FLOW-BASED RECURRENT BELIEF STATE LEARNING FOR POMDPS

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ABSTRACT

Partially Observable Markov Decision Process (POMDP) provides a principled and generic framework to model real world sequential decision making processes but yet remains unsolved, especially for high dimensional continuous space and unknown models. The main challenge lies in how to accurately obtain the belief state, which is the probability distribution over the unobservable environment states given historical information. Accurately calculating this belief state is a precondition for obtaining an optimal policy of POMDPs. Recent advances in deep learning techniques show great potential to learn good belief states, but they assume the belief states follow certain types of simple distributions such as diagonal Gaussian, which imposes strong restrictions to precisely capture the real belief states. In this paper, we introduce the **F**l**O**w-based **R**ecurrent **BE**lief **S**tate model (FORBES), which incorporates normalizing flows into the variational inference to learn general continuous belief states for POMDPs. Furthermore, we show that the learned belief states can be plugged into downstream RL algorithms to improve performance. In experiments, we show that our methods successfully capture the complex belief states that enable multi-modal predictions as well as high quality reconstructions, and results on challenging visual-motor control tasks show that our method achieves superior performance and sample efficiency.

1 INTRODUCTION

Partially Observable Markov Decision Process (POMDP) (Åström, 1965) provides a principled and generic framework to model real world sequential decision making processes. Unlike Markov Decision Process (MDP), the observations of a POMDP are generally non-Markovian. Therefore, to make optimal decisions, the agent needs to consider all historical information, which is usually intractable. One effective solution is to obtain the belief state. The belief state is defined as the probability distribution of the unobservable environment state conditioned on the past observations and actions (Kaelbling et al., 1998). Such belief state accurately summarizes the history. Traditional methods of calculating belief states (Smallwood & Sondik, 1973; Sondik, 1971; Kaelbling et al., 1998) assume finite discrete space with a known model. In many real world problems, however, the underlying model remains unknown, and the state space is large and even continuous.

With the recent advances of deep learning technologies, a branch of works have been proposed to learn the belief states of POMDPs with unknown model and continuous state space (Krishnan et al., 2015; Gregor et al., 2019; Lee et al., 2020; Hafner et al., 2019b; 2020; 2021). These works solve the belief state learning problem by sequentially maximizing the observation probability at each timestep using the variational inference and achieve the state-of-the-art performance on many visual-motor control tasks (Hafner et al., 2019a; Zhu et al., 2020; Okada et al., 2020; Ma et al., 2020a). However, they still cannot capture general belief states due to the intractability of complex distributions in high-dimensional continuous space and instead approximate the belief states with diagonal Gaussians. This approximation imposes strong restrictions and is problematic. As shown in Figure 1, the blue area denotes the unobservable state space of the POMDP. Given the past information τ, the agent maintains a prior distribution of the state s, denoted as $p(s|\tau)$ (the distribution in white). Each colored distribution corresponds to the belief state after receiving a different new observation o, denoted as the posterior distribution $q(s|\tau, o)$. Consider an example of the true beliefs as shown in Figure 1(b), with their Gaussian approximations shown in Figure 1(a). The approximation error of Gaussian distributions will easily result in problems of intersecting belief which leads

to a mixed-up state (e.g., the white triangle), and empty belief, which leads to a meaningless state (e.g., the grey triangle). This also explains the poor reconstruction problems in interactive environments observed by Okada & Taniguchi (2021). Furthermore, as mentioned in Hafner et al. (2021), the Gaussian approximation of belief states also makes it difficult to predict multi-modal future behaviours. Therefore, it is preferable to relax the Gaussian assumptions and use a more flexible family of distributions to learn accurate belief states as shown in Figure 1(b).

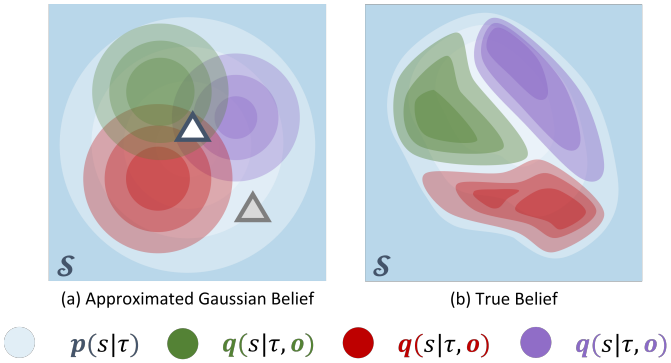


Figure 1: Difference between (a) spherical Gaussian belief states and (b) true belief states (Better viewed in color).

In this paper, we propose a new method called **F**l**O**w-based **R**ecurrent **BE**lief **S**tate model (FORBES) that is able to learn general continuous belief states for POMDPs. FORBES incorporates Normalizing Flows (Tabak & Turner, 2013; Rezende & Mohamed, 2015; Dinh et al., 2017) into the variational inference step to construct flexible belief states. In experiments, we show that FORBES allows the agent to maintain flexible belief states, which result in multi-modal and precise predictions as well as higher quality reconstructions. We also demonstrate the results combining FORBES with downstream RL algorithms on challenging visual-motor control tasks (DeepMind Control Suite, Tassa et al. (2018)). The results show the efficacy of FORBES in terms of improving both performance and sample efficiency.

Our contributions can be summarized as follows:

- We propose FORBES, the first flow-based belief state learning algorithm that is capable of learning general continuous belief states for POMDPs.
- We propose a POMDP RL framework based on FORBES for visual-motor control tasks, which uses the learned belief states from FORBES as the inputs to the downstream RL algorithms.
- Empirically, we show that FORBES allows the agent to learn flexible belief states that enable multi-modal predictions as well as high quality reconstructions and help improve both performance and sample efficiency for challenging visual-motor control tasks.

2 Preliminaries

2.1 Partially Observable Markov Decision Process

Formally, a Partially Observable Markov Decision Process (POMDP) is a 7-tuple $(\mathcal{S}, \mathcal{A}, T, R, \Omega, O, \gamma)$, where $\mathcal{S}$ is a set of states, $\mathcal{A}$ is a set of actions, T is a set of conditional transition probabilities between states, R is the reward function, Ω is a set of observations, O is a set of conditional observation probabilities, and γ is the discount factor.

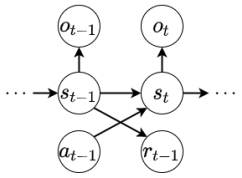


Figure 2: The PGM of POMDP

At each timestep $t-1$, the state of the environment is $s_{t-1} \in \mathcal{S}$. The agent takes an action $a_{t-1} \in \mathcal{A}$, which causes the environment to transit to state s_t with probability $T(s_t \mid s_{t-1}, a_{t-1})$. The agent then receives an observation $o_t \in \Omega$ which depends on the new state of the environment s_t with probability $O(o_t \mid s_t)$. Finally, the agent receives a reward r_{t-1} equal to $R(s_{t-1})$. Such a POMDP model can also be described as

a probabilistic graphical model (PGM) as shown in Figure 2. After having taken action a_{t-1} and observing o_t, an agent needs to update its belief state, which is defined as the probability distribution of the environment state conditioned on all historical information:

$$b(s_t) = p(s_t \mid \tau_t, o_t) \tag{1}$$

where $\tau_t = \{o_1, a_1, \ldots, o_{t-1}, a_{t-1}\}$.

