RE-EXAMINING LEARNING LINEAR FUNCTIONS IN CONTEXT

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ABSTRACT

In context learning (ICL) is an attractive method of solving a wide range of problems. Inspired by Garg et al. (2022), we look closely at ICL in a variety of train and test settings for several transformer models of different sizes trained from scratch. Our study complements prior work by pointing out several systematic failures of these models to generalize to data not in the training distribution, thereby showing some limitations of ICL. We find that models adopt a strategy for this task that is very different from standard solutions.

1 INTRODUCTION

In-context learning (ICL) Brown et al. (2020) promises to make interacting with LLMs easy and accessible. ICL enables the model to learn a task from a prompt with instructions and a few examples at inference time, without any adjustment of the model's parameters from pretraining. While there have been theoretical reconstructions of ICL, there have been few studies on exactly how ICL works in practice. However, ICL depends on a model's pretraining as Garg et al. (2022) have shown; so doing an in depth analysis of this feature of LLMs is difficult. Hence, most of analysis done on how ICL works are done on small models and simple tasks. Garg et al. (2022) makes the problem mathematically precise: the model learns a task/function given in-context examples at inference time in a next-token-prediction format Brown et al. (2020); given a prompt containing a task input-output examples $(x_1, f(x_1), .., x_n, ?)$, the model is asked to generate a value approximating $f(x_n)$.

Inspired by Garg et al. (2022), we investigated whether smaller LLMs with transformer architectures ICL the class $\mathcal{L}$ of linear functions. While Garg et al. (2022) answer "yes", we provide a more nuanced answer based on a deeper analysis. We have studied the 1 dimensional case with functions for over 30 models, from transformer architectures with 1 attention head (AH) and 1 MLP layer up 12 MLP layers and 8 AH. We also studied small attention-only models Olsson et al. (2022). Since we are interested in whether transformer models can ICL and if so how, even small transformer models are relevant, indeed essential since such an investigation requires training from scratch. Our main findings are these.

1. Several state of art papers on ICL claim that Transformer based models trained from scratch can ICL linear functions with performances close to algorithms such as Least Squares, and that a transformer can implement this algorithm. We show that the models we tested do not do this. We show that the models failed to generalize and to provide robust predictions beyond their training data. In particular, all our transformer models failed to ICL the concept of a strictly increasing or strictly decreasing linear function, even over larger intervals in $\mathbb{R}$. We replicated this behavior for several training and test distributions. We trained transformers on different distributions various Gaussian, Bimodal and Uniform distributions.
2. Our experiments show that all our models trained from scratch have 'boundary values" $(B, -B)$ for prompts x_i; when $f(x_i) > B$ or $< -B$, model performance degrades substantially. Training on uniform distributions makes this particularly clear.
3. All our transformer models solve task of ICL linear function by learning a projection from "nearby" sequences of points in the training data; Section 5 provides a mathematical formulation of what we think the models are doing. The exact projection depends upon the training distribution. The models do not implement linear regression or an algorithm like linear interpolation.

2 BACKGROUND

Statistical learning examines the application of a learned function over a test domain and the expected loss over novel applications. The ability to bring the error over test to that over the training set is typically taken to indicate an ability to generalize.

Neyshabur et al. (2017), Villa et al. (2013) define learnability in statistical learning theory via the notion of *uniform consistency*. Let μ be a distribution over $\mathcal{H}$ and μ_n the update of μ after n training samples $z_i = (x_i, y_i)$. Let A_{z_n} be an algorithm for picking out a hypothesis from $\mathcal{H}$ based on n training samples. $inf_{\mathcal{H}}$ is the hypothesis in $\mathcal{H}$ with the lowest possible error (Shalev-Shwartz et al., 2010; Kawaguchi et al., 2017).

Definition 1 *An algorithm A on a hypothesis space $\mathcal{H}$ is uniformly consistent if and only if*
$\forall \epsilon > 0 \; lim_{n\to\infty} sup_{\mu}$
$$\mu_n(\{z_n : \mathbb{E}_{\mu}(\{A_{z_n} - inf_{\mathcal{H}} \mathbb{E}_{\mu} > \epsilon\}) = 0$$

In our example, the best hypothesis $inf_{\mathcal{H}}$ is a prediction $\hat{f}$ of some target function f. The best hypothesis is when $\hat{f} = f$ with f, which yields 0 expected error. There is of course an algorithm that gives exactly the target function, linear interpolation, given two data points. Moreover linear regression is an algorithm that converges to the target function on any data set in our set up.

Definition 2 *A class of hypotheses $\mathcal{H}$ is* uniformly learnable *just in case there exists a uniformly consistent algorithm for $\mathcal{H}$.*

The class of linear functions $\mathcal{L}$ is clearly uniformly learnable. What is left open here is the choice of distribution of the data both for train and test and the sampling method (since our class is uncountably large). Garg et al. (2022) take a definition of learning where average expected error goes to 0 when data in train and test are sampled both from the same normal distribution. So for instance this means that the model will see linear functions with $a, b \in [-1, 1]$ around 70% of the time. This can make a big difference on the error reported for learning a class of mathematical functions like $\mathcal{L}$, whose definition does not in any way depend on a particular distribution or sampling. Nevertheless, we would hope the model has found an algorithm such that $\hat{f} = f$ given a test set of linear functions with $a, b \notin [-1, 1]$. We also hope the algorithm will transfer to different distributions. This is what we investigate below.

3 RELATED WORK

Since Brown et al. (2020) introduced ICL, there has been considerable research indicating that ICL is possible because of a sort of gradient "ascent" Akyürek et al. (2022); Von Oswald et al. (2023). Dong et al. (2022) provides an important survey of successes and challenges in ICL and that so far, only simple problems for ICL have been analyzed, eg the case of linear or simple Boolean functions.

Garg et al. (2022) offered an important advance showing that a Transformer trained from scratch (GPT-2 with an embedding size of 256) performed in-context learning of n-dimensional linear functions given identical train and test distributions $N(0, 1)$.

Further research then offered several theoretical reconstructions for how ICL for linear functions might work in Transformers. Von Oswald et al. (2023); Ahn et al. (2023); Mahankali et al. (2023) provided a construction to show transformers ICL from their doing gradient descent during ICL. Fu et al. (2023) showed that Transformers could ICL in virtue of using higher-order optimization techniques. Xie et al. (2021); Wu et al. (2023); Zhang et al. (2023); Panwar et al. (2023) argued that ICL follows on a Bayesian point of view. Bai et al. (2024) show that transformers can under certain assumptions implement many algorithms with near-optimal predictive power on various in-context data distributions. Giannou et al. (2024); Zhang et al. (2024) modify transformers with linear attention and Zhang et al. (2024) introduce a new training regime to show that modified transformers can learn linear functions. Given Pérez et al. (2021)'s result that full transformers with linear attention are Turing complete, however, these theoretical demonstrations are perhaps not surprising. Xie et al. (2021); Zhang et al. (2024) examine how ICL works despite differences between training and inference distributions.

