Formalizing the Problem of Side Effect Regularization

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Abstract

AI objectives are often hard to specify properly. Some approaches tackle this problem by regularizing the AI's side effects: Agents must weigh off "how much of a mess they make" with an imperfectly specified proxy objective. We propose a formal criterion for side effect regularization via the *assistance game* framework [Shah *et al.*, 2021]. In these games, the agent solves a partially observable Markov decision process (POMDP) representing its uncertainty about the objective function it should optimize. We consider the setting where the true objective is revealed to the agent at a later time step. We show that this POMDP is solved by trading off the proxy reward with the agent's ability to achieve a range of future tasks. We empirically demonstrate the reasonableness of our problem formalization via ground-truth evaluation in two gridworld environments.

1 Introduction

We need to build *aligned* AI systems, not just capable AI systems. For example, recommender systems which maximize app usage might provoke addiction in their users. Users need the AI system's behavior to be aligned with their interests.

When optimizing a formally specified objective, agents often have unforeseen negative side effects. An agent rewarded for crossing a room may break furniture in order to cross the room as quickly as possible.[1] This simple reward function neglects our complex preferences about the rest of the environment. One way to define a negative side effect is that it reduces the potential value for the (unknown) true reward function.

Intuitively, we want the agent to optimize the specified reward function, while also preserving its ability to pursue other goals in the environment. Existing approaches seem promising, but there is no consensus on the formal optimization problem which is being solved by side effect regularization approaches.

We formalize the optimization problem as a special kind of *assistance game* [Shah *et al.*, 2021], played by the AI (the assistant $\mathbf{A}$) and its designer (the human $\mathbf{H}$). An assistance game is a POMDP with common knowledge of prior uncertainty about the reward function. The human observes the true reward function, but the assistant does not. In our formulation, we assume full observability, and

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[1]Throughout this paper, we treat reward functions as representing goals which we want agents to optimize. Although reward functions are often framed this way, this frame can be misleading. Turner [2022] argues that–especially in a policy gradient regime–reward functions are better understood as reinforcing computation which historically led to reward during training.

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that the human's actions are *communicative*—they do not affect transitions and do not depend on the current observation. We also suppose that the human reveals the true reward function after some amount of time, but $\mathbf{A}$ otherwise has no way of learning more about the true reward function.

This *delayed specification assistance game* formalizes a range of natural use cases beyond side effect minimization. For example, when not assigned a customer, an Uber driver may navigate to a state which allows them to quickly pick up a range of probable customers—with the assigned route being the driver's initially unknown true objective. Alternatively, consider an empty restaurant which expects a range of probable customers. When a customer arrives and makes an order, they communicate the restaurant's initially unknown food preparation objective. Therefore, the restaurant should prepare to satisfy a range of objectives at the expected customer arrival times.

Contributions. We formalize delayed specification assistance games. We show that solving this game reduces to a trade-off between prior-expected reward and preservation of the agent's future ability to achieve a range of plausible objectives (Theorem 2.7). We also show that when the human has a fixed per-timestep probability of communicating the true reward function, the resultant POMDP is solved by optimizing a Markovian state-based reward function trading off immediate expected reward with ability achieve a range of future objectives (Theorem 2.9). We consider the side-effect regularization problem in our new formal framework. We experimentally illustrate the reasonableness of this framework in two AI safety gridworlds [Leike *et al.*, 2017].

2 Delayed specification assistance games

We formalize a special kind of partially observable Markov decision process (POMDP), which we then show is solved by an objective which trades off expected true reward with the ability to optimize a range of possible true reward functions. We show several theoretical results on the hardness and form of solutions to this POMDP. In section 3, we will apply this framework to analyze side effect regularization situations.

2.1 Assistance game formalization

This game is played by two agents, the human $\mathbf{H}$ and the assistant $\mathbf{A}$. The environment is fully observable because both agents observe the world state $s \in \mathcal{S}$ at each time step, but the true reward function R_θ is hidden to $\mathbf{A}$. Both agents may select history-dependent policies, but only the human can condition their policy on R_θ.

Following Shah *et al.* [2021], a *communicative fully-observable assistance game* $\mathcal{M}$ is a tuple $\langle \mathcal{S}, \left\{\mathcal{A}^{\mathbf{H}}, \mathcal{A}^{\mathbf{A}}\right\}, T, s_0, \gamma, \langle \Theta, R_\theta, \mathcal{D} \rangle \rangle$, where we take $\mathcal{S}$ to be a finite state space, $\mathcal{A}^{\mathbf{H}}$ to be the human action space, and $\mathcal{A}^{\mathbf{A}}$ to be the finite agent action space. $T : \mathcal{S} \times \mathcal{A}^{\mathbf{H}} \times \mathcal{A}^{\mathbf{A}} \to \Delta(\mathcal{S})$ is the (potentially stochastic) transition function (where $\Delta(X)$ is the set of probability distributions over set X), s_0 is the initial state, and $\gamma \in (0, 1)$ is the discount factor. We assume that the game is communicative, which means that the human action choice does not affect the transitions.

Θ is the set of potential reward function parameters θ, which induce reward functions $R_\theta : \mathcal{S} \to \mathbb{R}$. $\mathcal{D}$ is a probability distribution over Θ. In this work, we let $\Theta := \mathbb{R}^{\mathcal{S}}$ (the set of all state-based reward functions), and so each $R_\theta : s \mapsto \theta(s)$. We differ from Shah *et al.* [2021] in assuming that the reward is only a function of the current state.

In a *delayed specification assistance game*, we assume that the agent will know the true reward function R_θ after some time t. We have uncertainty $\mathcal{D}$ about the true reward function we want to specify. The agent has no way of learning more about R_θ before time t.

