IMPROVING MODEL ROBUSTNESS WITH LATENT DISTRIBUTION LOCALLY AND GLOBALLY

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ABSTRACT

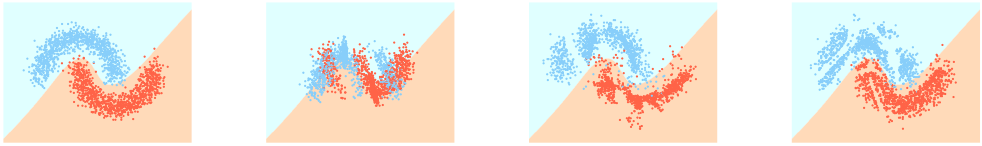
We propose a novel adversarial training method which leverages both the local and global information to defend adversarial attacks. Existing adversarial training methods usually generate adversarial perturbations locally in a supervised manner and fail to consider the data manifold information in a global way. Consequently, the resulting adversarial examples may corrupt the underlying data structure and are typically biased towards the decision boundary. In this work, we exploit both the local and global information of data manifold to generate adversarial examples in an unsupervised manner. Specifically, we design our novel framework via an adversarial game between a discriminator and a classifier: the discriminator is learned to differentiate the latent distributions of the natural data and the perturbed counterpart, while the classifier is trained to recognize accurately the perturbed examples as well as enforcing the invariance between the two latent distributions. We conduct a series of analysis on the model robustness and also verify the effectiveness of our proposed method empirically. Experimental results show that our method substantially outperforms the recent state-of-the-art (i.e. Feature Scattering) in defending adversarial attacks by a large accuracy margin (e.g. 17.0% and 18.1% on SVHN dataset, 9.3% and 17.4% on CIFAR-10 dataset, 6.0% and 16.2% on CIFAR-100 dataset for defending PGD20 and CW20 attacks respectively).

1 INTRODUCTION

Deep Neural Networks (DNNs) have achieved impressive performance on a broad range of datasets, yet can be easily fooled by adversarial examples or perturbations (LeCun et al., 2015; He et al., 2016; Gers et al., 1999). Adversarial examples have been shown to be ubiquitous beyond different tasks such as image classification (Goodfellow et al., 2014), segmentation (Fischer et al., 2017), and speech recognition (Carlini & Wagner, 2018). Overall, adversarial examples raise great concerns about the robustness of learning models, and have drawn enormous attention over recent years.

To defend adversarial examples, great efforts have been made to improve the model robustness (Kannan et al., 2018; You et al., 2019; Wang & Zhang, 2019; Zhang & Wang, 2019). Most of them are based on the adversarial training, i.e. training the model with adversarially-perturbed samples rather than clean data (Goodfellow et al., 2014; Madry et al., 2017; Lyu et al., 2015). In principle, adversarial training is a min-max game between the adversarial perturbations and classifier. Namely, the indistinguishable adversarial perturbations are designed to mislead the output of the classifier, while the classifier is trained to produce the accurate predictions for these perturbed input data. Currently, the adversarial perturbations are mainly computed by enforcing the output invariance in a supervised manner (Madry et al., 2017). Despite its effectiveness in some scenarios, it is observed recently that these approaches may still be limited in defending adversarial examples.

In particular, we argue that these current adversarial training approaches are typically conducted in a local and supervised way and fail to consider globally the overall data manifold information; such information however proves crucially important for attaining better generalization. As a result, the generated adversarial examples may corrupt the underlying data structure and would be typically biased towards the decision boundary. Therefore, the well-generalizing features inherent to the data distribution might be lost, which limits the performance of the DNNs to defend adversarial examples even if adversarial training is applied (Ilyas et al., 2019a; Schmidt et al., 2018). For illustration, we have shown a toy example in Figure 1. As clearly observed, adversarially-perturbed examples gen-



(a) Original Data (b) PGD (c) Feature Scattering (d) Ours

Figure 1: Illustrative example of different perturbation schemes. (a) Original data, Perturbed data using (b) PGD: a supervised adversarial generation method (c) Feature Scattering, and (d) the proposed ATLD method. The overlaid boundary is from the model trained on clean data.

erated by PGD, one of the most successful adversarial training method, corrupt the data manifold, which would inevitably lead to poor performance if the training is conducted based on these perturbed examples. On the other hand, the current state-of-the-art method Feature Scattering (Zhang & Wang, 2019) can partially alleviate this problem but still leads to corruptions on the data manifold.

To address this limitation, we propose a novel method called Adversarial Training with Latent Distribution (ATLD) which additionally considers the data distribution globally in an unsupervised fashion. In this way, the data manifold could be well preserved, which is beneficial to attain better model generalization. Moreover, since the label information is not required when computing the adversarial perturbations, the resulting adversarial examples would not be biased towards the decision boundary. This can be clearly observed in Figure 1(d).

Our method can be divided into two steps: first, we train the deep model with the adversarial examples which maximize the variance between latent distributions of clean data and adversarial counterpart rather than maximizing the loss function. We reformulate it as *a minimax game between a discriminator and a classifier*. The adversarial examples are crafted by the discriminator to make different implicitly the latent distributions of clean and perturbed data, while the classifier is trained to decrease the discrepancy between these two latent distributions as well as promoting accurate classification on the adversarial examples as Figure 2 shows. Then, during the inference procedure, we generate the specific perturbations through the discriminator network to diminish the impact of the adversarial attack as shown in Figure 6 in Appendix.

On the empirical front, with the toy examples, we show that our proposed method can preserve more information of the original distribution and learn a better decision boundary than the existing adversarial training method. We also test our method on three different datasets: CIFAR-10, CIFAR-100 and SVHN with the famous PGD, CW and FGSM attacks. Our ATLD method outperforms the state-of-the-art methods by a large margin. e.g. ATLD improves over Feature Scattering (Zhang & Wang, 2019) by 17.0% and 18.1% on SVHN for PGD20 and CW20 attacks. Our method also shows a large superiority to the conventional adversarial training method (Madry et al., 2017), boosting the performance by 32.0% and 30.7% on SVHN for PGD20 and CW20 attacks.

2 Related Work

Adversarial Training. Adversarial training is a family of techniques to improve the model robustness (Madry et al., 2017; Lyu et al., 2015). It trains the DNNs with adversarially-perturbed samples instead of clean data. Some approaches extend the conventional adversarial training by injecting the adversarial noise to hidden layers to boost the robustness of latent space (Ilyas et al., 2019b; You et al., 2019; Santurkar et al., 2019; Liu et al., 2019). All of these approaches *generate the adversarial examples by maximizing the loss function with the label information*. However, the structure of the data distribution is destroyed since the perturbed samples could be highly biased towards the non-optimal decision boundary (Zhang & Wang, 2019). Our proposed method has a similar training scheme with adversarial training by replacing clean data with the perturbed one. Nevertheless, our method generates the adversarial perturbations without the label information which weakens the impact of non-optimal decision boundary and can retain more information of the underlying data distribution.