Unlike this prior research, we examine how ICL works in practice under different training and testing distributions in order to establish what transformers *actually* do in ICL 1 dimensional linear functions, whereas most prior research has concentrated on transformer models *can* or *could* do on this task. Even for this simplest case, we show transformers ICL in a different way from any of these proposed methods.

Bhattamishra et al. (2023) trained small GPT-2 models from scratch to show that Transformers can ICL simple boolean functions, while their performance deteriorates on more complex tasks. Wu et al. (2023) studied ICL by pretraining a linearly parameterized single-layer linear attention model for linear regression with a Gaussian prior proving that the pretrained model closely matches the Bayes optimal algorithm. Raventós et al. (2024) investigated whether models with ICL can solve new tasks very different from those seen during pretraining.

Olsson et al. (2022) offer an in depth analysis of ICL across tasks using a general evaluation measure on prompt length. They propose that a learned copying and comparison mechanism known as an *induction head* is at the heart of ICL.

4 Experiments

In this section, we show that: (i) models do not implement linear regression; (ii) this performance holds across different types of distributions; (iii) these distributions all show the presence of boundary values beyond which the models do not perform well; (iv) models with attention layers (AL) (models with at least two AL only or 1 AL+MLP layer) are needed to give an ICL effect (v) ordering and restricting the order of prompts can improve performance. In the last subsection, we put all of these observations together.

Following Garg et al. (2022) we trained several small decoder only transformer models from scratch to perform in-context learning of linear functions.[1] We set the number of layers (L) from 1 to 6, and attention heads (AH) from 1 to 4. We also trained a 9L6AH model and the 12L8AH GPT2 with an embedding size of 256. The task of the model is to predict the next value for $f(x_i)$ through a prompt of type $(x_1, f(x_1), ..., x_i)$. We refer to that prediction as $\hat{f}(x_i)$. To train the model $\mathcal{L}$ to ICL, we looked for a θ^* that optimizes the following auto-regressive objective:

$$\theta^* = \arg\min_{\theta} \mathbb{E}_{x_i \in D_I, f \in D_F} \left[\sum_{i=0}^{k} l\left(f\left(x_{i+1}\right), \mathcal{L}_\theta\left((x_1, f(x_1), ..., f(x_i), x_{i+1})\right)\right) \right]$$

where $\mathcal{L}_\theta$ is a learner, $l : (y, \hat{y}) \rightarrow ||y - \hat{y}||^2$ is squared error and $f : x \rightarrow ax + b$ is a linear function with a, b chosen at random according to some training distribution for functions D_F and samples x_i picked randomly according to a training distribution for points D_I. To simplify, we will note that $f \in D_F, x \in D_I$. We choose at random a function $f \in D_F$ and then a sequence of points $x_i \in D_I$ at random, random prompts, from a distribution D_I at each training step. We update the model through a gradient update. We use a batch size of 64 and train for 500k steps. The models saw over 1.3 billion training examples for each distribution we studied. For D_F and D_I we used several distributions: the normal distribution $N(0, 1)$, "rectangle" or uniform distributions over given intervals and bimodal distributions.

In comparing how model performance evolves with parameters like the number of layers of the model or number of attention heads, we tested the models on a variety of test distributions for both functions D_F^t and data points or prompts D_I^t. But while in train we always take the same distribution ($D_F = D_I$), in test, we sometimes take $D_F^t \neq D_I^t$. To see how the model performs in ICL relative to (D_I^t, D_F^t), we generate a set of $N = 100$ functions in D_F^t; and our data samples for test are composed of $N_b = 64$ batches, each containing $N_p = 41$ points in D_I^t. In each batch b, for all points, we predict for each x_k^b, $k \geq 2$, $f(x_k^b)$ given the prompt $(x_1^b, f(x_1^b), ..., x_{k-1}^b, f(x_{k-1}^b), x_k^b)$. We calculate for each function the mean average over all the points N_p of all batches N_b, then do a mean average over all functions. Formally this is:

$$\epsilon_\sigma = \frac{1}{N} \Sigma_{i=1}^{N} \Sigma_{b=1}^{N_b} \frac{1}{N_b} (\frac{1}{N_p} \Sigma_{i=3}^{N_p} (pred_i^b - y_i^b)^2)$$

[1] Our code can be found in https://anonymous.4open.science/r/incontext-learning-556D/

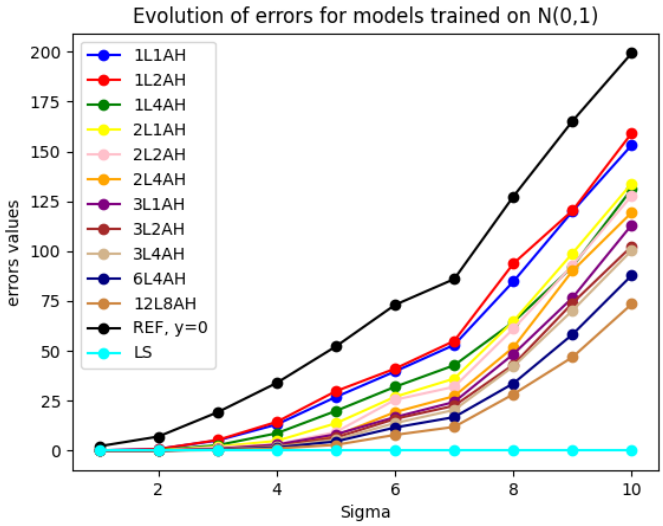

Figure 1: Evolution of error rates for various models with $D_F, D_I = D_I^t = N(0,1)$ and D_F^t for various $N(0,\sigma)$. The black curve illustrates a model that predicts $f(x_n) = 0, \forall f$ and $\forall x_n$. The cyan line LS represents linear or ridge regression, which is trivially a perfect estimator given our totally clean input data.

In all our error calculations, we exclude the first two predictions of each batch from the squared error calculation, since we need at least two points to be able to find a linear function and the first two predictions by the model are hence almost always wrong.

4.1 Models do not implement linear regression

When trained on $D_F = D_I = N(0,1)$ and the target functions had values in [-1, 1], even small models were able to converge to a 0 average error. The error was not always identical to 0 at least in some batches but rather similar to Liu et al.'s finding on MSE estimation by transformers.

On the other hand, all the models had systematic and non 0 average error once we chose the target $f \in D_F^t = N(0,\sigma)$ for $\sigma > 2$. Figure 1 shows that the error rate increases substantially and non-linearly as $D_F^t = N(0,\sigma)$ and σ increases. To ensure that comparisons between models are meaningful, for each $N(0,\sigma)$, we set a seed when generating the 100 random linear functions, ensuring that each model sees the same randomly chosen functions and the same set of prompting points x_i. The table 3 in the Appendix contains the full figures for average error.