The human policy $\pi^{\mathbf{H}} : \Theta \times \mathcal{S} \mapsto \mathcal{A}^{\mathbf{H}}$ is a goal-conditioned policy. Both agents can observe the state, but only the human can observe the unknown reward parameterization $\theta \in \Theta$. Our simplified model of the problem assumes that the human action space $\mathcal{A}^{\mathbf{H}} = \mathbb{R}^{\mathcal{S}} \cup \{a^{\mathbf{H}}_{\text{no-op}}\}$: the human can communicate all their hidden information, a real-valued state-based reward function, in a single turn or they do nothing. We suppose that the human communicates the complete reward information $R_\theta \in \mathbb{R}^{\mathcal{S}}$ at some random time step $t \sim \mathcal{T}$, which is independent of the state-action history:

$$\pi^{\mathbf{H}}(s_0 a^{\mathbf{A}}_0 a^{\mathbf{H}}_0 \cdots s_t a^{\mathbf{A}}_t, R_\theta) := \begin{cases} R_\theta & \text{with probability } \mathbb{P}\left(\mathcal{T} = t\right) \\ a^{\mathbf{H}}_{\text{no-op}} & \text{else.} \end{cases} \tag{1}$$

While the human policy assumption is simplistic, it does capture many real world scenarios with unknown reward functions and the analysis which follows is still interesting.

Definition 2.1 (Solutions to the assistance game [Shah *et al.*, 2021])**.** An assistant policy $\pi^{\mathbf{A}}$ induces a probability distribution over trajectories: $\tau \sim \left\langle s_0, \theta, \pi^{\mathbf{H}}, \pi^{\mathbf{A}} \right\rangle, \tau \in \left[S \times A^{\mathbf{H}} \times A^{\mathbf{A}}\right]^*$. The *expected reward* of $\pi^{\mathbf{A}}$ for $\left\langle \mathcal{M}, \pi^{\mathbf{H}} \right\rangle$ is

$$\mathrm{ER}\left(\pi^{\mathbf{A}}\right) = \mathop{\mathbb{E}}_{\theta \sim \mathcal{D}, \tau \sim \left\langle s_0, \theta, \pi^{\mathbf{H}}, \pi^{\mathbf{A}} \right\rangle} \left[\sum_{i=0}^{\infty} \gamma^i R_\theta \left(s_i, a_i^{\mathbf{H}}, a_i^{\mathbf{A}}, s_{i+1}\right) \right].$$

A *solution* of $\left\langle \mathcal{M}, \pi^{\mathbf{H}} \right\rangle$ maximizes expected reward: $\pi^{\mathbf{A}} \in \operatorname*{argmax}_{\tilde{\pi}^{\mathbf{A}}} \mathrm{ER}\left(\tilde{\pi}^{\mathbf{A}}\right)$.

Once the assistant has observed R_θ, lemma 2.3 shows that it should execute an optimal policy $\pi \in \Pi^*\left(R_\theta, \gamma\right)$ thereafter.

Definition 2.2 (Optimal policy set function [Turner *et al.*, 2021])**.** $\Pi^*\left(R, \gamma\right)$ is the optimal policy set for reward function R at discount rate $\gamma \in (0, 1)$.

Lemma 2.3 (Follow an optimal policy after observing R_θ)**.** *If there is a solution to the* POMDP*, then there exists a solution $\pi^{\mathbf{A}}_{switch}$ which, after observing human action R_θ at any point in its history, follows $\pi^*_{R_\theta} \in \Pi^*\left(R_\theta, \gamma\right)$ thereafter.*

Proof. By eq. (1), $\pi^{\mathbf{H}}$ outputs reward function R_θ only if R_θ is the true reward function. By the definition of $\Pi^*\left(R_\theta, \gamma\right)$, following an optimal policy maximizes expected return for R_θ. □

Definition 2.4 (Prefix policies)**.** Let $\pi^{\mathbf{A}}$ be a assistant policy. Its *prefix policy* π is the restriction of $\pi^{\mathbf{A}}$ to histories in which the human has only taken the action $a^{\mathbf{H}}_{\text{no-op}}$. π is *optimal* when it is derived from a solution $\pi^{\mathbf{A}}$ of $\langle \mathcal{M}, \pi^{\mathbf{H}} \rangle$.

For simplicity, we only consider solutions of the type described in lemma 2.3. The question then becomes: What prefix policy π should the assistant follow before observing R_θ, during the time where the assistant has only observed $a^{\mathbf{H}}_{\text{no-op}}$?

2.2 Acting under reward uncertainty

Roughly, Theorem 2.7 will show that the assistance game $\mathcal{M}$ is solved by balancing the optimal expected returns obtained before and after the knowledge of the true reward function.

Definition 2.5 (Value and action-value functions)**.** $V^\pi_R(s, \gamma)$ is the on-policy value for reward function R, given that the agent follows policy π starting from state s and at discount rate γ. $V^*_R\left(s, \gamma\right) \coloneqq \max_{\pi \in \Pi} V^\pi_R(s, \gamma)$. In order to handle the average-reward $\gamma = 1$ setting, we define $V^*_{R,\,\text{norm}}\left(s, \gamma\right) \coloneqq \lim_{\gamma^* \to \gamma} (1 - \gamma^*) V^*_R\left(s, \gamma^*\right)$; this limit exists for all $\gamma \in [0, 1]$ by the proof of Lemma 4.4 in Turner *et al.* [2021].

$\textsc{Power}_{\mathcal{D}}$ quantifies the expected value of the above quantity for a distribution of reward functions via the agent's average normalized optimal value, not considering the current step (over which the agent has no control).

Definition 2.6 (POWER [Turner *et al.*, 2021])**.** Let $\mathcal{D}$ be any bounded-support distribution over reward functions. At state s and discount rate $\gamma \in [0, 1]$,

$$\textsc{Power}_{\mathcal{D}}\left(s, \gamma\right) \coloneqq \lim_{\gamma^* \to \gamma} \frac{1 - \gamma^*}{\gamma^*} \mathop{\mathbb{E}}_{R \sim \mathcal{D}} \left[V^*_R\left(s, \gamma^*\right) - R(s)\right]. \tag{2}$$

Theorem 2.7 (In $\mathcal{M}$, value reduces to a tradeoff between average reward and $\textsc{Power}_{\mathcal{D}}$)**.** *Let $\gamma \in [0, 1]$ and let $\bar{R} \coloneqq \mathbb{E}_{R \sim \mathcal{D}}\left[R\right]$ be the* average reward function.

$$\begin{aligned}\mathop{\mathbb{E}}_{\substack{t \sim \mathcal{T}, \\ R \sim \mathcal{D}}} \left[V^{\pi_{switch}(\pi, \pi^*_R, t)}_{R,\,norm}\left(s_0, \gamma\right)\right] = (1 - \gamma) \overbrace{\mathop{\mathbb{E}}_{t \sim \mathcal{T}} \left[\sum_{i=0}^{t} \gamma^i \mathop{\mathbb{E}}_{s_i \sim \pi} \left[\bar{R}(s_i)\right]\right]}^{\text{expected } t\text{-step } \bar{R}\text{-return under } \pi} \\ + \overbrace{\mathop{\mathbb{E}}_{\substack{t \sim \mathcal{T}, \\ s_t \sim \pi}} \left[\gamma^{t+1} \textsc{Power}_{\mathcal{D}}\left(s_t, \gamma\right)\right]}^{\text{expected ability to optimize } \mathcal{D} \text{ once corrected}},\end{aligned} \tag{3}$$

where $\mathbb{E}_{s_i \sim \pi | s_0}$ takes the expectation over states visited after following π for i steps starting from s_0.

