GLOBAL NODE ATTENTIONS VIA ADAPTIVE SPECTRAL FILTERS

Anonymous authors
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ABSTRACT

Graph neural networks (GNNs) have been extensively studied for prediction tasks on graphs. Most GNNs assume local homophily, i.e., strong similarities in local neighborhoods. This assumption limits the generalizability of GNNs, which has been demonstrated by recent work on disassortative graphs with weak local homophily. In this paper, we argue that GNN's feature aggregation scheme can be made flexible and adaptive to data without the assumption of local homophily. To demonstrate, we propose a GNN model with a global self-attention mechanism defined using learnable spectral filters, which can attend to any nodes, regardless of distance. We evaluated the proposed model on node classification tasks over six benchmark datasets. The proposed model has been shown to generalize well to both assortative and disassortative graphs. Further, it outperforms all state-of-the-art baselines on disassortative graphs and performs comparably with them on assortative graphs.

1 INTRODUCTION

Graph neural networks (GNNs) have recently demonstrated great power in graph-related learning tasks, such as node classification (Kipf & Welling, 2017), link prediction (Zhang & Chen, 2018) and graph classification (Lee et al., 2018). Most GNNs follow a message-passing architecture where, in each GNN layer, a node aggregates information from its direct neighbors indifferently. In this architecture, information from long-distance nodes is propagated and aggregated by stacking multiple GNN layers together (Kipf & Welling, 2017; Velickovic et al., 2018; Defferrard et al., 2016). However, this architecture underlies the assumption of local homophily, i.e. proximity of similar nodes. While this assumption seems reasonable and helps to achieve good prediction results on graphs with strong local homophily, such as citation networks and community networks (Pei et al., 2020), it limits GNNs' generalizability. Particularly, determining whether a graph has strong local homophily or not is a challenge by itself. Further, strong and weak local homophily can be both exhibited in different parts of a graph, which makes the problem more complex and challenging.

Pei et al. (2020) proposed a metric to measure local node homophily based on how many neighbors of the node are from the same class. Using this metric, they categorized graphs as assortative (strong local homophily) or disassortative (weak local homophily), and showed that classical GNNs such as GCN (Kipf & Welling, 2017) and GAT (Velickovic et al., 2018) perform poorly on disassortative graphs. Liu et al. (2020) further showed that GCN and GAT are outperformed by a simple multi-layer perceptron (MLP) in node classification tasks on disassortative graphs. This is because naive local aggregation of homophilic models brings in more noise than information for such graphs. These findings indicate that these GNN models lose their expressive power when the fundamental assumption of local homophily does not hold.

Based on the above observation, we argue that a well-generalized GNN should perform well on graphs, regardless of their local homophily. Furthermore, since a real-world graph can exhibit both strong and weak homophily in different node neighborhoods, a powerful GNN model should be able to aggregate node features using different strategies accordingly. For instance, in disassortative networks where a node shares no similarity with any of its direct neighbors, such a GNN model should be able to ignore direct neighbors and reach farther to find similar nodes, or at least, resort to the node's attributes to make a prediction. Moreover, to be adaptive, such aggregation strategies should be learned from data rather than decided upfront.

To circumvent this issue, in this paper, we propose a novel GNN model with global self-attention mechanism, called GNAN. Most existing attention-based aggregation architectures perform self-attention to the local neighborhood of the current node (Velickovic et al., 2018), which may add local noises in aggregation. Unlike these works, we aim to design an aggregation method that can gather informative features from both close and far-distant nodes. To achieve this, we employ graph wavelets under a relaxed condition of localization, which enables us to learn attention weight for each node in the spectral domain. In doing so, the model can effectively capture not only local information but also global structure into node representations.

To further improve the generalizability of our model, instead of using predefined spectral kernels, we propose to use multi-layer perceptrons (MLP) to learn the desired spectral filters without limiting their shapes. Existing work on graph wavelet transform chooses wavelet filters heuristically, such as heat kernel, wave kernel and personalized page rank kernel (Klicpera et al., 2019b; Xu et al., 2019; Klicpera et al., 2019a). These are mostly low-pass filters, which mean these models implicitly treat high-frequency components as "noises" and have them discarded (Shuman et al., 2013; Hammond et al., 2011; Chang et al., 2020). However, this may hinder the learning ability of models since high frequencies can carry meaningful information about local discontinuities, as analyzed in (Shuman et al., 2013). Our model overcomes these limitations by incorporating fully learnable spectral filters into the proposed global self-attention mechanism.

From a computational perspective, learning global self-attention may impose high computational overhead, particularly when graphs are large. We alleviate this problem from two aspects. First, we sparsify nodes according to their wavelet coefficients, which enables attention weights to be distributed across the graph sparsely. Second, we observed that learned filters from different MLPs tend to converge to be of similar shapes. Thus, we use a single MLP to reduce redundancy among filters, where each dimension in the output corresponds to one adaptive spectral filter. In addition to these, following (Xu et al., 2019; Klicpera et al., 2019b), we use a fast algorithm to efficiently approximate graph wavelet transform, which has computational complexity $O(nm)$, where n is the order of Chebyshev polynomials and m is the number of edges in a graph.

To summarize, the main contributions of this work are as follows:

1. We propose a generalized GNN model which performs well on both assortative and disassortative graphs, regardless of local node homophily.
2. We exhibit that GNN's aggregation strategy can be trained via a fully learnable spectral filter, thereby enabling feature aggregation from both close and far nodes.
3. We show that, unlike commonly understood, higher-frequency on disassortative graphs provides meaningful information that helps improving prediction performance.

We conduct extensive experiments to compare GNAN with well-known baselines on node classification tasks. The experimental results show that GNAN significantly outperforms state-of-the-art methods in disassortative graphs where the local node homophily is weak, and performs comparably with the state-of-the-art methods on assortative graphs where local node homophily is strong. This work provides a general model for learning on different types of graphs.