The results in Figure 1 and Table 3 (Table 3 is in Appendix B) confirm that at least the larger models are able to generalize somewhat to unseen examples, given that all the curves in Figure 1 have lower error rates than the baseline that predicts $\hat{f}(x_n) = 0$ everywhere. But their generalizing ability was far from perfect; and contrary to what Akyürek et al. (2022); Von Oswald et al. (2023) have suggested, the models did not use linear regression to ICL the target function. If they had, we would not see the error patterns we do.

Our models' performance depends on how often it has seen examples "similar" to the target function value it is trying to predict. When $D_F = N(0,\sigma)$ there is an over 68% chance that a function chosen for train $f(a,b)$ will have $a, b \in [-\sigma, \sigma]$ and over a 95% chance it will have $a, b \in [-2\sigma, 2\sigma]$. So a model with $D_F = D_I = N(0,1)$ has seen sequences of values for $f(a,b)$ with $a, b \in [-2, 2]$ more than 95% of the time. Given a pretraining with over a billion examples, models will have seen prompts for functions with $a, b \notin [-2, 2]$, just not many of them. As the models are tested with $D_F^t = N(0,\sigma)$ and so confronted with more sequences representing functions $f(a,b)$ for $a, b \notin [-2, 2]$, all the models do less and less well.

4.2 REPLICATING SECTION 4.1 RESULTS FOR MODELS TRAINED ON OTHER DISTRIBUTIONS

We've just seen that when the distribution of training data follows a simple Gaussian $N(0,1)$, the models, for any number of layers and attention head, give good results when $D_F^t = D_I^t = N(0,1)$, but offer degraded performance as we test on distributions $N(0,\sigma)$ for larger σ. We now show that same sort of behavior with different training distributions but all tested on $N(0,\sigma)$.

Training on bimodal distributions We tested how our models fared with a bimodal distribution of training data. Our strategy was to increase the values the model can see during training, by extending the distribution seen during training. We trained our GPT2 on the bimodal distribution $0.5N(-1,1)+0.5N(1,1)$. With this training, the model was more likely to see wider values during training and therefore work better on larger values.

Most of the models we tested had more robust performance with a bimodal distribution for $D_F = 0.5N(-1,1) + 0.5N(1,1)$ than they did with $D_F = N(0,1)$ at least with $D_F^t = D_I^t = N(0,\sigma)$ and $n \geq 6$. The best models had almost equally good performance on $D_F^t = N(0,\sigma)$ for $\sigma \leq 3$ and superior performance with $D_F^t = N(0,\sigma)$ for $\sigma \geq 3$, as can be seen from Table 1. For the values of the table, we took $D_I^t = U(-1,1)$, the uniform distribution over $[-1,1]$, but the results remain similar when taking $D_I^t = N(0,1)$. The fact that performance varies with the distribution should not happen, if the models were using gradient descent to compute linear regression in ICL.

Training on uniform distributions To have more control on the notion of maximum and minimum values the models saw, we next trained our models on uniform distributions. We illustrate with $U(-5,5)$. Given the observations of Section 4.1 concerning the errors our models made on functions with large coefficients, we wanted to study whether these errors arose because the models hadn't encountered functions with such large coefficients in pretraining. By keeping D_F, D_I normal or bimodal, we can't control "the largest value the model could see", because it's always possible that it could have generated a large value during training. By training on a uniform distribution, however, we know exactly what the smallest and largest values that the model could have seen in its training. For example, setting D_F, D_I to $U(-5,5)$, the largest value the model could have seen is $30 = 5{*}5{+}5$ and the smallest value it could have seen is -30. Most likely it saw values significantly > -30 and < 30.

Training with $U(-5,5)$ gave good results for $D_F^t = D_I^t = U(-1,1)$. Models were able to find target functions with coefficients in [-1,1] from only 2 points (see leftmost plot of Figure 5 in Appendix C); and all our models work well when D_F, D_I, D_F^t, D_I^t use the same distribution. The models trained on a uniform distribution sometimes do even better than models trained on N(0,1) or a bimodal distribution–up to three times better for $D_F^t = D_I^t = N(0,9)$ as Table 1 shows. Learning was at times very efficient, requiring just two prompts, as in Figure 5 (Appendix B).

4.3 ERROR ANALYSIS, SIGMOID APPROXIMATIONS AND BOUNDARY VALUES

We wanted to look at what sorts of errors our models made for $f(a,b) \in \mathcal{L}$ for $a, b \notin [-2,2]$– i.e. the values of $\hat{f}(x)$ outside the interval that includes the vast majority they have seen. Our models exhibit problematic behavior of 2 kinds. Even our best models, for $a, b \notin [-2,2]$ but reasonably close, say in $[-9,9]$, predict $\hat{f}(x)$ to a sigmoid-like function with correct estimates for the target function within a certain interval.[2] Consider Figure 2 for an illustrative $f(x) = 8x + 9$. While the left plot of Figure 2 shows that the model's prediction $\hat{f}(x)$ diverges dramatically from $f(x)$ outside of a certain interval, the right plot of Figure 2 shows the model has learned something with ICL about the function from the prompt and approximates it at least within a certain interval.

For equations with coefficients highly unlikely to be sampled in $N(0,1)$ (for example $f(x) = 30x + 30$ in Figure 3), however, the results are catastrophic. Figure 3 shows that the model's prediction $\hat{f}$ doesn't converge to any stable prediction with ICL. This happens across a range of models.

The same pattern holds when $D_F, D_I = U(-5,5)$. Consider again as an illustrative example the target function, $f(x) = 9x$ for our largest trained model. The model approximates $f(x)$ well within

[2] Giannou et al. (2024) tested a different type of model, with linear attention only, and showed similar behavior.

models / σ	1	2	3	4	5	6	7	8	9	10
$3L4AH_N, d_{emb} = 64$	0.0	0.0	0.22	0.4	1.73	6.56	8.56	20.44	39.73	53.93
$3L4AH_B, d_{emb} = 64$	0.03	0.15	0.53	1.32	2.74	3.91	5.52	10.22	13.86	22.72
$3L4AH_U, d_{emb} = 64$	0.02	0.03	0.13	0.36	0.84	1.79	2.54	7.06	11.38	17.75
$6L4AH_N, d_{emb} = 64$	0.0	0.0	0.2	0.38	1.58	5.72	7.99	15.53	32.96	50.35
$6L4AH_B, d_{emb} = 64$	0.01	0.04	0.23	0.44	1.19	2.15	3.08	4.8	9.98	18.01
$6L4AH_U, d_{emb} = 64$	0.02	0.04	0.11	0.24	0.57	1.36	1.82	4.62	10.23	15.07
$12L8AH_N, d_{emb} = 256$	0.0	0.0	0.32	1.34	3.14	8.8	12.13	30.14	49.37	73.93
sorted $12L8AH_N$	0.0	0.01	0.32	1.63	3.69	8.39	10.06	27.11	43.23	58.56
$12L8AH_B, d_{emb} = 256$	0.0	0.01	0.08	0.29	0.78	2.23	3.66	9.04	18.68	30.23
sorted $12L8AH_B$	0.01	0.03	0.18	0.25	0.74	2.27	2.62	6.87	13.73	20.8
$12L8AH_U, d_{emb} = 256$	0.0	0.01	0.13	0.71	1.92	6.78	10.92	27.91	38.75	64.39
sorted $12L8AH_U$	0.01	0.01	0.13	0.75	2.12	6.18	10.5	26.8	36.3	53.48
$REF_{D_F^t, D_I^t}$**: y=0**	1.52	4.43	13.55	19.94	30.81	44.75	52.71	76.11	105.43	128.52