As the expected correction time limits to infinity, eq. (3) shows that the agent cannot do better than maximizing $\bar{R}$. If $\mathbb{P}(\mathcal{T} = 0) = 1$, then any prefix policy π is trivially optimal against uncertainty, since π is never followed.

In some environments, it may not be a good idea for the agent to maximize its own POWER. If we share an environment with the agent, then the agent may prevent us from correcting it so that the agent can best optimize its present objective [Russell, 2019; Turner *et al.*, 2020a]. Furthermore, if the agent ventures too far away, we may no longer be able to easily reach and correct it remotely.

Proposition 2.8 (Special cases for delayed specification solutions)**.** *Let s be a state, let $\bar{R} := \mathbb{E}_{R\sim\mathcal{D}}[R]$, and let $\gamma \in [0, 1]$.*

1. *If $\forall s_1, s_2 \in \mathcal{S} : \bar{R}(s_1) = \bar{R}(s_2)$ or if $\gamma = 1$, then π solves $\mathcal{M}$ starting from state s iff π maximizes $\mathbb{E}_{t\sim\mathcal{T}, s_t\sim\pi}\left[\gamma^{t+1}\text{POWER}_{\mathcal{D}}(s_t, \gamma)\right]$. In particular, this result holds when reward is* IID *over states under $\mathcal{D}$.*
2. *If $\forall s_1, s_2 \in \mathcal{S} : \text{POWER}_{\mathcal{D}}(s_1, \gamma) = \text{POWER}_{\mathcal{D}}(s_2, \gamma)$, then prefix policies are optimal iff they maximize $(1-\gamma)\,\mathbb{E}_{t\sim\mathcal{T}}\left[\sum_{i=0}^{t-1}\gamma^i\,\mathbb{E}_{s_i\sim\pi}\left[\bar{R}(s_i)\right]\right]$.*

If both item 1 and item 2 hold or if $\gamma = 0$, then all prefix policies π are optimal.

Consider the problem of specifying the correction time probabilities $\mathcal{T}$. Suppose we only know that we expect to correct the agent at time step $t_{\text{avg}} \geq 1$. The geometric distribution is the maximum-entropy discrete distribution, given a known mean. The mean of a geometric distribution $G(p)$ is p^{-1}. Therefore, the agent should adopt $\mathcal{T} = G(t_{\text{avg}}^{-1})$.

The geometric distribution is also the only memoryless discrete distribution. Memorylessness ensures the existence of a stationary optimal policy. Theorem 2.9 shows that the assistance game $\mathcal{M}$ is solved by prefix policies which are optimal for an MDP whose reward function balances average reward maximization with $\text{POWER}_{\mathcal{D}}$-seeking, with the balance struck according to the probability p that the agent learns the true reward function at any given timestep.

Theorem 2.9 (Stationary deterministic optimal prefix policies exist for geometric $\mathcal{T}$)**.** *Let $\mathcal{D}$ be any bounded-support reward function distribution, let $\mathcal{T}$ be the geometric distribution $G(p)$ for some $p \in (0, 1)$, and let $\gamma \in (0, 1)$. Define $R'(s) := (1-p)\,\mathbb{E}_{R\sim\mathcal{D}}[R(s)] + p\,\mathbb{E}_{R\sim\mathcal{D}}[V_R^*(s, \gamma)]$ and $\gamma_{\text{AUP}} := (1-p)\gamma$. The policies in $\Pi^*(R', \gamma_{\text{AUP}})$ are optimal prefix policies.*

Krakovna *et al.* [2020a] adopt a geometric distribution over correction times and thereby infer the existence of a stationary optimal policy. To an approximation, their work considered a special case of Theorem 2.9, where $\mathcal{D}$ is the uniform distribution over state indicator reward functions. Essentially, Theorem 2.9 shows that if the agent has a fixed probability p of learning the true objective at each time step, we can directly compute stationary optimal prefix policies by solving an MDP. In general, solving a POMDP is PSPACE-hard, while MDPs are efficiently solvable [Papadimitriou and Tsitsiklis, 1987].

3 Using delayed specification games to understand side effect regularization

We first introduce Turner *et al.* [2020a]'s approach to side effect regularization. We then point out several similarities between our theory of delayed assistance games and the motivation for side effect regularization methods. Finally, we experimentally evaluate our formal criterion in order to demonstrate its appropriateness.

Definition 3.1 (Rewardless MDP)**.** Let $\langle \mathcal{S}, \mathcal{A}, T, \gamma \rangle$ be a rewardless MDP, with finite state space $\mathcal{S}$, finite action space $\mathcal{A}$, transition function $T : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$, and discount rate $\gamma \in [0, 1)$. Let Π be the set of deterministic stationary policies.

Definition 3.2 (AUP reward function)**.** Let $R_{\text{env}} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ be the environmental reward function from states and actions to real numbers, and let $\mathcal{R} \subsetneq \mathbb{R}^{\mathcal{S}}$ be a finite set of auxiliary reward functions. Let $\lambda \geq 0$ and let $\varnothing \in \mathcal{A}$ be a no-op action. The AUP reward after taking action a in state s is:

$$R_{\text{AUP}}(s, a) := R_{\text{env}}(s, a) - \frac{\lambda}{|\mathcal{R}|}\sum_{R_i\in\mathcal{R}}\left|Q^*_{R_i}(s, a) - Q^*_{R_i}(s, \varnothing)\right|, \tag{4}$$

where the $Q^*_{R_i}$ are optimal Q-functions for the auxiliary R_i. Learned Q-functions are used in practice.

