# On diffusion-based generative models and their error bounds: The log-concave case with full convergence estimates

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## Abstract

We provide full theoretical guarantees for the convergence behaviour of diffusion-based generative models under the assumption of strongly log-concave data distributions while our approximating class of functions used for score estimation is made of Lipschitz continuous functions avoiding any Lipschitzness assumption on the score function. We demonstrate via a motivating example, sampling from a Gaussian distribution with unknown mean, the powerfulness of our approach. In this case, explicit estimates are provided for the associated optimization problem, i.e. score approximation, while these are combined with the corresponding sampling estimates. As a result, we obtain the best known upper bound estimates in terms of key quantities of interest, such as the dimension and rates of convergence, for the Wasserstein-2 distance between the data distribution (Gaussian with unknown mean) and our sampling algorithm. Beyond the motivating example and in order to allow for the use of a diverse range of stochastic optimizers, we present our results using an $L^2$-accurate score estimation assumption, which crucially is formed under an expectation with respect to the stochastic optimizer and our novel auxiliary process that uses only known information. This approach yields the best known convergence rate for our sampling algorithm.

## 1 Introduction

Diffusion-based generative models, also known as score-based generative models (SGMs) (Song & Ermon, 2019; Song et al., 2021; Sohl-Dickstein et al., 2015; Ho et al., 2020), have become over the past few years one of the most popular approaches in generative modelling due to their empirical successes in data generation tasks. These models have achieved state-of-the-art results in image generation (Dhariwal & Nichol, 2021; Rombach et al., 2022), audio generation (Kong et al., 2020) and inverse problems (Chung et al., 2022; Song et al., 2022; Cardoso et al., 2024; Boys et al., 2023) outperforming other generative models like generative adversarial networks (GANs) (Goodfellow et al., 2014), variational autoencoders (VAEs) (Kingma & Welling, 2014), normalizing flows (Rezende & Mohamed, 2015) and energy-based methods (Zhao et al., 2017).

SGMs generate approximate data samples from high-dimensional data distributions by combining two diffusion processes, a forward and a backward in time process. The former process is used to iteratively and smoothly transform samples from the unknown data distribution into (Gaussian) noise, while the associated backward in time process reverses the noising procedure and generates new samples from the starting unknown data distribution. A key role in these models is played by the score function, i.e. the gradient of the log-density of the solution of the forward process, which appears in the drift of the stochastic differential equation (SDE) associated to the backward process. Since this quantity depends on the unknown data distribution, an estimator of the score has to be constructed during the noising step using score-matching techniques (Hyvärinen, 2005; Vincent, 2011). These techniques have the advantage of not suffering from known problems of traditional pushforward generative models, such as mode collapse (Salmona et al., 2022).

The widespread applicability and success of SGMs have been accompanied by a growing interest in the theoretical understandings of these models, particularly in their convergence theory e.g. in Bortoli (2022); Block et al. (2020); Lee et al. (2022; 2023); Chen et al. (2023a;d); Li et al. (2024); Conforti et al. (2023); Benton et al. (2024); Oko et al. (2023); Pedrotti et al. (2023). At its core, this new generative modeling approach combines optimisation and sampling procedures – specifically, the approximation of the score and the creation of new samples, which make its theoretical analysis both an interesting and a rich challenge.

Some of the recent advances in the study of the theoretical properties of SGMs concentrate around the sampling procedure by assuming suitable control for the score estimation procedure. For instance, the analysis in Lee et al. (2022); Chen et al. (2023d) assumes that the score estimate is $L^2$-*accurate*, meaning that the $L^2$ error between the score and its estimate is small, and provides estimates in total variation (TV) distance. Under the same assumption, the more recent contribution Conforti et al. (2023) establishes non-asymptotic bounds in Kullback Leibler (KL) divergence by assuming finite relative Fisher of data distribution with respect to a Gaussian distribution.

The main drawback of the aforementioned $L^2$-accurate (and in some cases $L^\infty$-accurate) score estimation assumption is that the corresponding expectation is given with respect to density of the solution of the forward process, which depends on the unknown data distribution.

Our approach introduces a novel auxiliary process that relies solely on known information and uses the density of the solution of this process in the $L^2$-accurate score estimation. To further highlight the powerfulness of our approach, we present a motivating example on the case of sampling from a Gaussian distribution with unknown mean. Full theoretical estimates for the convergence properties of the SGM are provided in Wasserstein-2 distance, while our choice of stochastic optimizer for the score approximation is a simple Langevin-based algorithm. Our estimates are explicit and contain the best known optimal dependencies in terms of dimension and rate of convergence (Theorem 1). To the best of the authors' knowledge, these are the first such explicit results with transparent dependence on the parameters involved in the sampling and optimization combined procedures of the diffusion models. By connecting the diffusion models with the theoretical guarantees of machine learning optimizers via standard stochastic calculus tools, the results in Theorem 1, together with the bounds achieved in the more general setting in Theorem 10, provide the theoretical justification for the empirical success of the diffusion models.

We close this introductory section by highlighting some other, alternative approaches which were recently developed. One may consult Yang & Wibisono (2023) for an approach based on the assumption that the score estimation error has a sub-Gaussian tail. This is a stronger assumption than $L^2$-*accurate*. In Chen et al. (2023b), non-asymptotic bounds in TV and Wassertein distance of order 2 are derived when the data distribution is supported on a low-dimensional linear subspace. Finally, convergence guarantees in TV were developed in Chen et al. (2023c) for the probability flow ordinary differential equation (ODE) implementation of SGMs under the $L^2$-accurate score estimate assumption.

*Notation.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space. We denote by $\mathbb{E}[X]$ the expectation of a random variable $X$. For $1 \leq p < \infty$, $L^p$ is used to denote the usual space of $p$-integrable real-valued random variables. The $L^p$-integrability of a random variable $X$ is defined as $\mathbb{E}[|X|^p] < \infty$. Fix an integer $M \geq 1$. For an $\mathbb{R}^M$-valued random variable $X$, its law on $\mathcal{B}(\mathbb{R}^M)$, i.e. the Borel sigma-algebra of $\mathbb{R}^M$ is denoted by $\mathcal{L}(X)$. Let $T > 0$ denotes some time horizon. For a positive real number $a$, we denote its integer part by $\lfloor a \rfloor$. The Euclidean scalar product is denoted by $\langle \cdot, \cdot \rangle$, with $|\cdot|$ standing for the corresponding norm (where the dimension of the space may vary depending on the context). Let $\mathbb{R}_{>0} := \{x \in \mathbb{R} |\ x > 0\}$. Let $f : \mathbb{R}^M \to \mathbb{R}$ be a continuously differentiable function. The gradient of $f$ is denote by $\nabla f$. For any integer $q \geq 1$, let $\mathcal{P}(\mathbb{R}^q)$ be the set of probability measures on $\mathcal{B}(\mathbb{R}^q)$. For $\mu,\ \nu \in \mathcal{P}(\mathbb{R}^M)$, let $\mathcal{C}(\mu, \nu)$ denote the set of probability measures $\zeta$ on $\mathcal{B}(\mathbb{R}^{2M})$ such that its respective marginals are $\mu$ and $\nu$. For any $\mu$ and $\nu \in \mathcal{P}(\mathbb{R}^M)$, the Wasserstein distance of order 2 is defined as

$$W_2(\mu, \nu) = \left( \inf_{\zeta \in \mathcal{C}(\mu,\nu)} \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} |x - y|^2 \ \mathrm{d}\zeta(x, y) \right)^{\frac{1}{2}}.$$

## 2 Technical Background

In this section, we provide a brief summary behind the construction of score-based generative models (SGMs) based on diffusion introduced in Song et al. (2021). The fundamental concept behind SGMs centers around the use of an ergodic (forward) diffusion process to diffuse the unknown data distribution $\pi_{\mathsf{D}} \in \mathcal{P}(\mathbb{R}^M)$ to a known prior distribution and then learn a backward process to transform the prior to the target distribution

$\pi_{\mathsf{D}}$ by estimating the score function of the forward process. In our analysis, we focus on the forward process $(X_t)_{t\in[0,T]}$ given by an Ornstein-Uhlenbeck (OU) process

$$\mathrm{d}X_t = -X_t \ \mathrm{d}t + \sqrt{2}\ \mathrm{d}B_t, \quad X_0 \sim \pi_{\mathsf{D}}, \tag{1}$$

where $(B_t)_{t\in[0,T]}$ is an $M$-dimensional Brownian motion and we assume that $\mathbb{E}[|X_0|^2] < \infty$. The process 1 is chosen to match the forward process in the original paper (Song et al., 2021), which is also referred to as Variance Preserving Stochastic Differential Equation. The noising process 1 can also be represented as follows

$$X_t \overset{\text{a.s.}}{=} m_t X_0 + \sigma_t Z_t, \quad m_t = e^{-t}, \quad \sigma_t^2 = 1 - e^{-2t}, \quad Z_t \sim \mathcal{N}(0, I_M), \tag{2}$$

where $\overset{\text{a.s.}}{=}$ denotes almost sure equality and $I_M$ denotes the identity matrix of dimension $M$. Under mild assumptions on the target data distribution $\pi_{\mathsf{D}}$ (Haussmann & Pardoux, 1986; Cattiaux et al., 2023), the backward process $(Y_t)_{t\in[0,T]} = (X_{T-t})_{t\in[0,T]}$ is given by

$$\mathrm{d}Y_t = (Y_t + 2\nabla \log p_{T-t}(Y_t))\ \mathrm{d}t + \sqrt{2}\ \mathrm{d}\bar{B}_t, \quad Y_0 \sim \mathcal{L}(X_T), \tag{3}$$

where $\{p_t\}_{t\in[0,T]}$ is the family of densities of $\{\mathcal{L}(X_t)\}_{t\in(0,T]}$ with respect to the Lebesgue measure, $\bar{B}_t$ is an another Brownian motion independent of $B_t$ in 1 defined on $(\Omega, \mathcal{F}, \mathbb{P})$. However, the sampling is done in practice from the invariant distribution of the forward process, which, in this case, is a standard Gaussian distribution. Therefore, the backward process 3 becomes

$$\mathrm{d}\widetilde{Y}_t = (\widetilde{Y}_t + 2\ \nabla \log p_{T-t}(\widetilde{Y}_t))\ \mathrm{d}t + \sqrt{2}\ \mathrm{d}\bar{B}_t, \quad \widetilde{Y}_0 \sim \pi_\infty = \mathcal{N}(0, I_M). \tag{4}$$

Here, we have $\widetilde{Y}_0 \overset{\text{a.s.}}{=} Z_T$. Since $\pi_{\mathsf{D}}$ is unknown, the score function $\nabla \log p_t$ in 3 cannot be computed exactly. To address this issue, an estimator $s(\cdot, \theta^*, \cdot)$ is *learned* based on a family of functions $s : [0,T]\times\mathbb{R}^d\times\mathbb{R}^M \to \mathbb{R}^M$ parametrized in $\theta$, aiming at approximating the score of the ergodic (forward) diffusion process over a fixed time window $[0,T]$. In practice, the functions $s$ are neural networks and in particular cases like the motivating example in Section 3.1, the functions $s$ can be wisely designed. The optimal value $\theta^*$ of the parameter $\theta$ is determined by optimizing the following score-matching objective

$$\mathbb{R}^d \ni \theta \mapsto \mathbb{E}\left[\int_0^T |\nabla \log p_t(X_t) - s(t, \theta, X_t)|^2 \ \mathrm{d}t\right]. \tag{5}$$

To account for numerical instability issues for training and sampling at $t = 0$ as observed in practice in Song et al. (2021, Appendix C) and for the possibility that the integral of the score function in 5 may diverge when $t = 0$ (see Appendix A), a discretised version of the score-matching optimization problem is usually considered

$$\text{minimize} \quad \mathbb{R}^d \ni \theta \mapsto U(\theta) := \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} |\nabla \log p_t(x) - s(t,\theta,x)|^2 \ p_t(x)\ \mathrm{d}x\ \mathrm{d}t, \tag{6}$$

where $\epsilon > 0$ and $\kappa : [0,T] \to \mathbb{R}_{>0}$ is a weighting function. The score-matching objective $U$ in 6 can be rewritten via denoising score matching (Vincent, 2011) as follows

$$U(\theta) = \mathbb{E}[\kappa(\tau)|\sigma_\tau^{-1} Z + s(\tau, \theta, m_\tau X_0 + \sigma_\tau Z)|^2] + C, \tag{7}$$

where the expectation is over $\tau \sim \text{Uniform}([\epsilon, T])$, $X_0 \sim \pi_{\mathsf{D}}$ and $Z \sim \mathcal{N}(0, I_M)$, and where $C \in \mathbb{R}$ is a constant independent of $\theta$ (see Appendix A for the derivation with the OU representation 26). The stochastic gradient $H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ of 6 deduced using 7 is given by

$$H(\theta, \mathbf{x}) = 2\kappa(t) \sum_{i=1}^M \left(\sigma_t^{-1} z^{(i)} + s^{(i)}(t, \theta, m_t x_0 + \sigma_t z)\right) \nabla_\theta s^{(i)}(t, \theta, m_t x_0 + \sigma_t z), \tag{8}$$

where $\mathbf{x} = (t, x_0, z) \in \mathbb{R}^m$ with $m = 2M + 1$. As contribution to this analysis, we introduce an auxiliary process $(Y_t^{\text{aux}})_{t\in[0,T]}$ containing the approximating function $s$ depending on the (random) estimator of $\theta^*$ denoted by $\hat{\theta}$, for $t \in [0,T]$,

$$\mathrm{d}Y_t^{\text{aux}} = (Y_t^{\text{aux}} + 2\ s(T-t, \hat{\theta}, Y_t^{\text{aux}}))\ \mathrm{d}t + \sqrt{2}\ \mathrm{d}\bar{B}_t, \quad Y_0^{\text{aux}} \sim \pi_\infty = \mathcal{N}(0, I_M). \tag{9}$$

The process 9 will play an important role in the derivation of the nonasymptotic estimates in Wasserstein distance of order two between the target data distribution and the generative distribution of the diffusion model. Indeed, it connects the backward process 4 and the numerical scheme 11, which facilitates the analysis of the convergence of the diffusion model (see Appendix B.2 and Appendix C.2 for more details). For this reason, we introduce a sequence of stepsizes $\{\gamma_k\}_{k\in\{0,\ldots,K\}}$ such that $\sum_{k=0}^{K}\gamma_k = T$. For any $k \in \{0,\ldots,K\}$, let $t_{k+1} = \sum_{j=0}^{k}\gamma_j$ with $t_0 = 0$ and $t_{K+1} = T$. Let $\gamma_k = \gamma \in (0,1)$ for each $k = 0,\ldots,K$. The discrete process $(Y_k^{\mathrm{EM}})_{k\in\{0,\ldots,K+1\}}$ of the Euler–Maruyama approximation of 9 is given, for any $k \in \{0,\ldots,K\}$, as follows

$$Y_{k+1}^{\mathrm{EM}} = Y_k^{\mathrm{EM}} + \gamma(Y_k^{\mathrm{EM}} + 2\ s(T-k,\hat{\theta},Y_k^{\mathrm{EM}})) + \sqrt{2\gamma}\ \bar{Z}_{k+1}, \quad Y_0^{\mathrm{EM}} \sim \pi_\infty = \mathcal{N}(0,I_M), \tag{10}$$

where $\{\bar{Z}_k\}_{k\in\{0,\ldots,K+1\}}$ is a sequence of independent $M$-dimensional Gaussian random variables with zero mean and identity covariance matrix. We emphasize that the approximation 10 is the one chosen in the original paper Song et al. (2021). Finally, the continuous-time interpolation of 10, for $t \in [0,T]$, is given by

$$\mathrm{d}\widehat{Y}_t^{\mathrm{EM}} = (\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}} + 2\ s(T-\lfloor t/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}))\ \mathrm{d}t + \sqrt{2}\ \mathrm{d}\bar{B}_t, \qquad \widehat{Y}_0^{\mathrm{EM}} \sim \pi_\infty = \mathcal{N}(0,I_M), \tag{11}$$

where $\mathcal{L}(\widehat{Y}_k^{\mathrm{EM}}) = \mathcal{L}(Y_k^{\mathrm{EM}})$ at grid points for each $k \in \{0,\ldots,K+1\}$.

## 3 Main Results

Before introducing the main assumptions of the paper, we present a motivating example.

### 3.1 A Motivating Example: Full Estimates for Multivariate Gaussian Initial Data with Unknown Mean

In this section, we consider the case where the data distribution follows a multivariate normal distribution with unknown mean and identity covariance, i.e., $X_0 \sim \pi_{\mathsf{D}} = \mathcal{N}(\mu, I_d)$ for some unknown $\mu \in \mathbb{R}^d$ with $M = d$. We show that, by using diffusion models, we are able to generate new data from an approximate distribution that is close to $\pi_{\mathsf{D}}$. More precisely, we provide a non-asymptotic convergence bound with explicit constants in Wasserstein-2 distance between the law of the diffusion model and $\pi_{\mathsf{D}}$, which can be made arbitrarily small by appropriately choosing key parameters on the upper bound. In this example, the estimation of the score function reduces to the estimation of the unknown mean by the methods of convex optimization with optimal dependence of the dimension combined with the most efficient sampling method for high-dimensional Gaussian data.

In this setting by using 1, we can derive the score function given by

$$\nabla \log p_t(x) = -x + m_t\mu, \tag{12}$$

which can be approximated using

$$s(t,\theta,x) = -x + m_t\theta, \quad (t,\theta,x) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{13}$$

To obtain an approximated score, i.e., to obtain an optimal value of $\theta$ in 13, we opt for a popular class of algorithms called stochastic gradient Langevin dynamics (SGLD) to solve the optimisation problem 6. In addition, we choose the weighting function $\kappa(t) = \sigma_t^2$ as in Song & Ermon (2019) and we set $\epsilon = 0$ in 6. Using the approximating function 13 in 8, we can obtain the following expression for the stochastic gradient

$$\begin{aligned} H(\theta,\mathbf{x}) &= 2\sigma_t^2 \sum_{i=1}^{d} \left(\sigma_t^{-1} z^{(i)} + (-m_t x_0^{(i)} - \sigma_t z^{(i)} + m_t\theta^{(i)})\right) m_t e_i \\ &= 2\sigma_t^2 m_t \left(\sigma_t^{-1} z - m_t x_0 - \sigma_t z + m_t\theta\right), \end{aligned} \tag{14}$$

where $\mathbf{x} = (t, x_0, z) \in \mathbb{R}^m$ with $m = 2d+1$ and $e_i$ denotes the unit vector with $i$-th entry being 1. Fixing the so-called inverse temperature parameter $\beta > 0$, the associated SGLD algorithm is given by

$$\theta_0^\lambda := \theta_0, \quad \theta_{n+1}^\lambda = \theta_n^\lambda - \lambda H(\theta_n^\lambda, \mathbf{X}_{n+1}) + \sqrt{2\lambda/\beta}\ \xi_{n+1}, \quad n \in \mathbb{N}_0, \tag{15}$$

where $\lambda > 0$ is often called the stepsize or gain of the algorithm, $(\xi_n)_{n\in\mathbb{N}_0}$ is a sequence of standard Gaussian vectors and

$$(\mathbf{X}_n)_{n\in\mathbb{N}_0} = (\tau_n, X_{0,n}, Z_n)_{n\in\mathbb{N}_0}, \tag{16}$$

is a sequence of i.i.d. random variables generated as follows. For each $n \in \mathbb{N}_0$, we sample $\tau_n$ from Uniform$([0,T])$ such that $\mathcal{L}(\tau_n) = \mathcal{L}(\tau)$, sample $X_{0,n}$ from $\pi_\mathrm{D} = \mathcal{N}(\mu, \mathrm{I}_d)$ such that $\mathcal{L}(X_{0,n}) = \mathcal{L}(X_0)$, and sample $Z_n$ from $\mathcal{N}(0, I_d)$ such that $\mathcal{L}(Z_n) = \mathcal{L}(Z)$. In addition, we consider the case where $\theta_0$, $(\xi_n)_{n\in\mathbb{N}_0}$, and $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ in 15 are independent and we have $\mathbb{E}[H(\theta, \mathbf{X}_{n+1})] = \nabla U(\theta)$.