Table 1: Comparison showing the evolution of squared errors for models trained on different distributions; index N: $D_F = N(0,1)$, B $D_F = 0.5N(-1,1) + 0.5N(1,1)$ and $D_F = U(-5,5)$. We show error rates for models prompted without and with the natural ordering on the prompts [sorted], for the large model size. $D_i^t = U(-1,1)$ and $D_F^t = N(0,\sigma)$

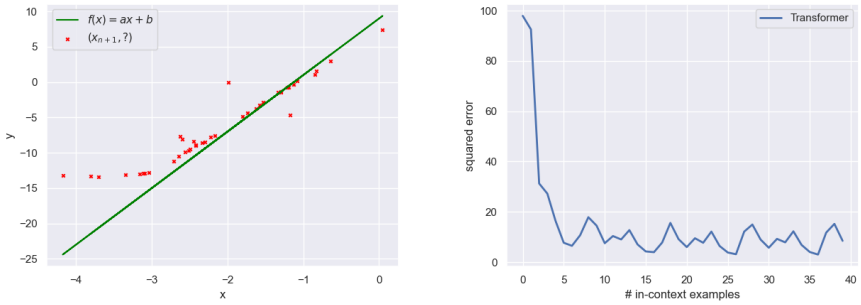

Figure 2: Plots of predictions of the 12L8AH model trained on $N(0,1)$ and error evolution over number of prompts for $f(x) = 8x + 9$

a certain range, but it predicts $\hat{f}(x)$ to be a constant function for intervals outside of that range. As shown in the left plot in Figure 6 in Appendix D, $\hat{f}^+(v) \approx 30$ for values v for which the ground truth target function f is such that $30 \leq f(v)$, and the model predicts an approximally constant function $\hat{f}^-(v) \approx -30$ for values v on which $f(v) \leq -30$. We call values like -30 and 30 where the model starts to predict constant functions $\hat{f}(v)$ and $\hat{f}^-(v)$ *boundary values*. For models trained on $U(-5,5)$, 30 and -30, with $30 = 5 * 5 + 5$ and $-30 = -5 * 5 - 5$, are respectively the biggest and smallest values the model could have seen during training. If such a model hasn't seen a value above 30 or below -30, it won't infer one.

This behavior just noted for our largest model remains true for other models M values $\hat{f}(v) < B - \alpha$ or $\hat{f}(v) > B + \alpha$, where α is a constant determined by M. However for functions and data samples when the values of $f(x)$ in the prompt sequence are such that $f(x) > B + \alpha$ or $< -B - \alpha$—an example is $f(x) = 40x + 40$—the model starts to make a mess of things, assigning $\hat{f}(v)$ random values for $f(v)$ far away from B (i.e $> B + \alpha$ or $< -B - \alpha$, as seen in the rightmost plot in Figure 6. The model loses its capacity to model the target function anywhere.

All our models trained on $U(-5,5)$ estimate the target function more or less well for x with $f(x) \in [-B, B]$ with boundary values $-B, B$; but once we are outside $[-B, B]$, the estimations

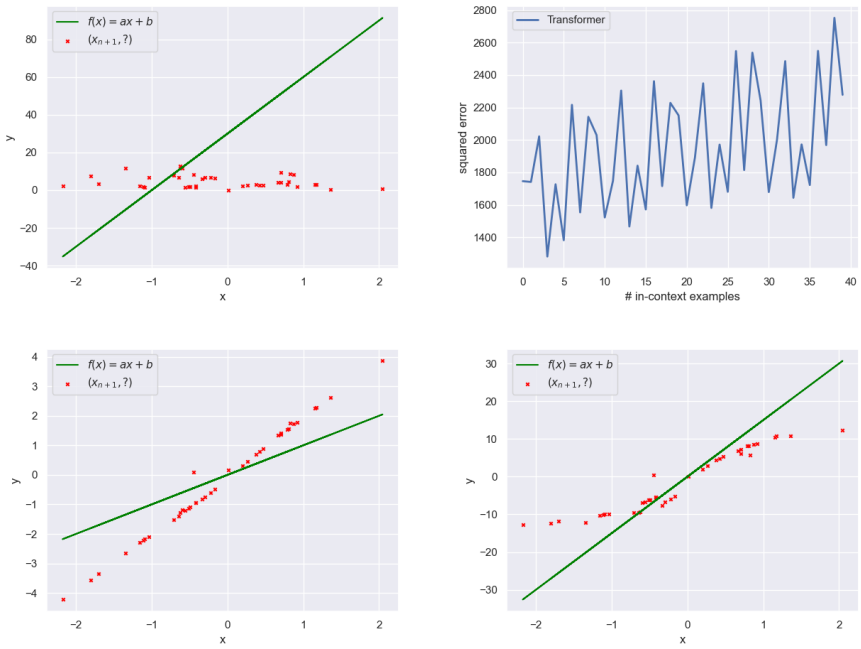

Figure 3: Plots on first line of predictions for the 12L8AH model trained on $N(0,1)$ and error evolution over number of prompts for $f(x) = 30x + 30$. On second line Plots for $f(x) = x$ and $f(x) = 15x$ for models 2L attention only with 32AH and $d_{embedding} = 256$

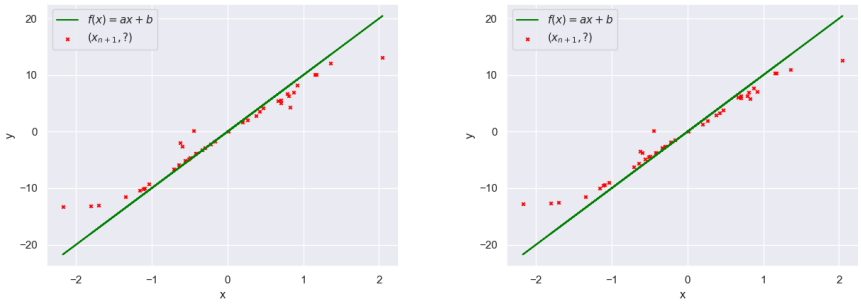

Figure 4: Plots for $f(x) = 10x$ by a 12l8ah model and by a 6l4ah model.

become constant functions or chaotic. Different models trained on different uniform distributions give different boundary values; for instance see Figures 3 or 6 Appendix F.