Manifold-based Adversarial Training. Song et al. (2017) propose to generate the adversarial examples by projecting on a proper manifold. Zhang & Wang (2019) leverage the manifold information in the forms of inter-sample relationship within the batch to generate adversarial adversarial perturbations. Virtual Adversarial Training and Manifold Adversarial Training are proposed improve model generalization and robustness against adversarial examples by ensuring the local smoothness of the data distribution (Zhang et al., 2018; Miyato et al., 2017). Some methods are designed to enforce the local smoothness around the natural examples by penalizing the difference between the outputs of adversarial examples and clean counterparts (Kannan et al., 2018; Chan et al., 2020; Jakubovitz & Giryes, 2018). All of these methods just *leverage the local information* of the distribution or manifold. Differently, our method generates the perturbations additionally considering the structure of distribution globally.

Unsupervised Domain Adversarial Training. Domain Adversarial Training shares a training scheme similar to our method where the classifier and discriminator compete with each other (Odena et al., 2017; Long et al., 2018; Ganin et al., 2016). However, its objective is to reduce the gap between the source and target distributions in the latent space. The discriminator is used to measure the divergence between these two distributions in the latent space. The training scheme of our method is also based on competition between the classifier and discriminator. Different from the previous framework, the discriminator of our method is used to capture the information of distributions of adversarial examples and clean counterparts in the latent space which helps generate the adversarial perturbations.

GAN-based Adversarial Training Methods. Several GAN-based methods leverage GANs to learn the clean data distribution and purify the adversarial examples by projecting them on clean data manifold before classification (Meng & Chen, 2017; Metzen et al., 2017). The framework of GAN can also be used to generate the adversarial examples (Baluja & Fischer, 2018). The generator produces the adversarial examples to deceive both the discriminator and classifier; the discriminator and classifier attempt to differentiate the adversaries from clean data and produce the correct labels respectively. Some adversary detector networks are proposed to detect the adversarial examples which can be well aligned with our method (Gong et al., 2017; Grosse et al., 2017). In these works, a pretrained network is augmented with a binary detector network. The training of the pretrained network and detector involves generating adversarial examples to maximize their losses. Differently, our method generates the adversarial examples just to minimize the loss of the discriminator and feed them as the training set to the classifier. Such adversarial examples are deemed to induce most different latent representations from the clean counterpart.

3 BACKGROUND

3.1 ADVERSARIAL TRAINING

Let us first introduce the widely-adopted adversarial training method for defending against adversarial attacks. Specifically, it solves the following minimax optimization problem through training.

$$\min_{\theta}\{\mathbb{E}_{(x,y)\sim\mathcal{D}}[\max_{x'\in S_x} L(x', y; \theta)]\}, \tag{1}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ are respectively the clean data samples and the corresponding labels drawn from the dataset $\mathcal{D}$, $L(\cdot)$ is the loss function of DNN with model parameter $\theta \in \mathbb{R}^m$, and $x' \in \mathbb{R}^n$ denotes perturbed samples in a feasible region $S_x \triangleq \{z : z \in B(x, \epsilon) \cap [-1.0, 1.0]^n\}$ with $B(z, \epsilon) \triangleq \{z : \|x - z\|_\infty \leq \epsilon\}$ being the ℓ_∞-ball at center x with radius ϵ.

Whilst the outer minimization can be conducted by training to find the optimal model parameters θ, the inner maximization essentially generates the strongest adversarial attacks on a given set of model parameters θ. In general, the solution to the minimax problem can be found by training a network minimizing the loss for worst-case adversarial examples, so as to attain adversarial robustness. Given a set of model parameters θ, the commonly adopted solution to the inner maximization problem can lead to either one-step (e.g., FGSM) or multi-step (e.g., PGD) approach (Madry et al., 2017). In particular, for a given single point x, the strongest adversarial example x' at the t-th iteration can be iteratively obtained by the following updating rule:

$$x^{t+1} = \Pi_{S_x}(x^t + \alpha \cdot \mathrm{sgn}(\nabla_x L(x^t, y; \theta))), \tag{2}$$

where $\Pi_{S_x}(\cdot)$ is a projection operator to project the inputs onto the region S_x, $\mathrm{sgn}(\cdot)$ is the sign function, and α is the updating step size. For the initialization, x^0 can be generated by randomly sampling in $B(x, \epsilon)$.

It appears in (1) that each perturbed sample x' is obtained individually by leveraging its loss function $L(x', y; \theta)$ with its label y. However, without considering the inter-relationship between samples, we may lose the global knowledge of the data manifold structure which proves highly useful for attaining better generalization. This issue has been studied in a recent work (Zhang & Wang, 2019) where a new method named feature scattering made a first step to consider the inter-sample relationship *within the batch*; unfortunately this approach did not take the full advantages of the global knowledge of the entire data distribution. In addition, relying on the maximization of the loss function, the adversarially-perturbed data samples may be highly biased towards the decision boundary, which potentially corrupts the structure of the original data distribution, especially when the decision boundary is non-optimal (see Figure 1 again for the illustration).

3.2 Divergence Estimation

To measure the discrepancy of two distributions, statistical divergence measures (e.g., Kullback-Leibler and Jensen-Shannon divergence) have been proposed. In general, given two distributions $\mathbb{P}$ and $\mathbb{Q}$ with a continuous density function $p(x)$ and $q(x)$ respectively, f-divergence is defined as $D_f(\mathbb{P}||\mathbb{Q}) \triangleq \int_{\mathcal{X}} q(x) f\left(\frac{p(x)}{q(x)}\right) dx$. The exact computation of f-divergence is challenging, and the estimation from samples has attracted much interest. For instance, leveraging the variational methods, Nguyen et al. (2010) propose a method for estimating f-divergence from only samples; Nowozin et al. (2016) extend this method by estimating the divergence with learning the parameters of discriminator. Specifically, the f-divergence between two distributions $\mathbb{P}$ and $\mathbb{Q}$ can be lower-bounded using Fenchel conjugate and Jensen's inequality (Nowozin et al., 2016).