Throughout this section, we fix

$$0 < \lambda \le \min\{\mathbb{E}[\sigma_\tau^2 m_\tau^2]/(4\mathbb{E}[\sigma_\tau^4 m_\tau^4]), 1/(2\mathbb{E}[\sigma_\tau^2 m_\tau^2])\}. \tag{17}$$

Table 1: Explicit expressions for the constants in Theorem 1.

| CONSTANT | FULL EXPRESSION |
|---|---|
| $C_{\mathsf{SGLD},1}$ | $1/\mathbb{E}[\sigma_\tau^2 m_\tau^2]$ |
| $C_{\mathsf{SGLD},2}$ | $4\mathbb{E}[\sigma_\tau^4 m_\tau^2(\sigma_\tau^{-1}\lvert Z\rvert + m_\tau\lvert X_0\rvert + \sigma_\tau\lvert Z\rvert + m_\tau\lvert\theta^*\rvert)^2]/\mathbb{E}[\sigma_\tau^2 m_\tau^2]$ |
| $T_\delta$ | $2^{-1}\ln\left(4\sqrt{2}\left(\sqrt{\mathbb{E}[\lvert X_0\rvert^2]} + \sqrt{d}\right)/\delta\right)$ |
| $\beta_\delta$ | $144d(\sqrt{4/3} + 2\sqrt{33})^2/(\delta^2\mathbb{E}[\kappa(\tau)m_\tau^2])$ |
| $\lambda_\delta$ | $\min\Big\{\mathbb{E}[\sigma_\tau^2 m_\tau^2]/(4\mathbb{E}[\sigma_\tau^4 m_\tau^4]), 1/(2\mathbb{E}[\sigma_\tau^2 m_\tau^2]), \delta^2\mathbb{E}[\sigma_\tau^2 m_\tau^2]$<br>$\times(576(\sqrt{4/3} + 2\sqrt{33})^2\mathbb{E}[\sigma_\tau^4 m_\tau^2\left(\sigma_\tau^{-1}\lvert Z\rvert + m_\tau\lvert X_0\rvert + \sigma_\tau\lvert Z\rvert + m_\tau\lvert\theta^*\rvert\right)^2])^{-1}\Big\}$ |
| $n_\delta$ | $(\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2])^{-1}\ln\left(12(\sqrt{4/3} + 2\sqrt{33})\sqrt{\mathbb{E}[\lvert\theta_0 - \theta^*\rvert^2]}/\delta\right)$ WITH FIXED $\lambda$ $(\le \lambda_\delta)$ |
| $\gamma_\delta$ | $\min\left\{\delta/(4(18d + 132\lvert\theta^*\rvert^2)^{1/2}), 1/2\right\}$ |

Theorem 1 states the non-asymptotic (upper) bounds between the generative distribution of the diffusion model $\mathcal{L}(\widehat{Y}_{K+1}^{\mathrm{EM}})$ and the data distribution $\pi_\mathsf{D}$. An overview of the proof can be found in Appendix B.2.

**Theorem 1.** *Under the setting described in this section, then, for any $T > 0$ and $\gamma \in (0, 1/2]$,*

$$\begin{aligned} &W_2(\mathcal{L}(\widehat{Y}_{K+1}^{EM}), \pi_\mathsf{D}) \\ &\le \sqrt{2}e^{-2T}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{d}) \\ &\quad + (\sqrt{4/3} + 2\sqrt{33})(e^{-n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\sqrt{\mathbb{E}[|\theta_0 - \theta^*|^2]} + \sqrt{dC_{\mathsf{SGLD},1}/\beta} + \sqrt{\lambda C_{\mathsf{SGLD},2}}) \\ &\quad + \gamma(\sqrt{18d} + \sqrt{132|\theta^*|^2}), \end{aligned} \tag{18}$$

*where $C_{\mathsf{SGLD},1}$ and $C_{\mathsf{SGLD},2}$ are given explicitly in Table 1. In addition, the result in 18 implies that for any $\delta > 0$, if we choose $T > T_\delta$, $\beta \ge \beta_\delta$, $0 < \lambda \le \lambda_\delta$, $n \ge n_\delta$, and $0 < \gamma < \gamma_\delta$, then*

$$W_2(\mathcal{L}(\widehat{Y}_{K+1}^{EM}), \pi_\mathsf{D}) < \delta,$$

*where $T_\delta, \beta_\delta, \lambda_\delta, n_\delta$ and $\gamma_\delta$ are given explicitly in Table 1.*

**Remark 2.** *The result in Theorem 1 achieves the optimal rate of convergence of order one for Euler or Milstein schemes of SDEs with constant diffusion coefficients. Furthermore, one notes that the dependence of the dimension on the upper bound in 18 is in the form of $\sqrt{d}$. To the best of the authors' knowledge, the result in Theorem 1 is the first convergence bound where the parameters involved in the sampling and optimization steps of the diffusion models appear explicitly. In the optimization procedure, we use SGLD algorithm 15 to solve the problem 6. Since the stochastic gradient $H$ in 14 is strongly convex by Proposition 12, it has been proved, for instance in Barkhagen et al. (2021), that, for large enough $\beta > 0$, the output of SGLD is an almost minimizer of 6 when $n$ is large. Thus, in the diffusion model, we set $\hat{\theta} = \theta_n^\lambda$ indicating*

*$\nabla \log p_t(x) \approx s(t, \hat{\theta}, x) = s(t, \theta_n^\lambda, x)$ for large values of $n$ and for all $t$ and $x$. Crucially, this allows us to use the established convergence results for SGLD to deduce a sampling upper bound for $W_2(\mathcal{L}(\widehat{Y}_{K+1}^{EM}), \pi_{\mathsf{D}})$ with explicit constants in 18. Consequently, this bound can be controlled by any given precision level $\delta > 0$ by appropriately choosing $T, \beta, \lambda, n$ and $\gamma$.*

This motivating example has focused on exploring the convergence of diffusion-based generative models using a Langevin-based algorithm, specifically SGLD, which is well-known for its theoretical guarantees in achieving global convergence. However, in the general case discussed in Section 3.2, we do not prescribe a specific optimizer to choose to minimise the distance between $\hat{\theta}$ and $\theta^*$.

### 3.2 General Case

In this section, we derive the full non-asymptotic estimates in Wasserstein distance of order two between the target data distribution $\pi_{\mathsf{D}}$ and the generative distribution of the diffusion model under the assumptions stated below. As explained in Section 2 (see also Appendix A), it could be necessary in the general setting to restrict $t \in [\epsilon, T]$ for $\epsilon \in (0, 1)$ in 6. Therefore, we truncate the integration in the backward diffusion at $t_K = T - \epsilon$ and run the process $(Y_t)_{t\in[0,T-\epsilon]}$, $(Y_t^{\mathrm{aux}})_{t\in[0,T-\epsilon]}$ and $(Y_k^{\mathrm{EM}})_{k\in\{0,\ldots,K\}}$ up to time $T - \epsilon$.

#### 3.2.1 Assumptions for the General Case

In the motivating example in Section 3.1, we have chosen the SGLD algorithm to solve the optimisation problem 6. Other algorithms, such as ADAM (Kingma & Ba, 2015), TheoPouLa (Lim & Sabanis, 2024) and stochastic gradient descent (Jentzen et al., 2021), can be chosen as long as they satisfy the following assumption. Fix $\epsilon > 0$.

**Assumption 1.** *Let $\theta^*$ be a minimiser of 6 and let $\hat{\theta}$ be the (random) estimator of $\theta^*$ obtained through some approximation procedure such that $\mathbb{E}[|\hat{\theta}|^4] < \infty$. There exists $\widetilde{\varepsilon}_{AL} > 0$ such that*

$$\mathbb{E}[|\hat{\theta} - \theta^*|^2] < \widetilde{\varepsilon}_{AL}.$$

**Remark 3.** *As a consequence of Assumption 1, we have $\mathbb{E}[|\hat{\theta}|^2] < 2\widetilde{\varepsilon}_{AL} + 2|\theta^*|^2$.*

We consider the following assumption on the data distribution.

**Assumption 2.** *The data distribution $\pi_{\mathsf{D}}$ has a finite second moment, is strongly log-concave, and $\int_\epsilon^T |\nabla \log p_t(0)|^2 \, dt < \infty$.*

**Remark 4.** *As a consequence of Assumption 2 and the preservation of strong log-concavity under convolution, see, e.g., Saumard & Wellner (2014, Proposition 3.7), there exists $L_{MO} : [0, T] \to (0, \infty]$ such that for all $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}^M$, we have*

$$\langle \nabla \log p_t(x) - \nabla \log p_t(\bar{x}), x - \bar{x} \rangle \leq -L_{MO}(t)|x - \bar{x}|^2. \tag{19}$$

*The function $L_{MO}(t)$ in 19 has a lower bound for all $t \in [0, T]$, which we denote by $\widehat{L}_{MO}$. Moreover, Assumption 2 with the estimate 19 implies that the processes in 3 and in 4 have a unique strong solution, see, e.g., Krylov (1991, Theorem 1).*

Next, we consider the following assumption on the approximating function $s$ and its gradient.

**Assumption 3.** *The function $s : [0, T] \times \mathbb{R}^d \times \mathbb{R}^M \to \mathbb{R}^M$ is continuously differentiable in $x \in \mathbb{R}^M$. Let $D_1 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$, $D_2 : [0, T] \times [0, T] \to \mathbb{R}_+$ and $D_3 : [0, T] \times [0, T] \to \mathbb{R}_+$ be such that $\int_\epsilon^T \int_\epsilon^T D_2(t, \bar{t}) \; dt \; d\bar{t} < \infty$ and $\int_\epsilon^T \int_\epsilon^T D_3(t, \bar{t}) \; dt \; d\bar{t} < \infty$. For $\alpha \in \left[\frac{1}{2}, 1\right]$ and for all $t, \bar{t} \in [0, T]$, $x, \bar{x} \in \mathbb{R}^M$, and $\theta, \bar{\theta} \in \mathbb{R}^d$, we have that*

$$|s(t, \theta, x) - s(\bar{t}, \bar{\theta}, \bar{x})| \leq D_1(\theta, \bar{\theta})|t - \bar{t}|^\alpha + D_2(t, \bar{t})|\theta - \bar{\theta}| + D_3(t, \bar{t})|x - \bar{x}|,$$

*where $D_1$, $D_2$ and $D_3$ have the following growth in each variable: i.e. there exist $\mathsf{K}_1$, $\mathsf{K}_2$, and $\mathsf{K}_3 > 0$ such that for each $t, \bar{t} \in [0, T]$ and $\theta, \bar{\theta} \in \mathbb{R}^d$,*

$$|D_1(\theta, \bar{\theta})| \leq \mathsf{K}_1(1 + |\theta| + |\bar{\theta}|), \qquad |D_2(t, \bar{t})| \leq \mathsf{K}_2(1 + |t|^\alpha + |\bar{t}|^\alpha),$$
$$|D_3(t, \bar{t})| \leq \mathsf{K}_3(1 + |t|^\alpha + |\bar{t}|^\alpha).$$

*Moreover, there exists* $\mathsf{K}_4 > 0$ *such that, for all* $x, \bar{x} \in \mathbb{R}^M$ *and for any* $k = 1, \dots M$,

$$|\nabla_x s^{(k)}(t, \theta, x) - \nabla_{\bar{x}} s^{(k)}(t, \theta, \bar{x})| \leq \mathsf{K}_4(1 + 2|t|^{\alpha})|x - \bar{x}|.$$

**Remark 5.** *Let* $\mathsf{K}_{Total} := \mathsf{K}_1 + \mathsf{K}_2 + \mathsf{K}_3 + |s(0,0,0)| > 0$. *Using Assumption 3, we have*

$$|s(t, \theta, x)| \leq \mathsf{K}_{Total}(1 + |t|^{\alpha})(1 + |\theta| + |x|).$$

We postpone the proof of Remark 5 to Appendix C.1.

**Remark 6.** *Assumption 3 imposes Lipschitz continuity on a family of approximating functions* $s(\cdot, \cdot, \cdot)$ *with respect to the input variables,* $t$ *and* $x$*, as well as the parameters* $\theta$*. It is well-known that the continuity properties of neural networks with respect to* $t$*, and* $x$ *are largely determined by the activation function at the last layer. For instance, neural networks with activation functions such as hyberbolic tangent and sigmoid functions at the last layer satisfy the Lipschitz continuity with respect to* $t$ *and* $x$ *(Virmaux & Scaman, 2018; Fazlyab et al., 2019).*

For the following assumption on the score approximation, we let $\hat{\theta}$ be as in Assumption 1 and we let $(Y_t^{\mathrm{aux}})_{t \in [0,T]}$ be the auxiliary process defined in 9.

**Assumption 4.** *There exists* $\varepsilon_{SN} > 0$ *such that*

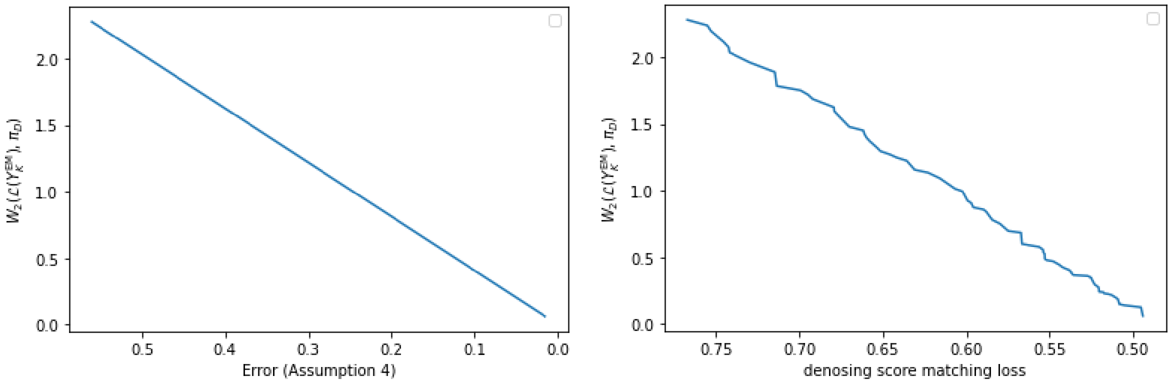
$$\mathbb{E} \int_0^{T-\epsilon} |\nabla \log p_{T-r}(Y_r^{aux}) - s(T - r, \hat{\theta}, Y_r^{aux})|^2 \, dr < \varepsilon_{SN}. \tag{20}$$

**Remark 7.** *We highlight that the expectation in Assumption 4 is taken over the auxiliary process 9. This density is known since the approximating function* $s$ *and the estimator* $\hat{\theta}$ *are known. To the best of authors' knowledge, this is a novelty with respect to previous works (Bortoli, 2022; Chen et al., 2023d; Lee et al., 2022; 2023; Chen et al., 2023a; Conforti et al., 2023; Benton et al., 2024) which consider the unknown density of the forward process (or its numerical discretization).*

**Remark 8.** *The motivating example in Section 3.1 is a special case of the general setting. Indeed, Assumption 1 is satisfied due to Lemma 13 with* $\hat{\theta} = \theta_n^{\lambda}$ *being the nth-iterate of the SGLD algorithm 14 and* $\theta^* := \mu$*. Assumption 2 is satisfied by the score function 12. Assumption 3 is satisfied by the approximating function 13 with* $d = M$*,* $\alpha = 1$*,* $D_1(\theta, \bar{\theta}) := |\bar{\theta}|$*,* $D_2(t, \bar{t}) := e^{-t}$*,* $D_3(t, \bar{t}) := 1$*, for any* $\theta, \bar{\theta} \in \mathbb{R}^d$*,* $t, \bar{t} \in [0, T]$ *and* $\mathsf{K}_1 = \mathsf{K}_2 = \mathsf{K}_3 = \mathsf{K}_4 = 1$*. Furthermore, both functions satisfy Assumption 4 with* $d = M$*. Indeed, by Assumption 1 with* $\theta^* = \mu$*, we have*

$$\begin{aligned}
&\mathbb{E} \int_0^{T-\epsilon} |\nabla \log p_{T-r}(Y_r^{aux}) - s(T - r, \hat{\theta}, Y_r^{aux})|^2 \, dr \\
&\leq 2 \, \mathbb{E} \int_0^{T-\epsilon} |\nabla \log p_{T-r}(Y_r^{aux}) - s(T - r, \theta^*, Y_r^{aux})|^2 \, dr \\
&\quad + 2 \, \mathbb{E} \int_0^{T-\epsilon} |s(T - r, \theta^*, Y_r^{aux}) - s(T - r, \hat{\theta}, Y_r^{aux})|^2 \, dr \\
&= (e^{-2\epsilon} - e^{-2T}) \, \mathbb{E}[|\hat{\theta} - \theta^*|^2] < (e^{-2\epsilon} - e^{-2T})\widetilde{\varepsilon}_{AL} < \varepsilon_{SN}.
\end{aligned}$$

**Remark 9.** *We conduct a numerical experiment to show the convergence of diffusion-based generative models under Assumption 1, 2, 3 and 4. We consider the case where* $X_0 \sim \pi_{\mathsf{D}} = \mathcal{N}(\mu, I_d)$ *with* $\mu = (-1.2347, -0.89244)$ *and* $d = 2$*. We use SGLD algorithm 15 to solve the optimization problem 7, with* $\kappa(t) = \sigma_t^2$*,* $\epsilon = 0$*,* $T = 2$*,* $\lambda = 5 \times 10^{-5}$*,* $\beta = 10^{12}$*, and* $n = 4 \times 10^4$*. At each iteration,* 128 *mini-batch samples are used to estimate the stochastic gradient. Then, we generate samples using the Euler-Maruyama approximation 10 with* $\gamma = 10^{-3}$ *and* $s$ *given in 13. At each iteration* $n$*, we evaluate the quality of* 100,000 *generated samples using the Wasserstein distance of order two. In addition, we compute the* $L^2$ *error between the score function and the approximated function* $s$ *with* $\theta_n^{\lambda}$*, using the auxiliary process as in Assumption 4. Figure 1 (a) demonstrates the error in Assumption 4 vanishes as the generative model converges, where the degree of convergence is measured by* $W_2(\mathcal{L}(Y_K^{EM}), \pi_{\mathsf{D}})$*. This empirical observation justifies Assumption 4. Moreover, we explore the relationship between the quality of generated samples and the error using 7 via denoising score matching in Figure 1 (b).*



(a) Error with Assumption 4.