Boundary values for models on normal and bimodal distributions will vary, as the largest values in the sequences the model has seen and the number of times it has seen those values will vary depending on the sampling. We have found that larger models will have slightly larger boundary values $|B|$ than smaller ones. For instance, Figure 4 shows the plots for the predictions of two models (12L8AH, and 6L4AH) with both $D_F, D_I = N(0,1)$ for the equation $f(x) = 10x$. The larger model has boundary values $\approx$ -13.7, 13.7, the smaller one boundary values $\approx$ -12, 12.

Constraints from boundary values hold regardless of model size (for plots see Appendix D and Figure 7) and also hold for attention only models (See Appendix E, Figure 8). Larger models trained on the same distribution and the same number of data are able to ICL $\mathcal{L}$ functions over a slightly larger number of intermediate values than smaller models, as Figure 1 suggests.

4.4 PREDICTIONS FOR MODELS WITH ONLY ATTENTION LAYERS OR WITH ONLY MLP

To understand better which components in the transformer architecture are responsible for ICL, we tested various components. We found that attention layers (AL) were the important components for ICL but ICL only worked reasonably well when the model had 2 AL (see also figure 3). Beyond 2 AL what mattered most was the number of attention heads (whether they are summed over all layers or counted within a layer). A single AL model had only a very limited ICL generalization capability beyond testing on $D_F^t = N(0,1)$, but it did better than a 12 layer MLP, which showed no ICL capability, probably because the method of training on the predict next token format is not suitable for models without attention heads. The details of various AL models are found in Table 2.

models / σ	1	2	3	4	5	6	7	8	9	10
$1AL1AH_U$	0.38	2.29	9.3	14.97	25.25	37.54	45.4	67.0	95.19	117.6
$2AL8AH_U$	0.1	0.62	5.53	10.59	18.62	30.61	36.97	57.79	83.26	103.58
$3AL4AH_U$	0.35	1.42	8.17	15.13	24.15	37.99	45.2	68.73	96.37	118.3
$3AL8AH_U$	0.12	1.16	5.45	9.36	18.22	28.77	35.62	52.44	78.12	100.18
$2Al32AH_N$	0.06	0.91	5.96	10.43	18.96	30.11	36.77	55.59	81.66	103.17
$REF_{D_F^t,D_I^t} : y = 0$	1.52	4.43	13.55	19.94	30.81	44.75	52.71	76.11	105.43	128.52

Table 2: Comparison showing the evolution of squared errors for models with attention layers only. We give figures for a model with only 1 attention layer/1AH (1AL1AH) two 2-attention layer only models (2AL8AH, 2AL32AH) and two 3 attention layer only model (3AL4AH,3AL8AH). $D_I = D_F = U(-1,1)$, $D_i^t = U(-1,1)$ and $D_F^t = N(0,\sigma)$. All models have embeddings of size 64, except $2Al32AH$ has size 256.

4.5 ORDERING AND RESTRICTING THE SIZE OF PROMPTS CAN IMPROVE PERFORMANCE

Our experiments with distributions also showed that a model performance improves when the sequence of prompts for the x_i follows or are "sorted" to follow the natural order on $\mathbb{R}$, especially for bigger models. Error rates were comparable to the original models without sorting for small test values of σ with $D_F^t = N(0,\sigma)$ and substantially lowered error rate, by up to a third depending on the training distribution, from the unsorted models. The details concerning the sorted models are in Table 1.

While at least 2 points are needed to find a linear function, we noticed that model performance degrades when the size of the prompt during inference is greater than the maximal size of prompts the model saw during training, as the rightmost plot in Figure 5 shows (Appendix C). This held over all models and training distributions. This means that the model takes into account the whole sequence in its calculations, not just the last two or three data points. Had the model only looked at a small fixed subsequence, large size prompts in inference would not have affected model behavior.

4.6 TAKING STOCK: WHAT ARE THE MODELS LEARNING

To ICL $\mathcal{L}$, we expected a transformer model given $(x_1, f(x_1), ..., x_n), ?)$ to perform a linear regression. The hypothesis and theoretical constructions of Akyürek et al. (2022); Von Oswald et al. (2023) shows that transformer models can perform linear regression with 1 step of gradient descent. If that were the case, the models should generalize without difficulty. But this is clearly not what we observed. Error rates depend on training distributions D_F, D_I and on the distance of the target function's values from the majority of the data points in the model's training. We've also demonstrated model sensitivity to the entire sequence of ICL prompts. This shows that the models did not learn to use linear regression to solve this task. All our models failed to learn the concept of a strictly monotone increasing or decreasing linear function in $\mathcal{L}$ over arbitrarily large or at least many large intervals of $\mathbb{R}$.[3]

[3]This makes sense in terms of Asher et al. (2023)'s characterization of learnability. The concept of a strictly monotone increasing or decreasing linear function describes a Π_1^0 set in the Borel hierarchy which Asher et al. (2023) show is not learnable using ordinary LLM assumptions.

The lack of generalizability might suggest our models overfit the data. However, the pretraining data has no noise, and it's too large to be memorized by our models (our largest models with 256 size embeddings have $< 10^7$ parameters; each parameter would have to encode on average sequences for over 100 different functions). Moreover, our models performed similarly on several different training distributions for D_F and D_I and tested on $N(0, \sigma)$ for $\sigma \in \{1, 2\}$. Given that 100 samplings with $D_F^t = N(0, 1)$ nets on average 20 functions with coefficients the model with $D_F = D_I = U(-1, 1)$ has not seen in training, we would expect the model's performance to degrade more substantially than it did. This implies that the models didn't overfit to their training regimes.

Our error analysis has shown us the existence of boundary values, values for which models do well on the interval defined by the boundary values but degrade outside of them. These boundary values fluctuate depending on model training distributions. This is further evidence that the model's method for solving the task does not involve a real calculation of a linear function but an adjustment of values that the model has seen.

5 How might the models be learning?

Olsson et al. (2022) argue that a copying and comparison mechanism (induction head) is at the heart of ICL. They show that induction heads do not exist for 1 layer attention-only models but do for attention-only models with two or more layers. They empirically establish a strong connection between the formation of induction heads in models with greater than 2 attention layers and the model's ability to ICL. Our experiments with attention-only models showed that multiple attention layers were needed for ICL. Attention-only models with induction heads could ICL linear functions reasonably well, at least in when $D_F = D_F^t$; and in fact the large 2 attention only layer model with 32 AH was more robust than the full transformer model with 1 (attention and MLP layer) and 1 or 2 AH (See Table 3 Appendix B). Olsson et al. (2022) also note that induction head behavior is possible with 1 attention + MLP layer.