$$\begin{aligned} D_f(\mathbb{P}||\mathbb{Q}) &= \int_{\mathcal{X}} q(x) \sup_{t \in \mathrm{dom} f^*} \{t \frac{p(x)}{q(x)} - f^*(t)\} dx \\ &\geq \sup_{T \in \tau} \left(\int_{\mathcal{X}} p(x) T(x) dx - \int_{\mathcal{X}} q(x) f^*(T(x)) dx \right) \\ &= \sup_{W} (\mathbb{E}_{x \sim \mathbb{P}}[g_f(V_W(x))] + \mathbb{E}_{x \sim \mathbb{Q}}[-f^*(g_f(V_W(x)))]), \end{aligned} \tag{3}$$

where $V_W : \mathcal{X} \to \mathbb{R}$ is a discriminator network with parameter W and $g_f : \mathbb{R} \to \mathrm{dom} f^*$ is an output activation function which is determined by the type of discriminator. f is a convex lower-semicontinous function and f^* is its conjugate defined by $f^*(t) = \sup_{u \in \mathrm{dom} f}[ut - f(u)]$. The objective of discriminator for GANs is a special case of (3) with the activation function $g_f(t) = -\log(1 + e^{-t})$ and $f^*(g) = -\log(2 - e^g)$. It approximates the Jense-Shannon divergence between real and fake distributions. Arjovsky et al. (2017) also develop a method to estimate the Wasserstein-distance by neural network. In this paper, these methods will be used to estimate the Jensen-Shannon divergence between latent distributions of adversarial and clean examples.

4 Adversarial Training with Latent Distribution

As discussed in Section 3.1, the conventional adversarial training methods relying on the knowledge of data labels and the local information to generate adversarial examples may be biased toward the decision boundary; such individual adversarial example generation does not capture the global knowledge of the data manifold.

To alleviate these limitations, we propose a novel method to compute the perturbed samples by leveraging the global knowledge of the whole data distribution and then disentangling them from the data labels and the loss function. Generally speaking, the perturbations are generated to enlarge the variance between latent distributions of clean and adversarial data.

Formally, we try to identify the perturbed example $x^{adv} \in B(x, \epsilon)$ that yields in the latent space a distribution P_θ that is the most different from the distribution Q_θ induced by the clean samples x, without resorting to the labels y. The latent space could be defined in any hidden layer, e.g. the layer before the softmax of a DNN. Specifically, the optimization problem of the adversarial training can

be reformulated as follows

$$\min_{\theta} \quad \{\mathbb{E}_{(x,y)\sim\mathcal{D}}[L(x^{adv}, y; \theta)] + D_f(P_\theta^* || Q_\theta)\} \tag{4}$$

$$\text{s.t.} \quad P_\theta^* = \arg\max_{P_\theta}[D_f(P_\theta || Q_\theta)] \tag{5}$$

where $x, y, \mathcal{D}, B(x, \epsilon), L(\cdot)$ are similarly defined as before, Q_θ and P_θ^* are the latent distributions respectively induced by the natural example x and the adversarial sample $x^{adv} \in B(x, \epsilon)$ with model parameter θ, and $D_f(\cdot)$ is the f-divergence measure of two distributions. Intuitively, the adversarial example x^{adv} is crafted to induce the worst distribution P_θ^* which maximizes the f-divergence between the latent distributions P_θ and Q_θ, while the model parameter θ is learned to minimize the classification loss on adversarial examples and f-divergence between the latent distributions P_θ^* and Q_θ induced by adversarial example x^{adv} and clean data x.

It is still challenging to solve the above optimization problem, since both the objective function and the constraint are entangled with the adversarial examples x^{adv} and the model parameters θ. To make the problem more tractable, we propose a novel Adversarial Training with Latent Distribution (ATLD) method. In the next subsection, by taking into account the entire data distribution globally, we first focus on the constraint and identify the adversarial samples x^{adv} through the maximization problem. We then solve the minimization of the objective function with the adversarial training procedure. To further enhance the performance, we add specific perturbations named Inference with Manifold Transformation (IMT) in Section 4.2 to input samples for enforcing them towards the more separable natural data manifold. Finally, we classify the transformed data points with the adversarially-trained model.

4.1 Generating Adversarial Examples for Training

First, we optimize the constraint (5) to generate the adversarial examples or its induced distribution P_θ^* for training. Intuitively, the adversarial example x^{adv} is crafted to maximize the divergence between the distributions induced by natural example x and adversarial counterpart x^{adv} in an unsupervised fashion because no knowledge of labels y is required. Together with the objective function in (4), our proposed adversarial training method is to minimize such divergence as well as the classification error for adversarial example x^{adv}.

However, it is a challenging task to evaluate the divergence between two latent distributions. To make it more tractable, we leverage *a discriminator* network for estimating the Jensen-Shannon divergence between these two distributions according to Section 3.2. To better deal with multi-modal distributions, the discriminator network additionally outputs the class label of the latent feature (Odena et al., 2017; Long et al., 2018). It is noted that the class label information is not used for generating adversarial examples. Hence the adversarial examples are still generated in an unsupervised way. Then, by using (3), the optimization problem in (4) and (5) can be approximated as follows in a tractable way.

$$\begin{aligned}
&\min_{\theta}\Big\{ \sup_{W} \sum_{i=1}^{N} [\underbrace{\log D_W^0(f_\theta(x_i^{adv})) + (1 - \log D_W^0(f_\theta(x_i))}_{L_d^0}] \\
&\quad + \underbrace{L(x_i^{adv}, y_i; \theta)}_{L_f} + \min_{W}[\underbrace{l(D_W^{1:C}(f_\theta(x_i)), y_i) + l(D_W^{1:C}(f_\theta(x_i^{adv})), y_i)}_{L_d^{1:C}}] \Big\}, \\
&\text{s.t.} \quad x_i^{adv} = \arg\max_{x_i' \in B(x_i, \epsilon)} \underbrace{[\log D_W^0(f_\theta(x_i')) + (1 - \log D_W^0(f_\theta(x_i))]}_{L_d^0}
\end{aligned} \tag{6}$$

where N denotes the number of training samples and D_W denotes the discriminator network with the sigmoid activation function and parameter W. D_W^0 is the first dimension for the output of D_W which is used to determine whether the latent feature is from adversary manifold (output the manifold label of the latent feature). $D_W^{1:C}$ is the remaining C dimensions for the output of D_W which is used to output the class label of the latent feature. C is the number of classes. f_θ is the map from the input space to last latent space of the classifier with model parameter θ. For ease of description, we represent the components in Eq. (6) as three parts: L_f, L_d^0 and $L_d^{1:C}$. L_d^0 and $L_d^{1:C}$

are the manifold loss and the classification loss for the discriminator network respectively. Let L_f represent the loss function of the classifier network. *The detailed derivation for Eq. (6) can be seen in Appendix.*

For our method, the Jensen-Shannon divergence between two latent distributions is approximated by $\sup_W \sum_{i=1}^{N} L_d^0(x_i, x_i^{adv}; W, \theta)$. During the training procedure, the classifier and the discriminator compete with each other and are updated iteratively. The classifier is trained to enforce the invariance between the latent distributions of the adversarial and the natural examples by minimizing the loss L_d^0. Simultaneously, it also tries to classify the adversarial examples as accurately as possible by minimizing L_f, while the discriminator is learned to differentiate these two distributions via minimizing the loss L_d^0. The classification loss of discriminator $L_d^{1:C}$ is also minimized for better dealing with multi-modal distributions. For each training iteration, the adversarial examples are crafted to make different the adversarial latent distribution and natural one by maximizing L_d^0. The detailed training procedure of our framework is depicted in Figure 2.