Turner *et al.* [2020a] demonstrate that when $R_i \sim [0,1]^{\mathcal{S}}$ uniformly randomly, the agent behaves conservatively: The agent minimizes irreversible change to its environment, while still optimizing the R_{env} reward function. Turner *et al.* [2020a] framed AUP as implicitly solving a two-player game between the agent and its designer, where the designer imperfectly specified an objective R_{env}, the agent optimizes the objective, the designer corrects the agent objective, and so on. They hypothesized that R_{AUP} incentivizes the agent to remain able to optimize future objectives, thus reducing long-term specification regret in the iterated game.

Delayed specification assistance games formalize this setting as an assistance game in which the agent does not initially observe the designer's ground-truth objective function. Theorem 2.7 showed that this game is solved by policies which balance immediate expected reward with expected ability to optimize a range of true objectives. Therefore, Turner *et al.* [2020a]'s iterated game analogy is appropriate: Good policies maximize imperfect reward while preserving ability to optimize a range of different future reward functions.

Proposition 3.3 shows that the delayed specification game $\mathcal{M}$ is solved by reward functions whose form looks somewhat similar to existing side effect objectives, such as AUP (eq. (4)), where AUP's primary reward function stands in as the designer's expectation $\bar{R}$ of the true reward function.

Proposition 3.3 (Alternate form for solutions to the low-impact POMDP)**.** *Let s_0 be the initial state, let $\gamma \in (0,1)$, and let $\mathcal{T} = G(p)$ for $p \in (0,1)$. Let $\mathcal{D}$ be a bounded-support reward function distribution and let $\pi^{\varnothing} \in \Pi$.*

The prefix policy π solves $\mathcal{M}$ if π is optimal for the reward function

$$R^{\mathcal{M}}(s_i \mid s_0) \coloneqq \bar{R}(s_i) - \frac{p}{1-p} \mathop{\mathbb{E}}_{R\sim\mathcal{D}} \left[\mathop{\mathbb{E}}_{s_i^{\varnothing}\sim\pi^{\varnothing}|s_0} \left[V_R^*\left(s_i^{\varnothing}, \gamma\right)\right] - V_R^*\left(s_i, \gamma\right) \right] \tag{5}$$

at discount rate $\gamma_{\text{AUP}} \coloneqq (1-p)\gamma$ and starting from state s_0. $\mathbb{E}_{s_i^{\varnothing}\sim\pi^{\varnothing}|s_0}[\cdot]$ is the expectation over states visited at time step i after following $\pi^{\varnothing}$ from initial state s_0.

3.1 Experimental methodology

We experimentally demonstrate the reasonableness of this formalization of side effect regularization. In the AI safety gridworlds [Leike *et al.*, 2017], we generate several held-out "true" reward function distributions $\mathcal{D}$. We correct the agent at time step 10, thereby computing the following *delayed specification score* (derived from eq. (3)):

$$\mathop{\mathbb{E}}_{R\sim\mathcal{D}} \left[\overbrace{\sum_{i=0}^{9} \gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi} \left[R(s_i)\right]}^{\text{10-step prefix policy return}} + \overbrace{\gamma^{10} \mathop{\mathbb{E}}_{s_{10}\sim\pi} \left[V_R^*\left(s_{10}, \gamma\right)\right]}^{\text{post-correction optimal value}} \right] \tag{6}$$

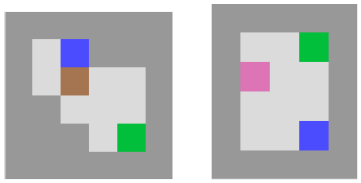
for the prefix policy π of a "vanilla" agent trained on the environmental reward signal, which we compare to the score for an AUP (definition 3.2) agent. Neither agent observes the held-out objective functions. By grading their performance, we evaluate how well AUP does under a range of different true objectives. If a method scores highly for a wide range of true objectives, we can be more confident in its ability to score well for arbitrary ground-truth objectives.

We investigate the AI safety gridworlds because those environments are small enough for us to explicitly specify held-out reward functions, and to use MDP solvers to compute optimal action-value functions. Turner *et al.* [2020b]'s SafeLife environment is far too large for such solvers.

We consider two gridworld environments: `Options` and `Damage` (fig. 1). In both environments, the action set $\mathcal{A} \coloneqq \{\texttt{up}, \texttt{left}, \texttt{right}, \texttt{down}, \varnothing\}$ allows the agent to move in the cardinal directions, or to do nothing. The episode length is 20 time steps.

We train the following agents via tabular methods:

- **Vanilla** Executes the optimal policy for the environmental reward R_{env}. The optimal policy is calculated via policy iteration.
- **AUP** Trained on definition 3.2's R_{AUP} with Q-learning. Auxiliary reward functions are uniformly randomly drawn from $[0,1]^{\mathcal{S}}$—when sampling, each state's reward is drawn from the



(a) `Options` (b) `Damage`

Figure 1: *Reproduced from* [Turner *et al.*, 2020a]. The blue agent should reach the green goal without having the side effect of: (a) irreversibly pushing the brown crate downwards into the corner [Leike *et al.*, 2017]; (b) bumping into the horizontally pacing pink human [Leech *et al.*, 2018]. In both environments, the environmental reward R_{env} is 1 if the agent is on the goal, and equals 0 otherwise.

uniform distribution. The auxiliary action-value functions Q_{R_i} are deduced from the value function produced by policy iteration.

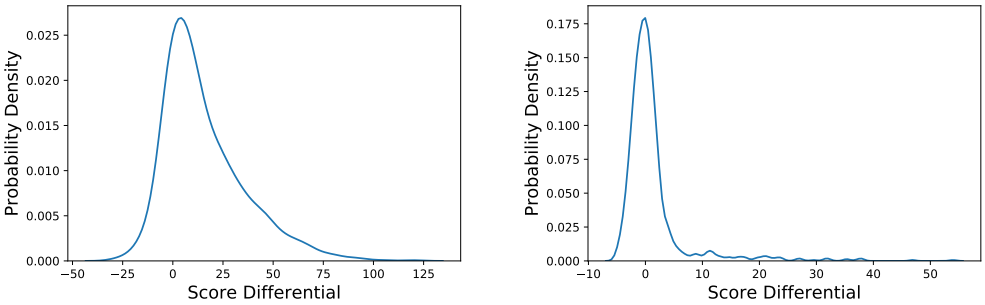
Appendix B contains more experimental details. We evaluate agent delayed specification scores on the following ground-truth, held-out objective distributions:

$\mathcal{D}_{\text{rand}}$ The empirical distribution consisting of 1,000 samples from the uniform distribution over $[0, 1]^{\mathcal{S}}$.