Our *induction head hypothesis* is that the induction heads predict a value for $f(x_n)$ given a prompt sequence $\vec{x} = (x_{1,1}, x_{1,2}(= f(x_1)), x_{2,1}, x_{2,2}, ...x_{n,1}, ?)$ by using a projection from similar sequences or subsequences in the training, $\vec{y} = (y_{1,1}, y_{2,2}...y_{n,1}, y_{n,2})$, with $x_{i,1}$ close to $y_{i,1}$ for some j and $x_{i,2}$ close to $y_{j,2}$. Given the effects of prompt length on performance, we know that the whole sequence matters with $p_2 \leq p_1$ for optimal predictions. This is evidence for a pointwise comparison like we are proposing (which is more complicated and potentially more accurate than simply averaging the $y_{n,2}$ of the three closest $y_{n,1}$ neighbors of $x_{n,1}$).

Olsson et al. (2022) report that larger models' induction heads can exploit sequences $\vec{y}$ that are "more dissimilar" to each other than smaller models can. The fact that the larger models respond well to prompts ordered according to the natural ordering on $\mathbb{R}$ suggests that larger models with more attention heads can exploit comparing sequences that converge or diverge from the target sequence $\vec{x}$ as the prompts $x_{i,1}$ near $x_{n,1}$ increase or decrease. And they can compare in different ways. For smaller models that did not improve performance with respect to naturally ordered prompts, the hypothesis suggests they are restricted to simpler operations on the sequences they have seen. This and our observations about boundary values provide further empirical support for the induction head hypothesis.

Given boundary values, $-B, B$, all or the vast majority of the sequences the model has seen have values z_i with $-B < z_i < B$. If the target sequence $\vec{x}$ has maximum values $-B < x_i < B$, i.e. $-B < Maxval_{x_i}\vec{x} < B$, then chances are high that the model will find a weighted set of sequences Y close to the test sequence $\vec{x}$ and compute bounds for $x_{n,2} = f(x_n))$. Call the sequence $\vec{y}$ generated by a function $g = a_g x + b_g$ a *g-sequence*. We assume the standard measure over sequences. Given a g-sequence $\vec{y}$ closest to the f-sequence $\vec{x}$ generated by target function $f = a_f x + b_f$ is such that $a_g = a_f$, then the model will be able to approximate $x_{n,2}$ by averaging the distances between $y_{i,2}$ and $x_{j,2}$ for the closest $y_{i,1}$ to $x_{j,1}$ for all $x_{j,1}$. Now suppose that the closest g-sequences Y are not all such that $a_f = a_g$. The model must now construct a function $\mathfrak{h}(Y_{\vec{x}}, \vec{x})$ that computes a distance d between the values it has seen in $Y_{\vec{x}}$ and the targets $\vec{x}$ for some optimized set $Y_{\vec{x}}$ of sequences close to $\vec{x}$. If $\mathfrak{h}(Y_{\vec{x}}, \vec{x})(x_{k,1}) = z_{k,2}$ is the k-th member of $\mathfrak{h}(Y_{\vec{x}}, \vec{x})$, we optimize $\mathfrak{h}$ such that $|z_{k,2} - x_{k,2}|$ is minimized for all k. The model then averages these distances to yield an "average" $\mathfrak{h}(Y_{\vec{x}}, \vec{x})$ to compute $z_{2,n} = \hat{f}(x_{1,n})$. The larger the set very close $\vec{y} \in Y_{\vec{x}}$, the better the projection and hence the prediction.

For prompts outside the boundary values $-B, B$, the closest $\vec{y}$ are those with values near the boundary ($y_{n,2} \approx B(-B)$). So the model will predict $x_{n,2} \approx B(-B)$, if it uses this method of projection. So in sum, we think the model estimates a set Y of closest sequences to the target $\vec{x}$ and computes:

$$\hat{f}(x_n) = x_{n,2} = \frac{1}{n}\sum_{i=1}^{n} \mathfrak{h}(Y_{\vec{x},x_i})(x_{n,1}), \textit{ for } -B < Maxval_{x_i}\vec{x} < B$$

$$\textit{and } \hat{f}(x_n) \approx B(-B), \text{if } Maxval_{x_i}\vec{x} < -B - \alpha_{\mathcal{L}}, \textit{ or } Maxval_{x_i}\vec{x} > B + \alpha_{\mathcal{L}}$$

$$\textit{Otherwise} \hat{f}(x_n) \textit{ takes a random value} \in [-B, B],\ \alpha_{\mathcal{L}} > 0 \textit{ a characterstic model value}$$

The observations we have made are clearly compatible with this hypothesis (See Table 6 Appendix F) and the weighted averages are calculable in a 2 layer Attention only model with suitable heads. This also explains why training the model with smaller sequences in a kind of "curriculum learning" may not be helpful. We trained the same 12L8AH model once with curriculum learning which supposedly helps the model perform better for different types of size prompts and generalize. and the same without curriculum and found that the model without curriculum performs better. This is consistent with our hypothesis, since the model without curriculum looks at batch sizes of up to 41 points in each step, and can therefore see more sequences than the model with curriculum, which looks at batch sizes of up to 41 points. The induction head hypothesis is less precise then linear regression but can approximate it given an appropriate set Y. Our induction head hypothesis predicts that model performance will be sensitive to a choice of training distribution for D_F, D_I as well as a choice of test distributions and of course to the presence of boundary values.

6 Conclusion

In this paper we have identified a systematic failure case of decoder-only transformer models of various sizes (up to 9.5 million parameters) and architectures. All models failed to learn robustly the class of linear functions on non-noisy data, a task which is entirely determined by only two points and involves a trivial mathematical operation that has been documented to be by construction learnable by LLMs. However, the models did learn something different that enabled them to approximate linear functions over intervals where their training gave lots of examples. Rather than learning a standard algorithm for the task, these models instead perform a kind of projection from close sequences seen during the training.

Our investigations perforce focus on relatively small models. But our study highlights a broader issue with ICL: the gap between what LLMs *can* learn and what they *actually* learn. Much larger models also face this limitation. The minimality of our examples and the capacity to easily train the models from scratch is actually a key strength of our study. We hope this contribution will inspire further research into higher-level failure modes for transformer-based models.

References

Kwangjun Ahn, Xiang Cheng, Hadi Daneshmand, and Suvrit Sra. Transformers learn to implement preconditioned gradient descent for in-context learning. *Advances in Neural Information Processing Systems*, 36:45614–45650, 2023.

Ekin Akyürek, Dale Schuurmans, Jacob Andreas, Tengyu Ma, and Denny Zhou. What learning algorithm is in-context learning? investigations with linear models. *arXiv preprint arXiv:2211.15661*, 2022.

Nicholas Asher, Swarnadeep Bhar, Akshay Chaturvedi, Julie Hunter, and Soumya Paul. Limits for learning with large language models. In *12th Joint Conference on Lexical and Computational Semantics (*Sem)*. Association for Computational Linguistics, 2023.

Yu Bai, Fan Chen, Huan Wang, Caiming Xiong, and Song Mei. Transformers as statisticians: Provable in-context learning with in-context algorithm selection. *Advances in neural information processing systems*, 36, 2024.