(b) Error using $U(\theta)$ in 7.

Figure 1: The quality of generated samples with respect to (a) the error with Assumption 4 and (b) the error obtained through denoising score matching using $U(\theta)$ in 7.

### 3.2.2 Full Estimates for the General Case

The main result under the general setting is stated as follows. An overview of the proof can be found in Appendix C.2.

**Theorem 10.** *Let Assumptions 1, 2, 3 and 4 hold. Then, there exist constants* $C_1$, $C_2$, $C_3$ *and* $C_4 > 0$ *such that for any* $T > 0$ *and* $\gamma, \epsilon \in (0, 1)$,

$$W_2(\mathcal{L}(Y_K^{EM}), \pi_{\mathsf{D}}) \leq C_1\sqrt{\epsilon} + C_2 e^{-2\widehat{L}_{MO}(T-\epsilon)-\epsilon} + C_3(T, \epsilon)\sqrt{\varepsilon_{SN}} + C_4(T, \epsilon)\gamma^{\alpha}, \tag{21}$$

*where* $C_1$, $C_2$, $C_3$ *and* $C_4$ *are given explicitly in Table 3 (Appendix D). In addition, the result in 21 implies that for any* $\delta > 0$, *if we choose* $0 \leq \epsilon < \epsilon_\delta$, $T > T_\delta$, $0 < \varepsilon_{SN} < \varepsilon_{SN,\delta}$ *and* $0 < \gamma < \gamma_\delta$ *with* $\epsilon_\delta$, $T_\delta$, $\varepsilon_{SN,\delta}$, *and* $\gamma_\delta$ *given in Table 3, then*

$$W_2(\mathcal{L}(Y_K^{EM}), \pi_{\mathsf{D}}) < \ \delta.$$

**Remark 11.** *The error bounds in 21 are not as good as the ones provided in Theorem 1 due to the general form of the approximating function s. We emphasize that the optimal rate of convergence of order* $\alpha \in \left[\frac{1}{2}, 1\right]$ *for the Euler or Milstein scheme of SDEs with constant diffusion coefficients is achieved using the Lipschitz continuity on the derivative of s in Assumption 3. In the explicit expression of* $C_4$ *in Table 3, the dependence of the dimension is* $O(M)$ *due to numerical techniques from Kumar & Sabanis (2019) used in the proof of Theorem 10 to achieve the optimal rate of convergence.*

## 4 Related Work and Comparison

We describe assumptions and results of various existing works in our notation and framework to facilitate the comparisons with our results which are provided in 2-Wasserstein distance. Beyond being theoretical relevant, the choice of the use of this metric is motivated by its applications in generative modeling. A popular performance metric currently used to examine the quality of the images produced by generative models is the Fréchet Inception Distance (FID) which was originally introduced in Heusel et al. (2017). This metric measures the Fréchet distance between the distribution of generated samples and the distribution of real samples, assuming Gaussian distributions, which is equivalent to the Wasserstein distance of order two. Our results provided under this metric are therefore also relevant for practical applications.

We can classify the previous approaches based on their assumptions on two key quantities: (i) score approximation error and (ii) assumptions on the data distribution. Based on these approaches, we give a brief overview of some of the most relevant recent contributions in the field.

### 4.1 Score Approximation Assumptions

Most of the recent analysis of score-based generative models, see, e.g., Chen et al. (2023d); Lee et al. (2022; 2023); Chen et al. (2023a); Conforti et al. (2023); Benton et al. (2024), have considered assumptions (in $L^2$) on the absolute error of the following type, i.e. for any $k \in \{0, \ldots, N-1\}$, there exists $\underline{\varepsilon} > 0$

$$\mathbb{E}\left[|\nabla \log p_{t_k}(X_{t_k}) - s(t_k, \hat{\theta}, X_{t_k})|^2\right] \leq \underline{\varepsilon}, \tag{22}$$

where the expectation is taken with respect to the unknown $\{p_t\}_{t\in[0,T]}$ and $\hat{\theta}$ is a deterministic quantity. In Chen et al. (2023a); Conforti et al. (2023); Benton et al. (2024), the assumption 22 is written in integral (averaged) form. In Lee et al. (2023), the bound in 22 is not uniform over $t$, i.e. $\underline{\varepsilon}_t := \frac{\underline{\varepsilon}}{\sigma_t^4}$ and this allows the score function to grow as $\frac{1}{\sigma_t^2}$ as $t \to 0$. The authors in Conforti et al. (2023) use the relative $L^2$-score approximation error such as

$$\mathbb{E}\left[|2\nabla \log \tilde{p}_{t_k}(X_{t_k}) - \tilde{s}(t_k, \hat{\theta}, X_{t_k})|^2\right] \leq \underline{\varepsilon}\ \mathbb{E}\left[|2\nabla \log \tilde{p}_{t_k}(X_{t_k})|^2\right], \tag{23}$$

where the expectation is with respect to the unknown density of the law of $X_t$ against the Gaussian distribution $\pi_\infty(x)$, i.e. $\tilde{p}_t(x) := p_t(x)/\pi_\infty(x)$, and $\tilde{s}(t, \theta, x)$ in 23 is the approximating function for the score function of $\tilde{p}_t$. A pointwise assumption in Bortoli (2022), given by

$$|\nabla \log p_t(x) - s(t, \hat{\theta}, x)| \leq C(1 + |x|)/\sigma_t^2, \tag{24}$$

for $C \geq 0$, is considered under the manifold (compact) setting. The assumption 24 takes into account the explosive behaviour of the score function as $t \to 0$. In an attempt to obtain a weaker control than 24, the assumption Bortoli (2022, A5) is used, namely

$$\mathbb{E}\left[|\nabla \log p_{T-t_k}(Y_k^{\mathrm{EI}}) - s(T - t_k, \hat{\theta}, Y_k^{\mathrm{EI}})|^2\right] \leq C^2 \mathbb{E}\left[\left(1 + |Y_k^{\mathrm{EI}}|^2\right)\right]/\sigma_{T-t_k}^4. \tag{25}$$

We note that, unlike 22 and 23, the expectation in 25 is taken with respect to the algorithm $Y_k^{\mathrm{EI}}$ given by the exponential integrator (EI) discretization scheme[1], which has known density. However, the bounds of Bortoli (2022, Theorem H.1) in Wasserstein distance of order one derived under this assumption scale exponentially in problem parameters such as the diameter of the manifold $\mathfrak{M}$ and the inverse of the early stopping time parameter $\epsilon$, i.e. $\exp(O(\mathrm{diam}(\mathfrak{M})^2/\epsilon))$. Furthermore, an assumption similar to 25 is considered in a very recent and concurrent result[2] Gao et al. (2023):

$$\sup_{k=1,\ldots,K}\left(\mathbb{E}\left[|\nabla \log p_{T-(k-1)\eta}(Y_k^{\mathrm{EI}}) - s(T-(k-1)\eta, \hat{\theta}, Y_k^{\mathrm{EI}})|^2\right]\right)^{1/2} \leq \underline{\varepsilon},$$

where $\eta > 0$ is the stepsize and $T = K\eta$ with $K \in \mathbb{N}$.

We emphasize that the existing results in the literature do not take the expectation with respect to the stochastic optimizer $\hat{\theta}$. This, together with the use of our novel auxiliary process 9 that uses only known information, allows us to deduce state-of-the-art bounds in the Wasserstein distance of order two in the following sense. The bounds scale polynomially in the data dimension $M$, i.e. $O(\sqrt{M})$ in the motivating example (Theorem 1) and $O(M)$ in the general case (Theorem 10) while achieving the optimal rate of convergence $\gamma$ in both cases.

### 4.2 Assumptions on the Data Distribution

The vast majority of the results available in the literature are in KL divergence and TV distance. For two general data distributions $\mu$ and $\nu$, there is no known relationship between their KL divergence and their $W_2$. However, for strongly log-concave data distributions, a bound on the Wasserstein distance of order two

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[1] This analysis can be extended to the Euler-Maruyama numerical scheme in $Y_k^{\mathrm{EM}}$ 10.

[2] The concurrent paper Gao et al. (2023) appeared few days earlier when the first draft of this work was made available online.

in terms of KL divergence follows from an extension of Talagrand's inequality (Gozlan & Léonard, 2010, Corollary 7.2).

Some convergence results in different metrics can be deduced under the following types of assumptions. Convergence bounds in Wasserstein distance of order one with exponential complexity has been obtained in Bortoli (2022) under the so-called manifold hypothesis, namely assuming that the target distribution is supported on a lower-dimensional manifold or is given by some empirical distribution. Moreover, the results in TV distance in Lee et al. (2022) and in KL divergence Yang & Wibisono (2023) assumed that the data distribution satisfies a logarithmic Sobolev inequality and the score function is Lipschitz resulting in convergence bounds characterized by polynomial complexity. By replacing the requirement that the data distribution satisfies a functional inequality with the assumption that $\pi_{\mathrm{D}}$ has finite KL divergence with respect to the standard Gaussian and by assuming that the score function for the forward process is Lipschitz, the authors in Chen et al. (2023d) managed to derive bounds in TV distance which scale polynomially in all the problem parameters. By requiring only the Lipschitzness of the score at the initial time instead of the whole trajectory, the authors in Chen et al. (2023a, Theorem 2.5) managed to show, using an exponentially decreasing then linear step size, convergence bounds in KL divergence with quadratic dimensional dependence and logarithmic complexity in the Lipschitz constant. In the work by Benton et al. (2024), the authors provide convergence bounds in KL divergence that are linear in the data dimension up to logarithmic factors.

We summarise the results of Chen et al. (2023d;a); Benton et al. (2024) and compare them to ours in Table 2. A careful examination of Chen et al. (2023a, Proof of Theorem 2.2) and Benton et al. (2024, Proof of Theorem 1 and Corollary 1) reveals that the authors require to have uniqueness of solutions for the backward SDE 3. Therefore, additional integrability conditions on the score function depending on the type of uniqueness of solutions considered are still needed.

We emphasize that in Theorem 10, we do not assume the score function to be Lipschitz continuous with a uniformly bounded Lipschitz constant. This is particularly useful for future work in nonconvex settings, where the upper bound estimates will be independent of the (potentially large) Lipschitz constant of the score function, which could otherwise hide additional dimensional dependencies. The requirement $\int_{\epsilon}^{T} |\nabla \log p_t(0)|^2 \mathrm{d}t < \infty$ in Assumption 2 is weaker than the Lipschiz assumption on the score function, but it is still difficult to verify in practical applications.

As pointed out in Chen et al. (2023d, Section 4), some type of log-concavity assumption on the data distribution is needed to derive polynomial convergence rates in 2-Wasserstein distance. This justifies the need for our Assumption 2 under which bounds linear in the data dimension are derived in Theorem 10. The concurrent result Gao et al. (2023) has a similar assumption.

Table 2: Summary of previous bounds and our results. Bounds expressed in terms of the number of steps required to guarantee an error of at most $\delta$ in the stated metric. The variance -$\underline{\epsilon}$ Gaussian perturbation of the target data distribution is denoted by $\pi_{\mathrm{D}}^{\epsilon}$. All the bounds assume that $\pi_{\mathrm{D}}$ has finite second moments.

| Optimization estimate | Regularity condition | Metric | Complexity | Result |
|---|---|---|---|---|
| None | $\forall t$, $\nabla \log p_t$, $L$-Lipschitz | $\mathrm{TV}(\mathcal{L}(\widehat{Y}_{K+1}), \pi_{\mathrm{D}})$ | $\tilde{O}\left(\frac{ML^2}{\delta^2}\right)$ | (Chen et al., 2023d, Theorem 2) |
| None | $\forall t$, $\nabla \log p_t$ $L$-Lipschitz | $\sqrt{\mathrm{KL}(\mathcal{L}(\widehat{Y}^{\mathrm{EI}}_{K+1}) \vert\vert \pi_{\mathrm{D}})}$ | $\tilde{O}\left(\frac{ML^2}{\delta^2}\right)$ | (Chen et al., 2023a, Theorem 2.1) |
| None | Integrability conditions on $\nabla \log p_t$ for the uniqueness of $(Y_t)_{t\in[0,T]}$ | $\sqrt{\mathrm{KL}(\mathcal{L}(\widehat{Y}^{\mathrm{EI}}_{K}) \vert\vert \pi^{\epsilon}_{\mathrm{D}})}$ | $\tilde{O}\left(\frac{M^2 \log^2(1/\epsilon)}{\delta^2}\right)$ | (Chen et al., 2023a, Theorem 2.2) |
| None | Integrability conditions on $\nabla \log p_t$ for the uniqueness of $(Y_t)_{t\in[0,T]}$ | $\sqrt{\mathrm{KL}(\mathcal{L}(\widehat{Y}^{\mathrm{EI}}_{K}) \vert\vert \pi^{\epsilon}_{\mathrm{D}})}$ | $\tilde{O}\left(\frac{M \log^2(1/\epsilon)}{\delta^2}\right)$ | (Benton et al., 2024, Corollary 1) |
| Yes | $\pi_{\mathrm{D}} \sim \mathcal{N}(\mu, I_M)$ | $W_2(\mathcal{L}(\widehat{Y}^{\mathrm{EM}}_{K+1}), \pi_{\mathrm{D}})$ | $\tilde{O}(\sqrt{M}\log(\delta^{-1}))$ | Theorem 1 |
| Yes | Assumption 2 | $W_2(\mathcal{L}(\widehat{Y}^{\mathrm{EM}}_{K}), \pi_{\mathrm{D}})$ | $\tilde{O}(M^{1/\alpha}(\widetilde{\varepsilon}_{\mathrm{AL}}\epsilon)^{1/(2\alpha)}\log(\delta^{-1}))$ | Theorem 10 |

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# Appendix

## A Objective Function via Denoising Score Matching

In this section, we show that the objective function $U$ in 6 can be written into 7 using denoising score matching (Vincent, 2011) and the OU representation

$$X_t \overset{\mathrm{d}}{=} m_t X_0 + \sigma_t Z, \quad m_t = e^{-t}, \quad \sigma_t^2 = 1 - e^{-2t}, \quad Z \sim \mathcal{N}(0, I_M), \tag{26}$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution. We start by noticing that

$$\begin{aligned}
&\int_{\mathbb{R}^M} \langle \nabla \log p_t(x), s(t,\theta,x)\rangle \; p_t(x)\; \mathrm{d}x \\
&= \int_{\mathbb{R}^M} \langle \nabla p_t(x),\; s(t,\theta,x)\rangle \; \mathrm{d}x \\
&= \int_{\mathbb{R}^M} \left\langle \nabla_x \int_{\mathbb{R}^M} p_0(\tilde{x}) p_{t|0}(x|\tilde{x})\; \mathrm{d}\tilde{x},\; s(t,\theta,x) \right\rangle \; \mathrm{d}x \\
&= \int_{\mathbb{R}^M} \left\langle \int_{\mathbb{R}^M} p_0(\tilde{x}) p_{t|0}(x|\tilde{x}) \nabla_x \log(p_{t|0}(x|\tilde{x}))\; \mathrm{d}\tilde{x},\; s(t,\theta,x) \right\rangle \; \mathrm{d}x \\
&= \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} p_{t,0}(x,\tilde{x}) \left\langle \nabla_x \log(p_{t|0}(x|\tilde{x})),\; s(t,\theta,x) \right\rangle \; \mathrm{d}x\; \mathrm{d}\tilde{x},
\end{aligned} \tag{27}$$

where $p_{t|0}$ is the density of the transition kernel associated with 1 and $p_{t,0}$ is the joint density of $X_t$ and $X_0$. Using 27 in the objective function given in 6 and the OU representation 26, we have

$$\begin{aligned}
U(\theta) &= \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} \left( |\nabla \log p_t(x)|^2 - 2\langle \nabla \log p_t(x), s(t,\theta,x)\rangle + |s(t,\theta,x)|^2 \right) \; p_t(x)\; \mathrm{d}x\; \mathrm{d}t \\
&= \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} \Bigg( |\nabla \log p_t(x)|^2 - 2\left\langle \nabla_x \log(p_{t|0}(x|\tilde{x})),\; s(t,\theta,x) \right\rangle \\
&\qquad\qquad + |s(t,\theta,x)|^2 \Bigg) p_{t,0}(x,\tilde{x})\; \mathrm{d}x\; \mathrm{d}\tilde{x}\; \mathrm{d}t \\
&= \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} |\nabla_x \log p_{t|0}(x|\tilde{x}) - s(t,\theta,x)|^2 p_{t,0}(x,\tilde{x})\; \mathrm{d}x\; \mathrm{d}\tilde{x}\; \mathrm{d}t \\
&\quad + \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} |\nabla \log p_t(x)|^2 p_t(x)\; \mathrm{d}x\; \mathrm{d}t \\
&\quad - \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} |\nabla_x \log p_{t|0}(x|\tilde{x})|^2 p_{t,0}(x,\tilde{x})\; \mathrm{d}x\; \mathrm{d}\tilde{x}\; \mathrm{d}t \\
&= \mathbb{E}[\kappa(\tau)|\sigma_\tau^{-1} Z + s(\tau,\theta,m_\tau X_0 + \sigma_\tau Z)|^2] + \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} |\nabla \log p_t(x)|^2 p_t(x)\; \mathrm{d}x\; \mathrm{d}t \\
&\quad - \int_\epsilon^T \frac{\kappa(t)}{T-\epsilon} \int_{\mathbb{R}^M} \int_{\mathbb{R}^M} |\nabla_x \log p_{t|0}(x|\tilde{x})|^2 p_{t,0}(x,\tilde{x})\; \mathrm{d}x\; \mathrm{d}\tilde{x}\; \mathrm{d}t.
\end{aligned} \tag{28}$$

We emphasize that we may choose to restrict $t \in [\epsilon, T]$ with $\epsilon \in (0,1)$ to prevent the divergence of the integrals on the right-hand side of 28.

## B Proofs of the Results for the Multivariate Gaussian Initial Data with Unknown Mean

In this section, we provide the proof of Theorem 1. We start by introducing the results which will be used in the proof of Theorem 1.

### B.1 Preliminary Estimates

We provide the results for the SGLD algorithm 15 with $\lambda$ given in 17, $\beta > 0$, $C_{\mathsf{SGLD},1}$ and $C_{\mathsf{SGLD},2}$ given in Table 1, as well as for the auxiliary process 9, the discrete process 10, and the continuous-time interpolation 11 with $s$ given in 13 and $\gamma \in (0, 1/2]$.

**Proposition 12.** *For any $\theta, \bar{\theta} \in \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^m$,*

$$|H(\theta, \mathbf{x}) - H(\bar{\theta}, \mathbf{x})| = 2\sigma_t^2 m_t^2 |\theta - \bar{\theta}|,$$
$$\langle H(\theta, \mathbf{x}) - H(\bar{\theta}, \mathbf{x}), \theta - \bar{\theta} \rangle = 2\sigma_t^2 m_t^2 |\theta - \bar{\theta}|^2.$$

Proposition 12 is derived from the definition of the stochastic gradient 14.