2 PRELIMINARIES

Let $\mathcal{G} = (V, E, A, \boldsymbol{x})$ be an undirected graph with N nodes, where V, E, and A are the node set, edge set, and adjacency matrix of $\mathcal{G}$, respectively, and $\boldsymbol{x} : V \mapsto \mathbb{R}^m$ is a graph signal function that associates each node with a feature vector. The normalized Laplacian matrix of $\mathcal{G}$ is defined as $\boldsymbol{L} = \boldsymbol{I} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$, where $\boldsymbol{D} \in \mathbb{R}^{N\times N}$ is the diagonal degree matrix of $\mathcal{G}$. In spectral graph theory, the eigenvalues $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$ and eigenvectors $\boldsymbol{U}$ of $\boldsymbol{L} = \boldsymbol{U}\Lambda\boldsymbol{U}^H$ are known as the graph's spectrum and spectral basis, respectively, where $\boldsymbol{U}^H$ is the Hermitian transpose of $\boldsymbol{U}$. The graph Fourier transform of $\boldsymbol{x}$ is $\hat{\boldsymbol{x}} = \boldsymbol{U}^H\boldsymbol{x}$ and its inverse is $\boldsymbol{x} = \boldsymbol{U}\hat{\boldsymbol{x}}$.

The spectrum and spectral basis carry important information on the connectivity of a graph (Shuman et al., 2013). Intuitively, lower frequencies correspond to global and smooth information on the graph, while higher frequencies correspond to local information, discontinuities and possible noise (Shuman et al., 2013). One can apply a spectral filter g as in Equation 1 and use graph Fourier transform to manipulate signals on a graph in various ways, such as smoothing and denoising (Schaub &

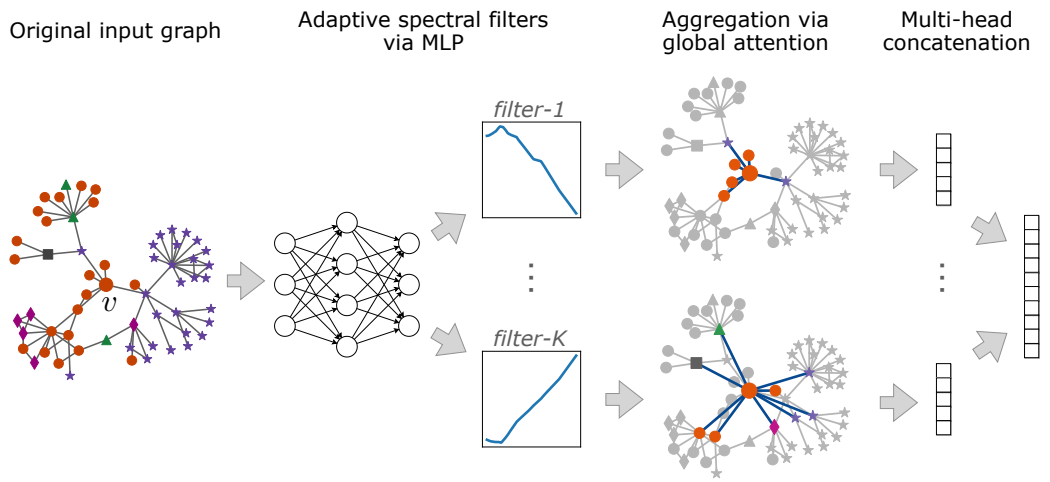

Figure 1: Illustration of a spectral node attention layer on a 3-hop ego network of the central node v from the CITESEER dataset. Node classes are indicated by shape and color. Passing the graph through two learned spectral filters (adaptive spectral filters) place attention scores on nodes, including node v itself. Nodes with positive attention scores are presented in color. Node features are aggregated for node v according to attention scores (aggregation via global attention). The low-pass filter attend to local neighbors (filter 1), while the high-pass filter skips the first hop and attend the nodes in the second hop (filter K). The resulting embeddings from multiple heads are then concatenated before being sent to the next layer (multi-head concatenation). Note that we have visualized learned filters from experiments.

Segarra, 2018), abnormally detection (Miller et al., 2011) and clustering (Wai et al., 2018). Spectral convolutions on graphs is defined as the multiplication of a signal $\boldsymbol{x}$ with a filter $g(\Lambda)$ in the Fourier domain, i.e.

$$g(\boldsymbol{L})\boldsymbol{x} = g(\boldsymbol{U}\Lambda\boldsymbol{U}^H)\boldsymbol{x} = \boldsymbol{U}g(\Lambda)\boldsymbol{U}^H\boldsymbol{x} = \boldsymbol{U}g(\Lambda)\hat{\boldsymbol{x}}. \tag{1}$$

When a spectral filter is parameterized by a scale factor, which controls the radius of neighbourhood aggregation, Equation 1 is also known as the Spectral Graph Wavelet Transform (SGWT) (Hammond et al., 2011; Shuman et al., 2013). For example, Xu et al. (2019) uses a small scale parameter $s < 2$ for a heat kernel, $g(s\lambda) = e^{-\lambda s}$, to localize the wavelet at node.

3 PROPOSED APPROACH

Graph neural networks (GNNs) learn lower-dimensional embeddings of nodes from graph structured data. In general, given a node, GNNs iteratively aggregate information from its neighbor nodes, and then combine the aggregated information with its own information. An embedding of node v at the kth layer of GNN is typically formulated as

$$\boldsymbol{m}_v = \text{aggregate}(\{\boldsymbol{h}_u^{(k-1)} | u \in \mathcal{N}_v\})$$
$$\boldsymbol{h}_v^{(k)} = \text{combine}(\boldsymbol{h}_v^{(k-1)}, \boldsymbol{m}_v),$$

where $\mathcal{N}_v$ is the set of neighbor nodes of node v, $\boldsymbol{m}_v$ is the aggregated information from the neighbors, and $\boldsymbol{h}_u^{(k)}$ is the embedding of the node u at the kth layer ($\boldsymbol{h}_u^{(0)} = \boldsymbol{x}_u$). The embedding of the node $\boldsymbol{h}_v^n$ at the final layer is then used for some prediction tasks. In most GNNs, $\mathcal{N}_v$ refers to a set of one-hop neighbors of node v. Therefore, one needs to stack multiple aggregation layers to collect the information from more than one-hop neighborhood with this structure.