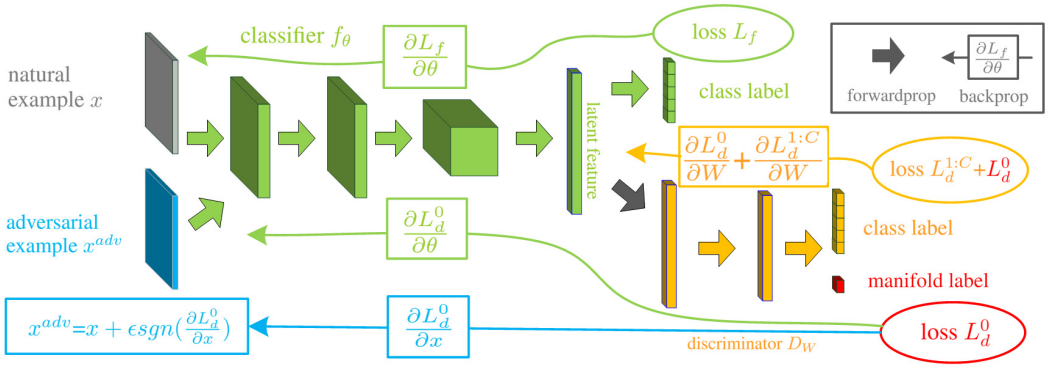


Figure 2: Overall architecture of ATLD and its training procedure. 1) The natural example is fed into the network, and the discriminator outputs its prediction. The manifold loss L_d^0 is computed with the prediction and true label and generates the adversarial example x^{adv} (blue arrow). 2) Both the clean and adversarial sample are fed into the network to train the classifier (green arrow) and the discriminator (yellow arrow) iteratively.

Remarks. It is worth noting that the labeled information is not required for generating adversarial examples. Therefore, our method prevents the perturbed examples from highly biasing towards the decision boundary and more information of the original distribution structure is preserved. In addition, since the discriminator is trained with the whole dataset (both clean and adversarial examples), it captures the global information of data manifold. Consequently, by training with adversarial examples generated according to the manifold loss of the discriminator, our method can improve the model robustness against adversarial examples with the global structure of data distribution.

4.2 Inference with Manifold Transformation

To enhance the generalization of ATLD, we further develop a new inference method with manifold transformation. Although adversarially-trained models can well recognize adversarial examples, there are still potential examples which are easily misclassified especially for unseen data. In other words, the generalization to adversarial examples is hard to achieve due to the more complex distribution of adversarial examples (Schmidt et al., 2018; Zhai et al., 2019). To alleviate this problem, our proposed inference method first pushes adversarial examples towards the manifold of natural examples which is simpler and further away from the decision boundary than the adversarial distribution. Then the more separable adjusted examples are classified by the adversarially-trained model. Specifically, the input sample is fed into our adversarially-trained model and the discriminator outputs the probability of such a sample lying on the adversarial manifold. If this probability is higher than a certain threshold, we compute the transformed example x^t by adding the specific perturbation r^* to the input sample x to reduce such a probability. This perturbation can be computed as:

$$r^* = \arg \min_{\|r\|_\infty \leq \epsilon} \log D_W^0(f_\theta(x + r)). \tag{7}$$

Intuitively, the reduction of probability of this data point lying on adversarial manifold indicates that this point moves towards the benign example manifold after adding perturbation r^*. In other words,

it becomes more separable since the benign example manifold is further away from the decision boundary. When the probability of the image lying on adversary manifold is lower than threshold, we still add such a perturbation to input image to make it more separable but with smaller magnitude. In the experiment part, we show this perturbation can move the adversarial examples away from the decision boundary. The whole inference procedure can be seen in Figure 5 in Appendix.

5 EXPERIMENTS

We conduct experiments on the widely-used datasets, e.g., CIFAR-10, SVHN, and CIFAR-100. Following the Feature Scattering method (Zhang & Wang, 2019), we leverage the wideresnet (Zagoruyko & Komodakis, 2016) as our basic classifier and discriminator model structure. During the training phase, the initial learning rate is empirically set to 0.1 for all three datasets. We train our model 400 epochs with the transition epoch $60, 90$ and the decay rate 0.1. The input perturbation budget is set to $\epsilon = 8$ with the label smoothing rate as 0.5.

We evaluate the various models on white-box attacks and black-box attacks. Under the white-box attacks, we compare the accuracy of the proposed method with several competitive methods, including: (1) the original wideresnet (Standard) trained with natural examples; (2) Traditional Adversarial Training with PGD (AT) (Madry et al., 2017); (3) Triplet Loss Adversarial training (TLA) (Mao et al., 2019); (4) Layer-wise Adversarial Training (LAT): injecting adversarial perturbation into the latent space (Sinha et al., 2019); (5) Bilateral: adversarial perturb on examples and labels both (Wang & Zhang, 2019); (6) Feature-scattering: generating adversarial examples with considering interrelationship of samples (Zhang & Wang, 2019). These comparison algorithms present the most competitive performance on defending adversarial attack. Under the black-box attacks, we compare four different algorithms used to generate the test time attacks: Vanilla training with natural examples, adversarial training with PGD, Feature Scattering, and our proposed model.

5.1 DEFENDING WHITE-BOX ATTACKS

We show the classification accuracy under several white-box attacks on CIFAR-10, CIFAR-100, SVHN in this section. We first report the accuracy on CIFAR-10 in Table 1 with the attack iterations $T = 20, 40, 100$ for PGD (Madry et al., 2017) and CW (Carlini & Wagner, 2017). As observed, overall, our proposed method achieves a clear superiority over all the defence approaches on both the clean data and adversarial examples (except that it is slightly inferior to Feature Scattering in FGSM). We also observe one exception that the standard model performs the best on clean data. Our approach performs much better than the other baseline models on PGD and CW attack. Particularly, we improve the performance of the recent state-of-the-art method Feature Scattering almost 3.1% and 5.2% under PGD20 and CW20 attack respectively. With the implementation of Inference with Manifold Transformation (IMT), our approach (ATLD-IMT) is 8.9% and 17.4% higher than the Feature Scattering under PGD20 and CW20 attack respectively. However, the performance on clean data is declined from 93.3% to 86.4% since IMT appears to have a negative effect for classifying clean data. In order to reduce the impact of IMT on the natural data, a threshold is used to limit the perturbation of IMT based on the output of discriminator. The perturbation is halved if the output of discriminator is within the range of $[0.3, 0.7]$ (ATLD-IMT+). Under such setting, our approach could achieve high performance on adversarial attacks without sacrificing its accuracy on clean data.