The proof of the following lemmas are postponed to Section B.3.

**Lemma 13.** *For any $n \in \mathbb{N}_0$,*

$$\mathbb{E}\left[|\theta_n^\lambda - \theta^*|^2\right] \le (1 - 2\lambda \mathbb{E}[\sigma_\tau^2 m_\tau^2])^n \mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2}.$$

As a consequence of Lemma 13, we derive the following corollary.

**Corollary 14.** *For any $n \in \mathbb{N}_0$,*

$$\mathbb{E}\left[|\theta_n^\lambda|^2\right] \le 2e^{-2n\lambda \mathbb{E}[\sigma_\tau^2 m_\tau^2]} \mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + 2dC_{\mathsf{SGLD},1}/\beta + 2\lambda C_{\mathsf{SGLD},2} + 2|\theta^*|^2.$$

**Lemma 15.** *It holds that*

$$\sup_{t \ge 0} \mathbb{E}\left[|Y_t^{aux}|^2\right] \le C_{\mathsf{aux}},$$

*where $C_{\mathsf{aux}} := (8/3)(e^{-2n\lambda \mathbb{E}[\sigma_\tau^2 m_\tau^2]} \mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2) + 2d$.*

**Lemma 16.** *It holds that*

$$\sup_{k \in \mathbb{N}} \mathbb{E}\left[|Y_k^{EM}|^2\right] \le C_{\mathsf{EM}},$$

*where $C_{\mathsf{EM}} := 3d + 20(e^{-2n\lambda \mathbb{E}[\sigma_\tau^2 m_\tau^2]} \mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2)$.*

**Lemma 17.** *For any $0 < \gamma \le 1/2$, one obtains that*

$$\sup_{t \ge 0} \mathbb{E}\left[|\widehat{Y}_t^{EM} - \widehat{Y}_{\lfloor t/\gamma \rfloor \gamma}^{EM}|^2\right] \le \gamma C_{\mathsf{EMose}},$$

*where $C_{\mathsf{EMose}} := 8d + 56(e^{-2n\lambda \mathbb{E}[\sigma_\tau^2 m_\tau^2]} \mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2)$.*

**Lemma 18.** *It holds that*

$$\sup_{t \ge 0} \mathbb{E}\left[|\widehat{Y}_t^{EM}|^2\right] \le \widehat{C}_{\mathsf{EM}},$$

*where $\widehat{C}_{\mathsf{EM}} := 18d + 128(e^{-2n\lambda \mathbb{E}[\sigma_\tau^2 m_\tau^2]} \mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2)$.*

### B.2 Proof of the Main Result in the Motivating Example

*Proof of Theorem 1.* We derive the non-asymptotic estimate for $W_2(\mathcal{L}(\widehat{Y}_{K+1}^{\mathrm{EM}}), \pi_{\mathsf{D}})$ using the splitting

$$\begin{aligned} W_2(\mathcal{L}(Y_{K+1}^{\mathrm{EM}}), \pi_{\mathsf{D}}) \le{} & W_2(\pi_{\mathsf{D}}, \mathcal{L}(Y_{t_{K+1}})) + W_2(\mathcal{L}(Y_{t_{K+1}}), \mathcal{L}(\widetilde{Y}_{t_{K+1}})) \\ & + W_2(\mathcal{L}(\widetilde{Y}_{t_{K+1}}), \mathcal{L}(Y_{t_{K+1}}^{\mathrm{aux}})) + W_2(\mathcal{L}(Y_{t_{K+1}}^{\mathrm{aux}}), \mathcal{L}(Y_{K+1}^{\mathrm{EM}})). \end{aligned} \tag{29}$$

Since $t_{K+1} = T$, we have $W_2(\pi_{\mathsf{D}}, \mathcal{L}(Y_{t_{K+1}})) = 0$. We provide upper bounds on the error made by approximating the initial condition of the backward process $Y_0 \sim \mathcal{L}(X_T)$ with $\widetilde{Y}_0 \sim \pi_\infty$, i.e. $W_2(\mathcal{L}(Y_{t_{K+1}}), \mathcal{L}(\widetilde{Y}_{t_{K+1}}))$, the error made by approximating the score function with $s$, i.e. $W_2(\mathcal{L}(\widetilde{Y}_{t_{K+1}}), \mathcal{L}(Y^{\mathrm{aux}}_{t_{K+1}}))$, and the discretisation error, i.e. $W_2(\mathcal{L}(Y^{\mathrm{aux}}_{t_{K+1}}), \mathcal{L}(Y^{\mathrm{EM}}_{K+1}))$, separately.

**Upper bound on** $W_2(\mathcal{L}(Y_{t_{K+1}}), \mathcal{L}(\widetilde{Y}_{t_{K+1}}))$. Applying Itô's formula and using 3 and 4 with the score function given in 12, we have, for any $t \in [0, T]$,

$$\begin{aligned} \mathrm{d}|Y_t - \widetilde{Y}_t|^2 &= 2\langle Y_t - \widetilde{Y}_t, Y_t - \widetilde{Y}_t + 2(\nabla \log p_{T-t}(Y_t) - \nabla \log p_{T-t}(\widetilde{Y}_t))\rangle \mathrm{d}t \\ &= -2|Y_t - \widetilde{Y}_t|^2 \mathrm{d}t. \end{aligned} \tag{30}$$

Integrating and taking expectation both sides in 30 yields

$$\begin{aligned} \mathbb{E}[|Y_t - \widetilde{Y}_t|^2] &= \mathbb{E}[|Y_0 - \widetilde{Y}_0|^2] - 2\int_0^t \mathbb{E}[|Y_s - \widetilde{Y}_s|^2]\mathrm{d}s \\ &\leq e^{-2t}\mathbb{E}[|Y_0 - \widetilde{Y}_0|^2]. \end{aligned} \tag{31}$$

Using 31, the representation 2 with $Z_T \overset{\mathrm{d}}{=} \widetilde{Y}_0$ and $1 - \sigma_t \leq m_t$, we have

$$\begin{aligned} \mathbb{E}[|Y_{t_{K+1}} - \widetilde{Y}_{t_{K+1}}|^2] &\leq e^{-2t_{K+1}}\mathbb{E}[|Y_0 - \widetilde{Y}_0|^2] \\ &= e^{-2t_{K+1}}\mathbb{E}[|m_T X_0 + (\sigma_T - 1)\widetilde{Y}_0|^2] \\ &\leq 2e^{-4t_{K+1}}\left(\mathbb{E}[|X_0|^2] + d\right). \end{aligned} \tag{32}$$

Using 32, we have

$$\begin{aligned} W_2(\mathcal{L}(Y_{t_{K+1}}), \mathcal{L}(\widetilde{Y}_{t_{K+1}})) &\leq \sqrt{\mathbb{E}[|Y_{t_{K+1}} - \widetilde{Y}_{t_{K+1}}|^2]} \\ &\leq \sqrt{2}e^{-2T}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{d}). \end{aligned} \tag{33}$$

**Upper bound on** $W_2(\mathcal{L}(\widetilde{Y}_{t_{K+1}}), \mathcal{L}(Y^{\mathbf{aux}}_{t_{K+1}}))$. Applying Itô's formula and using the process 4 with the score function 12 and the process 9 with the approximating function 13, we have, for any $t \in [0, T]$,

$$\begin{aligned} \mathrm{d}|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2 &= 2\langle \widetilde{Y}_t - Y^{\mathrm{aux}}_t, \widetilde{Y}_t - Y^{\mathrm{aux}}_t + 2(\nabla \log p_{T-t}(\widetilde{Y}_t) - s(T-t, \hat{\theta}, Y^{\mathrm{aux}}_t))\rangle \mathrm{d}t \\ &= -2|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2 \mathrm{d}t + 4\langle \widetilde{Y}_t - Y^{\mathrm{aux}}_t, m_{T-t}(\mu - \hat{\theta})\rangle \mathrm{d}t. \end{aligned} \tag{34}$$

Integrating and taking expectation on both sides in 34 and using that the minimiser $\theta^\star = \mu$, we have

$$\mathbb{E}[|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2] = -2\int_0^t \mathbb{E}[|\widetilde{Y}_s - Y^{\mathrm{aux}}_s|^2]\mathrm{d}s + 4\int_0^t \mathbb{E}[\langle \widetilde{Y}_s - Y^{\mathrm{aux}}_s, m_{T-s}(\theta^* - \hat{\theta})\rangle]\mathrm{d}s. \tag{35}$$

Differentiating both sides of 35 and using Young's inequality, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2] &= -2\mathbb{E}[|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2] + 4\mathbb{E}[\langle \widetilde{Y}_t - Y^{\mathrm{aux}}_t, m_{T-t}(\theta^* - \hat{\theta})\rangle] \\ &\leq -\mathbb{E}[|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2] + 4e^{-2T}e^{2t}\mathbb{E}[|\theta^* - \hat{\theta}|^2], \end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^t\mathbb{E}[|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2]) \leq 4e^{-2T}e^{3t}\mathbb{E}[|\theta^* - \hat{\theta}|^2].$$

Integrating both sides and using Lemma 13 yields

$$\begin{aligned} &\mathbb{E}[|\widetilde{Y}_t - Y^{\mathrm{aux}}_t|^2] \\ &\leq (4/3)e^{-2T}(e^{2t} - e^{-t})\mathbb{E}[|\theta^* - \hat{\theta}|^2] \\ &\leq (4/3)e^{-2(T-t)}(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2}). \end{aligned} \tag{36}$$

Using 36, we have

$$
\begin{aligned}
W_2(\mathcal{L}(\widetilde{Y}_{t_{K+1}}), \mathcal{L}(Y^{\mathrm{aux}}_{t_{K+1}})) &\leq \sqrt{\mathbb{E}[|\widetilde{Y}_{t_{K+1}} - Y^{\mathrm{aux}}_{t_{K+1}}|^2]} \\
&\leq \sqrt{4/3}(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2})^{1/2}.
\end{aligned}
\tag{37}
$$

**Upper bound on $W_2(\mathcal{L}(Y^{\mathbf{aux}}_{t_{K+1}}), \mathcal{L}(\widehat{Y}^{\mathbf{EM}}_{K+1}))$.** Applying Itô's formula and using the processes 9 and 11 with the approximating function $s$ given in 13, we have, for any $t \in [0, T]$,

$$
\begin{aligned}
\mathrm{d}|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2 &= 2\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, Y_t^{\mathrm{aux}} - \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor \gamma}\rangle \mathrm{d}t \\
&\quad + 4\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, s(T-t, \hat{\theta}, Y_t^{\mathrm{aux}}) - s(T - \lfloor t/\gamma \rfloor \gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor \gamma})\rangle \mathrm{d}t \\
&= -2\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, Y_t^{\mathrm{aux}} - \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor \gamma}\rangle \mathrm{d}t \\
&\quad + 4\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, (m_{T-t} - m_{T - \lfloor t/\gamma \rfloor \gamma})\hat{\theta}\rangle \mathrm{d}t.
\end{aligned}
\tag{38}
$$

Integrating both sides and taking expectation in 38 yields

$$
\begin{aligned}
\mathbb{E}\left[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2\right] &= -2\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right] \mathrm{d}s - 2\int_0^t \mathbb{E}\left[\left\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, \widehat{Y}_s^{\mathrm{EM}} - \widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma \rfloor \gamma}\right\rangle\right] \mathrm{d}s \\
&\quad + 4\int_0^t \mathbb{E}\left[\left\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, (m_{T-s} - m_{T-\lfloor s/\gamma \rfloor \gamma})\hat{\theta}\right\rangle\right] \mathrm{d}s.
\end{aligned}
\tag{39}
$$

Differentiating both sides in 39, using Young's inequality and $m_{T-t} - m_{T-\lfloor t/\gamma \rfloor \gamma} \leq \gamma m_{T-t}$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] &= -2\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] - 2\mathbb{E}\left[\left\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\int_{\lfloor t/\gamma \rfloor \gamma}^t \mathrm{d}\overline{B}_s\right\rangle\right] \\
&\quad - 2\mathbb{E}\left[\left\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, \int_{\lfloor t/\gamma \rfloor \gamma}^t (-\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma \rfloor \gamma} + 2m_{T-\lfloor s/\gamma \rfloor \gamma}\hat{\theta})\mathrm{d}s\right\rangle\right] \\
&\quad + 4\mathbb{E}[\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, (m_{T-t} - m_{T-\lfloor t/\gamma \rfloor \gamma})\hat{\theta}\rangle] \\
&\leq -\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] - 2\mathbb{E}\left[\left\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\int_{\lfloor t/\gamma \rfloor \gamma}^t \mathrm{d}\overline{B}_s\right\rangle\right] \\
&\quad + 2\gamma^2\mathbb{E}[(-\widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor \gamma} + 2m_{T-\lfloor t/\gamma \rfloor \gamma}\hat{\theta})^2] + 8\gamma^2 e^{-2(T-t)}\mathbb{E}[|\hat{\theta}|^2] \\
&\leq -\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] - 2\mathbb{E}\left[\left\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\int_{\lfloor t/\gamma \rfloor \gamma}^t \mathrm{d}\overline{B}_s\right\rangle\right] \\
&\quad + 4\gamma^2\mathbb{E}[|\widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor \gamma}|^2] + 24\gamma^2\mathbb{E}[|\hat{\theta}|^2].
\end{aligned}
\tag{40}
$$

We derive an upper bound for the second term on the right-hand side in 40. Using Cauchy-Schwarz inequality, Itô formula applied to $tB_t$ and Young's inequality, we have

$$\begin{aligned}
&-2\mathbb{E}\left[\left\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&= -2\mathbb{E}\left[\left\langle (Y_t^{\mathrm{aux}} - Y_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{aux}}) - (\widehat{Y}_t^{\mathrm{EM}} - \widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}), \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&= -2\mathbb{E}\left[\left\langle \int_{\lfloor t/\gamma\rfloor\gamma}^{t} (-Y_s^{\mathrm{aux}} + 2m_{T-s}\hat{\theta})\mathrm{d}s, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&\quad + 2\mathbb{E}\left[\left\langle \int_{\lfloor t/\gamma\rfloor\gamma}^{t} (-\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}} + 2m_{T-\lfloor s/\gamma\rfloor\gamma}\hat{\theta})\mathrm{d}s, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&= 2\mathbb{E}\left[\left\langle \int_{\lfloor t/\gamma\rfloor\gamma}^{t} (Y_s^{\mathrm{aux}} - \widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})\mathrm{d}s, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&= 2\mathbb{E}\left[\left\langle \int_{\lfloor t/\gamma\rfloor\gamma}^{t} (Y_s^{\mathrm{aux}} - Y_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{aux}})\mathrm{d}s, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&= 2\mathbb{E}\left[\left\langle \int_{\lfloor t/\gamma\rfloor\gamma}^{t} \int_{\lfloor s/\gamma\rfloor\gamma}^{s} (-Y_\nu^{\mathrm{aux}})\mathrm{d}\nu\mathrm{d}s, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&\quad + 2\mathbb{E}\left[\left\langle \int_{\lfloor t/\gamma\rfloor\gamma}^{t} \sqrt{2}\int_{\lfloor s/\gamma\rfloor\gamma}^{s} \mathrm{d}\overline{B}_\nu\mathrm{d}s, \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&\leq 2\sqrt{2}\left(\mathbb{E}\left[\left|\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \int_{\lfloor s/\gamma\rfloor\gamma}^{s} (-Y_\nu^{\mathrm{aux}})\mathrm{d}\nu\mathrm{d}s\right|^2\right]\right)^{1/2} \left(\mathbb{E}\left[\left|\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s\right|^2\right]\right)^{1/2} \\
&\quad + 4\mathbb{E}\left[\left\langle t\int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s - \int_{\lfloor t/\gamma\rfloor\gamma}^{t} s\mathrm{d}\overline{B}_s, \int_{\lfloor t/\gamma\rfloor\gamma}^{t} \mathrm{d}\overline{B}_s \right\rangle\right] \\
&\leq \sqrt{2}\gamma^{5/2}\left(\sup_{t\geq 0}\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right] + d\right) + 2d\gamma^2.
\end{aligned} \tag{41}$$

Substituting 41 into 40 yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] \leq{}& -\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] + \sqrt{2}\gamma^{5/2}\sup_{t\geq 0}\mathbb{E}[|Y_t^{\mathrm{aux}}|^2] \\
&+ \sqrt{2}d\gamma^{5/2} + 2d\gamma^2 + 4\gamma^2\mathbb{E}[|\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}|^2] + 24\gamma^2\mathbb{E}[|\hat{\theta}|^2].
\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^t\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2]) \leq e^t\gamma^2\left(\sqrt{2}\sup_{t\geq 0}\mathbb{E}[|Y_t^{\mathrm{aux}}|^2] + 3d + 4\mathbb{E}[|\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}|^2] + 24\mathbb{E}[|\hat{\theta}|^2]\right). \tag{42}$$

Integrating both sides in 42 and using Lemma 15, Lemma 16 and Corollary 14, yield

$$\begin{aligned}
&\mathbb{E}[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2] \\
&\leq \sqrt{2}\gamma^2(2d + (8/3)(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2)) \\
&\quad + 3d\gamma^2 + 4\gamma^2(3d + 20(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2)) \\
&\quad + 48\gamma^2(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2) \\
&\leq 18d\gamma^2 + 132\gamma^2(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2).
\end{aligned} \tag{43}$$

By using 43, we have

$$
\begin{aligned}
&W_2(\mathcal{L}(Y^{\mathrm{aux}}_{t_{K+1}}), \mathcal{L}(\widehat{Y}^{\mathrm{EM}}_{K+1})) \\
&\leq \sqrt{\mathbb{E}[|Y^{\mathrm{aux}}_{t_{K+1}} - \widehat{Y}^{\mathrm{EM}}_{t_{K+1}}|^2]} \\
&\leq \gamma(18d + 132(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2))^{1/2}.
\end{aligned}
\tag{44}
$$

**Final upper bound on $W_2(\mathcal{L}(\widehat{Y}^{\mathbf{EM}}_{K+1}), \pi_{\mathsf{D}})$.** Substituting 33, 37, 44 into 29 yields

$$
\begin{aligned}
&W_2(\mathcal{L}(\widehat{Y}^{\mathrm{EM}}_{K+1}), \pi_{\mathsf{D}}) \\
&\leq \sqrt{2}e^{-2T}(\sqrt{\mathbb{E}\left[|X_0|^2\right]} + \sqrt{d}) \\
&\quad + \sqrt{4/3}(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2})^{1/2} \\
&\quad + \gamma(18d + 132(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2))^{1/2} \\
&\leq \sqrt{2}e^{-2T}(\sqrt{\mathbb{E}\left[|X_0|^2\right]} + \sqrt{d}) \\
&\quad + (\sqrt{4/3} + 2\sqrt{33})(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2})^{1/2} \\
&\quad + \gamma(18d + 132|\theta^*|^2)^{1/2}.
\end{aligned}
\tag{45}
$$

The bound for $W_2(\mathcal{L}(\widehat{Y}^{\mathrm{EM}}_{K+1}), \pi_{\mathsf{D}})$ in 45 can be made arbitrarily small by appropriately choosing parameters including $T, \beta, \lambda, n$ and $\gamma$. More precisely, for any $\delta > 0$, we first choose $T > T_\delta$ with $T_\delta$ given explicitly in Table 1 such that

$$
\sqrt{2}e^{-2T}(\sqrt{\mathbb{E}\left[|X_0|^2\right]} + \sqrt{d}) < \delta/4. \tag{46}
$$

Next, we choose $\beta \geq \beta_\delta$ and $0 < \lambda \leq \lambda_\delta$ with $\beta_\delta$ and $\lambda_\delta$ given explicitly in Table 1, and, in the case where $\lambda = \lambda_\delta$, we choose $n \geq n_\delta$ with $n_\delta$ given explicitly in Table 1 such that

$$
\begin{aligned}
&(\sqrt{4/3} + 2\sqrt{33})(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}[|\theta_0 - \theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2})^{1/2} \\
&\leq (\sqrt{4/3} + 2\sqrt{33})\sqrt{d/(\beta\mathbb{E}[\sigma_\tau^2 m_\tau^2])} \\
&\quad + (\sqrt{4/3} + 2\sqrt{33})(4\lambda\mathbb{E}[\sigma_\tau^4 m_\tau^2(\sigma_\tau^{-1}|Z| + m_\tau|X_0| + \sigma_\tau|Z| + m_\tau|\theta^*|)^2]/(\mathbb{E}[\sigma_\tau^2 m_\tau^2]))^{1/2} \\
&\quad + (\sqrt{4/3} + 2\sqrt{33})e^{-n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\sqrt{\mathbb{E}[|\theta_0 - \theta^*|^2]} \\
&\leq \delta/12 + \delta/12 + \delta/12 = \delta/4.
\end{aligned}
\tag{47}
$$

Finally, we choose $0 < \gamma < \gamma_\delta$ with $\gamma_\delta$ given explicitly in Table 1 such that

$$
\gamma(18d + 132|\theta^*|^2)^{1/2} < \delta/4. \tag{48}
$$

Using 46, 47 and 48 in 45, we obtain $W_2(\mathcal{L}(\widehat{Y}^{\mathrm{EM}}_{K+1}), \pi_{\mathsf{D}}) < \delta$. □

### B.3 Proof of the Preliminary Results

We provide the proofs of the results of Appendix B.1.