Adaptive spectral filters. Instead of stacking multiple layers, we introduce a spectral attention layer that rewires a graph based on spectral graph wavelets. A spectral graph wavelet $\boldsymbol{\psi}_v$ at node v is a modulation in the spectral domain of signal centered around the node v, given by an N-dimensional vector

$$\boldsymbol{\psi}_v = \boldsymbol{U}g(\Lambda)\boldsymbol{U}^H\delta_v, \tag{2}$$

where $g(\cdot)$ is a spectral filter and δ_v is a one-hot vector for node v.

The common choice of a spectral filter is heat kernel. A wavelet coefficient ψ_{vu} computed from a heat kernel can be interpreted as the amount of energy that node v has received from the node u in its local neighborhood. In this work, instead of using pre-defined localized kernels, we use multi-layer perceptrons (MLP) to learn spectral filters. With learnable spectral kernels, we obtain wavelet coefficients

$$\boldsymbol{\psi}_v = \boldsymbol{U}\text{diag}(\text{MLP}(\Lambda))\boldsymbol{U}^H\delta_v, \tag{3}$$

where MLP is a learnable spectral filter composed of two-hidden layers. Similar to that of a heat kernel, the wavelet coefficient with the learned filter ψ_{vu} can be understood as the amount of energy that distributed from node v to node u, under the conditions regulated by the spectral filter. Note that we use the terminology wavelet and spectral filter interchangeably as we have relaxed the wavelet definition from (Hammond et al., 2011) so that the learned filters are not necessarily localized in the spectral and spatial domain. Equation 3 requires the eigen-decomposition of a Laplacian matrix, which is expensive and infeasible for large graphs. We follow Xu et al. (2019); Klicpera et al. (2019b) to approximate graph wavelet transform using Chebyshev polynomials (Shuman et al., 2013).

Global self-attention. Unlike the previous work (Xu et al., 2019) where a coefficient is directly used to normalize the input feature of the node v, we employ spectral graph wavelets to give attention weights to the other nodes in the aggregation. We normalize attention weights through the softmax layer

$$\boldsymbol{a}_v = \text{softmax}(\boldsymbol{\psi}_v),$$

where $\boldsymbol{a}_v \in \mathbb{R}^N$ is an attention weight vector. With the attention weights, the update layer is then formalized as

$$\boldsymbol{h}_v^{(k)} = \sigma\left(\sum_{u=1}^{N} a_{vu}\boldsymbol{h}_u^{(k-1)}\boldsymbol{W}^{(k)}\right), \tag{4}$$

where $\boldsymbol{W}^{(k)}$ is a weight matrix shared across all nodes in the kth layer and σ is ELU nonlinear activation. Unlike heat kernel, the wavelet coefficient with a learned kernel is not localized. Hence, the aggregation function can actively aggregate information from far-distant nodes. Note that the update layer is not divided into aggregation and combine steps in our work. Instead, we compute the attention a_{vv} directly from a spectral filter.

Sparsified node attentions. With predefined localized spectral filters such as a heat kernel, most of the coefficients of wavelets are zero due to the locality. In our method, spectral filters are fully learned from data. As a result, attention weights obtained from the learned spectral filters do not impose any sparsity. This means that to perform the aggregation operation we need to retrieve all possible nodes in a graph, which is time consuming with large graphs. From our experiments, we found that most of attention weights are negligible after softmax. To sparsify attention weights, we discard the entries of wavelet bases that are below the threshold t, i.e.

$$\bar{\psi}_{vu} = \begin{cases} \psi_{vu} & \text{if } \psi_{vu} > t \\ -\infty & \text{if } \psi_{vu} \leq t, \end{cases} \tag{5}$$

where $\bar{\psi}$ is then fed into the softmax layer to compute attention weights. For the experiments, we set t to 0.

We adopt multi-head attention to model multiple spectral filters. Each attention head aggregates node information with a different spectral filter, and the aggregated embedding is concatenated before being sent to the next layer. We can allocate an independent MLP for each of the attention heads; however, we found independent MLPs tend to converge to similar shapes. Instead, we adopt a single MLP: $\mathbb{R}^N \rightarrow \mathbb{R}^{N\times M}$, where M is the number of attention heads, and each column of output corresponding to one adaptive spectral filter.

We name the multi-head spectral attention architecture as a global node attention network (GNAN). The design of GNAN is easily generalizable, and many existing GNNs can be expressed as special cases of GNAN (see Appendix C). Figure 1 illustrates how GNAN works with two attention heads learned from the CITESEER dataset. As shown in the illustration, the MLP models adaptive filters such as low-band pass and high-band pass filters. The low-band pass filter assigns high attention weights in local neighborhood, and the high-band pass filter assigns high attention weights on the far-distant nodes, which cannot be captured by a one-hop aggregation scheme.

Table 1: Dataset statistics. We categorize the datasets into assortative and disassortative based on their homophily (β).

Group	Dataset	#Nodes	#Edges	#Classes	#Features	Density	β
Assortative	CORA	2,708	5,429	7	1,433	1.44e-3	0.83
	CITESEER	3,327	4,732	6	3,703	8.23e-4	0.71
	PUBMED	19,717	44,338	3	500	2.28e-4	0.79
Disassortative	CORNELL	183	298	5	1,703	1.68e-2	0.11
	TEXAS	183	325	5	1,703	1.77e-2	0.06
	WISCONSIN	251	515	5	1,703	1.49e-2	0.16

4 EXPERIMENTS

To evaluate the performance of our proposed approach, we conduct experiments on node classification tasks with assortative graph datasets, where the labels of nodes exhibit strong homophily, and disassortative graph datasets, where the local homophily is weak and labels of nodes represent their structural roles. To quantify the assortativeness of graphs, we use the metric β introduced by Pei et al. (2020),

$$\beta = \frac{1}{|V|}\sum_{v\in V}\beta_v \quad \text{and} \quad \beta_v = \frac{|\{u \in \mathcal{N}_v | \ell(u) = \ell(v)\}|}{|\mathcal{N}_v|}, \tag{6}$$

where $\ell(v)$ refers to the label of node v. β measures the homophily of a graph, and β_v measures the homophily of node v in the graph. A graph has a strong local homophily if β is large and vice versa.