Similarly, the accuracy on CIFAR-100 and SVHN are shown in Table 2 with the attack iterations $T = 20, 100$ for both PGD and CW for conciseness. Although our method is slightly weaker than Feature Scattering under FGSM attack on CIFAR-100, overall, our proposed method ATLD achieves state-of-the-art performance over all the other approaches under various adversarial attacks. Furthermore, our ATLD-IMT version exceeds Feature Scattering by almost 19.2% and 23.8% against the attack of CW100 on CIFAR-100 and SVHN respectively. More details about the defense of white-box attacks under different attack budgets can be seen in Appendix.

5.2 DEFENDING BLACK-BOX ATTACKS

To further verify the robustness of ATLD, we conduct transfer-based black-box attack experiments on CIFAR-10. More black-box attack results on CIFAR-100 and SVHN are listed in Appendix. Four different models are used for generating test time attacks including the Vanilla Training model,

Table 1: Accuracy under different White-box Attack attack on CIFAR-10

Models	Clean	Accuracy under White-box Attack ($\epsilon = 8$)						
		FGSM	PGD20	PGD40	PGD100	CW20	CW40	CW100
Standard	**95.60**	36.90	0.00	0.00	0.00	0.00	0.00	0.00
AT	85.70	54.90	44.90	44.80	44.80	45.70	45.60	45.40
TLA	86.21	58.88	51.59	-	-	-	-	-
LAT	87.80	-	53.84	-	53.04	-	-	-
Bilateral	91.20	70.70	57.50	–	55.20	56.20	–	53.80
FS	90.00	78.40	70.50	70.30	68.60	62.40	62.10	60.60
ATLD	93.34	**87.91**	73.58	72.95	72.82	67.63	66.26	65.40
ATLD-IMT	86.42	84.62	79.48	**77.06**	74.20	**79.81**	77.14	**75.20**
ATLD-IMT+	90.78	84.37	**79.82**	76.71	**74.53**	79.31	**77.17**	74.46

Table 2: Accuracy under different White-box Attack attack on CIFAR-100 and SVHN

Models	CIFAR-100($\epsilon = 8$)						SVHN($\epsilon = 8$)					
	CLEAN	FGSM	PGD20	PGD100	CW20	CW100	CLEAN	FGSM	PGD20	PGD100	CW20	CW100
Standard	**79.00**	10.00	0.00	0.00	0.00	0.00	**97.20**	53.00	0.30	0.10	0.30	0.10
AT	59.90	28.50	22.60	22.30	23.20	23.00	93.90	68.40	47.90	46.00	48.70	47.30
LAT	60.94	-	27.03	26.41	-	-	60.94	-	27.03	26.41	-	-
Bilateral	68.20	60.80	26.70	25.30	-	22.10	94.10	69.80	53.90	50.30	-	48.90
FS	73.90	**61.00**	47.20	46.20	34.60	30.60	96.20	83.50	62.90	52.00	61.30	50.80
ATLD	74.90	57.88	48.43	48.29	40.52	40.36	96.90	**93.30**	68.42	55.46	66.32	52.62
ATLD-IMT	63.17	56.64	**53.26**	**53.35**	**50.80**	**49.77**	85.85	89.07	**79.93**	**74.70**	**79.44**	**74.59**
ATLD-IMT+	68.17	56.61	50.67	50.43	46.92	46.15	92.43	90.34	78.33	70.96	77.52	71.04

the Adversarial Training with PGD model, the Feature Scattering Training model and our model. As demonstrated by the results in Table 3, our proposed approach can achieve competitive performance almost in all the cases. Specifically, ATLD outperforms Feature Scattering significantly in 8 cases while it demonstrates comparable or slightly worse accuracy in the other 3 cases.

It deserves our attention that ATLD-IMT appears to have a negative impact on the black-box attacks though it stills performs much better than PGD. This may be explained in several aspects. On one hand, the distributions of adversarial examples produced by different models may differ significantly in the latent space; on the other hand, our discriminator lacks the ability to deal with the unseen distributions since the discriminator only distinguishes one type of adversarial examples from the natural data during training. We will leave the investigation of this topic as future work.

Table 3: Accuracy under black-box attack on CIFAR-10

Defense Models	Attacked Models											
	vanilla training			Adversarial Training			Feature Scattering			Ours		
	FGSM	PGD20	CW20	FGSM	PGD20	CW20	FGSM	PGD20	CW20	FGSM	PGD20	CW20
PGD	84.62	84.89	84.83	72.20	63.77	63.27	82.26	80.56	79.31	82.56	81.34	79.97
FS	**88.64**	89.25	89.31	77.18	**66.59**	**66.40**	82.80	81.01	78.34	82.99	81.56	79.92
ATLD	88.11	**91.36**	**91.52**	**80.13**	65.91	65.76	**85.97**	**84.43**	**82.15**	**84.47**	**82.24**	**80.34**
ATLD-IMT	86.53	89.93	89.91	79.16	65.61	65.07	85.76	83.75	81.40	83.90	81.98	79.77

6 Conclusion

We have developed a novel adversarial training method which leverages both the local and global information to defend adversarial attacks in this paper. In contrast, existing adversarial training methods mainly generate adversarial perturbations in a local and supervised fashion, which could however limit the model's generalization. We have established our novel framework via an adversarial game between a discriminator and a classifier: the discriminator is learned to differentiate globally the latent distributions of the natural data and the perturbed counterpart, while the classifier is trained to recognize accurately the perturbed examples as well as enforcing the invariance between the two latent distributions. Extensive empirical evaluations have shown the effectiveness of our proposed model when compared with the recent state-of-the-art in defending adversarial attacks in both the white-box and black-box settings.