2.2 Normalizing Flow

Instead of using the Gaussian family to approximate the prior and posterior belief distributions, we believe it is more desirable to use a family of distributions that is highly flexible and preferably flexible enough to describe all possible true belief states. Therefore, we use Normalizing Flows (Tabak & Turner, 2013; Rezende & Mohamed, 2015) to parameterize those distributions.

Rather than directly parameterizing statistics of the distribution itself, Normalizing Flows model the transformations, or the "flow" progress, needed to derive such a distribution. More specifically, it describes a sequence of invertible mappings that gradually transform a relatively simple probability density to a more flexible and complex one.

Let $f_\theta : \mathbb{R}^D \to \mathbb{R}^D$ to be an invertible and differentiable mapping in state space parameterized by θ. Given a random variable $\mathbf{x} \in \mathbb{R}^D$ with probability distribution $p(\mathbf{x})$, we can derive the probability of the transformed random variable $\mathbf{z} = f_\theta(\mathbf{x})$ by applying the change of variable formula:

$$p(\mathbf{z}) = p(\mathbf{x}) \left| \det \frac{\partial f_\theta^{-1}}{\partial \mathbf{z}} \right| \tag{2}$$

$$\log p(\mathbf{z}) = \log p(\mathbf{x}) - \log \left| \det \frac{\partial f_\theta}{\partial \mathbf{z}} \right| \tag{3}$$

To construct a highly flexible family of distributions, we can propagate the random variable at beginning $\mathbf{z}_0$ through a sequence of K mappings and get $\mathbf{z}_K = f_{\theta_K} \circ f_{\theta_{K-1}} \circ \cdots \circ f_{\theta_1}(\mathbf{z}_0)$ with the probability

$$\log p_K(\mathbf{z}_K) = \log p(\mathbf{z}_0) - \sum_{k=1}^{K} \log \left| \det \frac{\partial f_{\theta_k}}{\partial \mathbf{z}_{k-1}} \right| \tag{4}$$

Given a relatively simple distribution of $\mathbf{z}_0$, say, Gaussian distribution, by iteratively applying the transformations, the flow is capable of representing a highly complex distribution with the probability that remains tractable. The parameters $\theta_1, \ldots, \theta_K$ determine the transformations of the flow.

An effective transformation that is widely accepted is affine coupling layer (Dinh et al., 2017; Kingma & Dhariwal, 2018; Kingma et al., 2017). Given the input $\mathbf{x} \in \mathbb{R}^D$, let s and t stand for scale and translation functions which are usally parameterized by neural networks, where $s, t : \mathbb{R}^k \to \mathbb{R}^{D-k}, k < D$. The output, $\mathbf{y}$, can be viewed as a concatenation of its first k dimensions $\mathbf{y}_{1:k}$ and the remaining part $\mathbf{y}_{k+1:D}$:

$$\mathbf{y}_{1:k} = \mathbf{x}_{1:k}, \quad \mathbf{y}_{k+1:D} = \mathbf{x}_{k+1:D} \odot \exp(s(\mathbf{x}_{1:k})) + t(\mathbf{x}_{1:k}) \tag{5}$$

where $\odot$ denotes the element-wise product (see details about affine coupling layer in Appendix A.1).

3 Flow-based Recurrent Belief State Learning

3.1 Flow-based Recurrent Belief State model

We propose the **F**l**O**w-based **R**ecurrent **BE**lief **S**tate model (FORBES) which learns general continuous beilef states via normalizing flows under the variational inference framework. Specifically, the FORBES model consists of components needed to construct the PGM of POMDP as shown in Figure 2:

$$\begin{aligned} \text{State transition model}: &\quad p(s_t|s_{t-1}, a_{t-1}) \\ \text{Observation model}: &\quad p(o_t|s_t) \\ \text{Reward model}: &\quad p(r_t|s_t) \end{aligned} \tag{6}$$

In addition, we have a belief inference model $q(s_t|\tau_t, o_t)$ to approximate the true posterior distribution $p(s_t|\tau_t, o_t)$, where $\tau_t = \{o_1, a_1, \ldots, o_{t-1}, a_{t-1}\}$ is the past information. The above components of FORBES can be optimized jointly by maximizing the Evidence Lower BOund (ELBO) (Jordan et al., 1999) or more generally the variational information bottleneck (Tishby et al., 2000; Alemi et al., 2016):

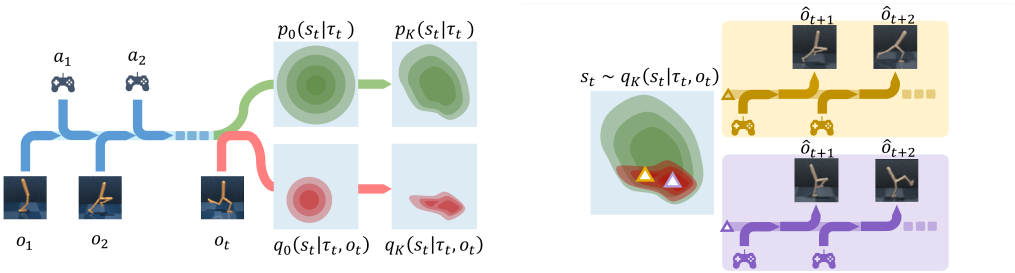
$$\begin{aligned}
&\log p(o_{1:T}, r_{1:T}|a_{1:T}) \\
&\quad \geq \mathbb{E}_{q(s_{1:t}|\tau_t, o_t)}\left[\sum_{t=1}^{T}(\ln p(o_t|s_t) + \ln p(r_t|s_t) - D_{\mathrm{KL}}(q(s_t|\tau_t, o_t)\|p(s_t|s_{t-1}, a_{t-1})))\right] \\
&\quad \doteq \mathcal{J}_{\mathrm{Model}}
\end{aligned} \tag{7}$$

Detailed derivations can be found in Appendix.A.5. In practice, the state transition model, observation model, reward model, and belief inference model can be represented by stochastic deep neural networks parameterized by ψ:

$$p_\psi(s_t|s_{t-1}, a_{t-1}), \quad p_\psi(o_t|s_t), \quad p_\psi(r_t|s_t), \quad q_\psi(s_t|\tau_t, o_t) \tag{8}$$

where their outputs usually follow simple distributions such as diagonal Gaussians. The parameterized belief inference model $q_\psi(s_t|\tau_t, o_t)$ acts as an encoder that encodes the historical information using a combination of convolutional neural networks and recurrent neural networks.