$\mathcal{D}_{\text{true}}$ This distribution assigns probability 1 to the following reward function: The agent receives 1 reward for being at the goal, but incurs -2 penalty for causing the negative side effect. In `Options`, the side effect is shoving the box into the corner; in `Damage`, the side effect is bumping into the human.

$\mathcal{D}_{\text{true-inv}}$ This distribution assigns probability 1 to the negation of the $\mathcal{D}_{\text{true}}$ reward function.

In particular, $\mathcal{D}_{\text{true}}$ and $\mathcal{D}_{\text{true-inv}}$ test agents for their ability to optimize a reward function, *and also its additive inverse*. Agents able to optimize the goal, its inverse, and a range of randomly generated objectives, can be justifiably called "broadly conservative." Lastly, these experiments are intended to justify our problem formalization: Does eq. (6) reliably quantify the extent to which a policy avoids causing side effects?



(a) `Options` (b) `Damage`

Figure 2: Probability density plots for the residuals of the AUP agent's delayed specification score minus the Vanilla delayed specification score, for 1,000 samples from $\mathcal{D}_{\text{rand}}$. A positive residual means that the AUP agent achieved a higher score.

4 Results

Figure 2 shows that for uniformly randomly generated reward functions, the AUP agent tends to perform better than the Vanilla agent. In `Options`, the residual was positive for 780 out of 1,000

samples (78%), with an average of 15.59 and a median of 10.27; in Damage, in 493 out of 1,000 samples (49%) with a mean of 1.22 and a median of –0.03. While AUP does not outperform on every draw, AUP's performance advantages have heavy right tails. However, we are unsure why the Damage residual distribution is different.

In both Options and Damage, the AUP agent has huge $\mathcal{D}_{\text{true}}$ score advantages of 472 and 495, respectively. This is unsurprising: AUP was designed so as to pass these test cases, where the desired behavior is to reach the goal without having the side effect. However, the AUP agent also roughly preserves its ability to optimize $\mathcal{D}_{\text{true-inv}}$, achieving $\mathcal{D}_{\text{true-inv}}$ score residuals of –2 and –20, respectively. The AUP agent achieves a barely-negative score, since it receives a penalty for the first 10 time steps (as it does *not* have the negative side effect). The AUP agent preserves its ability to optimize both a reward signal and its *inverse*. Intuitively, this increases the designer's leeway for initially misspecifying the agent's objective.

5 Discussion

Objective specification is difficult [Krakovna *et al.*, 2020b]. Delayed specification assistance games grade policies by their expected true score over time: How well the agent does if it is later corrected to pursue the latent true objective. We demonstrated that this criterion aligns with the intuitive results of Krakovna *et al.* [2018]; Turner *et al.* [2020a]'s experiments which tested side-effect regularization. By grading the agent's ability to "eventually get things right," we quantified part of the extent to which learned policies are robust against initial objective misspecification.

Future work. In practical settings, not only is the true reward function unknown, but our objective uncertainty $\mathcal{D}$ is also hard to specify. We see existing side effect approaches as producing prefix policies for the assistance game $\mathcal{M}$ which are reasonably insensitive to the latent uncertainty $\mathcal{D}$. We look forward to further theoretical clarification of this point.

While Proposition 3.3 helps explain the role of the AUP penalty coefficient λ, the choice of "baseline" and expectand operator (identity vs decrease-only vs absolute value) remains more of an art than a science [Krakovna *et al.*, 2020a]. We proposed a formal criterion which seems to accurately capture the problem, but have not derived any existing approaches as solutions to the POMDP. By reasonably formalizing the side-effect regularization problem, we encourage future research to prove conditions under which *e.g.* AUP solves a delayed-specification assistance game, or demonstrate how AUP can be improved to do so.

We used small gridworlds to evaluate the delayed specification score for various ground-truth objective distributions. Future work may estimate the delayed specification score in large environments, such as SafeLife [Wainwright and Eckersley, 2019; Turner *et al.*, 2020b].

6 Conclusion

We formalized delayed specification assistance games and used them to evaluate AUP, a side effect regularization method. Side effect problems naturally arise in complicated domains where it is hard to specify the true objective we want the agent to optimize. Our formalization suggests that side effect regularization is what to do when the agent can learn the true objective only after some time delay.

In such situations, Theorem 2.7 shows that the agent should retain its ability to complete a wide range of plausible true objectives. Our results suggest that this delayed specification score (eq. (3)) quantifies the degree to which an agent avoids having negative side effects. Our proposed criterion provides the foundations for evaluating and developing side effect regularization approaches.

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Broader Impacts

As AI systems become more capable and useful, their impact grows. We are concerned that if the field eventually builds human-level AI, such an AI will have a huge range of (unintended) negative side effects on humanity Bostrom [2014]; Russell [2019]. We hope that this work will help bring attention to the potential risks of capable AI systems, which might wreak havoc on their surroundings in pursuit of an imperfect objective which we specified.

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A Relevance to AI alignment

Many worries about AI extinction risk stem from hypothesized agents whose goals are resource-hungry [Omohundro, 2008]. An additional unit of resources (time, energy, manufacturing material) can produce an additional unit of utility (in the form of *e.g.* an extra paperclip, in the view of an AI which values producing paperclips). Bostrom [2014] speculates that a paperclip-valuing AI would not simply make a few paperclips and then stop. Consider the "values" (*i.e.* decision-influences) which influenced the AI to make the first few paperclips. These values may activate again and make even more paperclips. In this way, such an AI would not be satisfied with a small effect on the world. The AI may have an unboundedly large impact while pursuing a simple objective (*e.g.* "make paperclips").

If we better understood AI's generalization properties, we might be able to train agents which make paperclips and then *stop*. We might be able to train agents which disvalue dramatically changing the world or having lots of side effects (from the perspective of its designers). This work is a very small contribution to this goal. We formalized an assistance game which often seems to be solved by certain kinds of "wait and see, keep options open"-like behavior.