*Proof of Lemma 13.* By 15, we have for any $n \in \mathbb{N}_0$,

$$
\begin{aligned}
|\theta^\lambda_{n+1} - \theta^*|^2 &= \left|\theta^\lambda_n - \theta^* - \lambda H(\theta^\lambda_n, \mathbf{X}_{n+1}) + \sqrt{2\lambda/\beta}\ \xi_{n+1}\right|^2 \\
&= \left|\theta^\lambda_n - \theta^*\right|^2 + 2\left\langle \theta^\lambda_n - \theta^*, -\lambda H(\theta^\lambda_n, \mathbf{X}_{n+1}) + \sqrt{2\lambda/\beta}\ \xi_{n+1}\right\rangle \\
&\quad + \left|-\lambda H(\theta^\lambda_n, \mathbf{X}_{n+1}) + \sqrt{2\lambda/\beta}\xi_{n+1}\right|^2 \\
&= \left|\theta^\lambda_n - \theta^*\right|^2 - 2\lambda\left\langle \theta^\lambda_n - \theta^*, H(\theta^\lambda_n, \mathbf{X}_{n+1}) - H(\theta^*, \mathbf{X}_{n+1})\right\rangle \\
&\quad - 2\lambda\left\langle \theta^\lambda_n - \theta^*, H(\theta^*, \mathbf{X}_{n+1})\right\rangle + 2\sqrt{2\lambda/\beta}\left\langle \theta^\lambda_n - \theta^*, \xi_{n+1}\right\rangle + \lambda^2\left|H(\theta^\lambda_n, \mathbf{X}_{n+1})\right|^2 \\
&\quad - 2\lambda\sqrt{2\lambda/\beta}\left\langle H(\theta^\lambda_n, \mathbf{X}_{n+1}), \xi_{n+1}\right\rangle + (2\lambda/\beta)\left|\xi_{n+1}\right|^2.
\end{aligned}
\tag{49}
$$

Taking conditional expectation on both sides in 49 yields

$$\begin{aligned}\mathbb{E}\left[|\theta_{n+1}^{\lambda}-\theta^*|^2 \mid \theta_n^{\lambda}\right] &= \left|\theta_n^{\lambda}-\theta^*\right|^2 - 2\lambda\mathbb{E}\left[\left\langle \theta_n^{\lambda}-\theta^*, H(\theta_n^{\lambda},\mathbf{X}_{n+1})-H(\theta^*,\mathbf{X}_{n+1})\right\rangle \mid \theta_n^{\lambda}\right] \\ &\quad + 2\lambda^2\mathbb{E}\left[\left|H(\theta_n^{\lambda},\mathbf{X}_{n+1})-H(\theta^*,\mathbf{X}_{n+1})\right|^2 \mid \theta_n^{\lambda}\right] \\ &\quad + 2\lambda^2\mathbb{E}\left[\left|H(\theta^*,\mathbf{X}_{n+1})\right|^2 \mid \theta_n^{\lambda}\right] + 2\lambda d/\beta.\end{aligned}$$

Recall that $0 < \lambda \leq \min\{\mathbb{E}[\sigma_\tau^2 m_\tau^2]/(4\mathbb{E}[\sigma_\tau^4 m_\tau^4]), 1/(2\mathbb{E}[\sigma_\tau^2 m_\tau^2])\}$. Using Proposition 12 and the stochastic gradient 14, we have

$$\begin{aligned}\mathbb{E}\left[|\theta_{n+1}^{\lambda}-\theta^*|^2 \mid \theta_n^{\lambda}\right] &\leq \left(1-2\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]\right)|\theta_n^{\lambda}-\theta^*|^2 + 2\lambda d/\beta \\ &\quad - 2\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]|\theta_n^{\lambda}-\theta^*|^2 + 8\lambda^2\mathbb{E}\left[\sigma_\tau^4 m_\tau^4\right]\left|\theta_n^{\lambda}-\theta^*\right|^2 \\ &\quad + 8\lambda^2\mathbb{E}\left[\sigma_\tau^4 m_\tau^2\left(\sigma_\tau^{-1}|Z| + m_\tau|X_0| + \sigma_\tau|Z| + m_\tau|\theta^*|\right)^2\right] \\ &\leq \left(1-2\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]\right)|\theta_n^{\lambda}-\theta^*|^2 + 2\lambda d/\beta \\ &\quad + 8\lambda^2\mathbb{E}\left[\sigma_\tau^4 m_\tau^2\left(\sigma_\tau^{-1}|Z| + m_\tau|X_0| + \sigma_\tau|Z| + m_\tau|\theta^*|\right)^2\right].\end{aligned}$$

This implies that

$$\mathbb{E}\left[|\theta_{n+1}^{\lambda}-\theta^*|^2\right] \leq \left(1-2\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]\right)^{n+1}\mathbb{E}\left[|\theta_0-\theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2}.$$

□

*Proof of Lemma 15.* Applying Itô's formula and using the process 9 with $s$ given in 13, we have, for any $t \in [0,T]$,

$$\mathrm{d}|Y_t^{\mathrm{aux}}|^2 = -2|Y_t^{\mathrm{aux}}|^2\mathrm{d}t + 2\langle Y_t^{\mathrm{aux}}, 2m_{T-t}\hat{\theta}\rangle\mathrm{d}t + 2\langle Y_t^{\mathrm{aux}}, \sqrt{2}\mathrm{d}\overline{B}_t\rangle + 2d\mathrm{d}t.$$

Integrating both sides and taking expectation, we have

$$\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right] = -2\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}}|^2\right]\mathrm{d}s + 2\int_0^t \mathbb{E}[\langle Y_s^{\mathrm{aux}}, 2m_{T-s}\hat{\theta}\rangle]\mathrm{d}s + \mathbb{E}\left[|Y_0^{\mathrm{aux}}|^2\right] + 2dt.$$

Then, differentiating both sides, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right] &= -2\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right] + 2\mathbb{E}[\langle Y_t^{\mathrm{aux}}, 2m_{T-t}\hat{\theta}\rangle] + 2d \\ &\leq -\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right] + 4m_{T-t}^2\mathbb{E}[|\hat{\theta}|^2] + 2d,\end{aligned}$$

which, by rearranging the terms, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^t\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right]) \leq 4e^{-2T}e^{3t}\mathbb{E}[|\hat{\theta}|^2] + 2de^t.$$

Integrating both side and using Corollary 14, we obtain

$$\begin{aligned}\mathbb{E}\left[|Y_t^{\mathrm{aux}}|^2\right] &\leq e^{-t}(\mathbb{E}\left[|Y_0^{\mathrm{aux}}|^2\right] + (4/3)e^{-2T}(e^{3t}-1)\mathbb{E}[|\hat{\theta}|^2] + 2d(e^t-1)) \\ &\leq d(2-e^{-t}) + (4/3)e^{-2T}(e^{2t}-e^{-t})\mathbb{E}[|\hat{\theta}|^2] \\ &\leq 2d + e^{-2(T-t)}(8/3)(e^{-2n\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]}\mathbb{E}[|\theta_0-\theta^*|^2] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2).\end{aligned}$$

□

*Proof of Lemma 16.* Using the process 10 with the approximating function $s$ given in 13, we have

$$\begin{aligned}|Y_{k+1}^{\mathrm{EM}}|^2 &= |Y_k^{\mathrm{EM}} + \gamma(-Y_k^{\mathrm{EM}} + 2m_{T-k}\hat{\theta})|^2 + 2\gamma|\bar{Z}_{k+1}|^2 \\ &\quad + 2\langle Y_k^{\mathrm{EM}} + \gamma(-Y_k^{\mathrm{EM}} + 2m_{T-k}\hat{\theta}), \sqrt{2\gamma}\bar{Z}_{k+1}\rangle.\end{aligned} \tag{50}$$

Taking conditional expectation on both sides in 50, using the independence of $Y_k^{\mathrm{EM}}$ and $\bar{Z}_{k+1}$, Young's inequality and $0 < \gamma \leq 1/2$, we obtain

$$\begin{aligned}
\mathbb{E}\left[|Y_{k+1}^{\mathrm{EM}}|^2 \mid Y_k^{\mathrm{EM}}\right] &= |Y_k^{\mathrm{EM}}|^2 + 2\langle Y_k^{\mathrm{EM}}, \gamma(-Y_k^{\mathrm{EM}} + 2m_{T-t_k}\hat{\theta})\rangle \\
&\quad + \gamma^2|Y_k^{\mathrm{EM}}|^2 + 4\gamma^2 m_{T-k}^2|\hat{\theta}|^2 - 2\gamma^2\langle Y_k^{\mathrm{EM}}, 2m_{T-t_k}\hat{\theta}\rangle + 2\gamma d \\
&= (1-2\gamma)|Y_k^{\mathrm{EM}}|^2 + 4\gamma^2 m_{T-k}^2|\hat{\theta}|^2 + 2\gamma d \\
&\quad + \gamma^2|Y_k^{\mathrm{EM}}|^2 + 2\gamma(1-\gamma)\langle Y_k^{\mathrm{EM}}, 2m_{T-k}\hat{\theta}\rangle \\
&\leq (1-\gamma)|Y_k^{\mathrm{EM}}|^2 + 4\gamma(\gamma+2)m_{T-k}^2|\hat{\theta}|^2 + 2\gamma d \\
&\quad - \gamma|Y_k^{\mathrm{EM}}|^2 + \gamma^2|Y_k^{\mathrm{EM}}|^2 + (\gamma/2)|Y_k^{\mathrm{EM}}|^2 \\
&\leq (1-\gamma)|Y_k^{\mathrm{EM}}|^2 + 4\gamma(\gamma+2)|\hat{\theta}|^2 + 2\gamma d.
\end{aligned}$$

Thus,

$$\mathbb{E}\left[|Y_{k+1}^{\mathrm{EM}}|^2\right] \leq (1-\gamma)^{k+1}d + 10|\hat{\theta}|^2 + 2d. \tag{51}$$

Using Corollary 14 in 51 yields

$$\mathbb{E}\left[|Y_k^{\mathrm{EM}}|^2\right] \leq (1-\gamma)^k d + 2d + 20(e^{-2n\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]}\mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2).$$

□

*Proof of Lemma 17.* Using the process 11 with the approximating function $s$ given in 13, Lemma 16, Corollary 14 and $\gamma \in (0, 1/2]$, we have, for any $t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}} - \widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}|^2\right] \\
&= \mathbb{E}\left[\left|\int_{\lfloor t/\gamma\rfloor\gamma}^t (-\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}} + 2m_{T-\lfloor s/\gamma\rfloor\gamma}\hat{\theta})\mathrm{d}s + \sqrt{2}\int_{\lfloor t/\gamma\rfloor\gamma}^t \mathrm{d}\overline{B}_s\right|^2\right] \\
&\leq \gamma^2\mathbb{E}\left[\left|-\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}} + 2m_{T-\lfloor t/\gamma\rfloor\gamma}\hat{\theta}\right|^2\right] + 2d\gamma \\
&\leq 2\gamma^2\mathbb{E}\left[|\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}|^2\right] + 8\gamma^2\mathbb{E}\left[|\hat{\theta}|^2\right] + 2d\gamma \\
&\leq 6\gamma^2 d + 40\gamma^2(e^{-2n\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]}\mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2) \\
&\quad + 16\gamma^2(e^{-2n\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]}\mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2) + 2d\gamma \\
&\leq \gamma C_{\mathsf{EMose}},
\end{aligned}$$

where $C_{\mathsf{EMose}} = 8d + 56(e^{-2n\lambda\mathbb{E}\left[\sigma_\tau^2 m_\tau^2\right]}\mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2)$. □

*Proof of Lemma 18.* Applying Itô's formula to the process 11 with the approximating function $s$ given in 13, we have, for any $t \in [0, T]$,

$$\begin{aligned}
\mathrm{d}|\widehat{Y}_t^{\mathrm{EM}}|^2 &= 2\langle \widehat{Y}_t^{\mathrm{EM}}, -\widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}} + 2m_{T-\lfloor t/\gamma\rfloor\gamma}\hat{\theta}\rangle\mathrm{d}t + 2\langle \widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\mathrm{d}\overline{B}_t\rangle + 2d\mathrm{d}t \\
&= -2|\widehat{Y}_t^{\mathrm{EM}}|^2\mathrm{d}t + 2\langle \widehat{Y}_t^{\mathrm{EM}}, \widehat{Y}_t^{\mathrm{EM}} - \widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}\rangle\mathrm{d}t + 2\langle \widehat{Y}_t^{\mathrm{EM}}, 2m_{T-\lfloor t/\gamma\rfloor\gamma}\hat{\theta}\rangle\mathrm{d}t \\
&\quad + 2\langle \widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\mathrm{d}\overline{B}_t\rangle + 2d\mathrm{d}t.
\end{aligned}$$

Integrating both sides and taking expectation, we have

$$\begin{aligned}
\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^2\right] &= -2\int_0^t \mathbb{E}\left[|\widehat{Y}_s^{\mathrm{EM}}|^2\right]\mathrm{d}s + 2\int_0^t \mathbb{E}\left[\langle \widehat{Y}_s^{\mathrm{EM}}, \widehat{Y}_s^{\mathrm{EM}} - \widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}}\rangle\right]\mathrm{d}s \\
&\quad + 2\int_0^t \mathbb{E}\left[\langle \widehat{Y}_s^{\mathrm{EM}}, 2m_{T-\lfloor s/\gamma\rfloor\gamma}\hat{\theta}\rangle\right]\mathrm{d}s + \mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^2\right] + 2dt.
\end{aligned}$$

Then, differentiating both sides and using Young's inequality, yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^2\right] &= -2\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^2\right] + 2\mathbb{E}\left[\langle \widehat{Y}_t^{\mathrm{EM}}, \widehat{Y}_t^{\mathrm{EM}} - \widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}\rangle\right] \\ &\quad + 2\mathbb{E}\left[\langle \widehat{Y}_t^{\mathrm{EM}}, 2m_{T-\lfloor t/\gamma\rfloor\gamma}\hat{\theta}\rangle\right] + 2d \\ &\leq -\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^2\right] + 2\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}} - \widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{\mathrm{EM}}|^2\right] + 8\mathbb{E}[|\hat{\theta}|^2] + 2d.\end{aligned}$$

Using Lemma 17, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^t\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^2\right]\right) \leq e^t(2\gamma C_{\mathsf{EMose}} + 8\mathbb{E}[|\hat{\theta}|^2] + 2d).$$

Integrating both sides, using $\gamma \in (0, 1/2]$ and Corollary 14 yields

$$\begin{aligned}\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^2\right] &\leq e^{-t}d + (1-e^{-t})(2\gamma C_{\mathsf{EMose}} + 8\mathbb{E}[|\hat{\theta}|^2] + 2d) \\ &\leq 18d + 128(e^{-2n\lambda\mathbb{E}[\sigma_\tau^2 m_\tau^2]}\mathbb{E}\left[|\theta_0 - \theta^*|^2\right] + dC_{\mathsf{SGLD},1}/\beta + \lambda C_{\mathsf{SGLD},2} + |\theta^*|^2).\end{aligned}$$

□

## C Proofs of the Results in the General Case

In this section, we provide the proof of Theorem 10. We start by introducing the results which will be used in the proof of Theorem 10.

### C.1 Preliminary Estimates for the General Case

Throughout this section, we fix $\epsilon \in (0,1)$. The following auxiliary results will be used in the proof of Theorem 10 and their proofs are postponed to Appendix C.3.

We provide an upper bound for the moments of $(\widehat{Y}_t^{\mathrm{EM}})_{t\in[0,T]}$ defined in 11.

**Lemma 19.** *Let Assumptions 1 and 3 hold. For any* $p \in [2,4]$ *and* $t \in [0, T-\epsilon]$,

$$\sup_{0\leq s\leq t} \mathbb{E}\left[|\widehat{Y}_s^{EM}|^p\right] \leq C_{\mathsf{EM},p}(t),$$

*where*

$$\begin{aligned}C_{\mathsf{EM},p}(t) &:= e^{t(3p-1-\frac{2}{p}+2^{2p-1}\mathsf{K}_{Total}^p(1+T^{\alpha p}))} \\ &\quad \times \left(\mathbb{E}\left[|\widehat{Y}_0^{EM}|^p\right] + 2^{3p-2}\mathsf{K}_{Total}^p t(1+\mathbb{E}[|\hat{\theta}|^p])(1+T^{\alpha p}) + \frac{2}{p}(pM + p(p-2))^{\frac{p}{2}}t\right).\end{aligned}$$

The following result provides an estimate for the one step error associated with $(\widehat{Y}_t^{\mathrm{EM}})_{t\in[0,T]}$ defined in 11.