4.1 EXPERIMENTAL SETUP

Baseline methods. We compare GNAN against 11 benchmark methods: vanilla GCN (Kipf & Welling, 2017) and its simplified version SGC (Wu et al., 2019); two spectral methods, one using the Chebyshev polynomial spectral filters (Defferrard et al., 2016) and the other using the auto-regressive moving average (ARMA) filters (Bianchi et al., 2019); the graph attention model GAT (Velickovic et al., 2018); APPNP which allows adaptive neighbourhood aggregation using personalized page rank (Klicpera et al., 2019a); three sampling-based approaches, GraphSage (Hamilton et al., 2017), FastGCN Chen et al. (2018) and ASGCN (Huang et al., 2018); and Geom-GCN which targets prediction on disassortative graphs (Pei et al., 2020). We also include MLP in the baselines since it performs better than many GNN-based methods on disassortative graphs (Liu et al., 2020).

Datasets. We evaluate our method and baselines on node classification tasks over three citation networks: CORA, CITESEER and PUBMED (Sen et al., 2008), and three webgraphs from the WebKB dataset[1]. We divide these datasets into two groups, assortative and disassortative, based on their β. The dataset details are summarized in Table 1.

Hyper-parameter settings. For the citation networks, we follow the experimental setup for node classification from (Hamilton et al., 2017; Huang et al., 2018; Chen et al., 2018) and report the results averaged on 10 runs. For the webgraphs, we run each model on the 10 splits provided by (Pei et al., 2020) and take the average, where each split uses 60%, 20%, and 20% nodes of each classes for training, validation and testing, respectively. We have reported better results on GCN and GAT than Pei et al. (2020) by converting the graphs to undirected before training [2]. Geom-GCN uses node embeddings pre-trained from different methods such as Isomap (Tenenbaum et al., 2000), Poincare (Nickel & Kiela, 2017) and struc2vec (Ribeiro et al., 2017). We hereby report the best micro-F1 among all variants for Geom-GCN.

We use the best performing hyperparameters specified in each baseline method's original papers. For hyperparameters not specified by the original authors, we use the parameters from (Fey & Lenssen, 2019). We report the test accuracy from the epoch with the smallest validation loss and highest validation accuracy. Early termination is adopted for both validation loss and accuracy, so that the training is stopped when neither metric improves for 100 consecutive epochs. We use a two-layer

[1] http://www.cs.cmu.edu/afs/cs.cmu.edu/project/theo-11/www/wwkb/

[2] https://openreview.net/forum?id=S1e2agrFvS

Table 2: Micro-F1 results for node classification. The proposed model consistently outperforms the other variants of GNN on disassortative graphs and performs comparable on assortative graphs.

Method	CORA	CITESEER	PUBMED	CORNELL	TEXAS	WISCONSIN
GCN	87.4 ± 0.2	78.5 ± 0.5	87.8 ± 0.2	59.2 ± 3.2	64.1 ± 4.9	64.1 ± 6.3
Chebyshev	88.2 ± 0.2	79.4 ± 0.4	89.3 ± 0.3	76.5 ± 9.4	79.7 ± 5.0	82.5 ± 2.8
ARMA	85.2 ± 2.5	76.7 ± 0.5	86.3 ± 5.7	74.9 ± 2.9	82.2 ± 5.1	78.4 ± 4.6
GAT	87.6 ± 0.3	77.7 ± 0.3	83.0 ± 0.1	58.9 ± 3.3	60.0 ± 5.7	62.0 ± 5.2
SGC	87.2 ± 0.3	78.8 ± 0.4	81.1 ± 0.3	58.1 ± 4.6	58.9 ± 6.1	51.8 ± 5.9
GraphSAGE	86.3 ± 0.6	77.4 ± 0.5	89.2 ± 0.5	67.3 ± 6.9	82.7 ± 4.8	77.6 ± 4.6
APPNP	$\mathbf{88.4} \pm 0.3$	77.6 ± 0.6	86.0 ± 0.3	70.3 ± 9.3	79.5 ± 4.6	81.2 ± 2.5
FastGCN	86.1 ± 0.4	78.1 ± 0.7	86.0 ± 0.8	72.7 ± 6.8	68.4 ± 4.6	68.8 ± 5.4
ASGCN	86.8 ± 0.4	77.9 ± 0.3	88.0 ± 0.7	73.0 ± 6.1	68.1 ± 7.7	66.3 ± 6.5
Geom-GCN	86.3 ± 0.3	$\mathbf{81.4} \pm 0.3$	89.1 ± 0.1	75.4	73.5	80.4
MLP	72.1 ± 1.3	74.9 ± 1.8	88.6 ± 0.2	81.4 ± 6.3	79.2 ± 6.1	82.7 ± 4.5
GNAN*	87.4 ± 0.4	79.0 ± 1.1	$\mathbf{89.9} \pm 0.9$	$\mathbf{82.7} \pm 8.3$	$\mathbf{85.1} \pm 5.7$	$\mathbf{85.9} \pm 3.8$

GNAN where each head's filter is learned using a MLP of 2 hidden layers and then approximated by Chebyshev polynomials. Each layer of the MLP consists of a linear function and a ReLU activation. To avoid overfitting, dropout is applied in each GNAN layer on both attention weights and inputs equally.

4.2 RESULTS AND DISCUSSION

We use two evaluation metrics to evaluate the node classification performance: micro-F1 and macro-F1. The results with micro-F1 are summarized in Table 2, and the results with macro-F1 are provided in Table 3 in the appendix. Overall, on assortative citation networks, GNAN performs comparably with state-of-the-art methods, ranks first on PUBMED and the second on CORA and CITESEER in terms of micro-F1 scores. On disassortative graphs, GNAN outperforms all state-of-the-art methods by a margin of at least 2.4% and MLP by a margin of at least 1.3%. The results indicate that GNAN can learn spectral filters that work adaptively based on different characteristics of graphs.

Although our model performs well on both assortative and disassortative graphs, it is unclear how GNAN performs on disassortative nodes, whose neighbors are mostly of different classes, in an assortative graph. To further investigate how our model classifies disassortative nodes in assortative graphs in comparison with other state-of-the-art methods, we report an average classification accuracy on disassortative nodes at different levels of β_v in Figure 2. The nodes of CITESEER and PUBMED datasets are binned into five groups based on β_v. For example, all nodes with $0.3 < \beta_v \leq 0.4$ belong to the 0.4 bin. We have excluded CORA from the report since it has very few nodes with low β_v.