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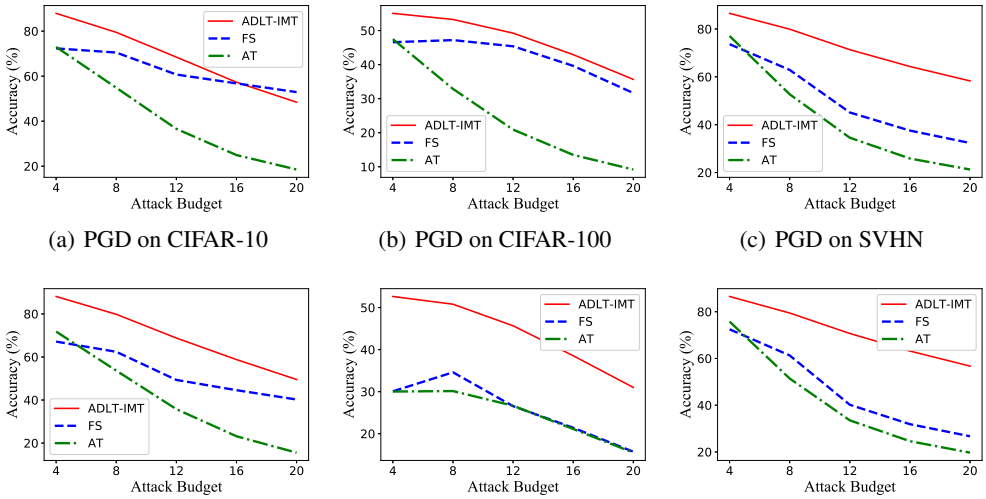
APPENDIX

A ADDITIONAL EXPERIMENT DETAILS

A.1 MODEL ROBUSTNESS AGAINST PGD AND CW ATTACKER UNDER DIFFERENT ATTACK BUDGETS

We further evaluate the model robustness against PGD and CW attacks under different attack budgets with a fixed attack step of 20. These results are shown in Figure 3. It is observed that the performance of Adversarial Training with the PGD method (AT) drops quickly as the attack budget increases. The Feature Scattering method (FS) can improve the model robustness across a wide range of attack budgets. The proposed approach ADLT-IMT further boosts the performance over Feature Scattering by a large margin under different attack budgets especially under CW attack, except that our ADLT-IMT is slightly inferior to Feature Scattering under PGD attack with budget $\epsilon = 20$ on CIFAR-10.



(a) PGD on CIFAR-10 (b) PGD on CIFAR-100 (c) PGD on SVHN

(d) CW on CIFAR-10 (e) CW on CIFAR-100 (f) CW on svhn

Figure 3: Model performance under PGD and CW attacks with different attack budgets.

A.2 BLACK-BOX RESULTS ON SVHN AND CIFAR-100

We conduct more evaluations on the transfer-based black-box attacks on SVHN and CIFAR-100. We report the results in Table 4. It can be observed that our proposed method overall outperforms Feature Scattering in most of the cases on SVHN. Surprisingly, the Adversarial Training method, i.e. the PGD, performs better than our method and Feature Scattering method in three cases. This also partially reveals the more challenging nature of defending black-box attacks than white-box attacks.

On CIFAR-100, it can be observed that our method and Feature Scattering are comparable. The performance of these two methods differs little though our method outperforms Feature Scattering significantly under PGD20 and CW20 against adversarial attacks generated from the Feature Scattering model.

Overall, though the proposed ATLD method may not lead to remarkably higher performance than the current state-of-the-art algorithms in defending black-box attacks (as we observed in the case of white-box attacks), it still generates overall better or comparable performance. We will again leave the further exploration of defending black-box attacks as our future work.

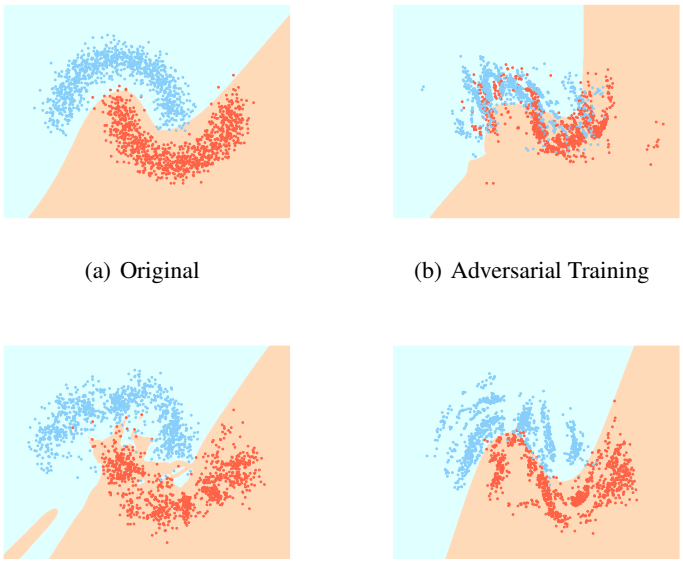
Table 4: Accuracy under black-box attack on SVHN and CIFAR-100

Defense Models	Attacked Models (SVHN)											
	Vanilla Training			Adversarial Training			Feature Scattering			Ours		
	FGSM	PGD20	CW20	FGSM	PGD20	CW20	FGSM	PGD20	CW20	FGSM	PGD20	CW20
PGD	**88.86**	89.17	89.11	75.56	62.54	62.74	90.76	88.62	88.76	90.98	**90.10**	**90.31**
FS	80.89	86.65	87.10	82.27	65.59	65.18	**96.59**	81.91	82.52	**93.11**	81.65	82.46
ATLD	82.61	**89.39**	**89.49**	**83.57**	**66.00**	**65.28**	96.42	**91.88**	**92.36**	91.75	86.46	87.37
ATLD-IMT	80.27	82.18	82.53	79.61	64.63	64.84	90.33	86.84	82.06	87.97	83.27	83.71

Defense Models	Attacked Models (CIFAR-100)											
	Vanilla Training			Adversarial Training			Feature Scattering			Ours		
	FGSM	PGD20	CW20	FGSM	PGD20	CW20	FGSM	PGD20	CW20	FGSM	PGD20	CW20
PGD	60.46	60.88	60.92	47.16	40.37	39.53	61.61	61.20	60.78	60.71	60.99	60.96
FS	66.51	**71.47**	**71.88**	**56.96**	44.91	**46.37**	**74.00**	62.07	62.07	**69.35**	**63.36**	63.50
ATLD	**67.30**	71.42	71.56	56.01	**44.92**	45.84	71.52	**67.72**	**68.32**	67.20	63.02	**63.69**
ATLD-IMT	57.77	61.40	60.66	48.15	39.07	41.30	63.09	61.48	60.78	61.32	60.95	59.93

A.3 Illustration of the overlaid boundary change of different methods

We conduct a toy example in Figure 4 to illustrate the effect on how the various methods would affect the decision boundary after the adversarial training is applied. In Figure 4, (a) shows the decision boundary trained with clean data; (b) shows the decision boundary adversarially trained with the perturbed samples by PGD; (c) presents the decision boundary given by the adversarial training of Feature Scattering; and (d) illustrates the decision boundary trained from our proposed ATLD. Clearly, both the PGD (Figure 4(b)) and the FS (Figure 4(c)) vary the original decision boundary significantly. Moreover, it can be observed that the adversarial training with PGD corrupts the data manifold completely. On the other hand, FS appears able to retain partially the data manifold information since it considers the inter-sample relationship locally. Nonetheless, its decision boundary appears non-smooth, which may hence degrade the performance. In contrast, as shown in Figure 4(d), our proposed method considers to retain the data manifold globally, which varies the decision boundary slightly. This may explain why our proposed ATLD method could outperform the other approaches.