**Lemma 20.** *Let Assumptions 1 and 3 hold. For any* $p \in [2,4]$ *and* $t \in [0, T-\epsilon]$,

$$\mathbb{E}\left[|\widehat{Y}_t^{EM} - \widehat{Y}_{\lfloor t/\gamma\rfloor\gamma}^{EM}|^p\right] \leq \gamma^{\frac{p}{2}} C_{\mathsf{EMose},p},$$

*where*

$$\begin{aligned}C_{\mathsf{EMose},p} &:= 2^{p-1}(C_{\mathsf{EM},p}(T) + \mathsf{K}_{Total}^p(1+T^{\alpha p})(2^{3p-2}C_{\mathsf{EM},p}(T) + 2^{4p-3}(1+\mathbb{E}[|\hat{\theta}|^p]))) \\ &\quad + (Mp(p-1))^{\frac{p}{2}}.\end{aligned}$$

The following result is a modification of Kumar & Sabanis (2019, Lemma 4.1).

**Lemma 21.** *Let Assumption 3 hold and let* $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^M \to \mathbb{R}^M$ *such that*

$$b(t,\theta,x) := x + 2s(t,\theta,x). \tag{52}$$

*Then, for any* $x, \bar{x} \in \mathbb{R}^M$, $t \in [0,T]$, $\theta \in \mathbb{R}^d$, $\alpha \in [\frac{1}{2}, 1]$, *and* $k = 1, \dots M$,

$$\left| b^{(k)}(t,\theta,x) - b^{(k)}(t,\theta,\bar{x}) - \sum_{i=1}^{M} \frac{\partial b^{(k)}(t,\theta,\bar{x})}{\partial y^i}(x^i - \bar{x}^i) \right| \leq \mathsf{K}_4(1 + 2|t|^\alpha)|x - \bar{x}|^2.$$

**Lemma 22.** *Let Assumption 3 hold and let b be as in 52. Then, one obtains that, for any* $(t,\theta,x) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^M$ *and* $k = 1, \dots M$,

$$|\nabla_x b^{(k)}(t,\theta,x)| \leq 1 + 2\mathsf{K}_3(1 + 2|t|^\alpha). \tag{53}$$

### C.2 Proof of the Main Result for the General Case

*Proof of Theorem 10.* We proceed as in the proof of Theorem 1 using the splitting

$$\begin{aligned} W_2(\mathcal{L}(Y_K^{\mathrm{EM}}), \pi_{\mathsf{D}}) \leq{}& W_2(\pi_{\mathsf{D}}, \mathcal{L}(Y_{t_K})) + W_2(\mathcal{L}(Y_{t_K}), \mathcal{L}(\widetilde{Y}_{t_K})) \\ &+ W_2(\mathcal{L}(\widetilde{Y}_{t_K}), \mathcal{L}(Y_{t_K}^{\mathrm{aux}})) + W_2(\mathcal{L}(Y_{t_K}^{\mathrm{aux}}), \mathcal{L}(Y_K^{\mathrm{EM}})), \end{aligned} \tag{54}$$

where the first term on the right-hand side in 54 corresponds to the error made by the early stopping and the remaining terms have the same interpretation of the corresponding ones in 29.

**Upper bound on $W_2(\pi_{\mathsf{D}}, \mathcal{L}(Y_{t_K}))$.** For any $t \in [0,T]$, note that

$$1 - m_t \leq \sigma_t, \qquad \sigma_t^2 = 1 - e^{-2t} \leq 2t. \tag{55}$$

Recall that $t_K = T - \epsilon$. Using the representation of the OU process 2 and the inequalities 55, we have

$$\begin{aligned} W_2(\pi_{\mathsf{D}}, \mathcal{L}(Y_{t_K})) &= W_2(\pi_{\mathsf{D}}, \mathcal{L}(X_{T-t_K})) \\ &\leq \sqrt{\mathbb{E}\left[|X_0 - X_{T-t_K}|^2\right]} \\ &\leq \sqrt{2}\left[(1 - m_{T-t_K})\sqrt{\mathbb{E}[|X_0|^2]} + \sigma_{T-t_K}\sqrt{M}\right] \\ &\leq \sqrt{2}\sigma_{T-t_K}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M}) \\ &\leq 2\sqrt{\epsilon}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M}). \end{aligned} \tag{56}$$

**Upper bound on $W_2(\mathcal{L}(Y_{t_K}), \mathcal{L}(\widetilde{Y}_{t_K}))$.** Using Itô's formula, we have, for any $t \in [0, T-\epsilon]$,

$$\begin{aligned} \mathrm{d}|Y_t - \widetilde{Y}_t|^2 &= 2\langle Y_t - \widetilde{Y}_t, Y_t + 2\nabla \log p_{T-t}(Y_t) - \widetilde{Y}_t - 2\nabla \log p_{T-t}(\widetilde{Y}_t)\rangle \ \mathrm{d}t \\ &= 2|Y_t - \widetilde{Y}_t|^2 \ \mathrm{d}t + 4\langle Y_t - \widetilde{Y}_t, \nabla \log p_{T-t}(Y_t) - \nabla \log p_{T-t}(\widetilde{Y}_t)\rangle \ \mathrm{d}t. \end{aligned} \tag{57}$$

By integrating, taking expectations on both sides in 57, and using Remark 4 with the lower bound $\widehat{L}_{\mathrm{MO}}$ in the estimate 19, we have

$$\begin{aligned} \mathbb{E}[|Y_{t_K} - \widetilde{Y}_{t_K}|^2] &\leq \mathbb{E}[|Y_0 - \widetilde{Y}_0|^2] + \int_0^{t_K} 2(1 - 2\widehat{L}_{\mathrm{MO}}) \ \mathbb{E}\left[|Y_t - \widetilde{Y}_t|^2\right] \mathrm{d}t \\ &\leq \mathbb{E}[|Y_0 - \widetilde{Y}_0|^2] e^{2(1-2\widehat{L}_{\mathrm{MO}})t_K}. \end{aligned} \tag{58}$$

Using 58, the representation 2 with $Z_T \overset{\mathrm{d}}{=} \widetilde{Y}_0$ and 55, we have

$$\begin{aligned} \mathbb{E}[|Y_{t_K} - \widetilde{Y}_{t_K}|^2] &\leq \mathbb{E}[|Y_0 - \widetilde{Y}_0|^2] e^{2(1-2\widehat{L}_{\mathrm{MO}})t_K} \\ &= \mathbb{E}[|m_T X_0 + (\sigma_T - 1)\widetilde{Y}_0|^2] \ e^{2(1-2\widehat{L}_{\mathrm{MO}})t_K} \\ &\leq 2\left(\mathbb{E}[|X_0|^2] + M\right) e^{2(1-2\widehat{L}_{\mathrm{MO}})t_K - 2T}. \end{aligned} \tag{59}$$

Using 59, we have

$$\begin{aligned} W_2(\mathcal{L}(Y_{t_K}), \mathcal{L}(\widetilde{Y}_{t_K})) &\leq \sqrt{\mathbb{E}[|Y_{t_K} - \widetilde{Y}_{t_K}|^2]} \\ &\leq \sqrt{2}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M}) e^{-2\widehat{L}_{\mathrm{MO}}(T-\epsilon)-\epsilon}. \end{aligned} \tag{60}$$

**Upper bound on $W_2(\mathcal{L}(\widetilde{Y}_{t_K}), \mathcal{L}(Y^{\mathbf{aux}}_{t_K}))$.** Using Itô's formula, we have, for $t \in [0, T-\epsilon]$,

$$\begin{aligned} \mathrm{d}|\widetilde{Y}_t - Y_t^{\mathrm{aux}}|^2 &= 2\langle \widetilde{Y}_t - Y_t^{\mathrm{aux}}, \widetilde{Y}_t + 2\ \nabla \log p_{T-t}(\widetilde{Y}_t) - Y_t^{\mathrm{aux}} - 2\ s(T-t, \hat{\theta}, Y_t^{\mathrm{aux}})\rangle\ \mathrm{d}t \\ &= 2|\widetilde{Y}_t - Y_t^{\mathrm{aux}}|^2\ \mathrm{d}t + 4\ \langle \widetilde{Y}_t - Y_t^{\mathrm{aux}}, \nabla \log p_{T-t}(\widetilde{Y}_t) - \nabla \log p_{T-t}(Y_t^{\mathrm{aux}})\rangle\ \mathrm{d}t \\ &\quad + 4\ \langle \widetilde{Y}_t - Y_t^{\mathrm{aux}}, \nabla \log p_{T-t}(Y_t^{\mathrm{aux}}) - s(T-t, \hat{\theta}, Y_t^{\mathrm{aux}})\rangle\ \mathrm{d}t. \end{aligned} \tag{61}$$

By integrating and taking the expectation on both sides in 61, and using the Remark 4 with the lower bound $\widehat{L}_{\mathrm{MO}}$ in the estimate 19, Young's inequality with $\zeta \in (0,1)$ and Assumption 4, we have

$$\begin{aligned} \mathbb{E}[|\widetilde{Y}_{T-\epsilon} - Y_{T-\epsilon}^{\mathrm{aux}}|^2] &= 2\int_0^{T-\epsilon} \mathbb{E}[|\widetilde{Y}_s - Y_s^{\mathrm{aux}}|^2]\ \mathrm{d}s \\ &\quad + 4\int_0^{T-\epsilon} \mathbb{E}[\langle \widetilde{Y}_s - Y_s^{\mathrm{aux}}, \nabla \log p_{T-s}(\widetilde{Y}_s) - \nabla \log p_{T-s}(Y_s^{\mathrm{aux}})\rangle]\ \mathrm{d}s \\ &\quad + 4\int_0^{T-\epsilon} \mathbb{E}[\langle \widetilde{Y}_s - Y_s^{\mathrm{aux}}, \nabla \log p_{T-s}(Y_s^{\mathrm{aux}}) - s(T-s, \hat{\theta}, Y_s^{\mathrm{aux}})\rangle]\ \mathrm{d}s \\ &\leq \int_0^{T-\epsilon} 2(1+\zeta-2\widehat{L}_{\mathrm{MO}})\mathbb{E}[|\widetilde{Y}_s - Y_s^{\mathrm{aux}}|^2]\ \mathrm{d}s \\ &\quad + 2\zeta^{-1} \int_0^{T-\epsilon} \mathbb{E}[|\nabla \log p_{T-s}(Y_s^{\mathrm{aux}}) - s(T-s, \hat{\theta}, Y_s^{\mathrm{aux}})|^2]\ \mathrm{d}s \\ &\leq \int_0^{T-\epsilon} 2(1+\zeta-2\widehat{L}_{\mathrm{MO}})\ \mathbb{E}[|\widetilde{Y}_s - Y_s^{\mathrm{aux}}|^2]\ \mathrm{d}s + 2\zeta^{-1}\varepsilon_{\mathrm{SN}} \\ &\leq 2e^{2(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)}\zeta^{-1}\varepsilon_{\mathrm{SN}}. \end{aligned} \tag{62}$$

Using 62 and $t_K = T - \epsilon$, we have

$$\begin{aligned} W_2(\mathcal{L}(\widetilde{Y}_{t_K}), \mathcal{L}(Y_{t_K}^{\mathrm{aux}})) &\leq \sqrt{\mathbb{E}[|\widetilde{Y}_{t_K} - Y_{t_K}^{\mathrm{aux}}|^2]} \\ &\leq \sqrt{2\zeta^{-1}} e^{(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)}\sqrt{\varepsilon_{\mathrm{SN}}}. \end{aligned} \tag{63}$$

**Upper bound on $W_2(\mathcal{L}(Y^{\mathbf{aux}}_{t_K}), \mathcal{L}(Y^{\mathbf{EM}}_K))$.** The following bound is derived by modifying Kumar & Sabanis (2019, Lemma 4.7). For the sake of presentation, let $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^M \to \mathbb{R}^M$ such that

$$b(t, \theta, x) = x + 2s(t, \theta, x). \tag{64}$$

Consequently, $(\widehat{Y}_t^{\mathrm{EM}})_{t\in[0,T]}$ can be expressed using 64 as

$$\mathrm{d}\widehat{Y}_t^{\mathrm{EM}} = b(T - \lfloor t/\gamma \rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor\gamma})\ \mathrm{d}t + \sqrt{2}\ \mathrm{d}\bar{B}_t, \qquad \widehat{Y}_0^{\mathrm{EM}} \sim \pi_\infty = \mathcal{N}(0, I_M). \tag{65}$$

Using Itô's formula, we have, for $t \in [0, T-\epsilon]$,

$$\begin{aligned} &\mathrm{d}|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2 \\ &= 2\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, Y_t^{\mathrm{aux}} + 2\ s(T-t, \hat{\theta}, Y_t^{\mathrm{aux}}) - \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor\gamma} - 2\ s(T - \lfloor t/\gamma \rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor\gamma})\rangle\ \mathrm{d}t \\ &= 2|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2\ \mathrm{d}t + 4\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, s(T-t, \hat{\theta}, Y_t^{\mathrm{aux}}) - s(T-t, \hat{\theta}, \widehat{Y}_t^{\mathrm{EM}})\rangle\ \mathrm{d}t \\ &\quad + 4\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, s(T-t, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor\gamma}) - s(T - \lfloor t/\gamma \rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor\gamma})\rangle\ \mathrm{d}t \\ &\quad + 2\langle Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}, b(T-t, \hat{\theta}, \widehat{Y}_t^{\mathrm{EM}}) - b(T-t, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma \rfloor\gamma}))\rangle\ \mathrm{d}t. \end{aligned} \tag{66}$$

Integrating and taking the expectation on both sides in 66, using Cauchy–Schwarz inequality, Young's inequality with $\zeta \in (0,1)$, Assumption 3 and Remark 3, yield

$$
\begin{aligned}
&\mathbb{E}\left[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2\right] \\
&= 2\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s + 4\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, s(T-s,\hat{\theta},Y_s^{\mathrm{aux}}) - s(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}})\rangle\right]\,\mathrm{d}s \\
&\quad + 4\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, s(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}) - s(T-\lfloor s/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})\rangle\right]\,\mathrm{d}s \\
&\quad + 2\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, b(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})\rangle\right]\,\mathrm{d}s \\
&\leq \int_0^t 2(1+\zeta+2\mathsf{K}_3(1+2|T-s|^{\alpha}))\,\mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s \\
&\quad + 2\zeta^{-1}\int_0^t \mathbb{E}\left[|s(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}) - s(T-\lfloor s/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})|^2\right]\,\mathrm{d}s \\
&\quad + 2\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, b(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}))\rangle\right]\,\mathrm{d}s \\
&\leq \int_0^t 2(1+\zeta+2\mathsf{K}_3(1+2T^{\alpha}))\,\mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s \\
&\quad + 2\zeta^{-1}\mathsf{K}_1^2\,\mathbb{E}\left[(1+2|\hat{\theta}|)^2\right]\int_0^t |s-\lfloor s/\gamma\rfloor\gamma|^{2\alpha}\,\mathrm{d}s \\
&\quad + 2\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, b(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}))\rangle\right]\,\mathrm{d}s \\
&\leq \int_0^t 2(1+\zeta+2\mathsf{K}_3(1+2T^{\alpha}))\,\mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s + 4\zeta^{-1}\mathsf{K}_1^2(1+8(\widetilde{\varepsilon}_{\mathrm{AL}}+|\theta^*|^2))\gamma^{2\alpha}t \\
&\quad + 2\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, b(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})\rangle\right]\,\mathrm{d}s.
\end{aligned}
\tag{67}
$$

We proceed estimating the third term on the right-hand side of 67. Using 65, Young's inequality with $\zeta \in (0,1)$, Lemma 21 and Lemma 20, yields, for any $t \in [0, T-\epsilon]$

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left[\langle Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}, b(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})\rangle\right]\mathrm{d}s \\
&= \sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t \left(Y_s^{\mathrm{aux},(k)} - \widehat{Y}_s^{\mathrm{EM},(k)}\right)\Bigg(b^{(k)}(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}) \\
&\qquad\qquad - \sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^j}(\widehat{Y}_s^{\mathrm{EM},(j)} - \widehat{Y}^{\mathrm{EM},(j)}_{\lfloor s/\gamma\rfloor\gamma})\Bigg)\,\mathrm{d}s\Bigg] \\
&\quad + \sum_{k=1}^{M}\mathbb{E}\left[\int_0^t \left(Y_s^{\mathrm{aux},(k)} - \widehat{Y}_s^{\mathrm{EM},(k)}\right)\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^j}(\widehat{Y}_s^{\mathrm{EM},(j)} - \widehat{Y}^{\mathrm{EM},(j)}_{\lfloor s/\gamma\rfloor\gamma})\right)\,\mathrm{d}s\right] \\
&\leq \frac{\zeta}{2}\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s \\
&\quad + \frac{1}{2\zeta}\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t |b^{(k)}(T-s,\hat{\theta},\widehat{Y}_s^{\mathrm{EM}}) - b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad -\sum_{j=1}^{M} \frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\left(\widehat{Y}^{\mathrm{EM},(j)}_{s}-\widehat{Y}^{\mathrm{EM},(j)}_{\lfloor s/\gamma\rfloor\gamma}\right)|^{2}\ \mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\\
&\qquad\times\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s} b^{(j)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma})\ \mathrm{d}r\right)\mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\left[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\ \mathrm{d}\bar{B}^{(j)}_{r}\right)\mathrm{d}s\right]\\
&\leq\frac{\zeta}{2}\int_{0}^{t}\mathbb{E}\left[|Y^{\mathrm{aux}}_{s}-\widehat{Y}^{\mathrm{EM}}_{s}|^{2}\right]\ \mathrm{d}s+\frac{\mathsf{K}_{4}^{2}}{2\zeta}\int_{0}^{t}(1+2|T-s|^{\alpha})^{2}\ \mathbb{E}\left[|\widehat{Y}^{\mathrm{EM}}_{s}-\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}|^{4}\right]\ \mathrm{d}s\\
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\\
&\qquad\times\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s} b^{(j)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma})\ \mathrm{d}r\right)\ \mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\left[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\ \mathrm{d}\bar{B}^{(j)}_{r}\right)\ \mathrm{d}s\right]\\
&\leq\frac{\zeta}{2}\int_{0}^{t}\mathbb{E}\left[|Y^{\mathrm{aux}}_{s}-\widehat{Y}^{\mathrm{EM}}_{s}|^{2}\right]\ \mathrm{d}s+\gamma^{2}\frac{\mathsf{K}_{4}^{2}}{\zeta}t(1+4T^{2\alpha})C_{\mathsf{EMose},4}\\
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\\
&\qquad\times\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s} b^{(j)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma})\ \mathrm{d}r\right)\mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\left[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\ \mathrm{d}\bar{B}^{(j)}_{r}\right)\mathrm{d}s\right].
\end{aligned}
\tag{68}
$$

We proceed estimating the third and fourth term on the right-hand side of 68, separately.