The results in Figure 2 show that all GNNs based on local aggregation schemes perform poorly when β_v is low. One may argue that the performance on disassortative graphs might improve by stacking multiple GNN layers together, which would obtain information from far-distant nodes. However, it turns out that this approach introduces an oversmoothing problem in local aggregation-based GNNs (Li et al., 2018). On the other hand, GNAN outperforms the other GNN-based methods on disassortative nodes, suggesting that adaptive spectral filters reduce local noise in aggregation while allowing distant nodes to be attended to.

To understand how adaptive spectral filters contribute to GNAN's performance on disassortative graphs, we conduct an ablation study on spectral frequency ranges. We divide the entire frequency range $(0 \sim 2)$ into a set of predefined sub-ranges exclusively and manually set the filter response to zero for each sub-range at a time to check the influence of a given range to the classification. By doing so, the frequencies within a selected range do not contribute to neither node attention nor feature aggregation, therefore helping to reveal the importance of the frequency range. We divide the entire range based on three different lengths of sub-ranges: 0.25, 0.5, and 1.0. The results of frequency ablation on the three assortative graphs are summarized in Figure 3. The ablation result from the sub-range of length 1 reveals the importance of high-frequency range $(1 \sim 2)$ on node classification of disassortative graphs. The performances are significantly dropped by ablating high-frequency range

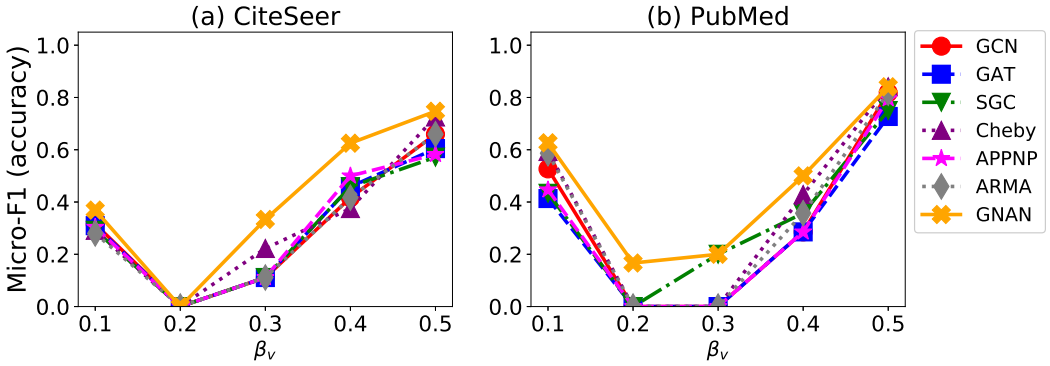

Figure 2: Micro-F1 results for classification accuracy on disassortative nodes ($\beta_v \leq 0.5$). GNAN shows better accuracy on classifying disassortative nodes than the other models.

on all datasets. Further investigation at the finer-level sub-ranges (0.5) shows that range $0.5 \sim 1.5$ has the most negative impact on performance, where the most important range varies across different datasets at the finest level (0.25). This finding matches our intuition that low-pass filters used in GNNs underlie the local node homophily assumption in a similar way as naive local aggregation. We suspect the choice of low-pass filters also relates to oversmoothing issues in spectral methods (Li et al., 2018), but we leave it for future work.

Attention head ablation. In GNAN, each head uses a filter to produce spectral attention weights. To delve the importance of a filter, we follow the ablation method used by (Michel et al., 2019). Specifically, we ablate one or more filters by manually setting the attention weights to zeros. We then measure the impact on performance using micro-F1. If the ablation results in a large decrease in performance, the ablated filters are considered important. We observe that all attention heads (filters) in GNAN are of similar importance, and only all attention heads combined produce the best performance. Please check Appendix B for the detailed results.

5 Related Work

Graph neural networks have been extensively studied recently. We categorize work relevant to ours into three perspectives and summarize the key ideas.

Attention on graphs. Graph attention networks (GAT) (Velickovic et al., 2018) was the first to introduce attention mechanisms on graphs. GAT assigns different importance scores to local neighbors via attention mechanism. Similar to other GNN variants, long-distance information propagation in GAT is realized by stacking multiple layers together. Therefore, GAT suffers from the oversmoothing issue (Zhao & Akoglu, 2020). Zhang et al. (2020) improve GAT by incorporating both structural and feature similarities while computing attention scores.

Spectral graph filters and wavelets. Some GNNs also use graph wavelets to extract information from graphs. Xu et al. (2019) applied graph wavelet transform defined by Shuman et al. (2013) in GNNs. Klicpera et al. (2019b) proposed a general GNN argumentation using graph diffusion kernels to rewire the nodes. Donnat et al. (2018) used heat wavelet to learn node embeddings in unsupervised ways and showed that the learned embeddings closely capture structural similarities between nodes. Other spectral filters used in GNNs can also be viewed as special forms of graph wavelets (Kipf & Welling, 2017; Defferrard et al., 2016; Bianchi et al., 2019). Coincidentally, Chang et al. (2020) also noticed useful information carried by high-frequency components from a graph Laplacian. Similarly, they attempted to utilize such components using node attentions. However, they resorted to the traditional choice of heat kernels and applied such kernels separately to low-frequency and high-frequency components divided by a hyperparameter. In addition to this, their work did not link high-frequency components to disassortative graphs.