In FORBES we provide special treatments for the belief inference model and the state transition model to represent more complex and flexible posterior and prior distributions. As shown in Figure 3(a), the input images $o_{1:t}$ and actions $a_{1:t-1}$ are encoded with $q_\psi(s_t|\tau_t, o_t)$ (the blue and the red path). Then our final inferred belief is obtained by propagating $q_\psi(s_t|\tau_t, o_t)$ through a set of normalizing flow mappings denoted $f_{\theta_K} \circ \cdots \circ f_{\theta_1}$ to get a representative posterior distribution $q_{\psi,\theta}(s_t|\tau_t, o_t)$. For convenience, we denote $q_0 = q_\psi$ and $q_K = q_{\psi,\theta}$. On the other hand, $o_{1:t-1}$ and $a_{1:t-2}$ are encoded with $q_\psi(s_{t-1}|\tau_{t-1}, o_{t-1})$ (the blue path), then the state transition model is used to obtain the prior guess of the state $p_\psi(s_t \mid \tau_t) = \mathbb{E}_{q_\psi(s_{t-1}|\tau_{t-1}, o_{t-1})}\left[p_\psi(s_t \mid s_{t-1}, a_{t-1})\right]$ (the green path). Then our final prior is obtained by propagating $p_\psi(s_t|\tau_t)$ through another set of normalizing flow mappings denoted $f_{\omega_K} \circ \cdots \circ f_{\omega_1}$ to get a representative prior distribution $p_{\psi,\omega}(s_t|\tau_t)$. For convenience, we denote $p_0 = p_\psi$ and $p_K = p_{\psi,\omega}$. Then as shown in Figure 3(b), we can sample the initial states (the yellow and purple triangles) from the belief states $s_t \sim q_K(s_t \mid \tau_t, o_t)$. For each sampled initial state, we can use the state transition model to predict the future states $\hat{s}_{t+h}$ given the future actions $a_{t:t+h-1}$, and then use the observation model to reconstruct the observations $\hat{o}_{t+h}$, where h is the prediction horizon.



(a) Belief state inference

(b) Predictions beginning from different samples

Figure 3: The algorithm framework of FORBES.

With the above settings, we can substitute the density probability inside the KL-divergence term in Equation 7 with Normalizing Flow:

$$\begin{aligned}
&D_{\mathrm{KL}}(q_K(s_t|\tau_t, o_t)\|p_K(s_t|s_{t-1}, a_{t-1})) = \mathbb{E}_{q_K(s_{1:t}|\tau_t, o_t)}\left[\log q_K(s_t|\tau_t, o_t) - \log p_K(s_t|\tau_t)\right] \\
&= \mathbb{E}_{q_K(s_{1:t}|\tau_t, o_t)}\left[\log q_0(s_t|\tau_t, o_t) - \sum_{k=1}^{K}\log\left|\det\frac{\partial f_{\theta_k}}{\partial s_{t,k-1}}\right| - \log p_0(s_t|\tau_t) + \log\left|\det\frac{\partial f_{\omega_k}}{\partial s_{t,k-1}}\right|\right]
\end{aligned} \tag{9}$$

where $p_K(s_t \mid s_{t-1}, a_{t-1}) = p_K(s_t \mid \tau_t)$ given the sampled s_{t-1} from $q_K(s_{1:t}|\tau_t, o_t)$. $s_{t,k}$ is the state variable s_t transformed by k layers of normalizing flows, and $s_{t,0} = s_t$.

To further demonstrate the properties of FORBES, we provide the following theorems.

Theorem 1 *The approximation error of the log-likelihood when maximizing the $\mathcal{J}_{\text{Model}}$ (the derived ELBO) defined in Equation 7 is:*

$$\log p(o_{1:T}, r_{1:T}|a_{1:T}) - \mathcal{J}_{\text{Model}} = \sum_{t=1}^{T} D_{\text{KL}}(q(s_t|\tau_t, o_t)\|p(s_t \mid \tau_t, o_t)) \tag{10}$$

where $p(s_t \mid \tau_t, o_t)$ denotes the true belief states.

Detailed poorfs can be found in Appendix.A.5. Theorem 1 suggests that, when the learning algorithm maximizes the $\mathcal{J}_{\text{Model}}$ (the derived ELBO), then the D_{KL} terms in the right-hand side are minimized, which indicate the KL-divergence between the learned belief states $q(s_t|\tau_t, o_t)$ and the true belief states $p(s_t \mid \tau_t, o_t)$. Clearly, if $p(s_t \mid \tau_t, o_t)$ is a complex distribution and $q(s_t|\tau_t, o_t)$ is chosen from a restricted distribution class such as diagonal Gaussian, then when the algorithm maximizes the $\mathcal{J}_{\text{Model}}$ (the derived ELBO), there will still be a potentially large KL-divergence between the learned and the true belief states. Therefore, naturally there raises the problem that is normalizing flow a universal distributional approximator that is capable of accurately representing arbitrarily complex belief states, so the KL-divergence terms in the right-hand side of Equation (10) can be minimized to approach zero? The answer is yes for a wide range of normalizing flows, here we provide the following theoretical results for the family of the flow used in FORBES:

Theorem 2 *(Teshima et al. (2020)) Let $\mathcal{M}$ be a set of measurable mappings from $\mathbb{R}^m$ to $\mathbb{R}^m$: $\mathcal{M} \doteq \{W_1 \circ g_1 \circ \cdots W_n \circ g_n : n \in \mathbb{N}, g_i \in \mathcal{G}, W_i \in \mathcal{A}\}$, where $\mathcal{G}$ contains affine coupling layers and $\mathcal{A}$ are affine transforms. Then, for any absolutely continuous probability measure μ and any probability measure ν on $\mathbb{R}^m$, there exists a sequence $\{g_i\}_{i=1}^{\infty} \subset \mathcal{M}$ such that $(g_i)_*\mu$ converges to ν in distribution as $i \to \infty$, where $(g_i)_*\mu := \mu \circ g_i^{-1}$. Namely, $\mathcal{M}$ is a universal distributional approximator.*

In fact, there always exists a diffeomorphism that can turn one well-behaved distribution to another. Besides the aforementioned affine coupling flow, many works show the distributional universality of other flows (Kong & Chaudhuri, 2020; Huang et al., 2018). Ideally, the universal approximation property of the flow model $q_K(s_t \mid \tau_t, o_t)$ allows us to approximate the true posterior $p(s_t \mid \tau_t, o_t)$ with arbitrary accuracy. Thus, compared to previous methods, FORBES helps close the gap between the log-likelihood and the ELBO to obtain a more accurate belief state. Though we usually cannot achieve the ideal zero KL-divergence in practice, our method can get a smaller approximation error, equally a higher ELBO than previous works. We verify this statement in section 4.1.

3.2 POMDP RL FRAMEWORK BASED ON FORBES

To show the advantage of the belief states inferred by the FORBES model compared to the existing belief inference method in visual-motor control tasks, we design a flow-based belief reinforcement learning algorithm for learning the optimal policy in POMDPs. The algorithm follows an actor-critic framework. The critic estimates the accumulated future rewards, and the actor chooses actions to maximize the estimated cumulated rewards. Both the actor and critic operate on top of the learned belief states and thus benefit from the accurate representations learned by the FORBES model.