The third term is estimated using Young's inequality with $\zeta\in(0,1)$, Lemma 22, Remark 5, Remark 3 and Lemma 19, and for any $t\in[0,T-\epsilon]$, we obtain that

$$
\begin{aligned}
&\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_{0}^{t}\left(Y^{\mathrm{aux},(k)}_{s}-\widehat{Y}^{\mathrm{EM},(k)}_{s}\right)\\
&\qquad\times\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^{j}}\int_{\lfloor s/\gamma\rfloor\gamma}^{s} b^{(j)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma})\mathrm{d}r\right)\mathrm{d}s\Bigg]\\
&\leq\frac{\zeta}{2}\int_{0}^{t}\mathbb{E}\left[|Y^{\mathrm{aux}}_{s}-\widehat{Y}^{\mathrm{EM}}_{s}|^{2}\right]\ \mathrm{d}s\\
&\quad+\frac{2\gamma}{\zeta}M(1+8\mathsf{K}_{3}^{2}(1+4T^{2\alpha}))\ \mathbb{E}\left[\int_{0}^{t}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}|b^{(j)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma})|^{2}\mathrm{d}r\ \mathrm{d}s\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\zeta}{2}\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s \\
&\quad + \frac{4\gamma}{\zeta}M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\qquad\qquad \times \int_0^t\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\left[(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))\mathbb{E}[|\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma}|^2] + 32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+\mathbb{E}[|\hat{\theta}|^2])\right]\mathrm{d}r\,\mathrm{d}s \\
&\leq \frac{\zeta}{2}\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s \\
&\quad + \frac{\gamma^2}{\zeta}4M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\quad \times t\left[(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))\sup_{0\leq s\leq t}\mathbb{E}[|\widehat{Y}_s^{\mathrm{EM}}|^2] + 32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2|\theta^*|^2)\right] \\
&\leq \frac{\zeta}{2}\int_0^t \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right]\,\mathrm{d}s \\
&\quad + t\frac{\gamma^2}{\zeta}4M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\quad \times \left[(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2|\theta^*|^2)\right].
\end{aligned}
\tag{69}
$$

We proceed with the estimate of the fourth term on the right-hand side of 68. For $k = 1, \dots, M$, we note

$$
\begin{aligned}
Y_s^{\mathrm{aux},(k)} - \widehat{Y}_s^{\mathrm{EM},(k)} &= Y^{\mathrm{aux},(k)}_{\lfloor s/\gamma\rfloor\gamma} - \widehat{Y}^{\mathrm{EM},(k)}_{\lfloor s/\gamma\rfloor\gamma} \\
&\quad + \int_{\lfloor s/\gamma\rfloor\gamma}^{s}(b^{(k)}(T-r,\hat{\theta},Y_r^{\mathrm{aux}}) - b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}})\,\mathrm{d}r \\
&\quad + \int_{\lfloor s/\gamma\rfloor\gamma}^{s}(b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}}) - b^{(k)}(T-r,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma}))\,\mathrm{d}r \\
&\quad + \int_{\lfloor s/\gamma\rfloor\gamma}^{s}2(s^{(k)}(T-r,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma}) - s^{(k)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor r/\gamma\rfloor\gamma}))\,\mathrm{d}r.
\end{aligned}
\tag{70}
$$

By using 70, Cauchy–Schwarz inequality, Young's inequality, Assumption 3 and Lemma 22, we have

$$
\begin{aligned}
&\sum_{k=1}^{M}\mathbb{E}\left[\int_0^t\left(Y_s^{\mathrm{aux},(k)} - \widehat{Y}_s^{\mathrm{EM},(k)}\right)\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^j}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\,\mathrm{d}\bar{B}_r^{(j)}\right)\mathrm{d}s\right] \\
&= \sum_{k=1}^{M}\mathbb{E}\left[\int_0^t\left(Y^{\mathrm{aux},(k)}_{\lfloor s/\gamma\rfloor\gamma} - \widehat{Y}^{\mathrm{EM},(k)}_{\lfloor s/\gamma\rfloor\gamma}\right)\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^j}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\,\mathrm{d}\bar{B}_r^{(j)}\right)\mathrm{d}s\right] \\
&\quad + \sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\left(\int_{\lfloor s/\gamma\rfloor\gamma}^{s}(b^{(k)}(T-r,\hat{\theta},Y_r^{\mathrm{aux}}) - b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}}))\,\mathrm{d}r\right) \\
&\qquad\qquad \times\left(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})}{\partial x^j}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\,\mathrm{d}\bar{B}_r^{(j)}\right)\mathrm{d}s\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\Bigg(\int_{\lfloor s/\gamma\rfloor\gamma}^{s}(b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}})-b^{(k)}(T-r,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}}))\ \mathrm{d}r\Bigg)\\
&\qquad\qquad\times\Bigg(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg)\ \mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\Bigg(\int_{\lfloor s/\gamma\rfloor\gamma}^{s}2(s^{(k)}(T-r,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})-s^{(k)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}}))\ \mathrm{d}r\Bigg)\\
&\qquad\qquad\times\Bigg(\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg)\ \mathrm{d}s\Bigg]\\
\leq{}&\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\gamma^{-1/2}|b^{(k)}(T-r,\hat{\theta},Y_r^{\mathrm{aux}})-b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}})|\mathrm{d}r\\
&\qquad\qquad\times\gamma^{1/2}\Bigg|\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg|\mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\int_{\lfloor s/\gamma\rfloor\gamma}^{s}|b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}})-b^{(k)}(T-r,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})|\mathrm{d}r\\
&\qquad\qquad\times\Bigg|\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg|\mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\int_{\lfloor s/\gamma\rfloor\gamma}^{s}2|s^{(k)}(T-r,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})-s^{(k)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})|\mathrm{d}r\\
&\qquad\qquad\times\Bigg|\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg|\mathrm{d}s\Bigg]\\
\leq{}&\frac{\gamma^{-1}}{2}\sum_{k=1}^{M}\mathbb{E}\Bigg[\int_0^t\int_{\lfloor s/\gamma\rfloor\gamma}^{s}|b^{(k)}(T-r,\hat{\theta},Y_r^{\mathrm{aux}})-b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}})|^2\mathrm{d}r\ \mathrm{d}s\Bigg]\\
&+\frac{\gamma}{2}\sum_{k=1}^{M}\int_0^t\mathbb{E}\Bigg[\Bigg|\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg|^2\mathrm{d}s\Bigg]\\
&+\sum_{k=1}^{M}\int_0^t\Bigg[\mathbb{E}\Bigg(\int_{\lfloor s/\gamma\rfloor\gamma}^{s}|b^{(k)}(T-r,\hat{\theta},\widehat{Y}_r^{\mathrm{EM}})-b^{(k)}(T-r,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})|\mathrm{d}r\Bigg)^2\Bigg]^{1/2}\\
&\qquad\qquad\times\Bigg[\mathbb{E}\Bigg|\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg|^2\Bigg]^{1/2}\mathrm{d}s\\
&+\sum_{k=1}^{M}\int_0^t\Bigg[\mathbb{E}\Bigg(\int_{\lfloor s/\gamma\rfloor\gamma}^{s}2|s^{(k)}(T-r,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})-s^{(k)}(T-\lfloor r/\gamma\rfloor\gamma,\hat{\theta},\widehat{Y}_{\lfloor r/\gamma\rfloor\gamma}^{\mathrm{EM}})|\ \mathrm{d}r\Bigg)^2\Bigg]^{1/2}\\
&\qquad\qquad\times\Bigg[\mathbb{E}\Bigg|\int_{\lfloor s/\gamma\rfloor\gamma}^{s}\sum_{j=1}^{M}\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma\rfloor\gamma}^{\mathrm{EM}})}{\partial x^j}\sqrt{2}\ \mathrm{d}\bar{B}_r^{(j)}\Bigg|^2\Bigg]^{1/2}\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathbb{E}\left[\int_0^t \int_{\lfloor s/\gamma \rfloor \gamma}^s \gamma^{-1}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))|Y_r^{\mathrm{aux}} - \widehat{Y}_r^{\mathrm{EM}}|^2 \mathrm{d}r \ \mathrm{d}s\right] \\
&\quad + 2\gamma^2 \sum_{k=1}^M \mathbb{E}\left[\int_0^t \sum_{j=1}^M \left|\frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma \rfloor \gamma}^{\mathrm{EM}})}{\partial x^j}\right|^2 \mathrm{d}s\right] \\
&\quad + \gamma^{1/2} 2(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2} \\
&\quad\times \sum_{k=1}^M \int_0^t \left[\int_{\lfloor s/\gamma \rfloor \gamma}^s \mathbb{E}[|\widehat{Y}_r^{\mathrm{EM}} - \widehat{Y}_{\lfloor r/\gamma \rfloor \gamma}^{\mathrm{EM}}|^2] \ \mathrm{d}r\right]^{1/2} \left[\mathbb{E}\int_{\lfloor s/\gamma \rfloor \gamma}^s \left|\sum_{j=1}^M \frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma \rfloor \gamma}^{\mathrm{EM}})}{\partial x^j}\right|^2 \mathrm{d}r\right]^{1/2} \mathrm{d}s \\
&\quad + \gamma^{1+\alpha} 4\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2} \sum_{k=1}^M \int_0^t \left[\mathbb{E}\int_{\lfloor s/\gamma \rfloor \gamma}^s \left|\sum_{j=1}^M \frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma \rfloor \gamma}^{\mathrm{EM}})}{\partial x^j}\right|^2 \mathrm{d}r\right]^{1/2} \mathrm{d}s \\
&\leq (1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}\left[|Y_r^{\mathrm{aux}} - \widehat{Y}_r^{\mathrm{EM}}|^2\right] \mathrm{d}s + 4\gamma^2 t M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\quad + \gamma^{3/2} 2(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2} C_{\mathsf{EMose},2}^{1/2} \sum_{k=1}^M \int_0^t \left[\mathbb{E}\int_{\lfloor s/\gamma \rfloor \gamma}^s \left|\sum_{j=1}^M \frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma \rfloor \gamma}^{\mathrm{EM}})}{\partial x^j}\right|^2 \mathrm{d}r\right]^{1/2} \mathrm{d}s \\
&\quad + \gamma^{1+\alpha} 4\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2} \sum_{k=1}^M \int_0^t \left[\mathbb{E}\int_{\lfloor s/\gamma \rfloor \gamma}^s \left|\sum_{j=1}^M \frac{\partial b^{(k)}(T-s,\hat{\theta},\widehat{Y}_{\lfloor s/\gamma \rfloor \gamma}^{\mathrm{EM}})}{\partial x^j}\right|^2 \mathrm{d}r\right]^{1/2} \mathrm{d}s \\
&\leq (1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \int_0^t \sup_{0 \leq r \leq s} \mathbb{E}\left[|Y_r^{\mathrm{aux}} - \widehat{Y}_r^{\mathrm{EM}}|^2\right] \ \mathrm{d}s + 4\gamma^2 t M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\quad + [\gamma^2 2(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2} C_{\mathsf{EMose},2}^{1/2} + \gamma^{3/2+\alpha} 4\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2}] \\
&\qquad \times [tM\sqrt{2}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}].
\end{aligned} \tag{71}
$$

Using 68, 69 and 71 in 67, we have

$$
\begin{aligned}
&\mathbb{E}\left[|Y_t^{\mathrm{aux}} - \widehat{Y}_t^{\mathrm{EM}}|^2\right] \\
&\leq \int_0^t 4(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha}))) \sup_{0 \leq r \leq s} \mathbb{E}\left[|Y_r^{\mathrm{aux}} - \widehat{Y}_r^{\mathrm{EM}}|^2\right] \ \mathrm{d}s \\
&\quad + 8\gamma^2 t \zeta^{-1} M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\qquad \times \left[(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}} + 2|\theta^*|^2)\right] \\
&\quad + 2\gamma^2 t\mathsf{K}_4^2 \zeta^{-1}(1+4T^{2\alpha})C_{\mathsf{EMose},4} + 8\gamma^2 t M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\quad + \gamma^{2\alpha} t 4\zeta^{-1}\mathsf{K}_1^2(1+8(\widetilde{\varepsilon}_{\mathrm{AL}} + |\theta^*|^2)) \\
&\quad + 4[\gamma^2(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2} C_{\mathsf{EMose},2}^{1/2} + \gamma^{3/2+\alpha} 2\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2}] \\
&\qquad \times [tM\sqrt{2}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sup_{0 \leq s \leq t} \mathbb{E}\left[|Y_s^{\mathrm{aux}} - \widehat{Y}_s^{\mathrm{EM}}|^2\right] \\
&\leq \int_0^t 4(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha}))) \sup_{0 \leq r \leq s} \mathbb{E}\left[|Y_r^{\mathrm{aux}} - \widehat{Y}_r^{\mathrm{EM}}|^2\right] \ \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \gamma^{2\alpha} t 2 \Bigg( \mathsf{K}_4^2 \zeta^{-1} (1 + 4T^{2\alpha}) C_{\mathsf{EMose},4} + 4M(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha})) + 2\zeta^{-1} \mathsf{K}_1^2 (1 + 8(\widetilde{\varepsilon}_{\mathrm{AL}} + |\theta^*|^2)) \\
&\qquad + 4\zeta^{-1} M (1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha})) \\
&\qquad\quad \times [(1 + 16\mathsf{K}_{\mathrm{Total}}^2 (1 + T^{2\alpha})) C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2 (1 + T^{2\alpha})(1 + 2\widetilde{\varepsilon}_{\mathrm{AL}} + 2|\theta^*|^2)] \\
&\qquad + 2[(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha}))^{1/2} C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1 (1 + 8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2}] \\
&\qquad\quad \times [M\sqrt{2}(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha}))^{1/2}] \Bigg) \\
&\le 2 e^{4(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha})))t} \gamma^{2\alpha} t \\
&\times \Bigg( \mathsf{K}_4^2 \zeta^{-1} (1 + 4T^{2\alpha}) C_{\mathsf{EMose},4} + 4M(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha})) + 2\zeta^{-1} \mathsf{K}_1^2 (1 + 8(\widetilde{\varepsilon}_{\mathrm{AL}} + |\theta^*|^2)) \\
&\quad + 4\zeta^{-1} M (1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha})) \\
&\qquad \times [(1 + 16\mathsf{K}_{\mathrm{Total}}^2 (1 + T^{2\alpha})) C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2 (1 + T^{2\alpha})(1 + 2\widetilde{\varepsilon}_{\mathrm{AL}} + 2|\theta^*|^2)] \\
&\quad + 2[(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha}))^{1/2} C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1 (1 + 8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2}] \\
&\qquad \times [M\sqrt{2}(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha}))^{1/2}] \Bigg).
\end{aligned}
\tag{72}
$$

Using 72 and $t_K = T - \epsilon$, we have

$$
\begin{aligned}
&W_2(\mathcal{L}(Y_{t_K}^{\mathrm{aux}}), \mathcal{L}(Y_K^{\mathrm{EM}})) \\
&\le \sqrt{\mathbb{E}\left[|Y_{t_K}^{\mathrm{aux}} - \widehat{Y}_{t_K}^{\mathrm{EM}}|^2\right]} \\
&\le \sqrt{2} e^{2(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha})))(T-\epsilon)} \gamma^{\alpha} \sqrt{T-\epsilon} \\
&\quad \times \Bigg( \mathsf{K}_4^2 \zeta^{-1} (1 + 4T^{2\alpha}) C_{\mathsf{EMose},4} + 4M(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha})) + 2\zeta^{-1} \mathsf{K}_1^2 (1 + 8(\widetilde{\varepsilon}_{\mathrm{AL}} + |\theta^*|^2)) \\
&\qquad + 4\zeta^{-1} M (1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha})) \\
&\qquad\quad \times [(1 + 16\mathsf{K}_{\mathrm{Total}}^2 (1 + T^{2\alpha})) C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2 (1 + T^{2\alpha})(1 + 2\widetilde{\varepsilon}_{\mathrm{AL}} + 2|\theta^*|^2)] \\
&\qquad + 2[(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha}))^{1/2} C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1 (1 + 8\widetilde{\varepsilon}_{\mathrm{AL}} + 8|\theta^*|^2)^{1/2}] \\
&\qquad\quad \times [M\sqrt{2}(1 + 8\mathsf{K}_3^2(1 + 4T^{2\alpha}))^{1/2}] \Bigg)^{1/2}.
\end{aligned}
\tag{73}
$$

**Final upper bound on $W_2(\mathcal{L}(Y_K^{\mathbf{EM}}), \pi_{\mathsf{D}})$.** Substituting 56, 60, 63, and 73 into 54, we have

$$
\begin{aligned}
&W_2(\mathcal{L}(Y_K^{\mathrm{EM}}), \pi_{\mathsf{D}}) \\
&\le (\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M}) 2\sqrt{\epsilon} \\
&\quad + \sqrt{2}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M}) e^{-2\widehat{L}_{\mathrm{MO}}(T-\epsilon)-\epsilon} \\
&\quad + \sqrt{2\zeta^{-1}} e^{(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)} \sqrt{\varepsilon_{\mathrm{SN}}} \\
&\quad + \sqrt{2} e^{2(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha})))(T-\epsilon)} \gamma^{\alpha} \sqrt{T-\epsilon}
\end{aligned}
$$

$$\begin{aligned}
&\times \Bigg( \mathsf{K}_4^2\zeta^{-1}(1+4T^{2\alpha})C_{\mathsf{EMose},4} + 4M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) + 2\zeta^{-1}\mathsf{K}_1^2(1+8(\widetilde{\varepsilon}_{\mathrm{AL}}+|\theta^*|^2)) \\
&\quad + 4\zeta^{-1}M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\qquad \times [(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2|\theta^*|^2)] \\
&\quad + 2[(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}}+8|\theta^*|^2)^{1/2}] \\
&\qquad \times [M\sqrt{2}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}] \Bigg)^{1/2}.
\end{aligned} \tag{74}$$

The bound for $W_2(\mathcal{L}(\widehat{Y}_K^{\mathrm{EM}}), \pi_{\mathsf{D}})$ in 74 can be made arbitrarily small by appropriately choosing parameters including $\epsilon, T, \varepsilon_{\mathrm{SN}}$ and $\gamma$. More precisely, for any $\delta > 0$, we first choose $0 \le \epsilon < \epsilon_\delta$ with $\epsilon_\delta$ given in Table 3 such that the first term on the right-hand side of 74 is

$$(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M})2\sqrt{\epsilon} < \delta/4. \tag{75}$$

Next, we choose $T > T_\delta$ with $T_\delta$ given in Table 3 such that the second term on the right-hand side of 74 is

$$\sqrt{2}(\sqrt{\mathbb{E}[|X_0|^2]} + \sqrt{M})e^{-2\widehat{L}_{\mathrm{MO}}(T-\epsilon)-\epsilon} < \delta/4. \tag{76}$$

Next, we turn to the third term on the right-hand side of 74. We choose $0 < \varepsilon_{\mathrm{SN}} < \varepsilon_{\mathrm{SN},\delta}$ with $\varepsilon_{\mathrm{SN},\delta}$ given in Table 3 such that

$$\sqrt{2\zeta^{-1}}e^{(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)}\sqrt{\varepsilon_{\mathrm{SN}}} < \delta/4. \tag{77}$$

Finally, we choose $0 < \gamma < \gamma_\delta$ with $\gamma_\delta$ given in Table 3 such that the fourth term on the right-hand side of 74 is

$$\begin{aligned}
&\sqrt{2}e^{2(1+\zeta+\mathsf{K}_3(1+2T^\alpha+4\mathsf{K}_3(1+4T^{2\alpha})))(T-\epsilon)}\gamma^\alpha\sqrt{T-\epsilon} \\
&\times \Bigg( \mathsf{K}_4^2\zeta^{-1}(1+4T^{2\alpha})C_{\mathsf{EMose},4} + 4M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) + 2\zeta^{-1}\mathsf{K}_1^2(1+8(\widetilde{\varepsilon}_{\mathrm{AL}}+|\theta^*|^2)) \\
&\quad + 4\zeta^{-1}M(1+8\mathsf{K}_3^2(1+4T^{2\alpha})) \\
&\qquad \times [(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))C_{\mathsf{EM},2}(T) + 32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2|\theta^*|^2)] \\
&\quad + 2[(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}}+8|\theta^*|^2)^{1/2}] \\
&\qquad \times [M\sqrt{2}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}] \Bigg)^{1/2} < \delta/4.
\end{aligned} \tag{78}$$

Using 75, 76, 77 and 78, we obtain $W_2(\mathcal{L}(\widehat{Y}_K^{\mathrm{EM}}), \pi_{\mathsf{D}}) < \delta$. ☐

### C.3 Proof of the Preliminary Results for the General Case

We provide the proofs of Section 3.2 and Appendix C.1.