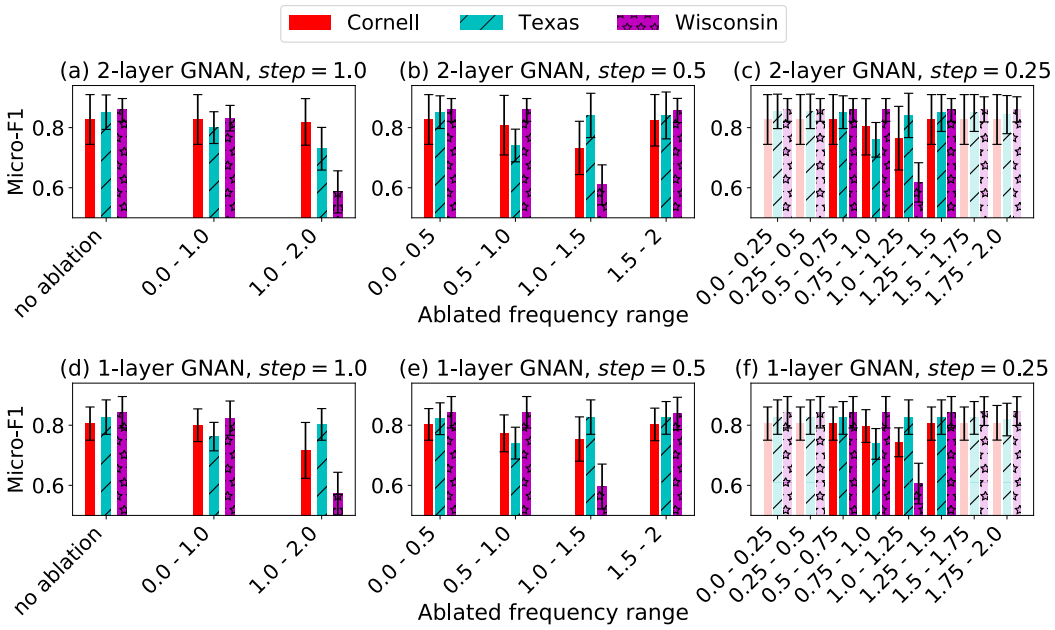

Figure 3: Micro-F1 with respect to an ablated frequency range on disassortative graphs. We divide the frequency range into a set of sub-ranges with different lengths. The results (a) and (d) reveal the importance of high-frequency range $(1 \sim 2)$. Further experiments show that there is a subtle difference in the most important range across datasets, but it ranges between $(0.75 \sim 1.25)$.

Prediction on disassortative graphs. Pei et al. (2020) have drawn attention to GCN and GAT's poor performance on disassortative graphs very recently. They tried to address the issue by essentially pivoting feature aggregation to structural neighborhoods from a continuous latent space learned by unsupervised methods. Another attempt to address the issue was proposed by Liu et al. (2020). They proposed to sort locally aggregated node embeddings along a one-dimensional space and used a one-dimensional convolution layer to aggregate embeddings a second time. By doing so, non-local but similar nodes can be attended to.

Although our method shares some similarities in motivation with the aforementioned work, it is fundamentally different in several aspects. To the best of our knowledge, our method is the first to learn spectral filters as part of supervised training on graphs. It is also the first architecture we know that computes node attention weights purely from learned spectral filters. As a result, in contrast to commonly used heat kernel, our method utilizes high-frequency components of a graph, which helps prediction on disassortative graphs.

6 Conclusion

In this paper, we study the node classification tasks on graphs where local node homophily is weak. We argue the assumption of local homophily is the cause of poor performance on disassortative graphs. In order to design more generalizable GNNs, we suggest that a more flexible and adaptive feature aggregation scheme is needed. To demonstrate, we have introduced the global node attention network (GNAN) which achieves flexible feature aggregation using learnable spectral graph filters. By utilizing the full graph spectrum adaptively via the learned filters, GNAN is able to aggregate features from nodes that are close and far. For node classification tasks, GNAN outperforms all benchmarks on disassortative graphs, and performs comparably on assortative graphs. On assortative graphs, GNAN also performs better for nodes with weak local homophily. Through our analysis, we find the performance gain is closely linked to the higher end of the frequency spectrum.

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APPENDIX

A FURTHER EXPERIMENT RESULTS

We provide the macro-F1 scores on the classification task in Table 3. The proposed model outperforms the other models on disassortative graphs and performs comparable on the assortative graphs.

Table 3: Macro-F1 for node classification task.

Method	CORA	CITESEER	PUBMED	CORNELL	TEXAS	WISCONSIN
GCN	86.0 ± 0.3	72.0 ± 1.6	86.8 ± 0.2	24.1 ± 9.1	34.0 ± 5.7	37.6 ± 9.2
Chebyshev	**86.8** ± 0.3	74.1 ± 1.0	88.7 ± 0.3	53.6 ± 17.6	64.1 ± 12.4	52.9 ± 7.7
ARMA	80.2 ± 6.8	66.4 ± 0.4	81.6 ± 13.9	48.5 ± 9.3	69.1 ± 12.3	53.3 ± 7.1
GAT	86.4 ± 0.4	69.2 ± 1.0	81.6 ± 0.1	19.0 ± 2.8	26.5 ± 6.8	30.0 ± 5.2
SGC	86.0 ± 0.3	74.7 ± 1.2	79.8 ± 0.3	21.9 ± 8.5	23.2 ± 7.5	35.9 ± 6.3
GraphSAGE	85.2 ± 0.1	74.2 ± 0.6	88.7 ± 0.6	49.2 ± 10.9	62.9 ± 9.6	63.7 ± 12.4
APPNP	87.0 ± 0.4	70.2 ± 1.4	84.8 ± 0.3	39.6 ± 16.6	61.0 ± 8.8	55.8 ± 5.7
FastGCN	84.5 ± 0.5	72.0 ± 2.3	84.6 ± 0.8	47.6 ± 13.0	38.9 ± 7.5	41.8 ± 10.6
ASGCN	85.7 ± 0.5	**75.4** ± 0.4	86.9 ± 0.7	54.3 ± 11.5	45.2 ± 10.0	43.4 ± 11.5
Geom-GCN	85.1 ± 0.3	76.9 ± 0.5	88.5 ± 0.1	–	–	–
MLP	67.2 ± 2.5	67.6 ± 3.5	88.1 ± 0.2	63.0 ± 12.3	61.7 ± 15.2	54.6 ± 11.5
GNAN	86.4 ± 0.4	74.1 ± 1.9	**89.2** ± 1.0	**66.6** ± 15.3	**72.0** ± 10.6	**63.5** ± 7.0

B ABLATION STUDY ON FILTERS

We provide the detailed version of Figure 3 (c) and (f) in Figure 4.