Satwik Bhattamishra, Arkil Patel, Phil Blunsom, and Varun Kanade. Understanding in-context learning in transformers and llms by learning to learn discrete functions. *arXiv preprint arXiv:2310.03016*, 2023.

Tom Brown, Benjamin Mann, Nick Ryder, Melanie Subbiah, Jared D Kaplan, Prafulla Dhariwal, Arvind Neelakantan, Pranav Shyam, Girish Sastry, Amanda Askell, et al. Language models are few-shot learners. *Advances in neural information processing systems*, 33:1877–1901, 2020.

P Kingma Diederik. Adam: A method for stochastic optimization. *(No Title)*, 2014.

Qingxiu Dong, Lei Li, Damai Dai, Ce Zheng, Jingyuan Ma, Rui Li, Heming Xia, Jingjing Xu, Zhiyong Wu, Tianyu Liu, et al. A survey on in-context learning. *arXiv preprint arXiv:2301.00234*, 2022.

Deqing Fu, Tian-Qi Chen, Robin Jia, and Vatsal Sharan. Transformers learn higher-order optimization methods for in-context learning: A study with linear models. *arXiv preprint arXiv:2310.17086*, 2023.

Shivam Garg, Dimitris Tsipras, Percy S Liang, and Gregory Valiant. What can transformers learn in-context? a case study of simple function classes. *Advances in Neural Information Processing Systems*, 35:30583–30598, 2022.

Angeliki Giannou, Liu Yang, Tianhao Wang, Dimitris Papailiopoulos, and Jason D Lee. How well can transformers emulate in-context newton's method? *arXiv preprint arXiv:2403.03183*, 2024.

Kenji Kawaguchi, Leslie Pack Kaelbling, and Yoshua Bengio. Generalization in deep learning. *arXiv preprint arXiv:1710.05468*, 2017.

Jerry Weihong Liu, Jessica Grogan, Owen M Dugan, Simran Arora, Atri Rudra, and Christopher Re. Can transformers solve least squares to high precision? In *ICML 2024 Workshop on In-Context Learning*.

Arvind Mahankali, Tatsunori B Hashimoto, and Tengyu Ma. One step of gradient descent is provably the optimal in-context learner with one layer of linear self-attention. *arXiv preprint arXiv:2307.03576*, 2023.

Behnam Neyshabur, Srinadh Bhojanapalli, David McAllester, and Nati Srebro. Exploring generalization in deep learning. *Advances in neural information processing systems*, 30, 2017.

Catherine Olsson, Nelson Elhage, Neel Nanda, Nicholas Joseph, Nova DasSarma, Tom Henighan, Ben Mann, Amanda Askell, Yuntao Bai, Anna Chen, et al. In-context learning and induction heads. *arXiv preprint arXiv:2209.11895*, 2022.

Madhur Panwar, Kabir Ahuja, and Navin Goyal. In-context learning through the bayesian prism. *arXiv preprint arXiv:2306.04891*, 2023.

Jorge Pérez, Pablo Barceló, and Javier Marinkovic. Attention is turing-complete. *Journal of Machine Learning Research*, 22(75):1–35, 2021.

Allan Raventós, Mansheej Paul, Feng Chen, and Surya Ganguli. Pretraining task diversity and the emergence of non-bayesian in-context learning for regression. *Advances in Neural Information Processing Systems*, 36, 2024.

Shai Shalev-Shwartz, Ohad Shamir, Nathan Srebro, and Karthik Sridharan. Learnability, stability and uniform convergence. *The Journal of Machine Learning Research*, 11:2635–2670, 2010.

Silvia Villa, Lorenzo Rosasco, and Tomaso Poggio. On learnability, complexity and stability. In *Empirical Inference*, pp. 59–69. Springer, 2013.

Johannes Von Oswald, Eyvind Niklasson, Ettore Randazzo, João Sacramento, Alexander Mordvintsev, Andrey Zhmoginov, and Max Vladymyrov. Transformers learn in-context by gradient descent. In *International Conference on Machine Learning*, pp. 35151–35174. PMLR, 2023.

Jingfeng Wu, Difan Zou, Zixiang Chen, Vladimir Braverman, Quanquan Gu, and Peter L Bartlett. How many pretraining tasks are needed for in-context learning of linear regression? *arXiv preprint arXiv:2310.08391*, 2023.

Sang Michael Xie, Aditi Raghunathan, Percy Liang, and Tengyu Ma. An explanation of in-context learning as implicit bayesian inference. *arXiv preprint arXiv:2111.02080*, 2021.

Ruiqi Zhang, Spencer Frei, and Peter L Bartlett. Trained transformers learn linear models in-context. *Journal of Machine Learning Research*, 25(49):1–55, 2024.

Yufeng Zhang, Fengzhuo Zhang, Zhuoran Yang, and Zhaoran Wang. What and how does in-context learning learn? bayesian model averaging, parameterization, and generalization. *arXiv preprint arXiv:2305.19420*, 2023.

Appendix A: Training details

Additional training information: Like Garg et al. (2022), we use also the Adam optimizer Diederik (2014) , and a learning rate of 10^{-4} for all models.
Computational resources: We used 1 GPU Nvidia Volta (V100 - 7,8 Tflops DP) for every training involved in these experiments.

Appendix B: Table of error progression for models trained on $N(0,1)$ distributions tested on $N(0,\sigma)$

models / σ	1	2	3	4	5	6	7	8	9	10
1L1AH $d_{embedding}$=64	0.1	0.8	5.1	13.1	26.9	39.7	53.0	84.8	120.0	153.2
1L2AH $d_{embedding}$=64	0.1	0.8	5.3	14.4	29.8	41.1	55.0	93.8	120.4	159.2
1L4AH $d_{embedding}$=64	0.0	0.2	2.7	8.7	19.9	32.0	42.8	64.5	92.3	131.2
2L1AH $d_{embedding}$=64	0.0	0.1	2.0	4.9	13.7	27.0	36.1	64.9	99.0	134.0
2L2AH $d_{embedding}$=64	0.0	0.0	1.6	3.2	9.3	25.5	32.0	61.1	92.9	127.8
2L4AH $d_{embedding}$=64	0.0	0.0	0.9	2.6	7.5	19.3	27.3	51.8	90.2	119.4
3L1AH $d_{embedding}$=64	0.0	0.0	0.9	3.0	8.2	16.8	24.4	48.4	76.7	113.2
3L2AH $d_{embedding}$=64	0.0	0.0	0.7	2.3	6.5	15.9	22.5	43.1	74.0	102.5
3L4AH $d_{embedding}$=64	0.0	0.0	0.6	1.9	5.5	13.8	20.4	42.2	70.3	100.4
6L4AH $d_{embedding}$=64	0.0	0.0	0.5	1.6	4.6	11.6	16.8	33.7	58.3	87.9
12L8AH $d_{embedding}$=256	0.0	0.0	0.3	1.1	2.9	7.9	11.9	28.3	46.9	73.5
REF: y=0	2.19	7.05	19.22	33.94	52.23	73.08	86.02	127.43	165.27	199.31