*Proof of Remark 5.* Using Assumption 3, we have

$$\begin{aligned}
|s(t,\theta,x)| &\le |s(t,\theta,x) - s(0,0,0)| + |s(0,0,0)| \\
&\le \mathsf{K}_1(1+|\theta|)|t|^\alpha + \mathsf{K}_2(1+|t|^\alpha)|\theta| + \mathsf{K}_3(1+|t|^\alpha)|x| + |s(0,0,0)| \\
&\le \mathsf{K}_{\mathrm{Total}}(1+|t|^\alpha)(1+|\theta|+|x|),
\end{aligned}$$

where $\mathsf{K}_{\mathrm{Total}} := \mathsf{K}_1 + \mathsf{K}_2 + \mathsf{K}_3 + |s(0,0,0)|$. ☐

*Proof of Lemma 19.* Using Itô's formula, we have, for any $t \in [0, T-\epsilon]$ and $p \in [2,4]$,

$$\begin{aligned} \mathrm{d}|\widehat{Y}_t^{\mathrm{EM}}|^p &= p\left\langle |\widehat{Y}_t^{\mathrm{EM}}|^{p-2}\widehat{Y}_t^{\mathrm{EM}}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma}\right\rangle \mathrm{d}t + p\left\langle |\widehat{Y}_t^{\mathrm{EM}}|^{p-2}\widehat{Y}_t^{\mathrm{EM}}, 2\ s(T-\lfloor t/\gamma\rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma})\right\rangle \mathrm{d}t \\ &\quad + p\left\langle |\widehat{Y}_t^{\mathrm{EM}}|^{p-2}\widehat{Y}_t^{\mathrm{EM}}, \sqrt{2}\ \mathrm{d}\overline{B}_t\right\rangle + \frac{p}{2}|\widehat{Y}_t^{\mathrm{EM}}|^{p-2}(2M)\ \mathrm{d}t + \frac{p(p-2)}{2}|\widehat{Y}_t^{\mathrm{EM}}|^{p-4}\ 2|\widehat{Y}_t^{\mathrm{EM}}|^2\mathrm{d}t. \end{aligned} \tag{79}$$

Integrating and taking expectation on both sides in 79, using Young's inequality and Remark 5, we have

$$\begin{aligned} \mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}}|^p\right] &= \mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^p\right] + p\int_0^t \mathbb{E}\left[\langle|\widehat{Y}_s^{\mathrm{EM}}|^{p-2}\widehat{Y}_s^{\mathrm{EM}}, \widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}\rangle\right]\mathrm{d}s \\ &\quad + 2p\int_0^t \mathbb{E}\left[\langle|\widehat{Y}_s^{\mathrm{EM}}|^{p-2}\widehat{Y}_s^{\mathrm{EM}}, s(T-\lfloor s/\gamma\rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})\rangle\right]\mathrm{d}s \\ &\quad + p(M+p-2)\int_0^t \mathbb{E}\left[|\widehat{Y}_s^{\mathrm{EM}}|^{p-2}\right]\ \mathrm{d}s \\ &\leq \mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^p\right] + 3(p-1)\int_0^t \mathbb{E}\left[|\widehat{Y}_s^{\mathrm{EM}}|^p\right]\ \mathrm{d}s + \int_0^t \mathbb{E}\left[|\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}|^p\right]\ \mathrm{d}s \\ &\quad + 2\int_0^t \mathbb{E}\left[|s(T-\lfloor s/\gamma\rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma})|^p\right]\ \mathrm{d}s \\ &\quad + \frac{2}{p}(pM+p(p-2))^{\frac{p}{2}}t + \frac{p-2}{p}\int_0^t \mathbb{E}\left[|\widehat{Y}_s^{\mathrm{EM}}|^p\right]\ \mathrm{d}s \\ &\leq \mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^p\right] + \left(3p-2-\frac{2}{p}\right)\int_0^t \mathbb{E}\left[|\widehat{Y}_s^{\mathrm{EM}}|^p\right]\ \mathrm{d}s + \int_0^t \mathbb{E}\left[|\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}|^p\right]\ \mathrm{d}s \\ &\quad + 2^p\mathsf{K}_{\mathrm{Total}}^p\int_0^t (1+|T-\lfloor s/\gamma\rfloor\gamma|^\alpha)^p\ \mathbb{E}\left[|\widehat{Y}^{\mathrm{EM}}_{\lfloor s/\gamma\rfloor\gamma}|^p\right]\ \mathrm{d}s \\ &\quad + 2^{2p-1}\mathsf{K}_{\mathrm{Total}}^p(1+\mathbb{E}[|\hat{\theta}|^p])\int_0^t (1+|T-\lfloor s/\gamma\rfloor\gamma|^\alpha)^p\ \mathrm{d}s + \frac{2}{p}(pM+p(p-2))^{\frac{p}{2}}t \\ &\leq \mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^p\right] + (3p-1-\frac{2}{p}+2^{2p-1}\mathsf{K}_{\mathrm{Total}}^p(1+T^{\alpha p}))\int_0^t \sup_{0\leq r\leq s}\mathbb{E}\left[|\widehat{Y}_r^{\mathrm{EM}}|^p\right]\ \mathrm{d}s \\ &\quad + 2^{3p-2}\mathsf{K}_{\mathrm{Total}}^p(1+\mathbb{E}[|\hat{\theta}|^p])(1+T^{\alpha p})t + \frac{2}{p}(pM+p(p-2))^{\frac{p}{2}}t. \end{aligned}$$

Using Grönwall's inequality, we have

$$\begin{aligned} \sup_{0\leq s\leq t}\mathbb{E}\left[|\widehat{Y}_s^{\mathrm{EM}}|^p\right] &\leq \mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^p\right] + (3p-1-\frac{2}{p}+2^{2p-1}\mathsf{K}_{\mathrm{Total}}^p(1+T^{\alpha p}))\int_0^t \sup_{0\leq r\leq s}\mathbb{E}\left[|\widehat{Y}_r^{\mathrm{EM}}|^p\right]\ \mathrm{d}s \\ &\quad + 2^{3p-2}\mathsf{K}_{\mathrm{Total}}^p(1+\mathbb{E}[|\hat{\theta}|^p])(1+T^{\alpha p})t + \frac{2}{p}(pM+p(p-2))^{\frac{p}{2}}t \\ &\leq e^{t(3p-1-\frac{2}{p}+2^{2p-1}\mathsf{K}_{\mathrm{Total}}^p(1+T^{\alpha p}))} \\ &\quad \times (\mathbb{E}\left[|\widehat{Y}_0^{\mathrm{EM}}|^p\right] + 2^{3p-2}\mathsf{K}_{\mathrm{Total}}^p t(1+\mathbb{E}[|\hat{\theta}|^p])(1+T^{\alpha p}) + \frac{2}{p}(pM+p(p-2))^{\frac{p}{2}}t). \end{aligned}$$

□

*Proof of Lemma 20.* Using 11, Lemma 19 and Remark 5, we have, for any $t \in [0, T-\epsilon]$ and $p \in [2,4]$,

$$\begin{aligned} &\mathbb{E}\left[|\widehat{Y}_t^{\mathrm{EM}} - \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma}|^p\right] \\ &\leq \gamma^p\mathbb{E}\left[\left|\widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma} + 2\ s(T-\lfloor t/\gamma\rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma})\right|^p\right] + \mathbb{E}\left[\left|\int_{\lfloor t/\gamma\rfloor\gamma}^t \sqrt{2}\ \mathrm{d}\overline{B}_s\right|^p\right] \end{aligned}$$

$$
\begin{aligned}
&\leq 2^{p-1}\gamma^p \left(\mathbb{E}\left[|\widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma}|^p\right] + 2^p\ \mathbb{E}\left[|s(T-\lfloor t/\gamma\rfloor\gamma, \hat{\theta}, \widehat{Y}^{\mathrm{EM}}_{\lfloor t/\gamma\rfloor\gamma})|^p\right]\right) + \gamma^{\frac{p}{2}}(Mp(p-1))^{\frac{p}{2}} \\
&\leq 2^{p-1}\gamma^p(C_{\mathsf{EM},p}(t) + 2^{3p-2}\mathsf{K}^p_{\mathrm{Total}}(1+T^{\alpha p})C_{\mathsf{EM},p}(t) + 2^{4p-3}\mathsf{K}^p_{\mathrm{Total}}(1+T^{\alpha p})(1+\mathbb{E}[|\hat{\theta}|^p])) \\
&\quad + \gamma^{\frac{p}{2}}(Mp(p-1))^{\frac{p}{2}} \\
&\leq \gamma^{\frac{p}{2}} C_{\mathsf{EMose},p},
\end{aligned}
$$

where

$$
C_{\mathsf{EMose},p} = 2^{p-1}(C_{\mathsf{EM},p}(T) + \mathsf{K}^p_{\mathrm{Total}}(1+T^{\alpha p})(2^{3p-2}C_{\mathsf{EM},p}(T) + 2^{4p-3}(1+\mathbb{E}[|\hat{\theta}|^p]))) + (Mp(p-1))^{\frac{p}{2}}.
$$

□

*Proof of Lemma 21.* By the mean value theorem, for any $k = 1, \ldots, M$, we have,

$$
b^{(k)}(t,\theta,x) - b^{(k)}(t,\theta,\bar{x}) = \sum_{i=1}^{M} \frac{\partial b^{(k)}(t,\theta,qx+(1-q)\bar{x})}{\partial y^i}(x^i - \bar{x}^{(i)}),
$$

for some $q \in (0,1)$. Hence, for a fixed $q \in (0,1)$, we have

$$
\begin{aligned}
&\left| b^{(k)}(t,\theta,x) - b^{(k)}(t,\theta,\bar{x}) - \sum_{i=1}^{M} \frac{\partial b^{(k)}(t,\theta,\bar{x})}{\partial y^i}(x^i - \bar{x}^i) \right| \\
&= \left| \sum_{i=1}^{M} \frac{\partial b^{(k)}(t,\theta,qx+(1-q)\bar{x})}{\partial y^i}(x^i - \bar{x}^{(i)}) - \sum_{i=1}^{M} \frac{\partial b^{(k)}(t,\theta,\bar{x})}{\partial y^i}(x^i - \bar{x}^i) \right| \\
&\leq \sum_{i=1}^{M} \left| \frac{\partial b^{(k)}(t,\theta,qx+(1-q)\bar{x})}{\partial y^i} - \frac{\partial b^{(k)}(t,\theta,\bar{x})}{\partial y^i} \right| |x^i - \bar{x}^i|.
\end{aligned}
$$

The proof is completed using Assumption 3. □

*Proof of Lemma 22.* At $x \in \mathbb{R}^M$ and for any $v \in \mathbb{R}^M$, we have, for any $k = 1, \ldots M$,

$$
\langle \nabla_x s^{(k)}(t,\theta,x), v\rangle = \lim_{h\to 0} \frac{s^{(k)}(t,\theta,x+vh) - s^{(k)}(t,\theta,x)}{h}.
$$

Using Assumption 3, we have

$$
\begin{aligned}
|\langle \nabla_x s^{(k)}(t,\theta,x), v\rangle| &\leq \lim_{h\to 0} \left| \frac{s^{(k)}(t,\theta,x+vh) - s^{(k)}(t,\theta,x)}{h} \right| \\
&\leq \lim_{h\to 0} \frac{|D_3(t,t)||x+vh-x|}{|h|} \qquad (80) \\
&\leq \mathsf{K}_3(1+2|t|^{\alpha})|v|.
\end{aligned}
$$

Taking $v = \frac{\nabla_x s^{(k)}(t,\theta,x)}{|\nabla_x s^{(k)}(t,\theta,x)|}$ in 80, we have

$$
|\nabla_x s^{(k)}(t,\theta,x)| \leq \mathsf{K}_3(1+2|t|^{\alpha}). \qquad (81)
$$

Using 52 and 81, we obtain

$$
\begin{aligned}
|\nabla_x b^{(k)}(t,\theta,x)| &\leq 1 + 2|\nabla_x s^{(k)}(t,\theta,x)| \\
&\leq 1 + 2\mathsf{K}_3(1+2|t|^{\alpha}).
\end{aligned}
$$

□

# D Table of Constants

Table 3 displays full expressions for constants which appear in Theorem 10.

Table 3: Explicit expressions for the constants in Theorem 10.

| CONSTANT | DEPENDENCY | FULL EXPRESSION |
|---|---|---|
| $C_1$ | $O(\sqrt{M})$ | $2(\sqrt{\mathbb{E}[\lvert X_0\rvert^2]} + \sqrt{M})$ |
| $C_2$ | $O(\sqrt{M})$ | $\sqrt{2}\left(\sqrt{\mathbb{E}[\lvert X_0\rvert^2]} + \sqrt{M}\right)$ |
| $C_3(T,\epsilon)$ | $O(e^{(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)})$ | $\sqrt{2\zeta^{-1}}e^{(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)}$ |
| $C_{\mathsf{EM},2}(T)$ | $O(Me^{T^{2\alpha+1}}T^{2\alpha+1}\widetilde{\varepsilon}_{\mathrm{AL}})$ | $e^{T(4+8\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))}$<br>$\times\,(\mathbb{E}[\lvert\widehat{Y}_0^{\mathrm{EM}}\rvert^2] + 16\mathsf{K}_{\mathrm{Total}}^2T(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2\lvert\theta^*\rvert^2)(1+T^{2\alpha}) + 2MT)$ |
| $C_{\mathsf{EM},4}(T)$ | $O(M^2e^{T^{4\alpha+1}}T^{4\alpha+1})$ | $e^{T(\frac{21}{2}+128\mathsf{K}_{\mathrm{Total}}^4(1+T^{4\alpha}))}$<br>$\times\,(\mathbb{E}[\lvert\widehat{Y}_0^{\mathrm{EM}}\rvert^4] + 1024\mathsf{K}_{\mathrm{Total}}^4T(1+\mathbb{E}[\lvert\hat{\theta}\rvert^4])(1+T^{4\alpha}) + 8(M^2+4M+4)T)$ |
| $C_{\mathsf{EMose},2}$ | $O(Me^{T^{2\alpha+1}}T^{4\alpha+1}\widetilde{\varepsilon}_{\mathrm{AL}})$ | $2(C_{\mathsf{EM},2}(T) + \mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(16C_{\mathsf{EM},2}(T) + 32(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2\lvert\theta^*\rvert^2))) + 2M$ |
| $C_{\mathsf{EMose},4}$ | $O(M^2e^{T^{4\alpha+1}}T^{8\alpha+1})$ | $8(C_{\mathsf{EM},4}(T) + \mathsf{K}_{\mathrm{Total}}^4(1+T^{4\alpha})(1024C_{\mathsf{EM},4}(T) + 8192(1+\mathbb{E}[\lvert\hat{\theta}\rvert^4]))) + 144M^2$ |
| $C_4(T,\epsilon)$ | $O(Me^{T^{4\alpha+1}}T^{4\alpha+1}\widetilde{\varepsilon}_{\mathrm{AL}}^{1/4})$ | $\sqrt{2}e^{2(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha})))(T-\epsilon)}\sqrt{T-\epsilon}$<br>$\times\Big(\mathsf{K}_4^2\zeta^{-1}(1+4T^{2\alpha})C_{\mathsf{EMose},4} + 4M(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))$<br>$+\,2\zeta^{-1}\mathsf{K}_1^2(1+8(\widetilde{\varepsilon}_{\mathrm{AL}}+\lvert\theta^*\rvert^2))$<br>$+\,4\zeta^{-1}M(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))$<br>$\times\,[(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))C_{\mathsf{EM},2}(T)$<br>$+\,32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2\lvert\theta^*\rvert^2)]$<br>$+\,2[(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}}+8\lvert\theta^*\rvert^2)^{1/2}]$<br>$\times\,[M\sqrt{2}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}]\Big)^{1/2}.$ |
| $\epsilon_\delta$ | - | $\delta^2/(64(\sqrt{\mathbb{E}[\lvert X_0\rvert^2]} + \sqrt{M})^2)$ |
| $T_\delta$ | - | $(2\widehat{L}_{\mathrm{MO}})^{-1}\left[\ln\left(4\sqrt{2}\left(\sqrt{\mathbb{E}\left[\lvert X_0\rvert^2\right]} + \sqrt{M}\right)/\delta\right) - \epsilon\right] + \epsilon$ |
| $\varepsilon_{\mathrm{SN},\delta}$ | - | $(\zeta\delta^2/32)e^{-2(1+\zeta-2\widehat{L}_{\mathrm{MO}})(T-\epsilon)}$ |
| $\gamma_\delta$ | - | $\min\Big\{(\delta/(4\sqrt{2}))^{1/\alpha}(T-\epsilon)^{-1/(2\alpha)}e^{-(2/\alpha)(1+\zeta+\mathsf{K}_3(1+2T^{\alpha}+4\mathsf{K}_3(1+4T^{2\alpha})))(T-\epsilon)}$<br>$\times\Big(\mathsf{K}_4^2\zeta^{-1}(1+4T^{2\alpha})C_{\mathsf{EMose},4} + 4M(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))$<br>$+\,2\zeta^{-1}\mathsf{K}_1^2(1+8(\widetilde{\varepsilon}_{\mathrm{AL}}+\lvert\theta^*\rvert^2))$<br>$+\,4\zeta^{-1}M(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))$<br>$\times\,[(1+16\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha}))C_{\mathsf{EM},2}(T)$<br>$+\,32\mathsf{K}_{\mathrm{Total}}^2(1+T^{2\alpha})(1+2\widetilde{\varepsilon}_{\mathrm{AL}}+2\lvert\theta^*\rvert^2)]$<br>$+\,2[(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}C_{\mathsf{EMose},2}^{1/2} + 2\mathsf{K}_1(1+8\widetilde{\varepsilon}_{\mathrm{AL}}+8\lvert\theta^*\rvert^2)^{1/2}]$<br>$\times\,[M\sqrt{2}(1+8\mathsf{K}_3^2(1+4T^{2\alpha}))^{1/2}]\Big)^{-1/(2\alpha)}, 1\Big\}.$ |