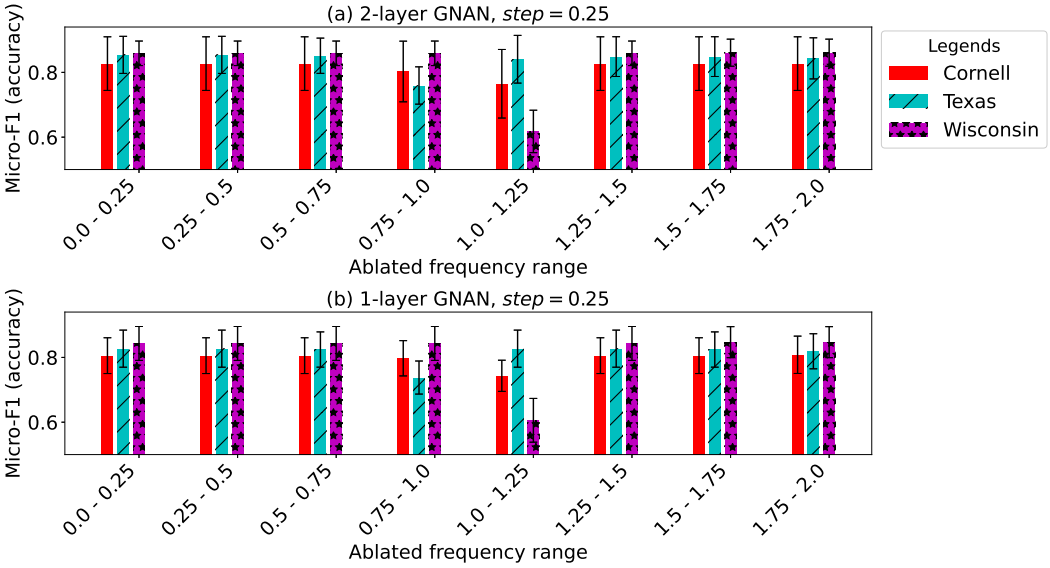

Figure 4: Full details of the performances on frequency ablation at 0.25 level.

We further ablated attention head to check the importance of each head in classification.

Ablating all but one spectral filter. In GNAN, each head uses a filter to produce spectral attention weights. To delve the importance of a filter, we follow the ablation method used by (Michel et al., 2019). Specifically, we ablate one or more filters by manually setting the attention scores to zeros. We then measure the impact on performance using micro-F1. If the ablation results in a large decrease in performance, the ablated fitlerbank(s)

is considered important. The results are summarized in Table 4a. All attention head (filters) in GNAN are of similar importance, and only all heads combined produces the best performance.

Ablating only one spectral filter. We then examine performance differences by ablating one filter only and keeping all other fitlerbanks Table 4b. Different with above, ablating just one fitlerbank only decreases performance by a small margin. Moreover, ablating some fitlerbanks does not impact prediction performance at all. This is an indicator of potential redundancies in the filters. We leave the redundancy reduction in the model for future work.

Table 4: Ablation study on attention head (filter). We use 12 attention heads for CORA and PUBMED, and 14 heads for CITESEER.

(a) Test accuracy by keeping only one filter

Dataset \ Head	1	2	3	4	5	6	7	8	9	10	11	12	13	14
CORA	-2.10%	-2.10%	-1.30%	-2.10%	-1.50%	-2.10%	-2.10%	-2.10%	-1.40%	-1.50%	-2.10%	-1.30%	-	-
CITESEER	-1.8%	-2.0%	-1.7%	-1.7%	-1.8%	-1.8%	-1.7%	-1.9%	-1.7%	-1.9%	-1.7%	-1.7%	-1.9%	-1.9%
PUBMED	-4.30%	-4.40%	-4.40%	-5.40%	-4.30%	-5.40%	-5.40%	-5.40%	-4.30%	-4.30%	-4.30%	-5.40%	-	-

(b) Test accuracy by ablating only one filter

Dataset \ Head	1	2	3	4	5	6	7	8	9	10	11	12	13	14
CORA	0.00%	0.00%	-0.30%	0.00%	-0.40%	0.00%	0.00%	0.00%	-0.40%	-0.40%	0.00%	-0.40%	-	-
CITESEER	-0.20%	-0.30%	-0.70%	-0.70%	-0.40%	-0.60%	-0.80%	0.00%	-0.60%	0.00%	-0.80%	-0.50%	0.00%	0.00%
PUBMED	-0.80%	-0.80%	-0.70%	0.00%	-0.80%	0.00%	0.00%	0.00%	-0.70%	-0.80%	-0.80%	0.00%	-	-

C CONNECTIONS TO OTHER METHODS

In this section, we connect GNAN to existing models. These connections show that GNAN is a generalized version of existing models.

C.1 CONNECTION TO GCN

A GCN (Kipf & Welling, 2017) layer can be expressed as

$$\boldsymbol{h}_i' = \mathrm{ReLU}(\sum_{j=1}^{N} \hat{a}_{ij} \boldsymbol{h}_j \boldsymbol{W})$$

where $\hat{a}_{ij}$ is the elements from the ith row of the symmetric adjacency matrix

$$\hat{\boldsymbol{A}} = \tilde{\boldsymbol{D}}^{-1/2} \tilde{\boldsymbol{A}} \tilde{\boldsymbol{D}}^{-1/2} \qquad \text{where} \qquad \tilde{A} = \boldsymbol{A} + \boldsymbol{I}_N,\ \tilde{D}_{ii} = \sum_j \tilde{A}_{ij}$$

So that

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in E \\ 0 & if\ e_{ij} \notin E \end{cases}$$

Therefore, GCN can be viewed as a single-headed case of Equation 4 with $\sigma = \mathrm{ReLU}$ and $a_{ij} = \hat{a}_{ij}$

C.2 CONNECTION TO POLYNOMIAL FILTERS

Polynomial filters localize in a node's K-hop neighbors utilizing K-order polynomials, most of them takes the following form:

$$g_\theta(\Lambda) = \sum_{k=0}^{K-1} \theta_k \Lambda^k$$

where θ_k is a learnable polynomial coefficient for each order. Thus a GNN layer using a polynomial filter becomes

$$\boldsymbol{h}'_j = \mathrm{ReLU}(\boldsymbol{U} g_\theta(\Lambda) \boldsymbol{U}^T \boldsymbol{h}_j)$$

which can be expressed using Equation 4 with $\boldsymbol{W}^{(k)} = \boldsymbol{I}_N$, $\sigma = \mathrm{ELU}$ and $a_{ij} = (\boldsymbol{U} g_\theta(\Lambda) \boldsymbol{U}^T)_{ij}$. Similar to our method, polynomial filters are normally formulated using ChebyShev recursive polynomials to reduce computational complexity.