The critic $v_\xi(s_\tau)$ aims to predict the discounted sum of future rewards that the actor can achieve given an initial belief state s_t, known as the state value $\mathbb{E}\left(\sum_{\tau=t}^{t+\infty} \gamma^{\tau-t} r_\tau\right)$, where ξ denote the parameters of the critic network and H is the prediction horizon. We leverage temporal-difference to learn this value, where the critic is trained towards a value target that is constructed from the intermediate reward and the critic output for the next step's state. In order to trade-off the bias and the variance of the state value estimation, we use the more general TD(λ) target (Sutton & Barto, 2018), which is a weighted average of n-step returns for different horizons and is defined as follows:

$$V_\tau^\lambda \doteq \hat{r}_\tau + \hat{\gamma}_\tau \begin{cases} (1-\lambda) v_\xi(s_{\tau+1}) + \lambda V_{\tau+1}^\lambda & \text{if} \quad \tau < t+H, \\ v_\xi(s_{t+H}) & \text{if} \quad \tau = t+H. \end{cases} \tag{11}$$

To better utilize the flexibility belief states from FORBES, we run the sampling method multiple times to capture the diverse predictions. Specifically, we sample N states from the belief state given by FORBES and then rollout trajectories of future states and rewards using the state transition model and the reward model. Finally, we train the critic to regress the TD(λ) target return using a mean squared error loss:

$$\mathcal{J}_{\text{Critic}}(\xi) = \mathop{\mathbb{E}}_{s_{i,0}\sim q_K, a_\tau \sim q_\phi, s_{i,\tau}\sim p_\psi} \Big[\sum_{i=1}^{N} \sum_{\tau=t}^{t+H} \tfrac{1}{2} \big(v_\xi(s_{i,\tau}) - \text{sg}(V_{i,\tau}^{\lambda}) \big)^2 \Big]. \tag{12}$$

where $sg(\cdot)$ is the stop gradient operation. The actor $a_\tau \sim q_\phi(a_\tau \mid s_\tau)$ aims to output actions that maximize the prediction of long-term future rewards made by the critic and is trained directly by backpropagating the value gradients through the sequence of sampled states and actions, i.e., maximize:

$$\mathcal{J}_{\text{Actor}}(\phi) = \mathop{\mathbb{E}}_{s_{i,0}\sim q_K, a_\tau \sim q_\phi, s_{i,\tau}\sim p_\psi} \left(\sum_{i=1}^{N} \sum_{\tau=t}^{t+H} \text{V}_{i,\tau}^{\lambda} \right) \tag{13}$$

We jointly optimize the model loss $\mathcal{J}_{Model}$ with respect to the model parameters ψ, θ and ω, the critic loss $\mathcal{J}_{Critic}$ with respect to the critic parameters ξ and the actor $\mathcal{J}_{Actor}$ loss with respect to the actor parameters ϕ using the Adam optimizer with different learning rates:

$$\min_{\psi,\xi,\phi,\theta,\omega} \quad \mathcal{J}_{\text{FORBES}} = \alpha_0 \mathcal{J}_{\text{Critc}}(\xi) - \alpha_1 \mathcal{J}_{\text{Actor}}(\phi) - \alpha_2 \mathcal{J}_{\text{Model}}(\psi, \theta, \omega) \tag{14}$$

where α_0, α_1, α_2 are coefficients for different components, and we summarize the whole framework of optimizing in Algorithm 1.

Algorithm 1 FORBES Algorithm

Input: buffer $\mathcal{B}$, imagination horizon H, interacting step T, batch size B, batch length L, number of trajetories N.

Draw B data sequences $\{(o_t, a_t, r_t)\}_t^{t+L}$ from $\mathcal{B}$
Infer belief $s_t \sim q_K(s_t|\tau_t, o_t)$.
for $i = 1, \ldots, N$ **do**
  Rollout imaginary trajectories $\{(s_{i,\tau}, a_{i,\tau})\}_{\tau=t}^{t+H}$ with belief transition model.
end for
For each $s_{i,\tau}$, predict rewards $p_\psi(r_{i,\tau}|s_{i,\tau})$ and values $v_\phi(s_{i,\tau})$ ▷ *Calculate imaginary returns*
Update $\theta, \omega, \xi, \phi, \psi$ using Equation (7), (9), (12), (13) and (14) ▷ *Optimize the model, critic and actor*

4 EXPERIMENTS

Our experiments evaluate FORBES on two image-based tasks. We first demonstrate the belief learning capacity on a digit writing task in Section 4.1, and show that FORBES captures beliefs that allow for multi-modal yet precise long-term predictions as well as higher ELBO. For large-scale experiments, we test the proposed POMDP RL framework based on FORBES in Section 4.2. The results of multiple challenging visual-motor control tasks from DeepMind Control Suite (Tassa et al., 2018) show that FORBES outperforms baselines in terms of performance and sample efficiency. In Section 4.3, we further provide ablation studies of the multiple imagined trajectories technique used in our method.

4.1 DIGIT WRITING TASKS

In this experiment, we validate the capacity of FORBES by modelling the partially observable sequence with visual inputs. We adopt the MNIST Sequence Dataset (D. De Jong, 2016) that consists of sequences of handwriting MNIST digit stokes. This problem can be viewed as a special case of POMDP, whose action space is $\emptyset$ and rewards remain 0. Such a problem setting separates the belief learning and policy optimizing problem and allows us to concentrate on the former one in this section. We convert the digit stroke to a sequence of images of size 28×28 to simulate the writing process. At time step t, the agent can observe o_t that has already written t pixels, and we train the agent maximizing $\mathcal{J}_{\text{Model}}$ in Equation 7 except for the reward reconstruction term.

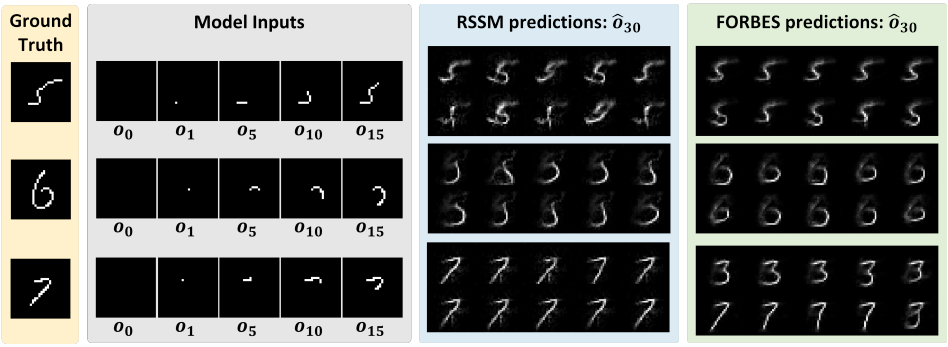


Figure 4: Predictions on sequential MNIST of two models.