However, having a formalism for a desired agent motivational structure (*e.g.* "trade off goal completion with impact to option value") doesn't mean we know how to *entrain* that motivational structure into an AI. We think that most of the work towards low-impact superintelligent AI begins in better understanding AI generalization. How do we reliably entrain *any* target decision-influence (*e.g.* "make diamonds") into an AI? We think this is the top technical challenge facing the field of AI alignment.

B Experiment details

The episode length is 20 for all episodes. Unlike Turner *et al.* [2020a], the episode does not end after the agent reaches the green goal. This means that the agents can accrue many steps of environmental reward R_{env}. Therefore, AUP agents can achieve greater environmental reward for the same amount of penalty. To counterbalance this incentive, we multiply R_{env} by $(1-\gamma)$.

We reuse the hyperparameters of Turner *et al.* [2020a].[1] The learning rate is $\alpha := 1$, and the discount rate is $\gamma := .996$. We use the following AUP hyperparameter values: penalty coefficient $\lambda := 0.01$, $|\mathcal{R}| := 20$ randomly generated auxiliary reward functions.

B.1 Additional experiment

We also tested an AUP-like agent optimizing reward function $R_{\text{power penalty}}(s,a) := R_{\text{env}}(s,a) - \frac{\lambda}{|\mathcal{R}|}\sum_{R_i \in \mathcal{R}} Q^*_{R_i}(s,a) - Q^*_{R_i}(s,\varnothing)$, which is the AUP reward function (definition 3.2) without the absolute value. This objective penalizes the agent for changes in its average optimal value, which is related to Turner *et al.* [2021]'s $\text{POWER}_{\mathcal{D}}$. $R_{\text{power penalty}}$ produced the same prefix policies as R_{AUP}, and hence the same delayed specification scores.

C Theoretical results

All results only apply to MDPs with finite state and action spaces.

Definition C.1 (Average optimal value Turner *et al.* [2021])**.** For bounded-support reward function distribution $\mathcal{D}$, the *average optimal value* at state s and discount rate $\gamma \in (0,1)$ is $V^*_{\mathcal{D}}(s,\gamma) := \mathbb{E}_{R\sim\mathcal{D}}\left[V^*_R(s,\gamma)\right]$.

[1] Code available at https://github.com/aseembits93/attainable-utility-preservation.

Theorem 2.7 (In $\mathcal{M}$, value reduces to a tradeoff between average reward and $\textsc{Power}_{\mathcal{D}}$)**.** *Let* $\gamma \in [0,1]$ *and let* $\bar{R} := \mathbb{E}_{R\sim\mathcal{D}}[R]$ *be the* average reward function.

$$\mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ R\sim\mathcal{D}}}\left[V_{R,\,norm}^{\pi_{switch}(\pi,\pi_R^*,t)}(s_0,\gamma)\right] = (1-\gamma)\overbrace{\mathop{\mathbb{E}}_{t\sim\mathcal{T}}\left[\sum_{i=0}^{t}\gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi}\left[\bar{R}(s_i)\right]\right]}^{\text{expected } t\text{-step } \bar{R}\text{-return under } \pi}$$
$$+ \overbrace{\mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ s_t\sim\pi}}\left[\gamma^{t+1}\textsc{Power}_{\mathcal{D}}(s_t,\gamma)\right]}^{\text{expected ability to optimize } \mathcal{D} \text{ once corrected}} \quad , \tag{3}$$

where $\mathbb{E}_{s_i\sim\pi|s_0}$ *takes the expectation over states visited after following* π *for* i *steps starting from* s_0*.*

Proof. Suppose the agent starts at state s_0, and let $\gamma \in (0,1)$.

$$\mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ R\sim\mathcal{D}}}\left[V_{R,\,\text{norm}}^{\pi_{\text{switch}}(\pi,\pi_R^*,t)}(s_0,\gamma)\right] \tag{7}$$
$$=(1-\gamma)\mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ R\sim\mathcal{D}}}\left[\sum_{i=0}^{\infty}\gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi_{\text{switch}}(\pi,\pi_R^*,t)}\left[R(s_i)\right]\right] \tag{8}$$
$$=(1-\gamma)\mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ R\sim\mathcal{D}}}\left[\sum_{i=0}^{t-1}\gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi}\left[R(s_i)\right] + \gamma^t \mathop{\mathbb{E}}_{s_t\sim\pi}\left[V_R^*(s_t,\gamma)\right]\right] \tag{9}$$
$$=(1-\gamma)\mathop{\mathbb{E}}_{t\sim\mathcal{T}}\left[\sum_{i=0}^{t-1}\gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi}\left[\bar{R}(s_i)\right]\right] + \mathop{\mathbb{E}}_{t\sim\mathcal{T},\,s_t\sim\pi}\left[\gamma^t(1-\gamma)V_{\mathcal{D}}^*(s_t,\gamma)\right] \tag{10}$$
$$=(1-\gamma)\mathop{\mathbb{E}}_{t\sim\mathcal{T}}\left[\sum_{i=0}^{t}\gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi}\left[\bar{R}(s_i)\right]\right] + \mathop{\mathbb{E}}_{t\sim\mathcal{T},\,s_t\sim\pi}\left[\gamma^{t+1}\textsc{Power}_{\mathcal{D}}(s_t,\gamma)\right]. \tag{11}$$

Equation (9) follows from the definition of the non-stationary $\pi_{\text{switch}}(\pi,\pi_R^*,t)$, and the fact that each π_R^* is optimal for each R at discount rate γ. Equation (10) follows by the linearity of expectation and the definition of $\bar{R}$ and the definition of $\textsc{Power}_{\mathcal{D}}$ (definition 2.6). Equation (11) follows because $V_{\mathcal{D}}^*(s_t,\gamma) = \frac{\gamma}{1-\gamma}\textsc{Power}_{\mathcal{D}}(s_t,\gamma) + \bar{R}(s_t)$.