(a) Original (b) Adversarial Training

(c) Feature Scattering (d) Ours

Figure 4: The overlaid decision boundary after the various adversarial training is applied

A.4 FURTHER DETAILS OF ATLD-IMT

We elaborate the training procedure of our IMT in this section. The overall architecture of ATLD-IMT is plotted in Figure 5. A test sample x is fed into the classifier, and the discriminator outputs the prediction. A special perturbation in IMT is then computed from the loss D_W and added back to x; in this way, the sample would be pushed towards the manifold of natural samples, which is supposed to be further away from the decision boundary. The prediction of the transformed x^t by the adversarially-trained classifier will then be output as the label of x.

To illustrate clearly the effect of our ATLD-IMT, we conduct additional toy experiments as shown in Figure 6 where we respectively plot the clean or natural data, perturbed data attacked by PGD, and adjusted data by ATLD-IMT in (a), (b), and (c). Moreover, the decision boundary is given by ATLD in all the three sub-figures. In (a), it deserves our attention that the boundary learned by ATLD could classify natural data well compared to the PGD and Feature Scattering as shown in Section A.3. As observed in (b), the perturbations generated by PGD will push the natural samples toward or even cross the decision boundary. Our proposed IMT can push the samples towards the manifold of natural examples as observed in (c). Since the manifold of natural examples would be more separable, this may further increase the classification performance as observed in the experiments.

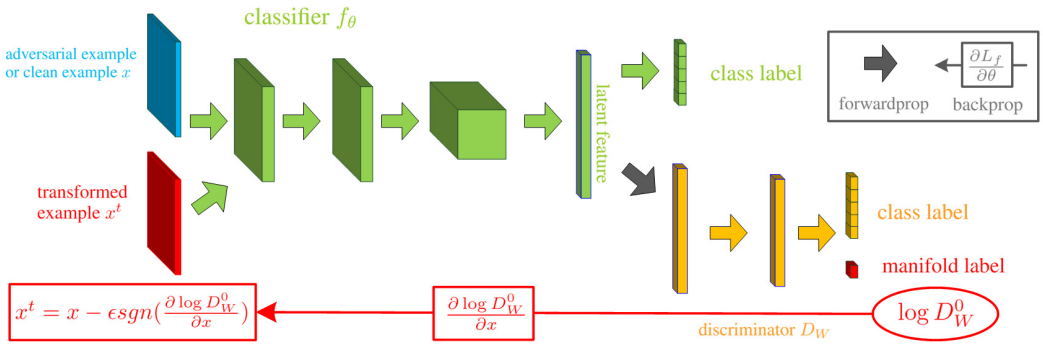
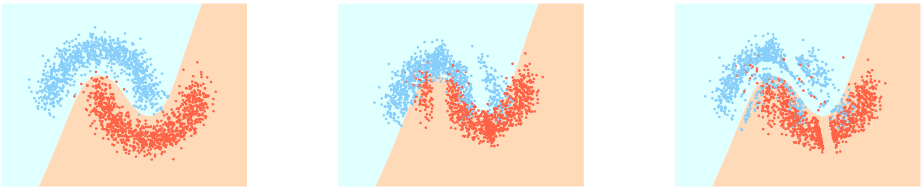


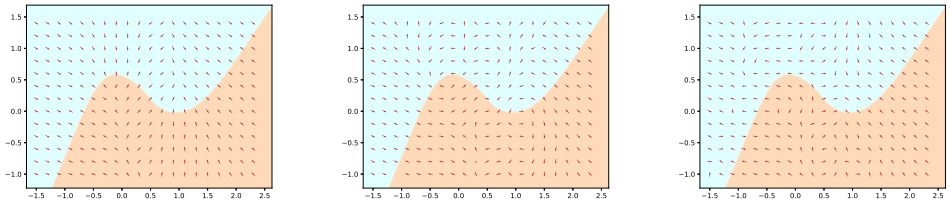
Figure 5: Detailed Procedure of IMT. 1) The natural example or adversarial example x is fed into the network, and the discriminator outputs its prediction. The loss $\log D_W$ is computed and the transformed example x^t (red arrow) is then generated. 2) The transformed sample is fed into the network and classified by the adversarially-trained network.



(a) Clean data (b) Perturbed data by PGD (c) Adjusted data by ATLD-IMT

Figure 6: Illustration of ATLD-IMT. The decision boundary is given by ATLD in all the three sub-figures, while (a) shows clean data, (b) draws perturbed data attacked by PGD, and (c) plots adjusted data by ATLD-IMT. Our proposed IMT can push the samples towards the manifold of natural examples as observed in (c). Since the manifold of natural examples would be more separable, this may further increase the classification performance.

A.5 Illustration of Vector Field of Different Perturbation Schemes



(a) Adversarial Training by PGD (b) Feature Scattering (c) Ours

Figure 7: Vector field illustration of different perturbation schemes. (a) PGD, (b) Feature Scattering, (c) the proposed ATLD method. The overlaid boundary is from the model trained on clean data. The figure plots the direction of adversarial perturbations at different points. It is worth noting that most directions of the adversarial perturbations for the conventional adversarial training methods point to the decision boundary. It indicates that the resulting adversarial examples are easily biased towards the decision boundary which potentially corrupts the structure of the underlying distribution. The perturbation directions for Feature Scattering and our method are influenced by decision boundary less.

B Detailed Derivation

In this section, we provide the details about the derivation for the main objective function (6) and elaborate how to compute the adversarial examples and the transformed examples.