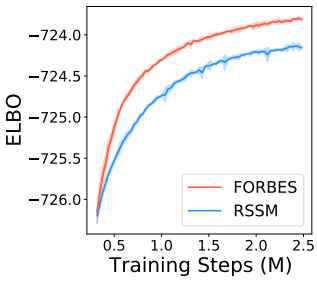


Figure 5: ELBO on digit writing

As shown in Figure 4, we randomly select three digits as examples (see Appendix A.6 for more results) and show the inputs as well as the prediction outputs of our model and the RSSM (Okada et al., 2020) baseline, which is the previous state-of-the-art method for learning continuous belief states of POMDPs. The leftmost column is the ground truth of the fully written digits. During the testing, we feed the initial 15 frames $\{o_1, o_2, \cdots, o_{15}\}$ to the model, and the columns in grey exhibit a part of the inputs. Then we sample several states from the inferred belief state and rollout via the learned state transition model (Equation (6)) for 15 steps and show the reconstruction results of the predictions. As shown in the blue and green columns on the right of Figure 4, though RSSM can also predict the future strokes in general, the reconstructions are relatively blurred and mix different digits up. It also fails to give diverse predictions. However, FORBES can make precise yet diverse predictions. Each prediction is clear and distinct from other digits. Given the beginning of the digit 7, FORBES successfully predicts both 7 and 3 since they have a similar beginning. The results can be partially explained via the mixed-up belief and the empty belief as shown in Figure 1, which support the claim that FORBES can better capture the complex belief states.

We also provide the quantitative results in Figure 5, which is the ELBO on test digits sequence set that is never seen during training. The results show that FORBES can achieve a tighter ELBO, which verifies the theoretical results in 3.1. Details of the network can be found in Appendix A.2.

4.2 Visual-motor control tasks

We experimentally evaluate the performance of FORBES on Reinforcement Learning on a variety of visual-motor control tasks from the DeepMind Control Suite (Tassa et al., 2018), illustrated in Figure 6. Across all the tasks, the observations are $64 \times 64 \times 3$ images. These environments provide different challenges. The Cartpole-Swingup task requires a long planning horizon and memorizing the state of the cart when it is out of view; Finger-Spinning includes contact dynamics between the finger and the object; Cheetah-Run exhibits high-dimensional state and action spaces; the Walker-Walk and Walker-Run are challenging because the robot has to learn to first stand up and then walk; Hopper Stand is based on a single-legged robot, which is sensitive to the reaction force on the ground and thus needs more accurate control. As for baselines, we include the scores for A3C Mnih et al. (2016) with state inputs (1e9 steps), D4PG Barth-Maron et al. (2018) (1e9 steps), PlaNet (Hafner et al., 2019b) (1e6 steps) and Dreamer Hafner et al. (2019a) with pixel inputs. All the scores of baselines are aligned with the ones reported in Hafner et al. (2019a) (see details in Appendix A.3). The details of the implementations can be found in Appendix A.2.

Our experiments empirically show that FORBES achieves superior performance and sample efficiency on challenging visual-motor control tasks. As illustrated in Figure 6, FORBES achieves higher scores than Dreamer (Hafner et al., 2019a) in most of the tasks and achieves better performance than PlaNet (Hafner et al., 2019b) within much fewer environment steps. We provide some insights into the results. As shown in Section 4.1, baselines with Gaussian assumptions may suffer

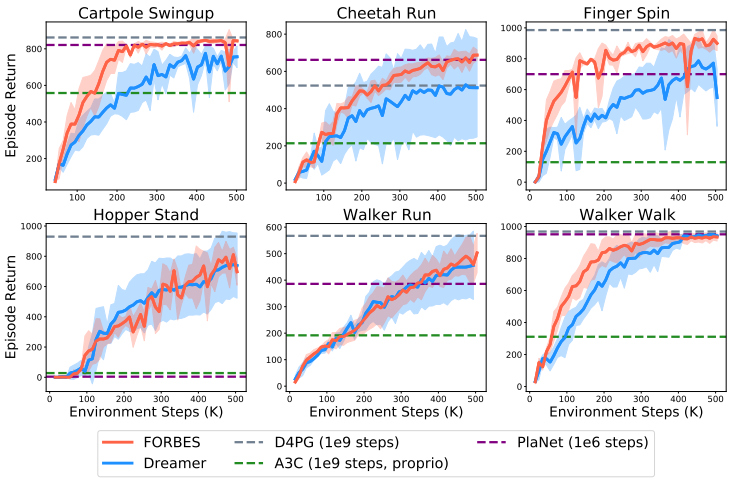


Figure 6: Performance on DeepMind Control Suite. The shaded areas show the standard deviation across 3 seeds. FORBES achieves better performance and sample efficiency in various challenging tasks.

from the mixed-up belief and empty belief issues, while FORBES can better capture the general belief states. Furthermore, multiple imagined trajectories can better utilize the diversity in the rollout. Therefore, the inner coherency within the model components allows the agent a better performance. We further discuss the role of multiple imagined trajectories in the next section.

4.3 Ablation Study

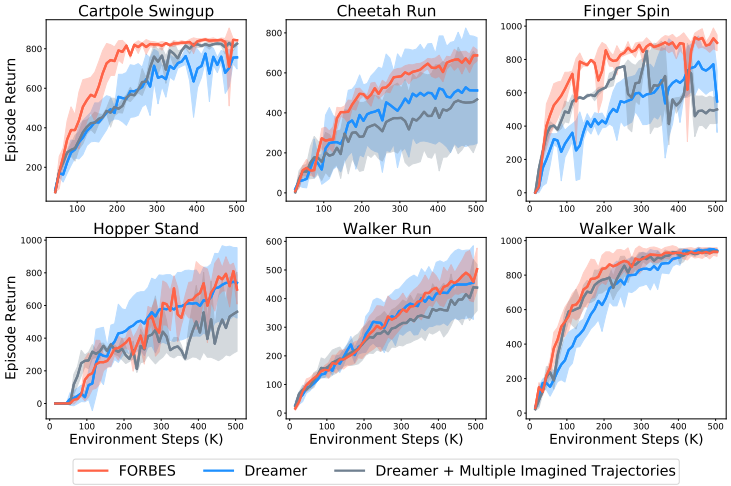


Figure 7: Comparison of the performance between FORBES and Dreamer with multiple imagined trajectories.

In order to verify that the outperformance of FORBES is not simply due to increasing the number of imagined trajectories, we conducted an ablation study in this section. We compare FORBES with the "Dreamer + multiple imagined trajectories" baseline by increasing the number of imagined trajectories in Dreamer to the same as in FORBES. As shown in Figure 7, no consistent and obvious gain can be observed after increasing the number of trajectories to Dreamer. The agent gains slight improvements in two environments and suffers from slight performance loss on other tasks. This result indicates that increasing the number of imagined trajectories may only be effective when the agent can make diverse predictions as in FORBES. The Gaussian assumptions lead to the lack of trajectory diversity, so that increasing the number of imagined trajectories will not effectively help.

5 RELATED WORK

POMDP: POMDP solving approaches can be divided into two categories based on whether their state, action and observation spaces are discrete or continuous. Discrete space POMDP solvers, in general, either approximate the value function using point-based methods (Kurniawati et al., 2008; Shani et al., 2013) or using Monte-Carlo sampling in the belief space (Silver & Veness, 2010; Kurniawati & Yadav, 2016) to make the POMDP problem tractable. Continuous space POMDP solvers often approximate the belief states as a distribution with few parameters (typically Gaussian) and solve the problem analytically either using gradients (Van Den Berg et al., 2012; Indelman et al., 2015) or using random sampling in the belief space (Agha-Mohammadi et al., 2014; Hollinger & Sukhatme, 2014). However, the classical POMDP methods mentioned above are all based on an accurately known dynamic model, which is a restricted assumption in many real world tasks.