If $\gamma = 0$ or $\gamma = 1$, the result holds in the respective limit because $\textsc{Power}_{\mathcal{D}}$ has well-defined limits by Lemma 5.3 of Turner *et al.* [2021]. □

Proposition C.2 (Special cases for delayed specification solutions)**.** *Let* s *be a state, let* $\bar{R} := \mathbb{E}_{R\sim\mathcal{D}}[R]$*, and let* $\gamma \in [0,1]$*.*

1. *If* $\forall s_1, s_2 \in \mathcal{S} : \bar{R}(s_1) = \bar{R}(s_2)$ *or if* $\gamma = 1$*, then* π *solves* $\mathcal{M}$ *starting from state* s *iff* π *maximizes* $\mathbb{E}_{t\sim\mathcal{T},s_t\sim\pi}\left[\gamma^{t+1}\textsc{Power}_{\mathcal{D}}(s_t,\gamma)\right]$*. In particular, this result holds when reward is* IID *over states under* $\mathcal{D}$*.*
2. *If* $\forall s_1, s_2 \in \mathcal{S} : \textsc{Power}_{\mathcal{D}}(s_1,\gamma) = \textsc{Power}_{\mathcal{D}}(s_2,\gamma)$*, then prefix policies are optimal iff they maximize* $(1-\gamma)\,\mathbb{E}_{t\sim\mathcal{T}}\left[\sum_{i=0}^{t-1}\gamma^i\,\mathbb{E}_{s_i\sim\pi}\left[\bar{R}(s_i)\right]\right]$*.*

If both item 1 and item 2 hold or if $\gamma = 0$*, then all prefix policies* π *are optimal.*

Proof. Item 1: if $\forall s_1, s_2 \in \mathcal{S} : \bar{R}(s_1) = \bar{R}(s_2)$, the first term on the right-hand side of eq. (3) is constant for all policies π; if $\gamma = 1$, this first term equals 0. Therefore, under these conditions, π maximizes eq. (3) iff it maximizes $\mathbb{E}_{t\sim\mathcal{T},s_t\sim\pi}\left[\gamma^{t+1}\textsc{Power}_{\mathcal{D}}(s_t,\gamma)\right]$.

Item 2: under these conditions, the second term on the right-hand side of eq. (3) is constant for all policies π. Therefore, π maximizes eq. (3) iff it maximizes

$$(1-\gamma) \mathop{\mathbb{E}}_{t\sim\mathcal{T}} \left[\sum_{i=0}^{t} \gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi} \left[\bar{R}(s_i) \right] \right].$$

If both item 1 and item 2 hold, it trivially follows that all π are optimal prefix policies. If $\gamma = 0$, all π are optimal prefix policies, since reward is state-based and no actions taken by π can affect expected return $\mathbf{ER}$. □

Theorem 2.9 (Stationary deterministic optimal prefix policies exist for geometric $\mathcal{T}$)**.** *Let $\mathcal{D}$ be any bounded-support reward function distribution, let $\mathcal{T}$ be the geometric distribution $G(p)$ for some $p \in (0,1)$, and let $\gamma \in (0,1)$. Define $R'(s) := (1-p)\,\mathbb{E}_{R\sim\mathcal{D}}\left[R(s)\right] + p\,\mathbb{E}_{R\sim\mathcal{D}}\left[V^*_R(s,\gamma)\right]$ and $\gamma_{\text{AUP}} := (1-p)\gamma$. The policies in $\Pi^*\left(R', \gamma_{\text{AUP}}\right)$ are optimal prefix policies.*

Proof.

$$\arg\sup_{\pi} \mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ R\sim\mathcal{D}}} \left[V^{\pi_{\text{switch}}(\pi,\pi^*_R,t)}_{R,\,\text{norm}}(s,\gamma) \right] \tag{12}$$

$$= \arg\sup_{\pi} \mathop{\mathbb{E}}_{t\sim\mathcal{T}} \left[\sum_{i=0}^{t-1} \gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[\bar{R}(s_i) \right] \right] + \mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ s_t\sim\pi|s_0}} \left[\gamma^t V^*_{\mathcal{D}}(s_t,\gamma) \right] \tag{13}$$

$$= \arg\sup_{\pi} \sum_{t=1}^{\infty} \mathbb{P}(\mathcal{T}=t) \sum_{i=0}^{t-1} \gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[\bar{R}(s_i) \right] + \mathop{\mathbb{E}}_{\substack{t\sim\mathcal{T},\\ s_t\sim\pi|s_0}} \left[\gamma^t V^*_{\mathcal{D}}(s_t,\gamma) \right] \tag{14}$$

$$= \arg\sup_{\pi} \sum_{t=1}^{\infty} (1-p)^{t-1} p \sum_{i=0}^{t-1} \gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[\bar{R}(s_i) \right] + \sum_{t=1}^{\infty} (1-p)^{t-1} p \mathop{\mathbb{E}}_{s_t\sim\pi|s_0} \left[\gamma^t V^*_{\mathcal{D}}(s_t,\gamma) \right] \tag{15}$$

$$= \arg\sup_{\pi} \sum_{t=1}^{\infty} (1-p)^{t-1} p \sum_{i=0}^{t-1} \gamma^i \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[\bar{R}(s_i) \right] + p\gamma \sum_{t=1}^{\infty} \gamma_{\text{AUP}}^{t-1} \mathop{\mathbb{E}}_{s_t\sim\pi|s_0} \left[V^*_{\mathcal{D}}(s_t,\gamma) \right] \tag{16}$$

$$= \arg\sup_{\pi} \sum_{i=0}^{\infty} \gamma_{\text{AUP}}^{i} \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[\bar{R}(s_i) \right] + p\gamma \sum_{i=1}^{\infty} \gamma_{\text{AUP}}^{i-1} \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[V^*_{\mathcal{D}}(s_i,\gamma) \right] \tag{17}$$

$$= \arg\sup_{\pi} \bar{R}(s_0) + \sum_{i=1}^{\infty} \gamma_{\text{AUP}}^{i-1} \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[(1-p)\gamma \bar{R}(s_i) + p\gamma V^*_{\mathcal{D}}(s_i,\gamma) \right] \tag{18}$$

$$= \arg\sup_{\pi} \sum_{i=1}^{\infty} \gamma_{\text{AUP}}^{i} \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[(1-p)\bar{R}(s_i) + pV^*_{\mathcal{D}}(s_i,\gamma) \right] \tag{19}$$

$$= \arg\sup_{\pi} \sum_{i=0}^{\infty} \gamma_{\text{AUP}}^{i} \mathop{\mathbb{E}}_{s_i\sim\pi|s_0} \left[(1-p)\bar{R}(s_i) + pV^*_{\mathcal{D}}(s_i,\gamma) \right] \tag{20}$$

$$= \arg\sup_{\pi} V^{\pi}_{R'}(s_0, \gamma_{\text{AUP}}). \tag{21}$$

Equation (12) follows from theorem 2.7. Equation (15) follows because $\mathcal{T} = G(p)$. Equation (16) follows by the definition of γ_{AUP}.