C.3 Connection to GAT

Our method is inspired by and closely related to GAT (Velickovic et al., 2018). To demonstrate the connection, we firstly define a matrix Φ where each column $\boldsymbol{\phi}_i$ is the transformed feature vector of node i concatenated with feature vector of another node (including i) in the graph.

$$\boldsymbol{\phi}_i = ||_{j=0}^{N} [\boldsymbol{W}\boldsymbol{h}_i || \boldsymbol{W}\boldsymbol{h}_j] \tag{7}$$

GAT multiplies each column of Φ with a learnable weight vector $\boldsymbol{\alpha}$ and masks the result with the adjacency $\boldsymbol{A}$ before feeding it to the nonlinear function $\mathrm{LeakyRelu}$ and $\mathrm{softmax}$ to calculate attention scores. The masking can be expressed as a Hadamard product with the adjacency matrix $\boldsymbol{A}$ which is the congruent of a graph wavelet transform with the filter $g(\Lambda) = \boldsymbol{I} - \Lambda$:

$$\Psi = \boldsymbol{A} = \boldsymbol{D}^{\frac{1}{2}} \boldsymbol{U} (\boldsymbol{I} - \Lambda) \boldsymbol{U}^T \boldsymbol{D}^{\frac{1}{2}} \tag{8}$$

And the GAT attention vector for node i become

$$\boldsymbol{a}_i = \mathrm{softmax}(\mathrm{LeakyReLU}(\boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{\phi}_{\mathrm{i}} \odot \bar{\boldsymbol{\psi}}_{\mathrm{i}})) \tag{9}$$

where $\bar{\boldsymbol{\psi}}_i$ is the ith row of Ψ after applying Equation 5 with $t = 0$, $\odot$ denotes the Hadamard product, as of (Velickovic et al., 2018).

In comparison with our method, GAT incorporate node features in the attention score calculation, while our method incorporate node features at the aggregation stage only. In other words, node attentions in our methods are purely computed from the graph wavelet transform.

C.4 Connection to Skip-gram methods

Skip-gram models in natural language processing are shown to be equivalent of matrix factorization (Levy & Goldberg, 2014). Recently Qiu et al. (2018) proved that many Skip-Gram Negative Sampling (SGNS) models used in node embedding, including DeepWalk (Perozzi et al., 2014), LINE (Tang et al., 2015b), PTE (Tang et al., 2015a), and node2vec (Grover & Leskovec, 2016), are essentially factorizing implicit matrices closely related to the normalized graph Laplacian. The implicit matrices can be presented as graph wavelet transform on the graph Laplacian. For simplicity, we hereby use DeepWalk, which is a generalized form of LINE and PTE, as an example. Qiu et al. (2018) shows DeepWalk factorizes the matrix

$$\log\left(\frac{\mathrm{vol}(\mathcal{G})}{T}(\sum_{r=1}^{T} \boldsymbol{P}^r)\boldsymbol{D}^{-1}\right) - \log(b) \tag{10}$$

where $\mathrm{vol}(\mathcal{G}) = \sum_i D_{ii}$ is the sum of node degrees, $\boldsymbol{P} = \boldsymbol{D}^{-1}\boldsymbol{A}$ is the random walk matrix, T is the skip-gram window size and b is the parameter for negative sampling. We know

$$\boldsymbol{P} = \boldsymbol{I} - \boldsymbol{D}^{-\frac{1}{2}} \boldsymbol{L} \boldsymbol{D}^{\frac{1}{2}} = \boldsymbol{D}^{-\frac{1}{2}} \boldsymbol{U} (\boldsymbol{I} - \Lambda) \boldsymbol{U}^T \boldsymbol{D}^{\frac{1}{2}}$$

So Equation 10 can be written using graph Laplacian as:

$$\log\left(\frac{\mathrm{vol}(\mathcal{G})}{T} \boldsymbol{D}^{-\frac{1}{2}} \sum_{r=1}^{T} (\boldsymbol{I} - \boldsymbol{L})^r \boldsymbol{D}^{\frac{1}{2}}\right) - \log(b)$$

Or, after eigen-decomposition, as:

$$\boldsymbol{M} = \log\left(\frac{\mathrm{vol}(\mathcal{G})}{Tb} \boldsymbol{D}^{-\frac{1}{2}} \boldsymbol{U} \sum_{r=1}^{T} (\boldsymbol{I} - \Lambda)^r \boldsymbol{U}^T \boldsymbol{D}^{\frac{1}{2}}\right) \tag{11}$$

where $\boldsymbol{U} \sum_{r=1}^{T} (\boldsymbol{I} - \Lambda)^r \boldsymbol{U}^T$, denoted as $\boldsymbol{\psi}_{sg}$, is a wavelet transform with filter $g_{sg}(\lambda) = \sum_{r=1}^{T} (1 - \lambda)^r$. Therefore, DeepWalk can be seen a special case of Equation 4 by using $\boldsymbol{\psi}$ directly to obtain a:

$$\boldsymbol{a}_i = \begin{cases} \psi_i & \text{if } i = k \\ 0 & \text{if } i \neq j \end{cases}$$

Assigning $\boldsymbol{H} = \boldsymbol{W} = \boldsymbol{I}$, $K = 1$ and $\sigma(X) = \log(\frac{\mathrm{vol}(\mathcal{G})}{Tb}\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{X}\boldsymbol{D}^{\frac{1}{2}})$. We have

$$\boldsymbol{h}_i' = \mathrm{FACTORIZE}(\sigma(\boldsymbol{a_i})) \tag{12}$$

where FACTORIZE is a matrix factorization operator of choice. Qiu et al. (2018) uses SVD in a generalized SGNS model, where the decomposed matrix $\boldsymbol{U}_d$ and Σ_d from $\boldsymbol{M} = \boldsymbol{U}_d \Sigma \boldsymbol{V}_d$ is used to obtain the node embedding $\boldsymbol{U}_d\sqrt{\Sigma_d}$.