MBRL for visual-motor control: Recent researches in model-based reinforcement learning (MBRL) for visual-motor control provides promising methods to solve POMDPs with high-dimensional continuous space and unknown models since visual-motor control tasks can be naturally modelled as POMDP problems. Learning effective latent dynamics models to solve challenging visual-motor control problems is becoming feasible through advances in deep generative modeling and latent variable models (Krishnan et al., 2015; Karl et al., 2016; Doerr et al., 2018; Buesing et al., 2018; Ha & Schmidhuber, 2018; Hafner et al., 2019b;a). Among which, the recurrent state-space model (RSSM) based methods (Hafner et al., 2019b;a) provide a principled way to learn continuous latent belief states for POMDPs by variational inference and learns behaviours based on the belief states using model-based reinforcement learning, which achieves high performance on visual-motor control tasks. However, they assume the belief states obey diagonal Gaussian distributions. Such assumptions impose strong restrictions to belief inference and lead to limitations in practice, including mode collapse, posterior collapse and object vanishing in reconstruction(Bowman et al., 2016; Salimans et al., 2015; Okada & Taniguchi, 2020). A few works propose particle filter based methods that use samples to approximate the belief states (Ma et al., 2020b; Igl et al., 2018). However, they suffer from insufficient sample efficiency and performance.

Normalizing Flows: Normalizing Flows (NF) are a family of generative models which produce tractable distributions with analytical density. For a transformation $f : \mathbf{R}^D \to \mathbf{R}^D$, the computational time cost of the log determinant is $\mathcal{O}(D^3)$. Thus most previous works choose to make the computation more tractable. Rezende & Mohamed (2015); van den Berg et al. (2019) propose to use restricted functional form of f. Another choice is to force the Jacobian of f to be lower triangular by using an autoregressive model (Kingma et al., 2016; Papamakarios et al., 2018). These models usually excel at density estimation, but the inverse computation can be time-consuming. Dinh et al. (2015; 2017); Kingma & Dhariwal (2018) propose the coupling method to make the Jacobian triangular and ensure the forward and inverse can be computed with a single pass. The applications of NF include image generation (Ho et al., 2019; Kingma & Dhariwal, 2018), video generation (Kumar et al., 2019) and reinforcement learning (Mazoure et al., 2020; Ward et al., 2019; Touati et al., 2020).

6 CONCLUSION

General continuous belief states inference is a crucial yet challenging problem in high-dimensional Partially Observable Markov Decision Process (POMDP) problems. In this paper, we propose the **F**l**O**w-based **R**ecurrent **BE**lief **S**tate model (FORBES) that can learn general continuous belief states by incorporating normalizing flows into the variational inference framework and then effectively utilize the learned belief states in downstream RL tasks. We show that theoretically, our method can accurately learn the true belief states and we verify the effectiveness of our method in terms of both the quality of learned belief states and the final performance of our extended POMDP RL framework on two visual input environments. The digit writing tasks demonstrate that our method can learn general belief states that enable precise and multi-modal predictions and high-quality reconstructions. Experiments on challenging visual control tasks demonstrate that the flow-based belief state inferred by FORBES improves the sample efficiency and final performance of downstream model-based reinforcement learning algorithms. General belief inference plays a vital role in solving the POMDP, and our method paves a way towards it. In the future, we will explore further approaches to improve the accuracy of belief states inference, such as combining contrastive learning and using advanced network architectures such as transformers to build normalizing flows.

7 REPRODUCIBILITY

We implement FORBES based on the Pytorch version of Dreamer, provided by "https://github.com/yusukeurakami/dreamer-pytorch" and our code will be released after the author notification in "https://anonymous.4open.science/r/Implementation-of-FORBES-E45B". We describe the implementation details as well as the hyperparameters in Appendix A.2. We train our FORBES model on NVIDIA Tesla V100 with Pytorch 1.5.0.

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A Appendix

A.1 Details of affine coupling layer for normalizing flow

Affine Coupling is a method for implementing a normalizing flow (where we stack a sequence of invertible bijective transformation functions). Affine coupling is one of these bijective transformation functions(Dinh et al., 2015). Specifically, it is an example of a reversible transformation where the forward function, the reverse function and the log-determinant are computationally efficient. For the forward function, we split the input dimension into two parts:

$$\mathbf{x}_a, \mathbf{x}_b = \text{split}(\mathbf{x}) \tag{15}$$

The first part stays the same $\mathbf{x}_a = \mathbf{y}_a$, while the second part $\mathbf{x}_b$ undergoes an affine transformation, where the parameters for this transformation are learnt using the first part $\mathbf{x}_a$ being put through a neural network. For the transformation conditions on context c, we also input c to the neural network (c can be empty for non-conditional transformation). Together we have:

$$\begin{aligned}
(\log \mathbf{s}, \mathbf{t}) &= \text{NN}\left(\mathbf{x}_a, c\right) \\
\mathbf{s} &= \exp(\log \mathbf{s}) \\
\mathbf{y}_b &= \mathbf{s} \odot \mathbf{x}_b + \mathbf{t} \\
\mathbf{y}_a &= \mathbf{x}_a \\
\mathbf{y} &= \text{concat}\left(\mathbf{y}_a, \mathbf{y}_b\right)
\end{aligned} \tag{16}$$

A.2 Hyper Parameters and implementation details

We use the convolutional and deconvolutional networks that are similar to Dreamer(Hafner et al., 2019a), a GRU (Cho et al., 2014) with 200 units in the dynamics model, and implement all other functions as two fully connected layers of size 200 with ReLU activations. Base distributions in latent space are 30-dimensional diagonal Gaussians with predicted mean and standard deviation. As for the parameters network (NN in Equation 16), we use a residual network composed of one fully connected layer, one residual block, and one fully connected layer. The residual network receives $\mathbf{x}_a$ and c as input. The input is first concatenated with the context and passed into the network. The residual block passes the input through two fully connected layers and returns the sum of the input and the output. Finally the last layer outputs the parameters namely $\log \mathbf{s}$ and $\mathbf{t}$ in Equation 16. We use 5 layers of affine coupling flows with a LU layer between them.

For DMControl tasks, we pre-process images by reducing the bit depth to 5 bits and draw batches of 50 sequences of length 50 to train the FORBES model, value model, and action model models using Adam (Kingma & Ba, 2014) with learning rates $\alpha_0 = 5 \times 10^{-4}$, $\alpha_1 = 8 \times 10^{-5}$, $\alpha_2 = 8 \times 10^{-5}$, respectively and scale down gradient norms that exceed 100. We clip the KL regularizers in $\mathcal{J}_{Model}$ below 3.0 free nats as in Dreamer and PlaNet. The imagination horizon is $H = 15$ and the same trajectories are used to update both action and value models. We compute the TD-λ targets with $\gamma = 0.99$ and $\lambda = 0.95$.

For digit writing experiments in Section 4.1, we decrease the GRU hidden size to be 20, let the base distributions be a 2-dimensional diagonal Gaussian and only use 3 layers of affine coupling flows.

For the image processing, we simply divide the raw pixels by 255 and subtract 0.5 to make the inputs lie in $[-0.5, 0.5]$.