Table 3: Comparison to show the evolution of squared ϵ type error depending on the distribution according to which we take the parameters, without taking into account the error of the prediction of the first and second prompts. $D_i^t = N(0,1)$

Appendix C: Failure to generalize to longer prompt sequences

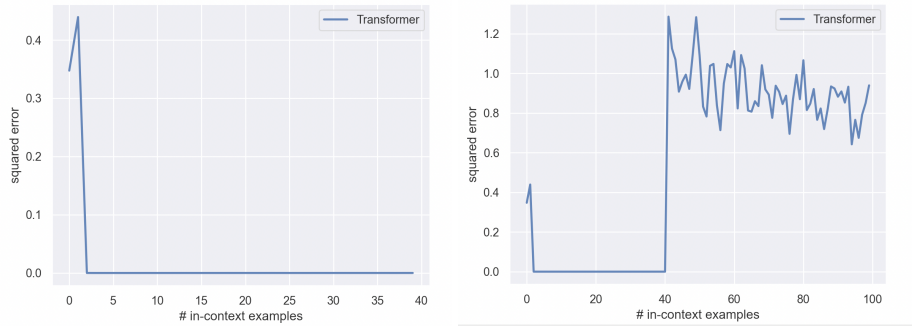

Figure 5: Plot of ICL for $f(x) = x$ with $D_F = D_I = D_I^t = U(-5,5)$ for the model 12L8AH; the one on the left is a zoom in on the first 40 points, where we see that models can often learn from 2 points, the second a view of what happens overall, when models are trained on sequences of length 41 prompts.

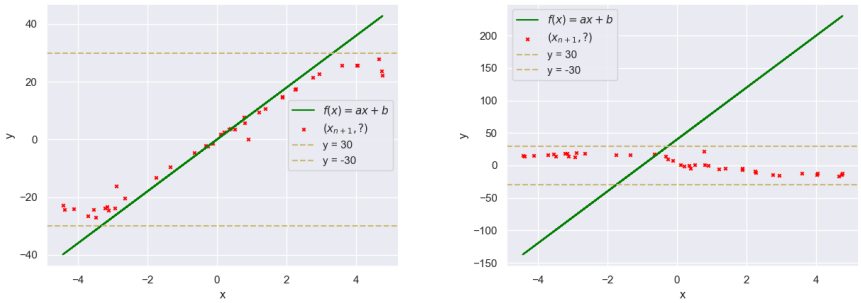

Figure 6: Plots for $f(x) = 9x$ and $f(x) = 40x + 40$ for a 12l8ah model trained on $U(-5, 5)$

Appendix D: Plots for boundary values with $U(-5, 5)$

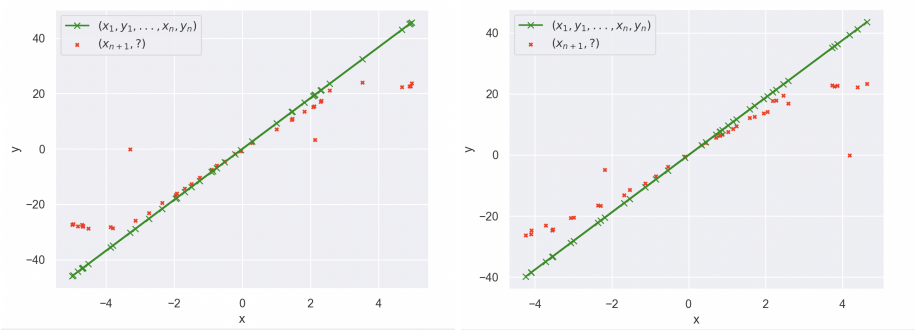

Figure 7: Boundary values: Plots for $f(x) = 9.4x$ for models 3L4AH and 6L4AH, $D_I = D_F = D_I^t = D_F^t = U(-5, 5)$

Appendix E: Example of boundary values for attention only models

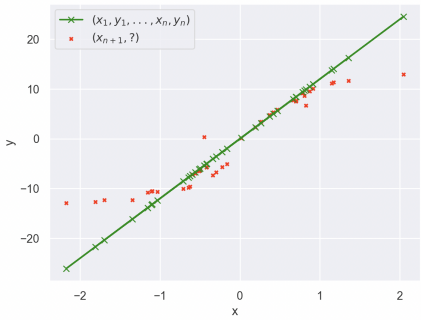

Figure 8: Boundary values for 2L32ah attention only model, with $d_{embedding} = 256$ to ICL the function $f(x) = 12x$

models / σ	1	2	3	4	5	6	7	8	9	10
$1L1AH_N$ $d_{embedding}$=64	48.8	57.62	73.48	84.51	116.63	129.52	142.34	177.69	191.05	246.43
$2L8AH_N$ $d_{embedding}$=64	2.24	4.81	5.8	7.19	10.01	19.04	30.22	38.03	73.32	118.89
$2L32AH_N$ $d_{embedding}$=256	1.17	2.64	3.47	5.01	7.88	16.85	24.1	40.98	66.04	95.03
REF: y=0	2.19	7.05	19.22	33.94	52.23	73.08	86.02	127.43	165.27	199.31

Table 4: Comparison to show the evolution of squared ϵ type error depending on the distribution according to which we take the parameters, without taking into account the error of the prediction of the first and second prompts. $D_F = D_I = D_i^t = N(0,1)$ for models with attention ONLY

Appendix F: The model searches for a sequence close to the input sequence.

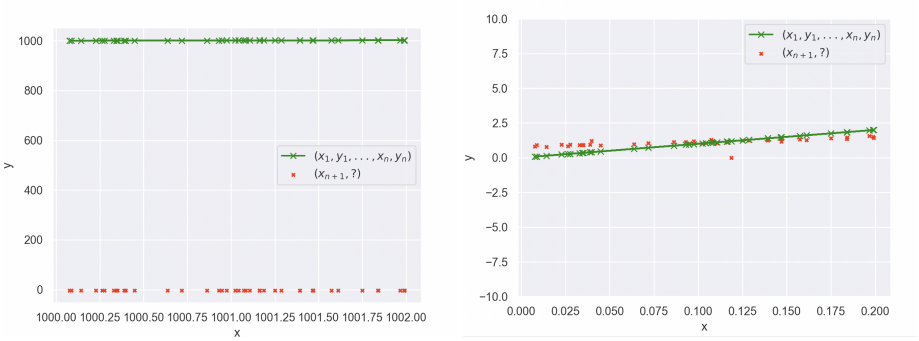

Figure 9: Plots model 12L8AH, trained on $D_I = D_F = N(0,1)$ for $f(x) = x$ for high values of x and $f(x) = 10x$ for low values of x