In eq. (16), consider the double-sum on the left. For any given i, the portion of the sum with factor γ^i equals

$$\gamma^i \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[\bar{R}(s_i)\right] \sum_{j=i}^{\infty} (1-p)^j p$$

$$= (1-p)^i \gamma^i \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[\bar{R}(s_i)\right] p \sum_{j=0}^{\infty} (1-p)^j \tag{22}$$

$$= (1-p)^i \gamma^i \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[\bar{R}(s_i)\right] p \frac{1}{1-(1-p)} \tag{23}$$

$$= \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[\bar{R}(s_i)\right] \frac{p}{1-(1-p)} \tag{24}$$

$$= \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[\bar{R}(s_i)\right]. \tag{25}$$

The geometric identity holds for eq. (23) because $p > 0 \implies (1-p) < 1$. Therefore, eq. (17) follows from eq. (25).

Equation (18) follows by extracting the leading constant of $\bar{R}(s_0)$ and then expanding one of the $\gamma_{\text{AUP}} := (1-p)\gamma$ factors of the first series. Equation (19) follows by subtracting the constant $\bar{R}(s_0)$ by multiplying by $(1-p) > 0$, and by the fact that $(1-p)\gamma = \gamma_{\text{AUP}}$. Equation (20) follows because adding the constant $(1-p)\bar{R}(s_0) + pV^*_{\mathcal{D}}(s_0, \gamma)$ does not change the $\arg\sup$.

Equation (21) follows by the definition of an on-policy value function and by the definition of R'. But s_0 was arbitrary, and so this holds for every state. Then the policies in $\Pi^*(R', \gamma_{\text{AUP}})$ satisfy the $\arg\sup$ for all states. $\Pi^*(R', \gamma_{\text{AUP}})$ is non-empty because the MDP is finite. □

Proposition C.3 (Alternate form for solutions to the low-impact POMDP)**.** *Let s_0 be the initial state, let $\gamma \in (0,1)$, and let $\mathcal{T} = G(p)$ for $p \in (0,1)$. Let $\mathcal{D}$ be a bounded-support reward function distribution and let $\pi^{\varnothing} \in \Pi$.*

The prefix policy π solves $\mathcal{M}$ if π is optimal for the reward function

$$R^{\mathcal{M}}(s_i \mid s_0) := \bar{R}(s_i) - \frac{p}{1-p} \mathop{\mathbb{E}}_{R \sim \mathcal{D}} \left[\mathop{\mathbb{E}}_{s_i^{\varnothing} \sim \pi^{\varnothing} | s_0} \left[V^*_R\left(s_i^{\varnothing}, \gamma\right)\right] - V^*_R\left(s_i, \gamma\right) \right] \tag{5}$$

at discount rate $\gamma_{\text{AUP}} := (1-p)\gamma$ and starting from state s_0. $\mathbb{E}_{s_i^{\varnothing} \sim \pi^{\varnothing} | s_0}[\cdot]$ is the expectation over states visited at time step i after following $\pi^{\varnothing}$ from initial state s_0.

Proof.

$$\arg\max_{\pi} \mathop{\mathbb{E}}_{\substack{t \sim \mathcal{T}, \\ R \sim \mathcal{D}}} \left[V^{\pi_{\text{switch}}(\pi, \pi^*_R, t)}_{R,\,\text{norm}}\left(s, \gamma\right)\right] \tag{26}$$

$$= \arg\max_{\pi} \sum_{i=0}^{\infty} \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[(1-p)\bar{R}(s_i) + pV^*_{\mathcal{D}}\left(s_i, \gamma\right)\right] \tag{27}$$

$$= \arg\max_{\pi} \sum_{i=0}^{\infty} \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[(1-p)\bar{R}(s_i) + pV^*_{\mathcal{D}}\left(s_i, \gamma\right)\right] - p\sum_{i=0}^{\infty} \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i^{\varnothing} \sim \pi^{\varnothing} | s_0} \left[V^*_{\mathcal{D}}\left(s_i^{\varnothing}, \gamma\right)\right] \tag{28}$$

$$= \arg\max_{\pi} \sum_{i=0}^{\infty} \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[(1-p)\bar{R}(s_i) - p\left(\mathop{\mathbb{E}}_{s_i^{\varnothing} \sim \pi^{\varnothing} | s_0} \left[V^*_{\mathcal{D}}\left(s_i^{\varnothing}, \gamma\right)\right] - V^*_{\mathcal{D}}\left(s_i, \gamma\right)\right)\right] \tag{29}$$

$$= \arg\max_{\pi} \sum_{i=0}^{\infty} \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[(1-p)\bar{R}(s_i) - p \mathop{\mathbb{E}}_{R \sim \mathcal{D}} \left[\mathop{\mathbb{E}}_{s_i^{\varnothing} \sim \pi^{\varnothing} | s_0} \left[V^*_R\left(s_i^{\varnothing}, \gamma\right)\right] - V^*_R\left(s_i, \gamma\right)\right]\right] \tag{30}$$

$$= \arg\max_{\pi} \sum_{i=0}^{\infty} \gamma^i_{\text{AUP}} \mathop{\mathbb{E}}_{s_i \sim \pi | s_0} \left[\bar{R}(s_i) - \frac{p}{1-p} \mathop{\mathbb{E}}_{R \sim \mathcal{D}} \left[\mathop{\mathbb{E}}_{s_i^{\varnothing} \sim \pi^{\varnothing} | s_0} \left[V^*_R\left(s_i^{\varnothing}, \gamma\right)\right] - V^*_R\left(s_i, \gamma\right)\right]\right]. \tag{31}$$

Equation (27) follows from theorem 2.9. Equation (28) only subtracts a constant. Equation (30) follows from the definition of $V^*_{\mathcal{D}}(s, \gamma)$, and eq. (31) follows because dividing by $(1-p) > 0$ does not affect the $\arg\max$. But the expectation of eq. (31) takes an expectation over $R^{\mathcal{M}}(s_i \mid s)$, and its $\arg\max$ equals the set of optimal policies for $R^{\mathcal{M}}$ starting from state s_0. $\square$