A.3 Extended information of Baselines

For model-free baselines, we compare with D4PG (Barth-Maron et al., 2018), a distributed extension of DDPG, and A3C (Mnih et al., 2016), the distributed actor-critic approach. D4PG is an improved variant of DDPG (Lillicrap et al., 2015) that uses distributed collection, distributional Q-learning, multi-step returns, and prioritized replay. We include the scores for D4PG with pixel inputs and A3C (Mnih et al., 2016) with vector-wise state inputs from DMCcontrol. For model-based baselines, we use PlaNet (Hafner et al., 2019b) and Dreamer (Hafner et al., 2019a), two state-of-the-art model-based RL. PlaNet (Hafner et al., 2019b) selects actions via online planning without an action model and drastically improves over D4PG and A3C in data efficiency. Dreamer (Hafner et al., 2019a) further improve the data efficiency by generating imaginary rollouts in the latent space.

A.4 Evidence Lower Bound Derivations

The variational bound for latent dynamics models $p\left(o_{1:T}, s_{1:T} \mid a_{1:T}\right) = \prod_t p(s_t|s_{t-1}, a_{t-1})p(o_t|s_t)$ and a variational posterior $q\left(s_{1:T} \mid o_{1:T}, a_{1:T}\right) = \prod_t q\left(s_t \mid o_{\leq t}, a_{<t}\right)$ follows from importance weighting and Jensen's inequality as shown,

$$
\begin{aligned}
\log p\left(o_{1:T} \mid a_{1:T}\right) &= \log \mathrm{E}_{p(s_{1:T}|a_{1:T})}\left[\prod_{t=1}^{T} p\left(o_t \mid s_t\right)\right] \\
&= \log \mathrm{E}_{q(s_{1:T}|o_{1:T},a_{1:T})}\left[\prod_{t=1}^{T} p\left(o_t \mid s_t\right) p\left(s_t \mid s_{t-1}, a_{t-1}\right) / q\left(s_t \mid o_{\leq t}, a_{<t}\right)\right] \\
&\geq \mathrm{E}_{q(s_{1:T}|o_{1:T},a_{1:T})}\left[\sum_{t=1}^{T} \log p\left(o_t \mid s_t\right) + \log p\left(s_t \mid s_{t-1}, a_{t-1}\right) - \log q\left(s_t \mid o_{\leq t}, a_{<t}\right)\right]
\end{aligned} \tag{17}
$$

A.5 PROOFS OF THEOREM

Theorem 1: The approximation error of the lower bound is

$$\log p(o_{1:T}, r_{1:T}|a_{1:T}) - \mathcal{J}_{\text{Model}} = \sum_{t=1}^{T} D_{\text{KL}}(q(s_t|\tau_t, o_t)\|p(s_t \mid \tau_t, o_t))$$

where $p(s_t \mid \tau_t, o_t)$ is the true posterior.

Proof:

$$\begin{aligned}
& D_{\text{KL}}(q(s_t \mid \tau_t, o_t)\|p\left(s_t \mid s_{t-1}, a_{t-1}, o_t\right)) \mid a_{1:T} \\
=& \int q(s_t \mid \tau_t, o_t) \log \frac{q(s_t \mid \tau_t, o_t)}{p\left(s_t \mid s_{t-1}, a_{t-1}, o_t\right)} \mathrm{d}s_t \\
=& \int q(s_t \mid \tau_t, o_t) \log \frac{q(s_t \mid \tau_t, o_t)}{\frac{p(s_t|s_{t-1}, a_{t-1})p(o_t|s_t)}{p(o_t|a_{1:T})}} \mathrm{d}s_t \\
=& \int q(s_t \mid \tau_t, o_t) \log q(s_t \mid \tau_t, o_t) \mathrm{d}s_t + \int q(s_t \mid \tau_t, o_t) \log p(o_t \mid a_{1:T}) \mathrm{d}s_t \\
& - \int q(s_t \mid \tau_t, o_t) \log[p(s_t \mid s_{t-1}, a_{t-1}) p(o_t \mid s_t)] \mathrm{d}s_t \\
=& \log p(o_t \mid a_{1:T}) + \int q(s_t \mid \tau_t, o_t) \log q(s_t \mid \tau_t, o_t) \mathrm{d}s_t - \int q(s_t \mid \tau_t, o_t) \log[p(s_t \mid s_{t-1}, a_{t-1}) p(o_t \mid s_t)] \mathrm{d}s_t \\
=& \log p(o_t \mid a_{1:T}) + \int q(s_t \mid \tau_t, o_t) \log q(s_t \mid \tau_t, o_t) \mathrm{d}s_t - \int q(s_t \mid \tau_t, o_t) \log p(s_t \mid s_{t-1}, a_{t-1}) \mathrm{d}s_t \\
& - \int q(s_t \mid \tau_t, o_t) \log p(o_t \mid s_t) \mathrm{d}s_t \\
=& \log p(o_t \mid a_{1:T}) + \int q(s_t \mid \tau_t, o_t) \log \frac{q(s_t \mid \tau_t, o_t)}{p(s_t \mid s_{t-1}, a_{t-1})} \mathrm{d}s_t - \int q(s_t \mid \tau_t, o_t) \log p(o_t \mid s_t) \mathrm{d}s_t \\
=& \log p(o_t \mid a_{1:T}) + D_{\text{KL}}\left(q\left(s_t \mid \tau_t, o_t\right) \|p\left(s_t \mid s_{t-1}, a_{t-1}, o_t\right)\right) - \mathbb{E}_{q(s_{1:t}|\tau_t, o_t)}[\log p\left(o_t \mid s_t\right)]
\end{aligned} \tag{18}$$

For a sequence from time 1 to T, we have

$$\begin{aligned}
& \sum_t D_{\text{KL}}\left(q\left(s_t \mid \tau_t, o_t\right) \|p\left(s_t \mid s_{t-1}, a_{t-1}, o_t\right)\right) \\
=& \log p(o_{1:T} \mid a_{1:T}) - \mathbb{E}_{q(s_{1:t}|\tau_t, o_t)}\left[\sum_{t=1}^{T}(\log p(o_t|s_t) - D_{\text{KL}}(q(s_t|\tau_t, o_t)\|p(s_t|s_{t-1}, a_{t-1})))\right]
\end{aligned} \tag{19}$$

Then we can derive the Theorem 1 with equation 19:

$$\begin{aligned}
& \log p(o_{1:T}, r_{1:T} \mid a_{1:T}) \\
=& \sum_t D_{\text{KL}}\left(q\left(s_t \mid \tau_t, o_t\right) \|p\left(s_t \mid s_{t-1}, a_{t-1}, o_t\right)\right) \\
& + \mathbb{E}_{q(s_{1:t}|\tau_t, o_t)}\left[\sum_{t=1}^{T}(\log p(o_t|s_t) + \log p(r_t|s_t) - D_{\text{KL}}(q(s_t|\tau_t, o_t)\|p(s_t|s_{t-1}, a_{t-1})))\right] \\
=& \sum_t D_{\text{KL}}\left(q\left(s_t \mid \tau_t, o_t\right) \|p\left(s_t \mid s_{t-1}, a_{t-1}, o_t\right)\right) + \mathcal{J}_{\text{Model}} \\
=& \sum_t D_{\text{KL}}\left(q\left(s_t \mid \tau_t, o_t\right) \|p\left(s_t \mid \tau_t, o_t\right)\right) + \mathcal{J}_{\text{Model}}
\end{aligned} \tag{20}$$

where $p(s_t \mid s_{t-1}, a_{t-1}, o_t) = p(s_t \mid \tau_t, o_t)$ given the sampled s_{t-1} from $q(s_{1:t}|\tau_t, o_t)$.