B.1 Derivation for Main Objective Function (6)

We start with minimizing the largest f-divergence between latent distributions P_θ and Q_θ induced by perturbed example x' and natural example x. And we denote their corresponding probability density functions as $p(z)$ and $q(z)$. According to Eq. (3), we have

$$\begin{aligned}
\min_\theta \max_{Q_\theta} D_f(P_\theta||Q_\theta) &= \min_\theta \max_{q(z)} \int_{\mathcal{Z}} q(z) \sup_{t\in \mathrm{dom} f^*} \{t\frac{p(z)}{q(z)} - f^*(t)\}dx \\
&\geq \min_\theta \max_{q(z)} \sup_{T\in\mathcal{T}} (\int_{\mathcal{Z}} p(z)T(z)dz - \int_{\mathcal{Z}} q(z)f^*(T(z))dz) \\
&= \min_\theta \max_{Q_\theta} \sup_W \Big\{\mathbb{E}_{z\sim P_\theta}[g_f(V_W(z))] + \mathbb{E}_{z\sim Q_\theta}[-f^*(g_f(V_W(z)))]\Big\} \\
&= \min_\theta \sup_W \Big\{\mathbb{E}_{x\sim\mathcal{D}}\{\max_{x'\in B(x,\epsilon)}[g_f(V_W(f_\theta(x')))] + [-f^*(g_f(V_W(f_\theta(x))))]\}\Big\}
\end{aligned} \tag{8}$$

To compute the Jensen-Shannon divergence between P_θ and Q_θ, we set $g_f(t) = -\log(1+e^{-t})$ and $f^*(g) = -\log(2-e^g)$. Then, we have

$$\min_\theta \max_{Q_\theta} D_{JS}(P_\theta||Q_\theta) \geq \min_\theta \sup_W \Big\{\mathbb{E}_{x\sim\mathcal{D}}\{\max_{x'\in B(x,\epsilon)}[\log D_W(f_\theta(x')))] + [1-\log D_W(f_\theta(x))))]\}\Big\} \tag{9}$$

where $D_W(x) = 1/(1+e^{-V_W(x)})$. (9) is equivalent to optimize the lower bound of Jensen-Shannon divergence between P_θ and Q_θ. With disentangling the computation of adversarial examples from Eq. (9) and further considering the classification loss for the classifier L_f and the discriminator $L_d^{1:C}$,

we can obtain the final objective:

$$\min_{\theta}\Big\{\sup_{W}\sum_{i=1}^{N}[\underbrace{\log D_W^0(f_\theta(x_i^{adv}))+(1-\log D_W^0(f_\theta(x_i))}_{L_d^0}]$$
$$+\underbrace{L(x_i^{adv},y_i;\theta)}_{L_f}+\min_{W}[\underbrace{l(D_W^{1:C}(f_\theta(x_i)),y_i)+l(D_W^{1:C}(f_\theta(x_i^{adv})),y_i)}_{L_d^{1:C}}]\Big\}, \tag{10}$$
$$\text{s.t.}\quad x_i^{adv}=\arg\max_{x_i'\in B(x_i,\epsilon)}\underbrace{[\log D_W^0(f_\theta(x_i'))+(1-\log D_W^0(f_\theta(x_i))}_{L_d^0}]$$

B.2 Computation for Adversarial Example and Transformed Example

To compute the adversarial example, we need to solve the following problem:

$$x_i^{adv}=\arg\max_{x_i'\in B(x_i,\epsilon)}\underbrace{[\log D_W^0(f_\theta(x_i'))+(1-\log D_W^0(f_\theta(x_i))}_{L_d^0}] \tag{11}$$

It can be reformulated as computing the adversarial perturbation as follows:

$$r_i^{adv}=\arg\max_{\|r\|_\infty\le\epsilon}[L_d^0(x_i+r_i,\theta)] \tag{12}$$

We first consider the more general case $\|r\|_p\le\epsilon$ and expand (12) with the first order Taylor expansion as follows:

$$r_i^{adv}=\arg\max_{\|r\|_p\le\epsilon}[L_d^0(x_i,\theta)]+\nabla_x\mathcal{F}^T r_i \tag{13}$$

where $\mathcal{F}=L(x_i,\theta)$. The problem (13) can be reduced to:

$$\max_{\|r_i\|_p=\epsilon}\nabla_x\mathcal{F}^T r_i \tag{14}$$

We solve it with the Lagrangian multiplier method and we have

$$\nabla_x\mathcal{F}r_i=\lambda(\|r_i\|_p-\epsilon) \tag{15}$$

Then we make the first derivative with respect to r_i:

$$\nabla_x\mathcal{F}=\lambda\frac{r_i^{p-1}}{p(\sum_j(r_i^j)^p)^{1-\frac{1}{p}}} \tag{16}$$

$$\nabla_x\mathcal{F}=\frac{\lambda}{p}(\frac{r_i}{\epsilon})^{p-1}$$

$$(\nabla_x\mathcal{F})^{\frac{p}{p-1}}=(\frac{\lambda}{p})^{\frac{p}{p-1}}(\frac{r_i}{\epsilon})^p \tag{17}$$

If we sum over two sides, we have

$$\sum(\nabla_x\mathcal{F})^{\frac{p}{p-1}}=\sum(\frac{\lambda}{p})^{\frac{p}{p-1}}(\frac{r_i}{\epsilon})^p \tag{18}$$

$$\|\nabla_x\mathcal{F}\|_{p^*}^{p^*}=(\frac{\lambda}{p})^{p^*}*1 \tag{19}$$

where p^* is the dual of p, i.e. $\frac{1}{p}+\frac{1}{p^*}=1$. We have

$$(\frac{\lambda}{p})=\|\nabla_x\mathcal{F}\|_{p^*} \tag{20}$$

By combining (17) and (20), we have

$$\begin{aligned} r_i^* =& \epsilon \text{sgn}(\nabla_x \mathcal{F})(\frac{|\nabla_x \mathcal{F}|}{\|\nabla_x \mathcal{F}\|_{p^*}})^{\frac{1}{p-1}} \\ =& \epsilon \text{sgn}(\nabla_x L_d^0)(\frac{|\nabla_x L_d^0|}{\|\nabla_x L_d^0\|_{p^*}})^{\frac{1}{p-1}} \end{aligned} \tag{21}$$

In this paper, we set p to ∞. Then we have

$$\begin{aligned} r_i^* &= \epsilon \lim_{p \to \infty} sgn(\nabla_x L_d^0)(\frac{|\nabla_x L_d^0|}{\|\nabla_x L_d^0\|_{p^*}})^{\frac{1}{p-1}} \\ &= \epsilon \text{sgn}(\nabla_x L_d^0)(\frac{|\nabla_x L_d^0|}{\|\nabla_x L_d^0\|_1})^0 \\ &= \epsilon \text{sgn}(\nabla_x L_d^0) \end{aligned} \tag{22}$$

Then we can obtain the adversarial example:

$$x_i^* = x_i + \epsilon \text{sgn}(\nabla_x L_d^0) \tag{23}$$

To compute the transformed example, we need to solve the following problem:

$$r^* = \arg \min_{\|r\|_\infty \leq \epsilon} \log D_W^0(f_\theta(x + r)). \tag{24}$$

With the similar method, we can easily get the transformed example x^t

$$x^t = x - \epsilon \text{sgn}(\nabla_x \log D_W^0). \tag{25}$$