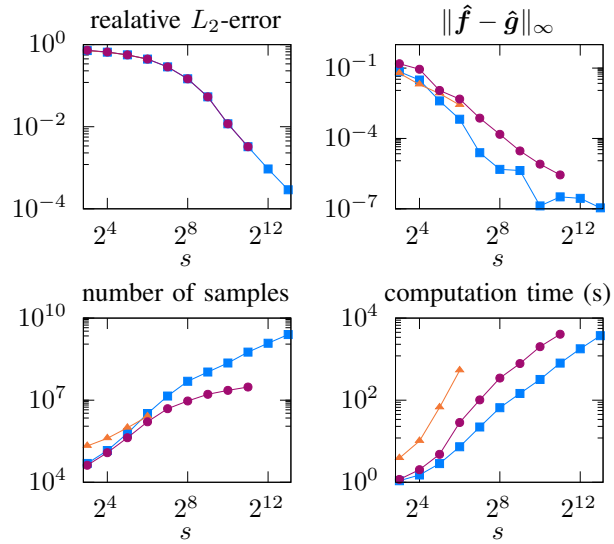

Fig. 1. Approximation results for the 10-dimensional test function for azure: full rank-1 lattices, orange: i.i.d. uniformly random points, and violet: subsampled rank-1 lattices.

Because of the product structure and the smoothness of the B-splines, we have $|\hat{f}_{\boldsymbol{k}}| \leq C\prod_{t=1}^d \max\{1, k_t\}^{-2}$ for some constant $C > 0$. For this reason we choose our initial search space $\Gamma \subset \mathbb{Z}^{10}$ to be a 10-dimensional hyperbolic cross of radius 2^8, i.e.,

$$\Gamma = \Big\{\boldsymbol{k} \in \mathbb{Z}^d : \prod_{t=1}^{10} \max\{1, |k_t|\} \leq 2^8\Big\}$$

with $|\Gamma| = 8\,827\,703\,433$ possible frequencies to choose from.

In our algorithm we choose the number of detection iterations $r = 5$ and the detection threshold to $\delta' = 10^{-12}$. To limit the number of detected frequencies, we set the target sparsity to $s \in \{2^3, \dots, 2^{13}\}$ and only consider the $s_{\text{local}} = \lceil 1.2s \rceil$ largest approximated Fourier coefficients in each dimension-incremental step. We then used three different sampling strategies for the dimension-incremental steps:

- full rank-1 lattices utilizing fast Fourier algorithms as in [22];
- i.i.d. uniformly random points utilizing good sampling complexity as in [17];
- subsampled rank-1 lattices combining both advantages

and obtained a frequency set $I \subset \mathbb{Z}^{10}$ and Fourier coefficients $\hat{\boldsymbol{g}} = (\hat{g}_{\boldsymbol{k}})_{\boldsymbol{k}\in I}$ of our approximation. All tests were performed 10 times in MATLAB® using 2 six core CPUs Intel® Xeon® CPU E5-2620 v3 @ 2.40GHz and 64 GB RAM. We stopped computations which exeeded a time limit of 1 hour.

In Figure 1 we depicted the medians of various quantities of the experiment. The two upper graphs show that the relative L_2-error as well as the error in the coefficients behaves the same with all three approaches. Note, that we did not need as many detection iterations as we proposed in the theory but $r = 5$ was sufficient. Next, we investigate the sampling complexity and computation time.

Sampling complexity. As discussed in the theoretical part, the reconstructing requirement of the rank-1 lattice blows up its size resulting in the most used sampling points. Because of computational infeasibility, we do not have many computations with the i.i.d. uniformly random points and cannot capture its behaviour (in our experience it should behave similar to the subsampled rank-1 lattice). The subsampled rank-1 lattices have better sampling complexity than the full rank-1 lattices and the graph suggests this advantage will increase for higher sparsity s.

Computation time. The fastest computation time can be seen with the full rank-1 lattices since the approximations only need one matrix-vector product each for which fast Fourier algorithms are utilized. The subsampled rank-1 lattices are slower by a constant factor of 10. Here, the same fast Fourier algorithm is used but the approximations use an iterative solver. We capped the maximal number of iterations by 10, which explains the constant factor. For the i.i.d. uniformly random points no fast Fourier algorithms are available making it the slowest approach.

The experiments confirms our theoretical findings of subsampled rank-1 lattices combining the computational and sampling advantages of full rank-1 lattices and random points, respectively.

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