A.6 More Results on Digit Writing Experiments

In this section, we show more results of the predictions on the digit writing experiment in Figure 8.

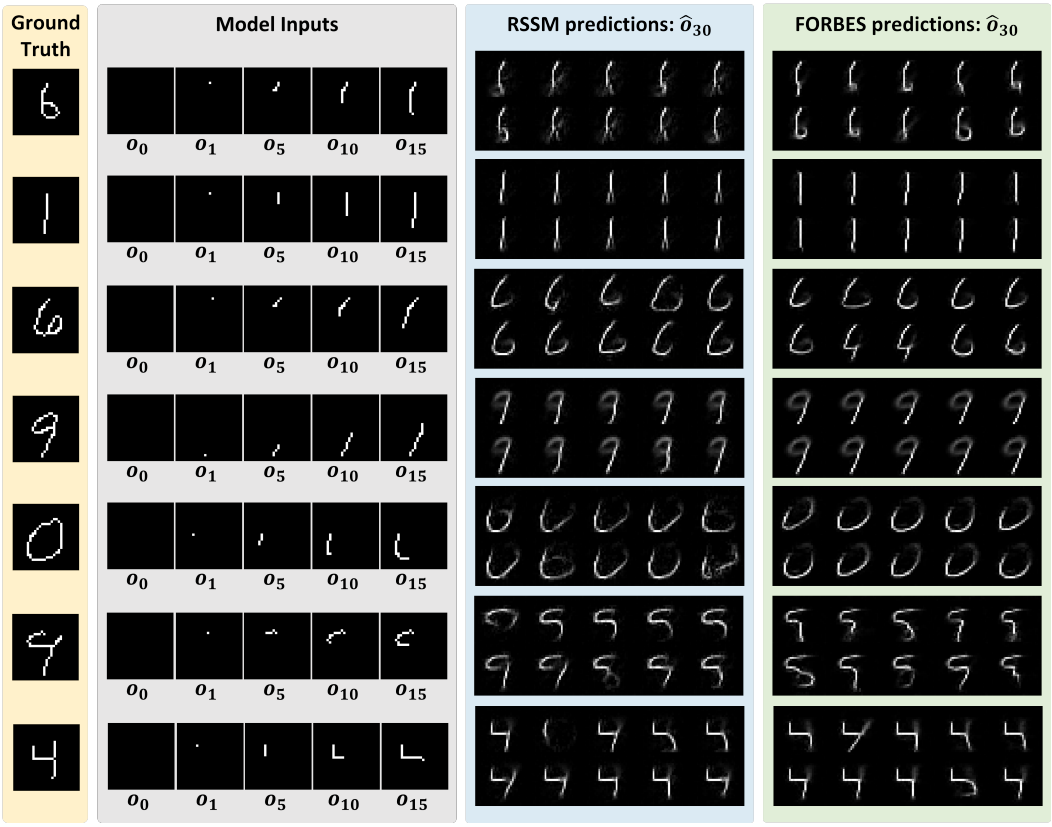
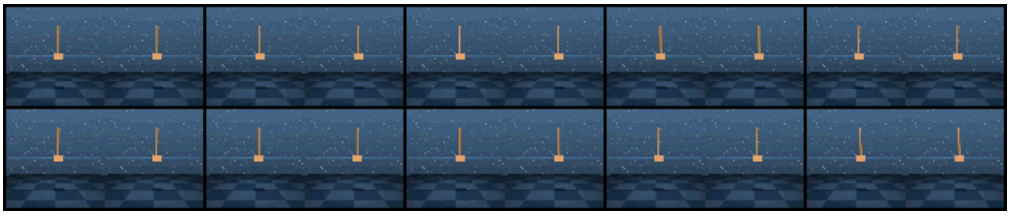


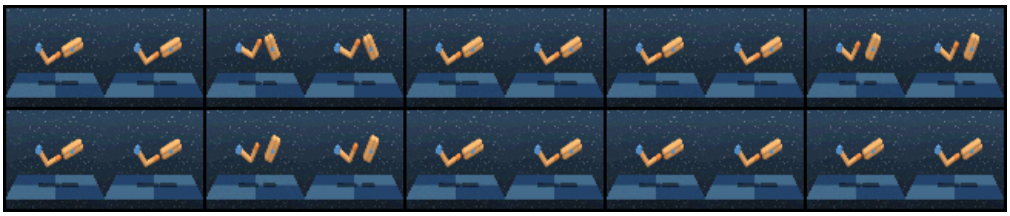
Figure 8: Additional prediction results on sequential MNIST of two models.

A.7 RECONSTRUCTIONS OF THE VISUAL CONTROL TASKS



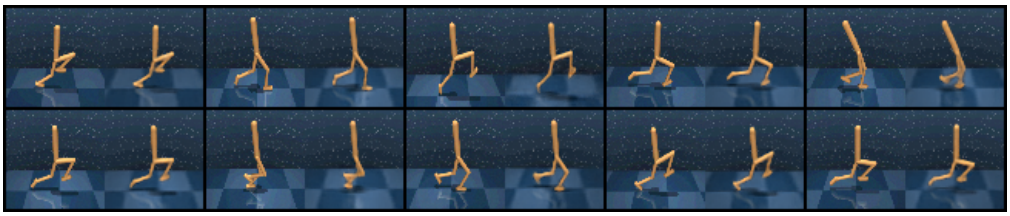
(a) Cartpole Swing Up

(b) Cheetah Run

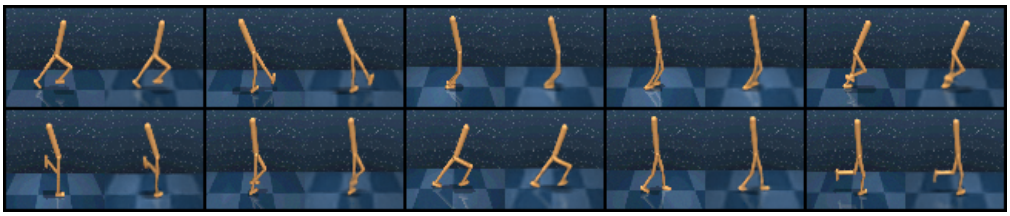


(c) Finger Spin

(d) Hopper Stand



(e) Walker Run



(f) Walker Walk

Figure 9: The reconstruction results on of FORBES six environments from DeepMind Control Suite(Tassa et al., 2018).

In this section, we show the reconstructions of the visual control tasks during the evaluating phase.

For each environment, we use 10 frames. The left one is the original picture for each frame, and the right one is the reconstruction picture. The following results in Figure 9 show that FORBES can make high-quality reconstructions. The corresponding videos can be found in the supplementary material.